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Relativistic quaternionic wave equation

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We study a one-component quaternionic wave equation which is relativistically covariant. Bilinear forms include a conserved four-vector current and an antisymmetric second rank tensor. Waves propagate within the light cone and there is a conserved quantity which looks like helicity. The principle of superposition is retained in a slightly altered manner. External potentials can be introduced in a way that allows for gauge invariance. There are some results for scattering theory and for two-particle wave functions as well as the beginnings of second quantization. However, we are unable to find a suitable Lagrangian or an energy-momentum tensor. © *2006 American Institute of Physics.* [DOI: 10.1063/1.2397555]

I. INTRODUCTION

Many attempts have been made to consider the extension of the usual quantum theory, based on the field of complex numbers, to quaternions. The 1936 paper by Birkhoff and von Neumann[1] opened the door to this possibility, and the 1995 book by Adler[2] covers many aspects that have been studied.

Here is a wave equation that appears to have escaped previous recognition:

$$\frac{\partial \psi}{\partial t} i = \mathbf{u} \cdot \nabla \psi + m \psi j. \tag{1.1}$$

The single wave function ψ is a function of the space time coordinates $\mathbf{x}, t$. The usual elementary quaternions i, j, k, are defined by

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1.2}$$

and

$$\mathbf{u} \cdot \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}. \tag{1.3}$$

Boldface type is used to designate a three-vector.

This combination Eq. (1.3) of elementary quaternions and space derivatives was originated by Hamilton[3] in 1846; its square is the negative of the Laplacian operator.

What one should note about Eq. (1.1) is that it employs quaternions which multiply the wave function on *both* the right side and the left side. This distinction arises from the noncommutativity of quaternion algebra and is central to the present study.

II. OTHER EQUATIONS

There are other quaternionic wave equations one can consider, based on the apparent structural similarities between quaternions and relativity. The simplest is

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$$\frac{\partial\psi}{\partial t} = \mathbf{u}\cdot\boldsymbol{\nabla}\psi, \tag{2.1}$$

which, when squared, appears as a four-dimensional Laplace equation, and not a wave equation.

Going to two-dimensions we construct

$$\begin{pmatrix} \frac{\partial}{\partial t} & \mathbf{u}\cdot\boldsymbol{\nabla} \\ \mathbf{u}\cdot\boldsymbol{\nabla} & -\frac{\partial}{\partial t} \end{pmatrix}\boldsymbol{\Psi} = m\boldsymbol{\Psi}. \tag{2.2}$$

When this equation is squared, we do get a wave equation, but it is for a tachyon. If one sets $m=0$ in this equation, it can be revised to appear as either two copies of the Weyl equation or the Maxwell equations (keeping only the imaginary components).

Various authors have shown that the familiar Dirac equation can be put into quaternionic form. This may be done by putting an i to the right of Ψ on one side of Eq. (2.2) (Ref. 4) or by the use of biquaternions in a one-component equation.[5] All of those representations involve eight real functions—as does the usual Dirac equation—while the basic equation of the current study [Eq. (1.1)] involves only four real functions.

There are two other known relativistic equations with four real components. One of these is the Majorana representation of the Dirac equation,

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi, \tag{2.3}$$

where all four of the gamma matrices can be made purely imaginary, so that one can take all four components of the Dirac wave function to be real functions of space-time. Indeed, if we write our quaternionic wave function as

$$\psi = \psi_0 + i\psi_1 + j\psi_2 + k\psi_3, \tag{2.4}$$

and arrange these four real functions as a column vector, then our Eq. (1.1) can be put in exactly this Majorana-Dirac form. One awkward feature of that formalism is that the usual Dirac Lagrangian becomes useless for an action principle, since every single term is identically zero.

The other comparison involves the Weyl equation (two complex components), which is usually reserved for massless particles. One can map quaternions onto a two-dimensional space of complex numbers. The correspondence can be expressed in terms of the familiar Pauli matrices $\mathbf{u}\rightarrow -i\sigma$, and the wave equation (1.1) can be written in a pseudo-Weyl form as

$$i\frac{\partial\boldsymbol{\Psi}}{\partial t} = -i\sigma\cdot\boldsymbol{\nabla}\boldsymbol{\Psi} + m\sigma_2\boldsymbol{\Psi}^*. \tag{2.5}$$

In this second example, one also has trouble with the usual Lagrangian in that the mass term is identically zero.

Both of these equations, Majorana-Dirac and modified Weyl, are used in building supersymmetry theories (see, for example, Ref. 6), but only after one introduces a second set of wave functions—with "dotted" spinor indices. Thus, they do return to eight real functions, which are, furthermore, not simply real functions but elements of a Grassmann algebra.

These comparisons leave me without a definitive answer to the question of whether the focal equation of this paper [Eq. (1.1)] is truly something new in theoretical physics. The work presented here will be to explore this quaternionic wave equation on its own terms and see what interesting things arise.

III. SOME PROPERTIES

In the usual quantum mechanics there is "gauge invariance of the first kind:" we can replace the complex wave function ψ by $\exp(i\theta)\psi$. This freedom is also noted by saying that there is a ray, not just one vector, in Hilbert space corresponding to each physical state. (The reader will note that this paper focuses entirely on the wave function approach to quantum theory and not the Hilbert space version.) For the quaternionic wave function we have a larger set of freedoms: $\psi \to q_1 \psi q_2$, where the two numbers q_1, q_2 are quaternions of unit magnitude. The one on the left induces a change of basis in the elementary quaternions **u** seen in Eq. (1.1), while the one on the right changes the particular choice of i and j acting to the right of ψ in that equation. Thus, instead of the usual $U(1)$ group, we appear to have $SU(2) \times SU(2)/Z_2$.

