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Discrete-Time $\mathcal{H}_2$ Guaranteed Cost Control

by

Richard Anthony Conway

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in

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in the

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of the

University of California, Berkeley

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Discrete-Time $H_2$ Guaranteed Cost Control

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Abstract

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Professor Roberto Horowitz, Chair

In this dissertation, we first use the techniques of guaranteed cost control [28] to derive an upper bound on the worst-case $\mathcal{H}_2$ performance of a discrete-time LTI system with causal unstructured norm-bounded dynamic uncertainty. This upper bound, which we call the $\mathcal{H}_2$ guaranteed cost of the system, can be computed either by solving a semi-definite program (SDP) or by using an iteration of discrete algebraic Riccati equation (DARE) solutions. We give empirical evidence that suggests that the DARE approach is superior to the SDP approach in terms of the speed and accuracy with which the $\mathcal{H}_2$ guaranteed cost of a system can be determined.

We then examine the optimal full information $\mathcal{H}_2$ guaranteed cost control problem, which is a generalization of the state feedback control problem in which the $\mathcal{H}_2$ guaranteed cost is optimized. First, we show that this problem can either be solved using an SDP or, under three regularity conditions, by using an iteration of DARE solutions. We then give empirical evidence that suggests that the DARE approach is superior to the SDP approach in terms of the speed and accuracy with which we can solve the optimal full information $\mathcal{H}_2$ guaranteed cost control problem.

The final control problem we consider in this dissertation is the output feedback $\mathcal{H}_2$ guaranteed cost control problem. This control problem corresponds to a nonconvex optimization problem and is thus “difficult” to solve. We give two heuristics for solving this optimization problem. The first heuristic is based entirely on the solution of SDPs whereas the second heuristic exploits DARE structure to reduce the number of complexity of the SDPs that must be solved. The remaining SDPs that must be solved for the second heuristic correspond to the design of filter gains for an estimator.

To show the effectiveness of the output feedback control design heuristics, we apply them to the track-following control of hard disk drives. For this example, we show that the heuristic that exploits DARE structure achieves slightly better accuracy and is more than 90 times faster than the heuristic that is based entirely on SDP solutions.

Finally, we mention how the results of this dissertation extend to a number of other
system types, including linear periodically time-varying systems, systems with structured uncertainty, and finite horizon linear systems.
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Chapter 1

Introduction

The $\mathcal{H}_2$ norm has long been the most widely-used measure of performance for stable discrete-time LTI systems. There are two reasons why this is the case. From a computational standpoint, the $\mathcal{H}_2$ norm is easy to calculate because it only requires the solution of a single discrete Lyapunov equation and standard matrix manipulations. From an intuitive standpoint, the squared $\mathcal{H}_2$ norm of an LTI system can be interpreted as the trace of the steady-state system output covariance under the assumption that the system is driven by zero-mean uncorrelated white Gaussian noise with unit covariance. Since many disturbances of interest can be modeled as Gaussian noise (either white or filtered), this makes the $\mathcal{H}_2$ norm a particularly useful measure of performance when the system and its disturbances are well-characterized.

However, it is often the case that the system and/or its disturbances are not well-characterized. In this case, it is customary to model the uncertainty in the system model and express the resulting model as a linear fractional transformation (LFT) of a known state space system and an unknown transfer function with a $\mathcal{H}_\infty$ norm bound which represents the uncertainty in the model.

In this framework, for LTI systems, we are interested in determining the worst-case $\mathcal{H}_2$ performance of the discrete-time system over all modeled uncertainty. In general, the unknown part of the system could have some structure, such as in $\mu$-synthesis. Necessary and sufficient conditions for robust generalized $\mathcal{H}_2$ performance over linear time-varying (LTV) uncertainty are derived in the frequency domain in [26]. The resulting conditions need to be checked at every frequency (or at least a fine grid of frequencies) and then integrated across frequency. In that paper, these conditions are then extended to state space systems and the resulting optimization problem is reduced to a convex optimization problem involving a finite number of linear matrix inequalities (LMIs). However, in both of these approaches, there is a significant amount of conservatism that arises because they do not make any assumptions on the causality of the unknown part of the system.

A related approach for guaranteeing robust performance of a system over model uncertainty is the approach used in guaranteed cost control [28]. The analysis results of this
framework are different than the previous framework in two ways. First, the guaranteed
cost control approach analyzes performance in the time domain instead of in the frequency
domain. Second, the techniques used in [28] only apply to systems with parametric uncer-
tainty.

After reviewing some basic mathematical ideas in Chapter 2, we use the techniques of
guaranteed cost control in Chapter 3 to derive an upper bound on the robust $H_2$ performance
of a discrete-time LTI system over a class of unstructured dynamic uncertainty. We refer
to this upper bound as the $H_2$ guaranteed cost of a system. The $H_2$ guaranteed cost of a
system is formulated in terms of the solution of a semi-definite program (SDP). We then
give empirical evidence that the speed and accuracy with which we can determine the $H_2$
guaranteed cost can be improved by exploiting the solutions of discrete algebraic Riccati
equations (DAREs). In Chapter 4, we show that the optimal full information $H_2$ guaranteed
cost control problem (i.e. a generalization of the state feedback $H_2$ guaranteed cost control
problem) can be solved using an SDP. As we did for the problem of computing the $H_2$
guaranteed cost of a system, we then give empirical evidence that the speed and accuracy
with which we can solve the optimal full information control problem can be improved by
exploiting the solutions of DAREs. In Chapter 5, we give two heuristics for designing output
feedback controllers that optimize the closed-loop $H_2$ guaranteed cost. The first heuristic is
based on the solution of SDPs and the second heuristic exploits the solution of DAREs. In
Chapter 6, we consider the application of the output feedback control design heuristics to
the design of a controller for a track-following hard disk drive controller. For this example,
we show that the heuristic that exploits DARE structure achieves slightly better accuracy
and is more than 90 times faster than the heuristic that is based entirely on SDP solutions.
Finally, in Chapter 7, we make some concluding remarks—including how the results of
this dissertation extend to linear periodically time-varying systems, systems with structured
uncertainty, and linear finite horizon systems—and mention some areas of future work.

1.1 Notation and Preliminaries

This section will detail the notation we will use throughout the paper. The identity matrix
of dimension $n$ will be denoted as $I_n$. The $m \times n$ zero matrix will be denoted as $0_{m \times n}$.
We will often omit the subscript of these quantities when the dimensions are clear from the
context.

The trace and determinant of a square matrix $S$ will be denoted respectively as $\text{tr}(S)$
and $\det(S)$. The kernel (i.e. null space) and image (i.e. range space) of the matrix $M$ will be
respectively denoted as $\text{Ker}(M)$ and $\text{Im}(M)$. The adjoint of $M$ (i.e. the complex conjugate
transpose of $M$) will be denoted as $M^*$. The Moore–Penrose pseudoinverse of $M$ will be
denoted as $M^\dagger$. Note that if $M = U \Sigma V^*$ where $\Sigma$ is invertible, $U^* U = I$, and $V^* V = I$, 

then

\[ M^\dagger = V \Sigma^{-1} U^*. \]  \hspace{1cm} (1.1)

We will denote the spectral norm (i.e. the maximum singular value) and the Frobenius norm of \( M \) respectively as \( \| M \| \) and \( \| M \|_F \). Note that the Frobenius norm can be expressed as

\[ \| M \|_F^2 = \text{tr}\{ M^* M \} = \text{tr}\{ M M^* \}. \]  \hspace{1cm} (1.2)

For two matrices \( M \) and \( N \) that have the same dimensions, we will use the inner product

\[ \langle M, N \rangle := \text{tr}\{ M^* N \}. \]  \hspace{1cm} (1.3)

The Hadamard product (i.e. element-wise multiplication) of \( M \) and \( N \) will be denoted as \( M \circ N \) and the operator “sum” will take the sum of all elements of a matrix. With this notation, note that

\[ \langle M, N \rangle = \text{sum}( \bar{M} \circ N) \]  \hspace{1cm} (1.4)

where \( \bar{M} \) is the complex conjugate of \( M \). The operator “\( \text{diag}\)” takes several matrices and stacks them diagonally:

\[ \text{diag}(M_1, \ldots, M_n) = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_n \end{bmatrix}. \]  \hspace{1cm} (1.5)

We will say that the matrix \( M \) is Schur (resp. anti-Schur) if all of its eigenvalues lie strictly inside (resp. strictly outside) the unit disk in the complex plane. Positive definiteness (resp. semi-definiteness) of a real symmetric matrix \( X \) will be denoted by \( X \succ 0 \) (resp. \( X \succeq 0 \)), and a • in a real symmetric matrix will represent a block which follows from symmetry.

A matrix pair \( (A, B) \) will be called stabilizable if \( \exists K \) such that \( A + BK \) is Schur. A matrix pair \( (C, A) \) will be called d-detectable if \( \exists L \) such that \( A + LC \) is Schur. Consider the LTI discrete-time system \( G \). We will write its state-space realization as

\[ G \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \]  \hspace{1cm} (1.6)

This realization will be called stabilizable if \( (A, B) \) is stabilizable; it will be called d-detectable if \( (C, A) \) is d-detectable. If \( G \) is stable and causal, we will denote its \( H_2 \) and \( H_\infty \) norms respectively as \( \| G \|_2 \) and \( \| G \|_\infty \). For two systems \( G_1 \) and \( G_2 \), we denote the lower linear fractional transformation (LFT) of \( G_1 \) by \( G_2 \) (shown in Fig. 1.1a) as \( \mathcal{F}_l(G_1, G_2) \). We will denote the upper LFT of \( G_1 \) by \( G_2 \) (shown in Fig. 1.1b) as \( \mathcal{F}_u(G_1, G_2) \). Partitioning \( G_1 \) as

\[ G_1 = \begin{bmatrix} G_{11}^{11} & G_{12}^{11} \\ G_{11}^{21} & G_{12}^{22} \end{bmatrix}. \]  \hspace{1cm} (1.7)
we will say that $F_l(G_1, G_2)$ (resp. $F_u(G_1, G_2)$) is well-posed if $I - G_1^{22}G_2$ (resp. $I - G_1^{11}G_2$) is invertible where $I$ is the identity operator.

The end of a mathematical proof will be denoted by the symbol ■. The symbol □ will be used to denote the end of formal remarks, algorithms, and formal mathematical statements (i.e. theorems, lemmas, corollaries, and propositions) whose proofs are omitted.

In an effort to simplify notation as much as possible, we will attach scope to the notation in this dissertation. The scope of all notation defined within a formal mathematical statement (i.e. a theorem, lemma, corollary, or proposition) or its proof will be limited to that mathematical statement and its proof. This means, for example, that notation defined in the proof of one theorem is independent from notation defined in later theorems or notation defined later in the chapter. The notation we introduce in this chapter (outside of formal mathematical statements) will be used throughout this dissertation. The scope of the notation defined in each of the chapters that follow will be limited to that respective chapter; i.e. the notation defined in any chapter is independent from notation defined in later chapters.
Chapter 2
Mathematical Preliminaries

In this chapter, we will present a number of basic mathematical concepts that will be used throughout the dissertation. Although there is a lot of literature for each one of the topics considered in this chapter, we will only present the absolute minimum amount of material for each topic. For the sake of completeness, we will prove the statements that are easily proven. The material in §§2.4–2.5 can be found, for example, in [21].

2.1 Schur Complements

Consider the matrix
\[
\hat{M} := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.
\] (2.1)

The quantity \( M_{11} - M_{12}M_{22}^{-1}M_{21} \) is called the Schur complement of \( M_{22} \) in the matrix \( \hat{M} \). Also, the quantity \( M_{22} - M_{21}M_{11}^{-1}M_{12} \) is called the Schur complement of \( M_{11} \) in the matrix \( \hat{M} \). We start with a useful property for Hermitian \( \hat{M} \).

Proposition 2.1.1. The following are equivalent

1. \( \hat{M} \succ 0 \)
2. \( M_{22} \succ 0 \) and \( M_{11} - M_{12}M_{22}^{-1}M_{12}^* \succ 0 \)
3. \( M_{11} \succ 0 \) and \( M_{22} - M_{12}^*M_{11}^{-1}M_{12} \succ 0 \).

Since \( \hat{M} \) trivially implies that \( M_{11} \succ 0 \) and \( M_{22} \succ 0 \), this is a special case of Corollary 2.3.2, which will be proven in §2.3. Thus, we will not give its proof here. We now give a result regarding matrix inverses, which is sometimes referred to as the quotient property of Schur complements [7].
Proposition 2.1.2. Let $M_{33}$ be an invertible matrix and define

\[
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} := \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} - \begin{bmatrix}
M_{13} & M_{31} \\
M_{23} & M_{33}
\end{bmatrix} M_{33}^{-1} \begin{bmatrix}
M_{31} & M_{32}
\end{bmatrix}.
\] (2.2)

Then

\[
M_{11} - [M_{12} M_{13}] M_{22} [M_{22} M_{33}]^{-1} [M_{21} M_{31}] = \tilde{M}_{11} - \tilde{M}_{12} \tilde{M}_{22}^{-1} \tilde{M}_{21}
\] (2.3)

whenever either side of the equation is well-defined.

Proof. Since

\[
\begin{bmatrix}
I & -M_{23} M_{33}^{-1} \\
0 & I
\end{bmatrix} \begin{bmatrix}
M_{22} & M_{23} \\
M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
I & 0 \\
-M_{33}^{-1} M_{32} & I
\end{bmatrix} = \begin{bmatrix}
\tilde{M}_{22} & 0 \\
0 & M_{33}
\end{bmatrix}
\] (2.4)

we see that the matrix $\begin{bmatrix}
M_{22} & M_{23} \\
M_{32} & M_{33}
\end{bmatrix}$ is invertible if and only if $\tilde{M}_{22}$ is invertible. It is then straightforward to verify that

\[
\begin{bmatrix}
M_{22} & M_{23} \\
M_{32} & M_{33}
\end{bmatrix}^{-1} = \begin{bmatrix}
I & -M_{23} M_{33}^{-1} \\
-M_{33}^{-1} M_{32} & I
\end{bmatrix} \tilde{M}_{22}^{-1} \begin{bmatrix}
I & -M_{23} M_{33}^{-1} \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & M_{33}^{-1}
\end{bmatrix}.
\] (2.5)

Plugging (2.5) into the left-hand side of (2.3) completes the proof.

In the previous proposition, we showed that the Schur complement of $\begin{bmatrix}
M_{22} & M_{23} \\
M_{32} & M_{33}
\end{bmatrix}$ in the matrix

\[
\tilde{M} := \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\] (2.6)

is the same as the Schur complement of $\tilde{M}_{22}$ in the matrix

\[
\tilde{M} := \begin{bmatrix}
\tilde{M}_{11} & \tilde{M}_{12} \\
\tilde{M}_{21} & \tilde{M}_{22}
\end{bmatrix} = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} - \begin{bmatrix}
M_{13} & M_{31} \\
M_{23} & M_{33}
\end{bmatrix} M_{33}^{-1} \begin{bmatrix}
M_{31} & M_{32}
\end{bmatrix}.
\] (2.7)

Note that $\tilde{M}$ is obtained by taking the Schur complement of $M_{33}$ in $\tilde{M}$. This means that we can evaluate expressions of the form in the left-hand side of (2.3) by taking successive Schur complements. In particular, we first take the Schur complement of $M_{33}$ in $\tilde{M}$ to obtain $\tilde{M}$ and then take the Schur complement of $\tilde{M}_{22}$ in $\tilde{M}$ to yield the left-hand side of (2.3).

A special case of Proposition 2.1.2 is the matrix inversion lemma [34].
Proposition 2.1.3 (Matrix Inversion Lemma). If $A$ and $D$ are invertible, then

$$(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}. \quad (2.8)$$

Proof. Note that

$$[-I \ 0] \begin{bmatrix} A & B \\ C & -D^{-1} \end{bmatrix}^{-1} [I \ 0] = [-I \ 0] \left( \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} -D^{-1} & C \\ B & A \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right)^{-1} [I \ 0] \quad (2.9)$$

$$\Rightarrow -[-I \ 0] \begin{bmatrix} A & B \\ C & -D^{-1} \end{bmatrix}^{-1} [I \ 0] = -[0 \ -I] \begin{bmatrix} -D^{-1} & C \\ B & A \end{bmatrix}^{-1} [0 \ I]. \quad (2.10)$$

Applying Proposition 2.1.2 to both sides of this expression finishes the proof. ■

2.2 Semi-Definite Programming

A semi-definite program is a optimization problem that is written as

$$\min_x \langle c, x \rangle \quad \text{s.t.} \quad \sum_{j=1}^{n} x_j F_{ij} \succeq G_i, \quad i = 1, \ldots, m \quad (2.11)$$

where $x = [x_1 \ \ldots \ x_n]^* \in \mathbb{R}^n$, $c = [c_1 \ \ldots \ c_n]^* \in \mathbb{R}^n$, $F_{ij} \in \mathbb{R}^{n_i \times n_i}$, and $G_i \in \mathbb{R}^{n_i \times n_i}$. These are convex optimization problems and can be solved using a variety of solvers, including SeDuMi [30], which uses a primal-dual path-following algorithm, and the mincx command in the Robust Control Toolbox for MATLAB, which uses a projective method. It should be noted that this description of an SDP could be generalized by adding affine equality constraints to (2.11). However, since we do not encounter problems of that form in this dissertation, we omit equality constraints for simplicity.

The Lagrangian dual of (2.11) is

$$\max_{Z_1, \ldots, Z_m} \sum_{i=1}^{m} \langle Z_i, G_i \rangle \quad \text{s.t.} \quad \sum_{i=1}^{m} \langle Z_i, F_{ij} \rangle = c_j, \quad j = 1, \ldots, n \quad \text{and} \quad Z_i \succeq 0, \quad i = 1, \ldots, m \quad (2.12)$$

where $Z_i \in \mathbb{R}^{n_i \times n_i}$. In many cases, (2.11) and (2.12) have the same value (see, e.g., §5.9 of [4]). Primal-dual path-following algorithms exploit this fact and attempt to solve both
problems simultaneously, which results in the algorithm simultaneously optimizing
\[ n + \sum_{i=1}^{m} \frac{n_i(n_i + 1)}{2} \] (2.13)
parameters. Thus, in addition to the number of optimization parameters in (2.11), the sizes of the constraints in (2.11) also play a role in determining how quickly primal-dual path-following algorithms are able to solve (2.11).

In this dissertation, we will often encounter optimization problems of the form
\[
\inf_{x} \langle c, x \rangle \\
\text{s.t.} \sum_{j=1}^{n} x_j F_{ij} \succ G_i, \quad i = 1, \ldots, m .
\] (2.14)

Although this is still a convex optimization problem, it is not an SDP because the constraints are strict. However, in many cases, it is equivalent to solving an SDP. The following proposition examines this relationship.

**Proposition 2.2.1.** If the optimization problem (2.14) is feasible, then it has the same value as (2.11). If instead (2.11) is infeasible, then (2.14) is also infeasible.

**Proof.** Define the sets
\[
X_0 := \left\{ x : \sum_{j=1}^{n} x_j F_{ij} \succeq G_i, \; i = 1, \ldots, m \right\}, \quad X_1 := \left\{ x : \sum_{j=1}^{n} x_j F_{ij} \succ G_i, \; i = 1, \ldots, m \right\}
\] (2.15)

Note that if \( X_0 = \emptyset \), then \( X_1 = \emptyset \). Thus, if (2.11) is infeasible, then (2.14) must also be infeasible. Now suppose that \( X_1 \neq \emptyset \). In this case, it can be shown that \( X_0 \) is the closure of \( X_1 \). In this case, we see that
\[
\inf_{x \in X_0} \langle c, x \rangle = \min_{x \in X_1} \langle c, x \rangle .
\] (2.16)

Since the left-hand and right-hand sides of this equality are respectively the values of (2.11) and (2.14), this concludes the proof.

Another useful property of (2.14) is given in the following remark.

**Remark 2.2.2.** If \( c \neq 0 \) and \( \bar{x} \) is feasible for (2.14), then \( \bar{x} \) can be perturbed in such a way that the cost function decreases while maintaining feasibility. Thus, \( \langle c, \bar{x} \rangle \) is strictly greater than the value of (2.14).  \(\square\)
CHAPTER 2. MATHEMATICAL PRELIMINARIES

The follow two propositions consider two simple feasibility problems that we will encounter in this dissertation.

**Proposition 2.2.3.** Let $M_{12}(V)$ be a matrix function of $V$. Then $\exists W$ and $V$ such that
\[
\begin{bmatrix}
M_{11} & M_{12}(V) \\
W - M_{22}
\end{bmatrix} \succ 0 \text{ if and only if } M_{11} \succ 0.
\]

**Proof.** ($\Rightarrow$) Trivial.

($\Leftarrow$) Let $M_{11} \succ 0$ and choose any value of $V$. By Schur complements, we see that
\[
W = I + M_{22} + M_{12}(V)^*M_{11}^{-1}M_{12}(V)
\]
satisfies the matrix inequality. ■

**Proposition 2.2.4.** $\exists K$ such that
\[
M_0 + M_L^*KM_R + M_R^*K^*M_L \succ 0 \quad (2.17)
\]
if and only if
\[
N_L^*M_0N_L \succ 0, \quad N_R^*M_0N_R \succ 0 \quad (2.18)
\]
where the columns of $N_R$ and $N_L$ respectively form bases for $\text{Ker}(M_R)$ and $\text{Ker}(M_L)$.

The idea of Proposition 2.2.4 originates in [8] and is now considered a standard result in semi-definite programming (see, e.g., [3]). This proposition is frequently used to eliminate matrix variables from SDPs with the proper structure. For example, if $M_0(x)$ is an affine function of the vector $x$, the SDP
\[
\begin{align*}
\min_{x,K} & \quad \langle c, x \rangle \\
\text{s.t.} & \quad M_0(x) + M_L^*KM_R + M_R^*K^*M_L \succ 0
\end{align*}
\quad (2.19)
\]
is equivalent to the SDP
\[
\begin{align*}
\min_x & \quad \langle c, x \rangle \\
\text{s.t.} & \quad N_L^*M_0(x)N_L \succ 0, \quad N_R^*M_0(x)N_R \succ 0
\end{align*}
\quad (2.20)
\]
where the columns of $N_R$ and $N_L$ respectively form bases for $\text{Ker}(M_R)$ and $\text{Ker}(M_L)$. Note that although (2.19) has fewer optimization parameters than (2.20), (2.20) has one more constraint than (2.19). Although each of the two constraints in (2.20) are smaller than the single constraint in (2.19), it is sometimes the case that (2.20) has more dual optimization parameters than (2.19). Therefore, the technique of matrix variable elimination sometimes increases the number of parameters being optimized by primal-dual path-following algorithms.

Although there are many factors that affect the amount of computation required to solve an SDP (e.g. the sparsity and structure of the $F_{ij}$ matrices), the number of parameters being optimized by a solver is a good indicator of how much computation will be required to solve the optimization problem. Thus, it is possible that using Proposition 2.2.4 to eliminate a matrix optimization parameter from an SDP will increase the amount of computation required to solve that SDP. In this dissertation, we only use the matrix variable elimination technique when it decreases the overall number of parameters that will be optimized by primal-dual path-following algorithms.
2.3 Inertia of Matrices

For a symmetric matrix, $X$, we define the functions $\nu_+(X)$, $\nu_0(X)$, and $\nu_-(X)$ to respectively be the number of positive, zero, and negative eigenvalues of $X$ counted with multiplicity. The inertia of the symmetric matrix $X$ is then defined as

$$\mathcal{N}(X) := (\nu_+(X), \nu_0(X), \nu_-(X)).$$  \hfill (2.21)

The following result is the fundamental result on inertia (e.g., [17]).

**Proposition 2.3.1 (Sylvester’s inertia theorem).** Let $X$ be a symmetric matrix and $M$ be an invertible matrix. Then $\mathcal{N}(X) = \mathcal{N}(M^*XM)$. \hfill \Box

Based on this result, we have the following corollary [15], which will be useful in §4.3.

**Corollary 2.3.2.** Let $X_{11}$ and $X_{22}$ be symmetric matrices and define $X := \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}$. Then each of the equations

$$\mathcal{N}(X) = \mathcal{N}(X_{11} - X_{12}X_{22}^{-1}X_{12}^*) + \mathcal{N}(X_{22})$$ \hfill (2.22)

$$\mathcal{N}(X) = \mathcal{N}(X_{22} - X_{12}^*X_{11}^{-1}X_{12}) + \mathcal{N}(X_{11})$$ \hfill (2.23)

hold when the relevant inverses exist.

**Proof.** Define $\Psi_1 := X_{11} - X_{12}X_{22}^{-1}X_{12}^*$, $\Psi_2 := X_{22} - X_{12}^*X_{11}^{-1}X_{12}$,

$$M_1 := \begin{bmatrix} I & 0 \\ -X_{22}^{-1}X_{12}^* & I \end{bmatrix}, \quad M_2 := \begin{bmatrix} I & -X_{11}^{-1}X_{12} \\ 0 & I \end{bmatrix}$$ \hfill (2.24)

and note that $M_1$ and $M_2$ are invertible. Thus, by Proposition 2.3.1,

$$\mathcal{N}(X) = \mathcal{N}(M_1^*XM_1) = \mathcal{N}(\text{diag}(\Psi_1, X_{22})) = \mathcal{N}(\Psi_1) + \mathcal{N}(X_{22})$$ \hfill (2.25)

$$\mathcal{N}(X) = \mathcal{N}(M_2^*XM_2) = \mathcal{N}(\text{diag}(X_{11}, \Psi_2)) = \mathcal{N}(\Psi_2) + \mathcal{N}(X_{11})$$ \hfill (2.26)

which concludes the proof. \hfill \Box

2.4 Linear Matrix Pencils

In this section, we review some basic theory on linear matrix pencils. These results are drawn from [21].

For $M, N \in \mathbb{C}^{n \times n}$, the family of matrices $\lambda N - M$ for $\lambda \in \mathbb{C}$ is called a linear matrix pencil. With an abuse of terminology, we will refer to them simply as matrix pencils. We will denote the matrix pencil $\lambda N - M$ as $(M, N)$. We define the characteristic polynomial of
the matrix pencil \((M, N)\) to be the polynomial \(\theta_{(M, N)}(\lambda) := \det(\lambda N - M)\). If \(\exists \lambda \in \mathbb{C} \) such that \(\theta_{(M, N)}(\lambda) \neq 0\), we say that the matrix pencil \((M, N)\) is regular.

For a regular matrix pencil, we will call the roots of \(\theta_{(M, N)}(\lambda)\) the finite eigenvalues of the matrix pencil \((M, N)\). When \(N\) is invertible, we have that

\[
\theta_{(M, N)}(\lambda) = \det(\lambda N - M) = \det(N) \det(\lambda I - N^{-1} M)
\]

(2.27)

which implies that the finite eigenvalues of \((M, N)\) are the same as the eigenvalues of the matrix \(N^{-1}M\). However, in this paper, we will not make any assumptions about the invertibility of \(N\). Thus, it is possible for \(\deg(\theta_{(M, N)}(\lambda)) < n\), where the “deg” operator returns the degree of a polynomial. In this case, we say that \((M, N)\) has an eigenvalue at \(\infty\) with multiplicity \(n - \deg(\theta_{(M, N)}(\lambda))\). With this convention, regular matrix pencils will always have exactly \(n\) eigenvalues, counting multiplicities. We will denote the set of eigenvalues of the regular matrix pencil \((M, N)\) as \(\text{Spec}(M, N)\). Note that \(\infty \in \text{Spec}(M, N)\) when \(\deg(\theta_{(M, N)}(\lambda)) < n\).

We will say that the matrix pencils \((M_1, N_1)\) and \((M_2, N_2)\) are strictly equivalent (written \((M_1, N_1) \sim (M_2, N_2)\)) if there exist invertible \(T\) and \(U\) such that \((M_1, N_1) = (TM_2U, TN_2U)\). Note that strict equivalence is an equivalence relation. Also, since

\[
\theta_{(M_1, N_1)}(\lambda) = \det(\lambda N_1 - M_1) = \det(\lambda TN_2U - TM_2U) = \det(T) \det(U) \det(\lambda N_2 - M_2) = \det(TU) \theta_{(M_2, N_2)}(\lambda)
\]

(2.28)

we see that the characteristic polynomial of \((M_1, N_1)\) is a nonzero multiple of the characteristic polynomial of \((M_2, N_2)\). Thus, \((M_1, N_1)\) is regular if and only if \((M_2, N_2)\) is regular. Moreover, strictly equivalent regular matrix pencils have the same eigenvalues and their respective multiplicities.

We will say that a subspace \(S\) is deflating for the matrix pencil \((M, N)\) if \(\dim(MS + NS) \leq \dim(S)\). Note that \(\{0\}\) is trivially a deflating subspace for any matrix pencil. We now present a fundamental result for deflating subspaces.

**Proposition 2.4.1.** The subspace \(S \neq \{0\}\) is a deflating subspace for the matrix pencil \((M, N)\) if and only if there exist nonsingular \([X_1 \quad X_2]\) and \([Y_1 \quad Y_2]\) such that \(\text{Im}(X_1) = S\) and

\[
\begin{bmatrix}
Y_1^* \\
Y_2^*
\end{bmatrix}
M
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= 
\begin{bmatrix}
M_{11} & M_{12} \\
0 & M_{22}
\end{bmatrix},
\begin{bmatrix}
Y_1^* \\
Y_2^*
\end{bmatrix}
N
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= 
\begin{bmatrix}
N_{11} & N_{12} \\
0 & N_{22}
\end{bmatrix}
\]

(2.29)

where \(M_{11}, N_{11} \in \mathbb{C}^{\dim(S) \times \dim(S)}\).

**Proof.** \((\Rightarrow)\) Let \(S\) be a deflating subspace and let \(T\) be any subspace such that \(MS + NS \subseteq T\) and \(\dim(S) = \dim(T)\). Choose \(X_1\) and \(Z_1\) so that their columns respectively form a basis for
$S$ and $T$ and then choose $X_2$ and $Z_2$ so that $[X_1 \ X_2]$ and $[Z_1 \ Z_2]$ are invertible. Define $[Y_1 \ Y_2]^* := [Z_1 \ Z_2]^{-1}$ and note that

$$I = \begin{bmatrix} Y_1^* & Z_1 \\ Y_2^* & Z_2 \end{bmatrix} \Rightarrow Y_2^* Z_1 = 0.$$  \hspace{1cm} (2.30)

Since $\text{Im}(MX_1) = MS \subseteq T = \text{Im}(Z_1)$, we choose $Q$ so that $MX_1 = Z_1 Q$. Therefore, $Y_2^* MX_1 = Y_2^* Z_1 Q = 0$. Similarly, $Y_2^* NX_1 = 0$.

$(\Leftarrow)$ Defining $[Z_1 \ Z_2] := [Y_1 \ Y_2]^{-*}$ where $Z_1$ has $\dim(S)$ columns, we see that

$$\begin{bmatrix} Y_1^* \\ Y_2^* \end{bmatrix} MX_1 = \begin{bmatrix} M_{11} \\ 0 \end{bmatrix} \Rightarrow MX_1 = [Z_1 \ Z_2] \begin{bmatrix} M_{11} \\ 0 \end{bmatrix} = Z_1 M_{11}. \hspace{1cm} (2.31)$$

Similarly, $NX_1 = Z_1 N_{11}$. Thus, $MS, NS \subseteq \text{Im}(Z_1) \Rightarrow MS + NS \subseteq \text{Im}(Z_1)$. Since $Z_1$ only has $\dim(S)$ columns, this concludes the proof. \hfill \blacksquare

In the previous proposition, it should be noted that knowledge of the deflating subspace $S$ was used to form the strict equivalence

$$\lambda N - M \sim \lambda \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} = \begin{bmatrix} \lambda N_{11} - M_{11} & \lambda N_{12} - M_{12} \\ 0 & \lambda N_{22} - M_{22} \end{bmatrix}. \hspace{1cm} (2.32)$$

This implies that $\theta_{(M,N)}(\lambda) = \alpha \theta_{(M_{11},N_{11})}(\lambda) \theta_{(M_{22},N_{22})}(\lambda)$ for some $\alpha \in \mathbb{C}\{0\}$. Thus, $(M, N)$ is regular if and only if $(M_{11}, N_{11})$ and $(M_{22}, N_{22})$ are regular. Also, we can find the eigenvalues of $(M, N)$ and their multiplicities by examining $(M_{11}, N_{11})$ and $(M_{22}, N_{22})$. In particular, $\text{Spec}(M, N) = \text{Spec}(M_{11}, N_{11}) \cup \text{Spec}(M_{22}, N_{22})$.

**Proposition 2.4.2.** If $\exists \Lambda$ such that $MX = NX\Lambda$, then $\text{Im}(X)$ is deflating for the matrix pencil $(M, N)$. Moreover, if $(M, N)$ is regular and $\text{Ker}(X) = \{0\}$, then $\text{Im}(X) \cap \text{Ker}(N) = \{0\}$.

**Proof.** Let $S = \text{Im}(X)$. First note that $\text{Im}(MX) = \text{Im}(NX\Lambda) \subseteq \text{Im}(NX)$, which implies $MS \subseteq NS \Rightarrow MS + NS = NS \Rightarrow \dim(MS + NS) = \dim(NS) \leq \dim(S)$. Thus, $\text{Im}(X) = S$ is deflating.

Now suppose that $(M, N)$ is regular. Choose $Y$ such that $\text{Im}(Y) = \text{Ker}(N) \cap \text{Im}(X)$. Since $\text{Im}(Y) \subseteq \text{Im}(X)$, choose $Q$ such that $Y = XQ$. Note that $NXQ = NY = 0$. Now choose $\lambda \in \mathbb{C}\{\text{Spec}(M, N) \cup \text{Spec}(\Lambda, I)\}$. Thus,

$$(\lambda N - M) [X(\lambda I - \Lambda)^{-1}Q] = NX(\lambda I - \Lambda)(\lambda I - \Lambda)^{-1}Q = NXQ = 0. \hspace{1cm} (2.33)$$

Since $\lambda \notin \text{Spec}(M, N)$, this implies that $(\lambda I - \Lambda)^{-1}Q = 0$. Since $\text{Ker}(X) = \{0\}$, we see that $(\lambda I - \Lambda)^{-1}Q = 0 \Rightarrow Q = 0 \Rightarrow Y = 0 \Rightarrow \text{Ker}(N) \cap \text{Im}(X) = \{0\}$. \hfill \blacksquare
Remark 2.4.3. Combining the last two propositions, we can say that if \( \exists X, \Lambda \) such that \( M X = N X \Lambda \) and \( \text{Ker}(X) = \{0\} \), then the pencil \( \Lambda N_{11} - M_{11} \) in Proposition 2.4.1 takes the form \( \lambda Y^* N X - Y^* M X = X^* N X (\lambda - \Lambda) \). Therefore, if \((M, N)\) is regular, the eigenvalues of \( \Lambda \) are eigenvalues of \((M, N)\) and their multiplicities in \((M, N)\) are not less than their respective multiplicities in \( \Lambda \). \( \Box \)

We now turn our attention to two special deflating subspaces. We will call \( S \) a Schur deflating subspace of \((M, N)\) if either \( S = \{0\} \) or \( \exists X \) and \( \Lambda \) such that \( M X = N X \Lambda \) where \( \Lambda \) is Schur and the columns of \( X \) are a basis of \( S \). Note that this definition is independent of the choice of basis. We will say that \( S \) is the maximal Schur deflating subspace of \((M, N)\) if \( T \subseteq S \) whenever \( T \) is a Schur deflating subspace of \((M, N)\). Note that a maximal Schur deflating subspace is necessarily unique.

Proposition 2.4.4. If \((M, N)\) is regular, then its maximal Schur deflating subspace exists.

Proof. Let \( S_1 \) and \( S_2 \) be Schur deflating subspaces of \((M, N)\). Since \( M \) and \( N \) are finite dimensional matrices (i.e. there is an upper bound on the dimension of deflating subspaces), it suffices to show that \( S_1 + S_2 \) is a Schur deflating subspace. If \( S_1 \subseteq S_2 \) or \( S_1 \supseteq S_2 \), this is trivial. If \( S_1 \cap S_2 = \{0\} \), let \( M X_1 = N X_1 \Lambda_1 \) and \( M X_2 = N X_2 \Lambda_2 \) where \( \Lambda_1 \) and \( \Lambda_2 \) are Schur and \( X_1 \) and \( X_2 \) are respectively bases for \( S_1 \) and \( S_2 \). Since the columns of \([X_1 \ X_2]\) are linearly independent and

\[
M \begin{bmatrix} X_1 & X_2 \end{bmatrix} = N \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}
\] (2.34)

we see that \( \text{Im}([X_1 \ X_2]) = S_1 + S_2 \) is a Schur deflating subspace.

Thus, it only remains to examine the case when \( S_1 \not\subseteq S_2, S_1 \not\supseteq S_2, \) and \( S_1 \cap S_2 \neq \{0\} \). Let \( X_0 \) be a basis for \( S_1 \cap S_2 \) and choose \( X_1 \) and \( X_2 \) so that \([X_0 \ X_1]\) and \([X_0 \ X_2]\) are respectively bases for \( S_1 \) and \( S_2 \). Note that the columns of \( X := [X_0 \ X_1 \ X_2] \) form a basis for \( S_1 + S_2 \). Now let

\[
M \begin{bmatrix} X_0 & X_1 \end{bmatrix} = N \begin{bmatrix} X_0 & X_1 \end{bmatrix} \begin{bmatrix} \Lambda_{00} & \Lambda_{01} \\ \Lambda_{10} & \Lambda_{11} \end{bmatrix}
\] (2.35)

\[
M \begin{bmatrix} X_0 & X_2 \end{bmatrix} = N \begin{bmatrix} X_0 & X_2 \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_{00} & \bar{\Lambda}_{02} \\ \bar{\Lambda}_{20} & \bar{\Lambda}_{22} \end{bmatrix}
\] (2.36)

where \( \begin{bmatrix} \Lambda_{00} & \Lambda_{01} \\ \Lambda_{10} & \Lambda_{11} \end{bmatrix} \) and \( \begin{bmatrix} \bar{\Lambda}_{00} & \bar{\Lambda}_{02} \\ \bar{\Lambda}_{20} & \bar{\Lambda}_{22} \end{bmatrix} \) are Schur. Since

\[
M \begin{bmatrix} X_0 & X_1 & X_2 \end{bmatrix} = N \begin{bmatrix} X_0 & X_1 & X_2 \end{bmatrix} \begin{bmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} \\ \Lambda_{10} & \Lambda_{11} & 0 \\ 0 & 0 & \Lambda_{22} \end{bmatrix}
\] (2.37)
we see that $S_1 + S_2$ is deflating for $(M, N)$.

To show that $S_1 + S_2$ is a Schur deflating subspace, it only remains to show that $\Lambda_{22}$ is Schur. Applying Proposition 2.4.2, we see that $\ker(N) \cap \text{Im}(X) = \{0\}$. Now we note that

$$N(X_0 \Lambda_{00} + X_1 \Lambda_{10}) = MX_0 = N(X_0 \bar{\Lambda}_{00} + X_2 \Lambda_{20})$$

$$\Rightarrow NX \begin{bmatrix} \Lambda_{00} - \bar{\Lambda}_{00} \\ \Lambda_{10} \\ -\Lambda_{20} \end{bmatrix} = 0 \Rightarrow X \begin{bmatrix} \Lambda_{00} - \bar{\Lambda}_{00} \\ \Lambda_{10} \\ -\Lambda_{20} \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \Lambda_{00} - \bar{\Lambda}_{00} \\ \Lambda_{10} \\ -\Lambda_{20} \end{bmatrix} = 0$$

Using the fact that $\Lambda_{20} = 0$, we see that $\begin{bmatrix} \Lambda_{00} \\ 0 \\ \Lambda_{22} \end{bmatrix}$ is Schur, which implies that $\Lambda_{22}$ is Schur.

\section{2.5 Discrete Algebraic Riccati Equations}

In this subsection, we give a brief review of discrete algebraic Riccati equations. These types of equations have been useful in the design of linear quadratic regulators, Kalman filters, and many other control and estimation schemes. These results are drawn from [21].

For given real matrices $A$, $B$, $Q = Q^*$, $R = R^*$, and $S$, we define

$$R_{(A,B,Q,R,S)}(P) := A^*PA + Q - (A^*PB + S)(B^*PB + R)^{-1}(B^*PA + S^*)$$

$$K_{(A,B,Q,R,S)}(P) := -(B^*PB + R)^{-1}(B^*PA + S^*)$$

$$A_{(A,B,Q,R,S)}(P) := A - B(B^*PB + R)^{-1}(B^*PA + S^*)$$

$$= A + BK_{(A,B,Q,R,S)}(P)$$

We will make the notation more compact in the remainder of the paper by respectively denoting these quantities as $\mathcal{R}_\phi(P)$, $\mathcal{K}_\phi(P)$, and $\mathcal{A}_\phi(P)$ where $\phi$ is an appropriately defined 5-tuple. The equation $\mathcal{R}_\phi(P) = P$ is called a discrete algebraic Riccati equation (DARE). If $\mathcal{R}_\phi(P) = P$ and $P = P^*$, then $P$ is called a solution of the DARE $\mathcal{R}_\phi(P) = P$. If $P$ is a solution of the DARE $\mathcal{R}_\phi(P) = P$ and $A_{\phi}(P)$ is Schur (resp. anti-Schur), then $P$ is called a stabilizing (resp. anti-stabilizing) solution. It is well-known that if a DARE has a stabilizing solution, then it is unique [21].

We now present a basic result on DAREs that allows us to “shift” the parameters of a DARE so that the last component of the parameter 5-tuple is zero.

\textbf{Proposition 2.5.1.} Let $R$ be invertible and define

$$\phi := (A, B, Q, R, S)$$

$$\tilde{\phi} := (A - BR^{-1}S^*, B, Q - SR^{-1}S^*, R, 0).$$

Then, the equalities $\mathcal{R}_{\tilde{\phi}}(P) = \mathcal{R}_\phi(P)$ and $\mathcal{A}_{\tilde{\phi}}(P) = \mathcal{A}_\phi(P) = (I + BR^{-1}B^*P)^{-1}(A - BR^{-1}S^*)$ hold for all symmetric $P$ such that $B^*PB + R$ is invertible.
Proof. It is straightforward to show that \( R\phi(P) = R\bar{\phi}(P) \) and \( \mathcal{A}\phi(P) = \mathcal{A}\bar{\phi}(P) \). Also, using the matrix inversion lemma, we see that \[
\mathcal{A}\bar{\phi}(P) = [I - B(B^*PB + R)^{-1}B^*P](A - BR^{-1}S^*)
= [I + BR^{-1}B^*P]^{-1}(A - BR^{-1}S^*)
\tag{2.46}
\]
which concludes the proof.

It should be noted that the “shifting” of the DARE parameters in the previous proposition does not affect the existence and values of the DARE solutions. Moreover, since it preserves the value of \( A \), it also does not affect the existence and values of stabilizing and anti-stabilizing DARE solutions.

With this result in place, we now explore the relationship between DAREs and two matrix pencils, one of which has a larger dimension than the other. We will refer to the matrix pencil with larger dimension as the dilated matrix pencil. We first give two results that explore the relationship between the DARE and the non-dilated matrix pencil.

**Proposition 2.5.2.** Let \( R \) be invertible and define \( \phi := (A, B, Q, R, S) \). If \( R\phi(P) = P \) then \( \Lambda = \mathcal{A}\phi(P) \) satisfies
\[
\begin{bmatrix}
A - BR^{-1}S^* & 0 \\
-Q + SR^{-1}S^* & I
\end{bmatrix}
\begin{bmatrix}
I \\
P
\end{bmatrix}
= \begin{bmatrix}
I & BR^{-1}B^* \\
0 & (A - BR^{-1}S^*)^*
\end{bmatrix}
\begin{bmatrix}
I \\
P
\end{bmatrix}
\Lambda .
\tag{2.47}
\]
Conversely, if \( A - BR^{-1}S^* \) is invertible and \( P, \Lambda \) satisfy (2.47), then \( R\phi(P) = P \) and \( \mathcal{A}\phi(P) = \Lambda \).

Proof. Define \( \bar{A} := A - BR^{-1}S^* \) and \( \bar{Q} := Q - SR^{-1}S^* \) and note that (2.47) is equivalent to the equations
\[
\bar{A} = (I + BR^{-1}B^*P)\Lambda \tag{2.48a}
\]
\[
-Q + P = \bar{A}^*PA . \tag{2.48b}
\]

Using Proposition 2.5.1 on the expression for \( R\phi(P) \), we see that
\[
R\phi(P) = \bar{A}^*P\bar{A} + \bar{Q} - \bar{A}^*PB(B^*PB + R)^{-1}B^*P\bar{A}
= \bar{A}^*P[\bar{A} - B(B^*PB + R)^{-1}B^*P\bar{A}] + \bar{Q}
\tag{2.49}
\]
whenever \( B^*PB + R \) is invertible. Using Proposition 2.5.1 to substitute for the expression in square brackets, we see that
\[
R\phi(P) = \bar{A}^*P\mathcal{A}\phi(P) + \bar{Q}
\tag{2.50}
\]
whenever \( B^*PB + R \) is invertible.
Suppose that \( R_\phi(P) = P \) and choose \( \Lambda = A_\phi(P) \). By (2.50), we see that (2.48b) holds. Also, by Proposition 2.5.1, we see that
\[
A_\phi(P) = (I + BR^{-1}B^*P)^{-1}A
\]
which implies that (2.48a) holds.

