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Studies of Auction Bidding with Budget-Constrained Participants

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Studies of Auction Bidding with Budget-Constrained Participants

By

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Abstract

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Consider a first-price, sealed-bid auction where participants have affiliated valuations and private budget constraints; that is, bidders have private multidimensional types. We offer sufficient conditions for the existence of a monotone equilibrium in this environment along with an equilibrium characterization. Hard budget constraints introduce two competing effects on bidding. The direct effect depresses bids as participants hit their spending limit. The strategic effect encourages more aggressive bidding by participants with large budgets. Together these effects can yield discontinuous equilibrium strategies stratifying competition along the budget dimension. The strategic consequences of private budget constraints can be a serious confound in interpreting bidding behavior in auctions. Evidence from an experimental auction market lends support to the strategic importance of budget constraints.

To my parents.

Contents

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Chapter 1

Introduction

Budget constraints are a fundamental feature of many auctions. For example, bidders may be buying real estate. Valuations for a house are partly idiosyncratic and all potential buyers face a private spending limit determined by a financial institution. Problems securing a large loan may prevent a buyer from placing a competitive bid on a desirable property.

Once bidders have multidimensional private information composed of a *valuation*, which determines their preferences, and a *budget*, which defines their feasible strategy set, even simple auctions feature nontrivial equilibrium behavior which has hitherto been poorly understood. Understanding equilibrium behavior is a critical first step in gauging a mechanism's economic properties, such as allocative efficiency and revenue potential.

For concreteness, consider a first-price, sealed-bid auction for one item. Bidders simultaneously submit their best offer to the seller and the highest bidder wins. Only the winner pays her own bid. In this straightforward setting, private budgets introduce two effects on equilibrium bidding. Budgets have a direct effect constraining bidders; however, they also introduce a strategic effect whereby high-budget bidders deliberately outbid high-valuation opponents who have the misfortune of being budget constrained. The strategic effect emerges as a discontinuity in the equilibrium strategy and it endogenously stratifies competition within the auction along the budget dimension. These effects profoundly change the strategic interaction of the first-price auction with consequences for its efficiency, revenue potential, and even bidders' perception of the winner's curse.

Once we recognize the direct and the strategic effects of private budgets, naive inference concerning a first-price auction's equilibrium becomes per-

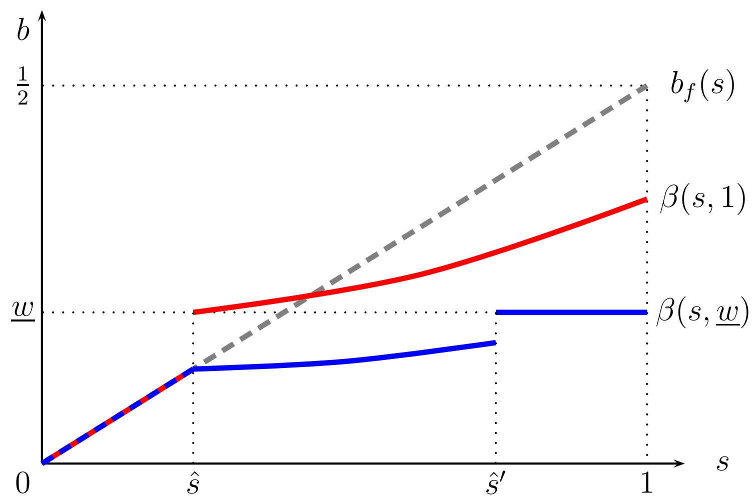


Figure 1.1: Equilibrium strategy in Example 1. High-budget bidders increase their bid discontinuously to $\underline{w}$ at $\hat{s}$. Low-budget bidders are restricted to bids below $\underline{w}$. $b_f(s)$ is the equilibrium strategy absent private budget constraints.

ilous. Although we show that an equilibrium in monotone strategies exists under general circumstances, we need to amend our intuition regarding equilibrium strategies accounting for the new dimension of competition. Example 1, which extends a textbook model,[1] serves to introduce the argument. As will be seen below, its intuition applies generally.

Example 1. Consider a first-price, sealed-bid auction for one item. There are two risk-neutral bidders and let $S_i \overset{i.i.d.}{\sim} U[0,1]$ be player i's private value for the item. Ties are resolved with a fair coin flip. If a bidder with value $S_i = s_i$ wins by bidding b_i, her utility is $s_i - b_i$; otherwise, it is zero. It is well known that the symmetric equilibrium[2] bidding strategy is $b_f(s_i) = s_i/2$.

Suppose, additionally, that bidders have a private budget of $w_i \in \{\underline{w}, 1\}$. A bidder of type $\theta_i = (s_i, w_i)$ cannot bid above w_i. Budgets are distributed independently such that $\Pr[W_i = \underline{w}] \equiv p \in (0,1)$. When $\underline{w} < 1/2$, $b_f(s_i) = s_i/2$ can no longer be an equilibrium. Instead, suppressing subscripts, the

[1] See Vickrey (1961); Krishna (2002); Milgrom (2004); Menezes & Monteiro (2005).
[2] Bayesian Nash equilibrium is the solution concept throughout.

symmetric equilibrium strategy is

$$\beta(s,w)=\begin{cases}\frac{s}{2} & \text{if } s\in[0,\hat{s}] \text{ and } w\in\{\underline{w},1\}\\ \frac{s^2(p-1)-2k_1}{2p(s-1)-2s} & \text{if } s\in(\hat{s},1] \text{ and } w=1\\ \frac{ps^2+2k_2}{2\hat{s}-2p\hat{s}+2sp} & \text{if } s\in(\hat{s},\hat{s}'] \text{ and } w=\underline{w}\\ \underline{w} & \text{if } s\in(\hat{s}',1] \text{ and } w=\underline{w}\end{cases}. \tag{1.1}$$

The constants $\{\hat{s},\hat{s}',k_1,k_2\}$ are defined in Appendix A.1. Figure 4.1 presents a sketch of $\beta(s,w)$ along with $b_f(s)$.[3]

Example 1 conveys two main points concerning equilibrium behavior. First, the equilibrium strategy features a prominent discontinuity at $\hat{s}$. The intuition for this discontinuity rests on the strategic tradeoff made by unconstrained bidders ($w_i=1$) in equilibrium. An unconstrained bidder always has the option of placing a bid above $\underline{w}$. Exercising this option substantially increases the probability of a win as she is guaranteed to outbid all constrained bidders. When a bidder's valuation is $\hat{s}$, the discontinuous increase in bid becomes worthwhile. With private budget constraints, some bidders therefore bid more aggressively than if budget constraints were absent. The intuition for the discontinuity at $\hat{s}'$ is similar; however, such bidders may tie with others rather than win outright.

Second, although the discontinuities imply higher bids by some types of bidders, on the margin bids are often less brazen than if budget constraints were absent. In Figure 4.1, the slope of $\beta(s,w)$ declines moving past $\hat{s}$. The intuition here hinges on the endogenous stratification of competition along the budget dimension. For instance, bidders placing a bid above $\underline{w}$ are competing on the margin only against other unconstrained bidders rather than all bidders. The precipitous decline in competition reduces the incentive to bid aggressively at distinct budget levels. A similar intuition holds for constrained bidders with value $s\in(\hat{s},\hat{s}']$. Only if a high-budget bidder is exceptionally pessimistic regarding the item's value does a low-budget bidder stand a chance of winning the auction.[4]

[3] Appendix A.1 confirms that $\beta(s,w)$ is an equilibrium strategy and generalizes this example to $S_i \overset{i.i.d.}{\sim} H$ and N bidders. The equilibrium's main features are robust to this extension. Some comparative statics for $\hat{s}$ are also presented.

[4] A corollary is that expected revenues in the auction decline for all $p>0$ and $\underline{w}<1/2$ (Lemma 9 in Appendix A.1). This conclusion is not immediate due to the non-monotonic adjustment of equilibrium strategies following the introduction of private budget constraints.

Although the above lessons appear to be specific to Example 1, or perhaps an artifact of the discrete budget distribution, they are a general phenomena that preceding research failed to identify.

Following a brief survey of preceding research, this study considers three complementary questions. All are crucial in delineating the strategic and economic implications of budget constraints in auctions. First, with private budget constraints does there exist a well-behaved—for instance, monotone—equilibrium? Employing a recent equilibrium existence result from Reny (2009), Chapter 2 offer sufficient conditions guaranteeing an equilibrium in monotone strategies. Apart from constraining a bidder's action, budget constraints focus attention on the economic tradeoffs in the bidding decision. A successful bidder wins the item but makes a payment from her budget while an unsuccessful bidder keeps her money to enjoy in some other manner. The trade-off between the two goods may be quite complex if bidders exhibit risk aversion or other preferences that are sensitive to the level of wealth, which is private information. Accommodating such effects requires a careful specification of a monotone strategy. Bidders with larger budgets tend to—but need not necessarily—bid more. Except for the private-values case, passing the equilibrium to a continuous action space depends on an endogenous tie-breaking rule, a common feature of auction models with multidimensional types.

Whereas the existence of a monotone equilibrium is reassuring, it is an empirically weak conclusion. Thus, Chapter 3 specialize to the general symmetric model with interdependent and affiliated valuations of Milgrom & Weber (1982) and we augment it with private budget constraints distributed on an interval. Fang & Parreiras (2002, 2003) analyze the second-price auction in this setting and it extends the analysis of Che & Gale (1998). Example 1's main features, such as discontinuous equilibrium strategies and endogenously stratified competition, can appear in this benchmark environment. For example, equilibrium bid distributions can be multimodal even when the underlying type-space is uniform and ex-post valuations have a unimodal distribution. The identified strategic effects seriously confound inference concerning model primitives as bidders react non-monotonically to the presence of private budgets. In addition to comparative static exercises, we sketch a simple test for the presence of private budget limits if only bid distributions are observable. One needs to vary the auction format holding the environment otherwise fixed.

Finally, in Chapter 4 we transplant the developed theory to a laboratory setting to document and study behavior in auctions with budget constraints. This exercise builds on the framework of Example 1 which is the simplest non-trivial setting where the interaction of private budget constraints and private valuations implies novel equilibrium responses. Although qualified by well-known empirical anomalies in auction experiments—such as overbidding relative to the risk-neutral Nash equilibrium prediction—the experimental evidence suggests that participants recognize the strategic implications of budget constraints in auctions and many key qualitative predictions of the equilibrium bidding model are evident in the data.

1.1 Literature

Researchers have recognized the possibility of budget constraints in auctions for over thirty years;[5] however, few studies treat both valuations and budgets as private information. Indeed, many studies examine budget constraints in multi-unit or sequential settings where a budget's strategic importance is transparent and where assumptions simplifying the information structure or the auction format are a practical necessity (Pitchik & Schotter, 1988; Benoît & Krishna, 2001; Pitchik, 2009; Brusco & Lopomo, 2008, 2009). Budgets feature in many internet-motivated applications, such as selling advertisements,[6] and accounts by Bulow *et al.* (2009), Salant (1997), and Cramton (1995) of wireless spectrum auctions give budget constraints a prominent role.

This study contends however that complex or multi-unit environments are not necessary to appreciate the strategic effects introduced by private budgets—single item settings are sufficient, although research here has been more limited. Early research by Che & Gale (1996a,b) considers first-price, second-price, and all-pay auctions where the sale item's common value is common knowledge but bidders have private budgets. In an important paper, Che & Gale (1998) discuss revenues in a class of auction mechanisms allowing for financial constraints; these results foreshadow Che & Gale (2006) which conducts revenue comparisons among auction formats when bidders have multidimensional independent types.

[5]Rothkopf (1977) and Palfrey (1980) are early discussions of multi-unit auctions with budgets.

[6]For example Ashlagi *et al.* (2010) propose a budget-accommodating modification of the generalized second-price auction used by internet search engines to sell advertisements. Edelman *et al.* (2007) and Varian (2009) provide an introduction to this auction format.

Fang & Parreiras (2002) were the first to characterize equilibrium bidding in the second-price auction with private budget constraints where valuations are affiliated and interdependent. Their environment forms the starting point of our analysis in Section 3. Kotowski (2010a) considers the same environment but analyzes the all-pay auction.

The optimal auction design problem in the spirit of Myerson (1981) incorporating budget constraints is particularly challenging and has been considered under various guises (Laffont & Robert, 1996; Monteiro & Page, 1998; Che & Gale, 1999, 2000; Maskin, 2000; Malakhov & Vohra, 2008). Pai & Vohra (2009) offer the most general description of optimal mechanisms. Designing multi-unit auctions accounting for budgets has also spurred a growing literature among computer scientists.[7]

While most models assume "hard" budget constraints, some authors have explored ex post default or asymmetries in bid financing ability (Zheng, 2001; Jaramillo, 2004; Rhodes-Kropf & Viswanathan, 2005).[8] While appropriate for specific applications, such models mask the strategic effects of budget constraints alone. Therefore, we treat budgets as rigid bounds for bids and payments. Cho *et al.* (2002) suggest that overcoming budget constraints can encourage collusion or the formation of bidding consortia. We assume non-collusive behavior.

[7]For example Abrams (2006), Andelman & Mansour (2004), Borgs *et al.* (2005), Dobzinski *et al.* (2008), Feldman *et al.* (2008), or Harsha *et al.* (2010).

[8]See also Che & Gale (1996b, 1998, 2006) for related analyses.

Chapter 2

The Environment and Equilibrium Existence

2.1 The Environment

Suppose there is a set of bidders $\mathcal{N} = \{1, 2, \ldots, N\}$ participating in a first-price, sealed-bid auction for one item. Players simultaneously submit bids and the highest bidder receives the item for sale. Only the highest bidder makes a payment equal to her bid to the auctioneer. In this section we define the model's environment: types, feasible actions, preferences, and our notion of a "monotone" strategy.

Types Each player has a private type $\theta_i = (s_i, w_i) \in \mathcal{S}_i \times \mathcal{W}_i = \Theta_i$. s_i is a player's *value-signal* and it is normalized to the unit interval, $\mathcal{S}_i = [0, 1]$. A value-signal is a player's information about the item for purchase. w_i is a player's *budget* and $\mathcal{W}_i = [\underline{w}_i, \bar{w}_i] \subset \mathbb{R}_+$.[1] Suppose $\underline{w}_i < \bar{w}_i$. Although specialized further in subsequent discussion, we will maintain that θ's distribution is atomless, has full support, and admits a strictly positive density, $f(\theta) > 0$ for all $\theta = (\theta_1, \ldots, \theta_N) \in \times_i \Theta_i$.

Feasible Bids A private budget introduces a natural constraint on the feasible actions of a bidder. In particular, a bidder cannot submit a bid in excess of her budget.

[1] For notation, profiles of values are in boldface as in $\mathbf{w} = (w_1, \ldots, w_N)$. As usual, $\mathbf{w}_{-i} = (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_N)$.

Assumption 1 (Feasible Bids)**.** *The set of feasible bids for a bidder of type* $\theta_i = (s_i, w_i)$ *is* $\mathcal{B}_i(w_i) = [r_i, w_i] \cup \{l\}$ *where* l *is a guaranteed losing bid and* $r_i \geq 0$ *is the reserve price.*[2] *Bids other than* l *are non-losing bids.*

The guaranteed losing bid ensures voluntary participation, an important consideration if there is a reserve price.

Although an action space where bidders can place non-losing bids from an interval, $[r_i, w_i]$, is the ultimate focus of our analysis, en route we will also consider a discretized version of the bidding set. Complications due to tied bids are the primary (technical) concern in many auction models and analyzing a model with finite bidding sets is a key step in establishing many features of an equilibrium. Following Reny & Zamir (2004) and McAdams (2006), among others, it will prove convenient to also consider the analogous discretized action set.

Assumption 2 (Disjoint Feasible Bids)**.** *For each* i*, let* $\mathcal{P}_i \subset \mathbb{R}_+$ *be a finite set such that for* $i \neq j$*,* $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$*. The set of feasible bids for a bidder of type* $\theta_i = (s_i, w_i)$ *is* $\mathcal{B}_i(w_i) = (\mathcal{P}_i \cap [r_i, w_i]) \cup \{l\}$*.*

The desired interpretation of $\mathcal{P}_i$ is as an arbitrarily fine finite grid of values.

Preferences A bidder's utility function is a mapping $u_i \colon \mathcal{S} \times \mathcal{W}_i \times \mathbb{R}_+ \to \mathbb{R}$. If bidder i of type (s_i, w_i) wins the auction, her payoff is $u_i(\mathbf{s}, w_i, b_i)$. If she does not receive the item, her payoff is $u_i(\mathbf{0}, w_i, 0)$. She makes zero payment but keeps her money (budget) to enjoy in some other manner. As a normalization, set $u_i(\mathbf{0}, 0, 0) = 0$. The signals $\mathbf{s}$ can be interpreted as indicating item quality. An item of zero quality is like not receiving the item at all. The function u_i satisfies the following conditions.

Assumption 3 (Utility)**.** *The function* $u_i(\mathbf{s}, w_i, b_i)$ *is bounded, differentiable, nondecreasing in* $\mathbf{s}$ *and* w_i*, and non-increasing in* b_i*. Furthermore,*

i) For feasible non-losing bids $b_i \geq b_i'$*,* $u_i(\mathbf{s}, w_i, b_i) - u_i(\mathbf{s}, w_i, b_i')$ *is nondecreasing in* $\mathbf{s}$ *and* w_i*;*

ii) There exists $\kappa_i > 0$ *such that* $\frac{\partial u_i}{\partial s_i} \in [\kappa_i, \infty)$ *and* $\frac{\partial u_i}{\partial w_i} \in [0, \infty)$*.*

[2] If $r_i > w_i$ then $\mathcal{B}_i(w_i) = \{l\}$. The losing bid is isolated and $l < r_i$. The bidder-specific reserve prices are common knowledge among the players.

Assumption 3(i) generalizes a standard assumption seen in Maskin & Riley (2000), Reny & Zamir (2004), or McAdams (2007), among others, by additionally assuming the utility difference is nondecreasing in w_i. Assumption 3(ii) asserts the existence of bounded partial derivatives of u_i with respect to a player's own type. This feature of utility will play an important role below when we make precise the notion of a monotone strategy.

Many utility function meet Assumption 3. Consider for example,

$$u_i(\mathbf{s}, w_i, b_i) = v_i(\mathbf{s}) + w_i - b_i \tag{2.1}$$

where $v_i\colon \mathcal{S} \to \mathbb{R}_+$ is a bidder's valuation, $v_i(\mathbf{0}) = 0$, and $\partial v_i/\partial s_i \in [\kappa_i, \infty)$. Section 3 will consider the model in detail when bidders' preferences are in this class. Assumption 3 allows for more general preferences as well. For example, to accommodate risk-aversion one may consider $\hat{u}_i(v_i(\mathbf{s}) + w_i - b_i)$ where $\hat{u}_i(\cdot)$ is concave with a bounded derivative. Further generalizations are also possible.

In many cases, wealth or a budget can interact with preferences over auction outcomes in non-trivial ways. For example, consider risk-aversion. Suppose that increasing a bidder's wealth decreases her aversion to risk. In a first-price auction, decreasing aversion to risk encourages a bidder to bid *less*: the insurance with respect to the auction outcome from a higher bid is less valuable.[3] Therefore, a bidder's budget can behave like (private) parameter controlling a bidder's displayed risk preferences and a wealthy bidder may justifiably bid less than a counterpart with the same value-signal but a lower budget. Example 6 in Appendix A.1 introduces wealth-dependent risk-preferences into the model of Example 1 and bidders with lower budgets bid more in equilibrium (at least over a range of value-signals). Situations like these demand a qualification to the usual notion of a monotone strategy, a task we turn to next.

Monotone Strategies In his analysis of the multi-unit auction with risk-averse bidders, Reny (2009) pioneered the weakening of the notion of a monotone strategy in games of incomplete information. Indeed, our analysis closely parallel's his construction as the analogy between our setting and a multi-unit auction is quite clear. A first-price auction with private budget constraints can be interpreted as an unusual multi-unit auction: the highest bidder wins

[3]See Krishna (2002, p. 38) or Matthews (1987) on the effects of risk-aversion in a first-price auction.

the item for sale and all bidders "win" their budget irrespective of their feasible bid.

Typically, a strategy is monotone if an increase in s_i or w_i implies a weakly higher bid. In our setting, however, an increase in w_i will not necessarily imply a higher bid. As a first step in (re)defining a monotone strategy, we introduce an alternative partial order on the type-space. This order is defined using a bidder's utility function and relates changes in a bidder's information to changes in her utility.

Definition 1 ($\geq_{\theta_i}$ Ordering)**.** Define the value $\alpha_i \in [0, \infty)$ as

$$\alpha_i = \frac{\sup \frac{\partial u_i}{\partial w_i} - \inf \frac{\partial u_i}{\partial w_i}}{\inf \frac{\partial u_i}{\partial s_i}}.$$

Define the partial order $\geq_{\theta_i}$ on Θ_i as $\theta_i \geq_{\theta_i} \theta_i' \iff w_i \geq w_i'$ and $s_i - \alpha_i w_i \geq s_i' - \alpha_i w_i'$. The asymmetric component, $>_{\theta_i}$, is defined analogously with at least one of the preceding inequalities strict.