A first calculation is to take another time derivative of Eq. (1.1) and arrive at the second-order wave equation,

$$\frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi - m^2 \psi, \tag{3.1}$$

which is the ordinary Klein-Gordon equation for a relativistic particle of mass m.

Now we look at some bilinear forms. The first is $\rho = \psi^* \psi$, where the complex conjugation operator (*) changes the sign of each imaginary quaternion (and requires the reversal of order in multiplication of any expression upon which it operates). The second is the vector $\mathbf{U} = \psi^* \mathbf{u} \psi$. While ρ is real, $\mathbf{U}$ is purely imaginary, and we can write $\mathbf{U} = i\mathbf{U}_1 + j\mathbf{U}_2 + k\mathbf{U}_3$ in terms of three real three-vectors.

Making use of the wave equation [Eq. (1.1)], we then calculate

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{U}_1, \tag{3.2}$$

which is the familiar statement of a conserved current. We shall return to $\mathbf{U}_2$ and $\mathbf{U}_3$ shortly. [If you ask what singled out $\mathbf{U}_1$ as the conserved current, it is the choice of the imaginary i sitting beside the time derivative in the wave equation (1.1).]

IV. SPACE-TIME SYMMETRIES

Now we look at the behavior of the wave equation (1.1) under familiar symmetry transformations. To achieve rotation of the spatial coordinates **x**, we make the transformation

$$\psi \to \mathrm{e}^R \psi, \quad R = \mathbf{u} \cdot \theta/2, \tag{4.1}$$

where θ is the axis and the angle of rotation.

For the Lorentz transformation, we start with the infinitesimal form

$$\psi \to \psi + B\psi i, \quad B = \mathbf{u} \cdot \mathbf{v}/2, \tag{4.2}$$

where **v** is the direction and amount of the velocity boost. Note the appearance of the imaginary i acting on the right of ψ in this transformation. I leave it as an exercise for the reader to show that this transformation of ψ does indeed induce the familiar Lorentz transformation of the space-time coordinates in the wave equation (1.1).

One can now readily show that the components of the conserved current (ρ and $\mathbf{U}_1$) tansform as a Lorentz four-vector. With a bit more work, one can also see that the other two vectors $\mathbf{U}_2$ and $\mathbf{U}_3$ transform as the components of an antisymmetric second rank tensor in four dimensions (also called a six-vector).

A useful notation for operators that may multiply quaternionic functions on the right or on the left is the following:[1]

$$(a||b)\psi = a\psi b, \quad (a||b)(c||d) = (ac||db), \tag{4.3}$$

which allows us to write the finite Lorentz transformation operator as $e^{(B\|i)}$.

[1]A similar notation was introduced by the authors of Ref. 4.

The generators of the Lorentz group may be constructed as

$$\mathbf{J} = \mathbf{x} \times \boldsymbol{\nabla} - \frac{1}{2}\mathbf{u}, \quad \mathbf{K} = \mathbf{x}\frac{\partial}{\partial t} + t\,\boldsymbol{\nabla} - \frac{1}{2}\mathbf{u}||i. \tag{4.4}$$

One can extend this to the full Poincare group by adding the displacement operators: $\partial_\mu = (\partial_t, \nabla)$.

In the Appendix is a more extensive study of various tensors that can be built from solutions of the wave equation.

V. MORE BILINEAR FORMS

Start by defining the derivative operator which acts in both directions, $\mathrm{d}_\mu = (\mathrm{d}_0, \mathbf{d}) = \frac{1}{2}(\vec{\partial}_\mu - \overleftarrow{\partial}_\mu)$. This is a covariant four-vector, but let us now see how things behave when we combine it with the Lorentz transformation of the wave function:

$$D_\mu \equiv \psi^* \mathrm{d}_\mu \psi \rightarrow D_\mu + \frac{1}{2}\{i, \psi^* \mathrm{d}_\mu \mathbf{v} \cdot \mathbf{u}\psi\}. \tag{5.1}$$

The expression D_μ is purely imaginary, and so we can write $D_\mu = D_{1,\mu}\, i + D_{2,\mu}\, j + D_{3,\mu} k$. I hope that the use of the subscripts (1,2,3), denoting which imaginary component they come from, does not cause confusion with the vector or tensor subscripts μ. The expression inside the anticommutator brackets (next to i) is real. This leads us to conclude that under the Lorentz transformation of ψ

$$D_{2,\mu} \text{ and } D_{3,\mu} \text{ are } \textit{unchanged}, \tag{5.2}$$

$$D_{1,\mu} \rightarrow D_{1,\mu} + \textit{more complicated} \text{ stuff}. \tag{5.3}$$

This means that under the full Lorentz transformation of both coordinates and wave function D_2 and D_3 behave simply as four-vectors. The quantity D_1, however, will be shown in the Appendix to be part of a higher rank tensor.