We now consider the converse statement. Note that, in this case, (2.48) holds. Since \( \bar{A} \) is invertible, (2.48a) implies that \( I + BR^{-1}B^*P \) must also be invertible. Thus, by (2.48a) and Proposition 2.5.1,
\[
\Lambda = (I + BR^{-1}B^*P)^{-1}\bar{A} = A_\phi(P).
\]
Plugging this into (2.48b) yields that \( P = \bar{A}^*PA_\phi(P) + \bar{Q} \), which implies by (2.50) that \( R_\phi(P) = P \).

**Proposition 2.5.3.** Let \( A \in \mathbb{R}^{n \times n} \) and \( R \) be invertible. Define \( \phi := (A, B, Q, R, S) \) and
\[
M := \begin{bmatrix} A - BR^{-1}S^* & 0 \\ -Q + SR^{-1}S^* & I \end{bmatrix}, \quad N := \begin{bmatrix} I & BR^{-1}B^* \\ 0 & (A - BR^{-1}S^*)^* \end{bmatrix}.
\]
If the DARE \( R_\phi(P) = P \) has a stabilizing solution \( P_0 \), then the matrix pencil \( (M, N) \) is regular and has exactly \( n \) eigenvalues (counting multiplicity) that satisfy \(|\lambda| < 1\).

**Proof.** Define \( \bar{A} := A - BR^{-1}S^* \), \( \bar{Q} := Q - SR^{-1}S^* \), and \( \Lambda := A_\phi(P_0) \). By Proposition 2.5.2, we see that
\[
\begin{bmatrix} A - BR^{-1}S^* & 0 \\ -Q + SR^{-1}S^* & I \end{bmatrix} \begin{bmatrix} I \\ P_0 \end{bmatrix} = \begin{bmatrix} I & BR^{-1}B^* \\ 0 & (A - BR^{-1}S^*)^* \end{bmatrix} \begin{bmatrix} I \\ P_0 \end{bmatrix} \Lambda
\]
We now define the invertible matrices
\[
T := \begin{bmatrix} (I + BR^{-1}B^*P_0)^{-1} & 0 \\ -\bar{A}^*P_0(I + BR^{-1}B^*P_0)^{-1} & I \end{bmatrix}, \quad U := \begin{bmatrix} I & 0 \\ P_0 & I \end{bmatrix}.
\]
Using the two equations in (2.54), we see that
\[
TMU = T \begin{bmatrix} \bar{A} \\ P_0 - \bar{Q} \end{bmatrix} = T \begin{bmatrix} (I + BR^{-1}B^*P_0)\Lambda & 0 \\ \bar{A}^*P_0\Lambda & I \end{bmatrix} = \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}.
\]
Also,
\[
TNU = \begin{bmatrix} I & (I + BR^{-1}B^*P_0)^{-1}BR^{-1}B^* \\ 0 & \bar{A}^*[I - P_0(I + BR^{-1}B^*P_0)^{-1}BR^{-1}B^*] \end{bmatrix}.
\]
Noting that
\[
(I + BR^{-1}B^*P_0)^{-1}BR^{-1} = BR^{-1}(I + B^*P_0BR^{-1})^{-1} = B(B^*P_0B + R)^{-1}
\]
we see using Proposition 2.5.1 that
\[ \bar{A}^*[I - P_0(I + BR^{-1}B^*P_0)^{-1}BR^{-1}B^*] = [\bar{A} - B(B^*P_0B + R)^{-1}B^*P_0\bar{A}]^* = \Lambda^* . \] (2.59)

Therefore, the matrix pencil \((M, N)\) is strictly equivalent to the matrix pencil
\[ \lambda \begin{bmatrix} I & (I + BR^{-1}B^*P)^{-1}BR^{-1}B^* \\ \Lambda^* & 0 \end{bmatrix} - \begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} . \] (2.60)

The matrix pencil \(\lambda I - \Lambda\) is regular because \(\theta_{\lambda, I}(\lambda)\) is the characteristic polynomial of the matrix \(\Lambda\). The matrix pencil \(\lambda\Lambda^* - I\) is regular because \(\theta_{(I, \Lambda^*)}(0) = (-1)^n\). Thus, the matrix pencil (2.60) is regular, which implies that \((M, N)\) is regular.

The characteristic polynomial of (2.60) is \(\det(\lambda I - \Lambda) = \det(\lambda\Lambda^* - I)\). Since \(\Lambda\) is Schur, the polynomial \(\det(\lambda I - \Lambda)\) has the zeros \(\lambda_1, \ldots, \lambda_n\), all of which satisfy \(|\lambda_i| < 1\). It now remains to show that \(\det(\hat{\lambda}\Lambda^* - I) \neq 0\), \(\forall|\hat{\lambda}| < 1\). We do this by contradiction; suppose that there exists \(\hat{\lambda} \in \mathbb{C}\) such that \(|\hat{\lambda}| < 1\) and \(\det(\hat{\lambda}\Lambda^* - I) = 0\). Clearly, \(\hat{\lambda} \neq 0\), which implies that \(0 = \det(-\hat{\lambda}^{-1}I)\det(\hat{\lambda}\Lambda^* - I) = \det(\hat{\lambda}^{-1}I - \Lambda^*)\). This implies that \(\hat{\lambda}^{-1}\) is an eigenvalue of \(\Lambda^*\). Since \(\Lambda^*\) is Schur, this implies that \(|\hat{\lambda}^{-1}| < 1\), which is a contradiction.

The following two propositions explore the relationship between the DARE and the dilated matrix pencil.

**Proposition 2.5.4.** Let \(R\) be invertible and define \(\phi := (A, B, Q, R, S)\). If \(\mathcal{R}_\phi(P) = P\) then \(\Lambda = \mathcal{A}_\phi(P)\) and \(K = \mathcal{K}_\phi(P)\) satisfy
\[ 
\begin{bmatrix} A & 0 & B \\ -Q & I & -S \\ S^* & 0 & R \end{bmatrix} \begin{bmatrix} I \\ P \\ K \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & A^* & 0 \\ 0 & -B^* & 0 \end{bmatrix} \begin{bmatrix} I \\ P \\ K \end{bmatrix} \Lambda . \] (2.61)

**Proof.** Note that verifying (2.61) is equivalent to verifying the equations
\[ A + BK_\phi(P) = A_\phi(P) \] (2.62a)
\[ -Q + P - SK_\phi(P) = A^*PA_\phi(P) \] (2.62b)
\[ S^* + RK_\phi(P) = -B^*PA_\phi(P) . \] (2.62c)

Equation (2.62a) trivially follows from the definition of \(A_\phi(P)\). Since \(\mathcal{R}_\phi(P) = P\), we see that
\[ P = A^*PA + Q + (A^*PB + S)K_\phi(P) = A^*P(A + BK_\phi(P)) + Q + SK_\phi(P) \] (2.63)
which implies that (2.62b) holds. From the definition of \(K_\phi(P)\), we see that
\[ 0 = B^*PA + S^* + (B^*PB + R)K_\phi(P) = B^*P(A + BK_\phi(P)) + S^* + RK_\phi(P) \] (2.64)
which implies that (2.62c) holds.
Proposition 2.5.5. Let $A \in \mathbb{R}^{n \times n}$, $R$ is invertible, $\phi := (A, B, Q, R, S)$. If $P_0$ is a stabilizing solution of the DARE $R\phi(P) = P$, then the matrix pencil
\[
\lambda \begin{bmatrix}
 I & 0 & 0 \\
 0 & A^* & 0 \\
 0 & -B^* & 0
\end{bmatrix} - \begin{bmatrix}
 A & 0 & B \\
 -Q & I & -S \\
 S^* & 0 & R
\end{bmatrix}
\] (2.65)
is regular and the dimension of its maximal Schur deflating subspace is $n$.

Proof. Multiplying (2.65) on the left by the invertible matrix
\[
\begin{bmatrix}
 I & 0 & -BR^{-1} \\
 0 & I & SR^{-1} \\
 0 & 0 & I
\end{bmatrix}
\] (2.66)
we see that (2.65) is strictly equivalent to
\[
\lambda \begin{bmatrix}
 I & BR^{-1}B^* & 0 \\
 0 & (A - BR^{-1}S^*)^* & 0 \\
 0 & -B^* & 0
\end{bmatrix} - \begin{bmatrix}
 A - BR^{-1}S^* & 0 & 0 \\
 -Q + SR^{-1}S^* & I & 0 \\
 S^* & 0 & R
\end{bmatrix}
\] (2.67)
Since $R$ is invertible, we see that $\theta_{(R,0)}(\lambda) = \det(R) \neq 0$, which implies that the matrix pencil $(R,0)$ is regular and has no finite eigenvalues. Thus, since (2.67) is block upper triangular, we see by Proposition 2.5.3 that the matrix pencil (2.67) is regular and all of its finite eigenvalues come from the matrix pencil
\[
\lambda \begin{bmatrix}
 I & BR^{-1}B^* \\
 0 & (A - BR^{-1}S^*)^*
\end{bmatrix} - \begin{bmatrix}
 A - BR^{-1}S^* & 0 \\
 -Q + SR^{-1}S^* & I
\end{bmatrix}
\] (2.68)
By Proposition 2.5.3, this matrix pencil has exactly $n$ eigenvalues which satisfy $|\lambda| < 1$. By Remark 2.4.3, this implies that the dimension of the maximal Schur deflating subspace of (2.65) is at most $n$. However, since $\ker([I \ P_0 \ K_{\phi}(P_0)^*]) = \{0\}$, we see from Proposition 2.5.4 that $\text{Im}([I \ P_0 \ K_{\phi}(P_0)^*])$ is a Schur deflating subspace of (2.65) with dimension $n$. Therefore, the dimension of the maximal Schur deflating subspace of (2.65) is $n$. 

2.6 Discrete Algebraic Riccati Inequalities

In this section, we consider the relationship between certain classes of DAREs and certain classes of LMIs. We begin with the following proposition, which is a standard result in the design of linear quadratic regulators and Kalman filters.

Proposition 2.6.1. Let $\phi = (A, B, Q, R, S)$. If $(A, B)$ is stabilizable and $\begin{bmatrix}
 Q & S \\
 S^* & R
\end{bmatrix} > 0$ then the DARE $R\phi(P) = P$ has a stabilizing solution $P_0 > 0$. 

\qed
For given real matrices $A$, $B$, $Q = Q^*$, $R = R^*$, and $S$, we now define

$$
\mathcal{L}_{(A,B,Q,R,S)}(P) := \begin{bmatrix}
A^T \! PA + Q - P & A^T \! PB + S \\
- & B^T \! PB + R
\end{bmatrix}
$$

with the vector space of symmetric matrices as its domain. As in the previous section, we will make the notation more compact in the remainder of the paper by denoting this quantity as $\mathcal{L}_\phi(P)$ were $\phi$ is an appropriately defined 5-tuple. Note that because $\mathcal{L}_\phi(P) - \mathcal{L}_\phi(0)$ is a linear function of $P$, the constraints $\mathcal{L}_\phi(P) \succ 0$ and $\mathcal{L}_\phi(P) \prec 0$ are LMIs. These LMIs are sometimes referred to as discrete algebraic Riccati inequalities (DARIs).

The next lemma and theorem relate the stabilizing solution of the DARE $\mathcal{R}_\phi(P) = P$ respectively to the DARIs $\mathcal{L}_\phi(P) \succ 0$ and $\mathcal{L}_\phi(P) \prec 0$.

**Lemma 2.6.2.** Let $\phi = (A, B, Q, R, S)$. The DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that $B^T \! P_0B + R \succ 0 \iff (A, B)$ is stabilizable and $\exists P$ such that $\mathcal{L}_\phi(P) \succ 0$. Moreover, $\mathcal{L}_\phi(P) \succ 0 \Rightarrow P_0 \succ P$.

**Proof.** ($\Rightarrow$) Let $P_0$ be the stabilizing solution of the DARE satisfying $B^T \! P_0B + R \succ 0$. This implies that $A_\phi(P_0)$ is Schur and $(A, B)$ is stabilizable. Now we choose a Lyapunov function for $A_\phi(P_0)$, i.e. let $X \succ 0$ satisfy $X - [A_\phi(P_0)]^T \! X [A_\phi(P_0)] \succ 0$ and then define $Y(t) := P_0 - tX$. Note that $\mathcal{R}_\phi(Y(0)) - Y(0) = 0$ and, after some algebra,

$$
\frac{d}{dt} \left( \mathcal{R}_\phi(Y(t)) - Y(t) \right) \bigg|_{t=0} = X - [A_\phi(P_0)]^T \! X [A_\phi(P_0)] \succ 0.
$$

Thus, for sufficiently small $t > 0$, we see that $\mathcal{R}_\phi(Y(t)) - Y(t) \succ 0$ and $B^T \! Y(t)B + R = (B^T \! P_0B + R) - tB^T \! XB \succ 0$. By Schur complements, these two conditions are equivalent to $\mathcal{L}_\phi(Y(t)) \succ 0$.

($\Leftarrow$) Choose any $\Delta = \Delta^*$ such that $\mathcal{L}_\phi(\Delta) \succ 0$ and define $\tilde{\phi} := (A, B, \tilde{Q}, \tilde{R}, \tilde{S})$, where

$$
\begin{bmatrix}
\tilde{Q} & \tilde{S} \\
\tilde{S}^T & \tilde{R}
\end{bmatrix} := \begin{bmatrix}
A^T \Delta A + Q - \Delta & A^T \Delta B + S \\
- & B^T \Delta B + R
\end{bmatrix} = \mathcal{L}_\phi(\Delta) \succ 0.
$$

By Proposition 2.6.1, the DARE $\mathcal{R}_{\tilde{\phi}}(\tilde{P}) = \tilde{P}$ has a stabilizing solution $\tilde{P}_0 \succ 0$. Defining $P_0 := \tilde{P}_0 + \Delta$, we see that $B^T \! P_0B + R = B^T \! \tilde{P}_0B + \tilde{R} \succ 0$. Similarly, $\mathcal{R}_{\tilde{\phi}}(P_0) = \mathcal{R}_{\tilde{\phi}}(\tilde{P}_0) + \Delta = P_0$ and $A_{\tilde{\phi}}(P_0) = A_{\tilde{\phi}}(\tilde{P}_0)$. This implies that $P_0$ is the stabilizing solution of the DARE $\mathcal{R}_{\tilde{\phi}}(P) = P$. Since $P_0 - \Delta = \tilde{P}_0 \succ 0$, we see that $P_0 \succ \Delta$. Since the choice of $\Delta$ was arbitrary, this concludes the proof.

**Theorem 2.6.3.** Let $\phi = (A, B, Q, R, S)$. The DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that $B^T \! P_0B + R \prec 0 \iff (A, B)$ is stabilizable and $\exists P$ such that $\mathcal{L}_\phi(P) \prec 0$. Moreover, $\mathcal{L}_\phi(P) \prec 0 \Rightarrow P_0 < P$. 

\[\square\]
Proof. Defining \( \phi := (A, B, -Q, -R, -S) \) we see that \( \mathcal{L}_\phi(-P) = -\mathcal{L}_\phi(P) \). Thus, \( \exists P \) such that \( \mathcal{L}_\phi(P) < 0 \) if and only if \( \exists \bar{P} \) such that \( \mathcal{L}_\phi(\bar{P}) > 0 \). By Lemma 2.6.2, we thus see that \((A, B)\) is stabilizable and \( \exists P \) such that \( \mathcal{L}_\phi(P) < 0 \) if and only if the DARE \( \mathcal{R}_\phi(P) = \bar{P} \) has a stabilizing solution \( \bar{P}_0 \) such that \( B^*\bar{P}_0B + \bar{R} > 0 \). Defining \( P_0 := -\bar{P}_0 \), we see after a little algebra that \( \mathcal{R}_\phi(P_0) = -\mathcal{R}_\phi(P_0) = P_0; \ A_\phi(P_0) = A_\phi(P_0), \) and \( B^*P_0B + R = -(B^*\bar{P}_0B + \bar{R}) < 0 \) which establishes the required equivalency.

Now let \( \mathcal{L}_\phi(P) < 0 \). This implies that \( \mathcal{L}_\phi(-P) = -\mathcal{L}_\phi(P) > 0 \), which in turn implies that \( -P < P_0 \) by Lemma 2.6.2. Therefore, \( P \succ P_0 \).

Since the stabilizing solution of a DARE is unique, Theorem 2.6.3 yields a test for checking whether or not the DARI \( \mathcal{L}_\phi(P) \ < 0 \) is feasible, under the condition that \((A, B)\) is stabilizable. First, try to find the stabilizing solution of the DARE \( \mathcal{R}_\phi(P) = P \) using a software package such as MATLAB. The LMI is feasible if and only if the stabilizing solution \( P_0 \) exists and satisfies \( B^*P_0B + R < 0 \).

We now give the last result of this section, which relates the solution of a DARE to the inverse of DARI solutions. This somewhat nonstandard result is a discrete-time version of a continuous-time result in [13].

**Theorem 2.6.4.** Let \( Q \) and \( R \) be invertible and define
\[
\phi := (A, B, C^*Q^{-1}C, R, 0) \tag{2.70}
\]
\[
\psi := (A^*, C^*, -BR^{-1}B^*, -Q, 0). \tag{2.71}
\]
If the DARE \( \mathcal{R}_\phi(P) = P \) has a stabilizing solution \( P_0 \), then \( P_0 \prec P \) for any \( P \succ 0 \) which satisfies \( \mathcal{L}_\psi(P^{-1}) \prec 0 \).

**Proof.** We first make the additional assumptions that \( A \) and \( P_0 \) are invertible; we will relax these assumptions in the final part of the proof. For convenience, we define \( X_0 := P_0^{-1} \). Now let \( \mathcal{A}_\phi(P_0) = (I + BR^{-1}B^*P_0)^{-1}A \) (by Proposition 2.5.1), we see that \( \mathcal{A}_\phi(P_0) \) is invertible. By Proposition 2.5.2,
\[
\begin{bmatrix}
A & 0
\end{bmatrix}
\begin{bmatrix}
I \\
-C^*Q^{-1}C & I \\
\end{bmatrix}
\begin{bmatrix}
P_0
\end{bmatrix}
= 
\begin{bmatrix}
I \\
0
\end{bmatrix}
\begin{bmatrix}
BR^{-1}B^* & A^*
\end{bmatrix}
\begin{bmatrix}
P_0
\end{bmatrix}
\mathcal{A}_\phi(P_0).
\tag{2.72}
\]
Defining \( \bar{\Lambda} := X_0^{-1}(\mathcal{A}_\phi(P_0))^{-1}X_0 \), we see that
\[
A = (I + BR^{-1}B^*P_0)\mathcal{A}_\phi(P_0) \Rightarrow A(\mathcal{A}_\phi(P_0))^{-1}X_0 = X_0 + BR^{-1}B^*
\tag{2.73}
\]
\[
\Rightarrow AX_0\bar{\Lambda} = X_0 + BR^{-1}B^*
\tag{2.74}
\]
and
\[
P_0 - C^*Q^{-1}C = A^*P_0A_\phi(P_0)
\tag{2.75}
\]
\[
\Rightarrow P_0(\mathcal{A}_\phi(P_0))^{-1}X_0 - C^*Q^{-1}C(\mathcal{A}_\phi(P_0))^{-1}X_0 = A^*
\tag{2.76}
\]
\[
\Rightarrow \bar{\Lambda} - C^*Q^{-1}CX_0\bar{\Lambda} = A^*.
\tag{2.77}
\]
Thus,
\[
\begin{bmatrix}
  A^* & 0 \\
 BR^{-1}B^* & I
\end{bmatrix}
\begin{bmatrix}
  I \\
  X_0
\end{bmatrix}
= \begin{bmatrix}
  I & -C^*Q^{-1}C \\
  0 & A
\end{bmatrix}
\begin{bmatrix}
  I \\
  X_0
\end{bmatrix}
\tilde{\Lambda}.
\] (2.78)

Since $\tilde{\Lambda}$ and $(A_\phi(P_0))^{-1}$ are related by a similarity transformation, we see that $\tilde{\Lambda}$ is anti-Schur. Thus, since $A^*$ is nonsingular, we apply Proposition 2.5.2 again to see that $X_0$ is the anti-stabilizing solution of the DARE $R_\psi(X) = X$.

Let $P > 0$ satisfy $L_\psi(P^{-1}) < 0$. We will now show that $P_0 < P$. For convenience we define $X := P^{-1} > 0$. By Schur complements applied to $L_\psi(P^{-1}) < 0$, we see that $CXC^* - Q < 0$ and $R_\psi(X) - X < 0$. With some algebra, it can be shown that
\[
(X_0 - X) - \tilde{\Lambda}^*(X_0 - X)\tilde{\Lambda} = (R_\psi(X) - X) + (L_0 - L)(CXC^* - Q)(L_0 - L)^*
\] (2.79)

where $L_0 := -AX_0C(CX_0C^* - Q)^{-1}$ and $L := -AXC^*(CX^* - Q)^{-1}$. Since the right-hand side of this equation is negative definite and $\tilde{\Lambda}$ is anti-Schur, we conclude by discrete Lyapunov equation theory that $X_0 - X \succ 0$. Thus, $0 < X < X_0 \Rightarrow 0 < P_0 < P$.

We now relax the invertibility assumptions that we made at the beginning of the proof. We will show that $P_0 < P$ by perturbing $P_0$ by a small amount to produce $P_\epsilon$ which satisfies $P_0 < P_\epsilon < P$. Note that, for $\Delta > 0$ with sufficiently large minimum eigenvalue, $\exists \tilde{C}$ such that $\Delta - P_0A_\phi(P_0) - (A_\phi(P_0))^*P_0 = \tilde{C}^*\tilde{C} > 0$. Choose such values of $\Delta$ and $\tilde{C}$. We now define
\[
A_\epsilon := A + \epsilon I, \quad \tilde{C}_\epsilon := \begin{bmatrix} C \\ \sqrt{\epsilon}C \end{bmatrix}, \quad \tilde{Q} := \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix},
\] (2.80)
\[
\phi_\epsilon := (A_\epsilon, B, \tilde{C}_\epsilon^*\tilde{Q}^{-1}\tilde{C}_\epsilon, R, 0),
\] (2.81)
\[
\psi_\epsilon := (A_\epsilon^*, \tilde{C}_\epsilon^* - BR^{-1}B^*, -\tilde{Q}, 0).
\] (2.82)

Note that $L_\psi(X)|_{\epsilon = 0} = \begin{bmatrix} L_\psi(X) & 0 \\ 0 & -I \end{bmatrix} < 0$. Thus, $L_\psi(X) < 0$ for sufficiently small $\epsilon > 0$.

Also note that $\phi_\epsilon|_{\epsilon = 0} = \phi$. Since the stabilizing solution of a DARE is analytic in its parameters [9], $\exists \epsilon > 0$ such that $\forall \epsilon \in (-\epsilon, \epsilon)$, the DARE $R_{\phi_\epsilon}(X) = X$ has a stabilizing solution $P_\epsilon$ and, moreover, $P_\epsilon$ is analytic in $\epsilon$. Note in particular that $P_{|\epsilon = 0} = P_0$. Implicitly differentiating the DARE $R_{\phi_\epsilon}(P_\epsilon) = P_\epsilon$ with respect to $\epsilon$ and denoting the derivative of $P_\epsilon$ as $P'_\epsilon$, we obtain after some algebra that
\[
P'_\epsilon = (A_{\phi_\epsilon}(P_\epsilon))^*P'_\epsilon(A_{\phi_\epsilon}(P_\epsilon)) + P_\epsilon(A_{\phi_\epsilon}(P_\epsilon))^*P_\epsilon + \tilde{C}^*\tilde{C}
\] (2.83)
\[
\Rightarrow P'_{|\epsilon = 0} = (A_{\phi}(P_0))^*(P'_{|\epsilon = 0})(A_{\phi}(P_0)) + \Delta.
\] (2.84)

Since $A_{\phi}(P_0)$ is Schur and $\Delta > 0$, we see by Lyapunov equation theory that $P'_{|\epsilon = 0} > 0$. Thus, $\exists \tilde{\epsilon} > 0$ such that $A_\epsilon$ and $P'_\epsilon$ are invertible and $L_\psi(X) < 0$, $\forall \epsilon \in (0, \tilde{\epsilon})$. By the first part of the proof, $0 < P_\epsilon < P$, $\forall \epsilon \in (0, \tilde{\epsilon})$. Therefore, since $P'_{|\epsilon = 0} > 0$, we obtain that $P_0 < P_\epsilon < P$ for sufficiently small $\epsilon > 0$. ■
Chapter 3

$H_2$ Guaranteed Cost Analysis

In this chapter, we use the techniques of guaranteed cost control [28] to analyze the robust $H_2$ performance of an uncertain discrete-time LTI system. In §3.1, we first review the computation of a $H_2$ norm of a known discrete-time LTI system using a SDP. We then use the $S$-procedure to formulate a SDP that yields an upper bound on the worst-case $H_2$ performance of a discrete-time LTI system with unstructured parametric uncertainty. We then show that this analysis result easily extends to systems with unstructured causal dynamic LTV uncertainty.

In §3.2, we show that the SDP from §3.1.2 (and §3.1.3) has inherent Riccati equation structure, which can be exploited to formulate a fast and reliable algorithm for determining an upper bound on the worst-case $H_2$ performance of an uncertain system. The results of §3.2 are the most novel results of this chapter and are of key importance in the remainder of the dissertation.

We then give empirical evidence in §3.3 that suggests that the using Riccati equation algorithm is faster and more reliably accurate than solving the corresponding SDP. In §3.4, we give alternate matrix inequality characterizations of several quantities in this chapter. Finally, we show in §3.5 that the quantities in this chapter are independent of the realization chosen for the relevant state-space system.

In this chapter, we will use the notation $R_\phi(P)$, $K_\phi(P)$, and $A_\phi(P)$ defined in §2.5 and the notation $L_\phi(P)$ defined in §2.6.

3.1 SDP Approach

3.1.1 Known Systems

We begin by considering a given discrete-time LTI system $\bar{G}$ with the state-space realization

$$\bar{G} \sim \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}.$$  \hspace{1cm} (3.1)
The following theorem establishes sharp upper bounds on the $\mathcal{H}_2$ norm of $\bar{G}$ under the assumption that its state-space realization is known.

**Theorem 3.1.1.** The following are equivalent:

1. $\|\bar{G}\|_2^2 < \gamma$.

2. $\exists P, W$ such that

   \[\gamma > \text{tr}\{W\}\]
   \[W - \bar{B}^*P\bar{B} - \bar{D}^*\bar{D} > 0\]
   \[P - \bar{A}^*P\bar{A} - \bar{C}^*\bar{C} > 0\]
   \[P > 0.\]

3. $\exists P, W, V$ such that

   \[\gamma > \text{tr}\{W\}\]
   \[\begin{bmatrix} P & V^* \\ V & W \end{bmatrix} - \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} > 0\]
   \[P > 0.\]

**Proof.** We first recall that the $\mathcal{H}_2$ norm can be easily computed using a discrete Lyapunov equation solution. In particular, if $\bar{A}$ is Schur, then

\[\|\bar{G}\|_2^2 = \text{tr}\{\bar{B}^*P_0\bar{B} + \bar{D}^*\bar{D}\}\]  
(3.4)

where $P_0$ is the solution of the discrete Lyapunov equation

\[P_0 = \bar{A}^*P_0\bar{A} + \bar{C}^*\bar{C}.\]  
(3.5)

Moreover, $P_0 \succeq 0$. If $\bar{A}$ is not Schur, then $\|\bar{G}\|_2^2 = \infty$.

(1 $\Rightarrow$ 3) Since $\bar{A}$ is Schur, choose $P_0$ to be the solution of the discrete Lyapunov equation (3.5) and $P$ to be the solution of the discrete Lyapunov equation

\[P = \bar{A}^*P\bar{A} + \bar{C}^*\bar{C} + \epsilon I\]  
(3.6)

where $\epsilon \in \mathbb{R}$. Since $\bar{C}^*\bar{C} + \epsilon I \succ 0$, $\forall \epsilon > 0$, we see that $P \succ 0$, $\forall \epsilon > 0$. Moreover, $P$ is a continuous function of $\epsilon$ and $P|_{\epsilon=0} = P_0$. We now define

\[V := \bar{B}^*P\bar{A} + \bar{D}^*\bar{C}\]  
(3.7)

\[W := \bar{B}^*P\bar{B} + \bar{D}^*\bar{D} + \epsilon I\]  
(3.8)
and note that \( V \) and \( W \) are continuous functions of \( \epsilon \). Also note that

\[
W|_{\epsilon=0} = \bar{B}^*P_0\bar{B} + \bar{D}^*\bar{D} \quad (3.9)
\]

\[
\Rightarrow \text{tr}\{W\}|_{\epsilon=0} = \|\bar{G}\|_2^2. \quad (3.10)
\]

Since \( \text{tr}\{W\} \) is a continuous function of \( \epsilon \) and

\[
\begin{bmatrix}
P & V^* \\
V & W
\end{bmatrix}
- 
\begin{bmatrix}
\bar{A} & \bar{B} \\
\bar{C} & \bar{D}
\end{bmatrix}^*
\begin{bmatrix}
P & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\bar{A} & \bar{B} \\
\bar{C} & \bar{D}
\end{bmatrix} = \epsilon I
\]

(3.11)

we see that condition (3.3) is met for small enough \( \epsilon > 0 \).

\( (3 \Rightarrow 2) \) Since

\[
\begin{bmatrix}
P & V^* \\
V & W
\end{bmatrix}
- 
\begin{bmatrix}
\bar{A} & \bar{B} \\
\bar{C} & \bar{D}
\end{bmatrix}^*
\begin{bmatrix}
P & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\bar{A} & \bar{B} \\
\bar{C} & \bar{D}
\end{bmatrix} = 
\begin{bmatrix}
P - \bar{A}^*P\bar{A} - \bar{C}^*\bar{C} & \bullet \\
V - \bar{B}^*P\bar{B} - \bar{D}^*\bar{D} & W - \bar{B}^*P\bar{B} - \bar{D}^*\bar{D}
\end{bmatrix} \quad (3.12)
\]

this is trivial.

\( (2 \Rightarrow 1) \) Since (3.2c)–(3.2d) hold, we see that \( \bar{A} \) is Schur. Letting \( P_0 \) be the solution of the discrete Lyapunov equation (3.5), we see by subtracting (3.5) from (3.2c) that

\[
(P - P_0) - \bar{A}^*(P - P_0)\bar{A} \succ 0 \quad (3.13)
\]

which implies that \( P - P_0 \succ 0 \). Therefore,

\[
\bar{B}^*P_0\bar{B} + \bar{D}^*\bar{D} \preceq \bar{B}^*P\bar{B} + \bar{D}^*\bar{D} \prec W \quad (3.14)
\]

\[
\Rightarrow \text{tr}\{\bar{B}^*P_0\bar{B} + \bar{D}^*\bar{D}\} < \text{tr}\{W\} < \gamma. \quad (3.15)
\]

This implies that \( \|\bar{G}\|_2^2 < \gamma \). ■

In the preceding theorem, we gave two LMI characterizations of the \( \mathcal{H}_2 \) performance of a given system. Although the equivalence of conditions 1 and 2 is considered standard in the literature (e.g. [23]), we included its proof for completeness. However, condition 3 appears to be a new characterization of the \( \mathcal{H}_2 \) norm of a known state-space system.

For the purposes of this dissertation, condition 3 is more suitable characterization of the \( \mathcal{H}_2 \) performance of a given system for two reasons. First, it will allow us to generalize the results of §3.1.2, which consider systems with unstructured norm-bounded parametric uncertainty, to the results in §3.1.3, which consider systems with causal LTV norm-bounded uncertainty. Second, it will allow us to use the matrix variable elimination technique to derive an optimal control scheme in Chapter 4, which will be important in our approach to the output feedback problem developed in Chapter 5.
3.1.2 Systems with Parametric Uncertainty

In the previous subsection, we gave a sharp upper bound on the $\mathcal{H}_2$ performance of a known system in terms of LMIs. We now remove the restriction that $\bar{G}$ is known. In particular, we let $\bar{G}$ be represented as an LFT of a known state-space system $G$ and an unknown real matrix $\Delta$ as shown in Fig. 3.1. We let $\bar{G}$ have the state-space realization

$$
\begin{bmatrix}
    x_{k+1} \\
    q_k \\
    p_k \\
    d_k \\
    w_k
\end{bmatrix} =
\begin{bmatrix}
    A & B_1 & B_2 \\
    C_1 & D_{11} & D_{12} \\
    C_2 & D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
    x_k \\
    d_k \\
    p_k \\
    w_k
\end{bmatrix}
$$

(3.16)

where $x_k \in \mathbb{R}^{n_x}$, $q_k \in \mathbb{R}^{n_q}$, $p_k \in \mathbb{R}^{n_p}$, $d_k \in \mathbb{R}^{d_x}$, and $w_k \in \mathbb{R}^{n_w}$. We will denote the state-space realization of $G$ given in (3.16) as $\Sigma_G$. We now define the parametric uncertainty set

$$
\Delta := \{ \Delta \in \mathbb{R}^{n_d \times n_q} : \|\Delta\| \leq 1 \}.
$$

(3.17)

Note that the system in Fig. 3.1 is robustly well-posed over $\Delta \in \bar{\Delta}$ if and only if $\|D_{11}\| < 1$.

We now examine the system in Fig. 3.1 under the restriction that $\Delta \in \bar{\Delta}$ and the interconnection is well-posed. In this case, the closed-loop system has the realization

$$
\bar{G} \sim \begin{bmatrix}
    A + B_1 \Phi_{1,\Delta} & B_2 + B_1 \Phi_{2,\Delta} \\
    C_2 + D_{21} \Phi_{1,\Delta} & D_{22} + D_{21} \Phi_{2,\Delta}
\end{bmatrix} =: \begin{bmatrix}
    A_\Delta & B_\Delta \\
    C_\Delta & D_\Delta
\end{bmatrix}
$$

(3.18)

where

$$
\Phi_{1,\Delta} := (I - \Delta D_{11})^{-1} \Delta C_1
$$

$$
\Phi_{2,\Delta} := (I - \Delta D_{11})^{-1} \Delta D_{12}
$$

(Although we have already defined notation for the state-space realization of $\bar{G}$ in (3.1), we will use this notation in this section to emphasize that the state-space entries of $\bar{G}$ are a function of $\Delta$.)

For convenience, we define the quantity

$$
\mathcal{M}_{\Sigma_G}(\tau, P, W, V) := \begin{bmatrix}
    P & \bullet & \bullet \\
    0 & \tau I & \bullet \\
    V & 0 & W
\end{bmatrix} - \begin{bmatrix}
    A & B_1 & B_2 \\
    C_1 & D_{11} & D_{12} \\
    C_2 & D_{21} & D_{22}
\end{bmatrix}^* \begin{bmatrix}
    P & 0 & 0 \\
    0 & \tau I & 0 \\
    0 & 0 & I
\end{bmatrix} \begin{bmatrix}
    A & B_1 & B_2 \\
    C_1 & D_{11} & D_{12} \\
    C_2 & D_{21} & D_{22}
\end{bmatrix}.
$$

(3.19)
The subscript $\Sigma_G$ on $\mathcal{M}$ indicates that $\mathcal{M}$ is parameterized by the matrices $A$, $B_1$, $B_2$, $C_1$, $C_2$, $D_{11}$, $D_{12}$, $D_{21}$, and $D_{22}$. The matrix $\mathcal{M}$ will play a pivotal role in the remainder of this dissertation.

We now present a lemma and a theorem that yield an upper bound on the worst-case $\mathcal{H}_2$ performance of the interconnection in Fig. 3.1.

**Lemma 3.1.2.** Let $\|D_{11}\| < 1$. The condition

$$
\begin{bmatrix}
P & V^* \\
V & W
\end{bmatrix} - 
\begin{bmatrix}
A_\Delta & B_\Delta \\
C_\Delta & D_\Delta
\end{bmatrix}^* 
\begin{bmatrix}
P & 0 \\
0 & I
\end{bmatrix} 
\begin{bmatrix}
A_\Delta & B_\Delta \\
C_\Delta & D_\Delta
\end{bmatrix} \succ 0, \quad \forall \Delta \in \Delta 
$$

holds if and only if $\exists \tau > 0$ such that $\mathcal{M}_{\Sigma_G}(\tau, P, W, V) \succ 0$.

**Proof.** To prove the equivalence of the required statements, we first define for convenience

$$
L :=
\begin{bmatrix}
P & V^* \\
0 & 0 \\
V & W
\end{bmatrix} - 
\begin{bmatrix}
A^* & C_2^* \\
B_1^* & D_{21}^* \\
B_2^* & D_{22}^*
\end{bmatrix} 
\begin{bmatrix}
P & 0 \\
0 & I
\end{bmatrix} 
\begin{bmatrix}
A & B_1 & B_2 \\
C_2 & D_{21} & D_{22}
\end{bmatrix} .
$$

(3.21)

It is easily verified that (3.20) holds if and only if

$$
\begin{bmatrix}
I & 0 \\
\Phi_{1,\Delta} & \Phi_{2,\Delta} \\
0 & I
\end{bmatrix}^* L \begin{bmatrix}
I & 0 \\
\Phi_{1,\Delta} & \Phi_{2,\Delta} \\
0 & I
\end{bmatrix} \succ 0, \quad \forall \Delta \in \Delta .
$$

(3.22)

Letting $v_1$ and $v_2$ be appropriately sized vectors, the previous condition holds if and only if

$$
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}^* L \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} \succ 0, \quad \forall \Delta \in \Delta, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq 0 .
$$

(3.23)

We now give an alternate characterization of $\Delta$. Defining $\xi = \Phi_{1,\Delta}v_1 + \Phi_{2,\Delta}v_2$, we see that

$$
\xi = (I - \Delta D_{11})^{-1}\Delta(C_1v_1 + D_{12}v_2) 
\Rightarrow \xi = \Delta(C_1v_1 + D_{11}\xi + D_{12}v_2) .
$$

(3.24)

Thus, given $v_1$, $v_2$, and $\xi$, $\exists \Delta \in \Delta$ such that $\xi = \Phi_{1,\Delta}v_1 + \Phi_{2,\Delta}v_2$ if and only if

$$
\|\xi\|^2 \leq \|C_1v_1 + D_{11}\xi + D_{12}v_2\|^2 .
$$

(3.25)

Since we assume that $\|D_{11}\| < 1$, the only value of $\xi$ that satisfies (3.26) when $v_1 = 0$ and $v_2 = 0$ is $\xi = 0$. Therefore, if (3.26) is satisfied and $[v_1^* \xi \ v_2^*]^* \neq 0$, it must be that $[v_1^* \ v_2^*]^* \neq 0$. 

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Using the characterization of $\Delta$ given in (3.26), we see that (3.23) holds if and only if
\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
v_1 \\
v_2
\end{bmatrix}^* L \begin{bmatrix}
\xi_1 \\
\xi_2 \\
v_1 \\
v_2
\end{bmatrix} > 0, \quad \forall \begin{bmatrix}
\xi_1 \\
\xi_2 \\
v_1 \\
v_2
\end{bmatrix} \neq 0 \text{ satisfying } \|\xi\|^2 \leq \|C_1 v_1 + D_{11} \xi + D_{12} v_2\|^2. \tag{3.27}
\]

We now use Lemma 3.1.2 to conclude that (3.20) holds. Defining
\[
H = \text{an upper bound on the robust squared performance of } \bar{G},
\]
the inequality
\[
\|\bar{G}\|^2 < \gamma, \quad \forall \Delta \in \Delta,
\]
which is in turn equivalent to the condition $\mathcal{M}_{\Sigma}(\tau, P, W, V) > 0$.

**Theorem 3.1.3.** If $\mathcal{M}_{\Sigma}(\tau, P, W, V) > 0$, $P > 0$, and $\tau > 0$, then the system (3.18) is robustly well-posed and $\|\bar{G}\|^2 < \text{tr}\{W\}$, $\forall \Delta \in \Delta$.

**Proof.** We first show that the system is robustly well-posed. Extracting the (2,2) block of the inequality $\mathcal{M}_{\Sigma}(\tau, P, W, V) > 0$, we see that
\[
\tau I > B_1^T P B_1 + \tau D_{11}^* D_{11} + D_{21}^T D_{21} \geq \tau D_{11}^* D_{11}
\]
\[
\Rightarrow \quad I > D_{11}^* D_{11} \Rightarrow \quad \|D_{11}\| < 1. \tag{3.31}
\]
We now use Lemma 3.1.2 to conclude that (3.20) holds. Defining $\gamma := \text{tr}\{W\} + \epsilon$, we see by Theorem 3.1.1 that $\|\bar{G}\|^2 < \gamma$, $\forall \Delta \in \Delta$, $\epsilon > 0$. This implies that $\|\bar{G}\|^2 \leq \text{tr}\{W\}$, $\forall \Delta \in \Delta$.

We now recover the strict inequality. Since $\mathcal{M}_{\Sigma}(\tau, P, W, V)$ is a continuous function of $W$, we see that $\mathcal{M}_{\Sigma}(\tau, P, W - \mu I, V) > 0$ for small enough $\mu > 0$. By the argument in the first half of the proof, $\|\bar{G}\|^2 \leq \text{tr}\{W - \mu I\}$, $\forall \Delta \in \Delta$ holds for small enough $\mu > 0$. Since $\mu > 0 \Rightarrow \text{tr}\{W - \mu I\} < \text{tr}\{W\}$, this concludes the proof.

With this theorem in place, the value of the optimization problem
\[
\inf_{\tau, P, W, V} \text{tr}\{W\}
\]
\[
\text{s.t. } \tau > 0, \quad P > 0, \quad \mathcal{M}_{\Sigma}(\tau, P, W, V) > 0 \tag{3.32}
\]
is an upper bound on the robust squared $\mathcal{H}_2$ performance of $\bar{G}$ over all $\Delta \in \Delta$. We now make the following definition.
Definition 3.1.4. The square root of the value of (3.32) will be called the $\mathcal{H}_2$ guaranteed cost of $G$.

Note that, in this definition, the $\mathcal{H}_2$ guaranteed cost is a property of $G$, not the interconnection shown in Fig. 3.1. Also note that if $\tau > 0, P \succ 0, W, V$ satisfy $\mathcal{M}_{\Sigma G}(\tau, P, W, V) \succ 0$, then $\text{tr}\{W\}$ is strictly greater than the squared $\mathcal{H}_2$ guaranteed cost of $G$ by Remark 2.2.2. Finally, we note that the $\mathcal{H}_2$ guaranteed cost of $G$ is an upper bound on the worst-case $\mathcal{H}_2$ norm of the interconnection in Fig. 3.1; in general it will be somewhat conservative.

Since the cost function in (3.32) is linear and the constraints are all LMIs, this is a convex optimization. However, this problem is not an SDP because the matrix inequalities are strict. Relaxing the strict inequalities to nonstrict inequalities yields the SDP

$$\min_{\tau, P, W, V} \text{tr}\{W\} \quad \text{s.t.} \quad \tau \geq 0, P \succeq 0, \mathcal{M}_{\Sigma G}(\tau, P, W, V) \succeq 0$$

(3.33)

which can be solved using a solver such as SeDuMi [30] or the mincx function in the Robust Control Toolbox for MATLAB. As mentioned in Proposition 2.2.1, this relaxation is only valid when (3.32) is feasible (or (3.33) is infeasible). Thus, it is important to determine when (3.32) is feasible.

Theorem 3.1.5. The following are equivalent:

1. $\exists \tau > 0, P \succ 0, W, V$ such that $\mathcal{M}_{\Sigma G}(\tau, P, W, V) \succ 0$.

2. $A$ is Schur and $\left\| \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \right\|_\infty < 1$.

3. $A$ is Schur and the DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that

$$B_1^*P_0B_1 + D_{11}^*D_{11} - I \prec 0$$

(3.34)

where $\hat{\phi} := (A, B_1, C_1^*C_1, D_{11}^*D_{11} - I, C_1^*D_{11})$.