Figure 2.1 sketches the "greater-than" and "less-than" sets for type $\hat{\theta}_i$. An increase in s_i always implies a higher type. If w_i increases, s_i must increase commensurably to imply a higher type. $\alpha_i = 0$ when bidders are risk-neutral as in (2.1). In this case $\geq_{\theta_i}$ reduces to the usual coordinate-wise ordering of $\mathbb{R}^2$.

With $\geq_{\theta_i}$ defined, we can introduce the class of strategies that bidders will ultimately follow in a monotone equilibrium.

Definition 2 (Strategies)**.** A *feasible (pure) strategy* for player i is a measurable function $\beta_i \colon \Theta_i \to \mathbb{R}_+ \cup \{l\}$ such that $\beta_i(s_i, w_i) \in \mathcal{B}_i(w_i)$. Let $\mathscr{S}_i$ be player i's set of admissible strategies. A pure strategy β_i is *nondecreasing* if $\theta_i \geq_{\theta_i} \theta_i' \implies \beta_i(\theta_i) \geq \beta_i(\theta_i')$. Let $\mathscr{I}_i \subset \mathscr{S}_i$ be player i's set of nondecreasing admissible strategies.

A pure strategy is nondecreasing if $\geq_{\theta_i}$-higher types bid more. By defining a nondecreasing strategy in this manner, we can accommodate the countervailing incentives introduced by changes in a bidder's budget. Our definition aligns changes in a bidder's *utility*—rather than changes in her information or type—with changes in her bid.

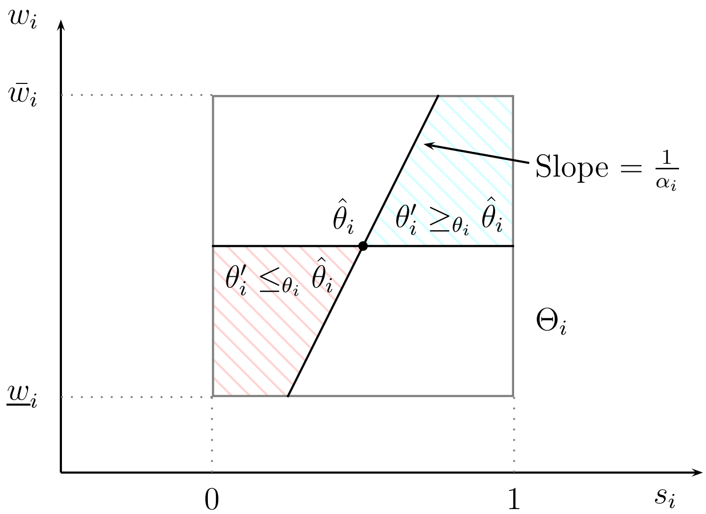


Figure 2.1: The $\geq_{\theta_i}$ type-space ordering. Sets greater-than and less-than $\hat{\theta}_i$ are indicated. $\geq_{\theta_i}$ is a weakening of the usual coordinate-wise partial order of $\mathbb{R}^2$.

2.2 Equilibrium Existence

In this section we consider the above environment and offer sufficient conditions for the existence of an equilibrium in nondecreasing strategies. As common in the literature, we consider the cases of finite (Assumption 2) and continuum (Assumption 1) action sets separately. Finite action sets allow us to identify sufficient conditions on the information structure and on preferences guaranteeing a monotone equilibrium. Continuum action sets introduce the added complexity of tied bids. Relevant ties are a serious concern when values are interdependent and their resolution demands a special tie-breaking protocol.

The proof of the following results is in Appendix A.2 and, noting the analogy described above, follows closely Reny (2009)'s analysis of multi-unit auctions. We consider two separate information structures in succession: independent types and affiliated types.

Proposition 1. *Consider the auction environment above meeting Assumptions 2–3. Suppose that bidder's types are mutually independent, i.e.* $\theta_i \perp\!\!\!\perp \theta_j$. *Then there exists an equilibrium in nondecreasing strategies.*

Once information is correlated across bidders we are more limited in the conclusion that we can draw. Specifically, $\geq_{\theta_i}$ must reduce to the usual partial order on $\mathbb{R}^2$ if we wish to accommodate affiliation in bidder's value signals.

Proposition 2. *Consider the auction environment above meeting Assumptions 2–3. Suppose*

i) Utility is quasilinear as in (2.1).

ii) Bidders' value-signals $\mathbf{s}$ *are affiliated and have joint density* $h(\mathbf{s})$.[4]

iii) Bidders' budgets are mutually independent and independent of value-signals.

iv) For all i *and* $\mathbf{s}_{-i}$, $v_i(s_i, \mathbf{s}_{-i})\lambda(\mathbf{s}_{-i}|s_i)$ *is nondecreasing in* s_i *where* $\lambda(\mathbf{s}_{-i}|s_i) \equiv \frac{h(\mathbf{s}_{-i}|s_i)}{1-H(\mathbf{s}_{-i}|s_i)}$ *and* $H(\mathbf{s}_{-i}|s_i) = \int_{\mathbf{y}_{-i}\leq \mathbf{s}_{-i}} h(\mathbf{y}_{-i}|s_i)d\mathbf{y}_{-i}$.

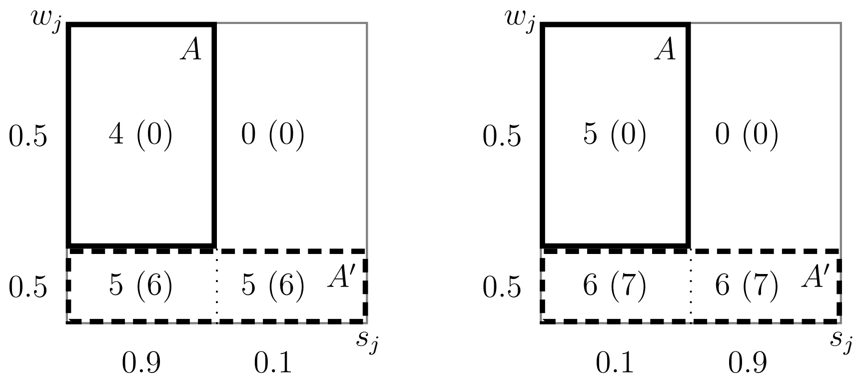
Then there exists an equilibrium in nondecreasing strategies.

Although affiliation is a standard assumption, its further qualification by condition (iv) requires context. Variants of (iv) appear in Fang & Parreiras (2002), Krishna & Morgan (1997), and Lizzeri & Persico (2000). The condition ensures the direct influence of a bidder's value-signal on preferences dominates its informational content concerning $\mathbf{s}_{-i}$. It is satisfied if types are independent. Under strict affiliation, it may fail. For example, if there are two bidders with valuations $v_i(s_i, s_j) = (s_i + s_j)/2$ and value-signals have joint density $h(s_i, s_j) \propto 4s_is_j + 1$ on $[0,1]^2$, then it holds. The condition fails if instead $h(s_i, s_j) \propto 4s_is_j + 0.01$. The following example highlights this assumption's precise role when types are affiliated.

Example 2. Suppose there are two bidders $\{i, j\}$ and fix a nondecreasing strategy β_j. Consider two bids placed by bidder i, $b > b'$. Let $A' = \{\theta_j \colon \beta_j(\theta_j) \leq b'\}$ be the set of j's types who are defeated by the bid b'. Let $A = \{\theta_j \colon b' < \beta_j(\theta_j) \leq b\}$ be the set of additional types defeated by the higher bid b. Let $U_i(b, \beta_j|\theta_i)$ be player i's expected utility when she is of type θ_i and bids b.

A common sufficient condition ensuring that nondecreasing strategies are best replies is that if $U_i(b, \beta_j|\theta_i) \geq U_i(b', \beta_j|\theta_i)$ for some θ_i, then this inequality continues to hold if θ_i increases. Without a limit on the degree

[4] See Milgrom & Weber (1982) for the properties of affiliated random variables.



(a) Low signal θ_i (b) High signal: $\theta_i' > \theta_i$

Figure 2.2: Failure of nondecreasing differences in Example 2. An increase in type from θ_i to θ_i' increases a bidder's payoffs conditional on winning. It also decreases the probability of winning by changing the conditional distribution of S_j.

of affiliation relative to a bidder's valuation, this condition can fail as illustrated by Figure 2.2. Figure 2.2 depicts Θ_j and the subsets A' (dashed enclosure) and A (solid enclosure). Budgets are independent of value-signals but (S_i, S_j) are positively correlated. The conditional distributions of W_j and S_j are noted along the axes. Θ_j is divided into four cells and within each cell C is bidder i's expected utility given $\theta_j \in C$ when bidding b (b'). For instance, the expected utility from bidding b' is zero (0) if $\theta_j \in A$ as $b' < \beta_j(\theta_j)$. On the other hand, $b > \beta_j(\theta_j)$ if $\theta_j \in A$ earning a payoff of 4 if θ_i is low and 5 if it is high. Thus, when θ_i is low,

$$\begin{aligned} U_i(b, \beta_j|\theta_i) &= (0.9)(0.5)4 + (0.5)(0.9 + 0.1)5 \\ &= 4.3 > 3 = (0.9 + 0.1)(0.5)6 = U_i(b', \beta_j|\theta_i). \end{aligned}$$

When θ_i increases to θ_i', higher realization of s_j become more likely reducing the probability of a win for player i. Thus,

$$\begin{aligned} U_i(b, \beta_j|\theta_i') &= (0.1)(0.5)5 + (0.5)(0.1 + 0.9)6 \\ &= 3.25 < 3.5 = (0.1 + 0.9)(0.5)7 = U_i(b', \beta_j|\theta_i'). \end{aligned}$$

The event A becomes much less likely making the higher bid inferior to the lower alternative. When types are independent, this reversal does not happen. Absent independence, condition (iv) in Proposition 2 ensures that the direct increase in expected utility is dominant precluding the preference reversal.

Continuum Action Spaces The equilibrium's extension to the continuum action space is qualified due to complications resulting from tied bids. Relevant ties render the bidder's utility function discontinuous, posing a problem for standard existence results. When values are private, however, these complications can be overcome.

Proposition 3. *Consider an auction environment above meeting Assumptions 1 and 3. Suppose valuations are private, i.e.* $u_i(\mathbf{s}, w_i, b_i) = u_i(s_i, w_i, b_i)$*. In the setting of Propositions 1 and 2 there exists an equilibrium in nondecreasing strategies under the uniform tie-breaking rule.*

Allowing interdependent values necessitates the use of an endogenous tie-breaking protocol depending both on submitted bids and player's announced types. That is, a bidder's strategy will specify a bid $\beta_i(\theta_i)$ and a message $\sigma_i(\theta_i)$ used to arbitrate ties it they occur. Araujo *et al.* (2008) recognize multidimensional types as a case where non-standard tie-breaking rules may be necessary because a bidder's payoffs depend in a complex manner on the precise set of opponents whom she may defeat at a given bid.[5] For example, a bidder may defeat opponents who are pessimistic regarding the item's value; however, she may defeat others who are exceptionally optimistic regarding its value but were unfortunate to be budget constrained. The conflict between the two effects does not suggest a universal preference for resolving ties in one's favor—a condition often needed to render simple tie-breaking rules, such as the uniform tie-breaking rule, sufficient for equilibrium existence.[6]

Proposition 4. *Consider an auction environment above meeting Assumptions 1 and 3. In the setting of Propositions 1 and 2 there exists an equilibrium in nondecreasing strategies with an endogenous tie-breaking rule where bidders truthfully announce their type.*

The auction's allocation rule, defined non-constructively in Appendix A.2, awards the item to the highest bidder if she is unique; otherwise, it resolves

[5]Jackson (2009) also has a thoughtful survey of this issue.

[6]Reny & Zamir (2004) confirm this fact when types are unidimensional. Athey (2001) relies on a similar conclusion.

ties by relying on bidders' announced types. Thus, the allocation rule is "standard" up to its treatment of relevant ties. Araujo & de Castro (2009) employ a similar construction. Whether pure-strategy equilibria in monotone strategies exist in a general setting absent a special tie-breaking protocol is an open question; however, as seen in the next section, in a specialized, symmetric environment with no reserve prices we can construct equilibria that feature no ties and thus the tie-breaking rule is irrelevant.

Chapter 3

Discontinuous Equilibrium Strategies in the Symmetric Model

Although monotonicity is an appealing equilibrium feature, it is an empirically weak conclusion. In special cases a sharper description is possible. Specifically, the key strategic consequences of budget constraints—discontinuous equilibrium strategies and endogenously stratified competition along the budget dimension—can appear when valuations are interdependent, value-signals are affiliated, and budgets distributed continuously on an interval. Within this specialized setting, this section concludes with some comparative statics and applications. For instance, if budgets are more binding on average, the implication for equilibrium strategies is ambiguous. Sometimes bidders may bid more, sometimes less. This conclusion contrasts with the second-price auction where the equivalent exercise unambiguously increases bids. A more detailed comparison of bid distributions between first- and second-price auctions allows an observer to infer the presence of budget constraints in an auction environment. Finally, setting a small reserve price, say ϵ, may decrease expected revenues as it can screen bidders along the wrong dimension.

Suppose there are two bidders with quasilinear preferences: $u_i(\mathbf{s}, w_i, b_i) = v_i(\mathbf{s}) + w_i - b_i$ as introduced in (2.1). Normalize $v_i(\mathbf{0}) = 0$ and $v_i(\mathbf{1}) = 1$. The type-space for each bidder is $\Theta_i = \mathcal{S}_i \times \mathcal{W}_i = [0, 1] \times [\underline{w}, \bar{w}]$ where $\underline{w} < 1 \leq \bar{w}$.[1]

[1] $\underline{w} \geq 1$ is a sufficient condition ensuring that some types will be unconstrained in

The bidders are ex ante symmetric with a differentiable valuation function $v_i(\mathbf{s}) = v(s_i, s_j)$. A bidder with budget w_i can bid $b_i \in [r, w_i] \cup \{l\}$.

To economize on subscripts, for notation let (S, W) be player 1's value-signal and budget respectively. (Y, Z) is player 2's value-signal and budget pair. Anticipating the model's symmetry, the exposition will be from player 1's perspective. The following assumption specifies the information structure and introduces notation.

Assumption 4. *The distribution of types satisfies the following:*

i) Value-signals are affiliated and have the joint density $h(s, y)$. $h(\cdot, \cdot)$ is bounded and $h(s, y) = h(y, s) > 0$. The conditional density of Y given $S = s$ is denoted by $h(y|s)$ and $H(y|s) = \int_0^y h(x|s)dx$. To simplify exposition, suppose $h(s, y)$ is differentiable.

ii) Budgets are independently and identically distributed and are independent of value-signals. For $w \leq v(\mathbf{1})$ the cumulative distribution function of a bidder's budget, $G(\cdot)$, is concave: $G''(w) \equiv g'(w) \leq 0$.

iii) For all y, $v(s, y)\lambda(y|s) \equiv v(s, y)\frac{h(y|s)}{1-H(y|s)}$ is nondecreasing in s.

Assumption 4 specializes the information structure introduced in Proposition 2 by imposing a symmetry requirement on the density $h(\mathbf{s})$ and a concavity condition on the budget distribution. By adding private budgets, this environment builds on Milgrom & Weber (1982)'s classic setting. Like Fang & Parreiras (2002), who analyze the second-price auction in an analogous setting, we restrict attention to two bidders for expositional clarity. Assumption 4(ii) is stronger than considered in Fang & Parreiras (2002), but it is equivalent to Assumption 5 from Che & Gale (1998) (see Lemma 29 in Appendix A.3). This assumption is not crucial for many results and it can be relaxed at the cost of considerable parsimony.[2]

The discussion below is organized into three cases depending on $\underline{w}$, the minimal budget. Thus, the exposition subsumes the comparative static of varying $\underline{w}$. Whether $\underline{w}$ is small, intermediate, or large is a model-specific question whose answer depends on the valuation function and on the distribution of types.

equilibrium.

[2]See Appendix A.4.2 for a relaxation of this assumption.

If $\underline{w} = 0$, or is sufficiently small, the equilibrium is primarily defined by a differential equation; however, identifying the appropriate boundary condition requires care. This case also serves to introduce concepts that help identify the remaining two scenarios. In particular, this section derives the model's key differential equation and establishes the qualitative behavior of its solutions of interest.

When $\underline{w}$ is large, the equilibrium is a natural constrained analogue of the equilibrium without budget constraints. Indeed, if $\underline{w} \to 1$ only dominated bids remain infeasible so recovering the unconstrained model's equilibrium for a large $\underline{w}$ is not surprising.

Finally, when $\underline{w}$ is intermediate, the offered equilibrium strategy features a discontinuity in the spirit of Example 1. In equilibrium, bidders endogenously separate into two groups—those with relatively large budgets and others with small budgets. However, unlike Example 1 the distribution of equilibrium bids has no mass points and has connected support. This case highlights the generality of the tension between the direct and strategic effects introduced by private budget constraints. Bids become depressed by budget limits; however, the chance to exploit others' possible budget constraint is a strategic option encouraging a more aggressive bidding strategy.

As it appears frequently in the following discussion, recall from Milgrom & Weber (1982) that in the corresponding model without budget constraints the equilibrium bidding strategy solves

$$b_f'(s) = [v(s,s) - b_f(s)] \frac{h(s|s)}{H(s|s)}, \qquad b_f(0) = 0. \tag{3.1}$$

3.1 Low Minimal Budgets

The following analysis applies when $\underline{w}$ and the reserve price are "low" (defined below), but for concreteness assume $\underline{w} = 0$ and set the reserve price to zero. Toward identifying the equilibrium strategy, consider the following working assumption.

Working Assumption 1. *Suppose a symmetric equilibrium bidding strategy is of the form* $\beta(s,w) = \min\left\{w, \bar{b}(s)\right\}$ *where* $\bar{b}(s) = \beta(s,\bar{w})$ *is the strategy of an unconstrained bidder. Moreover, assume* $\bar{b}(s)$ *is continuous, strictly increasing, and differentiable.*

Working Assumption 1 implies Leontief "isobid" curves in (s, w)-space with $\bar{b}(s)$ defining the locus of kink-points. We will confirm that there is a symmetric equilibrium of this form.

Assume that all players are following a strategy meeting Working Assumption 1. Viewing the auction as a revelation mechanism, if a bidder of type $(s, \bar{w})$ bids as if her type is $(x, \bar{w})$ then

$$
\begin{aligned}
U(\beta(x,\bar{w})|s,\bar{w}) = &\int_0^{\bar{b}(x)} \int_0^1 \left(v(s,y) - \bar{b}(x)\right) h(y|s)g(z)dydz \\
&+ \int_{\bar{b}(x)}^1 \int_0^x \left(v(s,y) - \bar{b}(x)\right) h(y|s)g(z)dydz.
\end{aligned}
$$

The first term is the contribution to expected utility from the defeating all opponents with a budget $z \leq \bar{b}(x)$. The second term is the contribution to expected utility of defeating all bidders with a budget $z > \bar{b}(x)$ but who have a value-signal less than x, and are therefore by Working Assumption 1 bidding less than $\bar{b}(x)$. Differentiating this expression with respect to x and setting

$$
\left.\frac{\partial U(\beta(x,\bar{w})|s,\bar{w})}{\partial x}\right|_{x=s} = 0,
$$

which must hold if $\bar{b}(s)$ is a best response for an unconstrained bidder, gives:

$$
\bar{b}'(s) = \frac{\left(G(\bar{b}(s)) - 1\right)\left(v(s,s) - \bar{b}(s)\right) h(s|s)}{g(\bar{b}(s)) \int_s^1 (v(s,y) - \bar{b}(s))h(y|s)dy + G(\bar{b}(s))H(s|s) - G(\bar{b}(s)) - H(s|s)}. \tag{3.2}
$$

Collecting hazard rate terms and simplifying notation gives:

$$
\bar{b}'(s) = \frac{\lambda(s|s)\left(\bar{b}(s) - v(s,s)\right)}{\gamma(\bar{b}(s))\left(\eta(s|s) - \delta(\bar{b}(s), s|s)\right)} \tag{3.3}
$$

where

$$
\begin{aligned}
\delta(b,x|s) &\equiv b + \frac{G(b)}{g(b)} + \frac{H(x|s)}{g(b)(1 - H(x|s))} \\
\eta(x|s) &\equiv \mathbb{E}\left[v(s,Y)|S = s, Y \geq x\right] = \int_x^1 v(s,y)\frac{h(y|s)}{1 - H(x|s)}dy \\
\gamma(z) &\equiv \frac{g(z)}{1 - G(z)} \\
\lambda(y|s) &\equiv \frac{h(y|s)}{1 - H(y|s)}
\end{aligned}
$$

Two observations concerning (3.3) follow. First, it is not clear that there is a solution to (3.3) satisfying Working Assumption 1. Second, (3.3) gives little guidance concerning boundary conditions. To appreciate this complication suppose $v(s, y) = (s+y)/2$ and say $s = 0$. A bid of $\epsilon > 0$ wins with probability greater than $G(\epsilon) > 0$ as some bidders have lower budgets. Consequently, $\mathbb{E}[v(0, Y)|S = 0, \text{“win”}] > 0$ and a bidder with a value-signal of zero can have an incentive to bid a positive amount if she can reliably defeat opponents who are more optimistic regarding the item's value. Budget constraints therefore attenuate the winner's curse.