Before proceeding, we note that $D_{\mu=0}$ can be reexpressed by using the wave equation (1.1):

$$D_{\mu=0} = -i\tau - \frac{j}{2}\boldsymbol{\nabla} \cdot \mathbf{U}_3 + k\left[m\rho + \frac{1}{2}\boldsymbol{\nabla} \cdot \mathbf{U}_2\right], \tag{5.4}$$

where $\tau \equiv \psi^* \mathbf{u} \cdot \mathbf{d}\psi$ is a real three-scalar. Under the Lorentz transformation of the wave function, we calculate $\tau \rightarrow \tau + \mathbf{v} \cdot \mathbf{D}_1$.

We have the identity

$$\partial^\mu D_\mu = 0; \tag{5.5}$$

and we will be interested in the following time derivatives, which are derived by using the wave equation (1.1):

$$\frac{\partial}{\partial t}\tau = -\boldsymbol{\nabla} \cdot \mathbf{D}_1, \tag{5.6}$$

$$\frac{\partial}{\partial t}D_\mu = i[2mD_{2,\mu} - \boldsymbol{\nabla} \cdot (\psi^* \mathbf{u}\mathrm{d}_\mu \psi)] - 2mjD_{1,\mu} + [i, \psi^* \mathrm{d}_\mu \mathbf{u} \cdot \mathbf{d}\psi], \tag{5.7}$$

$$\frac{\partial}{\partial t}\mathbf{U} = i[\boldsymbol{\nabla}\rho + 2\psi^* \mathbf{u} \times \mathbf{d}\psi + 2m\mathbf{U}_2] + j[-2m\mathbf{U}_1 + 2\mathbf{D}_3 - \boldsymbol{\nabla} \times \mathbf{U}_3] + k[-2\mathbf{D}_2 + \boldsymbol{\nabla} \times \mathbf{U}_2]. \tag{5.8}$$

See the Appendix for a more systematic discussion of tensor quantities.

VI. PLANE WAVES

One way of representing "plane-wave" solutions of the wave equation (1.1) is

$$\psi(\mathbf{x},t) = \exp(\eta \mathbf{u}\cdot \hat{p}\mathbf{p}\cdot\mathbf{x})\phi \exp((i\eta p + km)t), \tag{6.1}$$

where $\eta = \pm 1$. The set of possible momentum vectors $\mathbf{p} = \hat{p}p$ should cover only one-half of space to avoid overcounting of solutions. With this, one can construct the solution for the general initial value problem:

$$\psi(\mathbf{x},t) = \int d^3\mathbf{x}' \sum_\eta \int_H \frac{d^3p}{(2\pi)^3} \exp(\eta\mathbf{u}\cdot\hat{p}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}'))\psi(\mathbf{x}',t'=0)\exp((i\eta p + km)t), \tag{6.2}$$

where the subscript H reminds us that the integral covers only half of momentum space.

With the expansions

$$\exp(\eta\mathbf{u}\cdot\hat{p}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')) = \cos(\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')) + \eta\mathbf{u}\cdot\hat{p}\sin(\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')), \tag{6.3}$$

$$\exp((i\eta p + km)t) = \cos(\omega t) + (i\eta p + km)\sin(\omega t)/\omega, \tag{6.4}$$

where $\omega = \sqrt{p^2+m^2}$, we sum over η and reduce Eq. (6.2) to the following:

$$\psi(\mathbf{x},t) = \int d^3\mathbf{x}' \int_H \frac{d^3p}{(2\pi)^3} 2[\cos(\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}'))\psi(\mathbf{x}',0)(\cos(\omega t) + km\sin(\omega t)/\omega)$$
$$+ \mathbf{u}\cdot\mathbf{p}\sin(\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}'))\psi(\mathbf{x}',0)i\sin(\omega t)/\omega]. \tag{6.5}$$

Here we can recognize that the results of the integrals over $\mathbf{p}$ (which now may be extended to cover the full momentum space) give us functions of the invariant $R^2 = t^2 - (\mathbf{x}-\mathbf{x}')^2$, which vanish outside the light cone ($R^2 > 0$). Thus we do have relativistic causality for this quaternionic wave equation; something which we could have expected because the solutions satisfy the Klein-Gordon equation.

The Klein-Gordon equation also has the property that positive (negative) frequency solutions propagate only to positive (negative) frequency solutions. For the quaternionic equation, we have no way to talk about this distinction between positive and negative frequencies; however, we do find a substitute "selection rule" for wave propagation here.

First, we note the orthogonality relation

$$\int \frac{d^3x}{(2\pi)^3} \exp(-\eta'\mathbf{u}\cdot\hat{p}'\mathbf{p}'\cdot\mathbf{x})\exp(\eta\mathbf{u}\cdot\hat{p}\mathbf{p}\cdot\mathbf{x}) = \delta_{\eta,\eta'}\delta(\mathbf{p}\pm\mathbf{p}'), \tag{6.6}$$

where I have not required that both sets of momentum variables belong to the same half-space. Next, we use this orthogonality in Eq. (6.2), where we represent $\psi(\mathbf{x}',0)$ as any superposition of plane wave solutions with exclusively $\eta' = +1$ (or exclusively -1). The resulting $\psi(\mathbf{x},t)$ will contain only that same value for η. It is tempting to call this "helicity conservation" in the propagation of these quaternionic waves.