Proof. Since the equivalence of the last two conditions is standard in the literature (e.g. [19]), we only prove the equivalence of the first two conditions. Under the assumption that $A$ is Schur, note that

$$\left\| \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \right\|_\infty < 1 \iff \left\| \begin{bmatrix} A^* & C_1^* \\ B_1^* & D_{11}^* \end{bmatrix} \right\|_\infty < 1$$

(3.35)

$$\iff \exists \epsilon > 0 \text{ s.t. } \left\| \begin{bmatrix} A^* & C_1^* \\ B_1^* & D_{11}^* \end{bmatrix} \right\|_\infty^2 < \epsilon^{-2}$$

(3.36)

$$\iff \exists \epsilon > 0 \text{ s.t. } \left\| \begin{bmatrix} A^* & C_1^* \\ B_1^* & D_{11}^* \end{bmatrix} \right\|_\infty^2 < \epsilon^{-2}.$$
Defining \( \tau := \epsilon^{-2} \), we use the LMI representation of the \( \mathcal{H}_\infty \) norm (e.g. [25]) to see that this condition, along with \( A \) being Schur, is equivalent to the existence of \( P \succ 0 \) and \( \tau > 0 \) such that

\[
\begin{bmatrix}
  P & 0 \\
  0 & \tau I
\end{bmatrix} \succ
\begin{bmatrix}
  A & B_1 \\
  \sqrt{\tau} C_1 & \sqrt{\tau} D_{11} \\
  C_2 & D_{21}
\end{bmatrix}^* \begin{bmatrix}
  P & 0 & 0 \\
  0 & I & 0 \\
  0 & 0 & I
\end{bmatrix} \begin{bmatrix}
  A & B_1 \\
  \sqrt{\tau} C_1 & \sqrt{\tau} D_{11} \\
  C_2 & D_{21}
\end{bmatrix}
\]

(3.38)

Since this matrix inequality is the first two rows and columns of the condition \( \mathcal{M}_{\Sigma_G}(\tau, P, W, V) \succ 0 \), applying Proposition 2.2.3 concludes the proof.

### 3.1.3 Systems with Dynamic Uncertainty

In this subsection, we let \( \hat{G} \) be represented as an LFT of a known state space system and an unknown causal LTV map \( \Delta \) as shown in Fig. 3.1. We now define the space \( \ell^2_+ := \{ (u_0, u_1, \ldots) : \sum_{k=0}^{\infty} u_k^* u_k < \infty \} \) and its corresponding norm \( \| (u_0, u_1, \ldots) \|_{\ell^2_+}^2 := \sum_{k=0}^{\infty} u_k^* u_k \).

In this framework, we define the dynamic uncertainty set as

\[
\Delta_d := \{ \Delta : \Delta \text{ is a causal linear map,} \| \Delta \|_\infty \leq 1 \}
\]

(3.39)

where \( \| \Delta \|_\infty \) is the induced norm of \( \Delta \) from \( \ell^2_+ \) to \( \ell^2_+ \).

Since \( G \) is time-varying, we cannot use the \( \mathcal{H}_2 \) norm to measure its performance. If the system \( \hat{G} \) has \( n_w \) inputs, we use the \( \ell^2_+ \) semi-norm (i.e. a generalization of the \( \mathcal{H}_2 \) norm for time-varying systems) defined by

\[
\| \hat{G} \|_2^2 := \limsup_{N \to \infty} \frac{1}{N+1} \sum_{j=0}^{N} \sum_{i=1}^{n_w} \| \tilde{p}_{i,j} \|_{\ell^2_+}^2
\]

(3.40)

where \( \tilde{p}_{i,j} \) is the impulse response from the \( i \)th input when applied at time \( k = j \). It can be shown that this definition is equivalent to the definition given in [27]. We will also refer to this as the generalized \( \mathcal{H}_2 \) norm. Note that this definition corresponds to computing the squared \( \mathcal{H}_2 \) norm using the impulse response method applied at each possible time-step and then averaging all of these results. Thus, it represents an “RMS” \( \mathcal{H}_2 \) performance. We now characterize the robust generalized \( \mathcal{H}_2 \) performance of the interconnection (3.16).

**Theorem 3.1.6.** Let \( \hat{G} \) be the interconnection shown in Fig. 3.1 where \( \Delta \in \Delta_d \). If \( P > 0 \), \( \mathcal{M}_{\Sigma_G}(\tau, P, W, V) \succ 0 \), and \( \tau > 0 \), then \( \| \hat{G} \|_2^2 < \text{tr}\{W\} \), \( \forall \Delta \in \Delta_d \).

**Proof.** Fix \( \Delta \in \Delta_d \). Since \( \| \Delta \|_\infty \leq 1 \), we see by Theorem 3.1.5 and the small gain theorem that the interconnection is well-posed and bounded. We now close the loop to form the map
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from $(w_0, w_1, \ldots)$ to $(p_0, p_1, \ldots)$ and apply an impulse at time $k = j$ at the $i$th input and fix the values of $x^{i,j}_k, q^{i,j}_k, p^{i,j}_k, d^{i,j}_k$, and $w^{i,j}_k$ to correspond to the system response where $k$ is the time index. Since $\mathcal{M}_{\Sigma G}(\tau, P, W, V) \succ 0$, we see that

$$
\begin{bmatrix}
    x^{i,j}_k \\
    d^{i,j}_k \\
    w^{i,j}_k
\end{bmatrix}^* \begin{bmatrix}
    P & 0 & \tau I \\
    0 & \tau I & 0 \\
    V & 0 & W
\end{bmatrix} \begin{bmatrix}
    x^{i,j}_{k+1} \\
    q^{i,j}_{k} \\
    p^{i,j}_{k}
\end{bmatrix}^* \begin{bmatrix}
    P & 0 & 0 \\
    0 & \tau I & 0 \\
    0 & 0 & I
\end{bmatrix} \begin{bmatrix}
    x^{i,j}_{k+1} \\
    q^{i,j}_{k} \\
    p^{i,j}_{k}
\end{bmatrix} \geq \begin{bmatrix}
    x^{i,j}_{k} \\
    d^{i,j}_k \\
    w^{i,j}_k
\end{bmatrix}^* \begin{bmatrix}
    P & 0 & \tau I \\
    0 & \tau I & 0 \\
    V & 0 & W
\end{bmatrix} \begin{bmatrix}
    x^{i,j}_{k} \\
    d^{i,j}_k \\
    w^{i,j}_k
\end{bmatrix} \tag{3.41}
$$

where we have used the equations governing $G$. Let $\bar{N} \geq j$. Summing both sides from $k = 0$ to $\bar{N}$ yields, after a bit of simplification,

$$
e^*_i W e_i \geq (x^*_{N+1})^T P x^*_{N+1} + \sum_{k=0}^{\bar{N}} [(p^{i,j}_k)^* p^{i,j}_k + \tau ((q^{i,j}_k)^* q^{i,j}_k - (d^{i,j}_k)^* d^{i,j}_k)] \tag{3.42}
$$

Note that we have used the facts that $(w^{i,j}_k)^* V x^{i,j}_k = 0, \forall k$ and $x_k = 0, k \in \{0, \ldots, j\}$. Since $\|\Delta\|_\infty \leq 1$ and $\Delta$ is causal, we see that the sum quadratic constraint

$$
\sum_{k=0}^{\bar{N}} [(q^{i,j}_k)^* q^{i,j}_k - (d^{i,j}_k)^* d^{i,j}_k] \geq 0 \tag{3.43}
$$

is satisfied. Also, $P \succ 0 \Rightarrow x^*_{N+1} P x^*_{N+1} \geq 0$. Therefore, denoting the $i$th standard Euclidean basis vector as $e_i$ and defining $\bar{p}_{i,j} := (p^{i,j}_0, p^{i,j}_1, \ldots)$, we have that

$$
e^*_i W e_i \geq \sum_{k=0}^{\bar{N}} [(p^{i,j}_k)^* p^{i,j}_k], \forall \bar{N} \in \{j, j+1, \ldots\} \tag{3.44}
$$

$$\Rightarrow e^*_i W e_i \geq \|\bar{p}_{i,j}\|_{\ell^2+}^2 \tag{3.45}
$$

$$\Rightarrow \text{tr}\{W\} = \sum_{i=1}^{n_w} e^*_i W e_i \geq \sum_{i=1}^{n_w} \|\bar{p}_{i,j}\|_{\ell^2+}^2 \tag{3.46}
$$

Since this same argument applies regardless of $j$ (i.e. the time step at which we apply the impulses), we see that

$$
\text{tr}\{W\} \geq \limsup_{N \to \infty} \frac{1}{N+1} \sum_{j=0}^{N} \sum_{i=1}^{n_w} \|\bar{p}_{i,j}\|_{\ell^2+}^2 = \|\hat{G}\|_2^2. \tag{3.47}
$$

To recover the strict inequality, note that $\mathcal{M}_{\Sigma G}(\tau, P, W - \epsilon I, V) \succ 0$ for sufficiently small $\epsilon > 0$. Thus, applying the above argument yields $\text{tr}\{W - \epsilon I\} \geq \|\hat{G}\|_2^2 \Rightarrow \text{tr}\{W\} > \|\hat{G}\|_2^2$. Since $\Delta$ is arbitrary, this concludes the proof. $\blacksquare$
It should be noted that one of the key steps in the preceding proof was using the dynamic equations that govern $G$. This was possible because the system matrix

$$
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
$$

(3.48)
appeared explicitly in the LMI $\mathcal{M}_{\Sigma_G}(\tau,P,W,V) \succ 0$. If we had used condition 2 of Theorem 3.1.1 to build a characterization of robust $\mathcal{H}_2$ performance instead of condition 3, the system matrix would not have appeared explicitly in the relevant LMIs, which would leave us unable to prove a result analogous to the one in Theorem 3.1.6. Thus, this partially justifies our usage of the new characterization of nominal $\mathcal{H}_2$ performance given in Theorem 3.1.1. We will also see in Chapter 4 that using the new characterization allows us to solve optimal full information control problem in terms of DARE solutions.

In the preceding theorem, the LMI condition was exactly the same as the LMI condition in §3.1.2. Therefore, to find the best upper bound of this type on the robust generalized $\mathcal{H}_2$ performance of the interconnection in Fig. 3.1, we use the same approach in §3.1.2; we solve the SDP (3.33) to find the $\mathcal{H}_2$ guaranteed cost of $G$. In this case, however, the feasibility result in Theorem 3.1.5 is much more meaningful; if the interconnection in Fig. 3.1 is robustly stable over $\Delta \in \Delta_d$, then there will exist some finite upper bound on its robust generalized $\mathcal{H}_2$ performance.

### 3.2 Riccati Equation Approach

#### 3.2.1 $\tau$-Specific $\mathcal{H}_2$ Guaranteed Cost

In §3.1.2, we showed how to find the $\mathcal{H}_2$ guaranteed cost of $G$ using an SDP. In this subsection, we use the results of §2.6 to show that if we fix the value of $\tau > 0$, the optimization problem can be solved using the DARE $\mathcal{R}_\phi(P) = P$ where

$$
\phi := (A,B_1,Q,R,S), \quad Q := \tau C_1^*C_1 + C_2^*C_2,
R := \tau(D_{11}^*D_{11} - I) + D_{21}^*D_{21}, \quad S := \tau C_1^*D_{11} + C_2^*D_{21}.
$$

(3.49)

Note that this corresponds to solving the optimization problem

$$
J_{\tau}(G) := \inf_{P,W,V} \text{tr}\{W\} \quad \text{s.t.} \quad P \succ 0, \quad \mathcal{M}_{\Sigma_G}(\tau,P,W,V) \succ 0.
$$

(3.50)

With this in mind, we make the following definition.

**Definition 3.2.1.** $\sqrt{J_{\tau}(G)}$ will be called the $\tau$-specific $\mathcal{H}_2$ guaranteed cost of $G$.

As is implied by the notation, $J_{\tau}(G)$ is independent of the realization of $G$; this will be shown in §3.5. When (3.50) is infeasible, we use the convention that $J_{\tau}(G) = \infty$. We now give a theorem that examines the feasibility of (3.50).
Theorem 3.2.2. For given $\tau > 0$, the following are equivalent:

1. $\exists P \succ 0, W, V$ such that $\mathcal{M}_{\Sigma_G}(\tau, P, W, V) \succ 0$

2. $A$ is Schur and the DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that $B_1^*P_0B_1 + R \prec 0$

3. The DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0 \succeq 0$ such that $B_1^*P_0B_1 + R \prec 0$

Moreover, for any $P$ that satisfies the first condition, $P_0 \prec P$.

Proof. $(1 \Rightarrow 2)$ Looking at the upper left block of the inequality $\mathcal{M}_{\Sigma_G}(\tau, P, W, V) \succ 0$, we see that $P - A^*PA \succ Q \succeq 0$ which implies that $A$ is Schur. Note in particular that $(A, B_1)$ is trivially stabilizable. Removing the third row and column of $\mathcal{M}_{\Sigma_G}(\tau, P, W, V) \succ 0$, we see that $\mathcal{L}_\phi(P) \prec 0$. By Theorem 2.6.3, the DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that $B_1^*P_0B_1 + R \prec 0$. Moreover, $P_0 \prec P$.

$(2 \Rightarrow 3)$ The DARE can be written

$$P_0 - A^*P_0A = Q - (A^*P_0B_1 + S)(B_1^*P_0B_1 + R)^{-1}(B_1^*P_0A + S^*) \succeq Q \succeq 0. \quad (3.51)$$

Since $A$ is Schur, this implies that $P_0 \succeq 0$.

$(3 \Rightarrow 1)$ By Theorem 2.6.3, choose $P$ such that $\mathcal{L}_\phi(P) \prec 0$. Note that $P \succ P_0 \succeq 0$. Since the condition $\mathcal{L}_\phi(P) \prec 0$ is the first two rows and columns of the condition $\mathcal{M}_{\Sigma_G}(\tau, P, W, V) \succ 0$, we apply Proposition 2.2.3 to finish the proof. \[\blacksquare\]

For the purposes of numerical implementation, the second condition of the preceding theorem is the most useful. In particular, since the algorithm developed later in this section will require us to solve the DARE $\mathcal{R}_\phi(P) = P$ for many values of $\tau$, it is more efficient to check once that $A$ is Schur rather than verifying for each choice of $\tau$ that the corresponding stabilizing DARE solution satisfies $P_0 \succeq 0$. Moreover, when $P_0 \succeq 0$ is singular or nearly singular, small numerical errors could cause us in some instances to incorrectly conclude that $P_0 \not\succeq 0$.

Define the parameters

$$\psi := (B_2, B_1, Q_W, R, S_W)$$

$$Q_W := \tau D_{12}^* D_{12} + D_{22}^* D_{22}$$

$$S_W := \tau D_{12}^* D_{11} + D_{22}^* D_{21}. \quad (3.52)$$

We now give the solution of (3.50) in terms of a stabilizing DARE solution.

Theorem 3.2.3. Let $\tau > 0$. If $A$ is Schur and the DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that $B_1^*P_0B_1 + R \prec 0$, then $J_\tau(G) = \text{tr}\{\mathcal{R}_\psi(P_0)\}$. Otherwise, $J_\tau(G) = \infty$. 


Proof. Clearly, if the second condition of Theorem 3.2.2 is not met, then \( J_\tau(G) = \infty \). We now assume that the second condition of Theorem 3.2.2 is met. Define \( W_0 := R_\psi(P_0) \) and

\[
Q_V := \tau D_{12}^* C_1 + D_{22}^* C_2
\]

\[
V_0 := B_2^* P_0 A + Q_V - (B_2^* P_0 B_1 + S_W)(B_1^* P_0 B_1 + R)^{-1}(B_1^* P_0 A + S^*).
\]  

(3.53)

(3.54)

Now let \( \bar{P} \succ 0, \bar{W}, \bar{V} \) satisfy \( M_{\Sigma G}(\tau, \bar{P}, \bar{W}, \bar{V}) \succ 0 \). By Theorem 3.2.2, \( \bar{P} \succ P_0 \). Eliminating the first row and column of \( M_{\Sigma G}(\tau, \bar{P}, \bar{W}, \bar{V}) \), we see that

\[
0 \prec \begin{bmatrix}
\tau I & 0 \\
0 & W
\end{bmatrix} - \begin{bmatrix}
B_1 & B_2 \\
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}^* \begin{bmatrix}
P & 0 & 0 \\
0 & \tau I & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
B_1 & B_2 \\
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\]

\[
\prec \begin{bmatrix}
B_1 & B_2 \\
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}^* \begin{bmatrix}
P_0 & 0 & 0 \\
0 & \tau I & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
B_1 & B_2 \\
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-B_1^* P_0 B_1 - R \\
-B_2^* P_0 B_2 - S_W
\end{bmatrix} \bar{W} - \begin{bmatrix}
B_2^* P_0 B_2 - Q_V
\end{bmatrix}.
\]  

(3.55)

By Schur complements, this implies that \( \bar{W} \succ W_0 \), which in turn implies that \( J_\tau(G) \geq \text{tr}\{W_0\} \).

We now finish the proof by showing that \( J_\tau(G) \leq \text{tr}\{W_0\} \). Since (3.50) is feasible, we apply Proposition 2.2.1 to yield

\[
J_\tau(G) = \min_{P,W,V} \text{tr}\{W\} \quad \text{s.t.} \quad P \succeq 0, \ M_{\Sigma G}(\tau, P, W, V) \succeq 0.
\]  

(3.56)

Defining

\[
L := \begin{bmatrix}
I & \mathcal{K}_\phi(P_0) & 0 \\
0 & \mathcal{K}_\psi(P_0) & I \\
0 & I & 0
\end{bmatrix}
\]

(3.57)

we see after some algebra that

\[
LM_{\Sigma G}(\tau, P_0, W_0, V_0) L^* = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -(B_1^* P_0 B_1 + R)
\end{bmatrix} \succeq 0.
\]  

(3.58)

Since \( L \) is invertible, this implies that \( M_{\Sigma G}(\tau, P_0, W_0, V_0) \succeq 0 \). Also note that, by Theorem 3.2.2, \( P_0 \succeq 0 \). Therefore, \( J_\tau(G) \leq \text{tr}\{W_0\} \).

The following corollary, which gives necessary and sufficient conditions for \( J_\tau(G) \) being finite, will be useful later in this dissertation.
Corollary 3.2.4. Let $\tau > 0$. $J_\tau(G) \neq \infty$ if and only if the condition

$$\left\| \begin{bmatrix} \sqrt{\tau} I_{n_q} & 0 \\ 0 & I_{n_p} \end{bmatrix} G \begin{bmatrix} I_{n_d} \\ 0 \end{bmatrix} \right\|_\infty^2 < \tau$$  \hspace{1cm} (3.59)$$

holds.

Proof. Noting that the system of interest has the state-space realization

$$\begin{bmatrix} \sqrt{\tau} I_{n_q} & 0 \\ 0 & I_{n_p} \end{bmatrix} G \begin{bmatrix} I_{n_d} \\ 0 \end{bmatrix} \sim \begin{bmatrix} A & B_1 \\ \sqrt{\tau} C_1 & \sqrt{\tau} D_{11} \\ C_2 & D_{21} \end{bmatrix}$$ \hspace{1cm} (3.60)$$
we see by standard $\mathcal{H}_\infty$ theory (e.g. [27]) that (3.59) holds if and only if $A$ is Schur and the DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that $B_1^*P_0B_1 + R \prec 0$. By Theorem 3.2.3, this is equivalent to the condition

$$J_\tau(G) \neq \infty.$$ 

We now make a few notes on the numerical evaluation of $J_\tau(G)$. Once we have solved the DARE $\mathcal{R}_\phi(P) = P$ for its stabilizing solution $P_0$, we check the condition $B_1^*P_0B_1 + R \prec 0$ by performing the Cholesky factorization

$$ LL^* = -(B_1^*P_0B_1 + R).$$ \hspace{1cm} (3.61)$$
Using the Cholesky factor, we then express

$$ \mathcal{R}_\psi(P_0) = B_2^*P_0B_2 + \tau D_{12}^*D_{12} + D_{22}^*D_{22} + [L^{-1}(B_1^*P_0B_2 + S_W^*)]^*[L^{-1}(B_1^*P_0B_2 + S_W^*)].$$ \hspace{1cm} (3.62)$$
Using the fact that

$$\text{tr}\{B_2^*P_0B_2\} = \langle P_0B_2, B_2 \rangle = \langle P_0, B_2B_2^* \rangle = \text{sum}(P_0 \circ (B_2B_2^*))$$ \hspace{1cm} (3.63)$$
we see that

$$J_\tau(G) = \text{sum}(P_0 \circ (B_2B_2^*)) + \tau\|D_{12}\|_F^2 + \|D_{22}\|_F^2 + \|L^{-1}(B_1^*P_0B_2 + S_W^*)\|_F^2.$$ \hspace{1cm} (3.64)$$

This approach to computing $J_\tau(G)$ is more efficient than directly using the formula in Theorem 3.2.3 for two reasons. First, (3.64) minimizes the number of dense matrix-matrix multiplications by exploiting the inherent Frobenius norm and the Hadamard product structure. Second, (3.64) avoids inverting the dense matrix $B_1^*P_0B_1 + R$ and instead only has to invert the lower triangular matrix $L$. Avoiding the dense matrix inversion also improves the accuracy of this approach. In particular, since the condition number of $B_1^*P_0B_1 + R$ with respect to inversion is the square of the condition number of $L$, forming the product $L^{-1}(B_1^*P_0B_2 + S_W^*)$ tends to produce smaller numerical errors than those induced by forming the product $(B_1^*P_0B_1 + R)^{-1}(B_1^*P_0B_2 + S_W^*)$. 


In this approach, we do not exploit the fact that \( P_0 \succeq 0 \). Although there exist methods for continuous-time systems that directly compute the Cholesky factor of the \( P_0 \) without explicitly computing \( P_0 \) \cite{14,33}, these methods are not currently implemented by the Control Systems Toolbox for MATLAB. Although it is possible to write discrete-time versions of these algorithms ourselves, we have elected not to do so for the sake of simplicity.

With (3.64) in mind, we can evaluate \( J_\tau(G) \) using the following algorithm once we have verified that \( A \) is Schur.

**Algorithm 3.2.5.** The following algorithm computes \( J_\tau(G) \) under the assumption that \( A \) is Schur.

1. Find the stabilizing solution of the DARE \( P_0 = \mathcal{R}_\phi(P_0) \)
2. Compute the Cholesky factorization \( LL^* = -(B_1^*P_0B_1 + R) \)
3. \( \tilde{K} = L\backslash(B_1^*P_0B_2 + S_W^*) \)
4. \( J_\tau(G) = \sum(P_0 \circ (B_2B_2^*)) + \tau\|D_{12}\|_F^2 + \|D_{22}\|_F^2 + \|\tilde{K}\|_F^2 \)

If either of the first two steps fail, then \( J_\tau(G) = \infty \).

### 3.2.2 \( \mathcal{H}_2 \) Guaranteed Cost

In the previous subsection, we showed that the \( \tau \)-specific \( \mathcal{H}_2 \) guaranteed cost of \( G \) (i.e. \( \sqrt{J_\tau(G)} \)) could be determined using a single DARE. In this subsection, we develop an algorithm that finds the \( \mathcal{H}_2 \) guaranteed cost of \( G \) using an iteration of DARE solutions.

First note that we can equivalently reformulate (3.32) as the optimization

\[
\inf_{\tau > 0} J_\tau(G)
\]

where \( J_\tau(G) \) is defined in (3.50). This corresponds to breaking the convex optimization (3.32) into two optimizations; we first optimize over \( P,W,V \) to produce the cost function \( J_\tau(G) \) and then optimize over \( \tau \) to yield the squared \( \mathcal{H}_2 \) guaranteed cost of \( G \). A basic result about convex functions is that if \( f(x,y) \) is a convex function of \( x \) and \( y \), then \( \inf_y f(x,y) \) is a convex function of \( x \) (e.g. \cite{4}). Define

\[
f(\tau,(P,W,V)) := \begin{cases} 
\text{tr}\{W\}, & \tau > 0, \ P > 0, \ \mathcal{M}_{\Sigma_G}(\tau,P,W,V) > 0 \\
\infty, & \text{otherwise}
\end{cases}
\]

and note that \( f \) is convex. Since (3.32) can be written as

\[
\inf_{\tau,(P,W,V)} f(\tau,(P,W,V))
\]
we see that the function
\[
\inf_{P,W,V} f(\tau, (P,W,V)) \tag{3.68}
\]
is a convex function of \(\tau\). Since this function is the same as \(J_\tau(G)\) for \(\tau > 0\), we see that \(J_\tau(G)\) is a convex function of \(\tau > 0\), which implies that (3.65) is a nonlinear convex optimization.

Now we present a result which makes it especially easy to find values of \(\tau\) for which \(J_\tau(G) \neq \infty\).

**Proposition 3.2.6.** If the \(H_2\) guaranteed cost of \(G\) is finite, then the set of \(\tau > 0\) for which \(J_\tau(G) \neq \infty\) is the interval \((\tau, \infty)\) for some \(\tau \geq 0\).

**Proof.** The set of \(\tau > 0, P > 0, W, V\) for which \(\mathcal{M}_{\Sigma_G}(\tau, P, W, V) > 0\) is a convex open set. Therefore, the set
\[
S := \{\tau > 0 : \exists P > 0, W, V \text{ satisfying } \mathcal{M}_{\Sigma_G}(\tau, P, W, V) > 0\} \tag{3.69}
\]
is a convex open set. Note in particular that \(J_\tau(G) \neq \infty \Leftrightarrow \tau \in S\).

It now remains to show that \(S\) is unbounded above. Let \(\tau \in S\) and \(\alpha > 1\). Since \(J_\tau(G) \neq \infty\), choose \(P > 0, W, V\) such that \(\mathcal{M}_{\Sigma_G}(\tau, P, W, V) > 0\). Multiplying the inequality \(\mathcal{M}_{\Sigma_G}(\tau, P, W, V) > 0\) by \(\alpha\), we see that
\[
\begin{bmatrix}
\alpha P & \bullet & \bullet \\
0 & \alpha \tau I & \bullet \\
\alpha V & 0 & \alpha W
\end{bmatrix} \succ
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}^*
\begin{bmatrix}
\alpha P & 0 & 0 \\
0 & \alpha \tau I & 0 \\
0 & 0 & \alpha I
\end{bmatrix}
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} \tag{3.70}
\]
Thus, \(\mathcal{M}_{\Sigma_G}(\alpha \tau, \alpha P, \alpha W, \alpha V) > 0\) and \(\alpha P > 0\). This implies that \(J_{\alpha \tau}(G) \neq \infty\), which in turn implies that \(\alpha \tau \in S\). Since the choice of \(\alpha > 1\) was arbitrary, this concludes the proof. \(\blacksquare\)

We now analyze how \(J_\tau(G)\) varies as \(\tau\) is varied. Since the stabilizing solution of a DARE is analytic in its parameters \([9]\) and \(J_\tau(G)\) is an analytic function of the stabilizing DARE solution, we see immediately that \(J_\tau(G)\) is an analytic function of \(\tau\). Thus, we would like to construct an efficient algorithm for computing \((d/d\tau)(J_\tau(G))\).

Let \(P_0\) be the stabilizing solution of the DARE \(\mathcal{R}_\phi(P_0) = P_0\) where \(\phi\) is defined in (3.49). Implicitly differentiating the DARE \(\mathcal{R}_\phi(P_0) = P_0\) with respect to \(\tau\) yields
\[
\frac{dP_0}{d\tau} = A^* \frac{dP_0}{d\tau} A + C_1^* C_1 + \left( A^* \frac{dP_0}{d\tau} B_1 + C_1^* D_{11} \right) K_\phi(P_0) \\
+ K_\phi(P_0)^* \left( B_1^* \frac{dP_0}{d\tau} A + D_{11}^* C_1 \right) + K_\phi(P_0)^* \left( B_1^* \frac{dP_0}{d\tau} B_1 + D_{11}^* D_{11} - I \right) K_\phi(P_0) . \tag{3.71}
\]
Therefore, \( dP_0/d\tau \) satisfies the discrete Lyapunov equation

\[
\frac{dP_0}{d\tau} = A_\phi(P_0)\frac{dP_0}{d\tau} A_\phi(P_0) + (C_1 + D_{11}K_\phi(P_0))(C_1 + D_{11}K_\phi(P_0)) - K_\phi(P_0)^*K_\phi(P_0) .
\] (3.72)

Since \( A_\phi(P_0) \) is Schur (by definition of the stabilizing DARE solution), this discrete Lyapunov equation can always be uniquely solved for \( dP_0/d\tau \). With this in place, we can express after some algebra that

\[
\frac{d}{d\tau}(J_\tau(G)) = \text{tr}\left\{ \frac{dP_0}{d\tau} R_\psi(P_0) \right\} = \text{tr}\left\{ A_\psi(P_0) \frac{dP_0}{d\tau} A_\psi(P_0) - K_\psi(P_0)^*K_\psi(P_0) \right. \\
\left. + (D_{12} + D_{11}K_\psi(P_0))^*(D_{12} + D_{11}K_\psi(P_0)) \right\} (3.73)
\]

where \( \psi \) is defined in (3.52). Using the fact that

\[
\text{tr}\left\{ A_\psi(P_0) \frac{dP_0}{d\tau} A_\psi(P_0) \right\} = \left\langle \frac{dP_0}{d\tau} A_\psi(P_0), A_\psi(P_0) \right\rangle = \left\langle \frac{dP_0}{d\tau}, A_\psi(P_0)A_\psi(P_0)^* \right\rangle
\]

\[
= \text{sum}\left( \frac{dP_0}{d\tau} \circ (A_\psi(P_0)A_\psi(P_0)^*) \right) (3.74)
\]

we see that

\[
\frac{d}{d\tau}(J_\tau(G)) = \text{sum}\left( \frac{dP_0}{d\tau} \circ (A_\psi(P_0)A_\psi(P_0)^*) \right) + \|D_{12} + D_{11}K_\psi(P_0)\|_F^2 - \|K_\psi(P_0)\|_F^2 .
\] (3.75)

With this in place, we now give an algorithm for finding \( J_\tau(G) \) and its derivative at a given value of \( \tau > 0 \).

**Algorithm 3.2.7.** The following algorithm computes \( \frac{d}{d\tau}(J_\tau(G)) \).

1. Use Algorithm 3.2.5 to compute \( P_0, L, K, J_\tau(G) \)
2. \( K_\phi(P_0) = L^*\backslash(L\backslash(B_1^*P_0A + S^*)) \)
3. \( A_\phi(P_0) = A + B_1K_\phi(P_0) \)
4. \( Q_{\text{Lyap}} = (C_1 + D_{11}K_\phi(P_0))^*(C_1 + D_{11}K_\phi(P_0)) - K_\phi(P_0)^*K_\phi(P_0) \)
5. Solve the discrete Lyapunov equation \( \frac{dP_0}{d\tau} = A_\phi(P_0)^*\frac{dP_0}{d\tau} A_\phi(P_0) + Q_{\text{Lyap}} \)
6. \( K_\psi(P_0) = L^*\backslash K \)
7. \( A_\psi(P_0) = B_2 + B_1K_\psi(P_0) \)
8. \[ \frac{d}{d\tau}(J_\tau(G)) = \text{sum} \left( \frac{dP_0}{d\tau} \circ (A_\psi(P_0)A_\psi(P_0)^*) \right) + \|D_{12} + D_{11}K_\psi(P_0)\|_F^2 - \|K_\psi(P_0)\|_F^2. \]

The value and derivative of \( J_\tau(G) \) is also useful for generating a lower bound on the squared \( \mathcal{H}_2 \) guaranteed cost of \( G \). Consider Fig. 3.2, which shows a representative graph of \( J_\tau(G) \) in which \( \tau_0 \) is known to be an upper bound on the minimizing value of \( \tau \). By convexity, if \( \tau_1 \) is known to be a lower bound on the minimizing value of \( \tau \), the value and derivative of \( J_\tau(G) \) at \( \tau_0 \) gives us the lower bound \( \hat{J}_1 \). If instead, the value and derivative of \( J_\tau(G) \) at \( \tau_0 \) and \( \tau_2 \) are known, we have the lower bound \( \hat{J}_2 \). These lower bounds are respectively given by

\[
\hat{J}_1 = J_{\tau_0} - m_0(\tau_0 - \tau_1) \tag{3.76a}
\]
\[
\hat{J}_2 = \frac{m_2[m_0(\tau_0 - \tau_2) - (J_{\tau_0} - J_{\tau_2})]}{m_0 - m_2} + J_{\tau_2} \tag{3.76b}
\]

where \( m_i \) is the value of \( \frac{d}{d\tau}(J_\tau(G)) \) evaluated at \( \tau = \tau_i \). It should be noted that (3.76b) is less conservative when it is applicable.

With these results in place, we can easily solve (3.65) (and, hence, (3.32)) using the following algorithm.

**Algorithm 3.2.8.** The following algorithm computes the \( \mathcal{H}_2 \) guaranteed cost of \( G \).

1. **Check Feasibility:** Use the third condition of Theorem 3.1.5 to verify that (3.32) is feasible.

2. **Find Initial Interval:** Choose \( \alpha > 1 \). Starting from \( k = 0 \), iterate over \( k \) until a value of \( \tau = \alpha^k \) is found such that \( J_\tau(G) \neq \infty \) and \( (d/d\tau)(J_\tau(G)) > 0 \). Denote this value of value of \( \tau \) by \( \tau_u \); this corresponds to an upper bound on the optimal value of \( \tau \). If \( \tau_u = 1 \), then 0 is a lower bound on the optimal value of \( \tau \), otherwise \( \tau_u/\alpha \) is a lower bound.

3. **Bisection:** Solve the equation \( (d/d\tau)(J_\tau(G)) = 0 \) over \( \tau \) using bisection. Whenever \( J_\tau(G) = \infty \), this corresponds to a lower bound on the optimal value of \( \tau \).
In this algorithm, each evaluation of $J_\tau(G)$ is done using Algorithm 3.2.5 and each evaluation of $(d/d\tau)(J_\tau(G))$ when $J_\tau(G) \neq \infty$ is done using Algorithm 3.2.7.

It should be noted that the condition that $A$ is Schur is only checked once in this algorithm—in the first step. In our implementation, we use $\alpha = 100$. Also, except when the lower bound on the optimal value of $\tau$ is 0, we use the geometric mean instead of the arithmetic mean in the bisection step to better deal with large intervals in which the optimal value of $\tau$ could lie. In the bisection step, we use two stopping criteria; if we define the relative error as $\nu := 1 - \hat{J}/J_\tau(G)$ where $\hat{J}$ is the lower bound given in (3.76), we terminate the algorithm when either $\nu < 10^{-10}$ or the number of iterations (including the iterations required to find the initial interval) exceeds 30.

3.3 Numerical Experiments

In this section, we consider the application of the developed methodologies to randomly generated guaranteed cost performance problems. In particular, we consider three approaches: using the DARE approach in Algorithm 3.2.8, solving (3.33) using SeDuMi (parsed using YALMIP [22]), and solving (3.33) using the mincx command in the Robust Control Toolbox. The last two of these methods will be collectively called the LMI methods. It should be noted that YALMIP was not used when using mincx because YALMIP causes mincx to run more slowly. All numerical experiments were performed in MATLAB (with multithreaded computation disabled) on a computer with a 2.2 GHz Intel Core 2 Duo Processor and 2 GB of RAM.

To generate the random systems in our numerical experiments, we first generated a random stable discrete-time state space system using drss in MATLAB and then multiplied the system by the inverse of its $H_\infty$ norm (computed by the norm function). This system was then multiplied by a random number generated from a uniform distribution on $[-1, 1]$. The resulting system corresponded to generating random values of $A, B_1, C_1, D_{11}$ for a robustly stable system. The entries of $B_2, C_2, D_{12}, D_{21}$, and $D_{22}$ were generated randomly from independent normal distributions. For all of the numerical experiments, we chose the signal dimensions to be $n_q = 5, n_d = 6, n_p = 7, n_w = 8$.

In the first experiment, we tested the speed of the methodologies over several values of $n_x$. The results of this test are shown in Fig. 3.3. In particular, note that the DARE method is faster than the LMI methods for all of the randomly generated problems. For instance, for the 43rd-order system, it respectively took the DARE approach, the mincx approach, and the SeDuMi approach 1.27 seconds (averaged over several runs), 947.75 seconds, and 307.76 seconds to compute the $H_2$ guaranteed cost. Also note that the DARE method appears to have a complexity of $O(n_x^2)$ whereas the SeDuMi method appears to have a complexity of $O(n_x^5)$. The curve which corresponds to the mincx method is not smooth because the number of iterations required to solve the problem often changes dramatically from problem to problem, unlike the other two methods. Nonetheless, since the computational time required
for the mincx method is similar to that required by the SeDuMi method, it appears to have a complexity of roughly \( O(n_x^5) \) also. Thus, the difference in computational speed between the DARE approach and the other two approaches becomes more pronounced for larger values of \( n_x \).

In the second experiment, we tested the accuracy of the DARE approach compared to the LMI approaches for 400 randomly generated analysis problems with \( n_x = 20 \). To this end, we first define \( f_d \), \( f_m \), and \( f_s \) as the values of the squared \( H_2 \) guaranteed cost respectively computed using the DARE approach, mincx, and SeDuMi. The criterion we will be using to compare the accuracy of the relevant methods is the relative error, i.e. we use the criterion \( \nu_m := |1 - f_m/f_d| \) to compare the accuracy of the mincx approach to the DARE approach and the criterion \( \nu_s := |1 - f_s/f_d| \) to compare the accuracy of the SeDuMi approach to the DARE approach. For both comparisons, the results are split into two categories—cases in which the LMI approach reports a smaller cost and cases in which the DARE approach reports a smaller cost.

Fig. 3.4 shows the histogram of \( \nu_m \) for the cases in which mincx reports a smaller cost than the DARE approach. Note that all of these values are small, i.e. the mincx approach never significantly beats the DARE approach in terms of accuracy. Fig. 3.5 shows the histogram of \( \nu_m \) for the cases in which mincx reports a larger cost than the DARE approach. This figure
shows that the DARE approach often report a much smaller cost than the mincx approach. Since both of these methods require all of the iterates to be feasible, we conclude that the DARE approach had better accuracy in these cases and the difference in reported cost is due to mincx getting “stuck” due to numerical problems. Fig. 3.6 shows the histogram of $\nu_s$ for the cases in which SeDuMi reports a smaller cost than the DARE approach. Unlike the mincx approach, the SeDuMi approach generates significantly smaller estimates of the $\mathcal{H}_2$ guaranteed costs than the DARE approach for 14 cases. However, for each of these 14 cases, the cost of the nominal system computed by the norm function in MATLAB is larger than the $\mathcal{H}_2$ guaranteed cost determined by SeDuMi. Since this is a contradiction, we conclude that these results correspond to cases in which SeDuMi failed. In particular, since SeDuMi uses infeasible path-following algorithms, not all of the iterates are guaranteed to be feasible. In these 14 cases, the final iterate was not feasible. Since, in infinite precision, the feasibility of one iterate implies feasibility of the next iterate for these infeasible path-following algorithms, it is likely that SeDuMi failed to find any feasible iterates for these 14 cases. Fig. 3.7 shows the histogram of $\nu_s$ for the cases in which SeDuMi reports a larger cost than the DARE approach. This figure shows that, like the mincx method, the SeDuMi method sometimes gets “stuck” due to numerical problems. This suggests that the accuracy of the DARE method is superior to that of the LMI methods.
CHAPTER 3. $\mathcal{H}_2$ GUARANTEED COST ANALYSIS

3.4 Alternate Matrix Inequality Characterizations

In this section, we give alternate matrix inequality characterizations of the $\mathcal{H}_2$ guaranteed cost of $G$ and the $\tau$-specific $\mathcal{H}_2$ guaranteed cost of $G$. Although performing computations with these alternate characterizations is typically not as efficient or accurate as performing computations with the characterizations given earlier in this chapter, they will form the basis of our analysis in Chapters 4 and 5.

We begin by defining

$$M_{\Sigma G}(\tau, P, W, V) := \begin{bmatrix} P & \tau I & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \tau I & \bullet & \bullet & \bullet & \bullet & \bullet \\ V & 0 & W & \bullet & \bullet & \bullet & \bullet \\ PA & PB_1 & PB_2 & P & \bullet & \bullet & \bullet \\ \tau C_1 & \tau D_{11} & \tau D_{12} & 0 & \tau I & \bullet & \bullet \\ C_2 & D_{21} & D_{22} & 0 & 0 & I & \bullet \end{bmatrix}$$

(3.77)

$$M_{\Sigma G}^{mix}(\tau, P, W, V) := \begin{bmatrix} P & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \tau I & \bullet & \bullet & \bullet & \bullet & \bullet \\ V & 0 & W & \bullet & \bullet & \bullet & \bullet \\ A & B_1 & B_2 & P^{-1} & \bullet & \bullet & \bullet \\ C_1 & D_{11} & D_{12} & 0 & \tau^{-1}I & \bullet & \bullet \\ C_2 & D_{21} & D_{22} & 0 & 0 & I & \bullet \end{bmatrix}$$

(3.78)

$$M_{\Sigma G}^{inv}(\epsilon, \hat{P}, W, \hat{V}) := \begin{bmatrix} \hat{P} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \epsilon I & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hat{V} & 0 & W & \bullet & \bullet & \bullet & \bullet \\ A\hat{P} & \epsilon B_1 & B_2 & \hat{P} & \bullet & \bullet & \bullet \\ C_1\hat{P} & \epsilon D_{11} & D_{12} & 0 & \epsilon I & \bullet & \bullet \\ C_2\hat{P} & \epsilon D_{21} & D_{22} & 0 & 0 & I & \bullet \end{bmatrix}$$

(3.79)

Note that we require $P$ to be nonsingular and $\tau \neq 0$ for $M_{\Sigma G}^{mix}(\tau, P, W, V)$ to be well-defined.

**Proposition 3.4.1.** Given $\tau, P, W, V$, the following four conditions are equivalent:
1. \( \tau > 0, P > 0 \), and \( \mathcal{M}_{\Sigma G}^G(\tau, P, W, V) > 0 \)

2. \( \tilde{\mathcal{M}}_{\Sigma G}^G(\tau, P, W, V) > 0 \)

3. \( \mathcal{M}_{\Sigma G}^{mix}(\tau, P, W, V) > 0 \)

4. \( \mathcal{M}_{\Sigma G}^{inv}(\tau^{-1}, P^{-1}, W, VP^{-1}) > 0 \)

Proof. (1 \( \iff \) 2) By Schur complements, \( \tilde{\mathcal{M}}_{\Sigma G}^G(\tau, P, W, V) > 0 \) if and only if \( \tau > 0, P > 0 \) and

\[
\begin{bmatrix}
P & \bullet & \bullet \\
0 & \tau I & \bullet \\
V & 0 & W
\end{bmatrix} - \begin{bmatrix}
PA & PB_1 & PB_2 \\
\tau C_1 & \tau D_{11} & \tau D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}^* \begin{bmatrix}
P^{-1} & 0 & 0 \\
0 & \tau^{-1} I & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
P & PB_1 & PB_2 \\
\tau C_1 & \tau D_{11} & \tau D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} > 0. \tag{3.80}
\]

Since

\[
\begin{bmatrix}
P & PB_1 & PB_2 \\
\tau C_1 & \tau D_{11} & \tau D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} = \begin{bmatrix}
P & 0 & 0 \\
0 & \tau I & 0 \\
0 & 0 & I
\end{bmatrix} \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} \tag{3.81}
\]

we see that (3.80) is the condition \( \mathcal{M}_{\Sigma G}^G(\tau, P, W, V) > 0 \).

(1 \( \iff \) 3) By Schur complements, \( \mathcal{M}_{\Sigma G}^{mix}(\tau, P, W, V) > 0 \) if and only if \( \tau > 0, P > 0 \), and \( \mathcal{M}_{\Sigma G}^G(\tau, P, W, V) > 0 \).

(3 \( \Rightarrow \) 4) First note that \( P \) is nonsingular and \( \tau \neq 0 \). Therefore, the symmetric matrix

\[
L := \text{diag}(P, \tau I, I, I, I, I)
\tag{3.82}
\]

is invertible. Noting that \( L^{-1} \mathcal{M}_{\Sigma G}^{mix}(\tau, P, W, V) L^{-1} = \mathcal{M}_{\Sigma G}^{inv}(\tau^{-1}, P^{-1}, W, VP^{-1}) \), we use the invertibility of \( L \) to see that \( \mathcal{M}_{\Sigma G}^{mix}(\tau^{-1}, P^{-1}, W, VP^{-1}) > 0 \).

(3 \( \iff \) 4) First note that condition 4 implies that \( P \) is nonsingular and \( \tau \neq 0 \). Therefore, the matrix \( L \) defined in (3.82) is nonsingular. Noting that \( L \mathcal{M}_{\Sigma G}^{inv}(\tau^{-1}, P^{-1}, W, VP^{-1}) L = \mathcal{M}_{\Sigma G}^{mix}(\tau, P, W, V) \), we use the invertibility of \( L \) to see that \( \mathcal{M}_{\Sigma G}^{mix}(\tau, P, W, V) > 0 \). ■

We now give an alternate interpretation of condition 4 in the preceding proposition.