To address these issues it is convenient to develop an understanding of the global qualitative behavior of solutions to (3.3), especially when $s \in (0, 1)$ and $b \in (\underline{w}, \bar{w})$. To do so consider the slope field described by

$$b'(s, b) = \frac{\lambda(s|s)\,(b - v(s, s))}{\gamma(b)\,(\eta(s|s) - \delta(b, s|s))}. \tag{3.4}$$

$\bar{b}(s)$ is just a particular solution consistent with (3.4). We can recast the problem further and consider instead the system of autonomous differential equations:

$$\begin{aligned} \dot{s} &= \gamma(b)\,(\eta(s|s) - \delta(b, s|s)) \\ \dot{b} &= \lambda(s|s)\,(b - v(s, s)) \end{aligned} \;. \tag{3.5}$$

$\dot{s}$ and $\dot{b}$ are functions of the suppressed variable time (t).[3] We can recover (3.4), when it is defined, by noting

$$\frac{db}{ds} = \frac{db}{dt}\Big/\frac{ds}{dt} = \frac{\dot{b}(s, b)}{\dot{s}(s, b)}. \tag{3.6}$$

Interpreting the problem in this manner allows for the use of elementary techniques and results from phase-plane analysis to discern the qualitative behavior of candidate solutions.[4] The direction or velocity of motion as a function of time in (3.5) is immaterial for our purposes. Rather, we are interested in the paths taken by orbits. Such paths—if they are representable

[3]Both $\dot{b}$ and $\dot{s}$ are not defined at $s = 1$. However, they are defined for all $s < 1$; therefore, the following analysis can be used on the *interior* of the type-space to identify a candidate solution. This candidate can be extended using (3.3) to the boundary. To keep the exposition clear, we will suppress this technical detail.

[4]See for example Strogatz (1994); Cronin (2008); Agarwal & O'Regan (2008).

by a function $\bar{b}(s)$—correspond to solutions of (3.3) for different initial conditions. This section's remainder identifies orbits that when viewed as such functions meet the desiderata imposed on $\bar{b}(s)$ by Working Assumption 1.

To understand the qualitative behavior of a system of differential equations we are interested in two related objects: the *nullclines* and the *critical points*. The nullclines of (3.5) are loci in (s, b)-space where orbits are horizontal or vertical. These sets are

$$\nu = \left\{(s, b) \colon \dot{b}(s, b) = 0\right\}$$
$$\psi = \{(s, b) \colon \dot{s}(s, b) = 0\}$$

The next lemmas describe ν, ψ, and the behavior of the $(\dot{s}, \dot{b})$ system. As with all results in this section, their proofs are in Appendix A.3 unless noted otherwise.

Lemma 1 (ν)**.** *Let s_ν be such that $\underline{w} = v(s_\nu, s_\nu)$. Then on $[s_\nu, 1]$, ν is described by the continuous and strictly increasing function $\nu(s) = v(s, s)$. For $s \in [0, s_\nu)$, ν is empty.*

Lemma 2 (ψ)**.** *Define the correspondence $\psi(s) = \{b \colon (s, b) \in \psi\}$ and $\mathcal{S}_\psi = \{s \colon \psi(s) \neq \emptyset\}$. Then,*

a) If $\psi(s) \neq \emptyset$, $\psi(s)$ is single valued.[5]

b) The function

$$\tilde{\psi}(s) = \begin{cases} \psi(s) & \text{if } \psi \neq \emptyset \\ \underline{w} & \text{if } \psi = \emptyset \end{cases}$$

is continuous.

c) If $s_\psi \equiv \sup \mathcal{S}_\psi$, then $\lim_{\substack{s \to s_\psi \\ s \in \mathcal{S}_\psi}} \psi(s) = \underline{w}$.

In typical examples, $\mathcal{S}_\psi = [0, s_\psi]$ for some $s_\psi \leq 1$. If $\underline{w} = 0$, then $0 \in \mathcal{S}_\psi$. The following observation describes the $(\dot{s}, \dot{b})$ system between the nullclines.

Lemma 3. *If $b > \nu(s)$ then $\dot{b}(s, b) > 0$. If $b < \nu(s)$ then $\dot{b}(s, b) < 0$. If $b > \psi(s)$ then $\dot{s}(s, b) < 0$. If $b < \psi(s)$ then $\dot{s}(s, b) > 0$.*

[5] Appendix A.4.2 presents an example with a multi-valued $\psi(s)$ when $G(\cdot)$ is not concave.

Critical points occur where nullclines intersect. (s^*, b^*) is a *critical point* if $\dot{s}(s^*, b^*) = \dot{b}(s^*, b^*) = 0$. Let $\mathcal{C}^*$ denote the set of critical points.

Lemma 4. *If* $\underline{w} = 0$, $\mathcal{C}^* \neq \emptyset$.

The next lemma however ensures that orbits near critical points are well-behaved in generic cases. Specifically, around critical points (3.5) behaves like a linear system.

Lemma 5 (Saddle Point or Node)**.** *Let* (s^*, b^*) *be a critical point of* (3.5)*. Then* (s^*, b^*) *is either a node or a saddle point.*

As the critical points are nodes or saddles, there exist orbits that tend toward (or emanate from) each of the points. The direction of the orbits' motion as a function of time is irrelevant; rather, the key observation is that there are paths in (s, b)-space following (3.5) whose closure contains elements of $\mathcal{C}^*$.

Assumption 5. *For expositional clarity, we will consider the case where the set of critical points,* $\mathcal{C}^*$*, is at most a singleton.*

Appendix A.4.1 elaborates on the treatment when $\mathcal{C}^*$ is multivalued.[6] The set of critical points $\mathcal{C}^*$ may be empty if $\underline{w}$ is large. Section 3.2 subsumes this case.

Leveraging Assumption 5, the preceding analysis allows us to construct an accurate qualitative representation of the phase portrait of (3.5). Figure 3.1 sketches the situation. The nullclines $\psi(s)$ and $\nu(s)$ partition the space into four regions—$R_1, \ldots, R_4$—that meet at the critical point (s^*, b^*). Several sample orbits are plotted in the figure and thick arrows indicate the direction of flow in each region. The critical point is a saddle point. In cases when there are multiple critical points, they alternate between saddles and nodes. Generically, there is an odd number of critical points.

Figure 3.1 also informs our understanding of $\bar{b}(s)$. This function is hypothesized to be continuous, strictly increasing, and meeting (3.4). If such a solution to (3.3) exists, it must begin in region R_1, "pass through" the critical point (s^*, b^*) and continue in R_3. Only in regions R_1 and R_3 is $b'(s, b) = \dot{b}/\dot{s} > 0$. The analysis also confirms that the only possible orbits

[6] A sufficient condition for a single critical point is that $\int_s^1 [v(s, y) - v(s, s)] \frac{h(y|s)}{1 - H(s|s)} dy$ is decreasing in s and $H(s|s)$ is increasing.

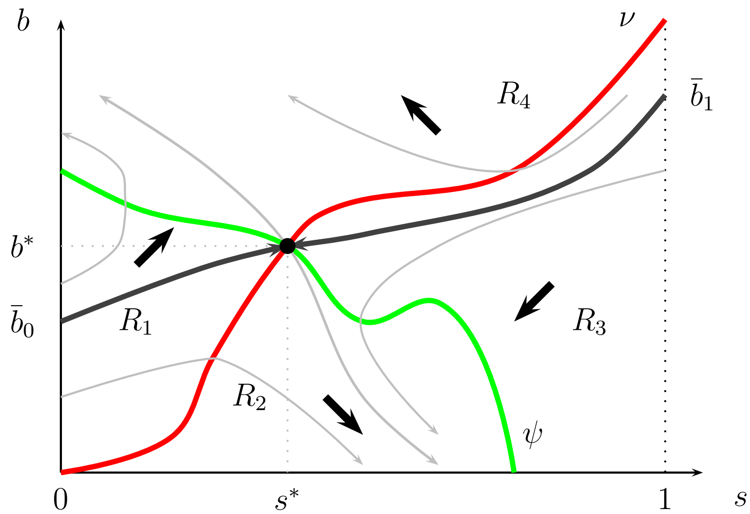


Figure 3.1: Phase-diagram of the $(\dot{s}, \dot{b})$ system defined in (3.5). By Assumption 5 there is a single critical point at (s^*, b^*). The two stable manifolds define the function $\bar{b}(s)$.

consistent with these requirements correspond to the two stable manifolds of (3.5), illustrated as orbits with initial conditions $(0, \bar{b}_0)$ and $(1, \bar{b}_1)$ in Figure 3.1. For notation, let $(s_t, b_t)_{[s_0, b_0]}$ denote the specific orbit of (3.5) in (s, b)-space at time t with initial condition (s_0, b_0). Considering the two stable manifolds, define

$$\bar{b} = \overline{\left\{(s_t, b_t)_{[0,\bar{b}_0]} \colon t \in [0, \infty)\right\} \cup \left\{(s_t, b_t)_{[1,\bar{b}_1]} \colon t \in [0, \infty)\right\}} \tag{3.7}$$

and let $\bar{b}(s)$ be the corresponding function. $\bar{b}(s)$ is well defined, continuous, strictly increasing and satisfies the differential equation (3.3) almost everywhere.[7] It has the endogenously determined boundary condition $\bar{b}(0) = \bar{b}_0 < \psi(0)$.

To summarize, there is a function $\bar{b}(s)$ consistent with the conditions outlined in Working Assumption 1. The remaining step is to verify that all bidders following $\beta(s, w)$ is an equilibrium.

[7]The exception is s^* where (3.3) is undefined. At s^* we can fill this gap by setting $\bar{b}'(s^*)$ equal to the slope of the negative eigenvector from the Jacobian matrix of (3.5) evaluated at the critical point.

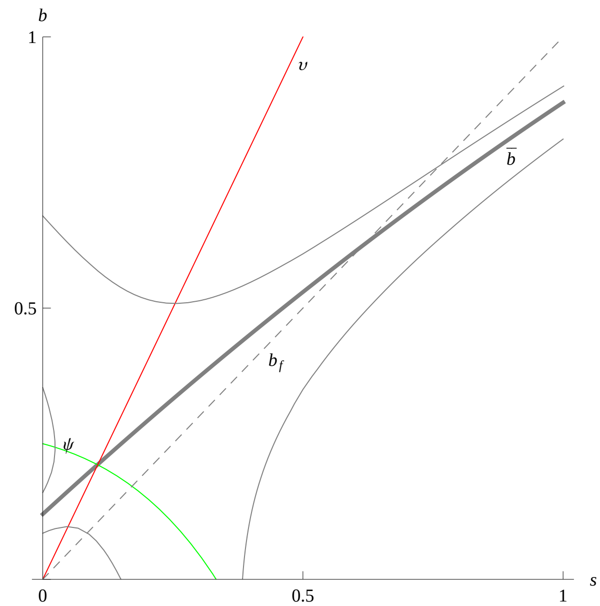


Figure 3.2: Equilibrium characterization in Example 3. There is a single critical point at the intersection of ν and ψ. $\bar{b}(s)$ is defined as the union of the stable manifolds passing through the critical point. For low value-signals, bidders with a large budget bid more than if budget constraints were absent.

Proposition 5 (Equilibrium of the First Price Auction: Case 1)**.** *Suppose* $\underline{w} = 0$*, then* $\beta(s,w) = \min\left\{w, \bar{b}(s)\right\}$ *where* $\bar{b}(s)$ *is defined by* (3.7) *is a symmetric equilibrium.*

Proof. See Appendix A.3. The proof is routine treating the auction as a revelation mechanism and showing that expected utility is concave in announced type. □

The following numerical example illustrates the concepts developed thus far. The example's parameters correspond to those chosen by Fang & Parreiras (2002) to illustrate their characterization of the second-price auction's equilibrium.

Example 3. Suppose bidders' valuations are given by $v(s_i, s_j) = s_i + s_j$ and that $W_i \overset{i.i.d.}{\sim} U[0,2]$, $S_i \overset{i.i.d.}{\sim} U[0,1]$. The reader can verify that the differential equation describing $\bar{b}(s)$ is

$$\bar{b}'(s) = \frac{2(2-\bar{b}(s))(2s-\bar{b}(s))}{(1+s)(3s-1)-4(s-1)\bar{b}(s)}.$$

There is no closed form solution for $\bar{b}(s)$. The ψ locus is defined by the function $\psi(s) = \frac{5}{4} + \frac{1}{s-1} + \frac{3s}{4}$. The critical point is at $s^* = \frac{1}{5}(5-2\sqrt{5}) \approx 0.105$. The eigenvalues of the Jacobian of $(\dot{s}, \dot{b})$ at the critical point are $\frac{-2(\sqrt{5}+\sqrt{205})}{5} \approx -6.62$ and $\frac{2(-\sqrt{5}+\sqrt{205})}{5} \approx 4.83$. Therefore, the critical point is a saddle point and the stable manifolds approach it with a slope $\frac{8\sqrt{5}}{3\sqrt{5}+\sqrt{205}} \approx 0.85$. Additional computation allows us to conclude that $\bar{b}_0 \approx 0.1202$ while the maximal bid is $\bar{b}_1 \approx 0.878$.

$\bar{b}(s)$ is plotted in Figure 3.2 along with $b_f(s) = s$, which is the equilibrium in the same model but with no budget constraints. The nullclines, ψ and ν, and representative orbits from the associated two-dimensional system are included for context. This example highlights the attenuation of the winner's curse: low value-signal bidders (who can afford to) bid considerably more than in the same model absent budget constraints. Winning the auction is not such bad news regarding the item's value.

When $\underline{w} > 0$, two new factors may complicate the preceding analysis. First, for a bidder placing a bid of $\bar{b}(0)$, the expected payoff may be negative: $\eta(0|0) - \bar{b}(0) < 0$. Such a bidder and those with budgets $w \in [\underline{w}, \bar{b}(0))$ may therefore be better off bidding zero. Second, the manifold tending to (s^*, b^*) from below may not originate on the b-axis. Instead, its initial condition will be of the form $(s_0, \underline{w})$ for some $s_0 > 0$. If this is the case, the strategy $\beta(s,w) = \min\{\bar{b}(s), w\}$ fails to specify the behavior of all types. In either case, Proposition 5 cannot apply; otherwise, we have the following definition and corollary.

Definition 3. The minimal budget $\underline{w}$ is *low* if a critical point exists, $s_0 = 0$, and $\eta(0|0) - \bar{b}(0) \geq 0$.

Although definition 3 is several steps removed from model primitives, nothing can be said regarding its latter two conditions absent a solution for $\bar{b}(s)$.

Corollary 1. *Suppose $\underline{w}$ is low. Then Proposition 5 continues to define an equilibrium.*

Proof. It is identical to the preceding analysis. As $\bar{b}(s)$ depends on $\underline{w}$ through $G(\cdot)$, the equilibrium strategy changes with $\underline{w}$. Its characterization as a strategy of the form $\beta(s, w) = \min\{w, \bar{b}(s)\}$ for the appropriate $\bar{b}(s)$ does not change. □

Analysis of subsequent cases, when $\underline{w}$ is large or intermediate, builds on the preceding results. In particular, the function

$$\bar{b}(s)\colon [s_0, 1] \to [\underline{w}, \bar{w}] \tag{3.8}$$

composed as the closure of the union of the upward-sloping (stable) manifolds passing through the critical point (s^*, b^*) will reappear. $\bar{b}(s)$ is reserved as notation for this function.

3.2 Large Minimal Budgets

If budget constraints are large relative to most valuations, a strong intuition suggests that they should matter strategically only for high valuation bidders. In Example 1, for instance, low-valuation bidders ($s < \hat{s}$) follow the bidding strategy $b_f(s)$ bidding *as if* budgets are strategically irrelevant. Their bids are far below the range of budgets. This section extends this observation to the general model. When $\underline{w}$ is sufficiently large all bidders with a value-signal below some threshold $\tilde{s}$ will bid according to the strategy $b_f(s)$, the equilibrium strategy identified by Milgrom & Weber (1982) and defined in (3.1). When $s > \tilde{s}$, bids in excess of $\underline{w}$ will be placed according to the function $\min\{w, \tilde{b}(s)\}$ where $\tilde{b}(s)$ is an appropriate solution to (3.3). Like the low $\underline{w}$ case, and in contrast with Example 1, the equilibrium strategy $\beta(s, w)$ for a large $\underline{w}$ will be continuous.

Definition 4. The minimal budget $\underline{w}$ is *sufficiently large* if $b_f^{-1}(\underline{w}) \notin \mathcal{S}_\psi$ or if $\psi = \emptyset$.

The specific term "sufficiently large" originates from the limiting case, $\underline{w} \geq b_f(1)$ which ensures Definition 4 is satisfied. However, the condition is often met in less trivial settings. Specifically suppose $\tilde{s} \equiv b_f^{-1}(\underline{w}) < 1$.

Proposition 6 (Equilibrium of the First-Price Auction: Case 2)**.** *Suppose $\underline{w}$ is sufficiently large. Let*

$$\beta(s, w) = \begin{cases} b_f(s) & \textit{if } s \leq \tilde{s} \\ \min\left\{w, \tilde{b}(s)\right\} & \textit{if } s > \tilde{s} \end{cases} \tag{3.9}$$

where $b_f(s)$ is defined in (3.1)*, $\tilde{s} \equiv b_f^{-1}(\underline{w})$, and $\tilde{b}(s)$ is the solution to*

$$\tilde{b}'(s) = \frac{\lambda(s|s)\left(\tilde{b}(s) - v(s,s)\right)}{\gamma(\tilde{b}(s))\left(\eta(s|s) - \delta(\tilde{b}(s), s|s)\right)}, \tag{3.10}$$

with boundary condition $\tilde{b}(\tilde{s}) = \underline{w}$. $\beta(s,w)$ is a symmetric equilibrium.

Proof. See Appendix A.3. The difference with the case of low minimal budgets is that the critical point (if it exists) is bypassed by $\tilde{b}(s)$. It is easy to see that if a critical point exists at (s^*, b^*) then $\tilde{s} > s^*$. □

As in the low $\underline{w}$ case, $\beta(s,w)$ is continuous and, abusing terminology, $\tilde{b}(s)$ extends $b_f(s)$ into the range of bids above $\underline{w}$. Bids above $\underline{w}$ defeat opponents who have a low value-signal *and* a relatively low budget. Therefore, the marginal effect on the probability of winning the auction changes as bids cross the $\underline{w}$ threshold. This change in reflected in the bidding strategy of unconstrained bidders. The corollary and example make this effect more precise.

Corollary 2. *There exists a $s' \leq 1$ such that for all bidders of type (s,w), with $s \in (\tilde{s}, s')$ and w sufficiently close to $\bar{w}$, $\beta(s,w) = \tilde{b}(s) \geq b_f(s)$.*

High-budget participants bid more when $s \in (\tilde{s}, s')$ because the marginal returns of higher bidding change. Bids above $\underline{w}$ defeat bidders who are budget constrained *and* who may have a high value-signal. Both effects encourage more aggressive bidding near $\underline{w}$.

Example 4. Suppose bidders' valuations are $v(s_i, s_j) = s_i + s_j$ and that $W_i \overset{i.i.d.}{\sim} U[\frac{1}{2}, 2]$, $S_i \overset{i.i.d.}{\sim} U[0,1]$. The only difference with respect to Example 3 is now $\underline{w} = \frac{1}{2}$. $\underline{w}$ is large enough to preclude any critical points. The differential equation describing $\tilde{b}(s)$ reduces to

$$\tilde{b}'(s) = \frac{2(2s - \tilde{b}(s))(2 - \tilde{b}(s))}{4(1-s)\tilde{b}(s) + s(3s+2) - 2}. \tag{3.11}$$

while $b_f(s) = s$; thus, $\tilde{s} = 1/2$. Figure 3.3 depicts the functions $\tilde{b}(s), b_f(s)$, and $\nu(s)$. The conclusion of Corollary 2 is clearly present.

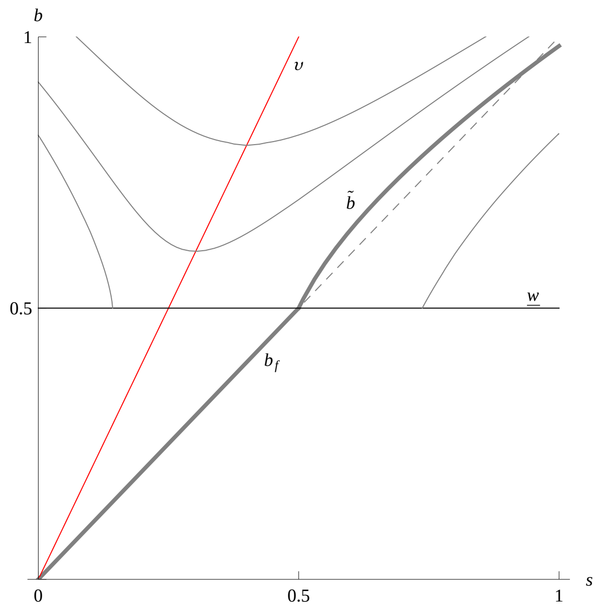


Figure 3.3: Equilibrium characterization in Example 4. $\underline{w} = 1/2$ is sufficiently large to preclude the existence of $\psi(s)$; therefore, there is no critical point. The function $\min\{\tilde{b}(s), w\}$ extends the equilibrium strategy into the range of bids above $\underline{w}$.