This interpretation is bolstered by the following observations. The operator $\mathbf{u}\cdot\nabla$, acting on a plane wave solution [Eq. (6.1)], has eigenvalue $-\eta p$. Furthermore, one can readily show, from Eq. (1.1), that

$$\frac{d}{dt}\int d^3x\,\psi^*\mathbf{u}\cdot\nabla\psi = 0. \tag{6.7}$$

VII. SUPERPOSITION

In the usual (complex) quantum theory, if we have two solutions to the Schrodinger (time dependent) equation, ψ_1 and ψ_2, then any linear combinmation $c_1\psi_1+c_2\psi_2$ is also a solution for arbitrary complex numbers c_1 and c_2. With our quaternionic wave equation (1.1), the idea of superposition requires a slightly different wording.

Note that the general plane wave solution [Eq. (6.1)] has an arbitrary amplitude ϕ positioned in the midst of certain quaternionic functions of space and time. Given any such solution, we find another solution by changing the amplitude: $\phi \rightarrow q\phi q'$, where q and q' are arbitrary quaternionic numbers. Furthermore, if we have one solution of Eq. (1.1)—ψ_1 with amplitude ϕ_1—and another solution—ψ_2 with amplitude ϕ_2—then we also have a solution by simply adding these two: $\psi_1+\psi_2$. This version of the principle of superposition is implicit in Eq. (6.2).

VIII. ADDING POTENTIALS

The original wave equation (1.1) can be extended by the introduction of external potentials, as follows:

$$\frac{\partial\psi}{\partial t}i = \mathbf{u}\cdot\nabla\psi + \mathrm{e}\varphi\psi - \mathrm{e}\mathbf{u}\cdot\mathbf{A}\psi i + m\psi \mathrm{e}^{ieW}j, \tag{8.1}$$

where φ, $\mathbf{A}$, W are real functions of space-time. The gauge transformation that leaves this equation invariant is

$$\psi \rightarrow \psi \mathrm{e}^{ie\chi}, \tag{8.2}$$

$$\varphi \rightarrow \varphi - \frac{\partial\chi}{\partial t}, \tag{8.3}$$

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi, \tag{8.4}$$

$$W \rightarrow W - 2\chi. \tag{8.5}$$

One can show that the previously discussed symmetries still hold, with $(\varphi,\mathbf{A})$ a Lorentz four-vector and W a scalar. This appearance of the four-vector potentials is (almost) exactly like the usual way of introducing electromagnetism into quantum theory; however, the explicit appearance of a gauge quantity W is something different.

The reflection symmetries of Eq. (8.1) are

$$\psi \rightarrow \psi j, \quad t,\mathbf{A},W \textit{ change} \text{ sign } (T), \tag{8.6}$$

$$\psi \rightarrow \psi k, \quad \mathbf{x},\varphi,W \textit{ change} \text{ sign } (CP), \tag{8.7}$$

$$\psi \rightarrow \psi i, \quad t,\mathbf{x},\mathbf{A},\varphi \textit{ change} \text{ sign } (TCP). \tag{8.8}$$

The current conservation equation (3.2) is still true for this extended wave equation (8.1), however, Eq. (6.7) must be modified. For the situation where $W=0$, we calculate

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\psi^*\mathbf{u}\cdot\mathbf{d}\psi) = -\nabla\cdot\mathbf{D}_1 + \mathrm{e}\rho\nabla\cdot\mathbf{A} - 2\mathrm{e}\mathbf{A}\cdot(\psi^*\mathbf{u}\times\mathbf{d}\psi), \tag{8.9}$$

where we have used the notations from Sec. V. From this we see that in the case where the only external potential is φ, then the space integral of τ, which we identified with helicity, is conserved.

IX. MORE ON PLANE WAVES

The plane wave solutions to the wave equation (1.1), which we set out in Sec. VI, contain an amplitude ϕ which we should study some more:

$$\psi(\mathbf{x},t) = \exp(\eta\mathbf{u}\cdot\hat{p}\mathbf{p}\cdot\mathbf{x})\phi\exp(\Omega t), \tag{9.1}$$

where $\Omega=\hat{\Omega}\omega=(i\eta p+km)$, $\omega=\sqrt{p^2+m^2}$.

We can ask to evaluate the various bilinear forms discussed earlier in the case of this plane wave solution. The easiest are

$$\rho=\phi^*\phi, \quad \tau=-\eta p\rho, \quad D_0=\Omega\rho, \tag{9.2}$$

but to do more we must be able to evaluate $\phi^*\mathbf{u}\cdot\hat{p}\phi$.

I now propose to classify the constants ϕ in a particular way. The set ϕ_α is defined such that it performs a specific rotation, as follows:

$$\hat{p}\cdot\mathbf{u}\phi_\alpha=\phi_\alpha\hat{\Omega}, \tag{9.3}$$

which sends one unit imaginary quaternion into another. With this type, the solution can be written as

$$\psi=\phi_\alpha\exp(\hat{\Omega}(\omega t+\eta\mathbf{p}\cdot\mathbf{x})), \tag{9.4}$$

which looks like the sort of plane waves we are used to. It should be noted that this definition of ϕ_α is not unique but leaves us with a $U(1)$ class of equivalent amplitudes,

$$\phi_\alpha\to\phi_\alpha\exp(\theta\hat{\Omega}), \tag{9.5}$$

just as in ordinary (complex) quantum theory.