**Proposition 3.4.2.** Given \( W \), \( \exists \tau, P, V \) such that \( \mathcal{M}_{\Sigma G}^{inv}(\tau^{-1}, P^{-1}, W, VP^{-1}) > 0 \) if and only if \( \exists \epsilon, \hat{P}, \hat{V} \) such that \( \mathcal{M}_{\Sigma G}^{inv}(\epsilon, \hat{P}, \hat{V}) > 0 \).

Proof. (\( \Rightarrow \)) Trivial.

(\( \Leftarrow \)) Since \( \mathcal{M}_{\Sigma G}^{inv}(\epsilon, \hat{P}, \hat{V}) > 0 \) implies that \( \hat{P} > 0 \) and \( \epsilon > 0 \), we see that \( \hat{P}^{-1} \) and \( \epsilon^{-1} \) exist. Choosing \( \tau = \epsilon^{-1}, P = \hat{P}^{-1}, \) and \( V = \hat{V} \hat{P}^{-1} \) completes the proof. ■
Proposition 3.4.1 and Proposition 3.4.2 give us three new characterizations of the $H_2$ guaranteed cost of $G$ and $J_\tau(G)$. In particular, the squared $H_2$ guaranteed cost of $G$ can be found by solving any of the three optimization problems

$$\inf_{\tau,P,W,V} \text{tr}\{W\} \quad \text{s.t.} \quad \bar{M}_{\Sigma_G}(\tau, P, W, V) \succ 0$$  \hspace{1cm} (3.83)

$$\inf_{\epsilon, P, W, V} \text{tr}\{W\} \quad \text{s.t.} \quad M_{\Sigma_G}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) \succ 0$$ \hspace{1cm} (3.84)

$$\inf_{\tau, P, W, V} \text{tr}\{W\} \quad \text{s.t.} \quad M_{\Sigma_G}^{\text{mix}}(\tau, P, W, V) \succ 0$$ \hspace{1cm} (3.85)

and $J_\tau(G)$ can be found for a given value of $\tau$ by solving any of the three optimization problems

$$J_\tau(G) = \inf_{P, W, V} \text{tr}\{W\} \quad \text{s.t.} \quad \bar{M}_{\Sigma_G}(\tau, P, W, V) \succ 0$$ \hspace{1cm} (3.86)

$$J_\tau(G) = \inf_{\hat{P}, W, V} \text{tr}\{W\} \quad \text{s.t.} \quad M_{\Sigma_G}^{\text{inv}}(\tau^{-1}, \hat{P}, W, \hat{V}) \succ 0$$ \hspace{1cm} (3.87)

$$J_\tau(G) = \inf_{P, W, V} \text{tr}\{W\} \quad \text{s.t.} \quad M_{\Sigma_G}^{\text{mix}}(\tau, P, W, V) \succ 0$$ \hspace{1cm} (3.88)

The constraints $\bar{M}_{\Sigma_G}(\tau, P, W, V) \succ 0$ and $M_{\Sigma_G}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) \succ 0$ are LMIs. Therefore, relaxing the strict inequalities in the optimization problems (3.83), (3.84), (3.86), and (3.88) to nonstrict inequalities results in SDPs. Of the SDP approaches to computing the $H_2$ guaranteed cost, the best one in terms of speed and accuracy is (3.33). This is due to the fact that the sizes of the LMI constraints in (3.83) and (3.84) are larger than the combined constraints in (3.33). The larger LMI constraints result in an increase in the number of dual optimization parameters, which generally increases the amount of computation required to solve the optimization problem and potentially increases the impact of numerical errors during the optimization process. Moreover, it is more numerically difficult to verify whether or not the larger LMI constraints are satisfied.

Although these alternate characterizations are not as useful as the characterizations given earlier in the chapter for computing the $H_2$ guaranteed cost and the $\tau$-specific $H_2$ guaranteed cost, the developments of Chapters 4 and 5 will be based exclusively on the alternate characterizations (3.83)–(3.88).

### 3.5 Realization Invariance

In this section we show that $J_\tau(G)$ is independent of the realization of $G$, provided that the realization is stabilizable and d-detectable. We also show that using a static scaling on the uncertainty of the system does not affect the value of the $H_2$ guaranteed cost.
Consider the systems and state-space realizations

\[ G_{\text{sim}} \sim \begin{bmatrix} T^{-1}AT & T^{-1}B_1 & T^{-1}B_2 \\ C_1T & D_{11} & D_{12} \\ C_2T & D_{21} & D_{22} \end{bmatrix} =: \Sigma_{G_{\text{sim}}} \] (3.89)

\[ G_{\text{uc}} \sim \begin{bmatrix} A & A_{1u} & B_1 & B_2 \\ 0 & A_{uc} & 0 & 0 \\ C_1 & C_{1u} & D_{11} & D_{12} \\ C_2 & C_{2u} & D_{21} & D_{22} \end{bmatrix} =: \Sigma_{G_{\text{uc}}} \] (3.90)

\[ G_{\text{uo}} \sim \begin{bmatrix} A & 0 & B_1 & B_2 \\ A_{u2} & A_{uo} & B_{u1} & B_{u2} \\ C_1 & 0 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & D_{22} \end{bmatrix} =: \Sigma_{G_{\text{uo}}} \] (3.91)

where \( A_{uc} \) and \( A_{uo} \) are Schur. Note that \( \Sigma_{G_{\text{uc}}} \) and \( \Sigma_{G_{\text{uo}}} \) are respectively uncontrollable and unobservable realizations of \( G \). Therefore, the systems \( G, G_{\text{sim}}, G_{\text{uc}}, \) and \( G_{\text{uo}} \) are the same when the initial value of the state variable for each system is zero.

Define the function

\[ \bar{J}_\tau(\Sigma_G) := \inf_{P,W,V} \text{tr}\{W\} \quad \text{s.t.} \quad P \succ 0, \quad \mathcal{M}_{\Sigma_G}(\tau, P, W, V) \succ 0. \] (3.92)

To show that \( J_\tau(G) \) is independent of the realization of \( G \), we will show that if \( \Sigma_1 \) and \( \Sigma_2 \) are both stabilizable and d-detectable state-space realizations \( G \), then \( \bar{J}_\tau(\Sigma_1) = \bar{J}_\tau(\Sigma_2) \). We begin with three lemmas that collectively prove that

\[ \bar{J}_\tau(\Sigma_G) = \bar{J}_\tau(\Sigma_{G_{\text{sim}}}) = \bar{J}_\tau(\Sigma_{G_{\text{uc}}}) = \bar{J}_\tau(\Sigma_{G_{\text{uo}}}) . \] (3.93)

**Lemma 3.5.1.** For fixed \( \tau > 0 \), \( \bar{J}_\tau(\Sigma_G) = \bar{J}_\tau(\Sigma_{G_{\text{sim}}}) \).

**Proof.** Define

\[ L := \text{diag}(T, I, I, T^{-*}, I, I) \] (3.94)

and note that \( L \) is invertible. Also note that

\[ L^* \mathcal{M}_{\Sigma_G}^{\text{mix}}(\tau, P, W, V)L = \mathcal{M}_{\Sigma_{G_{\text{sim}}}}^{\text{mix}}(\tau, T^*PT, W, VT) \] (3.95)

which implies that

\[ \mathcal{M}_{\Sigma_G}^{\text{mix}}(\tau, P, W, V) \succ 0 \quad \iff \quad \mathcal{M}_{\Sigma_{G_{\text{sim}}}}^{\text{mix}}(\tau, T^*PT, W, VT) \succ 0 . \] (3.96)

Therefore, by Proposition 3.4.1, \( \bar{J}_\tau(\Sigma_G) < \gamma \iff \bar{J}_\tau(\Sigma_{G_{\text{sim}}}) < \gamma \). This implies that \( \bar{J}_\tau(\Sigma_G) = \bar{J}_\tau(\Sigma_{G_{\text{sim}}}) \).
Lemma 3.5.2. For fixed $\tau > 0$, $\bar{J}_\tau(\Sigma_G) = \bar{J}_\tau(\Sigma_{G_{uc}})$.

Proof. In this proof, we will show that $\bar{J}_\tau(\Sigma_G) < \gamma \iff \bar{J}_\tau(\Sigma_{G_{uc}}) < \gamma$. This will allow us to conclude that $J_\tau(\Sigma_G) = J_\tau(\Sigma_{G_{uc}})$. Note that, by Proposition 3.4.1, $J_\tau(\Sigma_{G_{uc}}) < \gamma$ if and only if $\exists P, W, V$ such that $\text{tr}\{W\} < \gamma$ and $\mathcal{M}_{\Sigma_{G_{uc}}}^{\text{mix}}(\tau, P, W, V) > 0$. Partition $P$ and $V$ as

$$
P = \begin{bmatrix} P_{11} & \bullet \\ P_{21} & P_{22} \end{bmatrix}, \quad V = [V_1 \ V_2] \tag{3.97}
$$

where $P_{11}$ and $V_1$ have the same number of columns as $A$. Define

$$
\begin{align*}
L_1(P_{21}, V_2) &:= \begin{bmatrix} P_{21} & \bullet \\ V_2 & A^*_1 \end{bmatrix} = \begin{bmatrix} P_{21} & P_{22} \end{bmatrix}^{-1} \\
L_2(Q_{21}) &:= \begin{bmatrix} 0 & 0 & Q_{21} & 0 & 0 \end{bmatrix} \tag{3.98}
\end{align*}
$$

Symmetrically permuting the rows and columns of the inequality $\mathcal{M}_{\Sigma_{G_{uc}}}^{\text{mix}}(\tau, P, W, V) > 0$, we see that $J_\tau(\Sigma_{G_{uc}}) < \gamma$ if and only if $\exists P_{11}, P_{21}, P_{22}, W, V_1$, and $V_2$ such that $\text{tr}\{W\} < \gamma$ and

$$
\begin{bmatrix}
\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1) & \bullet & \bullet \\
L_1(P_{21}, V_2) & P_{22} & \bullet \\
L_2(Q_{21}) & A_{uc} & Q_{22}
\end{bmatrix} > 0. \tag{3.101}
$$

If $J_\tau(\Sigma_{G_{uc}}) < \gamma$, there exist $P_{11}, P_{21}, P_{22}, W, V_1$, and $V_2$ such that $\text{tr}\{W\} < \gamma$ and (3.101) is satisfied. Since (3.101) implies that $\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1) > 0$, we see that $J_\tau(\Sigma_{G_{uc}}) < \gamma \Rightarrow \bar{J}_\tau(\Sigma_G) < \gamma$.

Now suppose that $J_\tau(\Sigma_G) < \gamma$. Choose $P_{11}, W, V_1$ so that $\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1) > 0$ and $\text{tr}\{W\} < \gamma$. Now set $P_{12} = 0$ and $V_2 = 0$ and choose $P_{22}$ so that

$$
P_{22} \succ A^*_{uc}P_{22}A_{uc} + [L_1(0, 0)][\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1)]^{-1}[L_1(0, 0)]^*. \tag{3.102}
$$

Such a choice is possible because $A_{uc}$ is Schur. Also, since $\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1) > 0$, we see that $P_{22} > 0$. Since $P_{12} = 0$, we see that $Q_{12} = 0$, $L_2(Q_{21}) = 0$, and $Q_{22} = P_{22}^{-1}$. By Schur complements, (3.102) implies that

$$
\begin{bmatrix}
P_{22} - [L_1(0, 0)][\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1)]^{-1}[L_1(0, 0)]^* & \bullet \\
& \mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1)
\end{bmatrix} > 0. \tag{3.103}
$$

Using Schur complements on (3.103), we see that the chosen values of $P_{11}, P_{21}, P_{22}, W, V_1$, and $V_2$ satisfy (3.101). Therefore, $\bar{J}_\tau(\Sigma_G) < \gamma \Rightarrow \bar{J}_\tau(\Sigma_{G_{uc}}) < \gamma$. \qed

Lemma 3.5.3. For fixed $\tau > 0$, $\bar{J}_\tau(\Sigma_G) = \bar{J}_\tau(\Sigma_{G_{uc}})$. 

Proof. In this proof, we will show that $\tilde{J}_r(\Sigma_G) < \gamma \Leftrightarrow \tilde{J}_r(\Sigma_{G_{uo}}) < \gamma$. This will allow us to conclude that $\tilde{J}_r(\Sigma_G) = \tilde{J}_r(\Sigma_{G_{uo}})$. Note that, by Proposition 3.4.1, $\tilde{J}_r(\Sigma_{G_{uo}}) < \gamma$ if and only if $\exists P, W, V$ such that $\text{tr}\{W\} < \gamma$ and $\mathcal{M}^{mix}_{\Sigma_{G_{uo}}}(\tau, P, W, V) \succ 0$. Partition $P$ and $V$ as

$$P = \begin{bmatrix} P_{11} & \bullet \\ P_{21} & P_{22} \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

(3.104)

where $P_{11}$ and $V_1$ have the same number of columns as $A$. Define

$$\begin{bmatrix} Q_{11} & \bullet \\ Q_{21} & Q_{22} \end{bmatrix} := \begin{bmatrix} P_{11} & \bullet \\ P_{21} & P_{22} \end{bmatrix}^{-1}$$

(3.105)

$$L_1(P_{21}, V_2) := \begin{bmatrix} P_{21} & 0 & V_2^* & 0 & 0 \end{bmatrix}$$

(3.106)

$$L_2(Q_{21}) := \begin{bmatrix} A_{u2} & B_{u1} & B_{u2} & Q_{21} & 0 & 0 \end{bmatrix}.$$  

(3.107)

Symmetrically permuting the rows and columns of the inequality $\mathcal{M}^{mix}_{\Sigma_{G_{uo}}}(\tau, P, W, V) \succ 0$, we see that $\tilde{J}_r(\Sigma_{G_{uo}}) < \gamma$ if and only if $\exists P_{11}, P_{21}, P_{22}, W, V_1$, and $V_2$ such that $\text{tr}\{W\} < \gamma$ and

$$\begin{bmatrix} \mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1) & \bullet & \bullet \\ L_1(P_{21}, V_2) & P_{22} & \bullet \\ L_2(Q_{21}) & A_{u2} & Q_{22} \end{bmatrix} \succ 0.$$  

(3.108)

If $\tilde{J}_r(\Sigma_{G_{uo}}) < \gamma$, there exist $P_{11}, P_{21}, P_{22}, W, V_1$, and $V_2$ such that $\text{tr}\{W\} < \gamma$ and (3.108) is satisfied. Since (3.108) implies that $\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1) \succ 0$, we see that $\tilde{J}_r(\Sigma_{G_{uo}}) < \gamma \Rightarrow \tilde{J}_r(\Sigma_G) < \gamma$.

Now suppose that $\tilde{J}_r(\Sigma_G) < \gamma$. Choose $P_{11}, W, V_1$ so that $\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1) \succ 0$ and $\text{tr}\{W\} < \gamma$. Now set $P_{12} = 0$ and $V_2 = 0$ and choose $P_{22}$ so that

$$P_{22} \succ A_{u2}^* P_{22} A_{uo} + [L_1(0, 0)][\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1)]^{-1}[L_1(0, 0)]^*.$$  

(3.109)

Since $A_{uo}$ is assumed to be Schur and $\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1) \succ 0$, we see that $P_{22} \succ 0$. Since $P_{12} = 0$, we see that $Q_{12} = 0$, $L_1(P_{21}, V_2) = 0$, and $Q_{22} = P_{22}^{-1}$. By Schur complements, (3.109) implies that

$$\begin{bmatrix} P_{22} - [L_1(0, 0)][\mathcal{M}_{\Sigma_G}(\tau, P_{11}, W, V_1)]^{-1}[L_1(0, 0)]^* & \bullet \\ A_{u2} & Q_{22} \end{bmatrix} \succ 0.$$  

(3.110)

Using Schur complements on (3.110), we see that the chosen values of $P_{11}, P_{21}, P_{22}, W, V_1$, and $V_2$ satisfy (3.108). Therefore, $\tilde{J}_r(\Sigma_G) < \gamma \Rightarrow \tilde{J}_r(\Sigma_{G_{uo}}) < \gamma$.  

With these lemmas in place, we can now state and prove one of the two main results of this section.

**Theorem 3.5.4.** If $G$ is stable, the function $J_r(G)$ is independent of the realization chosen for $G$, provided that the realization of $G$ is stabilizable and $d$-detectable.
Figure 3.8: Uncertain system with static uncertainty scaling

Proof. Let $G$ have the state-space realizations $\Sigma_1$ and $\Sigma_2$. Although we assume that both realizations are stabilizable and d-detectable, we do not assume that they have the same order. If $\Sigma_1$ is not a controllable realization, transform the system into staircase controllability form then truncate the uncontrollable states. By Lemmas 3.5.1 and 3.5.2, neither one of these operations changes the value of $\bar{J}_r$. If the resulting realization is not observable, transform the system into staircase observability form then truncate the unobservable states. By Lemmas 3.5.1 and 3.5.3, neither one of these operations change the value of $\bar{J}_r$. Therefore, this process of reducing $\Sigma_1$ and $\Sigma_2$ to minimal realizations does not change the value of $\bar{J}_r$ for either realization. Since the minimal realizations of $\Sigma_1$ and $\Sigma_2$ are both minimal realizations of $G$, we see by Lemma 3.5.1 that they have the same value of $\bar{J}_r$. Therefore, $\bar{J}_r(\Sigma_1) = \bar{J}_r(\Sigma_2)$.

We now define for a fixed value of $\tau > 0$ the system and state-space realization

$$G_s \sim \begin{bmatrix} A & \frac{1}{\sqrt{\tau}} B_1 & B_2 \\ \sqrt{\tau} C_1 & D_{11} & \sqrt{\tau} D_{12} \\ C_2 & \frac{1}{\sqrt{\tau}} D_{21} & D_{22} \end{bmatrix} =: \Sigma_s .$$

(3.111)

Note that this corresponds to the static uncertainty scalings shown in Fig. 3.8. We now give the second main result of this section

**Theorem 3.5.5.** For fixed $\tau > 0$, $J_\tau(G) = J_1(G_s)$.

Proof. Define the invertible symmetric matrix

$$L := \text{diag} \left( I, \frac{1}{\sqrt{\tau}} I, I, I, \sqrt{\tau} I, I \right)$$

(3.112)

Note that $LM^{\text{mix}}_{\Sigma_G}(\tau, P, W, V)L = M^{\text{mix}}_{\Sigma_s}(1, P, W, V)$. This implies that $M^{\text{mix}}_{\Sigma_G}(\tau, P, W, V) > 0$ if and only if $M^{\text{mix}}_{\Sigma_s}(1, P, W, V) > 0$. Therefore, by (3.88), $J_\tau(G) < \gamma \iff J_1(G_s) < \gamma$, which implies that $J_\tau(G) = J_1(G_s)$. 

$\blacksquare$
Chapter 4

Full Information $H_2$ Guaranteed Cost Control

In this chapter, we show how to use the analysis results of the previous chapter to design controllers that optimize the $H_2$ guaranteed cost. The particular control design problem we consider is the full information (FI) control problem in which the controller has access to the state of the system and the disturbances acting on the system. We first present an SDP for determining an optimal controller and then show that, as in the previous chapter, this convex optimization can be efficiently solved using nonlinear convex optimization involving DARE solutions.

In this chapter, we will use the notation $N$ defined in §2.3, the notation $R_\phi(P)$, $K_\phi(P)$, and $A_\phi(P)$ defined in §2.5, the notation $L_\phi(P)$ defined in §2.6, the notation $M$ defined in §3.1.2, the notation $J_\tau$ defined in §3.2.1, and the notation $\bar{M}$ and $M^{inv}$ defined in §3.4.

4.1 Problem Formulation

In this section, we consider the optimal control (in terms of $J_\tau$) of the interconnection shown in Fig. 4.1. In this diagram, we let $G_{fi}$ have the state-space realization

$$G_{fi} \sim \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}.$$ (4.1)
CHAPTER 4. FULL INFORMATION $\mathcal{H}_2$ GUARANTEED COST CONTROL

This corresponds to letting

$$y_{fi} = \begin{bmatrix} x_{fi} \\ d \\ w \end{bmatrix}$$  \hspace{1cm} (4.2)

in Fig. 4.1 where $x_{fi}$ is the state variable of $G_{fi}$. The dimensions of the signals $x_{fi}$, $q$, $p$, $d$, $w$, and $u$ are respectively $n_x$, $n_q$, $n_p$, $n_d$, $n_w$, and $n_u$.

In this chapter, we will restrict the controller to lie in the set $\mathcal{K}$, which we define to be the set of controllers which are LTI and have finite order. We also define the set $\mathcal{K}_0$, which only contains controllers which are static gains. In this notation, the squared $\tau$-specific $\mathcal{H}_2$ guaranteed cost of the closed-loop system can be written as $J_\tau(\mathcal{F}_i(G_{fi}, K))$ when $K \in \mathcal{K}$.

In optimal full information $\mathcal{H}_2$ control and optimal full information $\mathcal{H}_\infty$ control, optimizing over dynamic controllers is equivalent to optimizing over static controllers. The next theorem shows that this is also the case for optimal full information $\mathcal{H}_2$ guaranteed cost control.

**Theorem 4.1.1.** The equality $\gamma = \gamma_0$ holds, where

$$\gamma := \inf_{K \in \mathcal{K}} J_\tau(\mathcal{F}_i(G_{fi}, K)), \quad \gamma_0 := \inf_{\bar{K} \in \mathcal{K}_0} J_\tau(\mathcal{F}_i(G_{fi}, \bar{K})).$$ \hspace{1cm} (4.3)

**Proof.** Since $\mathcal{K}_0 \subset \mathcal{K}$, we trivially have the inequality $\gamma \leq \gamma_0$. Thus, it only remains to prove that $\gamma \geq \gamma_0$. Since this is trivial if $\gamma$ is infinite, we assume that $\gamma$ is finite. Fix $\epsilon > 0$ and then choose $K \in \mathcal{K}$ so that $J_\tau(\mathcal{F}_i(G_{fi}, K)) < \gamma + \epsilon$. We now let $K$ have the realization

$$K \sim \begin{bmatrix} A^K & B^K_1 & B^K_2 & B^K_3 \\ C^K & D^K_1 & D^K_2 & D^K_3 \end{bmatrix}$$ \hspace{1cm} (4.4)

so that

$$\mathcal{F}_i(G_{fi}, K) \sim \begin{bmatrix} A & 0 & B_1 & B_2 \\ 0 & 0 & 0 & 0 \\ C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \end{bmatrix} + \begin{bmatrix} 0 & B_3 \\ I & 0 \\ 0 & D_{13} \\ 0 & D_{23} \end{bmatrix} \begin{bmatrix} B^K_1 & A^K & B^K_2 & B^K_3 \\ D^K_1 & C^K & D^K_2 & D^K_3 \end{bmatrix} =: \Sigma_{cl}.$$ \hspace{1cm} (4.5)
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Using the characterization of $J_\tau$ given by (3.86), choose $P, W, V$ such that $\text{tr}(W) < \gamma + \epsilon$ and $\tilde{M}_{\Sigma,cl}(\tau, P, W, V) > 0$. We now partition $P$ and $V$ respectively as

$$P = \begin{bmatrix} P_{11} & \bullet \\ P_{21} & P_{22} \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

(4.6)

where $P_{11}$ and $V_1$ have $n_x$ columns and define

$$\bar{D}^K_1 := D^K_1 - C^K P^{-1} P_{21}, \quad \bar{K} := \begin{bmatrix} \bar{D}^K_1 & D^K_2 & D^K_3 \end{bmatrix} \in \mathcal{K}_0$$

$$\bar{P} := P_{11} - P_{21} P^{-1} P_{21}, \quad \bar{V} := V_1 - V_2 P^{-1} P_{21}.$$  

(4.7)

Note that applying the controller $\bar{K}$ yields the closed-loop realization

$$\mathcal{F}(G_{fi}, \bar{K}) \sim \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} B_3 \\ D_{13} \\ D_{23} \end{bmatrix} \begin{bmatrix} \bar{D}^K_1 & D^K_2 & D^K_3 \end{bmatrix} =: \Sigma_{cl}.$$  

(4.8)

Defining

$$\Pi := \begin{bmatrix} I \\ -P_{22}^{-1} P_{21} \end{bmatrix} \quad L := \text{diag}(\Pi, I_{n_d}, I_{n_w}, \Pi, I_{n_q}, I_{n_p})$$

(4.9)

it is straightforward to show that

$$L^* \tilde{M}_{\Sigma_{cl}}(\tau, P, W, V) L = \tilde{M}_{\Sigma_{cl}}(\tau, \bar{P}, \bar{V}).$$  

(4.10)

Since $\tilde{M}_{\Sigma_{cl}}(\tau, P, W, V) > 0$ and $\text{Ker}(L) = \{0\}$, we therefore see that $\tilde{M}_{\Sigma_{cl}}(\tau, \bar{P}, \bar{V}) > 0$. Thus, since $\text{tr}\{W\} < \gamma + \epsilon$, we see that $J_\tau(\mathcal{F}(G_{fi}, \bar{K})) \leq \gamma + \epsilon$, which in turn implies that $\gamma_0 \leq J_\tau(\mathcal{F}(G_{fi}, \bar{K})) \leq \gamma + \epsilon$. Since the choice of $\epsilon$ was arbitrary, we conclude that $\gamma \geq \gamma_0$.  

With this theorem in place, we see that we only need to consider controllers of the form

$$K = [K_x \quad K_d \quad K_w]$$

(4.11)

where $K_x$, $K_d$, and $K_w$ are matrices of the appropriate size. With this restriction, the closed-loop system has the form

$$\mathcal{F}(G_{fi}, K) \sim \begin{bmatrix} A + B_3 K_x & B_1 + B_3 K_d & B_2 + B_3 K_w \\ C_1 + D_{13} K_x & D_{11} + D_{13} K_d & D_{12} + D_{13} K_w \\ C_2 + D_{23} K_x & D_{21} + D_{23} K_d & D_{22} + D_{23} K_w \end{bmatrix} =: \Sigma_{cl}.$$  

(4.12)

Therefore, characterizing the $\mathcal{H}_2$ guaranteed cost of the closed-loop system using (3.84), we are interested in solving the optimization problem

$$\inf_{\epsilon, \bar{P}, \bar{V}, \bar{K}_x, \bar{K}_d, \bar{K}_w} \text{tr}\{W\} \quad \text{s.t.} \quad \tilde{M}_{\Sigma_{cl}}^{\text{inv}}(\epsilon, \bar{P}, \bar{V}) > 0.$$  

(4.13)
We will call this optimization problem the optimal full information control problem. Since $\Sigma_{cl}$ is a function of $K_x$, $K_d$, and $K_w$, the matrix inequality in (4.13) is nonlinear in the optimization parameters. In this chapter, we first give two methods to solve (4.13) using SDPs. We then give an algorithm that exploits Riccati equation structure to improve the speed and accuracy with which we are able to solve this optimization problem.

4.2 SDP Approach

4.2.1 Using Nonlinear Change of Variables

In this subsection, we formulate the optimal full information control problem as a convex optimization problem with a single LMI constraint. The key idea in doing this is a nonlinear change of variables that generalizes the one given in [2].

Theorem 4.2.1. Solving (4.13) is equivalent to solving

$$\inf_{\epsilon, \hat{P}, W, \hat{V}, \hat{K}_x, \hat{K}_d, K_w} \text{tr}\{W\} \quad \text{s.t.}$$

$$\begin{bmatrix}
\hat{P} & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon I & 0 & 0 & 0 & 0 \\
A \hat{P} + B_3 \hat{K}_x & \epsilon B_1 + B_3 \hat{K}_d & B_2 + B_3 K_w & \hat{P} & 0 \\
C_1 \hat{P} + D_{13} \hat{K}_x & \epsilon D_{11} + D_{13} \hat{K}_d & D_{12} + D_{13} K_w & 0 & \epsilon I \\
C_2 \hat{P} + D_{23} \hat{K}_x & \epsilon D_{21} + D_{23} \hat{K}_d & D_{22} + D_{23} K_w & 0 & 0 & I \\
\end{bmatrix} \succ 0.$$ (4.14b)

Moreover, for any feasible iterate of (4.14), the controller

$$K = \begin{bmatrix} \hat{K}_x \hat{P}^{-1} & \epsilon^{-1} \hat{K}_d & K_w \end{bmatrix}$$

achieves the closed-loop performance $J(\epsilon^{-1})(F_i(G_{fi}, K)) < \text{tr}\{W\}$.

Proof. The key to this proof is the nonlinear change of variables

$$\hat{K}_x := K_x \hat{P}^{-1}, \quad \hat{K}_d := \epsilon K_d.$$ (4.16)

Denote the values of (4.13) and (4.14) respectively as $\gamma_0$ and $\gamma_1$. Note that plugging (4.16) into (4.14) and appropriately redefining the optimization parameters yields (4.13). Since this corresponds to increasing the value of (4.14) by restricting its optimization parameters to have a special form, we see that $\gamma_0 \geq \gamma_1$.

Now note that whenever $\epsilon, \hat{P}, W, \hat{V}, \hat{K}_x, \hat{K}_d, K_w$ satisfy (4.14b), $\hat{P} \succ 0$ and $\epsilon > 0$. This implies that the inverse change of variables

$$K_x = \hat{K}_x \hat{P}^{-1}, \quad K_d = \epsilon^{-1} \hat{K}_d.$$ (4.17)
CHAPTER 4. FULL INFORMATION $H_2$ GUARANTEED COST CONTROL

is well-defined. Plugging these expressions into the expression $M_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V})$ yields the left-hand side of (4.14b). Therefore, we see that $M_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \succ 0$, which implies that $J_{(\epsilon^{-1})(F_{l}(G_{fi}, K))} < \text{tr}\{W\}$. Since the choice of $\epsilon, \hat{P}, W, \hat{V}, \hat{K}_x, \hat{K}_d,$ and $K_w$ was arbitrary, this implies that $\gamma_0 \leq \gamma_1$. \hfill \blacksquare

If the strict inequality is relaxed to a non-strict inequality in the preceding theorem, (4.14) becomes a SDP. Thus, a reasonable way to solve the optimal full information control problem is a relax (4.14) to a SDP, solve the SDP using an appropriate solver, then reconstruct the controller using (4.15).

4.2.2 Using Matrix Variable Elimination

Starting from (4.13), we now use matrix variable elimination (Proposition 2.2.4) to formulate the optimal full information control problem as a convex optimization problem with a single LMI constraint. We first define the state-space realization

$$\Sigma_0 := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}.$$ \hfill (4.18)

Note that this corresponds to $F_l(G_{fi}, 0)$.

We now give two lemmas that examine the matrix inequality $M_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \succ 0$ for fixed $\epsilon, \hat{P}, W,$ and $\hat{V}$. Note that this matrix inequality is an LMI in $K$. The first lemma examines the feasibility of this LMI and the second lemma gives a specific value of $K$ which is guaranteed to satisfy the LMI if it is feasible.

**Lemma 4.2.2.** There exists $K$ such that $M_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \succ 0$ if and only if

$$\begin{bmatrix} I_{n_x+n_d+n_w} & 0 \\ 0 & N \end{bmatrix}^* M_{\Sigma_0}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \begin{bmatrix} I_{n_x+n_d+n_w} & 0 \\ 0 & N \end{bmatrix} \succ 0.$$ \hfill (4.19)

where the columns of $N$ are a basis for $\text{Ker}([B_3^* \hspace{0.5cm} D_{13}^* \hspace{0.5cm} D_{23}^*])$.

**Proof.** Defining $\hat{K} := [K_x \hat{P} \hspace{0.5cm} \epsilon K_d \hspace{0.5cm} K_w]$ and

$$M_1 := \begin{bmatrix} 0_{n_u \times (n_x+n_d+n_w)} & B_3^* & D_{13}^* & D_{23}^* \end{bmatrix}, \quad M_2 := \begin{bmatrix} I_{n_x+n_d+n_w} & 0_{(n_x+n_d+n_w) \times (n_x+n_d+n_w)} \end{bmatrix} \hfill (4.20)$$

we see that

$$M_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) = M_{\Sigma_0}^{inv}(\epsilon, \hat{P}, W, \hat{V}) + M_1^* \hat{K} M_2 + M_2^* \hat{K}^* M_1.$$ \hfill (4.21)
Using Proposition 2.2.4, we see for given \( \epsilon, \hat{P}, W, \hat{V} \) that there exists \( \hat{K} \) such that the matrix inequality \( \mathcal{M}_{\Sigma_0}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \) holds if and only if

\[
\begin{align*}
N_1^* \mathcal{M}_{\Sigma_0}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) N_1 & \succ 0 \\
N_2^* \mathcal{M}_{\Sigma_0}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) N_2 & \succ 0
\end{align*}
\tag{4.22a, b}
\]

where the columns of \( N_1 \) and \( N_2 \) respectively form bases for \( \text{Ker}(M_1) \) and \( \text{Ker}(M_2) \). Note that the matrices

\[
N_1 := \begin{bmatrix} I_{n_x + n_d + n_w} & 0 & N \end{bmatrix}, \quad N_2 := \begin{bmatrix} 0_{(n_x + n_d + n_w) \times (n_x + n_q + n_p)} & I_{n_x + n_d + n_w} \end{bmatrix}
\tag{4.23}
\]

satisfy this requirement. With this choice of \( N_1 \), (4.22a) becomes (4.19). Also note that (4.19) implies that

\[
\begin{bmatrix} \hat{P} & \bullet & \bullet \\
0 & \epsilon I & \bullet \\
\hat{V} & 0 & W \end{bmatrix} \succ 0
\tag{4.24}
\]

which in turn implies that \( \epsilon > 0 \) and \( \hat{P} \succ 0 \). Note that, with the choice of \( N_2 \) in (4.23), (4.22b) becomes

\[
\begin{bmatrix} \hat{P} & 0 & 0 \\
0 & \epsilon I & 0 \\
0 & 0 & I \end{bmatrix} \succ 0.
\tag{4.25}
\]

Thus, if the constraint (4.22a) is satisfied, then the constraint (4.22b) must also be satisfied. Therefore, the constraint (4.22b) is redundant in the system of constraints given by (4.22), which implies that there exists \( \hat{K} \) such that \( \mathcal{M}_{\Sigma_0}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \) if and only if (4.19) holds.

It now remains to show that there exists \( K \) such that \( \mathcal{M}_{\Sigma_0}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \) if and only if there exists \( \hat{K} \) such that \( \mathcal{M}_{\Sigma_0}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \). Note that the existence of \( K \) trivially implies the existence of \( \hat{K} \). Now suppose that \( \hat{K} \) satisfies \( \mathcal{M}_{\Sigma_0}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \). This implies that \( \epsilon > 0 \) and \( \hat{P} \succ 0 \), which implies that \( K \) can be constructed from \( \hat{K}, \hat{P}, \) and \( \epsilon \).

\[\blacksquare\]

**Lemma 4.2.3.** If (4.19) is satisfied, then the controller

\[
K^o := -(B_3^* \hat{P}^{-1} B_3 + \epsilon^{-1} D_{13}^* D_{13} + D_{23}^* D_{23})^\dagger [B_3^* \hat{P}^{-1} \epsilon^{-1} D_{13}^* D_{23}] \begin{bmatrix} A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22} \end{bmatrix}
\tag{4.26}
\]

satisfies \( \mathcal{M}_{\Sigma_0}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \).
Proof. Note that \( \epsilon > 0 \) and \( \hat{P} > 0 \). By Proposition 3.4.1, we see that
\[
\mathcal{M}^{inv}_{\Sigma cl}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \iff \mathcal{M}_{\Sigma cl}(\epsilon^{-1}, \hat{P}^{-1}, W, \hat{V}\hat{P}^{-1}) \succ 0.
\] (4.27)

Defining
\[
M_1 := \begin{bmatrix} \hat{P}^{-1} & \bullet & \bullet \\ 0 & \epsilon^{-1}I & \bullet \\ \hat{V}\hat{P}^{-1} & 0 & W \end{bmatrix}, \quad M_2 := \begin{bmatrix} \hat{P}^{-1} & \bullet & \bullet \\ 0 & \epsilon^{-1}I & \bullet \\ 0 & 0 & I \end{bmatrix}, \quad M_3 := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}, \quad M_4 := \begin{bmatrix} B_3 \\ D_{13} \\ D_{23} \end{bmatrix}
\] (4.28)

we write the condition \( \mathcal{M}_{\Sigma cl}(\epsilon^{-1}, \hat{P}^{-1}, W, \hat{V}\hat{P}^{-1}) \succ 0 \) as
\[
0 \prec M_1 - (M_3 + M_4 K)^* M_2 (M_3 + M_4 K) = M_1 - M_3^* M_3 M_3 - M_3^* M_2 M_4 K - K^* M_4^* M_3 M_4 - K^* M_4^* M_2 M_4 K.
\] (4.29)

Also note that, in this notation, \( K^o = -(M_4^* M_2 M_4)\dagger (M_4^* M_2 M_4) \).

Since \( M_2 \succ 0 \), we see that \( \text{Ker}(M_4^* M_2 M_4) = \text{Ker}(M_4) \) and \( M_4^* M_2 M_4 \succeq 0 \). We now perform the singular decomposition of \( M_4^* M_2 M_4 \):
\[
M_4^* M_2 M_4 = [U_1 \ U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = U_1 \Sigma U_1^*.
\] (4.30)

Note in particular that \( \text{Ker}(M_4^* M_2 M_4) = \text{Im}(U_2) \), which implies that \( M_4 U_2 = 0 \). Using the fact that \([U_1 \ U_2]\) is unitary, this in turn implies that
\[
M_4 = M_4 \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = [M_4 U_1 \ 0] \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} = M_4 U_1 U_1^*.
\] (4.31)

Noting that the Moore–Penrose pseudoinverse of \( M_4^* M_2 M_4 \) is given by \( U_1 \Sigma^{-1} U_1^* \), we see that
\[
M_4^* M_2 M_4 K_o = -(U_1 \Sigma U_1^*) (U_1 \Sigma^{-1} U_1^*) (M_4^* M_2 M_4) M_3 = -(U_1 U_1^* M_4^*) M_2 M_3
\] (4.32)
\[
\Rightarrow M_4^* M_2 M_4 K_o = -M_4^* M_2 M_3.
\] (4.33)

Using this identity, along with the fact that
\[
(M_4^* M_2 M_4)^\dagger (M_4^* M_2 M_4) (M_4^* M_2 M_4)^\dagger = (M_4^* M_2 M_4)^\dagger
\] (4.34)

we see that the identity
\[
(K - K^o)^* (M_4^* M_2 M_4) (K - K^o) = K^* M_4^* M_2 M_4 K + K^* M_4^* M_2 M_3 + M_4^* M_2 M_4 K
\] (4.35)
holds. Plugging this identity into (4.29) yields that $\mathcal{M}_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \succ 0$ if and only if
\[ M_1 - M_3^* M_2 M_3 + M_3^* M_2 M_4 (M_4^* M_2 M_4)^\dagger M_4^* M_2 M_3 \succ (K - K^o)^* (M_4^* M_2 M_4) (K - K^o). \] (4.36)

Define $\Phi_L$ and $\Phi_R(K)$ to respectively be the left-hand and right-hand sides of (4.36). Note that $\Phi_R(K) \succeq 0$, $\forall K$ and that $\Phi_L$ is not a function of $K$. By Lemma 4.2.2, choose $K$ such that $\mathcal{M}_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \succ 0$. This implies that (4.36) is satisfied. For this particular value of $K$, we see that $\Phi_L \succ \Phi_R(K) \succeq 0$, which implies that $\Phi_L \succ 0 = \Phi_R(K^o)$. Therefore, $\mathcal{M}_{\Sigma_{cl}}(\epsilon^{-1}, \hat{P}^{-1}, W, \hat{V}^{-1}) \succ 0$ is satisfied for $K = K^o$, which concludes the proof.

With this two lemmas in place, we can now formulate the optimal full information control problem as a convex optimization problem with a single LMI constraint.

**Theorem 4.2.4.** Solving (4.13) is equivalent to solving
\[
\inf_{\epsilon, \hat{P}, W, \hat{V}} \text{tr}\{W\} \quad \text{s.t.} \quad \begin{bmatrix} I_{n_x + n_d + n_w} & 0 \\ 0 & N \end{bmatrix}^* \mathcal{M}_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \begin{bmatrix} I_{n_x + n_d + n_w} & 0 \\ 0 & N \end{bmatrix} \succ 0 \tag{4.37a}
\]
where the columns of $N$ are a basis for $\text{Ker}([B_3^* \quad D_{13}^* \quad D_{23}^*])$. Moreover, whenever (4.37b) is satisfied, the controller
\[
K^o := -(B_3^* \hat{P}^{-1} B_3 + \epsilon^{-1} D_{13}^* D_{13} + D_{23}^* D_{23})^\dagger \begin{bmatrix} B_3^* \hat{P}^{-1} & \epsilon^{-1} D_{13}^* & D_{23}^* \end{bmatrix} \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}
\tag{4.38}
\]
satisfies $J_{(\epsilon^{-1})(\mathcal{F}_i(G_{fi}, K^o))} < \text{tr}\{W\}$.

**Proof.** Note that (4.13) is equivalent to
\[
\inf_{\epsilon, \hat{P}, W, \hat{V}} \left( \inf_{K_0, K_d, K_w} \text{tr}\{W\} \quad \text{s.t.} \quad \mathcal{M}_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \right). \tag{4.39}
\]
By Lemma 4.2.2, we see that
\[
\left( \inf_{K_0, K_d, K_w} \text{tr}\{W\} \quad \text{s.t.} \quad \mathcal{M}_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \succ 0 \right) = \begin{cases} \text{tr}\{W\}, & \text{(4.37b) is satisfied} \\ \infty, & \text{otherwise} \end{cases} \tag{4.40}
\]
which implies that (4.13) is equivalent to (4.37).

Now suppose that $\epsilon, \hat{P}, W,$ and $\hat{V}$ satisfy (4.37b). By Lemma 4.2.3, we see that $\mathcal{M}_{\Sigma_{cl}}^{inv}(\epsilon, \hat{P}, W, \hat{V}) \succ 0$ for the controller $K^o$. Therefore, by (3.84), we see that the condition $J_{(\epsilon^{-1})(\mathcal{F}_i(G_{fi}, K^o))} < \text{tr}\{W\}$ is satisfied.
Unlike the optimization problem (4.14), the optimization problem (4.37) does not have the controller parameters as optimization parameters. Comparing the number of optimization parameters in the two optimization problems, we see that (4.37) has fewer optimization parameters. Moreover, the size of the LMI constraint in (4.37) is smaller than the size of the LMI constraint in (4.14). Therefore, the optimization problem (4.37) is more attractive than (4.14) because it has fewer primal and dual optimization parameters.

The key idea in reducing the optimization problem in Theorem 4.2.1 to the optimization problem in Theorem 4.2.4 was eliminating the matrix variable $K$ from the optimization. This was possible only because the optimization problem (4.13) had a single matrix inequality constraint. If we had used condition 2 of Theorem 3.1.1 to build a characterization of robust $H_2$ performance, the optimization problem that would have corresponded to (4.13) would have had two LMI constraints that both depended on $K$. Thus, we would not have been able to eliminate the matrix variable $K$ from that optimization. Since the ideas in this section form the basis for the DARE approach considered in the next section, this justifies our usage of the new characterization of nominal $H_2$ performance given in this dissertation.

Relaxing the strict inequality to a non-strict inequality in (4.37) yields a SDP. Thus, a reasonable way to solve the optimal full information control problem is to relax (4.37) to a SDP, solve the SDP using an appropriate solver, then reconstruct the controller using (4.38).

### 4.3 Riccati Equation Approach

In the previous section, we showed that the optimal full information control problem could be solved using either of two SDPs. In this section, we will show under some mild assumptions that if we fix $\epsilon > 0$, what remains of the optimal full information control problem can be solved using a single DARE. In particular, using the characterization of $J_\tau$ given by (3.87), we see that

$$
\inf_{K_x, K_d, K_w} \left( \inf_{\hat{P}, \hat{W}, \hat{V}} \text{tr}\{W\} \quad \text{s.t.} \quad \mathcal{M}_{\Sigma_d}^{\text{inv}}(\epsilon, \hat{P}, W, \hat{V}) > 0 \right) = \inf_K J_{(\epsilon-1)}(\mathcal{F}(G_{fi}, K))
$$

which implies that solving the optimal full information control problem for fixed $\epsilon > 0$ corresponds to solving the optimization problem

$$
J_{fi, \epsilon} := \inf_K J_{(\epsilon-1)}(\mathcal{F}(G_{fi}, K)).
$$

Alternatively, it can be shown in the same manner as §4.2.2 that the optimal full information control problem for fixed $\epsilon > 0$ corresponds to solving the optimization problem

$$
J_{fi, \epsilon} = \left( \inf_{\hat{P}, \hat{W}, \hat{V}} \text{tr}\{W\} \quad \text{s.t.} \quad (4.37) \right).
$$
In §4.3.1–§4.3.2, we will give the value of this optimization problem and the optimal controller in terms of a stabilizing DARE solution. In §4.3.3, we will give an algorithm that solves the optimal full information control problem using an iteration of DARE solutions.