3.3 Intermediate Minimal Budgets

The intermediate case is defined in opposition to the low-$\underline{w}$ and high-$\underline{w}$ cases already considered. Therefore, suppose $\underline{w} > 0$, $b_f^{-1}(\underline{w}) \in \mathcal{S}_\psi$ and one of the following cases hold:

1. $\bar{b}(s)$ does not originate on the b-axis: $0 < s_0 \equiv \bar{b}^{-1}(\underline{w})$
2. $\bar{b}(s)$ originates on the b-axis, $s_0 \equiv 0$, but $\eta(0|0) - \bar{b}(0) < 0$.
3. There are no critical points.[8]

In both cases, the preceding theory is inadequate to describe equilibrium behavior. For instance, at $(s, w) = (b_f^{-1}(\underline{w}), \underline{w})$ the differential equation (3.9)

[8]We will assume in the discussion below that there is a critical point and thus $\bar{b}(s)$ exists. Accommodating case 3 is a straightforward modification of the argument below and is omitted for brevity.

evaluates to $\tilde{b}' < 0$ precluding an extension as in the high-$\underline{w}$ case. Such features prompt consideration of discontinuous bidding strategies similar to Example 1. While the discontinuities in Example 1 are partly due to the discrete set of budgets, supporting discontinuities where budgets can assume a continuum of values is a more involved matter. In contrast with Example 1, the distribution of equilibrium bids will exhibit no mass points and will have connected support; however, the underlying intuition will remain the same. To translate Example 1's intuition to this environment requires several new concepts.

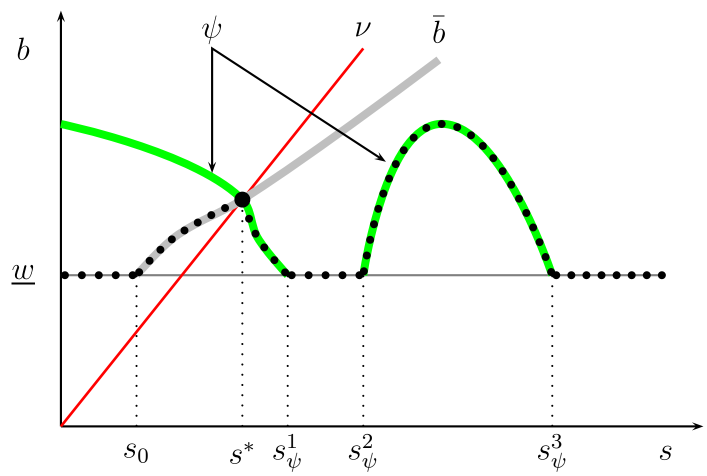


Figure 3.4: Definition of $\mu(s)$, denoted as the thick dotted curve. $\mu(s)$ "climbs" the $\bar{b}$ function until it reaches the critical point. For $s > s^*$ it follows $\min\{\psi(s), \underline{w}\}$. In the figure, $\psi(s)$ has two separate components.

We begin by specifying a function $\mu(s)$ that will define the initial "landing-point" in the discontinuous equilibrium strategy under construction. Recall the function $\tilde{\psi}(s)$ from Lemma 2 and define $\mu\colon [0,1] \to [\underline{w}, \bar{w}]$ as

$$\mu(s) = \begin{cases} \underline{w} & \text{if } s < s_0 \\ \min\{\bar{b}(s), \tilde{\psi}(s)\} & \text{if } s \geq s_0 \end{cases}. \tag{3.12}$$

$\bar{b}(s)$ was defined in (3.7) and passes through the critical point (s^*, b^*). As an example, Figure 3.4 presents $\mu(s)$—the heavy dotted curve—for the case of

$\mathcal{S}_\psi = [0, s_\psi^1] \cup [s_\psi^2, s_\psi^3]$.[9] To intuit the function's origin consider a high-budget bidder of type $(s, \bar{w})$ where $s > s_0$ and suppose she engages in the following thought experiment:

> (★) Suppose all other bidders follow the strategy
>
> $$\beta(y,z) = \begin{cases} b_f(y) & y \leq s \\ \min\left\{\bar{b}(y), z\right\} & y > s \end{cases}.$$
>
> Among all bids in the range $[\underline{w}, \bar{w}]$, what is the optimal bid?

$\mu(s)$ is the answer. To see why, suppose $s > s^*$. Trivially, bids above $\bar{b}(1)$ are dominated. From the analysis when $\underline{w}$ is low, among bids in the range $[\bar{b}(s), \bar{w}]$, $\bar{b}(s)$ is optimal. To find the optimal bid among bids in the range $[\underline{w}, \bar{b}(s)]$ we solve

$$b = \arg\max_{\check{b} \geq \underline{w}} \int_0^s \left(v(s,y) - \check{b}\right) h(y|s)dy + G(\check{b}) \int_s^1 \left(v(s,y) - \check{b}\right) h(y|s)dy.$$

The problem's first-order condition at an interior optimum leads to

$$0 = \eta(s|s) - b - \frac{G(b)}{g(b)} - \frac{H(s|s)}{g(b)(1 - H(s|s))} = \eta(s|s) - \delta(b, s|s) \implies b = \psi(s).$$

Otherwise, the solution is $b = \underline{w}$ at the boundary. As $\tilde{\psi}(s) < \bar{b}(s)$ it follows that $\mu(s)$ is the best response in the range $[\underline{w}, \bar{w}]$. The case of $s < s^*$ can be addressed similarly.

Although $\mu(s)$ is the best response among bids in $[\underline{w}, \bar{w}]$, it is not necessarily optimal among all bids. In particular, $b_f(s)$ may be superior when (★)'s solution is unrestricted.[10] To delineate the boundary between these

[9] The definition of $\mu(s)$ applies to the case of multiple critical points provided $\psi(s)$ is single-valued. The equilibrium strategy constructed below continues to apply with a minor amendment accounting for the additional critical points. See Appendix A.4.1.

[10] Bids in the range $(b_f(s), \underline{w})$ are easily seen to be suboptimal.

situations consider the inequality

$$\overbrace{\int_0^s (v(s,y)-\mu(s))\,h(y|s)dy + G(\mu(s))\int_s^1 (v(s,y)-\mu(s))\,h(y|s)dy}^{\text{Utility from bid of } \mu(s)} \\ \leq \underbrace{\int_0^s (v(s,y)-b_f(s))\,h(y|s)dy}_{\text{Utility from bid of } b_f(s)} \tag{3.13}$$

$$\iff \frac{G(\mu(s))(1-H(s|s))}{H(s|s)} \leq \frac{\mu(s)-b_f(s)}{\eta(s|s)-\mu(s)}$$

and define the set $\mathcal{Z} = \left\{s\colon (3.13) \text{ holds, and } s \in [s_0, b_f^{-1}(\underline{w})]\right\}$.

Lemma 6. *Suppose* $\underline{w}$ *is intermediate. Then* $\mathcal{Z} \neq \emptyset$ *and at* $\hat{s} \equiv \sup \mathcal{Z}$*,* (3.13) *holds with equality.*

Here the signal $\hat{s}$ will play the same role as in Example 1. This will be the value-signal of the lowest type who increases her bid discontinuously.

With $\hat{s}$ defined, we can state the proposed equilibrium strategy.[11] As the whole exceeds the sum of its parts, we defer providing intuition until afterwards. Let

$$\beta(s,w) = \begin{cases} b_f(s), & \text{if } s \leq \hat{s} \\ b_1(s) & \text{if } s > \hat{s}, w < \phi(s) \\ \min\left\{w, \hat{b}(s)\right\} & \text{if } s > \hat{s}, w \geq \phi(s) \end{cases} \tag{3.14}$$

Where the above terms are defined as follows:

- $\hat{s}$ is defined as above.
- $b_f(s)$ is the equilibrium bidding strategy from the first-price auction without budget constraints, defined by (3.1).
- $\hat{b}(s)$ is the solution to

$$\hat{b}'(s) = \frac{\lambda(s|s)\left(\hat{b}(s)-v(s,s)\right)}{\gamma(\hat{b}(s))\left(\eta(s|s)-\delta(\hat{b}(s),s|s)\right)} \tag{3.15}$$

[11] Both the low $\underline{w}$ and the large $\underline{w}$ cases can be accommodated as special cases of the strategy defined below.

subject to $\hat{b}(\hat{s}) = \mu(\hat{s})$.[12]

- $b_1(s)$ is the solution to

$$b_1'(s) = \frac{[v(s,s) - b_1(s)]h(s|s)G(\phi(s))}{H(\hat{s}|s) + \int_{\hat{s}}^{s} G(\phi(y))h(y|s)dy} \tag{3.16}$$

  subject to $b_1(\hat{s}) = b_f(\hat{s})$.

- $\phi(s)\colon [\hat{s}, 1] \to [\underline{w}, \mu(\hat{s})]$ is a non-increasing, continuous function such that:

  1. The following equation is satisfied:

$$\frac{\phi(s) - b_1(s)}{\eta(s|s) - \phi(s)} = \frac{G(\phi(s))(1 - H(s|s))}{H(\hat{s}|s) + \int_{\hat{s}}^{s} G(\phi(y))h(y|s)dy}. \tag{3.17}$$

  2. $\phi(\hat{s}) = \mu(\hat{s})$.
  3. $\phi(\hat{s}') = b_1(\hat{s}') = \underline{w}$ for some $\hat{s}' \in (\hat{s}, 1]$.

Figures 3.5 and 3.6 illustrate this strategy. Dashed curves indicate a discontinuity along the set $\mathcal{D} = \{(\hat{s}, w)\colon w \geq \mu(\hat{s})\} \cup \{(s, \phi(s))\colon s \in [\hat{s}, \hat{s}')\}$. Importantly, however, $\beta(\cdot, \underline{w})\colon [0,1] \to \mathbb{R}_+$ is continuous.

The intuition underlying this strategy echos that from Example 1. Bidders who increase their bid discontinuously at $\hat{s}$ are indifferent between the higher bid and a bid of $b_f(\hat{s})$. The larger bid is attractive for two reasons. First, a higher bid wins with greater probability. Indeed, a bid of $\hat{b}(\hat{s})$ defeats all competitors who have a value-signal $s < \hat{s}$, just like $b_f(\hat{s})$. It also defeats all competitors with a relatively low budget, $w < \hat{b}(\hat{s})$. Second, because a bid of $\hat{b}(\hat{s})$ bests many bidders with low budgets, it has the added benefit of attenuating the winner's curse. A defeated opponent may have received a high value-signal but was unlucky regarding her finances. These reasons compel a bidder with a value signal greater than $\hat{s}$ to favor the much higher bid.

[12]When there exists a critical point and $\hat{s} \leq s^*$, $\bar{b}(s)$ solves (3.15) with the initial condition $\hat{b}(\hat{s}) = \mu(\hat{s})$ and extends it to values $s > s^*$. From the arguments in the low $\underline{w}$ case, a strictly increasing and continuous $\hat{b}(s)$ exists. Appendix A.4.1 considers constructing $\hat{b}(s)$ when there are multiple critical points.

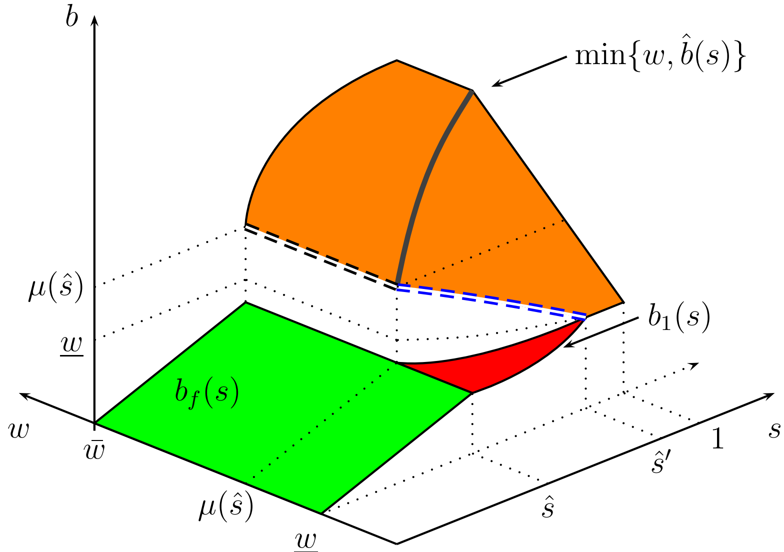


Figure 3.5: The strategy $\beta(s, w)$ when $\underline{w}$ is intermediate. The functions b_f, b_1, and $\hat{b}(s)$ are strictly increasing. Dashed curves indicate a discontinuity along the set $\mathcal{D} = \{(\hat{s}, w) \colon w \geq \mu(\hat{s})\} \cup \{(s, \phi(s)) \colon s \in [\hat{s}, \hat{s}')\}$. However, $\beta(\cdot, \underline{w})$ is continuous.

If bidders with a budget of $w \geq \hat{b}(\hat{s})$ choose to bid strictly above $\underline{w}$, bidders with budgets $w \in [\underline{w}, \hat{b}(\hat{s}))$ must somehow respond. Consider the problem now facing a bidder with a value-signal $\hat{s}_\epsilon = \hat{s} + \epsilon$ and a budget $\hat{w}_\epsilon = \hat{b}(\hat{s}) - \epsilon$. Such a bidder has an incentive to bid her budget and benefit from the two positive effects noted above. However, she does not compare bidding $b_f(\hat{s}_\epsilon)$ versus $\hat{w}_\epsilon$. The fact that all bidders with a budget in excess of $\hat{b}(\hat{s})$ are already outbidding her changes her incentives—she is facing less effective competition on the margin. The same effect is seen in Example 1 for bidders with a budget of $\underline{w}$ and a value-signal of $s \in (\hat{s}, \hat{s}')$. Therefore, she compares bidding $\hat{w}_\epsilon$ versus $b_1(\hat{s}_\epsilon)$ which accounts for this change in marginal incentives. $\{(s, \phi(s)) \colon s > \hat{s}\}$ is the set of types just indifferent between such bids as s increases. The subtle point, however, is that $b_1(\cdot)$ is not independent of the rate at which bidders prefer to bid their budget versus $b_1(s)$. As more bidders discontinuously increase their bid above $\underline{w}$, bidders with even lower budgets but higher value-signals have even less incentive to bid aggressively on the margin.

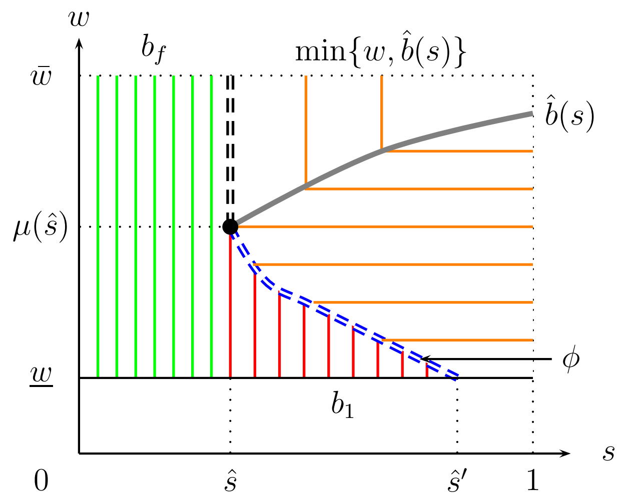


Figure 3.6: Level sets (isobid curves) of $\beta(s, w)$ when $\underline{w}$ is intermediate. The dashed curve indicates a discontinuity. $\beta(\cdot, \underline{w})$ is continuous.

Proposition 7 (Equilibrium of the First-Price Auction: Case 3)**.** *Suppose budgets are intermediate and value-signals are independent. Then* (3.14) *is a symmetric equilibrium.*

Proof. See Appendix A.3. Verifying that this is an equilibrium is routine. Preparatory work involves showing that functions ϕ and b_1 exist with the stated properties (they are introduced in a self-referential manner). Schauder's fixed-point theorem aids in this task. The independence of value-signals allows for a straightforward verification that the functions ϕ and b_1 have the properties described above.[13] □

The strategy $\beta(s, w)$ defined in (3.14) generates a bimodal distribution of bids with few bids near $\underline{w}$. This contrasts with Example 1 where, as an artifact of discrete set of possible budgets, $\underline{w}$ was a mass point of the bid distribution. After observing such a bimodal equilibrium bid distribution, an observer not accounting for the strategic implications of budget constraints may erroneously conclude that there are two distinct classes of

[13]The independence assumption is not necessary to confirm the proposed strategy is an equilibrium; only that it exists.

bidders: those with high values and those with low values. This is a serious error if attempting counterfactual analysis, for example to inform optimal reserve price setting. With private budgets even a uniform type distribution, and a unimodal distribution of ex post valuations, leads to this endogenous separation as highlighted by the following example.

Example 5. Suppose bidders' valuations are $v(s_i, s_j) = s_i + s_j$ and that $W_i \overset{i.i.d.}{\sim} U[\frac{1}{5}, 2]$, $S_i \overset{i.i.d.}{\sim} U[0,1]$. This is the setting of Examples 3 and 4 except now $\underline{w} = 1/5$. The ψ locus is $\psi(s) = \frac{3s}{4} + \frac{9}{10(s-1)} + \frac{5}{4}$. There is a critical point at $s^* = 1 - \frac{3}{5}\sqrt{2} \approx 0.151$. The differential equation describing $\hat{b}(s)$ is

$$\hat{b}'(s) = \frac{10(2 - \hat{b}(s))(\hat{b}(s) - 2s)}{20\hat{b}(s)(s-1) - 5s(3s+2) + 7}$$

and $b_f(s) = s$. The initial discontinuity is at $\hat{s} \approx 0.181$ where some bidders place a bid of $\mu(\hat{s}) \approx 0.286$—an increase in bid of approximately 58 percent.

Figure 3.7 depicts the functions $b_f, b_1, \hat{b}, \nu, \psi$, and ϕ. Several representative orbits from the associated two dimensional system are included to relate the example with previously derived results. Although $\bar{b}(s)$ is not shown, it is easy to see that for all $s > \hat{s}$, $\bar{b}(s) > \hat{b}(s)$ and $\bar{b}^{-1}(\underline{w}) = s_0 > 0$ confirming that this indeed is an intermediate level for $\underline{w}$.

Figure 3.8 presents a stacked histogram of the resulting bid distribution. Only 24 percent of types actually place a bid equal to their budget; nevertheless, the strategic consequences of budget constraints are pronounced. Without budget constraints, the distribution of bids would be uniform.

3.4 Comparisons, Comparative Statics, and Applications

In this section we place the previously derived equilibrium in context by considering several applications. In particular, we provide a comparison with the second-price auction's symmetric equilibrium strategy. This is a useful comparison if we wish to learn about the distribution of budget constraints from bidding behavior alone. We also note some subtle differences in the comparative static behavior of equilibrium strategies between the two formats, especially following changes to the budget distribution. Curiously, skewing the distribution of budgets to lower values may increase or decrease

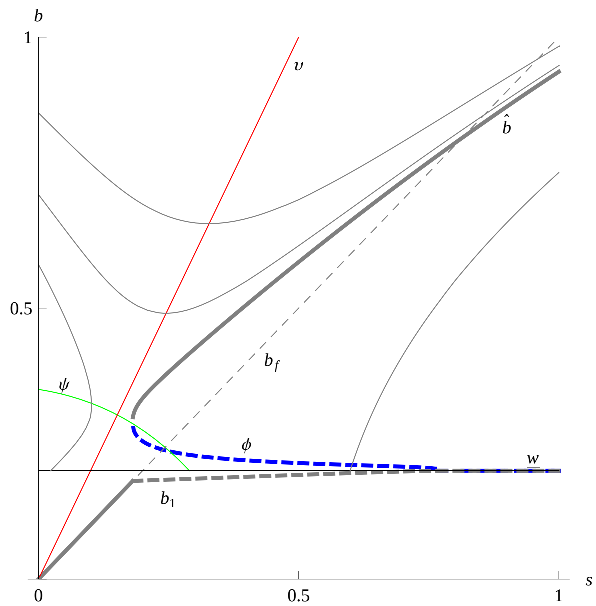


Figure 3.7: Equilibrium characterization in Example 5 where $\underline{w} = 1/5$. The equilibrium strategy bypasses the critical point; however, the strategy features a discontinuity.

equilibrium bids conditional on a player's type. The section concludes with a discussion of reserve prices. When budget constraints are present, even small reserve prices may decrease expected revenues.

The Second-Price Auction Fang & Parreiras (2002) study the second-price auction's symmetric equilibrium in the above environment. Referring to their characterization but adopting our notation,[14] the equilibrium strategy in the second-price auction is of the form

$$\beta_2(s, w) = \begin{cases} v(s, s) & s < \hat{s}_2 \\ \min\{b_2(s), w\} & s \geq \hat{s}_2 \end{cases}. \tag{3.18}$$

$b_2(s)$ is a solution for $s \in [0, 1]$ to the following boundary-value problem:

$$b_2'(s) = \frac{\lambda(s|s)}{\gamma(b_2(s))} \frac{[b_2(s) - v(s, s)]}{[\eta(s|s) - b_2(s)]}, \qquad b_2(1) = v(1, 1) = \eta(1|1).$$

[14]From Fang & Parreiras (2002), the function $\varphi(s, x)$ corresponds to $\eta(x|s)$ in this article.