With this α type of amplitude, we can now evaluate the plane wave values for the following bilinears:

$$\mathbf{U}\cdot\hat{p}=i\rho\eta\frac{p}{\omega}+k\rho\frac{m}{\omega}, \quad \mathbf{D}=\eta\mathbf{p}(\mathbf{U}\cdot\hat{p}). \tag{9.6}$$

Components of the vector $\mathbf{U}$ which are orthogonal to $\mathbf{p}$ will oscillate rapidly in space, thus any space average of them will be vanishingly small.

Two other categories for the amplitudes ϕ, called β and γ, can be defined as

$$\mathbf{u}\cdot\hat{p}\phi_\beta=\phi_\beta j, \tag{9.7}$$

$$\mathbf{u}\cdot\hat{p}\phi_\gamma=\phi_\gamma j\hat{\Omega}. \tag{9.8}$$

Note that the three numbers $\hat{\Omega}, j, j\hat{\Omega}$ are mutually anticommuting quaternions. If we calculate any of the bilinears involving $\mathbf{u}\cdot\hat{p}$ with either the β or γ type of amplitude, the result will be rapidly oscillating in time, thus any time average will be vanishingly small.

If we stay with the α type amplitudes, we get the following values, in the plane wave states, for various four-vectors that are defined in Sec. V or the Appendix:

$$j_\mu=\rho(1,\eta\mathbf{p}/\omega), \tag{9.9}$$

$$V_\mu=\rho(\eta p,\omega\hat{p}), \tag{9.10}$$

$$D_{2,\mu}=0, \tag{9.11}$$

$$D_{3,\mu} = \rho m(1, \eta \mathbf{p}/\omega). \tag{9.12}$$

The four-vectors j and D_3 look like what we would expect for the usual energy-momentum. The four-vector V, however, is spacelike, not timelike; it is similar to the spin vector $s_\mu = \epsilon_{\mu,\nu,\kappa,\lambda} P^\nu S^{\kappa,\lambda}$ in the usual theories, where s_0 is the helicity.

The plane wave solutions are characterized by a parameter $\mathbf{p}$ which we sometimes call "momentum." This is merely a linguistic habit carried over from conventional quantum theory (following de Broglie's rule that momentum equals Planck's constant divided by wavelength) and should not be confused with the physical quantity called momentum until and unless that connection is established.

X. SCATTERING

Lets start with the wave equation plus a source,

$$\frac{\partial}{\partial t}\psi i = \mathbf{u}\cdot\nabla\psi + m\psi j + s(\mathbf{x},t)i, \tag{10.1}$$

and write the retarded solution as

$$\psi(\mathbf{x},t) = \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3x' \int_H \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_\eta \exp(\eta\mathbf{u}\cdot\hat{p}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}'))s(\mathbf{x}',t')\exp(\Omega(t-t')). \tag{10.2}$$

For the general scattering problem, we replace the source s with $V\psi$ and add in the initial (free particle) solution $\psi_0(\mathbf{x},t)$. If the interaction V is independent of time, then we have an integral equation,

$$\psi(\mathbf{x},t) = \psi_0(\mathbf{x},t) + \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3x' \int_H \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_\eta \exp(\eta\mathbf{u}\cdot\hat{p}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')) \times V(\mathbf{x}')\psi(\mathbf{x}',t')\exp(\Omega(t-t')). \tag{10.3}$$

Now we make the "Born approximation" that $\psi = \psi_0$ under the integral and let the time t go to $+\infty$. Then, we find that the integral over t' gives us $\delta(\omega-\omega_0)$, which is usually read as conservation of energy. This result appears to be generally true, not just in the first Born approximation. One can now project this solution onto any plane wave solution and achieve the quaternionic version of the S matrix.

In the special case when the scattering potential V comes from the term φ in the extended wave equation (8.1), we also find—as a result of the integral over t'—that we have the selection rule $\eta = \eta_0$. This is consistent with the result noted after Eq. (8.9).

XI. SOME OTHER SOLUTIONS

We can write solutions for the extended wave equation (8.1) in some special cases.

One may ask whether there is a central potential, $\varphi(r)$, which leads to bound states. The easiest way to explore this is through "reverse engineering:" write down a plausible wave function and see what potential fits the wave equation. The form

$$\psi = [f(r) + \mathbf{u}\cdot\hat{r}g(r)]\phi(t) \tag{11.1}$$

leads to the requirements

$$r^4\frac{\mathrm{d}}{\mathrm{d}r}f^2 = -\frac{\mathrm{d}}{\mathrm{d}r}(r^4g^2), \quad \mathrm{e}\varphi = -f'/g, \quad \phi(t) = \phi_0\mathrm{e}^{kmt}. \tag{11.2}$$

If we try the asymptotic $(r\to\infty)$ behavior, $g\to\alpha r^{-\beta}$, we find a similar behavior for f, provided that $0<\beta<2$. The wave function is then normalizable for $\beta>1.5$, and the potential is $\mathrm{e}\varphi(r) = \sqrt{\beta(2-\beta)}/r$ at large r. Looking instead at $r\to 0$, one can do the same analysis and require $\beta < 1.5$; this suggests that we are dealing with something like a shielded Coulomb potential.