A few quantities which will be important in this chapter are the combinations of state-space matrices

\[
B_{[1,3]} := \begin{bmatrix} B_1 & B_3 \end{bmatrix}, \quad C := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad D_1 := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}, \quad D_2 := \begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix}, \quad D_3 := \begin{bmatrix} D_{13} \\ D_{23} \end{bmatrix}, \quad D_{[1,3]} := \begin{bmatrix} D_{11} & D_{13} \\ D_{21} & D_{23} \end{bmatrix}
\]

and the parameters

\[
Q := C_1^* C_1 + \epsilon C_2^* C_2, \quad S := C_1^* \begin{bmatrix} D_{11} & D_{13} \end{bmatrix} + \epsilon C_2^* \begin{bmatrix} D_{21} & D_{23} \end{bmatrix}, \quad \bar{Q} := D_{12}^* D_{12} + \epsilon D_{22}^* D_{22}, \quad \bar{S} := D_{12}^* \begin{bmatrix} D_{11} & D_{13} \end{bmatrix} + \epsilon D_{22}^* \begin{bmatrix} D_{21} & D_{23} \end{bmatrix}
\]

\[
\phi := (A, B_{[1,3]}, Q, R, S), \quad \psi := (B_2, B_{[1,3]}, \bar{Q}, R, \bar{S}).
\]

At various points in this section, we will make the following assumptions:

(A1) \( D_3^* D_3 \) is invertible

(A2) \( (A, B_3) \) is stabilizable

(A3) \( \dim \left( \text{Ker} \begin{bmatrix} A - \lambda I & B_3 \\ C & D_3 \end{bmatrix} \right) = 0, \forall \lambda \in \mathbb{C} \text{ satisfying } |\lambda| \geq 1. \)

It can be shown that condition (A3) is equivalent to the condition that \( (\hat{C}, \hat{A}) \) is d-detectable where

\[
\hat{A} := A - B_3 (D_3^* D_3)^{-1} D_3^* C, \quad \hat{C} := C - D_3 (D_3^* D_3)^{-1} D_3^* C.
\]

These regularity conditions are analogous to those required for design of a linear quadratic regulator or a full information \( \mathcal{H}_\infty \) controller using DAREs.
In this section, we will extensively study the DARE $\mathcal{R}_\phi(P) = P$. When this DARE has a stabilizing solution $P_0$ such that $B_3^* P_0 B_3 + D_{13}^* D_{13} + \epsilon D_{23}^* D_{23}$ is invertible, we will consider the controller

$$K^o := \begin{bmatrix} K_x^o & K_d^o & K_w^o \end{bmatrix}$$

$$:= - (B_3^* P_0 B_3 + D_{13}^* D_{13} + \epsilon D_{23}^* D_{23})^{-1} \begin{bmatrix} B_3^* P_0 & D_{13}^* & \epsilon D_{23}^* \end{bmatrix} \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}. \quad (4.47)$$

With this controller, the closed-loop system has the realization

$$\mathcal{F}_l(G_{fi}, K^o) \sim \begin{bmatrix} A^{cl} & B_1^{cl} & B_2^{cl} \\ C_1^{cl} & D_{11}^{cl} & D_{12}^{cl} \\ C_2^{cl} & D_{21}^{cl} & D_{22}^{cl} \end{bmatrix} := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} B_3 \\ D_{13} \\ D_{23} \end{bmatrix} \begin{bmatrix} K_x^o & K_d^o & K_w^o \end{bmatrix}. \quad (4.48)$$

We are now interested in the problem of determining the closed-loop performance associated with this controller for a fixed value of $\epsilon > 0$. Thus, we are interested in finding $J_{(\epsilon^{-1})}(\mathcal{F}_l(G_{fi}, K^o))$. To do so, we will apply the methods of §3.2.1, which use the parameters

$$Q^{cl} := \epsilon^{-1}(C^{cl}_1)^* C^{cl}_1 + (C^{cl}_2)^* C^{cl}_2 \quad S^{cl} := \epsilon^{-1}(C^{cl}_1)^* D_{11}^{cl} + (C^{cl}_2)^* D_{21}^{cl}$$

$$Q_W^{cl} := \epsilon^{-1}(D_{12}^{cl})^* D_{12}^{cl} + (D_{22}^{cl})^* D_{22}^{cl} \quad S_W^{cl} := \epsilon^{-1}(D_{12}^{cl})^* D_{11}^{cl} + (D_{22}^{cl})^* D_{21}^{cl} \quad (4.49)$$

$$R^{cl} := \epsilon^{-1}((D_{11}^{cl})^* D_{11}^{cl} - I) + (D_{21}^{cl})^* D_{21}^{cl} \quad \psi^{cl} := (B_2^{cl}, B_1^{cl}, Q_W^{cl}, R^{cl}, S_W^{cl})$$

We now give two results that collectively give an expression for $J_{(\epsilon^{-1})}(\mathcal{F}_l(G_{fi}, K^o))$.

**Lemma 4.3.1.** Let $\epsilon > 0$ and suppose $P_0$ is a stabilizing solution of the DARE $\mathcal{R}_\phi(P) = P$ such that the factorization

$$B_{13}^* P_0 B_{13} + R = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} * \begin{bmatrix} -I_{n_d} & 0 \\ 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \quad (4.50)$$

exists where $T_{11}$ and $T_{22}$ are invertible. Then $(B_1^{cl})^*(\epsilon^{-1} P_0) B_1^{cl} + R^{cl} < 0$, $\mathcal{R}_{\psi^{cl}}(\epsilon^{-1} P_0) = \epsilon^{-1} \mathcal{R}_\psi(P_0)$, and $\epsilon^{-1} P_0$ is the stabilizing solution of the DARE $\mathcal{R}_{\phi^{cl}}(P) = P$. Moreover, $P_0 \succeq 0 \Leftrightarrow A + B_3 K_x^o$ is Schur.

**Proof.** We first note that, by (4.50), $B_{13}^* P_0 B_{13} + D_{13}^* D_{13} + \epsilon D_{23}^* D_{23} = T_{22}^* T_{22}$, which implies by the invertibility of $T_{22}$ that $K^o$ and the definitions in (4.48)–(4.49) are well-defined. Now define the matrices

$$M_{21} := B_{11}^* P_0 A + D_{11}^* C_1 + \epsilon D_{21}^* C_2 \quad M_{31} := B_{3}^* P_0 A + D_{13}^* C_1 + \epsilon D_{23}^* C_2$$

$$M_{22} := B_{11}^* P_0 B_1 + D_{11}^* D_{11} - I + \epsilon D_{21}^* D_{21} \quad M_{32} := B_{3}^* P_0 B_1 + D_{13}^* D_{11} + \epsilon D_{23}^* D_{21}$$

$$M_{33} := B_{3}^* P_0 B_3 + D_{13}^* D_{13} + \epsilon D_{23}^* D_{23} \quad (4.51)$$


Therefore, by (4.52) and (4.54), we see that
\[ A \]
which implies by Proposition 2.1.2 that
\[ \text{Defining the matrices} \]
\[ \overline{M}_{21} \quad \overline{M}_{22} := [(B^cl)^*P_0A^cl + \epsilon(S^cl)^* (B^cl_1)^*P_0B^cl_1 + \epsilon R^cl] \]
gives that
\[ A_{\phi_{cl}}(\epsilon^{-1}P_0) = A^cl - B^cl_1(\epsilon^{-1}\overline{M}_{22})^{-1}(\epsilon^{-1}\overline{M}_{21}) = A^cl - B^cl_1\overline{M}_{22}^{-1}\overline{M}_{21}. \]

Using (4.48) and (4.49), we see after some algebra that
\[ \begin{bmatrix} A^cl & B^cl_1 \\ \overline{M}_{21} & \overline{M}_{22} \end{bmatrix} = \begin{bmatrix} A & B_1 \\ M_{21} & M_{22} \end{bmatrix} - \begin{bmatrix} B_3 \\ M_{32} \end{bmatrix} M_{33}^{-1} \begin{bmatrix} M_{31} & M_{32} \end{bmatrix} \]
which implies by Proposition 2.1.2 that
\[ A = \begin{bmatrix} B_1 & B_3 \end{bmatrix} \begin{bmatrix} M_{22} & M^*_{32} \\ M_{32} & M_{33} \end{bmatrix}^{-1} \begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} = A^cl - B^cl_1\overline{M}_{22}^{-1}\overline{M}_{21}. \]

Therefore, by (4.52) and (4.54), we see that \( A_{\phi}(P_0) = A_{\phi_{cl}}(\epsilon^{-1}P_0) \). Using similar arguments, we see that \( \mathcal{R}_\psi(P_0) = \epsilon\mathcal{R}_{\psi_{cl}}(\epsilon^{-1}P_0) \) and \( \mathcal{R}_{\phi}(P_0) = \epsilon\mathcal{R}_{\phi_{cl}}(\epsilon^{-1}P_0) \). From the latter of these statements, we see that \( P_0 = \epsilon\mathcal{R}_{\phi_{cl}}(\epsilon^{-1}P_0) \), which implies that \( \epsilon^{-1}P_0 \) is the stabilizing solution of the DARE \( \mathcal{R}_{\phi_{cl}}(P) = P \).

By Proposition 2.3.1, we see from (4.50) that \( \mathcal{N}(B^*_{[1,3]}P_0B_{[1,3]} + R) = (n_u, 0, n_d) \). Now note that, by Corollary 2.3.2,
\[ \mathcal{N}(B^*_{[1,3]}P_0B_{[1,3]} + R) = \mathcal{N}(M_{22} - M^*_{32}M_{33}^{-1}M_{32}) + \mathcal{N}(M_{33}) = \mathcal{N}(\overline{M}_{22}) + \mathcal{N}(M_{33}). \]
Since \( M_{33} = T^*_{22}T_{22} \succ 0 \), this implies that
\[ \mathcal{N}(\overline{M}_{22}) = (n_u, 0, n_d) - (n_u, 0, 0) = (0, 0, n_d). \]
Since \( (B^cl_1)^*(\epsilon^{-1}P_0)B^cl_1 + R^cl = \epsilon^{-1}\overline{M}_{22}, \) we see that \( (B^cl_1)^*(\epsilon^{-1}P_0)B^cl_1 + R^cl \prec 0 \).

We now note that the DARE \( \mathcal{R}_{\phi_{cl}}(P) = P \) has the form of the DARE examined in Theorem 3.2.2. Therefore, \( \epsilon^{-1}P_0 \succ 0 \iff A^cl \) is Schur.

**Theorem 4.3.2.** Suppose \( P_0 \) is a stabilizing solution of the DARE \( \mathcal{R}_{\phi}(P) = P \) such that \( A + B_3K^o_2 \) is Schur and the factorization (4.50) exists where \( T_{11} \) and \( T_{22} \) are invertible. Then the controller \( K^o \) achieves \( J_{(\epsilon^{-1})}(\mathcal{F}_l(G_{fi}, K^o)) = \epsilon^{-1}\text{tr}(\mathcal{R}_\psi(P_0)) \).

**Proof.** By Lemma 4.3.1, we see that \( \epsilon^{-1}P_0 \) is the stabilizing solution of the DARE \( \mathcal{R}_{\phi_{cl}}(P) = P \) and \( (B^cl_1)^*(\epsilon^{-1}P_0)B^cl_1 + R^cl \prec 0 \). Since \( A^cl \) is Schur, we see by Theorem 3.2.3 that
\[ J_{(\epsilon^{-1})}(\mathcal{F}_l(G_{fi}, K^o)) = \text{tr}(\mathcal{R}_{\psi_{cl}}(\epsilon^{-1}P_0)) \].
Since \( \mathcal{R}_{\psi_{cl}}(\epsilon^{-1}P_0) = \epsilon^{-1}\mathcal{R}_\psi(P_0) \) by Lemma 4.3.1, this concludes the proof.

\[ \square \]
4.3.1 Optimal Control for $\epsilon = 1$

In this subsection, we give necessary and sufficient conditions for $J_{f_i,1} \neq \infty$ under assumptions (A1)–(A3). Once we have done this, we show that the controller $K^o$ always achieves the optimal closed-loop cost, i.e. that $J_{f_i,1} = J_1(\mathcal{F}_l(G_{f_i}, K^o))$. To do this we will make use of the techniques in [13].

**Theorem 4.3.3.** Let $\epsilon = 1$. If (A1)–(A3) hold, the following are equivalent:

1. $J_{f_i,\epsilon} \neq \infty$
2. The DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0 \succeq 0$ such that the factorization (4.50) exists where $T_{11}$ and $T_{22}$ are invertible
3. The DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that $A + B_3K^*_1(P_0)$ is Schur and the factorization (4.50) exists where $T_{11}$ and $T_{22}$ are invertible

**Proof.** (1 $\Leftrightarrow$ 2) By Corollary 3.2.4, we know that $\exists K \in \mathcal{K}_o$ such that $J_1(\mathcal{F}_l(G_{f_i}, K))$ is finite if and only if $\exists K \in \mathcal{K}_o$ such that

$$
\left\| \mathcal{F}_l(G_{f_i}, K) \begin{bmatrix} I_{n_d} \\ 0 \end{bmatrix} \right\|_\infty < 1.
$$

Since assumptions (A1)–(A3) hold, we see by standard discrete-time $\mathcal{H}_\infty$ theory (e.g. [27]) that this is equivalent to condition 2.

(2 $\Leftrightarrow$ 3) This is trivial by Lemma 4.3.1. ■

The remainder of this section is devoted to showing that the controller $K^o$ always achieves the optimal closed-loop cost, i.e. that $J_{f_i,1} = J_1(\mathcal{F}_l(G_{f_i}, K^o))$. To this end, we give a sequence of lemmas which then allows us to prove the second main result of this section.

**Lemma 4.3.4.** Suppose assumption (A1) holds and $U$ is a matrix whose columns form an orthonormal basis for $\ker(D_3^*)$. Then the columns of the matrix

$$
\begin{bmatrix}
I \\
-D_3(D_3^*D_3)^{-1}B_3^* \\
0 \\
U
\end{bmatrix}
$$

form a basis for $\ker([B_3^* \; D_3^*])$.

**Proof.** Note that

$$
\begin{bmatrix}
B_3^* \\
D_3^*
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0
\Leftrightarrow
-B_3^*x = D_3^*y = D_3^*[y + D_3(D_3^*D_3)^{-1}B_3x - D_3(D_3^*D_3)^{-1}B_3x]
$$

$$
= D_3^*[y + D_3(\overline{D_3D_3})^{-1}B_3x] - B_3^*x.
$$

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Canceling the $B_3^*x$ from both sides of the equation, we see that this final condition holds if and only if $y + D_3(D_3^*D_3)^{-1}B_3x \in \text{Ker}(D_3^*)$. This in turn is equivalent to the existence of $n$ such that $y + D_3(D_3^*D_3)^{-1}B_3x = Un$. Therefore,

$$
[x\ y] \in \text{Ker}([B_3\ D_3^*]) \iff [x\ y] \in \text{Im} \left( \begin{bmatrix} I & 0 \\ -D_3(D_3^*D_3)^{-1}B_3^* & U \end{bmatrix} \right).
$$

(4.63)

Since the columns of (4.61) are linearly independent, this completes the proof. \[\blacksquare\]

**Lemma 4.3.5.** Suppose assumption (A1) holds and $U$ is a matrix whose columns form an orthonormal basis for $\text{Ker}(D_3^*)$. Then $UU^* = I - D_3(D_3^*D_3)^{-1}D_3^*$.

**Proof.** Choose $\hat{U}$ so that $[\hat{U}\ U]$ is orthogonal. This implies that $\hat{U}^*\hat{U} = I$ and $UU^* + \hat{U}\hat{U}^* = I$. Now note that

$$
D_3^*[\hat{U}\ U] = [L\ 0]
$$

(4.64)

for some matrix $L$. By assumption (A1), $\text{rank}(D_3) = n_u$, which implies that $\text{rank}(L) = n_u$, which in turn implies that $L$ must have at least $n_u$ columns. Since $L$ has $n_u$ rows, we see that $L$ cannot have more rows than columns. Since $\dim \text{Ker}(L) = 0$ (otherwise $U$ would not be a basis for $\text{Ker}(D_3^*)$), we see that $L$ must be square and invertible. Therefore, since

$$
D_3^* = [L\ 0][\hat{U}^*\ U^*] = L\hat{U}^*
$$

(4.65)

we see that

$$
D_3(D_3^*D_3)^{-1}D_3^* = \hat{U}L^*(L\hat{U}^*\hat{U}L^*)^{-1}L\hat{U}^* = \hat{U}L^*L^{-1}L\hat{U}^* = \hat{U}\hat{U}^* = I -UU^*
$$

(4.66)

which completes the proof. \[\blacksquare\]

**Lemma 4.3.6.** Let $\epsilon = 1$. If $R$ is invertible, $B_{[1,3]}^*P_0B_{[1,3]} + R$ is invertible, $0 \preceq P_0 \prec \hat{P}^{-1}$, and $\hat{P} + B_{[1,3]}R^{-1}B_{[1,3]}^* \succ 0$, then

$$
P_0 - P_0B_{[1,3]}(B_{[1,3]}^*P_0B_{[1,3]} + R)^{-1}B_{[1,3]}^*P_0 \preceq (\hat{P} + B_{[1,3]}R^{-1}B_{[1,3]}^*)^{-1}.
$$

(4.67)

**Proof.** Define $P_{\mu} := P_0 + \mu I$ and pick $\bar{\mu} > 0$ such that $P_0 \prec P_{\mu} \prec \hat{P}^{-1}$, $\forall \mu \in (0, \bar{\mu})$. With these definitions, we see that

$$
0 \prec \hat{P} \prec P_{\mu}^{-1}, \quad \forall \mu \in (0, \bar{\mu})
$$

$$
\Rightarrow \quad 0 \prec \hat{P} + B_{[1,3]}R^{-1}B_{[1,3]}^* \prec P_{\mu}^{-1} + B_{[1,3]}R^{-1}B_{[1,3]}^*, \quad \forall \mu \in (0, \bar{\mu})
$$

$$
\Rightarrow \quad (P_{\mu}^{-1} + B_{[1,3]}R^{-1}B_{[1,3]}^*)^{-1} \prec (\hat{P} + B_{[1,3]}R^{-1}B_{[1,3]}^*)^{-1}, \quad \forall \mu \in (0, \bar{\mu}.
$$

(4.68)
Using the matrix inversion lemma on the left-hand side of this expression, we see that
\[ P_{\mu} - P_{\mu}B_{[1,3]}(B_{[1,3]}^* P_{\mu} B_{[1,3]} + R)^{-1} B_{[1,3]}^* P_{\mu} \prec (\hat{P} + B_{[1,3]} R^{-1} B_{[1,3]}^*)^{-1}, \quad \forall \mu \in (0, \bar{\mu}). \] (4.69)

Therefore, since \( P_{\mu}|_{\mu=0} = P_0 \) and the left-hand side of the preceding inequality is a continuous function of \( \mu \) whenever it is well-defined, we see that (4.67) holds.

**Theorem 4.3.7.** Let \( \epsilon = 1 \). Suppose assumption (A1) holds and \( P_0 \) is a stabilizing solution of the DARE \( R_{\phi}(P) = P \) such that \( A + B_3 K_0 x \) is Schur and the factorization (4.50) exists where \( T_{11} \) and \( T_{22} \) are invertible. Then
\[ J_{fi,1} = J_1(\mathcal{F}_l(G_{fi}, K_0 x)) = \text{tr}\{R_{\psi}(P_0)\}. \] (4.70)

**Proof.** By Theorem 4.3.2 and the definition of \( J_{fi,\epsilon} \) given in (4.42), we immediately see that \( J_{fi,1} \leq \text{tr}\{R_{\psi}(P_0)\} \). It therefore suffices to show that \( J_{fi,1} \geq \text{tr}\{R_{\psi}(P_0)\} \). To do this, we will use the characterization of \( J_{fi,1} \) given by (4.43).

We first define for convenience
\[ \begin{bmatrix} \bar{C} & \bar{D}_1 & \bar{D}_2 \end{bmatrix} := U^* \begin{bmatrix} C & D_1 & D_2 \end{bmatrix}, \] (4.71)
\[ \begin{bmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 \end{bmatrix} := \begin{bmatrix} A & B_1 & B_2 \end{bmatrix} \begin{bmatrix} I & B_3 D_3^{-1} D_3^* & C & D_1 & D_2 \end{bmatrix} \] (4.72)
where the columns of \( U \) form a basis for \( \text{Ker}(D_3^*) \). By Lemma 4.3.4, we see that the columns of
\[ N := \begin{bmatrix} I \\ -D_3 (D_3^* D_3)^{-1} D_3^* & 0 \\ U \end{bmatrix} \] (4.73)
form a basis for \( \text{Ker}([B_3^* D_3^*]) \), which implies that this choice of \( N \) satisfies the requirements of Theorem 4.2.4. With this choice of \( N \), (4.37b) becomes (for \( \epsilon = 1 \))
\[ \begin{bmatrix} \hat{P} & \bullet & \bullet & \bullet & \bullet \\ 0 & I & \bullet & \bullet & \bullet \\ \hat{V} & 0 & W & \bullet & \bullet \\ \bar{A} \hat{P} & \bar{B}_1 & \bar{B}_2 & \hat{P} + B_3 (D_3^* D_3)^{-1} B_3^* & \bullet \\ \bar{C} \hat{P} & \bar{D}_1 & \bar{D}_2 & 0 & I \end{bmatrix} \succ 0 \] (4.74)

Let \( \hat{P}, W, \hat{V} \) satisfy (4.74). The remainder of the proof will show that \( \text{tr}\{W\} \geq \text{tr}\{R_{\psi}(P_0)\} \).

From the second and sixth rows of (4.74), we see that
\[ \begin{bmatrix} I & \bullet \\ \hat{D}_1 & I \end{bmatrix} \succ 0 \] (4.75)
which implies, by Schur complements, that $I - \tilde{D}_1^* \tilde{D}_1 \succ 0$. Using Lemma 4.3.5 we see that
\[
\tilde{D}_1^* \tilde{D}_1 - I = D_1^* U U^* D_1 - I = D_1^* (I - D_3 (D_3^* D_3)^{-1} D_3^*) D_1 - I \\
= (D_1^* D_1 - I) - D_1^* D_3 (D_3^* D_3)^{-1} D_3^* D_1 .
\] (4.76)

Since
\[
R = \begin{bmatrix} D_1^* D_1 - I & D_1^* D_3 \\ \bullet & D_3^* D_3 \end{bmatrix}
\] (4.77)
we see by Corollary 2.3.2 that
\[
\mathcal{N}(R) = \mathcal{N}(\tilde{D}_1^* \tilde{D}_1 - I) + \mathcal{N}(D_3^* D_3) = (0, 0, n_d) + (n_u, 0, 0) = (n_u, 0, n_d)
\] (4.78)
which implies that $R$ is invertible. By Proposition 2.1.2, Lemma 4.3.5, and (4.76), we see after some algebra that the following identities hold:
\[
\begin{align*}
A - B_{[1,3]} R^{-1} D_{[1,3]}^* C &= \tilde{A} - \tilde{B}_1 (\tilde{D}_1^* \tilde{D}_1 - I)^{-1} \tilde{D}_1^* \tilde{C} \\
B_2 - B_{[1,3]} R^{-1} D_{[1,3]}^* D_2 &= \tilde{B}_2 - \tilde{B}_1 (\tilde{D}_1^* \tilde{D}_1 - I)^{-1} \tilde{D}_1^* \tilde{D}_2 \\
B_{[1,3]} R^{-1} B_{[1,3]}^* &= B_3 (D_3^* D_3)^{-1} B_3^* + \tilde{B}_1 (\tilde{D}_1^* \tilde{D}_1 - I)^{-1} \tilde{B}_1 \\
I - D_{[1,3]} R^{-1} D_{[1,3]}^* &= U [I + \tilde{D}_1 (I - D_3^* D_3)^{-1} D_3^*] U^* .
\end{align*}
\] (4.79)

Also note that $Q = C^* C$ and $S = C^* D_{[1,3]}$. Thus, using (4.79) and the matrix inversion lemma, we see that
\[
Q - SR^{-1} S^* = C^* (I - D_{[1,3]} R^{-1} D_{[1,3]}^*) C = C^* U [I + \tilde{D}_1 (I - D_3^* D_3)^{-1} D_3^*] U^* C \\
= \tilde{C}^* (I - \tilde{D}_1 \tilde{D}_1^*)^{-1} \tilde{C} .
\] (4.80)

Therefore, defining
\[
\tilde{\phi} := (A - B_{[1,3]} R^{-1} D_{[1,3]}^* C, B_{[1,3]}, \tilde{C}^* (I - \tilde{D}_1 \tilde{D}_1^*)^{-1} \tilde{C}, R, 0) .
\] (4.81)
we see by Proposition 2.5.1 that $P_0$ is the stabilizing solution of the DARE $\mathcal{R}_{\tilde{\phi}}(P) = P$.

We will now use the fact that $P_0$ is the solution of the DARE $\mathcal{R}_{\tilde{\phi}}(P) = P$ along with Theorem 2.6.4 to prove that $0 \preceq P_0 \prec \hat{P}^{-1}$. We begin by eliminating the third row and column of (4.74), which yields that
\[
\begin{bmatrix}
\hat{P} & \bullet & \bullet & \bullet \\
0 & I & \bullet & \bullet \\
\hat{A} \hat{P} & \hat{B}_1 & \hat{P} + B_3 (D_3^* D_3)^{-1} B_3^* & \bullet \\
\hat{C} \hat{P} & \hat{D}_1 & 0 & I
\end{bmatrix} \succ 0.
\] (4.82)
By Schur complements, this implies that $\hat{P} > 0$ and

$$
\begin{bmatrix}
\hat{P} + B_3(D_3^*D_3)^{-1}B_3^* & \bullet \\
0 & I
\end{bmatrix}
- \begin{bmatrix}
\hat{A} \hat{P} & \hat{B}_1 \\
\hat{C} \hat{P} & \hat{D}_1
\end{bmatrix}
\begin{bmatrix}
\hat{P}^{-1} & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\hat{A} \hat{P} & \hat{B}_1 \\
\hat{C} \hat{P} & \hat{D}_1
\end{bmatrix}^* > 0.
$$

(4.83)

Defining

$$
L := \begin{bmatrix}
I & -\hat{B}_1(\hat{D}_1^*D_1 - I)^{-1}\hat{D}_1^* \\
0 & I
\end{bmatrix}
$$

(4.84)

we multiply (4.83) on the left and right respectively by $L$ and $L^*$ to obtain, after some algebra, that $L_\omega(P) < 0$ where

$$
\omega := \left( (\hat{A} - \hat{B}_1(\hat{D}_1^*D_1 - I)^{-1}\hat{D}_1^*\hat{C})^*, \hat{C}^* , \\
-B_3(D_3^*D_3)^{-1}B_3^* - \hat{B}_1(\hat{D}_1^*D_1 - I)^{-1}\hat{B}_1^*, \hat{D}_1\hat{D}_1^* - I, 0 \right). 
$$

(4.85)

Using (4.79), we see that

$$
\omega = \left( (A - B_1R^{-1}D_1^*C)^*, \hat{C}^*, -B_1R^{-1}B_1^*, \hat{D}_1\hat{D}_1^* - I, 0 \right).
$$

(4.86)

Therefore, since $P_0$ is the solution of the DARE $R_{\omega}(P) = P$, we see by Theorem 2.6.4 that $P_0 \prec \hat{P}^{-1}$. Also note that, since $A + B_3K_2^0$ is Schur, we see by Lemma 4.3.1 that $P_0 \succeq 0$. Therefore, $0 \preceq P_0 \prec \hat{P}^{-1}$.

With this in place, we can now show that $\text{tr}\{W\} > \text{tr}\{R_{\psi}(P_0)\}$. Returning to (4.74) we see, upon eliminating the first row and column of that inequality, that

$$
\begin{bmatrix}
I & \bullet & \bullet & \bullet \\
0 & W & \bullet & \bullet \\
\hat{B}_1 & \hat{B}_2 & \hat{P} + B_3(D_3^*D_3)^{-1}B_3^* & \bullet \\
\hat{D}_1 & \hat{D}_2 & 0 & I
\end{bmatrix}
\succeq 0.
$$

(4.87)

Using Schur complements, this implies that

$$
\begin{bmatrix}
I - \hat{D}_1^*\hat{D}_1 & \bullet & \bullet \\
-\hat{D}_1^*D_1 & W - \hat{D}_2^*\hat{D}_2 & \bullet \\
\hat{B}_1 & \hat{B}_2 & \hat{P} + B_3(D_3^*D_3)^{-1}B_3^*
\end{bmatrix}
\succeq 0.
$$

(4.88)

which in turn implies by Schur complements that

$$
\begin{bmatrix}
W - \hat{D}_2\hat{D}_2 - \hat{D}_2^*\hat{D}_1(I - \hat{D}_1^*\hat{D}_1)^{-1}\hat{D}_1^*\hat{D}_2 \\
\hat{B}_2 + \hat{B}_1(I - \hat{D}_1^*\hat{D}_1)^{-1}\hat{D}_1^*\hat{D}_2 & \hat{P} + B_3(D_3^*D_3)^{-1}B_3^* - \hat{B}_1(I - \hat{D}_1^*\hat{D}_1)^{-1}\hat{B}_1^*
\end{bmatrix}
\succeq 0.
$$

(4.89)
By Proposition 2.5.1, we see that
\[ W - D_2^*D_2 + D_2^*D_{[1,3]}R^{-1}D_{[1,3]}^*D_2 \]
\[ B_2 - B_{[1,3]}R^{-1}D_{[1,3]}^*D_2 \]
\[ \hat{P} + B_{[1,3]}R^{-1}B_{[1,3]}^* \]
\[ > 0. \]  
(4.90)

Using Schur complements one last time, we see that
\[ W > D_2^*D_2 - D_2^*D_{[1,3]}R^{-1}D_{[1,3]}^*D_2 \]
\[ + (B_2 - B_{[1,3]}R^{-1}D_{[1,3]}^*D_2)(\hat{P} + B_{[1,3]}R^{-1}B_{[1,3]}^*)^{-1}(B_2 - B_{[1,3]}R^{-1}D_{[1,3]}^*D_2). \]
(4.91)

By Lemma 4.3.6, this implies that \( W > \mathcal{R}_\psi(P_0) \) where
\[ \psi := (B_2 - B_{[1,3]}R^{-1}D_{[1,3]}^*D_2, B_{[1,3]}, D_2^*D_2 - D_2^*D_{[1,3]}R^{-1}D_{[1,3]}^*D_2, R, 0). \]
(4.92)

By Proposition 2.5.1, we see that \( \mathcal{R}_\psi(P_0) = \mathcal{R}_\psi(P_0) \), which implies that \( W > \mathcal{R}_\psi(P_0) \). This in turn implies that \( \text{tr}\{W\} > \text{tr}\{\mathcal{R}_\psi(P_0)\} \). Since the choice of \( \hat{P}, W, \hat{V} \) was arbitrary, this concludes the proof.

**4.3.2 Optimal Control for Fixed \( \epsilon > 0 \)**

In \S 4.3.1, we showed how to solve the optimal full information control problem with the restriction that \( \epsilon = 1 \). In this subsection, we solve the optimal full information control problem for fixed \( \epsilon > 0 \). To this end, we will use Theorem 3.5.5 to reformulate this problem as a problem in the form considered in \S 4.3.1.

We begin by defining the scaled plant \( \tilde{G}_{fi} \) by its state-space realization

\[
\tilde{G}_{fi} \sim \begin{bmatrix}
A & \sqrt{\epsilon}B_1 & B_2 & B_3 \\
\frac{1}{\sqrt{\epsilon}}C_1 & D_{11} & \frac{1}{\sqrt{\epsilon}}D_{12} & \frac{1}{\sqrt{\epsilon}}D_{13} \\
\frac{1}{\sqrt{\epsilon}}C_2 & \sqrt{\epsilon}D_{21} & D_{22} & D_{23} \\
\bar{I} & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0
\end{bmatrix}.
\]
(4.93)

Defining the scaled controller
\[
\tilde{K} := [K_x \sqrt{\epsilon}K_d \ K_w]
\]
(4.94)

we see that
\[
\mathcal{F}_i(\tilde{G}_{fi}, \tilde{K}) \sim \begin{bmatrix}
A & \sqrt{\epsilon}B_1 & B_2 \\
\frac{1}{\sqrt{\epsilon}}C_1 & D_{11} & \frac{1}{\sqrt{\epsilon}}D_{12} \\
\frac{1}{\sqrt{\epsilon}}C_2 & \sqrt{\epsilon}D_{21} & D_{22}
\end{bmatrix} + \begin{bmatrix}
B_3 \\
\frac{1}{\sqrt{\epsilon}}D_{13} \\
\frac{1}{\sqrt{\epsilon}}D_{23}
\end{bmatrix} [K_x \sqrt{\epsilon}K_d \ K_w] =: \Sigma_{\epsilon, \tilde{K}}.
\]
(4.95)
Thus, by Theorem 3.5.5, we see that
\[ J_{(\epsilon^{-1})}(F_i(G_{fi}, K)) = J_1(F_i(\bar{G}_{fi}, \bar{K})) \]  
(4.96)
Since infimizing over \( K \) is equivalent to infimizing over \( \bar{K} \), we see by infimizing both sides of (4.96) over their respective controllers that
\[ J_{fi,\epsilon} = \inf_{\bar{K}} J_1(F_i(\bar{G}_{fi}, \bar{K})). \]  
(4.97)
Thus, we have reformulated the problem of finding \( J_{fi,\epsilon} \) for \( \epsilon > 0 \) as a problem of the form considered in §4.3.1. To solve the optimization problem on the right-hand side of (4.97), the parameters of interest are
\[ \bar{\phi} := (A, B_{[1,3]}^T, \epsilon^{-1}Q, \epsilon^{-1}TRT, \epsilon^{-1}ST) \]  
(4.98)
\[ \bar{\psi} := (B_2, B_{[1,3]}^T, \epsilon^{-1}\bar{Q}, \epsilon^{-1}\bar{TRT}, \epsilon^{-1}\bar{ST}) \]  
(4.99)
where \( T := \text{diag}(\sqrt{\epsilon}I_{n_d}, I_{n_u}) \).

The following lemma shows that assumptions (A1)–(A3) are not affected by the scaling performed on \( G_{fi} \) to produce \( \bar{G}_{fi} \).

**Lemma 4.3.8.** The following statements hold:

1. (A1) holds for the scaled system \( \bar{G}_{fi} \) \( \Leftrightarrow \) (A1) holds for the unscaled system \( G_{fi} \)
2. (A2) holds for the scaled system \( \bar{G}_{fi} \) \( \Leftrightarrow \) (A2) holds for the unscaled system \( G_{fi} \)
3. (A3) holds for the scaled system \( \bar{G}_{fi} \) \( \Leftrightarrow \) (A3) holds for the unscaled system \( G_{fi} \)

**Proof.** Since \( \epsilon > 0 \), we see that
\[ [D_{13}^* D_{23}^*] \begin{bmatrix} \epsilon^{-1}I & 0 \\ 0 & I \end{bmatrix} [D_{13} D_{23}] > 0 \]  
(4.100)
which proves statement 1. Statement 2 is trivial. To prove statement 3, note that
\[
\begin{align*}
\text{Ker} \begin{bmatrix} A - \lambda I & B_3 \\ \frac{1}{\sqrt{\epsilon}}C_1 & \frac{1}{\sqrt{\epsilon}}D_{13} \end{bmatrix} &= \text{Ker} \begin{bmatrix} I_{n_x} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\epsilon}}I_{n_y} & 0 \\ 0 & 0 & I_{n_p} \end{bmatrix} \begin{bmatrix} A - \lambda I & B_3 \\ C & D_3 \end{bmatrix} \\
&= \text{Ker} \begin{bmatrix} A - \lambda I & B_3 \\ C & D_3 \end{bmatrix}
\end{align*}
\]  
(4.101)
which implies that
\[ \dim \left( \text{Ker} \begin{bmatrix} A - \lambda I & B_3 \\ \frac{1}{\sqrt{\epsilon}}C_1 & \frac{1}{\sqrt{\epsilon}}D_{13} \end{bmatrix} \right) = \dim \left( \text{Ker} \begin{bmatrix} A - \lambda I & B_3 \\ C & D_3 \end{bmatrix} \right) \]  
(4.102)
holds for all \( \lambda \in \mathbb{C} \).
With this lemma in place, we now give necessary and sufficient conditions for the condition $J_{f_i, \epsilon} \neq \infty$.

**Theorem 4.3.9.** Let $\epsilon > 0$ and suppose that (A1)–(A3) hold. Then $J_{f_i, \epsilon} \neq \infty$ if and only if the DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $P_0$ such that $A + B_3K_x^0$ is Schur and the factorization (4.50) exists where $T_{11}$ and $T_{22}$ are invertible.

**Proof.** (\(\Rightarrow\)) This is trivial by Theorem 4.3.2.

(\(\Leftarrow\)) By (4.97), we see that

$$\inf_{\mathcal{F}_1(G_{f_i}, \bar{K})} J_{f_i}(\phi, \bar{K}) \neq \infty. \quad (4.103)$$

Also note that, by Lemma 4.3.8, assumptions (A1)–(A3) hold for the scaled system $\bar{G}_{f_i}$. Therefore, by Theorem 4.3.3, the DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution $\bar{P}_0$ such that the matrix

$$A - B_3 (B_3^* \bar{P}_0 B_3 + \epsilon^{-1} D_{13}^* D_{13} + D_{23}^* D_{23})^{-1} (B_3^* \bar{P}_0 A + \epsilon^{-1} D_{13}^* C_1 + D_{23}^* C_2) \quad (4.104)$$

is Schur and the factorization

$$(B_{[1, 3]}T)^* \bar{P}_0 (B_{[1, 3]}T) + (\epsilon^{-1} T R T)^* = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}^* \begin{bmatrix} -I_{nd} & 0 \\ 0 & I_{nu} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}. \quad (4.105)$$

exists where $T_{11}$ and $T_{22}$ are invertible.

Define $P_0 := \epsilon \bar{P}_0$ and $T_{22} := \sqrt{\epsilon} T_{22}$. Note in particular that $T_{22}$ is invertible. With a little algebra, it is straightforward to show that $\epsilon^{-1} \mathcal{R}_\phi(P_0) = \mathcal{R}_\phi(\epsilon^{-1} P_0)$ and $A_\phi(P_0) = A_\phi(\epsilon^{-1} P_0)$. Thus,

$$\epsilon^{-1} P_0 = \bar{P}_0 = \mathcal{R}_\phi(\bar{P}_0) = \mathcal{R}_\phi(\epsilon^{-1} P_0) = \epsilon^{-1} \mathcal{R}_\phi(P_0) \quad (4.106)$$

which implies that $P_0 = \mathcal{R}_\phi(P_0)$. Moreover, since $A_\phi(P_0) = A_\phi(\bar{P}_0)$ is Schur, we see that $P_0$ is the stabilizing solution of the DARE $\mathcal{R}_\phi(P) = P$. Also note that the matrix in (4.104) is equal to $A + B_3K_x^0$, which implies that $A + B_3K_x^0$ is Schur. Finally, we note that

$$(B_{[1, 3]}T)^* \bar{P}_0 (B_{[1, 3]}T) + (\epsilon^{-1} T R T)^* = \left( \frac{1}{\sqrt{\epsilon}} T \right) \left( B_{[1, 3]}^* P_0 B_{[1, 3]} + R \right) \left( \frac{1}{\sqrt{\epsilon}} T \right)^* \quad (4.107)$$

which implies that (4.105) can be written

$$B_{[1, 3]}^* P_0 B_{[1, 3]} + R = (\sqrt{\epsilon} T^{-1}) \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}^* \begin{bmatrix} -I_{nd} & 0 \\ 0 & I_{nu} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} (\sqrt{\epsilon} T^{-1}) \quad (4.108)$$

which concludes the proof.
The following theorem gives the value of $J_{f_{i,\epsilon}}$ and shows that the controller $K^o$ is optimal.

**Theorem 4.3.10.** Suppose assumption (A1) holds and $P_0$ is a stabilizing solution of the DARE $\mathcal{R}_{\phi}(P) = P$ such that $A + B_3K^o_x$ is Schur and the factorization (4.50) exists where $T_{11}$ and $T_{22}$ are invertible. Then

$$J_{f_{i,\epsilon}} = J_{(\epsilon^{-1})}(\mathcal{F}_I(G_{f_i}, K^o)) = \epsilon^{-1} \text{tr} \{ \mathcal{R}_{\phi}(P_0) \} \, .$$

**(Proof.** It is straightforward to show that $\mathcal{R}_{\phi}(\epsilon^{-1}P_0) = \epsilon^{-1}\mathcal{R}_{\phi}(P_0)$ and $\mathcal{A}_{\phi}(\epsilon^{-1}P_0) = \mathcal{A}_{\phi}(P_0)$. Therefore $\epsilon^{-1}P_0$ is the stabilizing solution of the DARE $\mathcal{R}_{\phi}(P) = P$. Moreover

$$(B_{[1,3]}T)^*(\epsilon^{-1}P_0)(B_{[1,3]}T) + (\epsilon^{-1}TRT) = \left( \frac{1}{\sqrt{\epsilon}} T \right) (B_{[1,3]}^*P_0B_{[1,3]} + R) \left( \frac{1}{\sqrt{\epsilon}} T \right)
= \begin{bmatrix} T_{11} & 0 \\ T_{21} & \frac{1}{\sqrt{\epsilon}} T_{22} \end{bmatrix}^* \begin{bmatrix} -I_{n_d} & 0 \\ 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ T_{21} & \frac{1}{\sqrt{\epsilon}} T_{22} \end{bmatrix} \, .$$

(4.110)

Defining the controller

$$\tilde{K}^o := \begin{bmatrix} \bar{K}^o_x & \bar{K}^o_d & \bar{K}^o_w \end{bmatrix} := -(B_{[1,3]}^*(\epsilon^{-1}P_0)B_3 + \epsilon^{-1}D_{13}^*D_{13} + D_{23}^*D_{23})^{-1}
\times \begin{bmatrix} B_{[1,3]}^*(\epsilon^{-1}P_0) & \frac{1}{\sqrt{\epsilon}} D_{13}^* & D_{23}^* \\ \frac{1}{\sqrt{\epsilon}} D_{13} & C_1 & \frac{1}{\sqrt{\epsilon}} D_{12} \\ C_2 & \frac{1}{\sqrt{\epsilon}} D_{21} & D_{22} \end{bmatrix}$$

(4.111)

we see after some algebra that

$$\begin{bmatrix} \bar{K}^o_x & \bar{K}^o_d & \bar{K}^o_w \end{bmatrix} = \begin{bmatrix} K^o_x & \sqrt{\epsilon}K^o_d & K^o_w \end{bmatrix} \, .$$

(4.112)

Since $A + B_3K^o_x$ is Schur, this implies that $A + B_3K^o_x$ is Schur. Also, by Lemma 4.3.8, assumption (A1) holds for the scaled system $G_{f_i}$. Thus, applying Theorem 4.3.7 to the scaled system, we see that

$$\inf_K J_I(\mathcal{F}_I(\bar{G}_{f_i}, \bar{K})) = J_I(\mathcal{F}_I(\bar{G}_{f_i}, \bar{K}^o)) = \text{tr} \{ \mathcal{R}_{\phi}(\epsilon^{-1}P_0) \} \, .$$

(4.113)

With some algebra, it is straightforward to show that $\mathcal{R}_{\phi}(\epsilon^{-1}P_0) = \epsilon^{-1}\mathcal{R}_{\phi}(P_0)$. Therefore, by (4.96)–(4.97) and (4.112), this completes the proof. ■

With these results in place, we now make a few notes of the numerical evaluation of $J_{f_{i,\epsilon}}$ and the optimal controller $K^o$. Once we have solved the DARE $\mathcal{R}_{\phi}(P) = P$ for its stabilizing solution, we would like to check whether or not it permits the factorization (4.50) with invertible $T_{11}$ and $T_{22}$. We first make the partition

$$\begin{bmatrix} M_{22} & \bullet \\ M_{32} & M_{33} \end{bmatrix} = B_{[1,3]}^*P_0B_{[1,3]} + R \, .$$

(4.114)
where \( M_{22} \in \mathbb{R}^{n_d \times n_d} \). If the factorization (4.50) exists, then
\[
\begin{bmatrix}
M_{22} & \bullet \\
M_{32} & M_{33}
\end{bmatrix} =
\begin{bmatrix}
T_{11} & 0 \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
-\mathbb{I}_{n_d} & 0 \\
0 & \mathbb{I}_{n_u}
\end{bmatrix}
\begin{bmatrix}
T_{11} & 0 \\
T_{21} & T_{22}
\end{bmatrix} =
\begin{bmatrix}
T_{11}T_{21} - T_{11}^*T_{11} & \bullet \\
T_{22}T_{21} & T_{22}^*T_{22}
\end{bmatrix}.
\tag{4.115}
\]

From this, we see that \( M_{33} \succ 0 \), which implies that performing the Cholesky factorization \( T_{22}^*T_{22} = M_{33} \) yields a suitable value of \( T_{22} \). Once we have done this, we choose \( T_{21} := T_{22}^{-1}M_{32} \). By Corollary 2.3.2, note that
\[
\mathcal{N}(B_{[1,3]}^*P_0B_{[1,3]} + R) = \mathcal{N}(M_{22} - M_{32}^*M_{33}^{-1}M_{32}) + \mathcal{N}(M_{33})
\tag{4.116}
\]
Since \( \mathcal{N}(B_{[1,3]}^*P_0B_{[1,3]} + R) = (n_u, 0, n_d) \) by Proposition 2.3.1, we see that
\[
\mathcal{N}(M_{22} - M_{32}^*M_{33}^{-1}M_{32}) = (n_u, 0, n_d) - (n_u, 0, 0) = (0, 0, n_d)
\tag{4.117}
\]
which implies that \( M_{32}^*M_{33}^{-1}M_{32} - M_{22} \succ 0 \). Noting that
\[
M_{32}^*M_{33}^{-1}M_{32} - M_{22} = (T_{22}^*T_{21})^*(T_{22}^*T_{22})^{-1}(T_{22}^*T_{21}) - M_{22} = T_{21}^*T_{21} - M_{22}
\tag{4.118}
\]
we see that the Cholesky factorization
\[
T_{11}^*T_{11} = T_{21}^*T_{21} - M_{22}.
\tag{4.119}
\]
is guaranteed to exist and yields a suitable value of \( T_{11} \). By construction, this choice of \( T_{11}, T_{21}, \) and \( T_{22} \) satisfy (4.50). Also, \( T_{11} \) and \( T_{22} \) are invertible upper triangular matrices. Once this factorization has been performed, we express
\[
K_x^o = -T_{22}^{-1}T_{22}^*(-B_3^*P_0A + D_{13}^*C_1 + \epsilon D_{23}^*C_2)
K_d^o = -T_{22}^{-1}T_{22}^*M_{32} = -T_{22}^{-1}T_{21}
K_w^o = -T_{22}^{-1}T_{22}^*(B_3^*P_0B_2 + D_{13}^*D_{12} + \epsilon D_{23}^*D_{22})
\tag{4.120}
\]

To find the cost \( J_{f_i,\epsilon} \), we use the ideas in Lemma 4.3.1. With some algebra, it can be shown that
\[
B_1^{cl}P_0B_1^{cl} + \epsilon R^{cl} = -T_{11}^*T_{11}.
\tag{4.121}
\]
Exploiting this, we see that
\[
\epsilon R_{\psi,cl}(\epsilon^{-1}P_0) = (B_2^{cl})^*P_0B_2^{cl} + \epsilon Q^{cl} + \|(B_2^{cl})^*P_0B_1^{cl} + \epsilon S_1^{cl}\|_F[T_{11}^*T_{11}]^{-1}\|(B_2^{cl})^*P_0B_1^{cl} + \epsilon S_1^{cl}\|_F^2.
\tag{4.122}
\]
Dividing both sides by \( \epsilon \) and taking the trace of both sides, we see that
\[
J_{f_i,\epsilon} = \epsilon^{-1}\left(\sum(P_0 \circ [B_2^{cl}(B_2^{cl})^*]) + \|D_{12}\|_F^2 + \epsilon\|D_{22}\|_F^2 + \|T_{11}^{-1}(B_1^{cl})^*P_0B_2^{cl} + \epsilon (S_1^{cl})^*\|_F^2\right).
\tag{4.123}
\]

With these formulas in place, we now give an algorithm to find \( J_{f_i,\epsilon} \), the corresponding optimal controller, and the corresponding optimal closed-loop system.
Algorithm 4.3.11. The following algorithm computes $J_{f_i, \epsilon}$, the corresponding optimal controller, and the corresponding optimal closed-loop system under assumptions (A1)–(A3).