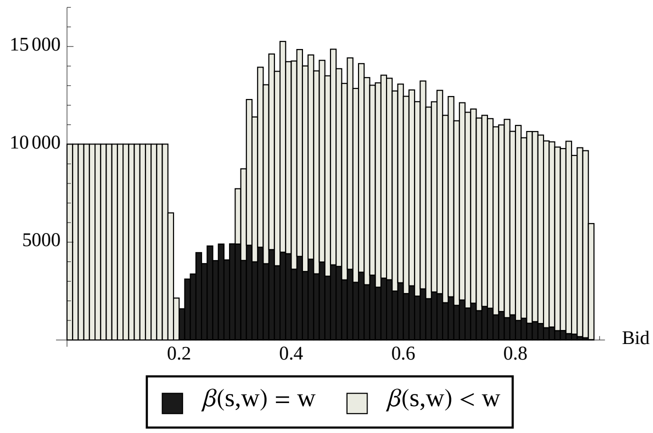


Figure 3.8: Stacked histogram of equilibrium bid distribution in Example 5. Bids place by constrained bidders ($\beta(s, w) = w$) and unconstrained bidders ($\beta(s, w) < w$) are indicated separately. There are few bids placed near $\underline{w}$ as bidders with sufficiently large budgets increase their bids discontinuously to values in excess of $\underline{w}$.

When $\hat{s}_2 > 0$ it is a point of discontinuity in the bidding strategy. If this is the case, $\lim_{s \to \hat{s}_2^+} b_2(s) = \underline{w}$. As in the first-price auction, the discontinuity's presence depends on the prior distribution of types and on the valuation function. Unlike the first-price auction, the second-price auction lacks critical points in the type-space's interior. More importantly, budget constraints in a second-price auction do not introduce the same strategic effects as seen in the first-price auction and there is no endogenous stratification of competition along the budget dimension. If bids discontinuously increase from $v(s, s)$ up to $\underline{w}$ at $s = \hat{s}_2$, all bidders with signals $s > \hat{s}_2$ will bid at least $\underline{w}$. The next lemma compares equilibrium strategies between first- and second-price auctions. It confirms the standard result that a bidder shades her bid more in the former.

Lemma 7. *Fix an auction environment as in sections 3.1–3.3. Let $\beta_1(s, w)$ and $\beta_2(s, w)$ be symmetric equilibrium strategies for the first-price and second-price auctions respectively. Then,*

i) $\beta_1(s, w) \leq \beta_2(s, w)$ and the inequality is strict if $\beta_1(s, w) < w$.

ii) If $\beta_1(s, \bar{w})$ and $\beta_2(s, \bar{w})$ have discontinuities at $\hat{s}_1$ and $\hat{s}_2$ respectively, then $\hat{s}_2 \leq \hat{s}_1$.

Exploiting the weak inequality in Lemma 7(i) allows for a simple test for the presence of budget constraints in an auction environment, even if bids or bid distributions are the only observables. If a first-price and second-price auction are conducted simultaneously for the same item[15] and a bidder submits the same bid across formats, her bid is driven by the budget constraint.

When such laboratory-style, bidder-level data are unavailable one can compare the distributions of submitted bids in both formats. To illustrate the idea, fix an auction environment such that $\underline{w} = 0$ and suppose $v(s, y)$ is a function of both arguments. Let $\beta_1(s, w)$ and $\beta_2(s, w)$ be the equilibrium strategies for the first-price and second-price auctions respectively. When $\underline{w} = 0$ neither β_1 nor β_2 exhibit discontinuities and $\bar{b}(0) > 0$. Thus, for all $w \leq \bar{b}(0)$, $\beta_1(s, w) = \beta_2(s, w) = w$. Consequently, the distribution of bids less than $\bar{b}(0)$ will be the same across auction formats and it will reflect the distribution of budgets in this range. Without budget constraints, bidders would bid $v(s, s)$ in the second-price auction's symmetric equilibrium. As $b_f(s) < v(s, s)$ the distribution of first-price auction bids in this range would be different if binding budgets were not present.

Such tests are unusual insofar as they demand observing bidding across auction formats while maintaining the environment otherwise fixed. However, they do in principle allow identifying private budget constraints, as distinct from valuations, based on bidding behavior alone.

Comparative Statics Several additional comparative statics can help foster intuition for the first-price auction equilibrium's properties. The following exercises mirror those offered by Fang & Parreiras (2002) for the second-price auction. The non-strategic release of value-enhancing information will encourage higher bids. However, unlike in the second-price auction, moving to a stochastically lower distribution of budgets can have an ambiguous effect on the equilibrium bid of a participant.

Public Signals Consider the situation where a bidder's final valuation depends on (S, Y) and on some public signal T which is observable to all

[15] For example, participants could submit format-contingent bids.

participants prior to bidding. Suppose (S, Y, T) are affiliated and

$$v(s, y|t) \equiv \mathbb{E}[\pi(S, Y, T)|S = s, Y = y, T = t].$$

π can be understood as a primitive mapping of signals to ex-post valuations.[16] Thus, higher realizations of T are good news regarding the sale item's final value. Similarly define

$$\eta(x|s, t) \equiv \mathbb{E}\left[v(S, Y|T)|Y \geq x, S = s, T = t\right].$$

By affiliation, v and η are increasing of their arguments while the conditional cumulative distribution $H(y|s, t)$ and the conditional hazard rate $\lambda(y|s, t)$ are now also decreasing in t.

As t can be treated as a model parameter, the equilibrium characterization remains unchanged with statements conditional on t replacing the preceding analysis' unconditional statements. In particular, the model's main differential equation (3.3) is now

$$\frac{d}{ds}b(s, t) = \frac{\lambda(s|s, t)\left(b(s, t) - v(s, s|t)\right)}{\gamma(b(s, t))\left(\eta(s|s, t) - \delta(b(s, t), s|s, t)\right)}. \tag{3.19}$$

The specific solution to (3.19) passing through the critical point, denoted here as $\bar{b}(s, t)$, admits comparative statics as t changes. The following lemma confirms an intuitive conclusion: good news—i.e. large realized values of T—encourages bidders to increase their bids.

Lemma 8. *Suppose that for all y, $v(s, y|t)\lambda(y|s, t)$ is non-decreasing in (s, t).*[17] *Let $t_0 < t_1$. Then for all $s \in [0, 1]$ such that $\bar{b}(s, t_0)$ and $\bar{b}(s, t_1)$ are defined, $\bar{b}(s, t_0) \leq \bar{b}(s, t_1)$.*

Financial Constraints Budget constraints offer a new dimension along which the auction environment may change. Sections 3.1–3.3 focused on changing the support of the budget distribution. Here we consider changing $G(\cdot)$ holding $[\underline{w}, \bar{w}]$ fixed. Specifically, consider two distributions of budgets, G_0 and G_1, with the same support such that

$$w' > w \implies \frac{g_0(w')}{g_1(w')} \geq \frac{g_0(w)}{g_1(w)}. \tag{3.20}$$

[16] Assume that $v(s, y|t)$ enjoys the previous properties of convenience, such as differentiability, boundedness, etc.

[17] This condition is satisfied if $v(s, y|t) = (s + y + t)/3$ and $h(s, y, t) \propto syt + 10$ on $[0, 1]^3$. It is always satisfied if signals are independent.

Thus, G_0 likelihood ratio dominates G_1.[18] Intuitively, financial constraints are more severe when budgets are distributed according to G_1 rather than G_0.

In the second-price auction, changing the budget distribution from G_0 to G_1 increases unconstrained bidders' bids (Fang & Parreiras, 2002). In the first-price auction, in contrast, increasing the severity of budget constraints in the sense of (3.20) has *ambiguous* effects on bidding: an unconstrained bidder may bid more or less. As likelihood ratio dominance is a restrictive stochastic ordering implying both hazard-rate dominance and first-order stochastic dominance, this ambiguity can be interpreted as a negative conclusion from an empirical perspective.

To confirm an ambiguous effect, it is sufficient to study a parametric example and to examine the bids placed by an unconstrained bidder near the critical point (s^*, b^*); indeed, in typical cases this is the only point where we can easily pin-down the value of an unconstrained bidder's bid. Specifically, consider the model of Example 3 but with three alternative distributions of budgets on $[0, 2]$:

$$G_0(w) = \frac{w}{2}, \qquad G_1(w) = \sqrt{\frac{w}{2}}, \qquad \text{and } G_{1'}(w) = \frac{w(4-w)}{4}.$$

G_0 likelihood ratio dominates both G_1 and $G_{1'}$. Comparatively, budgets are more likely to be low when budgets are distributed according to G_1 or $G_{1'}$. The critical point (s^*, b^*) occurs at the intersection of $\nu(s)$ and $\psi(s)$. $\nu(s)$ is invariant to changes in $G(\cdot)$; however, each distribution G_k implies a distinct $\psi_k(s)$:

$$\begin{aligned}
\psi_0(s) &= \frac{3s^2 + 2s - 1}{4s - 4} \\
\psi_1(s) &= \frac{9s^3 - 7s^2 + 3s + 3 - 4\sqrt{9s^5 - 11s^4 + 3s^3 + 3s^2}}{18\left(s^2 - 2s + 1\right)} \\
\psi_{1'}(s) &= \frac{9 - 6s - 3s^2 - \sqrt{9s^4 - 36s^3 + 48s^2 - 84s + 57}}{6 - 6s}
\end{aligned}$$

Table 3.1 summarizes approximate values for the critical point in each case and Figure 3.9 sketches the overall situation. The critical points are

[18] Likelihood ratio dominance implies hazard rate dominance, $\frac{g_1(w)}{1-G_1(w)} \geq \frac{g_0(w)}{1-G_0(w)}$, and first-order stochastic dominance, $G_1(w) \geq G_0(w)$.

Table 3.1: Critical Points for Different Budget Distributions

Distribution	s_k^*	b_k^*
G_0	0.1056	0.2111
G_1	0.0864	0.1728
$G_{1'}$	0.1264	0.2528

strictly ordered: $s_1^* < s_0^* < s_{1'}^*$. In a neighborhood of s_0^* the sign of the change in bid can be either positive or negative. Unconstrained players bid less when they are distributed according to G_1 instead of G_0: $\bar{b}_1(s_0^*) < \bar{b}_0(s_0^*)$. The converse is true when budgets follow $G_{1'}$ instead of G_0: $\bar{b}_{1'}(s_0^*) > \bar{b}_0(s_0^*)$.

The supporting intuition centers on two conflicting effects. When values are interdependent, budget limits attenuate the winner's curse encouraging higher bids. Defeating an opponent because she was budget constrained is good news. The countervailing force concerns the marginal effects of higher bids on the probability of winning. There is less incentive to bid aggressively because competition is curtailed at higher bid levels. Budgets directly reduce the probability that an opponent is capable of placing a relevant competing bid. Both effects become magnified as budget distributions are skewed toward lower values and the final alignment of incentives may favor one or the other.

Reserve Prices As in the case of the second-price auction, a general expression for expected revenues is not available but we can draw some revenue conclusions by examining equilibrium strategies alone. Specifically, as a final application we consider the setting of a common reserve price. The main lesson is one of caution: setting even a small reserve price may exclude the wrong types of bidders and thus *decrease* revenue. The following is an immediate corollary to Proposition 5 when $\underline{w}$ is low.

Corollary 3. *Suppose $\underline{w}$ is low and let $\beta(s,w) = \min\{\bar{b}(s), w\}$ be the equilibrium strategy absent any reserve prices. Let $r \in (\underline{w}, \bar{b}(0)]$ be a reserve price. Then*

$$\beta_r(s,w) = \begin{cases} l & \text{if } w < r \\ \min\left\{\bar{b}(s), w\right\} & \text{if } w \geq r \end{cases}$$

is an equilibrium following the introduction of a reserve price. The auction's expected revenue has decreased.

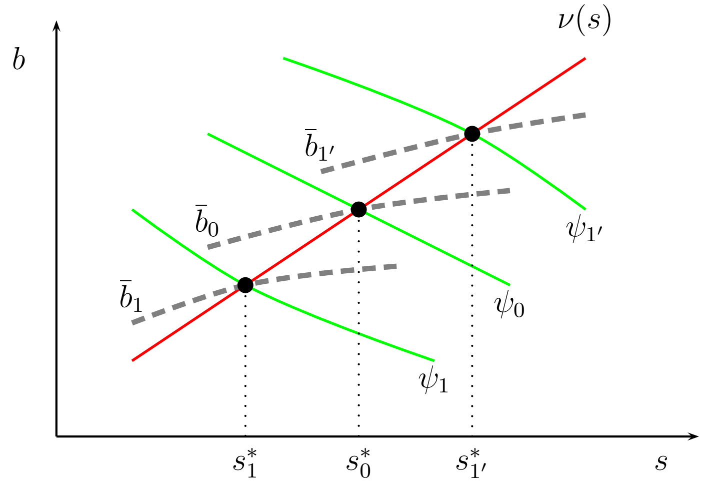


Figure 3.9: Changing budget distributions shifts the critical point. Thick dashed curves denote the $\bar{b}_k(s)$ functions in various cases. $\nu(s)$ is invariant to changes in $G_k(\cdot)$. Locally near the critical point, changing G_k in the manner considered has an ambiguous effects on the equilibrium strategy of an unconstrained bidder.

Setting $r < \bar{b}(0)$ leaves unchanged the bid of all bidders with a larger budget. Such bidders were already defeating all opponents with budgets $w < r$ and they are unmoved by the latter's exclusion from the auction. The expected revenue declines because the reserve price is reducing the probability of sale by screening low-budget bidders while not improving the expected price paid conditional on sale to a participant with a budget $w \geq r$. Absent budget constraints, in contrast, a small reserve price screens out low-valuation bidders and increases the bid of all participants through its uplift of the lowest-valuation participating bidder's equilibrium bid. Bidders with a value-signal of $s = 0$ continue to place a non-losing bid according to β_r provided they have a sufficiently large budget.

When private budget constraints exist, setting the revenue-maximizing reserve price is a delicate exercise. For instance, in the introductory model of Example 1 with budgets $w \in \{\underline{w}, 1\}$, the revenue-maximizing reserve price varies discontinuously with the model's parameters. If budgets are low with low probability, a reserve price of $r^* = 1/2$ is optimal. This is also the revenue-maximizing reserve price in the model absent budget constraints. If the probability of a bidder being budget constrained is sufficiently large, the optimal reserve price will be considerably lower, $r^* \leq \underline{w} < 1/2$. Selling the item more often at a possibly lower price is on balance the better decision if binding budget constraints are prevalent.

Chapter 4

Experiments

Whereas the preceding theory offers a rich set of predictions concerning equilibrium behavior, whether auction participants recognize the nuanced strategic implications of private budget constraints reduces to an empirical question. This chapter reexamines the introductory model of Example 1 in an experimental setting. Recall that Example 1 builds around a simple generalization of Vickrey (1961)'s classic first-price sealed-bid auction model. It is the simplest non-trivial setting where budget constraints matter for bidding behavior. A bidder will face a hard budget constraint with probability p; otherwise, the bidder's budget will be sufficient for all (undominated) bids. For simplicity, budgets are independent of valuations and both budget and valuations are private information. This framework, which can equally well be applied to other auction forms, offers a tractable setting to explore the effects of private budgets in auctions.

Mirroring nearly all first-price auction experiments, the experimental analysis cast doubt on the equilibrium bidding model *per se*. For example, overbidding relative to the risk-neutral equilibrium strategy is observed. On the other hand, the data lend support for the equilibrium's main qualitative features. In particular, the presence of budget constraints is recognized as a major strategic opportunity for unconstrained bidders. Moreover, the bidding strategy employed by unconstrained participants further accounts for the stratification of competition implied by budget constraints.

Although auction theory has recognized the possibility of budget constraints in auctions, to my knowledge their effects have not been explored in the laboratory. The only exception is Pitchik & Schotter (1988) which focuses on equilibrium selection in a sequential auction setting rather than on the strategic issues introduced by budget constraints in the standard model. Kagel (1995) and Kagel & Levin (2011) comprehensively survey the experimental auction literature, including research on first-price auctions.

Section 4.1 is a self-contained revisiting of Example 1. Equilibrium predictions are derived and include a congruence of equilibrium strategies among both constrained and unconstrained bidders with low valuations. For bidders with low valuations, budget limits are not strategically relevant. Equilibrium strategies also feature a prominent discontinuity when bidders who do not face a budget constraint outbid all constrained types. Finally, at relatively high bid levels, there exists a stratification of competition implying changed marginal returns to increased bids. Section 4.2 introduces the experimental environment which dutifully mirrors standard auction experiments and simultaneously allows for the collection of a rich individual level data-set. Finally, section 4.3 considers multiple tests of the equilibrium model's main predictions. In particular, we focus on the discontinuities in equilibrium strategies, the congruence of equilibrium behavior among high- and low-budget bidders at very low valuations, and the change in bids implies by the resultant stratification of competition between high and low-budget bidders. Whereas budgets clearly do impact behavior, subject's reactions often fail to correspond to the model's quantitative predictions. That said, the strategic issues introduced by budget limits impact behavior in a manner qualitatively aligned with theory.

4.1 The Model

The model under study is a generalization of Vickrey (1961)'s classic first-price, sealed-bid auction with independent private values. For simplicity, and looking forward to the experiment, assume there are 2 bidders, $i \in \{1, 2\}$, and each bidder has a private value $s_i \overset{i.i.d.}{\sim} U[0, 1]$. The bidder submitting the highest bid wins the auction and ties are resolved with a fair coin flip.[1] Bidders are assumed to be risk-neutral and winners receive a payoff of $s_i - b_i$ where b_i is the bidder's own bid. Losing bidders receive a payoff of zero. As we focus on the symmetric Bayesian Nash equilibrium, subscripts will be omitted unless confusion may result. When players do not face a budget

[1] When the model is augmented with budget constraints, ties among high-bidders occur with positive probability in equilibrium. Therefore, the tie-breaking assumption is important.

constraint, the equilibrium bidding strategy is

$$b_f(s) = \frac{s}{2}.$$

Consider the following generalization. Suppose players also have an absolute spending limit, a budget, known only to themselves. Player's budgets are independent and can assume one of two values $w_i \in \{\underline{w}, 1\}$. To preclude degenerate cases, suppose $\underline{w} < \frac{1}{2}$ and let $\text{Prob}[w_i = \underline{w}] = p \in (0,1)$. Therefore, a player's type $\theta = (s, w)$ is composed of both a valuation and a budget. While θ is private, its distribution is common knowledge. The presence of the budget constraints noticeably modifies the symmetric equilibrium bidding strategy.

Proposition 8. *The following strategy is a symmetric Bayesian Nash equilibrium of the first-price auction with private budget constraints:*

$$\beta(s,w) = \begin{cases} \frac{s}{2} & \textit{if } s \in [0, \hat{s}] \textit{ and } w \in \{\underline{w}, 1\} \\ \frac{s^2(p-1)-2k_1}{2p(s-1)-2s} & \textit{if } s \in (\hat{s}, 1] \textit{ and } w = 1 \\ \frac{ps^2+2k_2}{2\hat{s}-2p\hat{s}+2sp} & \textit{if } s \in (\hat{s}, \hat{s}'] \textit{ and } w = \underline{w} \\ \underline{w} & \textit{if } s \in (\hat{s}', 1] \textit{ and } w = \underline{w} \end{cases} . \tag{4.1}$$

The closed-form values of the constants $\{\hat{s}, \hat{s}', k_1, k_2\}$ are presented in Appendix A.1.

Proof. See Appendix A.1. Verifying that β is an equilibrium is routine; however, it is not immediate that β exists as both $\hat{s}$ and $\hat{s}'$ are defined as values at which a bidder is indifferent between the low and the high bids. □

Figure 4.1 sketches an example of $\beta(s, w)$, along with $b(s)$ for comparison. $\beta(s, w)$ displays several novel and testable features.

First, constrained and unconstrained bidders follow the same bidding strategy for valuations $s \leq \hat{s}$. For these bidders their budget is never binding and is ancillary information.

Second, at $\hat{s}$ the equilibrium strategy of an unconstrained bidder displays a discontinuity. The intuition for the jump is straightforward. By bidding more than $\underline{w}$, an unconstrained bidder can increase discontinuously his chance of winning the auction by outbidding all constrained types. With a valuation of $\hat{s}$ he is just indifferent between competing only with low valuation bidders

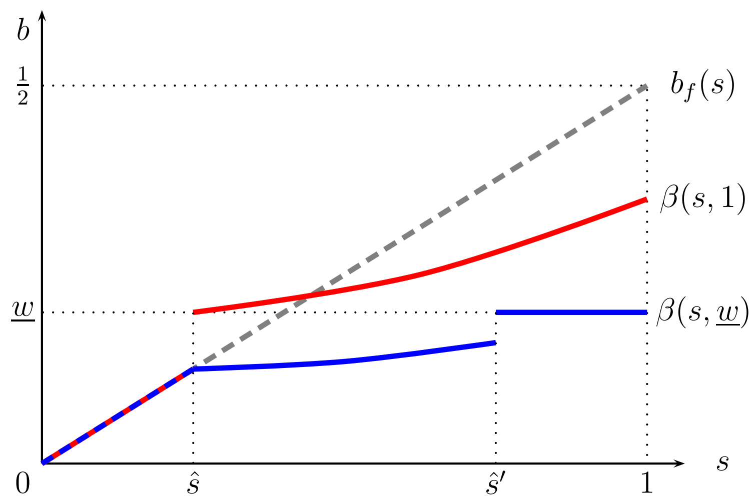


Figure 4.1: Equilibrium strategy in Proposition 8. High-budget bidders increase their bid discontinuously to $\underline{w}$ at $\hat{s}$. Low-budget bidders are restricted to bids below $\underline{w}$. $b_f(s)$ is the equilibrium strategy absent private budget constraints.

with a bid of $\hat{s}/2$ and paying a premium $\underline{w} - \hat{s}/2$ to boost his chance of winning by this discrete amount.