There are familiar procedures for taking the nonrelativistic limit of the Klein-Gordon or Dirac equation. Here is the best I could do with the present relativistic equation. First, write $\psi = \psi_{nr}\exp(k\omega t)$, where $\omega=\sqrt{m^2+p^2}\approx m-\boldsymbol{\nabla}^2/2m$. Next, multiply the equation from the right with $i\exp(-k\omega t)$. Finally, drop all terms that oscillate rapidly in time, as $\exp(\pm 2k\omega t)$. The resulting version of the full extended Eq. (8.1) is

$$\frac{\partial\psi_{nr}}{\partial t}k \approx H_{nr}\psi_{nr}, \tag{11.3}$$

$$H_{nr} = -\nabla^2/2m + m(1-\cos(\mathrm{e}W)) - \mathrm{e}\mathbf{u}\cdot\mathbf{A}\|k, \tag{11.4}$$

which looks like an ordinary Schrodinger equation except that the single imaginary is called k instead of i, and there is also the unfamiliar term with $\mathbf{A}$. What looks like an effective potential energy term (coming from the gauge quantity W) is positive, thus incapable of producing bound states, although it might conceivably yield metastable states through the oscillation of the cosine function.

XII. TWO-PARTICLE EQUATION

Previous studies of quaternionic quantum theory have gotten into trouble when they try to write wave functions for multiparticle systems. In the ordinary (complex) theory, one simply makes a direct product of one-particle wave functions, and because all the numbers there commute, one can manipulate such a product to achieve various sensible results. In the quaternion case, that approach leads to a horrid mess. (See, for example, Ref. 2, Chap. 9.)

The present work suggests a somewhat different approach. Consider this construction with plane waves:

$$\psi(1,2) = \exp(\eta_1\mathbf{u}\cdot\hat{p}_1\mathbf{p}_1\cdot\mathbf{x}_1)\psi(2)\exp(\Omega_1 t_1), \tag{12.1}$$

$$\psi(2) = \exp(\eta_2\mathbf{u}\cdot\hat{p}_2\mathbf{p}_2\cdot\mathbf{x}_2)\phi\exp(\Omega_2 t_2), \tag{12.2}$$

which might be described as a "nested" product. The symbol ϕ here is a quaternionic constant, which can depend on all the parameters of this two-particle wave function. Note that we have written this with independent time variables for the two particles.

These two-particle wave functions, with all their momentum-helicity labels, form a complete orthogonal set of functions in the space of $\mathbf{x}_1$ and $\mathbf{x}_2$. Note, however, that this product is ordered in a way that was meaningless in ordinary (complex) quantum theory but requires some extra bookkeeping in the quaternionic case.

Let us introduce some more compact notation for such wave functions:

$$\psi^{\mathrm{op}}(1) \equiv \exp(\eta_1\mathbf{u}\cdot\hat{p}_1\mathbf{p}_1\cdot\mathbf{x}_1)\|\exp(\Omega_1 t_1)\ , \tag{12.3}$$

where the $\|$ symbol separates those things that are to act on the left from what is to act on the right of whatever follows. Then the two-particle wave function Eq. (12.1) can be written simply as

$$\psi(1,2) = \psi^{\mathrm{op}}(1)\psi^{\mathrm{op}}(2)\phi; \tag{12.4}$$

and we can also write the operator of the wave equation as

$$\mathcal{D} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \| i - m \| k. \tag{12.5}$$

Next we have the propagators,

$$G(\mathbf{x}-\mathbf{x}', t-t') \equiv \sum_{p,\eta} \exp(\eta \mathbf{u} \cdot \hat{p}\mathbf{p} \cdot (\mathbf{x}-\mathbf{x}')) \| \exp((i\eta p + km)(t-t')) \ , \tag{12.6}$$

where $\Sigma_{p,\eta} = \int_H [\mathrm{d}^3 p/(2\pi)^3] \Sigma_\eta$, and this leads to

$$\mathcal{D} G(\mathbf{x}-\mathbf{x}', t-t') = 0, \quad G(\mathbf{x}-\mathbf{x}', 0) = \delta^3(\mathbf{x}-\mathbf{x}'), \tag{12.7}$$

$$G_+(x-x') \equiv \theta(t-t') G(\mathbf{x}-\mathbf{x}', t-t'), \tag{12.8}$$

$$\mathcal{D} G_+(x-x') = \delta^4(x-x'). \tag{12.9}$$

The coordinate x stands for the full space-time coordinates $t, \mathbf{x}$. Now Eq. (10.2) can be briefly written as

$$\psi(x) = \int \mathrm{d}^4 x' G_+(x-x') s(x'). \tag{12.10}$$

Following that construction, we now write down a general two-particle quaternionic wave function as follows:

$$\psi(x_1, x_2) = \int \mathrm{d}^4 x_1' G_+(x_1 - x_1') \int \mathrm{d}^4 x_2' G_+(x_2 - x_2') s(x_1', x_2'). \tag{12.11}$$

Acting on this with two of those differential operators gives

$$\mathcal{D}_2 \mathcal{D}_1 \psi(x_1, x_2) = s(x_1, x_2). \tag{12.12}$$

This is a two-particle wave equation of the Bethe-Salpeter type, involving separate times as well as separate space coordinates. The term s might be left as an external source or might be used to represent some interaction, such as $V(1,2)\psi(1,2)$. Note that the order in which the two differential operators are applied is significant.

It seems easy now to extend this to any number of particles. This appears to be a significant advance over previous studies of quaternionic wave equations, although there are still many issues to be faced.