1. Find the stabilizing solution of the DARE $R_\phi(P) = P$

2. Compute the Cholesky factorization $T_2^* T_2 = B_3^* P_0 B_1 + D_{13}^* D_{11} + \epsilon D_{23}^* D_{21}$

3. Form the optimal controller gains:

$$K_0^o = -T_2^* (T_2^* (B_3^* P_0 A + D_{13}^* C_1 + \epsilon D_{23}^* C_2))$$

$$K_d^o = -T_2^* T_2$$

$$K_w^o = -T_2^* (T_2^* (B_3^* P_0 B_2 + D_{13}^* D_{12} + \epsilon D_{23}^* D_{22}))$$

4. Form the closed-loop state-space matrices

$$A_{cl}^o = A + B_3 K_{x^o}, \quad B_{1}^o = B_1 + B_3 K_{d^o}, \quad B_{2}^o = B_2 + B_3 K_{w^o}$$

$$C_{1}^o = C_1 + D_{13} K_{x^o}, \quad D_{11}^o = D_{11} + D_{13} K_{d^o}, \quad D_{12}^o = D_{12} + D_{13} K_{w^o}$$

$$C_{2}^o = C_2 + D_{23} K_{x^o}, \quad D_{21}^o = D_{21} + D_{23} K_{d^o}, \quad D_{22}^o = D_{22} + D_{23} K_{w^o}$$

5. Verify that $A_{cl}^o$ is Schur

6. Form the closed-loop state-space matrices

$$A_{cl} = A + B_3 K_{x^o}, \quad B_{1} = B_1 + B_3 K_{d^o}, \quad B_{2} = B_2 + B_3 K_{w^o}$$

$$C_{1} = C_1 + D_{13} K_{x^o}, \quad D_{11} = D_{11} + D_{13} K_{d^o}, \quad D_{12} = D_{12} + D_{13} K_{w^o}$$

$$C_{2} = C_2 + D_{23} K_{x^o}, \quad D_{21} = D_{21} + D_{23} K_{d^o}, \quad D_{22} = D_{22} + D_{23} K_{w^o}$$

7. Verify that $A_{cl}$ is Schur

8. $\bar{K} = T_{11}^* \left( (B_{1}^o A^o P_0 B_2 + (D_{11}^o A^o P_0 B_2 + D_{12}^o A^o P_0 B_2 + \epsilon D_{21}^o A^o P_0 B_2 + D_{22}^o A^o P_0 B_2) \right)^*$

9. $J_{f_i, \epsilon} = \epsilon^{-1} \left( \sum P_0 \circ [B_2^o (B_{2}^o)^*] + \| D_{12}^o \|^2_F + \| D_{22}^o \|^2_F + \| K^* \|^2_F \right)$

If steps 1, 2, or 4 fail or if $A_{cl}$ is found to be not Schur in step 7, then $J_{f_i, \epsilon} = \infty$ and there is no optimizing controller.

4.3.3 Optimal $H_2$ Guaranteed Cost Control

Now that we have shown how to find $J_{f_i, \epsilon}$ and the corresponding optimal controller for fixed $\epsilon > 0$ by using a single DARE, we will develop an algorithm that solves the optimal full information control problem using an iteration of DARE solutions.

First note that we can use the characterization of $J_{f_i, \epsilon}$ given in (4.43) to equivalently reformulate (4.37) as the optimization

$$\inf_{\epsilon > 0} J_{f_i, \epsilon} \quad (4.124)$$
This corresponds to breaking the convex optimization (4.37) into two optimizations; we first optimize over \( \hat{P}, W, \hat{V} \) to produce the cost function \( J_{fi,\epsilon} \), and then optimize over \( \epsilon \) to yield the optimal cost of the full information control problem. By the same reasoning given in §3.2.2, we see that \( J_{fi,\epsilon} \) is a convex function of \( \epsilon \). This in turn implies that (4.124) is a nonlinear convex optimization.

Now we present a result which makes it especially easy to find values of \( \epsilon \) for which the optimal \( J_{(\epsilon^{-1})} \) is finite.

**Proposition 4.3.12.** If \( \exists \epsilon > 0 \) such that \( J_{fi,\epsilon} \neq \infty \), then the set of \( \epsilon > 0 \) for which \( J_{fi,\epsilon} \neq \infty \) is the interval \((0, \epsilon)\) for some \( \epsilon \in (0, \infty) \).

**Proof.** We first use the characterization of \( J_{fi,\epsilon} \) given by (4.43). Since the set of \( \epsilon, \hat{P}, W, \hat{V} \) that satisfy (4.37b) is open and convex, we see that the set

\[
S := \{ \epsilon > 0 : \exists \hat{P}, W, \hat{V} \text{ satisfying (4.37b)} \}
\]

is open and convex. Note in particular that \( \epsilon \in S \iff J_{fi,\epsilon} \neq \infty \).

Let \( \hat{\epsilon} \in S \) and choose \( \hat{K} \) such that \( J_{(\hat{\epsilon}^{-1})}(F_l(G_{fi}, \hat{K})) \neq \infty \). By Proposition 3.2.6, we see that \( J_{(\epsilon^{-1})}(F_l(G_{fi}, \hat{K})) \neq \infty, \forall \epsilon^{-1} \geq \hat{\epsilon}^{-1} \). Equivalently, \( J_{(\epsilon^{-1})}(F_l(G_{fi}, \hat{K})) \neq \infty, \forall \epsilon \in (0, \hat{\epsilon}] \).

Using the characterization of \( J_{fi,\epsilon} \) given by (4.42), we see that \( J_{fi,\epsilon} \neq \infty, \forall \epsilon \in (0, \hat{\epsilon}] \). \( \blacksquare \)

We now analyze how \( J_{fi,\epsilon} \) varies as \( \epsilon \) is varied. Since the stabilizing solution of a DARE is analytic in its parameters \([9]\) and \( J_{fi,\epsilon} \) is an analytic function of the stabilizing DARE solution, we see immediately that \( J_{fi,\epsilon} \) is an analytic function of \( \epsilon \). Thus, we would like to construct an efficient algorithm for computing \( dJ_{fi,\epsilon}/d\epsilon \).

Let \( P_0 \) be the stabilizing solution of the DARE \( \mathcal{R}_\phi(P) = P \) such that \( A + B_3K_\phi^* \) is Schur and the factorization (4.50) exists where \( T_{11} \) and \( T_{22} \) are invertible. Implicitly differentiating the DARE \( \mathcal{R}_\phi(P_0) = P_0 \) with respect to \( \epsilon \) yields

\[
\frac{dP_0}{d\epsilon} = A^*\frac{dP_0}{d\epsilon}A + C_2^*C_2 + \left( A^*\frac{dP_0}{d\epsilon}B_{[1,3]} + C^*_2 [D_{21} \ D_{23}] \right) K_\phi(P_0)
+ K_\phi(P_0)^* \left( B_{[1,3]^*} \frac{dP_0}{d\epsilon}A + \left[ D_{21} \ D_{23} \right]^*C_2 \right)
+ K_\phi(P_0)^* \left( B_{[1,3]^*} \frac{dP_0}{d\epsilon}B_{[1,3]} + \left[ D_{21} \ D_{23} \right]^* \left[ D_{21} \ D_{23} \right] K_\phi(P_0) \right). \tag{4.126}
\]

Therefore, \( dP_0/d\epsilon \) satisfies the discrete Lyapunov equation

\[
\frac{dP_0}{d\epsilon} = A_\phi(P_0)^* \frac{dP_0}{d\epsilon} A_\phi(P_0) + \left( C_2 + \left[ D_{21} \ D_{23} \right] K_\phi(P_0) \right)^* \left( C_2 + \left[ D_{21} \ D_{23} \right] K_\phi(P_0) \right). \tag{4.127}
\]

Since \( A_\phi(P_0) \) is Schur (by definition of the stabilizing DARE solution), this discrete Lyapunov equation can always be uniquely solved for \( dP_0/d\epsilon \). Moreover, by using the dlyap chol
function in MATLAB, we can directly solve for upper triangular \( U \) that satisfies \( U^*U = dP_0/d\epsilon \). Similarly, we can express after some algebra that

\[
\frac{d}{d\epsilon}(\mathcal{R}_\psi(P_0)) = A_\psi(P_0)^* \frac{dP_0}{d\epsilon} A_\psi(P_0) + (D_{22} + [D_{21} \quad D_{23}] \mathcal{K}_\psi(P_0))^*(D_{22} + [D_{21} \quad D_{23}] \mathcal{K}_\psi(P_0))
\]

(4.128)

which implies that

\[
\text{tr}\left\{ \frac{d}{d\epsilon}(\mathcal{R}_\psi(P_0)) \right\} = \| U A_\psi(P_0) \|_F^2 + \| D_{22} + [D_{21} \quad D_{23}] \mathcal{K}_\psi(P_0) \|_F^2.
\]

(4.129)

We now use the expression for \( J_{fi,\epsilon} \) given in Theorem 4.3.10 and the chain rule to express

\[
\frac{dJ_{fi,\epsilon}}{d\epsilon} = \frac{d}{d\epsilon} \left( \epsilon^{-1} \text{tr}\{\mathcal{R}_\psi(P_0)\} \right) = \epsilon^{-1} \text{tr}\left\{ \frac{d}{d\epsilon}(\mathcal{R}_\psi(P_0)) \right\} - \epsilon^{-2} \text{tr}\{\mathcal{R}_\psi(P_0)\}
\]

(4.130)

\[
= \epsilon^{-1} \left( \| U A_\psi(P_0) \|_F^2 + \| D_{22} + [D_{21} \quad D_{23}] \mathcal{K}_\psi(P_0) \|_F^2 - J_{fi,\epsilon} \right).
\]

The expressions for evaluating \( dJ_{fi,\epsilon}/d\epsilon \), depend on the quantities \( \mathcal{K}_\phi(P_0) \) and \( \mathcal{K}_\psi(P_0) \). We now show how to efficiently compute these quantities. We first partition \( \mathcal{K}_\phi(P_0) \) and \( \mathcal{K}_\psi(P_0) \) as

\[
\mathcal{K}_\phi(P_0) = \begin{bmatrix} K_{dx} \\ \bar{K}_x \end{bmatrix} \quad \mathcal{K}_\psi(P_0) = \begin{bmatrix} K_{dw} \\ \bar{K}_w \end{bmatrix}
\]

(4.131)

where \( K_{dx} \) and \( K_{dw} \) have \( n_d \) columns. It is easily verified that

\[
(B_{[1,3]}^*P_0B_{[1,3]} + R)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & T_{22}^{-1}T_{22}^{-*} \end{bmatrix} - \begin{bmatrix} I \\ -T_{22}^{-1}T_{21} \end{bmatrix} T_{11}^{-1}T_{11}^{-*} \begin{bmatrix} I \\ -T_{22}^{-1}T_{21} \end{bmatrix}^*.
\]

(4.132)

Using this expression along with the expressions in (4.120), we see that

\[
\begin{bmatrix} K_{dx} \\ \bar{K}_x \end{bmatrix} = \begin{bmatrix} 0 \\ K_\phi \end{bmatrix} + \begin{bmatrix} I \\ K_\phi \end{bmatrix} T_{11}^{-1}T_{11}^{-*} \left( (B_{11}^c)^*P_0A + (D_{11}^c)^*C_1 + \epsilon(D_{21}^c)^*C_2 \right).
\]

(4.133)

Note in particular that \( \bar{K}_x = K_\phi^o + K_d^o K_{dx} \). Similarly, it can be shown that

\[
K_{dw} = T_{11}^{-1}T_{11}^{-*} \left( (B_{11}^c)^*P_0B_2 + (D_{11}^c)^*D_{12} + \epsilon(D_{21}^c)^*D_{22} \right)
\]

(4.134)

\[
\bar{K}_w = K_w^o + K_d^o K_{dw}.
\]

(4.135)

With this in place, we now state the algorithm for finding \( dJ_{fi,\epsilon}/d\epsilon \).

**Algorithm 4.3.13.** The following algorithm computes \( dJ_{fi,\epsilon}/d\epsilon \).
1. Use Algorithm 4.3.11 to compute the following quantities: $P_0, T_{11}, T_{21}, T_{22}, K_x^o, K_d^o, K_w^o, A^{cl}, B_1^{cl}, B_2^{cl}, C_1^{cl}, C_2^{cl}, D_{11}^{cl}, D_{12}^{cl}, D_{21}^{cl}, D_{22}^{cl}, \hat{K}$, and $J_{fi,\epsilon}

2. Compute the quantities

\[ K_{dx} = T_{11} \setminus (T_{11}^* [(B_1^{cl})^* P_0 A + (D_{11}^{cl})^* C_1 + \epsilon (D_{21}^{cl})^* C_2]) \]
\[ K_{dw} = T_{11} \setminus (T_{11}^* [(B_1^{cl})^* P_0 B_2 + (D_{11}^{cl})^* D_{12} + \epsilon (D_{21}^{cl})^* D_{22}]) \]

3. Compute the quantities

\[ \bar{K}_x = K_x^o + K_d^o K_{dx} \]
\[ \bar{K}_w = K_w^o + K_d^o K_{dw} \]

4. Compute the quantities

\[ \mathcal{A}_\phi(P_0) = A + B_1 K_{dx} + B_3 \bar{K}_x \]
\[ \mathcal{A}_\psi(P_0) = B_2 + B_1 K_{dw} + B_3 \bar{K}_w \]
\[ \check{C} = C_2 + D_{21} K_{dx} + D_{23} \bar{K}_x \]
\[ \check{D} = D_{22} + D_{21} K_{dw} + D_{23} \bar{K}_w \]

5. Using the MATLAB function dlyapcho1, solve for the Cholesky factor $U$ in the discrete Lyapunov equation $U^* U = \mathcal{A}_\phi(P_0)^* (U^* U) \mathcal{A}_\phi(P_0) + \check{C}^* \check{C}$

6. $dJ_{fi,\epsilon}/d\epsilon = \epsilon^{-1}(\|U \mathcal{A}_\psi(P_0)\|_F^2 + \|\check{D}\|_F^2 - J_{fi,\epsilon})$}

The value and derivative of $J_{fi,\epsilon}$ is also useful for generating a lower bound on the optimal squared $\mathcal{H}_2$ guaranteed cost of the closed-loop system. Consider Fig. 4.2, which shows a representative graph of $J_{fi,\epsilon}$ in which $\epsilon_0$ is known to be a lower bound on the minimizing value of $\epsilon$. By convexity, if $\epsilon_1$ is known to be a upper bound on the minimizing value of $\epsilon$, the value and derivative of $J_{fi,\epsilon}$ at $\epsilon_0$ gives us the lower bound $\hat{J}_1$. If instead, the value and derivative of $J_{fi,\epsilon}$ at $\epsilon_2$ are known, we have the lower bound $\hat{J}_2$. These lower bounds are respectively given by

\[ \hat{J}_1 = J_{fi,\epsilon_0} + \bar{m}_0 (\epsilon_1 - \epsilon_2) \]
\[ \hat{J}_2 = \bar{m}_2 [\bar{m}_0 (\epsilon_2 - \epsilon_0) - (J_{fi,\epsilon_2} - J_{fi,\epsilon_0})] + J_{fi,\epsilon_2} \]
where $\tilde{m}_i$ is $dJ_{fi,\epsilon}/d\epsilon$ evaluated at $\epsilon = \epsilon_i$. It should be noted that the second of these lower bounds is less conservative when it is applicable.

With these results in place, we can easily solve (4.124) (and, hence, (4.13)) using the following algorithm:

**Algorithm 4.3.14.** The following algorithm computes the optimal $\mathcal{H}_2$ guaranteed cost of the closed-loop system along with an optimal controller.

1. **Check Regularity Conditions:** Verify that assumptions (A1)–(A3) hold.

2. **Find Initial Interval:** Choose $\alpha > 1$. Check if $J_{fi,\epsilon} \neq \infty$ and $dJ_{fi,\epsilon}/d\epsilon < 0$ when $\epsilon = 1$. If so, start from $k = 1$ and increment $k$ until either or these conditions fail to be met when $\epsilon = \alpha^k$. Denoting the corresponding value of $\epsilon$ as $\epsilon_u$, there exists an optimal value of $\epsilon$ in the interval $(\alpha^{-1}\epsilon_u, \epsilon_u)$.

   If instead either $J_{fi,\epsilon} = \infty$ or $dJ_{fi,\epsilon}/d\epsilon > 0$ when $\epsilon = 1$, start from $k = 1$ and increment $k$ until $J_{fi,\epsilon} \neq \infty$ and $dJ_{fi,\epsilon}/d\epsilon < 0$ when $\epsilon = \alpha^{-k}$. Denoting the corresponding value of $\epsilon$ as $\epsilon_l$, there exists an optimal value of $\epsilon$ in the interval $(\epsilon_l, \alpha \epsilon_l)$.

3. **Bisection:** Solve $dJ_{fi,\epsilon}/d\epsilon = 0$ over $\epsilon$ using bisection. Whenever $J_{fi,\epsilon} = \infty$ for a particular value of $\epsilon$, this value of $\epsilon$ is an upper bound on the optimal value of $\epsilon$.

   In this algorithm, each evaluation of $J_{fi,\epsilon}$ is done using Algorithm 4.3.11 and each evaluation of $dJ_{fi,\epsilon}/d\epsilon$ when $J_{fi,\epsilon} \neq \infty$ is done using Algorithm 4.3.13.

   In our implementation, we verify that assumption (A1) holds by checking that $\text{rank}(D_3^3) = n_u$ using the `rank` function in MATLAB. To verify that assumption (A2) holds, we first compute the controllability staircase form using the `ctrbf` function in MATLAB, then verify the stability of the uncontrollable modes. To verify that assumption (A3) holds, we use the `zero` command in MATLAB to find the transmission zeros of the system

$$G_{(A3)} \sim \begin{bmatrix} A & B_3 \\ C & D_3 \end{bmatrix}. \quad (4.137)$$

In step 2, we use $\alpha = 100$ and, in the bisection step, we use the geometric mean instead of the arithmetic mean. We use two stopping criteria in our implementation. If we define the relative error as $\nu_{fi} := 1 - J_{fi} / J_{fi,\epsilon}$ where $J_{fi}$ is the lower bound computed by (4.136), we terminate the algorithm when either $\nu_{fi} < 10^{-10}$ or the number of iterations (including the iterations required to find the initial interval) exceeds 30.

It should be noted that, in this algorithm, we do not explicitly check whether or not $\exists \epsilon > 0$ such that $J_{fi,\epsilon} \neq \infty$. This feasibility condition is equivalent to the solvability of a full information $\mathcal{H}_\infty$ control problem. However, there is no guarantee that this full information $\mathcal{H}_\infty$ control problem will meet the relevant regularity conditions. This means that there might exist a feasible full information $\mathcal{H}_\infty$ controller, even though the relevant DARE does
not have a stabilizing solution with the correct properties. Thus, if we fail to find a feasible full information $\mathcal{H}_\infty$ controller using the DARE approach, it does not necessarily imply that $\exists \epsilon > 0$ such that $J_{fi,\epsilon} \neq \infty$.

### 4.4 Numerical Experiments

In this section, we consider the application of the developed methodologies to randomly generated FI $\mathcal{H}_2$ guaranteed cost control problems. In particular, we consider three approaches—using the DARE approach outlined in §4.3, solving (4.14) using SeDuMi (parsed using YALMIP [22]), and solving (4.14) using the mincx command in the Robust Control Toolbox. The last two of these methods will be collectively called the LMI methods. It should be noted that YALMIP was not used when using mincx because YALMIP causes mincx to run more slowly. All numerical experiments were performed in MATLAB (with multithreaded computation disabled) on a computer with a 2.2 GHz Intel Core 2 Duo Processor and 2 GB of RAM.

To generate each random system in our numerical experiments, we first generated a random stable discrete-time state space system using drss in MATLAB, designed an optimal (non-robust) FI $\mathcal{H}_2$ controller, and then multiplied the closed loop system by the inverse of its $\mathcal{H}_\infty$ norm (computed by norm). This system was then multiplied by a random number generated from a uniform distribution on $[-1,1]$. The resulting system corresponded to generating random values of $A, B_1, C_1$, and $D_{11}$ for a robustly stable system. The FI $\mathcal{H}_2$ control step was used as a heuristic to make the control design problem less well-conditioned. (In particular, this tends to result in systems which are “closer” to not being robustly stabilizable.)

We then generated random values of $B_2, B_3, C_2, D_{12}, D_{13}, D_{21}, D_{22}, D_{23}, K_x$, and $K_d$ from independent normal distributions. Finally, we set

\[
\begin{align*}
A &\leftarrow A + B_3 K_x, \\
C_1 &\leftarrow C_1 + D_{13} K_x, \\
B_1 &\leftarrow B_1 + B_3 K_d, \\
D_{11} &\leftarrow D_{11} + D_{13} K_d.
\end{align*}
\]

Note that this corresponds to “shifting” the system by a randomly chosen control scheme; although the resulting system is not guaranteed to be stable, it is guaranteed that an FI control scheme exists which robustly stabilizes the system. For all of the numerical experiments, we chose the signal dimensions to be $n_q = 8, n_p = 7, n_d = 6, n_w = 5, n_u = 4$.

In the first experiment, we tested the speed of the methodologies over several values of $n_x$, the dimension of the plant state. The results of this test are shown in Fig. 4.3. In particular, note that the DARE method is faster than the LMI methods for all of the randomly generated problems. For instance, for the 24th-order system, it respectively took the DARE approach, the mincx approach, and the SeDuMi approach 0.27 seconds, 14.59 seconds, and 457.60 seconds to compute the optimal achievable performance and construct the optimal controller. Also note that the DARE method appears to have a complexity of $O(n_x^3)$ whereas the SeDuMi method appears to have a complexity of $O(n_x^4)$. The curve which
Figure 4.3: Time required to solve randomly-generated FI $H_2$ guaranteed cost control problems

corresponds to the $\text{mincx}$ method is not smooth because the number of iterations required to solve the problem often changes dramatically from problem to problem, unlike the other two methods. Nonetheless, solving the problem using $\text{mincx}$ appears to have a complexity of at least $O(n_x^4)$ also. Thus, the difference in computational speed between the DARE approach and the other two approaches becomes more pronounced for larger values of $n_x$.

In the second experiment, we tested the accuracy of the DARE approach compared to the LMI approaches for 100 randomly generated analysis problems with $n_x = 20$. To this end, we first define $f_d$, $f_m$, and $f_s$ as the squared $H_2$ guaranteed cost performance (determined using Algorithm 3.2.8) for the optimal closed loop systems respectively computed using the DARE approach, $\text{mincx}$, and SeDuMi. The criterion we will be using to compare the accuracy of the relevant methods is the relative error, i.e. we use the criterion $\nu_m := f_m/f_d - 1$ to compare the accuracy of the $\text{mincx}$ approach to the DARE approach and the criterion $\nu_s := f_s/f_d - 1$ to compare the accuracy of the SeDuMi approach to the DARE approach. For 98 of the random systems used in this paper, $|\nu_m| < 10^{-10}$, i.e. $\text{mincx}$ almost always achieved comparable accuracy to the DARE approach. For the other two systems, the relative error was 0.03% and 4%. (Note that these two cases correspond to $\text{mincx}$ getting “stuck” in a suboptimal solution.) SeDuMi, however, tended not to be quite as accurate. As shown in Fig. 4.4, the SeDuMi method frequently gets “stuck” in a suboptimal solution due to numerical problems. For one of the random systems, SeDuMi returned a controller which did not robustly stabilize the system. Also, for 99 of the random systems used in this paper, $\nu_s > 0$ (i.e. SeDuMi achieved inferior accuracy). For the one remaining system, $\nu_s = -13\%$. However, upon closer examination, we found that this was a numerical error arising in the analysis algorithm, not the synthesis algorithm developed in this paper. When we made a small perturbation on the initial condition for the analysis algorithm, it certified that both closed loop systems achieved comparable performance. This suggests that the accuracy of the DARE method is superior to that of the LMI methods.
Figure 4.4: Histogram of $\nu_s > 0$
Chapter 5

Output Feedback $\mathcal{H}_2$ Guaranteed Cost Control

In this section, we show how to use the results of Chapters 3–4 to design output feedback controllers which optimize the $\mathcal{H}_2$ guaranteed cost. We first present a non-convex optimization problem for determining an optimal controller and a solution heuristic which is based on the solution of a sequence of SDPs. We then give an algorithm which exploits DARE solutions to give a more computationally efficient heuristic for finding an optimal controller.

In this chapter, we will use the notation $\mathcal{R}_\phi(P)$, $\mathcal{K}_\phi(P)$, and $\mathcal{A}_\phi(P)$ defined in §2.5, the notation $J_\tau$ defined in §3.2.1, the notation $\bar{\mathcal{M}}$ defined in §3.4, and the notation $J_{f_t, \epsilon}$ defined in §4.3.

5.1 Problem Formulation

We now consider an optimal $\mathcal{H}_2$ guaranteed cost control problem of the form shown in Figure 5.1 in which $G$ has the known state-space realization

$$G \sim \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & 0 \end{bmatrix}.$$  \quad (5.1)$$

The state variable of $G$ will be denoted as $x$ and its dimension will be denoted as $n_x$. The dimensions of the signals $q$, $p$, $y$, $d$, $w$, and $u$ are respectively $n_q$, $n_p$, $n_y$, $n_d$, $n_w$, and $n_u$. Note that the control structure considered in this chapter is similar to the control structure considered in Chapter 4; the only difference is the choice of information that is available to the controller. In particular, we do not make any restrictions in this chapter on the structure of $y$ in Figure 5.1.
In this dissertation, we will restrict the controller to lie in the set $\mathcal{K}$, which we define to be the set of controllers which are LTI and have finite order. We will denote the state-space realization of $\mathcal{K}$ as

$$\mathcal{K} \sim \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}.$$  (5.2)

The state variable of $\mathcal{K}$ will be denoted as $x_c$ and its dimension will be denoted as $n_c$.

Using the characterization of $\mathcal{H}_2$ guaranteed cost performance given in (3.65), we are interested in solving the optimal control problem

$$\inf_{\mathcal{K} \in \mathcal{K}} \inf_{\tau > 0} J_{\tau}(\mathcal{F}_i(G, K)).$$  (5.3)

We will refer to this optimal control problem as the output feedback control problem. Because we have not fixed the order of the controller, this formulation cannot be put directly into a matrix inequality form in which the dimension of the matrix inequality is fixed.

### 5.2 Sequential Semi-Definite Programming Approach

#### 5.2.1 Nonlinear Change of Variables

In this section, we show that the optimization problem (5.3) is equivalent to the optimization problem

$$\inf_{\tau, X, Y, W, \hat{V}_1, \hat{V}_2, A_c, B_c, C_c, D_c} \text{tr}\{W\} \quad \text{s.t.}$$  (5.4a)

$$\begin{bmatrix} X & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ I & Y & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \tau I & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hat{V}_1 & \hat{V}_2 & 0 & W & \bullet & \bullet & \bullet & \bullet \\ M_{11} & M_{12} & M_{13} & M_{14} & X & \bullet & \bullet & \bullet \\ M_{21} & M_{22} & M_{23} & M_{24} & I & Y & \bullet & \bullet \\ M_{31} & M_{32} & M_{33} & M_{34} & 0 & 0 & \tau I & \bullet \\ M_{41} & M_{42} & M_{43} & M_{44} & 0 & 0 & 0 & I \end{bmatrix} \succ 0$$  (5.4b)
where

\[
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22} \\
M_{31} & M_{32} \\
M_{41} & M_{42}
\end{bmatrix} =
\begin{bmatrix}
AX + B_3 \hat{C}_c & A + B_3 \hat{D}_c C_3 \\
\hat{A}_c & YA + B_c C_3 \\
\tau(C_1 X + D_{13} \hat{C}_c) & \tau(C_1 + D_{13} \hat{D}_c C_3) \\
C_2 X + D_{23} \hat{C}_c & C_2 + D_{23} \hat{D}_c C_3
\end{bmatrix}
\tag{5.5}
\]

\[
\begin{bmatrix}
M_{13} & M_{14} \\
M_{23} & M_{24} \\
M_{33} & M_{34} \\
M_{43} & M_{44}
\end{bmatrix} =
\begin{bmatrix}
B_1 + B_3 \hat{D}_c D_{31} & B_2 + B_3 \hat{D}_c D_{32} \\
YB_1 + \hat{B}_c D_{31} & YB_2 + \hat{B}_c D_{32} \\
\tau(D_{11} + D_{13} \hat{D}_c D_{31}) & \tau(D_{12} + D_{13} \hat{D}_c D_{32}) \\
D_{21} + D_{23} \hat{D}_c D_{31} & D_{22} + D_{23} \hat{D}_c D_{32}
\end{bmatrix}
\tag{5.6}
\]

To do this, we will first show that if there exists an LTI controller of any order that achieves \(J_\tau(F_l(G, K)) < \gamma\), then the value of (5.4) must be less than \(\gamma\). We then show that if the (5.4) is less than \(\gamma\), then we can always reconstruct a controller of order \(n_c\) that achieves the closed-loop performance \(J_\tau(F_l(G, K)) < \gamma\). To this end, we will use the linearizing change of variables and congruence transformation given in [29].

**Theorem 5.2.1.** Let \(\tau > 0\). If \(\exists K \in \mathcal{K}\) such that \(J_\tau(F_l(G, K)) < \gamma\), then \(\exists X, Y, W, \hat{V}_1, \hat{V}_2, \hat{A}_c, \hat{B}_c, \hat{C}_c,\) and \(\hat{D}_c\) such that \(\text{tr}\{W\} < \gamma\) and (5.4b) is satisfied.

**Proof.** Let \(K \in \mathcal{K}\) satisfy \(J_\tau(F_l(G, K)) < \gamma\). Without loss of generality, assume that \(n_c \geq n_x\). Note that the closed-loop system has the realization

\[
F_l(G, K) \sim \begin{bmatrix}
A & 0 & B_1 & B_2 \\
0 & I & 0 & 0 \\
0 & 0 & D_{11} & D_{12} \\
C_1 & 0 & D_{21} & D_{22}
\end{bmatrix} + \begin{bmatrix}
0 & B_3 \\
I & 0 \\
0 & D_{13} \\
0 & D_{23}
\end{bmatrix} \begin{bmatrix}
A_c & B_c \\
C_c & D_c
\end{bmatrix} \begin{bmatrix}
0 & I & 0 & 0 \\
C_3 & 0 & D_{31} & D_{32}
\end{bmatrix} =: \Sigma_{cl}.
\tag{5.7}
\]

Using the characterization of \(J_\tau\) given by (3.86), fix \(P, W,\) and \(V\) such that \(\text{tr}\{W\} < \gamma\) and \(\mathcal{M}_{\Sigma_{cl}}(\tau, P, W, V) > 0\). Note in particular that \(P > 0\). We now partition \(P, P^{-1},\) and \(V\) respectively as

\[
P = \begin{bmatrix}
Y & N^* \\
N^* & U
\end{bmatrix}, \quad P^{-1} = \begin{bmatrix}
X & M^* \\
M^* & T
\end{bmatrix}, \quad V = \begin{bmatrix}
V_1 & V_2
\end{bmatrix}
\tag{5.8}
\]

where \(X, Y,\) and \(V_1\) have \(n_x\) columns. Note that, since \(N^* \in \mathbb{R}^{n_x \times n_x}\) and \(n_c \geq n_x\), if the chosen value of \(N^*\) does not have the property that \(\text{Ker}(N^*) = \{0\}\), it can always be perturbed by an arbitrarily small amount to produce a value of \(N^*\) that satisfies \(\text{Ker}(N^*) = \{0\}\). Since the set of \(\tau, P, W, V\) that satisfy \(\mathcal{M}_{\Sigma_{cl}}(\tau, P, W, V) > 0\) is open, we see that this inequality is not violated for small enough perturbations on \(N^*\). Thus, we assume without loss of generality that \(\text{Ker}(N^*) = \{0\}\).
Since $PP^{-1} = I$, we see that $YX + NM^* = I$ and $N^*X + UM^* = 0$. Thus, defining

$$\Pi := \begin{bmatrix} X & I \\ M^* & 0 \end{bmatrix}$$

(5.9) we see that

$$P\Pi = \begin{bmatrix} I & Y \\ 0 & N^* \end{bmatrix}.$$  (5.10)

Since $\ker(N^*) = \{0\}$, we see by (5.10) that $\ker(P\Pi) = \{0\}$, which implies that $\ker(\Pi) = \{0\}$. Thus, defining the matrix

$$\Pi := \begin{bmatrix} X & I \\ M^* & 0 \end{bmatrix}$$

(5.9) we see that

$$P\Pi = \begin{bmatrix} I & Y \\ 0 & N^* \end{bmatrix}.$$  (5.10)

we see after some algebra involving (5.10) that the condition $L^*\mathcal{M}_{el}(\tau, P, W, V)L \succ 0$ is the condition (5.4b).

Theorem 5.2.2. If $X$, $Y$, $W$, $\hat{V}_1$, $\hat{V}_2$, $\hat{A}_c$, $\hat{B}_c$, $\hat{C}_c$, and $\hat{D}_c$ satisfy $\text{tr}\{W\} < \gamma$ and (5.4b), then the controller

$$K^o \sim \begin{bmatrix} N^{-1}[\hat{A}_c - YAX - \hat{B}_cC_3X - YB_3(\hat{C}_c - \hat{D}_cC_3X)]M^* & N^{-1}(\hat{B}_c - YB_3\hat{D}_c) \\ (C_c - D_2C_3X)M^* & D_c \end{bmatrix}$$

(5.14) achieves the performance $J_*(\mathcal{F}_l(G, K)) < \gamma$ where $M$ and $N$ are any invertible matrices chosen so that $MN^* = I - XY$.

Proof. Note that the closed-loop system has the realization

$$\mathcal{F}_l(G, K) \sim \begin{bmatrix} A & 0 & B_1 & B_2 \\ 0 & 0 & 0 & 0 \\ C_1 & 0 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} 0 & B_3 \\ I & 0 \\ 0 & D_{13} \\ 0 & D_{23} \end{bmatrix} \begin{bmatrix} A^o_c & B^o_c \\ C^o_c & D^o_c \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \\ D_{31} & D_{32} \end{bmatrix} := \Sigma_{el}.$$  (5.15)

Define

$$P := \begin{bmatrix} Y & N \\ N^* & -N^*XM^{-*} \end{bmatrix}, \quad V := [\hat{V}_2 \ (\hat{V}_1 - \hat{V}_2X)M^{-*}], \quad \Pi := \begin{bmatrix} X & I \\ M^* & 0 \end{bmatrix}$$  (5.16)
and note that
\[
\Pi = \begin{bmatrix} I & Y \\ 0 & N^* \end{bmatrix} \tag{5.17}
\]
and \(\Pi\) is nonsingular. This implies that the matrix
\[
L := \text{diag}(\Pi, I_{n_d}, I_{n_w}, \Pi, I_{n_q}, I_{n_p})
\]
is nonsingular. Thus, \(\bar{M}_{\Sigma, cl}(\tau, P, W, V) \succ 0\) if and only if \(L^* M_{\Sigma, cl}(\tau, P, W, V)L \succ 0\). With some algebra involving (5.17), it is straightforward to show that the condition \(L^* M_{\Sigma, cl}(\tau, P, W, V)L \succ 0\) is the condition (5.4b), which implies that \(M_{\Sigma, cl}(\tau, P, W, V) \succ 0\). Using the characterization of \(J_\tau\) given by (3.86), we see that \(J_\tau(F_l(G, K)) < \gamma\). ■

In order to construct a controller from feasible \(X, Y, W, \hat{V}_1, \hat{V}_2, \hat{A}_c, \hat{B}_c, \hat{C}_c, \) and \(\hat{D}_c\), it is necessary to form the factorization \(MN^* = I - XY\). Although we use a QR decomposition to do this in our implementation, a pivoted LU decomposition would be equally suitable.

The optimization (5.4) is a nonconvex optimization because the matrix inequality is nonlinear in the optimization parameters; the products \(\tau X, \tau \hat{C}_c\) and \(\tau \hat{D}_c\) appear in the matrix inequality. Thus, (5.4b) is a bilinear matrix inequality (BMI) and the optimization (5.4) is a BMI optimization problem. Since it is generally difficult to find a global optimum of a BMI optimization problem [31], we do not follow this approach. Instead, we note that if the value of \(\tau\) is fixed or the values of \(X, \hat{C}_c,\) and \(\hat{D}_c\) are fixed, then (5.4b) becomes an LMI. In either of these cases, if the strict inequality in (5.4) is relaxed to a non-strict inequality, the optimization becomes a SDP. Thus, for a given initial guess for \(\tau\), a reasonable heuristic for solving (5.4) is to alternate between solving (5.4) for fixed \(\tau\) and solving (5.4) for fixed \(X, \hat{C}_c,\) and \(\hat{D}_c\).

There are two challenges in using this approach. The first difficulty we encounter is the difficulty of selecting the initial value of \(\tau\); since BMI optimization problems are non-convex, the selection of a “good” initial iterate is especially critical. The second difficulty that we might encounter is difficulty when reconstructing the controller using (5.14). In particular, since the controller reconstruction depends on both \(M^{-1}\) and \(N^{-1}\), we see that the controller reconstruction will be ill-conditioned if \(I - XY\) is ill-conditioned with respect to inversion. We will show in the next two sections that it is possible to deal with both of these problem using semi-definite programming.

### 5.2.2 Initial Controller Design

We now examine the problem of finding an initial value of \(\tau\). To deal with this problem, we follow the approach used in [20] in which the solution of two SDPs yield initial values of all optimization parameters. Since we are only interested in the initial value of \(\tau\), we will not explicitly construct initial values for the remaining optimization parameters in this dissertation.

The first convex optimization is a state feedback control design. In particular, we choose the controller to have the form \(u = K_x\dot{x}\), where \(K_x \in \mathbb{R}^{n_u \times n_x}\). Using the same techniques as
in §4.2.1, it can be shown that choosing the state feedback controller that optimizes the $\mathcal{H}_2$ guaranteed cost of the closed-loop system corresponds to solving the convex optimization problem

$$\inf_{\varepsilon, \hat{P}, W, \hat{V}, \hat{K}_x} \text{tr}\{W\} \quad \text{s.t.} \quad \begin{bmatrix} \hat{P} & \bullet & \bullet & \bullet & \bullet \\ 0 & \varepsilon I & \bullet & \bullet & \bullet \\ \hat{V} & 0 & W & \bullet & \bullet \\ A\hat{P} + B_3\hat{K}_x & \varepsilon B_1 & B_2 & \hat{P} & \bullet \\ C_1\hat{P} + D_{13}\hat{K}_x & \varepsilon D_{11} & D_{12} & 0 & \varepsilon I & \bullet \end{bmatrix} \succ 0.$$ \hspace{1cm} (5.18)

For any feasible values of $\varepsilon$, $\hat{P}$, $W$, $\hat{V}$, and $\hat{K}_x$, the controller $K_x = \hat{K}_x\hat{P}^{-1}$ achieves the closed-loop squared $\mathcal{H}_2$ guaranteed cost of $\text{tr}\{W\}$. Thus, to find a reasonable value of $K_x$, we relax (5.18) to an SDP, solve that SDP, and choose the state feedback gain $K_x$ to be $K_{sf} := \hat{K}_x\hat{P}^{-1}$.