Third, constrained bidders will only bid their entire budget for relatively high valuations, $s > \hat{s}'$. This implies that ties happen with positive probability in equilibrium. Indeed, the precise value of $\hat{s}'$ depends on the tie breaking procedure employed. A move from bidding less than $\underline{w}$ to a bid of $\underline{w}$ increases discontinuously a player's probability of winning and $\hat{s}'$ is the lowest type who can benefit from this jump in bid.

Fourth, there is a clear stratification of competition among high- and low-budget bidders. This stratification is most clearly exhibited in the discontinuity already mentioned; however, it is also reflected by the change in the equilibrium strategy's slope at valuations $s > \hat{s}$. Bids placed by bidders with valuations $s > \hat{s}$ are competing on the margin only with bids placed by opponents with similar budget levels and the strategies need to adjust to reflect this decline in effective competition.

Finally, it is worth noting that $\beta(s, w)$ nests $b_f(s) = \frac{s}{2}$ as a limiting case (for instance if $p \to 0$ or $\underline{w} \to \frac{1}{2}$).

Beyond the striking individual level behavior, the generalized model also makes several predictions concerning the auction's aggregate performance.

Proposition 9. *For all* $p > 0$ *and* $\underline{w} < 1/2$ *the expected revenue of the first-price auction with private budget constrains is less that the expected revenue when there are no budget constraints (say* $p = 0$*).*

Proof. See Lemma 9 in Appendix A.1. The conclusion is not immediate because of the non-monotonic response of bidding strategies following the introduction of private budgets. □

Conclusions concerning the decline in the auction's allocative efficiency are trivial. With positive probability, the item will be allocated to a relatively low-valuation bidder as a high-valuation opponent may be budget-constrained.

Based on this characterization of equilibrium, we focus on two related features of the model.

1. To what extent do individual and aggregate bidding behavior correspond to the equilibrium model? This includes investigating the congruity of the bidding strategy of constrained and unconstrained bidders with low valuations ($s \leq \hat{s}$) and the discontinuity in unconstrained bidder's bidding strategy at $\hat{s}$.[2] Both of these features help identify whether bidders can recognize the strategic importance of budget constraints and when such constraints do and do not matter.

2. How robust are the model's predictions about aggregate outcomes such as revenue and efficiency? Whereas theory predicts a decline in expected revenue and efficiency, whether this decline is empirically true is not a priori obvious. The relatively complicated bidding problem may encourage the use of heuristic strategies moving these variables in either direction or at best ameliorating their decline.

Investigating these features allows us to gauge how budget constraints enter the decision making calculus of bidders and presents a first step beyond the raw equilibrium analysis in incorporating such realistic constraints into more applied auction environments.

[2]The discontinuity in the constrained bidder's strategy at $\hat{s}'$ will not be a focus of the experimental analysis. The parameters selected for the experiment make this second discontinuity relatively small.

Table 4.1: Summary of Experimental Sessions

Session	Treatment Order	Rounds per Treatment	Subjects	Const. Subjects	Mean Earnings
1	1–2	5–5	22	18	$ 25.78
2	1–2	5–5	18	10	$ 19.39
3	2–1	5–5	18	16	$ 16.91

4.2 Experiment Procedures

Three experimental sessions were conducted at the Xlab at the University of California, Berkeley. Subjects were undergraduate students recruited by e-mail who have not participated in this experiment previously.[3] The sessions consisted of 22, 18, and 18 participants. This subject pool size is comparable to other recent auction studies (Kagel & Levin, 2009; Neugebauer & Perote, 2008). Per Xlab policy, subjects received a $5 show-up payment in addition to any earnings in the experiment. Average payments were $21.04 and the experiment lasted approximately 60 minutes with 20 additional minutes needed to process payments.[4] Payments for the experiment were calibrated to approximate a subject's average hourly wage of $15.

In each session, subjects participated in two treatments. In both treatments, valuations were independently distributed uniformly on $[0, 10]$ and were rounded to the nearest hundredth. In Treatment 1 bidders did not face a strategically relevant budget constraint nor any uncertainty about the budget of others. Specifically, subjects were informed that their budget was 10. This ensures that all undominated bids are feasible for all types of bidders. Treatment 2 introduces private budget constraints of $w_i = 3.50$ with probability $p = 0.7$; otherwise, $w_i = 10$. Subjects remained at the same budget level for the duration of the treatment and budgets were assigned randomly by computer. Table 4.1 summarizes the experimental sessions.

At the start of the experiment subjects were seated at computer termi-

[3]Many subjects have likely participated in previous Xlab experiments.

[4]In session 1 the round that was randomly selected to determine payment was from Treatment 1 in which no subject faced a budget constraint; therefore, the earnings are noticeably greater.

nals and presented with instructions for the experiment.[5] Subjects read the instructions; subsequently, the moderator read the instructions aloud. Subjects had the opportunity to ask clarifying questions. During the experiment, subjects participated in 10 rounds of bidding. In two sessions, 5 rounds of Treatment 1 were followed by 5 rounds of Treatment 2. Between the two treatments, there was a minor pause as the moderator publicly announced the new way budgets are henceforth determined. The third session reversed the treatment ordering. Subjects were not informed ahead of time concerning the number of planned rounds of bidding nor of the expected change in treatments midway through the session. There were no "practice" rounds. Each round consisted of the following:

1. Subjects were randomly matched to another bidder. Subjects remained anonymous.

2. Each subject was presented on their computer screen some information about the auction environment, such as the distribution of valuations and the number of bidders. In Treatment 1, subjects learned that all bidders have a budget of 10. In Treatment 2, subjects were informed about the distribution of the other player's budget. This information was also prominently displayed on a whiteboard to ensure that subjects were symmetrically informed.

3. Each subject was presented with 10 realized valuations and was asked to place a bid for each valuation as a separate auction.[6] Subjects were also informed of their own budget constraint.[7] In each of the 10 auctions per round, bids could be any value between zero and a subject's budget constraint. A bidding screen from this interface was included in the instructions and was explained at the start of the experiment (See Appendix B, Figure B.1).

4. A new round began once all players have submitted bids. Subjects received no feedback between rounds of bidding and were informed of this no feedback feature at the experiment's beginning.

[5]Abridged instructions are in Appendix B.1.

[6]The valuations were not ordered.

[7]Subjects were constrained or unconstrained and remained so for all rounds in Treatment 2.

At the experiment's end, the computer selected one auction from one round to determine payment. Subjects were informed of their budget, bid, and valuation from the selected auction and whether they won or not. The competing bidder's identity, bid, valuation, and budget was not revealed. The experiment's unit of account was the experimental currency unit (ECU). Subjects received $x_i + w_i - b_i$ ECU if they won the selected auction. A subject earned the value of the financial asset and made a payment from their budget. Otherwise they earned w_i ECU. The exchange rate was 1 ECU = 2 USD.

A novel feature of this experimental design is the hybrid random-matching and strategy method interface which elicited bids for several values simultaneously. Strategy method designs are relatively rare in auction experiments with Filiz-Ozbay & Ozbay (2007)'s pen-and-paper design being a recent exception. The design greatly increases the collected data's richness without significantly impacting costs while maintaining the spirit of traditional auction experiments built around round-by-round random matching.[8] Pezanis-Christou & Sadrieh (2003) is another computerized auction study employing a strategy method design; however, this experiment restricted subjects to submitting piecewise-linear bidding functions. Adopting a design in this spirit would impose too much structure on subjects' responses obfuscating the equilibrium's interesting elements.

The zero-feedback between rounds, although also unusual in auction experiments, minimizes potentially conflating issues such as learning, supergame effects, or regret. In most experiments subjects are informed of the previous round's outcome—winning bids, earnings, etc.—and they can sometimes review outcomes from earlier rounds. Recent experimental evidence suggests that observed overbidding in first-price auction experiments relative to the risk-neutral Nash equilibrium strategy, a phenomenon surveyed comprehensively by Kagel (1995) and Kagel & Levin (2008), can be partially attributed to the "usual" feedback offered during experiments. Neugebauer & Selten (2006) suggest that such an information structure biases learning while Filiz-Ozbay & Ozbay (2007) and Engelbrecht-Wiggans & Katok (2007) contend that it leads to anticipated regret. These effects in turn manifest themselves as relative overbidding.[9] Experiments by Neugebauer & Perote

[8]In a more complicated auction-like setting Kotowski (2010b) found no statistically significant difference between bids elicited using this hybrid design and a standard design where subjects placed only one bid every round.

[9]Risk aversion has also been suggested to rationalize observed overbidding in auction experiments (see the December 1992 issue of the *American Economic Review*). Some non-

(2008) suggest that providing no feedback between rounds leads to *slight* underbidding relative to the risk-neutral Nash equilibrium with bids converging to the equilibrium as the experiment progresses. Looking forward to the experimental results, and in response to this recent literature, the phenomenon of bidding closer to the risk-neutral Nash equilibrium prediction is *not* observed in this study despite the strategy method design arguably offering subjects a better platform in which to engage in introspective learning.[10] Thus, rather than interpreting the limit on feedback as a means to get bidding more in line with equilibrium's predictions, this study uses it to get a cleaner test of the theory limiting previously identified confounds.

The introduction of budget constraints on an auction-by-auction basis also addresses the subtle issue of "cash-balance effects" seen in some auction studies.[11] In many experiments, such as Rose & Levin (2008) or Andreoni *et al.* (2007), subjects carry an earnings account between rounds with wins contributing to and losses subtracting from this total. Although more salient in common-value settings where the winner's curse leads to losses for auction winners, Ham *et al.* (2005) note that the cash balances carried by bidders between rounds can bias bidding even in private-value experiments. With per auction budgets, a player's cash balance is a parameter under the experimenter's control.

With the adopted parameters of $p = 0.7$ and $\underline{w} = 3.50$, $\hat{s} = 4.93691$ and $\hat{s}' = 6.05617$. The equilibrium bidding strategy of an unconstrained bidder is:

$$\beta(s,10) = \begin{cases} \frac{s}{2} & \text{if } s \in [0, 4.93691] \\ \frac{0.03s^2+5.20556}{0.06s+1.4} & \text{if } s \in (4.93691, 10] \end{cases}.$$

The constrained bidder's equilibrium strategy is:

$$\beta(s,3.5) = \begin{cases} \frac{s}{2} & \text{if } s \in [0, 4.93691] \\ \frac{0.07s^2+0.731193}{0.14s+0.296215} & \text{if } s \in (4.93691, 6.05617] \\ 3.5 & \text{if } s \in (6.05617, 10] \end{cases}.$$

equilibrium bidding models, such as Level-k thinking, have also been proposed to explain these effects (Crawford & Iriberri, 2007). This experiment does not seek to explain sources of overbidding; rather, it considers it to be a feature of the data and seeks to understand the impact of budget constraints taking overbidding as given.

[10]For instance, by submitting several bids at once it may be easier to implement a monotonic bidding strategy or to better see how one's bids stand in relation to each other.

[11]Selecting only one auction for payment also controls for this effect.

This parameterization represents a compromise ensuring a prominence of the equilibrium's novel elements, and the "naturalness" of the visible parameters.

4.3 Experiment Results

The experiments are tailored to investigate the novel features of the equilibrium bidding strategy. This section begins by reviewing data from Treatment 1, which is the standard first-price auction without budget constraints. The purpose of this review is to ensure a comparability of the collected data across the sessions and with other experiments. We examine novel elements introduced by private budget constrains in subsection 4.3.2.

4.3.1 Bidding in Treatment 1

Recall that Treatment 1 corresponds to the standard two-player first-price auction with an equilibrium strategy $b_f(s) = \frac{s}{2}$. Figure 4.2 presents a scatter plot of value-bid pairs pooling across all sessions and rounds. For reference, the equilibrium bidding strategy and a 45-degree line are also displayed. A cursory examination of the scatter plot suggests that overbidding relative to the risk-neutral Nash equilibrium bidding strategy is a characteristic of the data. This is a standard observation in first-price auction experiments. Table 4.2 presents descriptive statistics of the submitted bids across sessions in Treatment 1.

As the equilibrium bidding strategy is linear in type, the linear regression

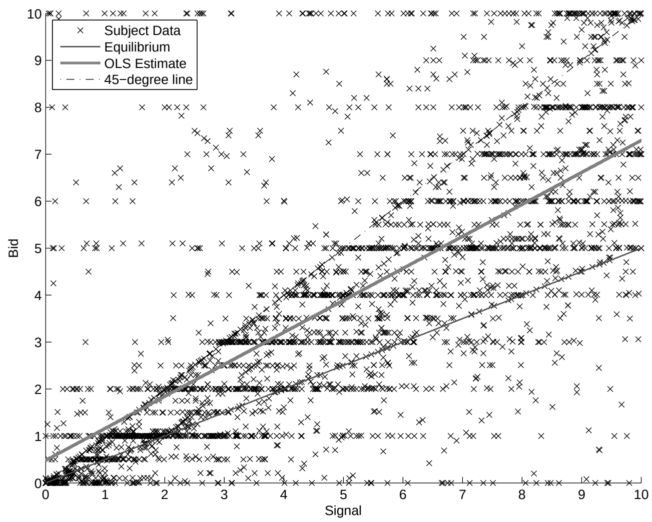
$$b_{i,t} = \beta_0 + \beta_1 s_{i,t} + \epsilon_{i,t}. \tag{4.2}$$

is a natural candidate to better organize the data.[12] $b_{i,t}$ is subject i's bid in auction t as a function of his or her signal $s_{i,t}$ and an idiosyncratic error $\epsilon_{i,t}$ Table 4.3 reports the results of this regression with subject, round, and subject × round fixed effects. For reference, the same regressions are performed using data from Palfrey & Pevnitskaya (2008) who perform a first-price auction experiment with two bidders and uniformly distributed valuations using a more traditional experimental design.[13]

[12] A Lowess regression yields a fitted curve that is very well approximated by a straight line; therefore, I will focus on linear regressions in discussing the data.

[13] In Palfrey & Pevnitskaya (2008) valuations are uniformly distributed on $[0, 700]$, subjects bid on one auction per round, and received feedback after each round. These data are rescaled to correspond to the scale seen here.

Figure 4.2: Bids in Treatment 1 (All Sessions)



Of the four specifications, Model (4) which includes a full set of subject × round fixed effects is the preferred specification given the heterogeneity seen in individual bidding strategies. Across all fixed-effect specifications, the estimates of the slope of the bidding function, β_1, are consistent with secular overbidding relative to the risk-neutral strategy—a common finding. The estimated parameters are comparable to those seen in the data from Palfrey & Pevnitskaya (2008).

An unusual feature of the collected data is the high preponderance of subjects submitting bids in excess of their valuations. In Treatment 1, 17.86 % of bids were in excess of a subject's valuations. Such behavior is difficult to reconcile with any standard interpretation of rational economic decision making. Despite this feature, no data are excluded from the empirical analysis and we interpret such departures from equilibrium behavior as noise that is characteristic of the data.

Table 4.2: Descriptive Statistics of Submitted Bids in Treatment 1

Session	Observations	Mean	St.Dev.	Skewness	Min	Max
All	2900	3.93	2.83	0.50	0	10
1	1100	3.90	2.76	0.36	0	10
2	900	3.71	2.74	0.61	0	10
3	900	4.17	2.99	0.51	0	10

4.3.2 Bidding in Treatment 2

Figures 4.3 and 4.4 present scatter plots of value-bid pairs for unconstrained and constrained bidders pooling across all sessions. For reference, the equilibrium bidding strategy and a 45-degree line are also plotted. In total, 44 subjects were constrained bidders and 14 subjects were unconstrained. Table 4.4 presents descriptive statistics of bids submitted by constrained and unconstrained bidders in Treatment 2.

An examination of the scatter plots and table suggest that budget constraints influence the bidding behavior of constrained and unconstrained bidders. Beyond the purely mechanical effect of limiting bids submitted by constrained bidders, the possible budget constraint attenuated the bids of many unconstrained bidders due to the reduced competition in Treatment 2. The presence of a discontinuity in the unconstrained bidders' strategy is a plausible conclusion suggested by Figure 4.3 which will be investigated below.

Discontinuities in Bidding Strategies

A key equilibrium prediction is the discontinuity in both constrained and unconstrained bidders' strategies. Here we focus on the unconstrained bidder's strategy. As noted above, Figure 4.3 is suggestive of the presence of a structural break in the bidding strategy of a unconstrained bidder.

As the theoretical model posits a break at $\hat{s} = 4.93691$, a simple test considers the following piecewise-linear model of bidding for unconstrained bidders:

$$b_{i,t} = \beta_0 + \beta_1 s_{i,t} + \delta_{i,t} \left(\beta_2 + \beta_3 s_{i,t}\right) + \epsilon_{i,t} \tag{4.3}$$

where $\delta_{i,t} = \mathbf{1}(s_{i,t} > \hat{s})$ is an indicator for valuations above $\hat{s}$. Although the

Table 4.3: Bidding in Treatment 1. OLS parameter estimates. Heteroskedasticity-consistent standard errors in parenthesis. (* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$)

	Sample		(1)	(2)	(3)	(4)
(A)	All Sessions	β_0	0.480***	-0.065	0.769***	0.637*
	[Obs = 2900]		(0.073)	(0.166)	(0.112)	(0.289)
		β_1	0.681***	0.672***	0.680***	0.684***
			(0.013)	(0.010)	(0.013)	(0.009)
(B)	Session 1	β_0	0.647***	0.183	0.834***	0.847**
	[Obs = 1100]		(0.121)	(0.183)	(0.169)	(0.324)
		β_1	0.637***	0.620***	0.635***	0.641***
			(0.023)	(0.019)	(0.022)	(0.018)
(C)	Session 2	β_0	0.64***	-0.726***	1.260***	-0.790***
	[Obs = 900]		(0.138)	(0.109)	(0.225)	(0.224)
		β_1	0.632***	0.648***	0.628***	0.659***
			(0.024)	(0.016)	(0.024)	(0.012)
(D)	Session 3	β_0	0.112	0.189	0.209	0.630***
	[Obs = 900]		(0.112)	(0.171)	(0.183)	(0.137)
		β_1	0.781***	0.758***	0.781***	0.763***
			(0.021)	(0.016)	(0.183)	(0.016)
(E)	Palfrey & Pevnitskaya (2008)	β_0	0.283***	0.235	0.471***	-
	[Obs = 240]		(0.065)	(0.315)	(0.129)	-
		β_1	0.664***	0.657***	0.664***	-
			(0.0174)	(0.016)	(0.017)	-
Subject FE			-	•	-	-
Round FE			-	-	•	-
Subject × Round FE			-	-	-	•

Figure 4.3: Bids in Treatment 2 by Unconstrained Bidders (All Sessions)

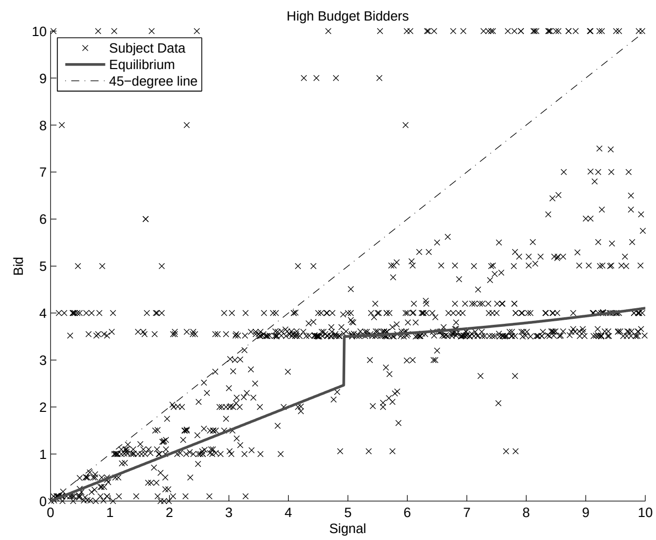


Figure 4.4: Bids in Treatment 2 by Constrained Bidders (All Sessions)

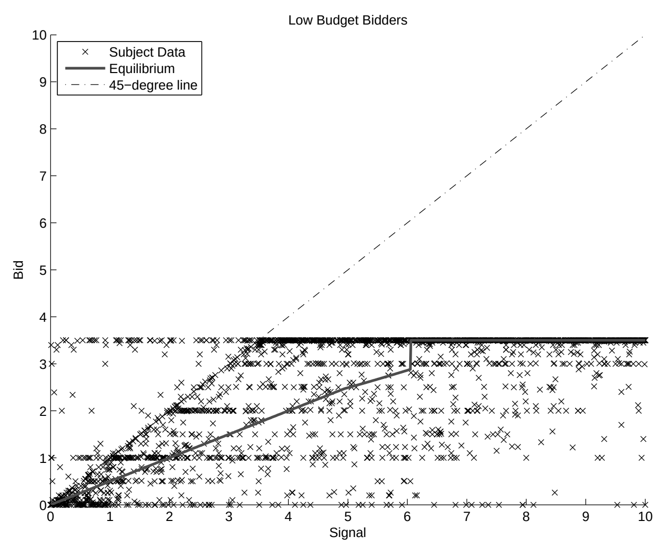


Table 4.4: Descriptive Statistics of Submitted Bids in Treatment 2

Unconstrained Bidders

Session	Observations	Mean	St.Dev.	Min	Max
All	700	3.519	2.279	0	10
1	200	3.341	1.561	0.05	7.48
2	400	3.375	2.435	0	10
3	100	4.450	2.620	0.1	10

Constrained Bidders

Session	Observations	Mean	St.Dev.	Min	Max
All	2200	2.385	1.271	0	3.50
1	900	2.550	1.190	0	3.50
2	500	2.430	1.254	0	3.50
3	800	2.171	1.338	0	3.50

equilibrium strategy of an unconstrained bidder is *not* linear for valuations above $\hat{s}$, it is nearly so and (4.3) offers a useful and easily interpretable approximation.[14] Table 4.5 reports the results of this regression with various fixed-effect controls.