XIII. NO LAGRANGIAN

If I use the interacting wave equation (8.1), and think that ψ^* is something independent of ψ, then the following would be suggested as a Lagrangian density:

$$\mathcal{L} = i\psi^* \frac{\partial \psi}{\partial t} i - i\psi^* \mathbf{u} \cdot \boldsymbol{\nabla} \psi - i\psi^* \mathrm{e}\varphi\psi + i\psi^* \mathrm{e}\mathbf{u} \cdot \mathbf{A}\psi i - im\psi^* \psi \mathrm{e}^{ieW} j. \tag{13.1}$$

Varying $i\psi^*$ gives immediately the full wave equation for ψ. Before varying ψi on the right, we do a few things: partially integrate in space and time; and move i from left side to right side in the second and third terms and rearrange the i and j coefficients in the last term (this is justified because those $\psi^* \cdots \psi$ expressions are real). Then we get the adjoint wave equation.

But that prescription is not what the usual action principle allows. The familiar game from complex qm does not work here. If one varies each of the four real functions which make up both quaternionic functions ψ and ψ^*, then we actually get 12 equations from the action principle. This is due to the fact that this Lagrangian is imaginary, that is, it consists of three imaginary parts and each of those parts must vanish after the variation. If we write

$$\mathcal{L} = i\mathcal{L}_1 + j\mathcal{L}_2 + k\mathcal{L}_3, \tag{13.2}$$

we find that the first term, $\mathcal{L}_1$, is Lorentz invariant [see Eq. (A9)]; but what should we do with the other two terms?

Our difficulty with a Lagrangian is different from the difficulty noted earlier for the Dirac-Majorana equation or for the pseudo-Weyl equation. But we do have a problem here.

XIV. DISCUSSION

Several advances have been made in trying to develop a sensible quantum theory based on quaternions, rather than complex numbers. So far, this work has been limited to the wave equation formalism.

We have noted the lack of a conserved energy-momentum tensor (see the Appendix) as well as the lack of a Lagrangian. Nevertheless, we can write down the time-development operator as

$$U(t) = \mathrm{e}^{Ht}, \quad H = -\mathbf{u}\cdot\nabla \| i + m \| k. \tag{14.1}$$

This operator H commutes with the angular momentum operator $\mathbf{J}$, but whether we want to call it the Hamiltonian is unclear. Perhaps these questions wait for a full model of how this quaternionic wave system interacts with other physical systems.

Another approach that may be relevant to that problem, as well as to improving our treatment of many-particle systems, is the method of second quantization. We are led to write down a quaternionic quantum field operator as

$$\psi(\mathbf{x},t) = \sum_{p,\eta} \exp(\eta\mathbf{u}\cdot\hat{p}\mathbf{p}\cdot\mathbf{x})a_{p,\eta}\exp((i\eta p + km)t) \tag{14.2}$$

involving some kind of annihilation/creation operators $a_{p,\eta}$. With this we immediately get

$$N = \frac{1}{(2\pi)^3}\int \mathrm{d}^3x\rho = \sum_{p,\eta} a^\dagger_{p,\eta}a_{p,\eta}, \tag{14.3}$$

$$h = \frac{1}{(2\pi)^3}\int \mathrm{d}^3x\tau = -\sum_{p,\eta} a^\dagger_{p,\eta}a_{p,\eta}\eta p. \tag{14.4}$$

Can one be sure that the matrix product $a^\dagger a$ is real? If these are matrices in a Fock space of the sort we are familiar with, with nonzero elements only on one line parallel to the central diagonal, then this product is real.

It remains unclear to this author whether the equation studied in this paper is merely an alternative mathematical formulation of things already well known or whether it may have consequential applications to some as-yet unidentified physics.

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APPENDIX: GENERAL TENSORS

We can construct Lorentz covariant tensors of any rank, as follows. Start with the direct product of the "two-way" derivative operators:

$$\mathrm{d}_{\mu}^{n} = \mathrm{d}_{\mu_1}\mathrm{d}_{\mu_2}\cdots\mathrm{d}_{\mu_n}, \tag{A1}$$

where the subscript μ now stands for the set of indices $\mu_1\cdots\mu_n$. This expression is manifestly a covariant tensor of rank n as far as the coordinate transformations are concerned, and our task is to package these between the wave functions, which transform under infinitesimal Lorentz transformations as given in Eq. (4.2).

We note that the packages $\psi^*\mathrm{d}_\mu^n\psi$ are real for n even and imaginary for n odd, while these conditions are reversed when we add the quaternions $\mathbf{u}$ inside the package.

We find the following constructions for covariant tensors of rank $n+1$, $Q_{\mu,\nu}^{(n+1)}=Q_{\mu,\nu}=(Q_{\mu,0},\mathbf{Q}_\mu)$:

$$Q_{\mu,\nu} = \left(\psi^*\mathrm{d}_\mu^n\psi, \left\{\frac{-i}{2}, \psi^*\mathrm{d}_\mu^n\mathbf{u}\psi\right\}\right) \quad \text{for } n \text{ even}, \tag{A2}$$

$$Q_{\mu,\nu} = \left(\left\{\frac{-i}{2}, \psi^*\mathrm{d}_\mu^n\psi\right\}, -\psi^*\mathrm{d}_\mu^n\mathbf{u}\psi\right) \quad \text{for } n \text{ odd}. \tag{A3}$$

In the case $n=0$ this is just the four-vector current, previously written as $j_\mu=(\rho,\mathbf{U}_1)$. All these tensors are real.