In the second convex optimization, we make the restriction $n_c = n_x$ and only consider controllers of the form

$$K \sim \begin{bmatrix} A_c \\ K_{sf} \\ B_c \end{bmatrix}.$$ \hspace{1cm} (5.19)

If the state variable of $K$ is interpreted as an estimate of $x$, this restriction can be interpreted as imposing a “separation structure” on the controller, i.e. we impose that the controller can be decomposed into a state estimator and a state feedback gain. We then form the following realization of $\mathcal{F}_t(G, K)$ whose state is given by $[x^* \ (x - x_c)^*]^*$:

$$\mathcal{F}_t(G, K) \sim \begin{bmatrix} A_{cl} \\ A_{cl} - A_c - B_cC_3 \\ A_c - B_3K_{sf} \\ -B_3K_{sf} \\ B_1 \\ B_1 - B_cD_{31} \\ B_2 \\ B_2 - B_cD_{32} \end{bmatrix}.$$

$$\begin{bmatrix} B_{11} \\ B_{12} \\ B_{21} \\ B_{22} \end{bmatrix}.$$ \hspace{1cm} (5.20)

Using the characterization of $\mathcal{H}_2$ guaranteed cost given by (3.83) along with the restriction $P = \text{diag}(X, Y)$ and the change of variables

$$[\tilde{A}_c \quad \tilde{B}_c] := Y \begin{bmatrix} A_c & B_c \end{bmatrix}.$$ \hspace{1cm} (5.21)
we see that we can optimize the closed-loop performance using the optimization problem

\[
\inf_{\tau, X, Y, W, V_1, V_2, \tilde{A}_c, \tilde{B}_c} \quad \text{tr}\{W\} \quad \text{s.t.} \quad \begin{bmatrix}
X & \cdots & \cdots & \cdots & \cdots \\
0 & Y & \cdots & \cdots & \cdots \\
0 & 0 & \tau I & \cdots & \cdots \\
V_1 & V_2 & 0 & W & \cdots & \cdots \\
M_{11} & M_{12} & M_{13} & M_{14} & X & \cdots & \cdots \\
M_{21} & M_{22} & M_{23} & M_{24} & 0 & Y & \cdots & \cdots \\
M_{31} & M_{32} & M_{33} & M_{34} & 0 & 0 & \tau I & \cdots & \cdots \\
M_{41} & M_{42} & M_{43} & M_{44} & 0 & 0 & 0 & I \\
\end{bmatrix} \succ 0
\] (5.22)

where

\[
\begin{bmatrix}
\tilde{M}_{11} & \tilde{M}_{12} \\
\tilde{M}_{21} & \tilde{M}_{22} \\
\tilde{M}_{31} & \tilde{M}_{32} \\
\tilde{M}_{41} & \tilde{M}_{42} \\
\end{bmatrix} := \begin{bmatrix}
X(A + B_3 K_{sf}) & -XB_3 K_{sf} \\
Y(A + B_3 K_{sf}) - \tilde{A}_c - \tilde{B}_c C_3 & \tilde{A}_c - YB_3 K_{sf} \\
\tau(C_1 + D_{13} K_{sf}) & -\tau D_{13} K_{sf} \\
C_2 + D_{23} K_{sf} & -D_{23} K_{sf} \\
\end{bmatrix}
\] (5.23)

\[
\begin{bmatrix}
\tilde{M}_{13} & \tilde{M}_{14} \\
\tilde{M}_{23} & \tilde{M}_{24} \\
\tilde{M}_{33} & \tilde{M}_{34} \\
\tilde{M}_{43} & \tilde{M}_{44} \\
\end{bmatrix} := \begin{bmatrix}
XB_1 & XB_2 \\
YB_1 - \tilde{B}_c D_{31} & YB_2 - \tilde{B}_c D_{32} \\
\tau D_{11} & \tau D_{12} \\
D_{21} & D_{22} \\
\end{bmatrix}
\] (5.24)

Note in particular that the value of $K_{sf}$ is not optimized in (5.22). Although it is not used by our algorithm developed here, it is worth mentioning that if $\tau$, $X$, $Y$, $W$, $V_1$, $V_2$, $\tilde{A}_c$, and $\tilde{B}_c$ are feasible, then $Y$ is invertible, which implies that the controller

\[
K \sim \begin{bmatrix}
Y^{-1} \tilde{A}_c & Y^{-1} \tilde{B}_c \\
K_{sf} & 0 \\
\end{bmatrix}
\] (5.25)

achieves the squared $H_2$ guaranteed cost $\text{tr}\{W\}$ or better.

There are two sources of conservatism in (5.22). First, the form of the controller given in (5.19) is rather restrictive. Second, we have made the restriction $P = \text{diag}(X, Y)$. In return, however, we have obtained a convex formulation for optimizing the closed-loop performance in the output feedback control problem. Thus, we relax (5.22) to an SDP and then solve that SDP. The value of $\tau$ which results from solving the SDP is a suitable initial value of $\tau$ for the BMI optimization problem (5.4).
5.2.3 Conditioning of the Controller Reconstruction Step

As mentioned in §5.2.1, we would like to make the matrix \( I - XY \) well-conditioned with respect to inversion. Define
\[
F := \begin{bmatrix} X & I \\ I & Y \end{bmatrix}.
\]

Since \( F > 0 \) for any feasible iterate of (5.4), we see by Schur complements that \( Y > 0 \) and \( X - Y^{-1} > 0 \) for any feasible iterate of (5.4). In particular, \( Y \) and \( X - Y^{-1} \) are invertible.

Exploiting the invertibility of these matrices yields
\[
F^{-1} = \begin{bmatrix} (X - Y^{-1})^{-1} & (I - YX)^{-1} \\ (I - XY)^{-1} & Y^{-1} + Y^{-1}(X - Y^{-1})^{-1}Y^{-1} \end{bmatrix}.
\]

Since \( (I - XY)^{-1} \) explicitly appears in the expression for \( F^{-1} \), we see that if \( F \) is easy to invert then \( I - XY \) must also be easy to invert. Since \( F \) is a positive definite matrix for any feasible iterate of (5.4), we would therefore like to make the ratio of its largest eigenvalue to its smallest eigenvalue (i.e. its condition number with respect to inversion) as small as possible.

To minimize the condition number of \( F \), we use the approach given in [3]. First note that the condition number of \( F > 0 \) is less than \( \kappa \) if and only if there exists \( t > 0 \) such that \((1/t)I \prec F \prec (\kappa/t)I\), which is in turn equivalent to the existence of \( t > 0 \) such that \( I \prec tF \prec \kappa I \). In this expression, the quantities \( tX \) and \( tY \) appear, which means that the matrix inequalities \( I \prec tF \prec \kappa I \) are nonlinear in \( t, \kappa, X, \) and \( Y \). However, the matrix inequalities \( I \prec tF \prec \kappa I \) are LMIs in the quantities \( t, \kappa, tX, \) and \( tY \). Based on this observation, we will scale all of the variables and constraints by \( t \). Multiplying (5.4b) by \( t \) yields the inequality
\[
\begin{bmatrix}
tX & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
tI & tY & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & t\tau I & 0 & \cdots & \cdots & \cdots \\
tM_{11} & tM_{12} & tM_{13} & tM_{14} & tX & 0 & \cdots & \cdots & \cdots \\
tM_{21} & tM_{22} & tM_{23} & tM_{24} & tI & tY & 0 & \cdots & \cdots & \cdots \\
t\tau M_{31} & t\tau M_{32} & t\tau M_{33} & t\tau M_{34} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
tM_{41} & tM_{42} & tM_{43} & tM_{44} & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots
\end{bmatrix} \succ 0
\]

which in linear in the quantities \( t, tX, tY, tW, t\hat{V}_1, t\hat{V}_2, t\hat{A}_c, t\hat{B}_c, t\hat{C}_c, \) and \( t\hat{D}_c \). We also note that placing an upper bound on the closed-loop squared \( \mathcal{H}_2 \) guaranteed cost can be achieved using the constraint \( \text{tr}\{tW\} < t\gamma \) for some fixed \( \gamma \). Thus, a reasonable way to improve the
conditioning of the controller reconstruction is to solve

$$\inf_{t, \kappa, tX, tY, tW, t\hat{V}_1, t\hat{V}_2, t\hat{A}_c, t\hat{B}_c, t\hat{C}_c, t\hat{D}_c} \kappa \quad \text{s.t.}$$

$$\begin{bmatrix} tW \\ tI \end{bmatrix} < t\gamma, \quad I < \begin{bmatrix} tX & tI \\ tI & tY \end{bmatrix} < \kappa I$$  \hspace{1cm} (5.29)

where $\gamma$ is some acceptable level of squared $H_2$ guaranteed cost performance for the closed-loop system. This optimization minimizes the condition number of $S$ subject to the closed-loop system meeting a constraint on its $H_2$ guaranteed cost. Note that we have fixed the value of $\tau$ and the acceptable level of $H_2$ guaranteed cost performance in this optimization. Also note that we do not explicitly include the constraint $t > 0$ in (5.29) because (5.28) implies that $t > 0$.

When the strict inequalities in (5.29) are relaxed to non-strict inequalities, it becomes an SDP in the variables $t, \kappa, tX, tY, tW, t\hat{V}_1, t\hat{V}_2, t\hat{A}_c, t\hat{B}_c, t\hat{C}_c, t\hat{D}_c$. Thus, improving the conditioning of the controller reconstruction process for a fixed value of $\tau$ and a fixed $H_2$ guaranteed cost can be solved using this SDP.

### 5.2.4 Solution Methodology

This section gives a heuristic for solving (5.3) using the results presented so far in this chapter.

**Algorithm 5.2.3.** The following algorithm is a heuristic for solving (5.3).

1. **Find Initial Value of $\tau$**
   - (a) **State Feedback Controller Design:** Solve (5.18) using an SDP solver. From the resulting set of optimization parameter values, reconstruct the state feedback gain $K_{sf} := \hat{K}_x \hat{P}^{-1}$.
   - (b) **“Separation Principle” Controller Design:** Solve (5.22) using an SDP solver.

2. **Controller Design**
   - (a) **Controller Design (Fixed $\tau$):** Fix $\tau$ to be the value obtained in the previous optimization. Solve (5.4) using an SDP solver.
   - (b) **Controller Design (Fixed $X, \hat{C}_c, \hat{D}_c$):** Fix $X, \hat{C}_c,$ and $\hat{D}_c$ to be the values obtained in the previous optimization. Solve (5.4) using an SDP solver. If the stopping criteria have not been met (see below), return to step 2a.

3. **Conditioning:** Choose a value of $\beta > 0$. Fix $\tau$ to be the value which yielded the smallest cost $\gamma_0$ in the preceding optimizations\(^1\) and fix $\gamma$ to be $(1 + \beta)\gamma_0$. Solve (5.29) using an SDP solver.

---

\(^1\)Although the value of the optimization problem should decrease every time step 2a or 2b is executed, numerical inaccuracies might cause this not to be the case.

In our implementation of the algorithm, although we choose the default value of $\beta$ to be 0.05, we allow this to be specified by the user. This allows the user to trade off closed-loop performance versus conditioning in the controller reconstruction step. We use two stopping criteria in step 2. If 6 total optimizations have been performed in step 2 or if the cost has decreased less than 1% in the last two optimizations, we exit step 2 and move on to step 3.

5.3 Riccati Equation and Semi-Definite Programming Approach

In this section, we exploit the solution of DAREs to reduce the computation required to design an output feedback controller. We assume throughout this section that the following regularity conditions hold:

(A1) $D_{13}^*D_{13} + D_{23}^*D_{23}$ is invertible

(A2) $(A, B_3)$ is stabilizable

(A3) $\dim \left( \text{Ker} \begin{bmatrix} A - \lambda I & B_3 \\ C_1 & D_{13} \\ C_2 & D_{23} \end{bmatrix} \right) = 0, \ \forall \lambda \in \mathbb{C} \text{ satisfying } |\lambda| \geq 1.$

It should be noted that these are the same regularity conditions considered in §4.3 for the optimal full information control problem.

For now, we fix $\tau > 0$ and consider the problem of optimizing $J_\tau(F_l(G, K))$. The approach we follow is similar to the approach taken in solving the discrete-time $\mathcal{H}_\infty$ control problem [19], which is in turn based on the approach taken in solving the continuous-time $\mathcal{H}_\infty$ control problem [10]. However, since $\mathcal{H}_2$ guaranteed cost control does not have a frequency domain or operator domain interpretation, we must directly manipulate Riccati equations to establish the relevant results. This will result in the proofs of our results being less elegant than the ones in [10, 19]

5.3.1 Reduction to Output Estimation Problem

In this subsection, we show that the solution of an optimal full information control problem reduces the optimal output feedback control problem (for fixed $\tau > 0$) to a problem that is analogous to the output estimation problem in the $\mathcal{H}_\infty$ literature. We begin by defining $G_{f_1}$ as in (4.1) and $J_{f_1, \epsilon}$ as in (4.42). With these definitions in mind, we have the following proposition.
Proposition 5.3.1. Let $\epsilon^{-1} = \tau > 0$. If $J_{f_i,\epsilon} = \infty$, then $J_r(\mathcal{F}_l(G, K)) = \infty$, $\forall K \in \mathcal{K}$.

Proof. We prove the contrapositive of the above statement. Let $J_r(\mathcal{F}_l(G, K)) \neq \infty$ and define $\hat{K} := K[C_3 \ D_{31} \ D_{32}]$. It is straightforward to show that $\mathcal{F}_l(G, K) = \mathcal{F}_l(G, \hat{K})$. This implies that $J_r(\mathcal{F}_l(G_{f_i}, K)) \neq \infty$, which implies that $J_{f_i,\epsilon} \neq \infty$. ■

With the previous proposition in place, we will assume for the remainder of this section that $J_{f_i,\epsilon} \neq \infty$. We will see later in this subsection that the optimal full information controller for $G_{f_i}$ plays an important role in optimizing the output feedback controller. Before continuing, we would thus like to find the optimal full information controller for $G_{f_i}$ using the methods of §4.3. To this end, we first define $B_{[1,3]} := [B_1 B_3]$ and let $Q, R, S, \bar{Q}, \bar{S}, \phi$, and $\psi$ be as defined in (4.45). Since we are assuming that $J_{f_i,\epsilon} \neq \infty$, we also let $K_x^o, K_d^o,$ and $K_w^o$ be as defined in (4.47), where $P_0$ is the stabilizing solution of the DARE $\mathcal{R}_\phi(P) = P$ such that $A + B_3 K_x^o$ is Schur and the factorization

$$B_{[1,3]}^* P_0 B_{[1,3]} + R = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}^* \begin{bmatrix} I_{P,d} & 0 \\ 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$$

exists where $T_{11}$ and $T_{22}$ are invertible. (This value of $P_0$ is guaranteed to exist by Theorem 4.3.9.) We also let $A^{cl}, B_1^{cl}, B_2^{cl}, C_1^{cl}, C_2^{cl}, D_1^{cl}, D_1^{cl}, D_2^{cl}, D_2^{cl},$ and $D_2^{cl}$ be as defined in (4.48).

With this in place, we define the notation

$$K_{dx} := T_{11}^{-1} T_{11}^{-1} [(B_1^{cl})^* P_0 A + (D_1^{cl})^* C_1 + \epsilon (D_1^{cl})^* C_2]$$

$$K_{dw} := T_{11}^{-1} T_{11}^{-1} [(B_1^{cl})^* P_0 B_2 + (D_1^{cl})^* D_1 + \epsilon (D_1^{cl})^* D_2]$$

$$K_x := K_x^o + K_d^o K_{dx}$$

$$K_w := K_w^o + K_d^o K_{dw}.$$  \hspace{1cm} (5.31)

As in §4.3.3, we have that

$$\mathcal{K}_\phi(P_0) = \begin{bmatrix} K_{dx} \\ K_x \end{bmatrix}, \hspace{1cm} \mathcal{K}_\psi(P_0) = \begin{bmatrix} K_{dw} \\ K_w \end{bmatrix}.$$  \hspace{1cm} (5.32)

We now consider Fig. 5.2b where the state-space realizations of $G_1$ and $G_2$ are respectively given by

$$G_1 \sim \begin{bmatrix} A + B_3 K_x^o & B_1 + B_3 K_d^o & B_2 + B_3 K_w^o & B_3 T_{22}^{-1} \\ C_1 + D_{13} K_x^o & D_{11} + D_{13} K_d^o & D_{12} + D_{13} K_w^o & D_{13} T_{22}^{-1} \\ C_2 + D_{23} K_x^o & D_{21} + D_{23} K_d^o & D_{22} + D_{23} K_w^o & D_{23} T_{22}^{-1} \\ -T_{11} K_{dx} & T_{11} & -T_{11} K_{dw} & 0 \end{bmatrix}$$

$$G_2 \sim \begin{bmatrix} A + B_1 K_{dx} & B_1 T_{11}^{-1} & B_2 + B_1 K_{dw} & B_3 \\ -T_{22} K_x & -T_{22} K_d T_{11}^{-1} & -T_{22} K_w & T_{22} \\ 0 & 0 & 0 & 0 \\ C_3 + D_{31} K_{dx} & D_{31} T_{11}^{-1} & D_{32} + D_{31} K_{dw} & 0 \end{bmatrix}.$$  \hspace{1cm} (5.33)
Denote the state variables of $G_1$ and $G_2$ respectively as $x^1$ and $x^2$. Combining $G_1$ and $G_2$ into a single block in Fig. 5.2b yields, after some algebra involving (5.31)–(5.32), the state-space realization (with state variable $[(x^2 - x^1)^* (x^2)^*]^*$) given by

$$
\begin{bmatrix}
A_{\phi}(P_0) & 0 & 0 & 0 & 0 \\
B_1K_{dx} & A & B_1 & B_2 & B_3 \\
C_1 + D_{13}K_x & C_1 & D_{11} & D_{12} & D_{13} \\
C_2 + D_{23}\bar{K}_x & C_2 & D_{21} & D_{22} & D_{23} \\
-D_{31}K_{dx} & C_3 & D_{31} & D_{32} & 0
\end{bmatrix}.
$$

(5.35)

Since $A_{\phi}(P_0)$ is Schur, we see that the states $(x^2 - x^1)^*$ are uncontrollable, but exponentially decay to zero. Therefore, this realization is equivalent to the realization of $G$, which implies that the block diagrams in Figs. 5.2a–5.2b are equivalent. Note in particular that $x^2 - x^1$ converges to zero and, when $G$ and the interconnection between $G_1$ and $G_2$ are driven by the same input signals, $x^2$ converges to $x$. This implies that $x^1$ and $x^2$ both converge to $x$ when $G$ and the interconnection between $G_1$ and $G_2$ are driven by the same input signals.

With this in place, we can place an interpretation on the signals in Fig. 5.2b. In particular, we see from the realization of $G_1$ that

$$
r = T_{11}[d - (K_{dx}x + K_{dw}w)].
$$

(5.36)

From the realization of $G_2$, we see that $n = 0$ and

$$
v = T_{22}[u - (\bar{K}_x x + K_0^oT_{11}^{-1}r + \bar{K}_w w)] = T_{22}[u - (K_0^o x + K_0^o d + K_0^o w)].
$$

(5.37)

Since $n = 0$, it does not play a role in the dynamics of the system. However, we will see later in this subsection that it serves a structural role. We also see that $T_{22}^{-1}v$ is the difference between the control being applied to $G$ and the optimal full information control. From this standpoint, we see that we would like to make $v$ “small” in some sense. We will formalize this idea later in this section.
We now define $G_3 := \mathcal{F}_l(G_2, K)$, and note that the block diagrams in Figs. 5.2b–5.2c are equivalent, which implies that all three block diagrams in Fig. 5.2 are equivalent. With this in mind, we note that analyzing the $\mathcal{H}_2$ guaranteed cost performance of the closed-loop system $\mathcal{F}_l(G, K)$ for a given value of $K$ is equivalent to analyzing the $\mathcal{H}_2$ guaranteed cost performance of the system $\mathcal{F}_l(G_1, G_3)$. The following theorem analyzes the $\mathcal{H}_2$ guaranteed cost performance of $\mathcal{F}_l(G_1, G_3)$.

**Theorem 5.3.2.** For fixed $\epsilon^{-1} = \tau > 0$,

$$ J_r(\mathcal{F}_l(G_1, G_3)) = J_{f_i, \epsilon} + \epsilon^{-1} J_1(G_3) . $$  

(5.38)

Due to the length and technical nature of the proof, we defer the proof of this theorem to §5.3.4. It should be noted that (5.38) could only be written in such a compact form due to the placeholder signal $n$; without that placeholder, $G_3$ would only have one output and $J_r(G_3)$ would not be well-defined.

Using Theorem 5.3.2, we thus see that the output feedback control problem (for fixed $\tau > 0$) can be written as

$$ \inf_K J_r(\mathcal{F}_l(G_1, G_3)) = \inf_K J_r(\mathcal{F}_l(G_1, \mathcal{F}_l(G_2, K))) = \inf_K \{ J_{f_i, \epsilon} + \epsilon^{-1} J_1(\mathcal{F}_l(G_2, K)) \} $$

$$ = J_{f_i, \epsilon} + \epsilon^{-1} \inf_K J_1(\mathcal{F}_l(G_2, K)) . $$

(5.39)

(5.40)

The remaining optimal control problem

$$ \inf_K J_1(\mathcal{F}_l(G_2, K)) $$

(5.41)

is analogous to the output estimation problem in the $\mathcal{H}_\infty$ literature.

Note that (5.41) is a control design problem in which $K$ is trying to keep $v$ and $n$ “small” in the closed-loop system. However, since $n = 0$ regardless of $K$, we see that this control design problem is only trying to keep $v$ “small”. As mentioned earlier in this subsection, this corresponds to trying to make the control action applied to $G$ “close” to the optimal full information control signal.

### 5.3.2 Reduction to Full Control Problem

For $\mathcal{H}_\infty$ control, the output estimation problem is solved by applying duality—transposing the closed-loop system transfer function matrix—to transform the problem into a disturbance feedforward problem, which is then reduced to a full information control problem. However, in our approach, this approach does not work because there is no known duality result. Thus, we now diverge slightly from the general approach taken in [19].
At this point, we restrict the class of controllers to ones which can be expressed as $K = \mathcal{F}_l(\hat{K}, \tilde{K})$ where

$$
\hat{K} \sim \begin{bmatrix}
  \mathcal{A}_\phi(P_0) & 0 & -I & B_3 \\
  K_x & 0 & 0 & I \\
  -(C_3 + D_{31}K_{dx}) & I & 0 & 0 \\
\end{bmatrix}.
$$

(5.42)

We denote the state variable of $\hat{K}$ as $\hat{x}$. For this control structure, the block diagrams in Figs. 5.3a and 5.3b are equivalent. Combining $G_2$ and $\hat{K}$ in Fig. 5.3b into a single block, $G_4$, yields Fig. 5.3c. Thus, the three block diagrams in Fig. 5.3 are equivalent for this control structure. We denote the state of $G_4$ as $x^4$.

The obvious question is whether or not this control structure worsens the level of achievable performance of the closed-loop system. The state-space realization of $G_4$ (with state variable $[(x^2 - \hat{x})^* (-\hat{x})^*]^*$) is given by

$$
G_4 \sim \begin{bmatrix}
  A + B_1K_{dx} & 0 & B_1T_{11}^{-1} & B_2 + B_1K_{dw} & I & 0 \\
  0 & \mathcal{A}_\phi(P_0) & 0 & 0 & I & -B_3 \\
  -T_{22}K_x & 0 & -T_{22}K_{dx} & -T_{22}K_w & 0 & T_{22} \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  C_3 + D_{31}K_{dx} & 0 & D_{31}T_{11}^{-1} & D_{32} + D_{31}K_{dw} & 0 & 0 \\
\end{bmatrix}.
$$

(5.43)

Since $\mathcal{A}_\phi(P_0)$ is Schur, we remove the unobservable state $-\hat{x}$ to yield

$$
G_4 \sim \begin{bmatrix}
  A + B_1K_{dx} & B_1T_{11}^{-1} & B_2 + B_1K_{dw} & I & 0 \\
  -T_{22}K_x & -T_{22}K_{dx} & -T_{22}K_w & 0 & T_{22} \\
  0 & 0 & 0 & 0 & 0 \\
  C_3 + D_{31}K_{dx} & D_{31}T_{11}^{-1} & D_{32} + D_{31}K_{dw} & 0 & 0 \\
\end{bmatrix}.
$$

(5.44)

Thus, we see that if we make the restriction $u_1 = B_3u_2$ in Fig. 5.3c, we exactly recover the control problem shown in Fig. 5.3a. This means that choosing this special control structure does not affect the achievable performance of the closed-loop system. The remaining optimal control problem

$$
\inf_{\tilde{K}} J_1(\mathcal{F}_l(G_4, \tilde{K}))
$$

(5.45)
is analogous to the full control problem in the $\mathcal{H}_\infty$ literature when $G_4$ is realized as in (5.44).

In making the system $\mathcal{F}_l(G_4, \tilde{K})$ stable, we see that its state will remain “small”. This implies that the state of $G_4$ remains “small”, which implies that $\hat{x}$ remains “close” to $x^2$. Thus, the state of $\tilde{K}$ can be regarded as an estimate of the state of $G_2$.

### 5.3.3 Solution Methodology

In the previous subsection, we reduced the optimal output feedback control problem (for fixed $\tau > 0$) to the optimal full control problem (5.45). It is not currently known whether or not the optimal full control problem can be solved using Riccati equations. Therefore, to solve this problem, we will resort to the SDP approach. In particular, we will show that the optimization problem (5.45) is equivalent to the optimization problem

$$\inf_{P,W,V,\hat{L}_x,L_v} \text{tr}\{W\} \quad \text{s.t.}$$

$$\begin{bmatrix}
P & \bullet & \bullet & \bullet \\
0 & T_{11}^*T_{11} & \bullet & \bullet \\
V & 0 & W & \bullet & \bullet \\
P\hat{A} + \hat{L}_x\hat{C} & PB_1 + \hat{L}_xD_{31} & P\hat{B} + \hat{L}_x\hat{D} & P & \bullet \\
T_{22}(L_v\hat{C} - \hat{K}_x) & T_{22}(L_vD_{31} - \hat{K}_d) & T_{22}(L_v\hat{D} - \hat{K}_w) & 0 & I \\
\end{bmatrix} \succ 0 \quad (5.46b)$$

where

$$\begin{bmatrix}
\hat{A} & \hat{B} \\
\hat{C} & \hat{D}
\end{bmatrix} := \begin{bmatrix}
A + B_1K_{dx} & B_2 + B_1K_{dw} \\
C_3 + D_{31}K_{dx} & D_{32} + D_{31}K_{dw}
\end{bmatrix} \quad (5.47)$$

To do this, we will first show that if there exists an LTI controller of any order that achieves $J_1(\mathcal{F}_l(G_4, \tilde{K})) < \gamma$, then the value of (5.46) must be less than $\gamma$. We then show that if the (5.46) is less than $\gamma$, then we can always reconstruct a static controller that achieves the closed-loop performance $J_1(\mathcal{F}_l(G_4, \tilde{K})) < \gamma$.

**Theorem 5.3.3.** Let $\tau > 0$. If there exists LTI $\tilde{K}$ of finite order such that $J_1(\mathcal{F}_l(G_4, \tilde{K})) < \gamma$, then $\exists P, W, V, \hat{L}_x, \text{ and } L_v$ such that $\text{tr}\{W\} < \gamma$ and (5.46b) is satisfied.

**Proof.** We first let $\tilde{K}$ have the realization

$$\tilde{K} \sim \begin{bmatrix}
A^{fc} & B^{fc} \\
C_1^{fc} & D_1^{fc} \\
C_2^{fc} & D_2^{fc}
\end{bmatrix} \quad (5.48)$$
so that

\[
\mathcal{F}_I(G_4, \bar{K}) \sim \begin{bmatrix}
\bar{A} & 0 & B_1 T_{11}^{-1} & \bar{B} \\
0 & 0 & 0 & 0 \\
-\bar{T}_{22} K_x & 0 & -\bar{T}_{22} K_d T_{11}^{-1} & -\bar{T}_{22} K_w \\
0 & 0 & 0 & 0 \\
\end{bmatrix} + \begin{bmatrix}
C_{1}^{f^c} & D_{1}^{f^c} \\
A_{2}^{f^c} & B_{2}^{f^c} \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & I & 0 & 0 \\
\tilde{C} & 0 & D_{31} T_{11}^{-1} & \bar{D} \\
\end{bmatrix} =: \Sigma_{f^c}. \tag{5.49}
\]

Using the characterization of \( J_1 \) given by (3.87), choose \( P^{f^c}, W \) and \( V^{f^c} \) such that \( \text{tr}\{W\} < \gamma \) and \( \mathcal{M}^{\text{inv}}_{f^c}(1, P^{f^c}, W, V^{f^c}) > 0 \). We now partition \( P^{f^c} \) and \( V^{f^c} \) respectively as

\[
P^{f^c} = \begin{bmatrix}
P_{11}^{f^c} & \cdots \\
P_{21}^{f^c} & P_{22}^{f^c} \\
\end{bmatrix}, \quad V^{f^c} = \begin{bmatrix}
V_{1}^{f^c} & V_{2}^{f^c} \\
\end{bmatrix} \tag{5.50}
\]

where \( P_{11}^{f^c} \) and \( V_{1}^{f^c} \) have \( n_x \) columns and define

\[
P := (P_{11}^{f^c} - (P_{21}^{f^c})^*(P_{22}^{f^c})^{-1} P_{21}^{f^c})^{-1}, \quad V := [V_{1}^{f^c} - V_{2}^{f^c} (P_{22}^{f^c})^{-1} P_{21}^{f^c}] P \tag{5.51}
\]

\[
\tilde{L}_x := P[D_{1}^{f^c} - (P_{21}^{f^c})^*(P_{22}^{f^c})^{-1} B^{f^c}], \quad \tilde{L}_v := D_{2}^{f^c} \tag{5.52}
\]

\[
\tilde{K}^{f^c} := \begin{bmatrix}
P^{-1} \tilde{L}_x \\
L_v \\
\end{bmatrix} \tag{5.53}
\]

Note that applying the controller \( \tilde{K}^{f^c} \) yields the closed-loop realization

\[
\mathcal{F}_I(G_4, \tilde{K}^{f^c}) \sim \begin{bmatrix}
\bar{A} & 0 & B_1 T_{11}^{-1} & \bar{B} \\
0 & 0 & 0 & 0 \\
-\bar{T}_{22} K_x & 0 & -\bar{T}_{22} K_d T_{11}^{-1} & -\bar{T}_{22} K_w \\
0 & 0 & 0 & 0 \\
\end{bmatrix} + \begin{bmatrix}
P^{-1} \tilde{L}_x \\
\tilde{C} & 0 & D_{31} T_{11}^{-1} & \bar{D} \\
\end{bmatrix} =: \Sigma_{f^c}. \tag{5.54}
\]

Define

\[
L := \text{diag}\left( \begin{bmatrix}
I & \cdots \\
-(P_{22}^{f^c})^{-1} P_{21}^{f^c} & I \\
-(P_{22}^{f^c})^{-1} P_{21}^{f^c} & I \\
I & 0 \\
\end{bmatrix} P, \ T_{11}, \ I_n, \ \begin{bmatrix}
I & \cdots \\
-(P_{22}^{f^c})^{-1} P_{21}^{f^c} & I \\
-(P_{22}^{f^c})^{-1} P_{21}^{f^c} & I \\
I & 0 \\
\end{bmatrix} P, \ \begin{bmatrix}
I_n \\
0 \\
\end{bmatrix} \right) \tag{5.55}
\]

and note that \( \text{Ker}(L) = \{0\} \). Thus, \( L^* \mathcal{M}^{\text{inv}}_{f^c}(1, P^{f^c}, W, V^{f^c}) L > 0 \). However, we see after some algebra that this condition is exactly the condition (5.46b). \( \blacksquare \)

**Theorem 5.3.4.** If \( P, W, V, \tilde{L}_x, \) and \( L_v \) satisfy \( \text{tr}\{W\} < \gamma \) and (5.46b), then the controller

\[
\tilde{K}^o := \begin{bmatrix}
P^{-1} \tilde{L}_x \\
L_v \\
\end{bmatrix} \tag{5.56}
\]

achieves the performance \( J_1(\mathcal{F}_I(G_4, \tilde{K})) < \gamma \).
Proof. Note that applying the controller $\tilde{K}^o$ yields the closed-loop realization

$$\mathcal{F}_l(G_4, \tilde{K}^o) \sim \begin{bmatrix} \dot{A} + P^{-1}\dot{L}_x \dot{C} & (B_1 + P^{-1}\dot{L}_x D_{31}) T_{11}^{-1} \dot{B} + P^{-1}\dot{L}_x \dot{D} \\ \frac{T_{22}(L_v C - K_x)}{0} & \frac{T_{22}(L_v D_{31} - K_d)}{0} \frac{T_{22}(L_v D - K_w)}{0} \end{bmatrix} =: \Sigma_{fc} . \quad (5.57)$$

Define $L := \text{diag}(I, T_{11}^{-1}, I, I, I)$ and note that $L$ is invertible. Defining $Z$ to be the left-hand side of (5.46b), we see that $L^*ZL \succ 0$, which implies that $\text{diag}(L^*ZL, I) \succ 0$. However, this last condition is exactly the condition that $\bar{M}_{\Sigma_{fc}}(1, P, W, V) \succ 0$. By the characterization of $J_1$ given in (3.86), we conclude that $J_1(\mathcal{F}_l(G^o, \tilde{K})) < \gamma$. ■

Putting the result of Theorem 5.3.4 together with Theorem 5.3.2 and the controller structure $K = \mathcal{F}_l(\hat{K}, \tilde{K})$ yields the following result: for any $P, W, V, \hat{L}_x, L_v$ that satisfy (5.46b), an output feedback controller which achieves $J_r(\mathcal{F}_l(G, K)) \leq J_{f, \epsilon} + \epsilon^{-1}\text{tr}\{W\}$ is given by

$$K \sim \begin{bmatrix} A + B_1 K_{dx} + B_3 \hat{K}_x + (P^{-1}\dot{L}_x - B_3 L_v)(C_3 + D_{31} K_{dx}) & P^{-1}\dot{L}_x - B_3 L_v \\ L_v(C_3 + D_{31} K_{dx}) - K_x & L_v \end{bmatrix} . \quad (5.58)$$

If the strict inequality in (5.46) is relaxed to a non-strict inequality, the optimization becomes a SDP. Thus, a reasonable way to solve the output feedback control problem (for fixed $\tau$) is to relax (5.46) to a SDP, solve the SDP using an appropriate solver, then reconstruct the output feedback controller using (5.58).

We now present a result which makes it especially easy to find values of $\tau > 0$ for which there exists a $K$ satisfying $J_r(\mathcal{F}_l(G, K)) \neq \infty$.

**Proposition 5.3.5.** If $\exists \tau > 0, K$ such that $J_r(\mathcal{F}_l(G, K)) \neq \infty$, then the set of $\tau > 0$ for which $\exists K$ satisfying $J_r(\mathcal{F}_l(G, K)) \neq \infty$ is the interval $(\tau, \infty)$ for some $\tau > 0$. □

The proof of this proposition is analogous to the proof of Proposition 4.3.12. With these results in place, we now give a heuristic for solving (5.3).

**Algorithm 5.3.6.** The following algorithm is a heuristic for solving the optimal output feedback control problem (5.3).

1. **Find Initial Value of $\tau$**
   
   (a) **Full Information Controller Design:** Using Algorithm 4.3.14, design an optimal full information controller.

   (b) **Find Feasible Value of $\tau$:** Choose $\alpha > 0$. For the final values determined during the last full information controller design, solve (5.46) using an SDP solver. If the optimization was feasible, reconstruct the corresponding output feedback controller $K$ using (5.58). If the optimization was not feasible, set $\tau \leftarrow \alpha \tau$, design a full information controller for $\epsilon = \tau^{-1}$ using Algorithm 4.3.11, and redo this step.
(c) **Closed-Loop System Analysis (Fixed K):** Form the closed-loop system 
\( F_l(G, K) \) and analyze its \( \mathcal{H}_2 \) guaranteed cost performance using Algorithm 3.2.8.

2. **Controller Design**

(a) **Output Feedback Controller Design (Fixed \( \tau \)):** For the value of \( \tau > 0 \) found in the previous closed-loop system analysis step, solve (5.46) using an SDP solver and reconstruct the corresponding controller \( K \) using (5.58).

(b) **Closed-Loop System Analysis (Fixed \( K \)):** Form the closed-loop system 
\( F_l(G, K) \) and analyze its \( \mathcal{H}_2 \) guaranteed cost performance using Algorithm 3.2.8. Return to step 2a.

In our implementation, we use \( \alpha = 100 \). We use two stopping criteria in this algorithm. If the number of output feedback controller optimizations (i.e. the number of times steps 1b and 2a have been executed) exceeds 30 or if \( J_{i+1}^{[i]} / J_{i}^{[i]} - 1 < 10^{-4} \) where \( J_{i}^{[i]} \) is the cost reported the \( i \)th time step 2b executes, we terminate the algorithm. We also terminate the algorithm if the SDP solver claims infeasibility in step 2a.

5.3.4 **Proof of Theorem 5.3.2**

In this section, we give the proof of Theorem 5.3.2. We assume throughout that \( \epsilon^{-1} = \tau > 0 \) is fixed and \( J_{f_i,\epsilon} \neq \infty \). Also, we will use the quantity \( \tilde{P} := \tau P_0 \) throughout.

Suppose that \( G_3 \) has the realization
\[
G_3 \sim \begin{bmatrix}
\tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\
C & D_1 & D_2 \\
0 & 0 & 0
\end{bmatrix}.
\] (5.59)

Note that although we are exploiting the fact that the second output of \( G_3 \) is zero, we are not explicitly exploiting any other structure of \( G_3 \). With this in place, we are interested in evaluating \( J_\tau(F_l(G_1, G_3)) \). We first express
\[
F_l(G_1, G_3) \sim \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & \hat{B}_{11} & \hat{B}_{12} \\
\hat{A}_{21} & \hat{A}_{22} & \hat{B}_{21} & \hat{B}_{22} \\
\hat{C}_{11} & \hat{C}_{12} & \hat{D}_{11} & \hat{D}_{12} \\
\hat{C}_{21} & \hat{C}_{22} & \hat{D}_{21} & \hat{D}_{22}
\end{bmatrix}
\begin{bmatrix}
A^{cl} - B_3 \tilde{D}_1 K_{dx} & B_3 T_{22}^{-1} \tilde{C} & B_1^{cl} + B_3 \tilde{D}_1 & B_2^{cl} + B_3 (T_{22}^{-1} \tilde{D}_2 - \tilde{D}_1 K_{dw}) \\
-\tilde{B}_1 T_{11} K_{dx} & A & \tilde{B}_1 T_{11} & \tilde{B}_2 - \tilde{B}_1 T_{11} K_{dw} \\
\tilde{C}_{11} - D_{13} \tilde{D}_1 K_{dx} & \tilde{C}_{11} T_{22}^{-1} \tilde{C} & D_{11}^{cl} + D_{13} \tilde{D}_1 & D_{12}^{cl} + D_{13} (T_{22}^{-1} \tilde{D}_2 - \tilde{D}_1 K_{dw}) \\
\tilde{C}_{21} - D_{23} \tilde{D}_1 K_{dx} & \tilde{C}_{21} T_{22}^{-1} \tilde{C} & D_{21}^{cl} + D_{23} \tilde{D}_1 & D_{22}^{cl} + D_{23} (T_{22}^{-1} \tilde{D}_2 - \tilde{D}_1 K_{dw})
\end{bmatrix}
\] (5.60)
where \( \tilde{D}_1 := T_{22}^{-1}\hat{D}_1\hat{T}_{11} \). In order to determine \( J_\tau(\mathcal{F}_i(G_1, G_3)) \) using the methods of \$3.2.1\$, we define the parameters

\[
\begin{align*}
\left[\hat{Q}_{11} \hat{Q}_{12}ight] :&= \tau \left[\hat{C}_{11}^* \hat{C}_{12}^* \right] \left[\hat{C}_{11} \hat{C}_{12}\right] + \left[\hat{C}_{21}^* \hat{C}_{22}^* \right] \left[\hat{C}_{21} \hat{C}_{22}\right] \quad \text{(5.61)} \\
\left[\hat{S}_1 \hat{S}_2\right] :&= \tau \left[\hat{C}_{11}^* \hat{C}_{12}^* \right] \hat{D}_{11} + \left[\hat{C}_{21}^* \hat{C}_{22}^* \right] \hat{D}_{21} \quad \text{(5.62)} \\
\hat{R} :&= \tau(\hat{D}_{11}^*\hat{D}_{11} - I) + \hat{D}_{21}^*\hat{D}_{21} \quad \text{(5.63)} \\
\hat{\phi} :&= \left(\begin{array}{c}
\hat{A}_{11} \\
\hat{A}_{21}
\end{array}\right), \quad \begin{array}{c}
\hat{B}_{11} \\
\hat{B}_{21}
\end{array}\), \quad \begin{array}{c}
\hat{Q}_{11} \\
\hat{Q}_{12}
\end{array}, \quad \begin{array}{c}
\hat{D}_{11} \\
\hat{D}_{21}
\end{array}, \quad \hat{R}, \quad \begin{array}{c}
\hat{S}_{11} \\
\hat{S}_{21}
\end{array}
\right) \quad \text{(5.64)} \\
\hat{\psi} :&= \left(\begin{array}{c}
\hat{B}_{12} \\
\hat{B}_{22}
\end{array}\right), \quad \begin{array}{c}
\hat{B}_{11} \\
\hat{B}_{21}
\end{array}, \quad \tau\hat{D}_{12}^*\hat{D}_{12} + \hat{D}_{22}^*\hat{D}_{22}, \quad \hat{R}, \quad \tau\hat{D}_{12}^*\hat{D}_{11} + \hat{D}_{22}^*\hat{D}_{21} \bigg) \quad \text{(5.65)}
\end{align*}
\]

With this notation in place, we see by Theorem 3.2.3 that \( J_\tau(\mathcal{F}_i(G_1, G_3)) \neq \infty \) if and only if \( \left[\hat{A}_{11} \hat{A}_{12}\right] \) is Schur and the DARE \( \mathcal{R}_\phi(P) = P \) has a stabilizing solution \( \hat{P} \) such that

\[
\begin{bmatrix}
\hat{B}_{11} \\
\hat{B}_{21}
\end{bmatrix}^* \hat{P} \begin{bmatrix}
\hat{B}_{11} \\
\hat{B}_{21}
\end{bmatrix} + \hat{R} \prec 0 \quad \text{(5.66)}
\]

Moreover, when \( \hat{P} \) exists,

\[
J_\tau(\mathcal{F}_i(G_1, G_3)) = \text{tr}\{\mathcal{R}_\phi(\hat{P})\} \quad \text{(5.67)}
\]

To study the DARE \( \mathcal{R}_\phi(P) = P \), we will use the methods of \$2.5\$. Thus, we will be interested in the matrix pencil

\[
\begin{bmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & \hat{A}_{11}^* & \hat{A}_{21}^* & 0 \\
0 & 0 & \hat{A}_{12}^* & \hat{A}_{22}^* & 0 \\
0 & 0 & -\hat{B}_{11}^* & -\hat{B}_{21}^* & 0
\end{bmatrix} \lambda 
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & 0 & 0 & \hat{B}_{11} \\
\hat{A}_{21} & \hat{A}_{22} & 0 & 0 & \hat{B}_{21} \\
-\hat{Q}_{11} & -\hat{Q}_{12} & I & 0 & -\hat{S}_{11} \\
-\hat{Q}_{12} & -\hat{Q}_{22} & 0 & I & -\hat{S}_{21} \\
\hat{S}_{11}^* & \hat{S}_{21}^* & 0 & 0 & \hat{R}
\end{bmatrix} = 0 \quad \text{(5.68)}
\]

From the statement of Theorem 5.3.2, we know that we are also interested in the value of \( J_1(G_3) \). In order to determine \( J_\tau(G_3) \) using the methods of \$3.2.1\$, we define the parameters

\[
\begin{align*}
\hat{\phi} &:= (\hat{A}, \hat{B}_1, \hat{C}^*\hat{C}, \hat{D}_1^*\hat{D}_1 - I, \hat{C}^*\hat{D}_1) \quad \text{(5.69)} \\
\hat{\psi} &:= (\hat{B}_2, \hat{B}_1, \hat{D}_2^*\hat{D}_2, \hat{D}_1^*\hat{D}_1 - I, \hat{D}_2^*\hat{D}_1) \quad \text{(5.70)}
\end{align*}
\]
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With this notation in place, we see by Theorem 3.2.3 that $J_1(G_3) \neq \infty$ if and only if $\tilde{A}$ is Schur and the DARE $\mathcal{R}_{\tilde{y}}(P) = P$ has a stabilizing solution $P$ such that $\tilde{B}_1^* \tilde{P} \tilde{B}_1 + D_1^* D_1 - I \prec 0$. Moreover, when $\tilde{P}$ exists,

$$J_1(G_3) = \text{tr}\{\mathcal{R}_{\tilde{y}}(\tilde{P})\} \quad \text{(5.71)}$$

Before examining the DARE $\mathcal{R}_{\tilde{y}}(P) = P$, we first establish a few useful identities.