The estimated bidding strategy offers support for a break in the bidding strategy of unconstrained bidders. An F-test for parameter constancy ($\beta_2 = 0$ and $\beta_3 = 0$) is rejected across all specifications.

Going beyond identifying the presence of a break, the equilibrium model offers several qualitative predictions concerning how the bidding strategy should behave within various ranges of the type space. In particular, the predictions that $\beta_1 = 0.5$, $\beta_2 > 0$, and $\beta_3 < 0$ are all individually supported by the above analysis. From $\beta_2 > 0$ we can conclude that unconstrained bidders increase their bid at relatively high valuations. On the other hand, $\beta_3 < 0$ implies a reduction in the slope of the bidding strategy at relatively high valuations. This is reflective of the decline in relative competition at higher bid levels due to the stratification of competition introduced by budget constraints.

[14]The slope of an unconstrained bidder's equilibrium strategy ranges continuously from 0.0508 to 0.1769.

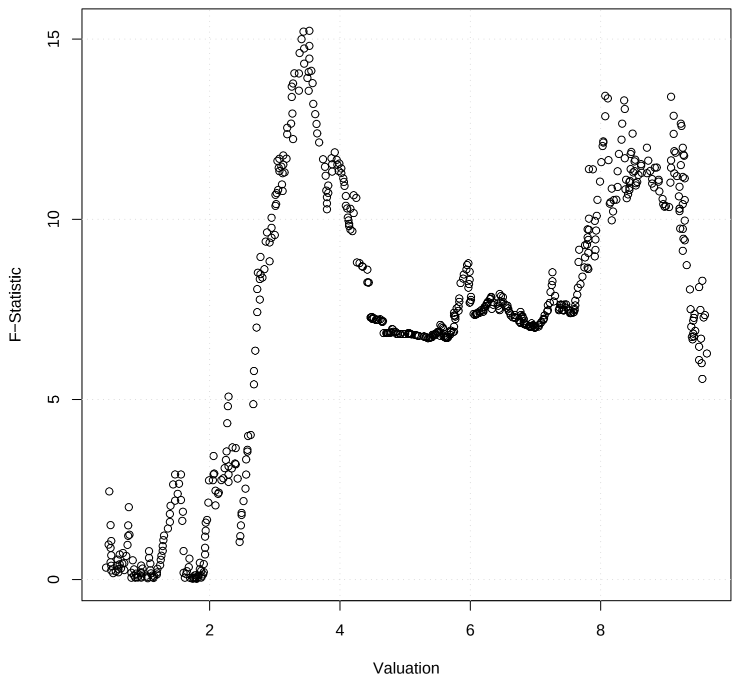
Table 4.5: Estimates for Unconstrained Bidder's Strategy. Heteroskedasticity-consistent standard errors in parenthesis. (OBS = 700, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$)

Variable	(1)	(2)	(3)	(4)
β_0	0.990***	1.62***	0.943**	1.230*
	(0.238)	(0.265)	(0.322)	(0.521)
β_1	0.565***	0.547***	0.566***	0.53***
	(0.076)	(0.048)	(0.077)	(0.047)
β_2	1.25*	1.360***	1.220*	1.27***
	(0.516)	(0.343)	(0.512)	(0.305)
β_3	-0.257**	-0.237***	-0.255**	-0.212***
	(0.098)	(0.063)	(0.098)	(0.059)
Subject FE	-	•	-	-
Round FE	-	-	•	-
Subject-Round FE	-	-	-	•
Regressors	4	17	8	73
R^2	0.31	0.74	0.31	0.79

While the preceding results are supportive of the qualitative implications of private budget constraints, the model's quantitive predictions continue to exhibit the shortcomings typical of the risk-neutral Nash equilibrium bidding model when applied to laboratory data. For example, with Subject-Round fixed effects, the estimate of the strategy's slope above $\hat{s}$ is $\hat{\beta}_1 + \hat{\beta}_3 = 0.53 - 0.212 = 0.318$. The 95% (asymptotic) confidence interval for this value is $CI_{0.95} = [0.247, 0.389]$. This confidence interval is above the slope's theoretical maximum value of 0.1769; therefore, unconstrained bidders are (still) too aggressive relative to the risk-neutral Nash equilibrium benchmark. However, this finding is consistent with the empirical regularity of overbidding relative to the risk-neutral model.

Whereas the preceding analysis suggests that the parameters characterizing the bidding strategy employed by low- and high-valuation bidders is not constant, one cannot conclude that the structural break occurs precisely at $\hat{s} \approx 4.93$. As overbidding relative to the risk-neutral Nash equilibrium bidding strategy has already been identified in the data, it is natural to con-

Figure 4.5: F-Tests for a Structural Break in the Unconstrained Bidder's Strategy (All Sessions)



sider discontinuities arising at a relatively lower signals than $\hat{s} \approx 4.93$. To identify such locations we can consider a series of F-tests. Figure 4.5 plots the value of the F-statistic as a function of the hypothesized break point in (4.3) in the model without any fixed effect controls. The maximum value, $\sup F = 15.23$, occurs at $\hat{\hat{s}} = 3.53$. Break points around 3.5 are a natural candidate to consider as that value is focal in the experiment's design.

To conclude this subsection, it is clear that unconstrained bidders recognize the strategic opportunity their extra budget affords them and their bidding behavior does display the qualitative effects identified by the equilibrium analysis. The model, however, continues to exhibit the quantitive shortcomings of the risk-neutral Nash equilibrium bidding model when applied to auction experiments.

Strategy Congruence Among Low Valuation Bidders

The coincidence of equilibrium bidding strategies for low-valuation bidders is one of the model's key features. In equilibrium, both constrained and unconstrained bidders with $s < \hat{s}$ bid according to the strategy $\beta(s, w) = b_f(s) = \frac{s}{2}$. Moreover, this corresponds to the equilibrium bidding strategy in the model without budget constraints. For low-valuation bidders, the presence of budget constraints in the environment is strategically irrelevant.

To test for the congruence between bidding strategies among low valuation bidders we will first restrict our sample to observations where the subject received a valuation below a cutoff: $s_{i,t} \leq \bar{c}$. We consider two possible cutoff values:

(a) In sample (a), we restrict attention to the case where $s_{i,t} \leq 3.5 = \underline{w}$. Whereas the equilibrium bidding model predicts the congruity of bidding strategies for a greater range of valuations, up to $\hat{s}$, restricting attention to this restricted sample limits any "mechanical" effects that the introduction of budget constraints may have. All undominated bids are feasible for both constrained and unconstrained bidders.

(b) In sample (b), we restrict attention to the case where $s_{i,t} \leq \hat{s} \approx 4.93$. As clear from the discussion in previous sections, this sample selection is motivated entirely by the theoretical predictions of the equilibrium bidding model.

In either sample we posit that bidding behavior will be consistent across treatments and, guided by theory, it will be a linear function of the valuation. Therefore, we first consider the specification

$$b_{i,t} = \beta_0 + \beta_1 s_{i,t} + \delta_{i,t} (\beta_2 + \beta_3 s_{i,t}) + \epsilon_{i,t} \tag{4.4}$$

where $\delta_{i,t}$ is an indicator variable for Treatment 2. Additionally one may posit that the introduction of budget constraints into the bidding environment may encourage an anchoring effect among high or low budget bidders implying systematically different bidding behavior among the two groups in Treatment 2. Therefore, we also consider the model

$$b_{i,t} = \beta_0 + \beta_1 s_{i,t} + \delta_{i,t} (\beta_2 + \beta_3 s_{i,t} + \beta_4 \bar{w}_{i,t}) + \epsilon_{i,t} \tag{4.5}$$

Table 4.6: Estimated parameters from Models 4.4(a & b) and 4.5(a & b). Heteroskedasticity-consistent standard errors in parenthesis. (* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$)

Variable	4.4(a)	4.5(a)	4.4(b)	4.5(b)
β_0	0.136	0.129	0.164	0.172
	(0.122)	(0.120)	(0.117)	(0.117)
β_1	0.630***	0.630***	0.645***	0.645***
	(0.047)	(0.047)	(0.028)	(0.028)
β_2	-0.159	-0.146	-0.125	-0.140
	(0.116)	(0.112)	(0.097)	(0.095)
β_3	-0.067	-0.067	-0.079	-0.078*
	(0.057)	(0.057)	(0.034)	(0.034)
β_4	-	-0.055	-	0.062
	-	(0.148)	-	(0.124)
Subject FE	•	•	•	•
Observations	2008	2008	2813	2813
R^2	0.52	0.52	0.57	0.57

where $\bar{w}_{i,t}$ is an indicator variable equal to 1 if subject i in auction t had a high budget level. In both models (4.4) and (4.5)—and across both samples—strategy congruence implies coefficients of zero for parameters β_2, β_3, and β_4.

Table 4.6 presents the result of OLS estimates of these models with a collection of subject fixed-effects. The estimated slope parameter, β_1 is essentially invariant across both samples and across both models. Moreover, with the exception of the β_3 parameter in model 4.5(b) we fail identify significant differences in the bidding of low-valuation subjects across treatments. The estimates of the β_4 parameter do not support the hypothesis that introduction of budget constraints affected the bidding of high- and low-budget bidders in disparate ways in Treatment 2.

The significant coefficient estimate for β_3 in model 4.5(b) can be partly explained by the mechanical effect of budget constraints. Taking as given the preponderance of experimental subject to overbid relative to the risk neutral Nash equilibrium strategy ($\beta_2 \approx 0.64$), there is a greater likelihood a subject may be "constrained" in the expanded sample (b). In sample (a), where budget constraints are genuinely ancillary information and all undominated bids are feasible for all bidders there is no significant change in bidding behavior.

Aggregate Predictions

Although the equilibrium bidding model gives predictions at the individual level, and its qualitative implications are reflected in the data, the model also offers aggregate predictions concerning the efficiency and the revenue of the auction. Arguably, the ability to predict these facts is the model's most important feature. To measure efficiency, I focus on ordinal efficient outcomes; that is, whether the winner of the auction had the greatest valuation.[15] Revenue is measured by the selling price. Tables 4.7 and 4.8 report means of these variables from the experimental sessions along with 95% confidence intervals.

Focusing first on revenue, both treatments confirm the presence of overbidding with average revenue comfortably exceeding the theoretical benchmarks. The revenue in Treatment 2 decline due to two effects. First, the direct effect of budget constrains mechanically constrains many bidders. Second, the strategic effect notwithstanding, the stratification of competitions encourages less aggressive bidding on the margin by high-budget, high-valuation bidders. Together these empirically salient points serve to depress revenue.

In contrast to the decline in revenue between treatments, the allocative efficiency of the auction did not change between treatments in an economically significant manner. This does suggest a reassuring hypothesis that, given the departures from equilibrium bidding already present, the additional effect of budgets on allocative efficiency may be small. A caveat in this interpretation is that the experiment's parameters are not necessarily tailored to examine efficiency effects and the theoretical decline of 15% may be too subtle to detect given the noise characteristic of bidding in auction experiments.

4.4 Discussion

This chapter is the first to investigate individual bidding behavior in first-price, independent-private value-auctions where bidders face a private budget

[15] This is the same as the measure $\mathcal{E}_A\%$ in Güth *et al.* (2005).

Table 4.7: Average Seller Profits and Allocative Efficiency in Treatment 1

Session	Auctions	Revenue		Efficiency	
		Mean	$CI_{0.95}$	Mean	$CI_{0.95}$
Theory	-	3.33	-	1.00	–
All	1450	5.60	[5.47, 5.73]	0.787	[0.766, 0.808]
1	550	5.54	[5.34, 5.75]	0.760	[0.724, 0.796]
2	450	5.31	[5.07, 5.55]	0.793	[0.756, 0.831]
3	450	5.96	[5.71, 6.22]	0.813	[0.777, 0.849]

Table 4.8: Average Seller Profits and Allocative Efficiency in Treatment 2

Session	Auctions	Revenue		Efficiency	
		Mean	$CI_{0.95}$	Mean	$CI_{0.95}$
Theory	-	3.05	-	0.856	–
All	1450	3.50	[3.42, 3.58]	0.752	[0.729, 0.774]
1	550	3.39	[3.32, 3.46]	0.716	[0.679, 0.754]
2	450	3.83	[3.66, 4.01]	0.798	[0.761, 0.835]
3	450	3.30	[3.15, 3.45]	0.749	[0.709, 0.789]

constraint. It therefore complements the main theoretical component of this study and serves as a first, controlled step in bringing the model to data.

Although the equilibrium bidding model inherits the quantitative shortcomings of the risk-neutral equilibrium model from usual auction experiments (without budget constraints), its qualitative predictions are consistent with the response of subjects to this novel change in the auction environment. The strategic, and not simply mechanical, effects of budgets are recognized by bidders and many qualitative predictions are supported by the data. Low-valuation bidders do not react strategically to the introduction of budget constraints. High-budget bidders on the other hand adjust their bidding strategy taking advantage of their fortunate position.

Chapter 5

Conclusion

Private budget constraints are a key feature of many economic environments and, as argued above, have a deep strategic effect which organizes economic interaction. Although monotone equilibria, properly understood, exist under general circumstances, describing equilibrium strategies in a first-price auction demands a marked amendment of existing intuition. Private budgets have both direct and strategic effects on equilibrium bids and they can subtly interact with preferences changing bidder's incentives.

Budget constraints also change the nature of competition in the first-price auction. Within a general environment competition becomes endogenously stratified along the budget dimension. The option to exploit other bidders' budget constraints introduces a strategic effect that breaks the natural relationship between valuations and bids and suggest many possibilities for further research. For instance, the degree to which the inefficiencies introduced by budget limits can be ameliorated by resale is an important issue awaiting resolution. Similarly, extending the analysis to more complex auction settings or addressing deeper technical matters within the standard environment, such as equilibrium uniqueness, are open questions. Untangling the more subtle relationship between bids, valuations, *and* budgets stands as a new challenge for theoretical and empirical researchers alike.

Appendix A

Technical Appendix

A.1 Example 1: Further Discussion

This appendix presents a detailed treatment of the model from in Example 1.

A.1.1 Example 1: The Uniform Two-Bidder Case

When signals are distributed uniformly, the model admits a closed-form expression for equilibrium strategies. Reproducing the strategy:

$$\beta(s,w) = \begin{cases} \frac{s}{2} & \text{if } s \leq \hat{s}, w \in \{\underline{w}, 1\} \\ \frac{s^2(p-1)-2k_1}{2p(s-1)-2s} & \text{if } \hat{s} < s \text{ and } w = 1 \\ \frac{ps^2+2k_2}{2\hat{s}-2p\hat{s}+2sp} & \text{if } s \in (\hat{s}, \hat{s}'] \text{ and } w = \underline{w} \\ \underline{w} & \text{if } \hat{s}' < s \text{ and } w = \underline{w} \end{cases}. \tag{A.1}$$

When $p \neq 1/2$, the constants $\{\hat{s}, \hat{s}', k_1, k_2\}$ are:

$$\hat{s} = \frac{\underline{w}(p-1) + p - \sqrt{-2\underline{w}^2 p + \underline{w}^2 + (\underline{w}-1)^2 p^2}}{2p-1},$$
$$\hat{s}' = \frac{(2\underline{w} - 2\underline{w}p + 2p - 2)\sqrt{-2\underline{w}^2 p + \underline{w}^2 + (\underline{w}-1)^2 p^2}}{(\underline{w}-1)(1-2p)^2} + \frac{2\underline{w}^2 p^2 - 4\underline{w}^2 p + 2\underline{w}^2 - \underline{w} - 2p^2 + 2p}{(\underline{w}-1)(1-2p)^2},$$
$$k_1 = (\hat{s} - p(\hat{s}-1))\,\underline{w} + \frac{\hat{s}^2(p-1)}{2},$$
$$k_2 = \frac{(1-p)\hat{s}^2}{2}.$$

If $p = 1/2$, the constants are

$$\hat{s} = \frac{\underline{w}}{1-\underline{w}} \qquad \text{and} \qquad \hat{s}' = \frac{\underline{w}(\underline{w}^2 + \underline{w} - 1)}{(\underline{w}-1)^3}.$$

$\{k_1, k_2\}$ are unchanged. Verification that (A.1) is an equilibrium follows from Appendix A.1.2 which analyzes this model's generalization.

Lemma 9. *For all $p \in [0,1]$ and $\underline{w} \in [0,1/2]$ the expected revenue of the first-price auction with private budget constrains is less that the expected revenue when there are no budget constraints.*

Lemma 9 is not trivial as some types of bidders increase their bids due to private budget constraints. Moreover, the distribution of a player's bids absent budget constraints does not (first-order) stochastically dominate the distribution of bids once budgets are introduced. Finally, the revenue inequality is tight (say at $p = 0$).

Proof. As types are independent, it is sufficient to show that the expected payment of a bidder declines once private budgets are introduced. With no budget constraints, the expected payment of a bidder is

$$\int_0^1 s b_f(s)\, ds = \int_0^1 \frac{s^2}{2} ds = \frac{1}{6}.$$

Using a standard envelope theorem argument to express equilibrium expected utility, we can write the expected payment of a high budget bidder with value-signal s as

$$m(s,1)=\begin{cases} s^2-\int_0^s xdx & \text{if } s\le\hat{s} \\ (p+(1-p)s)s-\int_0^{\hat{s}} xdx-\int_{\hat{s}}^s(p+(1-p)x)dx & \text{if } s>\hat{s} \end{cases}.$$

Similarly, for a low budget bidder:

$$m(s,\underline{w})=\begin{cases} s^2-\int_0^s xdx & \text{if } s\le\hat{s} \\ \quad(ps+(1-p)\hat{s})s-\int_0^{\hat{s}} xdx & \\ \qquad -\int_{\hat{s}}^s(px+(1-p)\hat{s})dx & \text{if } s\in(\hat{s},\hat{s}'] \\ \quad(ps+(1-p)\hat{s})s-\int_0^{\hat{s}} xdx & \\ \qquad -\int_{\hat{s}}^{\hat{s}'}(px+(1-p)\hat{s})dx & \text{if } s>\hat{s}' \\ \qquad +\int_{\hat{s}'}^s p\left(\frac{\hat{s}'+1}{2}\right)+(1-p)\hat{s}dx & \end{cases}$$

A bidder's ex-ante expected payment is thus a function of p and $\underline{w}$

$$\Pi(p,\underline{w})=p\int_0^1 m(s,\underline{w})ds+(1-p)\int_0^1 m(s,1)ds.$$

Some careful algebra allows us to express $\Pi(p,\underline{w})$ as

$$\begin{aligned}
&\Pi(p,\underline{w})\\
&=\int_0^{\hat{s}}\frac{s^2}{2}ds+\int_{\hat{s}}^1 p(1-p)\hat{s}ds+\int_{\hat{s}}^{\hat{s}'}(p^2+(1-p)^2)\frac{s^2}{2}ds\\
&\quad+\int_{\hat{s}'}^1\left[p^2\frac{\hat{s}^2}{2}+(1-p)^2\frac{s^2}{2}\right]ds\\
&=\int_0^{\hat{s}}\frac{s^2}{2}ds+\int_{\hat{s}}^1 p(1-p)\hat{s}ds+\int_{\hat{s}}^1(1-p)^2\frac{s^2}{2}+\int_{\hat{s}}^{\hat{s}'}p^2\frac{s^2}{2}ds+\int_{\hat{s}'}^1 p^2\frac{\hat{s}^2}{2}ds\\
&\le\int_0^{\hat{s}}\frac{s^2}{2}ds+\int_{\hat{s}}^1 p(1-p)\hat{s}ds+\int_{\hat{s}}^1(1-p)^2\frac{s^2}{2}+\int_{\hat{s}}^{\hat{s}'}p^2\frac{s^2}{2}ds+\int_{\hat{s}'}^1 p^2\frac{s^2}{2}ds\\
&=\frac{\hat{s}^3}{6}+p(1-p)(1-\hat{s})\hat{s}+(1-p)^2\frac{(1-\hat{s}^3)}{6}+(1-p)^2\frac{(1-\hat{s}^3)}{6}+p^2\frac{(1-\hat{s}^3)}{6}\\
&=\frac{1}{6}(1-2(p-1)p(\hat{s}-1)^3)
\end{aligned}$$

The inequality follows from changing $\hat{s}$ to s in the final term of the expression. As $2(p-1)p(\hat{s}-1)^3\ge 0$, it follows that $\Pi(p,\underline{w})\le 1/6$. □

A.1.2 Example 1: The General Case

Consider the setting of Example 1. Suppose that there are $N \geq 2$ bidders and values are independently and identically distributed according to $H\colon [0,1] \to [0,1]$ which is a strictly increasing distribution admitting a continuous density $h(\cdot)$. Noting symmetry, subscripts are omitted. The equilibrium strategy in this model without budget constraints is

$$b_f(s) = s - \frac{\int_0^s H(y)^{N-1} dy}{H(s)^{N-1}}. \tag{A.2}$$

As a regularity condition, suppose that $\underline{w} < b_f(1)$; hence, $b_f(s)$ is not feasible upon the introduction of budget constraints.