In addition, for n odd, we have the tensors of rank n,

$$R_{2,\mu} = \left\{\frac{-j}{2}, \psi^*\mathrm{d}_\mu^n\psi\right\}, \quad R_{3,\mu} = \left\{\frac{-k}{2}, \psi^*\mathrm{d}_\mu^n\psi\right\}, \tag{A4}$$

which generalize the previously noted four-vectors D_2, D_3. For n even, we have the tensors of rank $n+2$, $S_{\mu,\nu,\lambda}=-S_{\mu,\lambda.\nu}$:

$$S_{\mu,0,\alpha} = -S_{\mu,\alpha,0} = \left\{\frac{-j}{2}, \psi^*\mathrm{d}_\mu^n\mathbf{u}_\alpha\psi\right\}, \quad S_{\mu,\alpha,\beta} = \epsilon_{\alpha,\beta,\gamma}\left\{\frac{-k}{2}, \psi^*\mathrm{d}_\mu^n\mathbf{u}_\gamma\psi\right\}, \tag{A5}$$

where $\alpha,\beta,\gamma=1,2,3$. This generalizes the previously noted six-vector $(\mathbf{U}_2,\mathbf{U}_3)$.

We can make lower rank tensors by contracting indices:

$$g^{\mu_1,\mu_2}Q_{\mu,\nu}^{(n+1)} = -\left(m^2 + \frac{1}{4}\partial^\lambda\partial_\lambda\right)Q_{\mu',\nu}^{(n-1)}, \tag{A6}$$

where μ_1 and μ_2 are in the set μ, and the set μ' has these two indices removed. An alternative is to contract one of the μ indices with the ν index. We find, for solutions of the free wave equation (1.1), the following:

$$g^{\mu_1,\nu}Q_{\mu,\nu}^{(n+1)} = -mR_{3,\mu'}^{(n-1)} \quad \text{for } n \text{ even}, \tag{A7}$$

$$g^{\mu_1,\nu}Q_{\mu,\nu}^{(n+1)} = 0 \quad \text{for } n \text{ odd}. \tag{A8}$$

If we are looking to find a Lorentz scalar, a tensor of rank zero, take a closer look at the second rank tensor $Q_{\mu,\nu}$. The contraction is

$$Q_\mu^\mu = \left\{\frac{-i}{2}, \psi^*\mathrm{d}_0\psi\right\} + \psi^*\mathbf{u}\cdot\mathbf{d}\psi = D_{1,0} + \tau \tag{A9}$$

and this is exactly zero for the free equation (1.1) but not for the extended Eq. (8.1), where it equals $-ej_\mu A^\mu$.

Now we look at the contraction of such tensors with the derivative operator. In what follows we shall limit ourselves to solutions of the free equation (1.1). It is transparent that $\partial^{\mu_1}Q_{\mu,\nu}=0$ for any μ_1 in the set of labels μ. The same holds true for the tensors R and S. Furthermore, by using the wave equation, one can show that

$$\partial^{\nu}Q_{\mu,\nu}=0 \quad \text{for } n \text{ even}, \tag{A10}$$

$$\partial^{\nu}Q_{\mu,\nu}=2mR^{(n)}_{2,\mu} \quad \text{for } n \text{ odd}. \tag{A11}$$

For tensors of rank 1, we have just the previously identified j_μ, $D_{2,\mu}$, and $D_{3,\mu}$, all of which are conserved.

At rank 2, the usual desire is for a conserved symmetric tensor, which one can call the energy-momentum tensor. The closest we come here is the $Q_{\mu,\nu}$, which is not symmetric and is conserved only on the first index. Nevertheless, this does allow us to write integral quantities which are conserved (their time derivatives vanish), as follows:

$$V_\nu=(V_0,\mathbf{V})\equiv\int \mathrm{d}^3x Q^{(2)}_{0,\nu}, \tag{A12}$$

$$V_0=-\int \mathrm{d}^3x\psi^*\mathbf{u}\cdot\mathbf{d}\psi, \tag{A13}$$

$$\mathbf{V}=\int \mathrm{d}^3x\left(\mathbf{D}_1+m\mathbf{U}_3-\frac{1}{2}\nabla\times\mathbf{U}_1\right). \tag{A14}$$

This is not what we would identify as the energy-momentum, as noted at the end of Sec. IX.

For the second rank antisymmetric tensor we have

$$\partial^{\mu}S_{\mu,\nu}=-2mj_\nu+2D_{3,\nu}, \tag{A15}$$

$$\widetilde{S}_{\mu,\nu}=\epsilon_{\mu,\nu,\kappa,\lambda}S^{\kappa,\lambda}/2, \tag{A16}$$

$$\partial^{\mu}\widetilde{S}_{\mu,\nu}=-2D_{2,\nu}. \tag{A17}$$

[1] G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936).
[2] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford University Press, New York, 1995).
[3] *The Mathematical Papers of Sir William Rowan Hamilton*, edited by H. Halberstam and R. E. Ingram (Cambridge University Press, Cambridge, 1967), Vol. III, p. 432.
[4] S. De Leo and P. Rotelli, Mod. Phys. Lett. A **11**, 357 (1996).
[5] C. Lanczos, Z. Phys. **57**, 447 (1929), e-print physics/0508002.
[6] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University Press, Princeton, 1992).