**Lemma 5.3.7.** The following identities hold:

1. $B_3^* \tilde{P} A_1^d + \tau D_{13}^* C_1^{cl} + D_{23}^* C_2^{cl} = 0 \quad \text{(5.72)}$
2. $B_3^* \tilde{P} B_1^d + \tau D_{13}^* D_{11} + D_{23}^* D_{21} = 0 \quad \text{(5.73)}$
3. $B_3^* \tilde{P} B_2^d + \tau D_{13}^* D_{12} + D_{23}^* D_{22} = 0 \quad \text{(5.74)}$
4. $B_3^* \tilde{P} B_3 + \tau D_{13}^* D_{13} + D_{23}^* D_{23} = \tau T_{22}^* T_{22} \quad \text{(5.75)}$
5. $(B_1^d)^* \tilde{P} B_1^d + \tau (D_{11}^{cl})^* D_{11}^{cl} - \tau I + (D_{21}^{cl})^* D_{21}^{cl} = -\tau T_{11}^* T_{11} \quad \text{(5.76)}$
6. $T_{11}^* T_{11}^{-*} [(B_1^{cl})^* \tilde{P} A_{11}^d + \tau (D_{11}^{cl})^* C_1^{cl} + (D_{21}^{cl})^* C_2^{cl}] = \tau K_{dx} \quad \text{(5.77)}$
7. $T_{11}^* T_{11}^{-*} [(B_1^{cl})^* \tilde{P} B_{12}^d + \tau (D_{11}^{cl})^* D_{12}^d + (D_{21}^{cl})^* D_{22}^{cl}] = \tau K_{dw} \quad \text{(5.78)}$
8. $(A_1^{cl})^* \tilde{P} A_1^{cl} + \tau (C_1^{cl})^* C_1^{cl} + (C_2^{cl})^* C_2^{cl} + \tau K_{dx}^{*} T_{11}^* T_{11} K_{dx} = \tilde{P} \quad \text{(5.79)}$
9. $(B_2^{cl})^* \tilde{P} B_2^{cl} + \tau (D_{12}^{cl})^* D_{12}^{cl} + (D_{22}^{cl})^* D_{22}^{cl} + \tau K_{dw}^{*} T_{11}^* T_{11} K_{dw} = \epsilon^{-1} \mathcal{R}_{\tilde{y}}(P_0) \quad \text{(5.80)}$

**Proof.** To prove (5.72), we note that

$$B_3^* \tilde{P} A_1^d + \tau D_{13}^* C_1^{cl} + D_{23}^* C_2^{cl} = \tau [(B_3^* P_0 A + D_{13}^* C_1 + \epsilon D_{23}^* C_2) + (B_3^* P_0 B_3 + D_{13}^* D_{13} + \epsilon D_{23}^* D_{23}) K_{x}] \quad \text{(5.81)}$$

Applying the definition of $K_{x}$ proves (5.72). Equations (5.73)–(5.74) are proved similarly. Equation (5.75) follows trivially from the factorization (5.30). Note that the left-hand side of (5.76) is given by

$$B_1^* P B_1^d + \tau D_{11}^* D_{11}^d - \tau I + D_{21}^* D_{21}^d + (K_o^d)^* (B_3^* P B_1^d + \tau D_{13}^* D_{11}^d + D_{23}^* D_{21}^d) \quad \text{(5.82)}$$

Using (5.73), we see that the left-hand side of (5.76) is given by

$$B_1^* P B_1^d + \tau D_{11}^* D_{11}^d - \tau I + D_{21}^* D_{21}^d = \tau [(B_1^* P_0 B_3 + D_{11}^* D_{11} - I + \epsilon D_{21}^* D_{21}) + (B_1^* P_0 B_3 + D_{11}^* D_{13} - \epsilon D_{21}^* D_{23}) K_{d}] \quad \text{(5.83)}$$

Note that, by the factorization (5.30), $K_o^d = -(T_{22}^* T_{22})^{-1}(T_{22}^* T_{11}) = -T_{22}^{-1} T_{21}$. Therefore, using (5.30), we see that the left-hand side of (5.76) is given by

$$\tau [(T_{21}^* T_{21} - T_{11}^* T_{11}) + (-T_{22}^* T_{22})] \quad \text{(5.84)}$$
which proves (5.76). Noting that the left-hand side of (5.77) is
\[
\tau T_{11}^{-1} T_{11}^* \left[ \left( (B_1^{cl})^* P_0 A + (D_{11}^{cl})^* C_1 + \epsilon (D_{21}^{cl})^* C_2 \right) 
+ \left( (B_1^{cl})^* P_0 B_3 + (D_{11}^{cl})^* D_{13} + \epsilon (D_{21}^{cl})^* D_{23} \right) K_{22}^* \right] \quad (5.85)
\]
we see by (5.73) that (5.77) holds. Equation (5.78) is proved similarly. To prove (5.79) we note that, by Lemma 4.3.1, the DARE can be written as
\[
\tilde{P} = (A^{cl})^* \tilde{P} A^{cl} + \tau (C_1^{cl})^* C_1^{cl} + (C_2^{cl})^* C_2^{cl} - [(A^{cl})^* \tilde{P} B_1^{cl} + \tau (C_1^{cl})^* D_{11}^{cl} + (C_2^{cl})^* D_{21}^{cl}] 
\times [-\tau T_{11}^* T_{11}]^{-1} [(B_1^{cl})^* \tilde{P} A^{cl} + \tau (D_{11}^{cl})^* C_1^{cl} + (D_{21}^{cl})^* C_2^{cl}] \quad (5.86)
\]
holds. Note that we have used (5.76) to short the expression for the DARE. Using (5.77), this DARE can be written as
\[
\tilde{P} = (A^{cl})^* \tilde{P} A^{cl} + \tau (C_1^{cl})^* C_1^{cl} + (C_2^{cl})^* C_2^{cl} + K_{dx}^* [(B_1^{cl})^* \tilde{P} A^{cl} + \tau (D_{11}^{cl})^* C_1^{cl} + (D_{21}^{cl})^* C_2^{cl}] \quad (5.87)
\]
Using (5.77) again yields (5.79). To prove (5.80) we note that, by Lemma 4.3.1, the quantity \( \epsilon^{-1} R_\psi(P_0) \) can be written as
\[
(B_2^{cl})^* \tilde{P} B_2^{cl} + \tau (D_{12}^{cl})^* D_{12}^{cl} + (D_{22}^{cl})^* D_{22}^{cl} - [(B_2^{cl})^* \tilde{P} B_1^{cl} + \tau (D_{12}^{cl})^* D_{11}^{cl} + (D_{22}^{cl})^* D_{21}^{cl}] 
\times [-\tau T_{11}^* T_{11}]^{-1} [(B_1^{cl})^* \tilde{P} A^{cl} + \tau (D_{11}^{cl})^* C_1^{cl} + (D_{21}^{cl})^* C_2^{cl}] \quad (5.88)
\]
Note that we have used (5.76) to shorten the expression for \( \epsilon^{-1} R_\psi(P_0) \). Using (5.78), \( \epsilon^{-1} R_\psi(P_0) \) can be written as
\[
(B_2^{cl})^* \tilde{P} B_2^{cl} + \tau (D_{12}^{cl})^* D_{12}^{cl} + (D_{22}^{cl})^* D_{22}^{cl} + K_{dx}^* [(B_1^{cl})^* \tilde{P} B_2^{cl} + \tau (D_{12}^{cl})^* D_{12}^{cl} + (D_{21}^{cl})^* D_{22}^{cl}] \quad (5.89)
\]
Using (5.78) again yields (5.80).

With these identities in place, we now investigate the DARE \( R_\psi(P) = P \). The next two lemmas begin to show the structure of the stabilizing solution of the DARE \( R_\psi(P) = P \).

**Lemma 5.3.8.** The subspace
\[
\Im \begin{bmatrix} I_{n_z} & 0_{n_z \times n_x} & P \\ 0_{n_z \times n_x} & K_{dx} \end{bmatrix} \quad (5.90)
\]
is a Schur deflating subspace for the matrix pencil (5.68).
Proof. First note that it is sufficient to show that there exists a Schur matrix $\Lambda$ such that
\begin{align*}
\hat{A}_{11} + \hat{B}_{11} K_{dx} &= \Lambda \\
\hat{A}_{21} + \hat{B}_{21} K_{dx} &= 0 \\
-\hat{Q}_{11} + \hat{P} - \hat{S}_1 K_{dx} &= \hat{A}^{*}_{11} \hat{P} \Lambda \\
-\hat{Q}_{12} - \hat{S}_2 K_{dx} &= \hat{A}^{*}_{12} \hat{P} \Lambda \\
\hat{S}^*_1 + \hat{R} K_{dx} &= -\hat{B}_{11}^* \hat{P} \Lambda .
\end{align*}

Equation (5.91) yields the value of $\Lambda$. Thus
\begin{equation}
\Lambda = \hat{A}_{11} + \hat{B}_{11} K_{dx} = A + [B_1 \ B_3] [K_{dx} \ K_x] = A_{\phi}(P_0) \tag{5.96}
\end{equation}
which implies that $\Lambda$ is Schur. Verifying (5.92) is trivial. Using (5.72)–(5.73), we see after some algebra that
\begin{equation}
-\hat{Q}_{11} + \hat{P} - \hat{S}_1 K_{dx} - \hat{A}^{*}_{11} \hat{P} \Lambda = \hat{P} - \left[ (A^d)^* \hat{P} A^d + \tau (C_{11}^d)^* C_1 + (C_{21}^d)^* C_2 \\
+ [(A^d)^* \hat{P} B_{11}^d + \tau (C_{11}^d)^* D_{11} + (C_{21}^d)^* D_{21}^d] K_{dx} \right] . \tag{5.97}
\end{equation}
Using (5.77), we see that the right-hand side of (5.97) is
\begin{equation}
\hat{P} - \left[ (A^d)^* \hat{P} A^d + \tau (C_{11}^d)^* C_1 + (C_{21}^d)^* C_2 + \tau K_{dx} T_{11}^* T_{11} K_{dx} \right] . \tag{5.98}
\end{equation}
This expression is zero by (5.79), which implies that (5.93) holds. Using (5.72)–(5.73), we see after some algebra that
\begin{equation}
-\hat{Q}_{12} - \hat{S}_2 K_{dx} - \hat{A}^{*}_{12} \hat{P} \Lambda = 0 \tag{5.99}
\end{equation}
which implies that (5.94) holds. Using (5.72)–(5.73) and (5.76), we see after some algebra that
\begin{equation}
\hat{S}^*_1 + \hat{R} K_{dx} + \hat{B}_{11}^* \hat{P} \Lambda = [(B_{11}^d)^* \hat{P} A^d + \tau (D_{11}^d)^* C_1 + (D_{21}^d)^* C_2] - \tau T_{11}^* T_{11} K_{dx} . \tag{5.100}
\end{equation}
The right-hand side of this expression is zero by (5.77), which implies that (5.95) holds. \hfill \blacksquare

Lemma 5.3.9. If the DARE $\mathcal{R}_{\hat{\phi}}(P) = P$ has a stabilizing solution $\hat{P}$, then $\hat{P} = \text{diag}(P, \hat{P}_{22})$ for some $\hat{P}_{22} \in \mathbb{R}^{n_x \times n_x}$.

Proof. By Proposition 2.5.4, we see that the subspace $\mathcal{S} := \text{Im}([I \ \hat{P} K_{\hat{\phi}}(\hat{P})^*]^*)$ is a Schur deflating subspace of dimension $n_x + n_c$ for the matrix pencil (5.68). By Proposition 2.5.5, $\mathcal{S}$ must be the maximal Schur deflating subspace of that pencil. Partitioning
\begin{equation}
\hat{P} = \begin{bmatrix} \hat{P}_{11} & \bullet \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} , \quad K_{\hat{\phi}}(P) = \begin{bmatrix} \hat{K}_1 & \hat{K}_2 \end{bmatrix} \tag{5.101}
\end{equation}
CHAPTER 5. OUTPUT FEEDBACK $\mathcal{H}_2$ GUARANTEED COST CONTROL

where $\hat{P}_{11}$ and $\hat{K}_1$ have $n_x$ columns, we see by Lemma 5.3.8 that

$$\operatorname{Im} \begin{bmatrix} I \\ 0 \\ K_{dx} \end{bmatrix} \subset \operatorname{Im} \begin{bmatrix} I & 0 \\ 0 & I \\ \hat{P}_{11} & \hat{P}_{21} \end{bmatrix} \Rightarrow \exists Z_1, Z_2 \text{ s.t.} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ \hat{P}_{11} & \hat{P}_{21} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}. \tag{5.102}$$

From this, we immediately see that $Z_1 = I$ and $Z_2 = 0$, which implies that $\hat{P}_{11} = \hat{P}$ and $\hat{P}_{21} = 0$. \hfill \blacksquare

Before giving the exact form of the stabilizing solution of the DARE $\mathcal{R}_{\hat{\phi}}(P) = P$, we give another set of identities. These identities give explicit formulas for the each quantity that appears in the expressions $\mathcal{R}_{\hat{\phi}}(P)$ and $\mathcal{R}_{\hat{\phi}}(P)$ when $P$ has the form $P = \text{diag}(\hat{P}, \text{e}^{-1}X)$ for some matrix $X$.

**Lemma 5.3.10.** Let $X \in \mathbb{R}^{n_c \times n_c}$. The following identities hold:

$$\begin{aligned}
\hat{A}_{11}^* \hat{P} \hat{A}_{11} + \hat{A}_{21}^*(\text{e}^{-1}X) \hat{A}_{21} + \hat{Q}_{11} = \hat{P} + \text{e}^{-1}K_{dx}^* T_{11} (\hat{B}_1^* X \hat{B}_1 + \hat{D}_1^* \hat{D}_1 - I) T_{11} K_{dx} & \tag{5.103} \\
\hat{A}_{11}^* \hat{P} \hat{A}_{12} + \hat{A}_{21}^*(\text{e}^{-1}X) \hat{A}_{22} + \hat{Q}_{12} = -\text{e}^{-1}K_{dx}^* T_{11} (\hat{B}_1^* X \hat{A} + \hat{D}_1^* \hat{C}) & \tag{5.104} \\
\hat{A}_{12}^* \hat{P} \hat{A}_{12} + \hat{A}_{22}^*(\text{e}^{-1}X) \hat{A}_{22} + \hat{Q}_{22} = -\text{e}^{-1}(\hat{A}^* X \hat{A} + \hat{C}^* \hat{C}) & \tag{5.105} \\
\hat{A}_{11}^* \hat{P} \hat{B}_{11} + \hat{A}_{21}^*(\text{e}^{-1}X) \hat{B}_{21} + \hat{S}_1 = -\text{e}^{-1}K_{dx}^* T_{11} (\hat{B}_1^* X \hat{B}_1 + \hat{D}_1^* \hat{D}_1 - I) T_{11} & \tag{5.106} \\
\hat{A}_{12}^* \hat{P} \hat{B}_{11} + \hat{A}_{22}^*(\text{e}^{-1}X) \hat{B}_{21} + \hat{S}_2 = -\text{e}^{-1}(\hat{A}^* X \hat{B}_1 + \hat{C}^* \hat{D}_1) T_{11} & \tag{5.107} \\
\hat{B}_{11}^* \hat{P} \hat{B}_{11} + \hat{B}_{21}^*(\text{e}^{-1}X) \hat{B}_{21} + \hat{R} = -\text{e}^{-1}T_{11} (\hat{B}_1^* X \hat{B}_1 + \hat{D}_1^* \hat{D}_1 - I) T_{11} & \tag{5.108} \\
\end{aligned}$$

$$\begin{aligned}
\hat{B}_{12}^* \hat{P} \hat{B}_{12} + \hat{B}_{22}^*(\text{e}^{-1}X) \hat{B}_{22} + \tau \hat{D}_{12} \hat{D}_{12} + \hat{D}_{22}^* \hat{D}_{22} = (B_{12}^c)^* \hat{P} B_{12}^{cl} + \tau (D_{12}^{cl})^* D_{12}^{cl} + (D_{22}^{cl})^* D_{22}^{cl} & \\
+ \text{e}^{-1} \left[ B_{12}^c X \hat{B}_{22} + \hat{D}_{22} \hat{K}_{dw} T_{11} (\hat{B}_1^* X \hat{B}_1 + \hat{D}_1^* \hat{D}_1) T_{11} K_{dw} \right. & \\
- \left. \hat{K}_{dw} T_{11} (\hat{B}_1^* X \hat{B}_{22} + \hat{D}_1^* \hat{D}_2) - (\hat{B}_2^c X \hat{B}_1 + \hat{D}_2^c \hat{D}_1) T_{11} K_{dw} \right] & \tag{5.109} \\
\end{aligned}$$

$$\begin{aligned}
\hat{B}_{12}^* \hat{P} \hat{B}_{11} + \hat{B}_{22}^*(\text{e}^{-1}X) \hat{B}_{21} + \tau \hat{D}_{12} \hat{D}_{11} + \hat{D}_{22}^* \hat{D}_{21} & \\
= \text{e}^{-1} \left[ B_{12}^c X \hat{B}_1 + \hat{D}_2 \hat{D}_1 - \hat{K}_{dw} T_{11} (\hat{B}_1^* X \hat{B}_1 + \hat{D}_1^* \hat{D}_1 - I) \right] T_{11} & \tag{5.110} \\
\end{aligned}$$

**Proof.** With some algebra involving (5.72) and (5.75), we see that the left-hand side of (5.103) is given by

$$[(A_{12}^c)^* \hat{P} A_{12} + \tau (C_{12}^c)^* C_{12}^c + (C_{22}^c)^* C_{22}^c] + \text{e}^{-1} K_{dx}^* T_{11} (\hat{B}_1^* X \hat{B}_1 + \hat{D}_1^* \hat{D}_1 - I) T_{11} K_{dx}. \tag{5.111}$$
CHAPTER 5. OUTPUT FEEDBACK $\mathcal{H}_2$ GUARANTEED COST CONTROL

Using (5.79) to substitute for the term in square brackets proves (5.103). Equation (5.104) is shown using (5.72) and (5.75). Equation (5.105) is shown using (5.75). With some algebra involving (5.72) and (5.75), we see that the left-hand side of (5.106) is given by

$$[(A^{cl})^*PB_1^{cl} + \tau(C^{cl}_1)^*D_1^{cl} + (C^{cl}_2)^*D_2^{cl}] - \epsilon^{-1}K^{*}_{dx}T^{*}_{11}(\hat{B}_1^*X\hat{B}_1 + \hat{D}_1^*\hat{D}_1)T_{11}K_{dx}. \quad (5.112)$$

Using (5.77) to substitute for the term in square brackets proves (5.106). Equation (5.107) is shown using (5.73) and (5.75). Equation (5.108) is shown using (5.73) and (5.75)–(5.76). Equation (5.109) is shown using (5.74) and (5.75). With some algebra involving (5.73)–(5.75), we see that the left-hand side of (5.103) is given by

$$[(B^{cl}_2)^*PB_1^{cl} + \tau(D^{cl}_1)^*D_1^{cl} + (D^{cl}_2)^*D_2^{cl}] + \epsilon^{-1}(\hat{B}_2^*X\hat{B}_1 + \hat{D}_2^*\hat{D}_1 - K^{*}_{dx}(\hat{B}_1^*X\hat{B}_1 + \hat{D}_1^*\hat{D}_1))T_{11}. \quad (5.113)$$

Using (5.78) to substitute for the term in square brackets proves (5.110).

With the preceding results in place, we can now give a result which relates the stabilizing solutions of the DAREs $\mathcal{R}_{\hat{A}}(P) = P$ and $\mathcal{R}_{\hat{A}}(P) = P$.

**Lemma 5.3.11.** The matrix $\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$ is Schur and the DARE $\mathcal{R}_{\hat{A}}(P) = P$ has a stabilizing solution $\hat{P}$ such that

$$\begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{bmatrix}^* \hat{P} \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{bmatrix} + \hat{R} < 0 \quad (5.114)$$

if and only if $\hat{A}$ is Schur and DARE $\mathcal{R}_{\hat{A}}(P) = P$ has a stabilizing solution $\hat{P}$ such that $\hat{B}_1^*\hat{P}\hat{B}_1 + \hat{D}_1^*\hat{D}_1 - I < 0$. Moreover, $\hat{P} = \text{diag}(\hat{P}, \epsilon^{-1}\hat{P})$.

**Proof.** ($\Rightarrow$) By Lemma 5.3.9, we see that

$$\hat{P} = \begin{bmatrix} \hat{P} & 0 \\ 0 & \epsilon^{-1}\hat{P} \end{bmatrix} \quad (5.115)$$

for some $\hat{P} \in \mathbb{R}^{n_x \times n_x}$. By Theorem 3.2.2, we see that $\hat{P} \succeq 0$, which implies that $\hat{P} \succeq 0$. By (5.108), we see that $\hat{B}_1^*\hat{P}\hat{B}_1 + \hat{D}_1^*\hat{D}_1 - I < 0$. Using Lemma 5.3.10, we see after some algebra that $\mathcal{R}_{\hat{A}}(\hat{P})$ and $\mathcal{R}_{\hat{A}}(\hat{P})$ can respectively be expressed as

$$\mathcal{A}_{\hat{A}}(\hat{P}) = \begin{bmatrix} \mathcal{A}_{\hat{A}}(P_0) & * \\ 0 & \mathcal{A}_{\hat{A}}(\hat{P}) \end{bmatrix}, \quad \mathcal{R}_{\hat{A}}(P) = \begin{bmatrix} \hat{P} & 0 \\ 0 & \epsilon^{-1}\mathcal{R}_{\hat{A}}(\hat{P}) \end{bmatrix} \quad (5.116)$$
where the $\star$ represents a term that is not important to the analysis. Since $A_\phi(\hat{P})$ is Schur, we see that $A_\phi(\bar{P})$ is Schur. Also, since $R_\phi(\bar{P}) = \hat{P}$, we see that $R_\phi(\bar{P}) = \bar{P}$. Applying Theorem 3.2.2 tells us that $\hat{A}$ is Schur.

$(\Leftarrow)$ First note that, by Theorem 3.2.2, $\bar{P} \succeq 0$. By Theorem 4.3.3, we see that $P_0 \succeq 0$, which implies that $\bar{P} \succeq 0$. We now choose $\bar{P} = \text{diag}(\bar{P}, \epsilon^{-1}\bar{P})$ and note that $\bar{P} \succeq 0$. By (5.108), we see that (5.114) holds. This implies that the expressions for $A_\phi(\bar{P})$ and $R_\phi(\bar{P})$ are well-defined. As before, we use Lemma 5.3.10 to see that (5.116) holds. This in turn implies that $A_\phi(\bar{P})$ is Schur and $R_\phi(\bar{P}) = \hat{P}$. Applying Theorem 3.2.2 tells us that $\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$ is Schur.

With these results of this subsection in place, we can now show that Theorem 5.3.2 holds. We restate the theorem here for convenience.

**Theorem.** For fixed $\epsilon^{-1} = \tau > 0$,

$$J_\tau(F_i(G_1, G_3)) = J_{f_i, \epsilon} + \epsilon^{-1} J_1(G_3).$$

(5.118)

**Proof.** Suppose that $J_\tau(F_i(G_1, G_3)) \neq \infty$. Theorem 3.2.3 tells us that $\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$ is Schur and the DARE $R_\phi(P) = P$ has a stabilizing solution $\bar{P}$ such that (5.114) holds. Moreover, $J_\tau(F_i(G_1, G_3)) = \text{tr}\{R_\phi(\bar{P})\}$. By Lemma 5.3.11, we see that $\hat{A}$ is Schur and $\bar{P} = \text{diag}(\bar{P}, \epsilon^{-1}\bar{P})$ where $\bar{P}$ is the stabilizing solution of the DARE $R_\phi(P) = P$ that satisfies $\tilde{B}_1^* \bar{P} \tilde{B}_1 + \tilde{D}_1^* \tilde{D}_1 - I \prec 0$. Using Lemma 5.3.10, we see after some algebra that

$$R_\psi(\bar{P}) = (B_2^d)^* \bar{P} B_2^d + \tau(D_{12}^d)^* D_{12}^d + (D_{22}^d)^* D_{22}^d + \tau K_{dw}^* T_{11}^* T_{11} K_{dw} + \epsilon^{-1} R_\psi(\bar{P}).$$

(5.119)

By (5.80), we see that

$$R_\psi(\bar{P}) = \epsilon^{-1} R_\psi(P_0) + \epsilon^{-1} R_\psi(\bar{P}).$$

(5.120)

Taking the trace of both sides of this equation yields

$$R_\psi(\bar{P}) = \text{tr}\{\epsilon^{-1} R_\psi(P_0)\} + \epsilon^{-1} \text{tr}\{R_\psi(\bar{P})\}.$$ 

(5.121)

Note that $J_{f_i, \epsilon} = \text{tr}\{\epsilon^{-1} R_\psi(P_0)\}$ by Theorem 4.3.10. Also, $J_1(G_3) = \text{tr}\{R_\psi(\bar{P})\}$ by Theorem 3.2.3. Thus, (5.118) holds when $J_\tau(F_i(G_1, G_3)) \neq \infty$. 


Now suppose instead that \( J_\tau(\mathcal{F}_l(G_1, G_3)) = \infty \). By Theorem 3.2.3, we see that either
\[
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix}
\]
is not Schur or the DARE \( \mathcal{R}_\phi(P) = P \) does not have a stabilizing solution \( \hat{P} \) such that (5.114) holds. By Lemma 5.3.11, we see that either \( \tilde{A} \) is not Schur or the DARE \( \mathcal{R}_{\tilde{\phi}}(P) = P \) does not have a stabilizing solution \( \tilde{P} \) such that \( \tilde{B}_1^* \tilde{B}_1 + \tilde{D}_1^* \tilde{D}_1 - I \prec 0 \). This implies that \( J_1(G_3) = \infty \) by Theorem 3.2.3. Thus, (5.118) holds when \( J_\tau(\mathcal{F}_l(G_1, G_3)) = \infty \). \( \blacksquare \)
Chapter 6

$\mathcal{H}_2$ Guaranteed Cost Control of Hard Disk Drives

For several decades now, the areal storage density of hard disk drives (HDDs) has been doubling roughly every 18 months, in accordance with Kryder’s law. As the storage density is pushed higher, the concentric tracks on the disk that contain data must be pushed closer together, which requires much more accurate control of the read/write head. Currently available hard drives can store 2 TB of data on a 3.5” drive with three platters. This corresponds to an areal data density of 600 gigsbits/in$^2$. The current goal of the magnetic recording industry is to achieve an areal storage density of 4 terabits/in$^2$. It is expected that the track width required to achieve this data density is 25 nm. To achieve this specification for track-following control, in which the read/write head is maintained as close to the center of a given data track as possible, the $3\sigma$ value of the closed-loop position error signal (PES) should be less than 2.5 nm.

To help achieve this goal, the use of a secondary actuator has been proposed to give increased precision in read/write head positioning. There are three classes of secondary actuators: actuated suspensions [11], actuated sliders [16], and actuated heads [32]. Each of these proposed secondary actuator classes correspond to a different actuator location in Fig. 6.1. In the actuated head configuration, a microactuator (MA) actuates the read/write head with respect to the slider mounted at the tip of the suspension. In the actuated slider configuration, an MA directly actuates the head/slider assembly with respect to the suspension. For both of these configurations, it is difficult to design an MA which can be easily incorporated into the manufacture and assembly of a HDD on a large scale. In the actuated suspension configuration, the MA actuates the suspension with respect to the E-block. This type of MA is the least difficult to design and has been incorporated into some consumer products. We will thus be using this secondary actuation scheme in this chapter.

Since there tend to be large variations in HDD dynamics due to variations in manufacture and assembly, it is not enough to achieve the desired level of performance for a single plant; the controller must guarantee the desired level of performance for a large set of HDDs.
Thus, we are interested in finding a controller which gives robust performance over a set of HDDs. As mentioned earlier, the relevant performance metric in a HDD is the standard deviation of the PES. Since the squared $\mathcal{H}_2$ norm of a system can be interpreted as the sum of variances of the system outputs under the assumption that the system is driven by independent white zero mean Gaussian signals with unit covariance, the $\mathcal{H}_2$ norm is the most relevant performance metric for HDDs.

In this chapter, we will apply the techniques of Chapter 5 to the design of output feedback HDD controllers.

### 6.1 Hard Disk Drive Model

In this section, we present the HDD model we will be using in this chapter. The HDD we are considering has the PZT-actuated suspension shown in Fig. 6.2, which is a Vector model suspension provided to us by Hutchinson Technology Inc. In our setup, we use a laser Doppler vibrometer (LDV) to measure the absolute radial displacement of the slider. The control circuits include a Texas Instrument TMS320C6713 DSP board and an in-house made analog board with a 12-bit ADC, a 12-bit DAC, a voltage amplifier to drive the MA, and a current amplifier to drive the voice coil motor. The DSP sampling period is $1.4 \times 10^{-5}$ s and the controller delay, which includes ADC and DAC conversion delay and DSP computation
delay, is $6 \mu s$. A hole was cut through the case of the drive to allow the LDV laser to shine on the slider. It should be noted that these modifications affect the dynamics of the drive and thus have a detrimental effect on the attainable performance of the servo system.

The block diagram of our HDD setup is shown in Fig. 6.3 and the relevant signals and their units are listed in Table 6.1. In this block diagram, we treat the dynamics from the two control inputs to the head displacement as a single block to take into account the knowledge that both actuators can excite the same vibration modes in the suspension. Exploiting this knowledge allows us to form a model which does not have redundant states resulting from including two copies of the suspension vibration modes. The block $\Delta$ represents the unmodeled HDD dynamics, which we model as an unknown stable causal LTI system which satisfies $\|\Delta\|_\infty \leq 1$. This block, along with $W_\Delta$, represents output dynamic multiplicative uncertainty on $G_p$.

To construct a discrete-time model of our system, we used the methodology of [24]. To find the model of $G_p$, we first obtained frequency responses of our system from $u_v$ to $y$ and $u_p$ to $y$. Using weighted least squares, we separately fit a continuous-time model to each of these frequency responses, which we then combined into a single model and used common mode identification [5] to eliminate redundant copies of the suspension vibration modes. We then discretized this model with the $6 \mu s$ delay on each of its two inputs to yield the model

![HDD block diagram](image-url)

**Table 6.1: HDD signals**

<table>
<thead>
<tr>
<th>Signal</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Disturbances on the head position [18]</td>
<td>nm</td>
</tr>
<tr>
<td>$u_p$</td>
<td>PZT actuator control signal</td>
<td>V</td>
</tr>
<tr>
<td>$u_v$</td>
<td>Voice coil motor control signal</td>
<td>V</td>
</tr>
<tr>
<td>$w_a$</td>
<td>Airflow disturbances</td>
<td>(normalized)</td>
</tr>
<tr>
<td>$w_n$</td>
<td>PES sensor noise</td>
<td>(normalized)</td>
</tr>
<tr>
<td>$w_r$</td>
<td>Disturbances on the head position</td>
<td>(normalized)</td>
</tr>
<tr>
<td>$y$</td>
<td>PES</td>
<td>nm</td>
</tr>
<tr>
<td>$y_h$</td>
<td>Head displacement relative to the track center</td>
<td>nm</td>
</tr>
</tbody>
</table>
for $G_p$. This model is given by

$$G_p(z) = \left[ \begin{array}{c} -0.6858 \\ 20.94 \end{array} \right] z^{-1} + \sum_{i=1}^{6} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left( zI - \left[ \begin{array}{c} a_{i,1} \\ a_{i,2} \end{array} \right] \right)^{-1} B_i$$

where the model parameters are as listed in Table 6.2. Because there are two poles at $z = 0$—one for each input channel—the state-space model of $G_p$ has 14 states. The poles at $z = 0$ were introduced by the discretization of the continuous-time input delay. The six vibration modes in (6.1) are ordered from lowest to highest resonance frequency. The Bode plot of this model is shown in Fig. 6.4.

The weighting for the dynamic multiplicative uncertainty of $G_p$, given by

$$W_\Delta = \frac{0.9733 - z}{z - 0.465} ,$$

was chosen so that the uncertain model enveloped the experimental frequency response of $G_p$. The Bode magnitude plot of $W_\Delta$ is shown in Fig. 6.5a. Since $\Delta$ is a SISO system, upper and lower bounds on the magnitude of each input/output pair in $G_p$ can be easily computed at each frequency. Doing so yields the upper and lower bounds on the Bode magnitude plots of $G_p$ shown in Fig. 6.6. The values $\sigma_a = 0.04854$ and $\sigma_n = 1.3$ were determined by

---

### Table 6.2: Model parameters for $G_p$

<table>
<thead>
<tr>
<th>mode, $i$</th>
<th>$\begin{bmatrix} a_{i,1} \ a_{i,2} \end{bmatrix}$</th>
<th>$B_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\begin{bmatrix} 1.99607 \ -0.996286 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 3.982 \ -6.262 \end{bmatrix}$</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{bmatrix} 1.474 \ -0.9680 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1.415 \ 1.368 \end{bmatrix}$</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} 1.381 \ -0.9762 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -0.8049 \ 2.082 \end{bmatrix}$</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} 0.5387 \ -0.9632 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0.1366 \ -1.520 \end{bmatrix}$</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} 0.04209 \ -0.9353 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -0.06772 \ -0.1826 \end{bmatrix}$</td>
</tr>
<tr>
<td>6</td>
<td>$\begin{bmatrix} -1.653 \ -0.9527 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -0.09272 \ -0.1086 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
CHAPTER 6. $\mathcal{H}_2$ GUARANTEED COST CONTROL OF HARD DISK DRIVES

Figure 6.4: Bode plot of $G_p$

Figure 6.5: Bode magnitude plots of $W_\Delta$ and $G_r$
Figure 6.6: Nominal Bode magnitude plots for $G_p$ along with pointwise upper and lower bounds over modeled uncertainty.

matching the power spectrum density of the open loop slider motion respectively at low and high frequency. The disturbances on the head position are characterized by

$$G_r(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left( zI - \begin{bmatrix} 1.964 & 1 \\ -0.975 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} -0.2574 \\ 0.25 \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} \left( zI - \begin{bmatrix} 0.9956 & -0.0745 \\ 0 & 0.9956 \end{bmatrix} \right)^{-1} \begin{bmatrix} -0.9533 \\ 0.919 \end{bmatrix}. \quad (6.3)$$

Figure 6.5b shows the Bode magnitude plot of $G_r$. In addition to capturing the effect of disturbances on the head position, this model of $G_r$ also captures the low-frequency drift in the LDV position measurements resulting from integration of velocity measurements. The second-order mode near 1 kHz in this model captures the effect of disk modes between 1 kHz and 3 kHz.

These disturbances, although realistic for our experimental setup, are larger than the disturbances typically found in a HDD. First of all, the measurement noise of the LDV is somewhat larger than the measurement noise of the PES. Moreover, as we previously mentioned, the LDV has a significant low-frequency drift. These two factors along with the mechanical modifications of the drive significantly deteriorate the achievable level of closed-loop performance.

With some manipulation, the blocks in Fig. 6.3 can be grouped to form the LFT representation in Fig. 6.7. In this form $G_H$ has 19 states. For the remainder of this chapter, we will use the balanced realization of $G_H$ for analysis and control design.

### 6.2 Output Feedback Control

In Chapter 5, we developed two methodologies for designing output feedback controllers—one based on solving a sequence of SDPs (Algorithm 5.2.3) and another which exploits the
solution of Riccati equations to yield simplified SDPs (Algorithm 5.3.6). We will respectively call these approaches the SSDP approach and the DARE/SDP approach.

In this section, we will design controllers for the HDD model presented in §6.1. In an effort to boost the PES performance of the closed-loop system (at the expense of the control effort), we will deemphasize the control effort in the cost function by applying our solution heuristics to the plant

\[ \hat{G}_H := \text{diag}(1, 1, 0.01, 0.01, 1)G_H. \]  

(6.4)

We first applied the SSDP approach in §5.2 to the design of an output feedback controller for \( \hat{G}_H \). Doing so yielded the results shown in Table 6.3, which breaks down the \( \mathcal{H}_2 \) guaranteed cost and the cumulative optimization time at each optimization step in Algorithm 5.2.3. The first thing to note is that the SSDP approach took just over 1 hour to design a controller for this system. The next thing to note is that, at steps 1b and 3, there are degradations in performance. However, these are both expected. At step 1a, since the designed controller has direct access to the state of \( \hat{G}_H \), it is likely that the cost reported after this step is smaller than is achievable by an output feedback controller. In step 3, we allowed the cost to increase by up to 2% in order to optimize the conditioning of the controller reconstruction process; the \( \mathcal{H}_2 \) guaranteed cost was a design constraint rather than the objective function.

Table 6.3: Closed-loop performance and cumulative optimization time after each optimization step in SSDP approach

<table>
<thead>
<tr>
<th>Step</th>
<th>( \mathcal{H}_2 ) Guaranteed Cost</th>
<th>Cumulative Optimization Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>6.255</td>
<td>10.811</td>
</tr>
<tr>
<td>1b</td>
<td>9.730</td>
<td>239.89</td>
</tr>
<tr>
<td>2a</td>
<td>8.095</td>
<td>561.74</td>
</tr>
<tr>
<td>2b</td>
<td>8.095</td>
<td>1188.12</td>
</tr>
<tr>
<td>2a</td>
<td>8.095</td>
<td>1552.77</td>
</tr>
<tr>
<td>2b</td>
<td>8.095</td>
<td>2091.08</td>
</tr>
<tr>
<td>3</td>
<td>8.257</td>
<td>3610.92</td>
</tr>
</tbody>
</table>
Before performing step 3, we tried to reconstruct a controller which achieved the $\mathcal{H}_2$ guaranteed cost 8.095. However, the condition number of the matrix $I - XY$ (i.e. the ratio of its largest and smallest singular values) was $3.802 \times 10^{22}$. This resulted in large numerical inaccuracies when reconstructing the controller, which in turn resulted in an unstable closed-loop system. After performing step 3, the condition number of the matrix $I - XY$ was improved to $7.933 \times 10^8$. With the resulting values of the optimization parameters, a controller was reconstructed which achieves the $\mathcal{H}_2$ guaranteed cost 7.958 (as computed by Algorithm 3.2.8). Interestingly, this is 4% better than the cost reported by the solver in step 3. Figure 6.8 shows the Bode magnitude plot of nominal closed-loop sensitivity function from $r$ to $y_h$ along with pointwise upper and lower bounds on its Bode magnitude plot over modeled uncertainty. The nominal Bode magnitude plot has the peak value 1.25dB and the upper bound on the Bode magnitude plot has the peak value 5.57dB. Thus, even in the worst case, the Bode magnitude plot of the sensitivity function from $r$ to $y_h$ has a low peak.

After designing a controller using the SSDP approach, we designed a controller using the DARE/SDP approach. The algorithm took 39.03 seconds to run and reported a closed-loop $\mathcal{H}_2$ guaranteed cost of 7.747. By construction, this value of the closed-loop $\mathcal{H}_2$ guaranteed cost is exactly equal to the $\mathcal{H}_2$ guaranteed cost computed by analyzing the closed-loop performance using Algorithm 3.2.8. Thus, this controller performs 2% better than the controller designed using the SSDP approach. Also note that the DARE/SDP approach was more than 90 times faster than the SSDP approach.

Figure 6.9 shows the Bode magnitude plot of nominal closed-loop sensitivity function from $r$ to $y_h$ along with pointwise upper and lower bounds on its Bode magnitude plot over modeled uncertainty. The nominal Bode magnitude plot has the peak value 1.01dB and the upper bound on the Bode magnitude plot has the peak value 4.16dB. Thus, like the closed-loop system designed using the SSDP approach, the Bode magnitude plot of the sensitivity

![Figure 6.8: Bode magnitude plot of the nominal closed-loop sensitivity function from r to y_h for the controller designed using the SSDP approach along with its pointwise upper and lower bounds over all modeled uncertainty.](image)
Figure 6.9: Bode magnitude plot of the nominal closed-loop sensitivity function from $r$ to $y_h$ for the controller designed using the DARE/SDP approach along with its pointwise upper and lower bounds over all modeled uncertainty.

Table 6.4: Worst-case standard deviation of closed-loop signals over 3000 random closed-loop samples with controller designed using the DARE/SDP approach.

<table>
<thead>
<tr>
<th>Signal</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_h$</td>
<td>2.466nm</td>
</tr>
<tr>
<td>$u_v$</td>
<td>0.316V</td>
</tr>
<tr>
<td>$u_p$</td>
<td>0.279V</td>
</tr>
</tbody>
</table>

function from $r$ to $y_h$ has a low peak in both the nominal and worst case.

It should be noted that the $H_2$ guaranteed cost performance of a system is an upper bound on the worst-case $H_2$ performance of the system over all unmodeled uncertainty—not necessarily the actual worst-case performance. It is thus useful to perform a Monte Carlo analysis of the closed-loop system. Using the function `usample` in the Robust Control Toolbox for MATLAB, we first chose 3000 random samples of the closed-loop system with $\Delta$ restricted to be a stable causal 3rd-order system satisfying $\|\Delta\|_\infty \leq 1$. For each of the 3000 systems, we then found the standard deviation of each of the outputs by computing the relevant $H_2$ norm. The worst-case standard deviation of each signal is summarized in Table 6.4. We see that these results are significantly smaller than predicted by $H_2$ guaranteed cost analysis of the closed-loop system. There are two likely reasons for this. The first reason is that the random samples of $\Delta$ do not adequately represent the modeled uncertainty. This is likely because, as mentioned in §3.1.3, the $H_2$ guaranteed cost of a system guarantees robust performance over norm-bounded causal LTV uncertainty, whereas we only considered LTI norm-bounded uncertainty of order 3 in our Monte Carlo analysis. The second reason is that requiring (3.20) to hold introduces significant conservatism into the analysis of robust $H_2$ performance.
Chapter 7

Conclusions and Future Work

In this dissertation, we have developed mathematical tools for robust $\mathcal{H}_2$ analysis and synthesis. In particular, we first developed an upper bound on the worst-case $\mathcal{H}_2$ performance of a system with norm-bounded causal LTV uncertainty which could be determined using either an SDP or an iteration of Riccati equation solutions. We then showed that the corresponding optimal full information control problem (a generalization of the optimal state feedback control problem) could also be solved using either an SDP or an iteration of Riccati equation solutions. For both of these algorithms, we gave empirical evidence that suggests that the Riccati equation approach is faster and generally more accurate than the SDP approach.

We then considered the optimal output feedback control problem. Although this was formulated as a non-convex optimization problem, we gave two heuristics for solving this optimal control problem. The first heuristic was based entirely on the solution of SDPs. The second heuristic exploited Riccati equation structure to reduce the amount of computation. Although this second heuristic also required the solution of SDPs, it required fewer SDP solutions than the first heuristic and, moreover, each SDP had fewer optimization parameters and a smaller constraint dimension than the SDPs encountered in the first heuristic.

These two heuristics for optimal output feedback control were applied to a track-following HDD control problem in which the HDD had a PZT-actuated suspension. It was shown that although both methods yielded controllers with a reasonable level of robust performance, the algorithm that exploited Riccati equation structure was more than 90 times faster and yielded a slightly better value of the cost function.

The results of this dissertation easily extend to a number of other types of systems. In particular, we can extend the analysis and synthesis results to linear periodically time-varying systems and/or systems with structured uncertainty by using the techniques in [6]. Also, the results in this dissertation can be interpreted in the context of dynamic game theory.
For instance, letting $G$ be the LTI system shown in Fig. 7.1, it can be shown that

$$J_\tau(G) = \lim_{N \to \infty} \sup_{d_0, \ldots, d_N} \frac{1}{1 + N} \sum_{k=0}^{N} \{p_k^*p_k + \tau(q_k^*q_k - d_k^*d_k)\}$$  \hspace{1cm} (7.1)$$

where $w$ is white zero-mean unit covariance uncorrelated Gaussian noise, the state of $G$ at time $k = 0$ is zero, and $d_k$ is restricted to being a causal function of all other vectors. In this framework, we see that the results of this paper can be extended to linear finite horizon systems using dynamic game theory \[^1\].

In examining the performance of the controller designed in §6.2, we saw that the worst-case performance of the closed-loop system determined using a Monte-Carlo analysis significantly differed from its upper bound determined using the methods of this dissertation. This suggests that more effort is required in reducing the conservatism of the analysis result. A promising tool for doing this is dynamic multiplier theory, as used for the analysis of the robust $\mathcal{H}_2$ performance of continuous-time systems in \[^12\]. This approach would correspond to using dynamic uncertainty scalings to reduce the conservatism of the analysis result.

In the second heuristic for optimal output feedback control, we were not able to solve all relevant problems using only Riccati equations; we had to use the solution of an SDP to solve the optimal full control problem at the end of §5.3.2. At this point, it is unknown whether or not the optimal full control problem can be solved using only the solution of a single Riccati equation. There are thus two areas of future work related to efficiently solving this subproblem. The first area is in determining whether or not the optimal full control problem can be solved using a single Riccati equation. The second area is in finding fast heuristics for solving the optimal full control problem. These heuristics will be based on Riccati equation solutions. This second area is useful in that it is an easier problem to analyze. Thus, until a method for solving the optimal full control problem using a single Riccati equation is found (if it exists), these heuristics will be useful in reducing the amount of computation required in designing an output feedback controller.

Another area of future work is the comparison of the techniques in this dissertation to other techniques for robust $\mathcal{H}_2$ analysis and synthesis. In particular, since the currently known techniques for analyzing the robust $\mathcal{H}_2$ performance of a system are only capable of finding upper bounds on the robust $\mathcal{H}_2$ performance of the system, each technique will generate conservative estimates of the robust $\mathcal{H}_2$ performance. Thus, it would be meaningful to compare the conservatism of the currently known analysis techniques as well as the assumptions required to use each technique. Similarly, it would be meaningful to compare

Figure 7.1: System representation for dynamic game approach to $\mathcal{H}_2$ guaranteed cost

\[ \begin{array}{c}
q \\
p \\
G \\
d \\
w
\end{array} \]
the output feedback controller design techniques that attempt to optimize the closed-loop robust $\mathcal{H}_2$ performance of a system.
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