We will consider strategies in a similar class to the claimed equilibrium for Example 1. For notation, let $h_n(s) \equiv \frac{d}{ds} H(s)^n = nH(s)^{n-1}h(s)$ and $\xi(k) \equiv \binom{N-1}{k} p^{N-1-k}(1-p)^k$. Consider the following functions $a_j\colon [0,1] \to [0,1]$,

$$a_1(s) = H(s)^{N-1} \tag{A.3}$$

$$a_2(s) = \sum_{k=0}^{N-1} \xi(k) H(s)^k \tag{A.4}$$

$$a_3(s) = \sum_{k=0}^{N-1} \xi(k) H(s)^{N-1-k} H(\hat{s})^k \tag{A.5}$$

$$a_4(s) = \sum_{k=0}^{N-1} \sum_{m=0}^{N-1-k} \binom{N-1-k}{m} \frac{\xi(k) H(\hat{s})^k H(\hat{s}')^{N-1-k-m} \left(1 - H(\hat{s}')\right)^m}{m+1} \tag{A.6}$$

Lemmas 10 and 11 below will define $\hat{s}, \hat{s}' \in [\underline{w}, 1]$. Now let $a\colon [0,1] \times \{\underline{w}, 1\} \to [0,1]$ be

$$a(s,w) = \begin{cases} a_1(s) & \text{if } s \in [0,\hat{s}], w \in \{\underline{w}, 1\} \\ a_2(s) & \text{if } s \in (\hat{s}, 1], w = 1 \\ a_3(s) & \text{if } s \in (\hat{s}, \hat{s}'], w = \underline{w} \\ a_4(s) & \text{if } s \in (\hat{s}', 1], w = \underline{w} \end{cases}$$

And finally set[1]

$$\beta(s,w) = s - \frac{\int_0^s a(y,w) dy}{a(s,w)}. \tag{A.7}$$

[1]For completeness, $\beta(0,w) = 0$.

When all bidders follow the strategy $\beta(s,w)$, $a(s,w)$ is the probability that a bidder of type (s,w) wins the auction. (This probability incorporates the uniform tie breaking rule.)

Lemma 10. *There exists a unique* $\hat{s} \in (\underline{w}, b_f^{-1}(\underline{w}))$ *such that* $\int_0^{\hat{s}} a_1(y)dy = a_2(\hat{s})\,(\hat{s} - \underline{w})$.

Proof. Let $\chi(s) = \int_0^s H(y)^{N-1}dy$ and $\tau(s) = \sum_{k=0}^{N-1} \xi(k) H(s)^k (s - \underline{w})$. Evidently, $\tau(\underline{w}) = 0 < \chi(\underline{w})$. Let $b_f(\tilde{s}) = \underline{w}$. Then, using $\underline{w} = \tilde{s} - \frac{\int_0^{\tilde{s}} H(y)^{N-1}dy}{H(\underline{s})^{N-1}}$,

$$\tau(\tilde{s}) > \sum_{k=0}^{N-1} \xi(k) H(\tilde{s})^{N-1}(\tilde{s} - \underline{w}) = H(\tilde{s})^{N-1}(\tilde{s} - \underline{w}) = \int_0^{\tilde{s}} H(y)^{N-1}dy = \chi(\tilde{s}).$$

By continuity, there exists $\hat{s} \in (\underline{w}, \tilde{s})$ such that $\tau(\hat{s}) = \chi(\hat{s})$. To establish uniqueness of $\hat{s}$, differentiate $\tau(s)$ and $\chi(s)$:

$$\begin{aligned}\tau'(\hat{s}) &= \sum_{k=0}^{N-1} \xi(k) h_k(\hat{s})(\hat{s} - \underline{w}) + \sum_{k=0}^{N-1} \xi(k) H_k(\hat{s}) \\ &> \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k \geq \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^{N-1} = H(\hat{s})^{N-1} = \chi'(\hat{s})\end{aligned}$$

Thus, $\hat{s}$ is unique as $\tau(s)$ always crosses $\chi(s)$ from below. ◻

Henceforth, define $\hat{s}$ as in Lemma 10.

Lemma 11. *Let*

$$\hat{a}_4(x) = \sum_{k=0}^{N-1} \sum_{m=0}^{N-1-k} \frac{\binom{N-1-k}{m} \xi(k) H(\hat{s})^k H(x)^{N-1-k-m} \left(1 - H(x)\right)^m}{m+1}.$$

Then $\exists z \in (\hat{s}, 1)$ *such that* $\int_0^{\hat{s}} a_1(y)dy + \int_{\hat{s}}^{z} a_3(y)dy = \hat{a}_4(z)\,(z - \underline{w})$.

Note that $\hat{a}_4$ is a function of x while $a_4(\cdot)$, defined in (A.6), is a constant value. This lemma helps defines the $\hat{s}'$ in (A.6).

Proof. Let $\tilde{\chi}(z) = \int_0^{\hat{s}} a_1(y)dy + \int_{\hat{s}}^{z} a_3(y)dy$ and let $\tilde{\tau}(z) = \hat{a}_4(z)\,(z - \underline{w})$. Building on the proof of Lemma 10,

$$\begin{aligned}
&\tilde{\chi}(\hat{s})\\
&= \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k \,(\hat{s} - \underline{w})\\
&> \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k \,(\hat{s} - \underline{w}) \underbrace{\sum_{m=0}^{N-1-k} \frac{\binom{N-1-k}{m} H(\hat{s})^{N-1-k-m}(1-H(\hat{s}))^m}{m+1}}_{<1} = \tilde{\tau}(\hat{s}).
\end{aligned}$$

Trivially, $\tilde{\chi}(\hat{s}) < \tilde{\chi}(1)$. Furthermore, $\tau(1) = \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k (1 - \underline{w})$. Applying Lemma 10,

$$\begin{aligned}
&\tilde{\tau}(1) - \tilde{\chi}(1)\\
&= \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k (1-\underline{w}) - \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k (\hat{s} - \underline{w})\\
&\quad - \int_{\hat{s}}^{1} \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k H(y)^{N-1-k} dy\\
&= \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k (1-\hat{s}) - \int_{\hat{s}}^{1} \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k \underbrace{H(y)^{N-1-k}}_{<1} dy\\
&> \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k (1-\hat{s}) - \int_{\hat{s}}^{1} \sum_{k=0}^{N-1} \xi(k) H(\hat{s})^k dy = 0
\end{aligned}$$

Thus, $\tilde{\tau}(1) > \tilde{\chi}(1) > \tilde{\chi}(\hat{s}) > \tilde{\tau}(\hat{s})$. Hence, $\exists z \in (\hat{s}, 1)$ such that $\tilde{\chi}(z) = \tilde{\tau}(z)$. □

Let $\mathcal{Z}$ be the set of values satisfying Lemma 11. Define $\hat{s}' \equiv \inf \mathcal{Z}$.

Lemma 12. *Let $\hat{s}$ and $\hat{s}'$ be defined according to Lemmas 10 and 11. The strategy $\beta(s, w)$ defined in* (A.7) *is a symmetric equilibrium strategy in the generalization of Example 1 to N bidders and distribution of signals H.*

Proof. Consider the direct mechanism corresponding to the first-price auction when all bidder follow the strategy $\beta(s, w)$. By showing no player has incentive to misreport his type we rule out all deviations in β's range.

It is readily verified that when all players follow the strategy (A.7), a player's utility from a truthful announcement is

$$U(\beta(s,w)|s,w) = \int_0^s a(y,w)dy.$$

There are five cases to consider.

i) $\mathbf{s < \hat{s}, w = 1}$. Suppose that the player reports to be a type $x \leq \hat{s}$ instead of type s. Then,

$$\begin{aligned} U(\beta(x,w)|s,w) &= a(x,w)s - a(x,w)x + \int_0^x a(y,w)dy \\ &= \int_x^s a(x,w)dy + \int_s^x a(y,w)dy + U(\beta(s,w)|s,w) \\ &= \int_x^s a(x,w) - a(y,w)dy + U(\beta(s,w)|s,w) \qquad \text{(A.8)} \end{aligned}$$

When $x \leq y \leq s$, $a(y,w) \geq a(x,w)$. Therefore $\int_x^s a(x,w) - a(y,w)dy \leq 0$. When $x > s$, -$\int_s^x a(x,w) - a(y,w)dy \leq 0$. Thus, no such deviations are profitable for a type $(s,1)$.

Suppose instead that the player deviates to a report of $(x,1)$, $x > \hat{s}$. In this case (A.8) becomes

$$\begin{aligned} &U(\beta(x,1)|s,w) \\ &= \int_{\hat{s}}^s a(x,1) - a(y,1)dy + \int_x^{\hat{s}} a(x,1) - a(y,1)dy + U(\beta(s,1)|s,w) \\ &= U(\beta(s,1)|s,w) - \int_{\hat{s}}^s a(x,1) - a(y,1)dy - \int_x^{\hat{s}} a(x,1) - a(y,1)dy \end{aligned}$$

As $a(x,w) \geq a(y,w)$ $\forall y \leq x$, the preceding expression is less than $U(\beta(s,1)|s,w)$. Therefore, this is not a profitable deviation. Deviations to misreports of $(x,\underline{w})$ where $x > \hat{s}$ are ruled out analogously.

ii) $\mathbf{s \in (\hat{s}, 1], w = 1}$. An argument identical to that above demonstrates that such a player will not misreport his type to $(x,1)$ for any x.

Claims of being a type $(x, \underline{w})$ give a utility of

$$
\begin{aligned}
&U(\beta(x, \underline{w}); s, 1) \\
&= \int_x^s a(x, \underline{w}) - a(y, 1) dy + U(\beta(s, 1); s, 1) \\
&= \int_x^s \sum_{k=0}^{N-1} \xi(k) \left(H(x)^{N-1-k} H(\hat{s})^k - H(y)^k \right) dy + U(\beta(s, 1)|s, 1) \\
&\leq \int_x^s \sum_{k=0}^{N-1} \xi(k) \left(H(x)^{N-1} - H(y)^k \right) dy + U(\beta(s, 1)|s, 1) \qquad \text{(A.9)}
\end{aligned}
$$

As $H(y)^{N-1} \leq H(y)^k$, the first term of (A.9) is negative for all $x \in (\hat{s}, \hat{s}']$. Therefore, there is no profitable deviation for such a player. Reporting to be a type $(x, \underline{w})$ where $x > \hat{s}'$ is ruled out by bidding $\epsilon > 0$ more to break a possible tie.

iii) $\mathbf{s} < \hat{\mathbf{s}}, \mathbf{w} = \underline{\mathbf{w}}$. By the same argument as given in cases (i) and (ii) such a player will not have an incentive to misreport his type. Deviations to $x \leq \hat{s}$ are easy to rule out. Consider a report of $(x, \underline{w})$, $x \in (\hat{s}, \hat{s}']$. The expected utility from such a report is given by

$$
\begin{aligned}
&U(\beta(x, \underline{w})|s, \underline{w}) \\
&= \int_{\hat{s}}^s a(x, \underline{w}) - a(y, \underline{w}) dy + \int_x^{\hat{s}} a(x, 1) - a(y, 1) dy + U(\beta(s, \underline{w})|s, \underline{w}) \\
&= U(\beta(s, \underline{w})|s, \underline{w}) - \int_s^{\hat{s}} a(x, \underline{w}) - a(y, \underline{w}) dy - \int_{\hat{s}}^x a(x, \underline{w}) - a(y, \underline{w}) dy
\end{aligned}
$$

As $\sum_{k=0}^{N-1} \xi(k) H(x)^{N-1-k} H(\hat{s})^k > H(y)^{N-1}$ for $y \leq \hat{s}$ and $H(x)^{N-1-k} \geq H(y)^{N-1-k}$ for $x \geq y$, the player cannot profitably deviate in this manner. A simple adaptation of the above argument rules out deviations to $x > \hat{s}'$.

iv) $\mathbf{s} \in (\hat{\mathbf{s}}, \hat{\mathbf{s}}'], \mathbf{w} = \underline{\mathbf{w}}$. Deviations to $x \in [0, \hat{s}']$ are ruled out as in the previous cases. A player will not wish to report a type $(x, \underline{w})$ with $x > \hat{s}'$

as

$$U(\beta(x,\underline{w})|s,\underline{w})$$
$$= \int_{\hat{s}'}^{s} a(x,\underline{w}) - a(y,\underline{w})dy + \int_{x}^{\hat{s}'} a(x,\underline{w}) - a(y,\underline{w})dy + U(\beta(s,\underline{w})|s,\underline{w})$$
$$= U(\beta(s,\underline{w})|s,\underline{w}) - \int_{s}^{\hat{s}'} a(x,\underline{w}) - a(y,\underline{w})dy - \int_{\hat{s}'}^{x} \underbrace{a(x,\underline{w}) - a(y,\underline{w})}_{=0} dy.$$

and $a(x,\underline{w}) > a(y,\underline{w})\ \forall y \in (\hat{s},\hat{s}']$.

v) $\mathbf{s} \in (\hat{\mathbf{s}}', \mathbf{1}], \mathbf{w} = \underline{\mathbf{w}}$. It is sufficient to verify that such a player will not wish to mimic a type $x \leq \hat{s}'$.

$$U(\beta(x,\underline{w})|s,\underline{w})$$
$$= U(\beta(s,\underline{w}); s,\underline{w}) + \int_{\hat{s}'}^{s} \underbrace{a(x,\underline{w}) - a(y,\underline{w})}_{=0} dy - \int_{\hat{s}'}^{x} \underbrace{a(x,\underline{w}) - a(y,\underline{w})}_{>0} dy$$

Therefore, no type has any incentive to mimic the bid of any other type. Bids outside of $\mathrm{Im}[\beta(s,w)]$ are dominated by bids of $\beta(1,1)$ or $\beta(\hat{s}',1)$. □

Corollary 4. *When $N = 2$ and $H(s) = s$,* (A.1) *is an equilibrium of the first-price auction with budget constraints.*

Proof. It can be verified that (A.1) can be expressed as in Lemma 12. An alternative derivation of the equilibrium strategies is possible by solving the appropriate differential equations with carefully chosen boundary conditions. □

The following are comparative static results for the general case.

Lemma 13. *As $N \to \infty$, $\hat{s} \to \underline{w}$.*

Proof. Let $\hat{s}_N$ denote $\hat{s}$ when there are N bidders. For all N, $\underline{w} \leq \hat{s}_N \leq \underline{s}_N$ where $\tilde{s}_N$ solves $\underline{w} = \tilde{s}_N - \int_0^{\tilde{s}_N} \left(\frac{H(y)}{H(\tilde{s}_N)}\right)^{N-1} dy$. But, $0 \leq \tilde{s}_N - \underline{w} \leq \int_0^1 \left(\frac{H(y)}{H(\underline{w})}\right)^{N-1} dy \xrightarrow{N\to\infty} 0$. □

With Lemma 13 we can conclude that as the number of bidders grows without bound, the discontinuity in an unconstrained bidder's strategy vanishes. A similar result holds for the discontinuity in $\beta(s,\underline{w})$. Indeed, the overall limiting strategy is $\beta(s,w) = \min\{s,w\}$.

Lemma 14. *$\hat{s}$ is increasing in $\underline{w}$.*

Proof. Implicitly differentiating the indifference condition

$$\int_0^{\hat{s}} a_1(y)dy = a_2(\hat{s})\left(\hat{s} - \underline{w}\right),$$

$$\begin{aligned} &\frac{d}{d\hat{s}}\int_0^{\hat{s}} H(y)^{N-1}dy\frac{d\hat{s}}{d\underline{w}} \\ &= \frac{d}{d\hat{s}}\sum_{k=0}^{N-1}\xi(k)H(\hat{s})^{N-1}\frac{d\hat{s}}{d\underline{w}}\left(\hat{s}-\underline{w}\right) + \sum_{k=0}^{N-1}\xi(k)H(\hat{s})^k\left(\frac{d\hat{s}}{d\underline{w}} - 1\right). \end{aligned}$$

Rearranging the expression gives

$$\frac{d\hat{s}}{d\underline{w}} = \frac{-\sum_{k=0}^{N-1}\xi(k)H(\hat{s})^k}{H(\hat{s})^{N-1} - \sum_{k=0}^{N-1}\xi(k)H(\hat{s})^k - (\hat{s}-\underline{w})\sum_{k=0}^{N-1}\xi(k)h_k(\hat{s})}.$$

Noting that $H(\hat{s})^{N-1} - \sum_{k=0}^{N-1}\xi(k)H(\hat{s})^k = \sum_{k=0}^{N-1}\xi(k)\left(H(\hat{s})^{N-1} - H(\hat{s})^k\right) < 0$ and $\hat{s} > \underline{w}$ allows us to conclude that the denominator is negative. Thus, $\frac{d\hat{s}}{d\underline{w}} > 0$. □

A.1.3 Budget-Dependent Risk Aversion in Example 1

Example 6. Consider the two-budget-level model of Example 1. Suppose $p = \frac{1}{2}$ and that $w_i \in \{\frac{1}{4}, \frac{3}{4}\}$. A bidder with a budget of $\underline{w} = 1/4$ is constrained while a bidder with a budget of 3/4 will be unconstrained.

Unlike Example 1 suppose that bidders are risk-averse. Following a win, a bidder of type $\theta_i = (s_i, w_i)$ receives a payoff of $u_i(\theta_i, b_i) = (s_i - b_i)^{w_i + \frac{1}{4}}$. Payoffs following a loss are zero. This utility function captures the idea that bidders with a large budget are less risk-averse.[2] A bidder with a budget of $w_i = 3/4$ is risk-neutral while a bidder with a budget of $w_i = 1/4$ is strictly risk-averse. Introducing risk-preference heterogeneity in this form follows Cox *et al.* (1988).

[2]The utility function allows a modest degree of tractability and is adopted with this reason in mind. It is intended only as an illustration of an equilibrium in non-monotone strategies. Moreover, it does not satisfy Assumption 3.

The following is an equilibrium strategy profile:

$$\beta(s,3/4) = \begin{cases} \frac{s}{2} & s \leq \hat{s} \\ \frac{2s^2+13\sqrt{11}-42}{4(s+1)} & \hat{s} < s \end{cases}$$

$$\beta(s,1/4) = \begin{cases} \frac{2s}{3} & s < \hat{s}'' \\ \frac{16s^3+24\left(\sqrt{11}-3\right)s^2+33\left(19\sqrt{11}-63\right)}{24\left(s+\sqrt{11}-3\right)^2} & s \in [\hat{s}'', \hat{s}') \\ \frac{1}{4} & \hat{s}' \leq s \end{cases}$$

where $\hat{s} = \sqrt{11} - 3$, $\hat{s}'' = \frac{3}{4}(\sqrt{11} - 3)$, $\hat{s}' \approx 0.29839$. There is no analytic expression for $\hat{s}'$, but it is easy to solve for numerically.

Figure A.1 presents a sketch of the strategy $\beta(s, w)$. Just like Example 1 the equilibrium strategy features discontinuities at $\hat{s}$ and $\hat{s}'$ and competition becomes stratified along the budget dimension. Unlike Example 1, the equilibrium strategy highlights the nuanced role played by risk-aversion, which is controlled by the (private) parameter w_i. It is well known that equilibrium bids tend to increase with bidder's risk aversion (Krishna, 2002, p. 38). This effect is present for low-budget bidders, who are also more risk-averse. For value-signals $s < \hat{s}'$ low-budget bidders bid more than their large-budget counterparts. The equilibrium strategy is not increasing in (s, w) and thus not monotone as normally understood.

Proof. To confirm that $\beta(s, w)$ is an equilibrium strategy we will consider possible deviations by constrained and unconstrained bidders. Clearly, we need only consider possible deviations into the range of $\beta(s, w)$.

Unconstrained Bidders $(w = 3/4)$ Consider a bidder of type $s < \hat{s}$. Among bids in the range $[0, \hat{s}/2]$, the optimal bid will solve:

$$\max_{x \leq \hat{s}} \left(p \cdot \frac{3}{4}x + (1-p)x\right)\left(s - \frac{x}{2}\right)$$

Clearly, $x = s$ is the solution. A bid of $\underline{w} = 1/4$ is strictly dominated by a bid of $\underline{w} + \epsilon$ as the probability of winning increases discontinuously. The optimal bid (strictly) above 1/4 can be determined by solving $\max_{x \geq \hat{s}} U(x, s)$ where $U(x, s) = (p + (1-p)x)(s - \beta(x, 3/4))$. It is straightforward to calculate that

$$\frac{\partial U}{\partial x} = \frac{s-x}{2} \leq 0 \qquad \forall x \geq \hat{s}$$

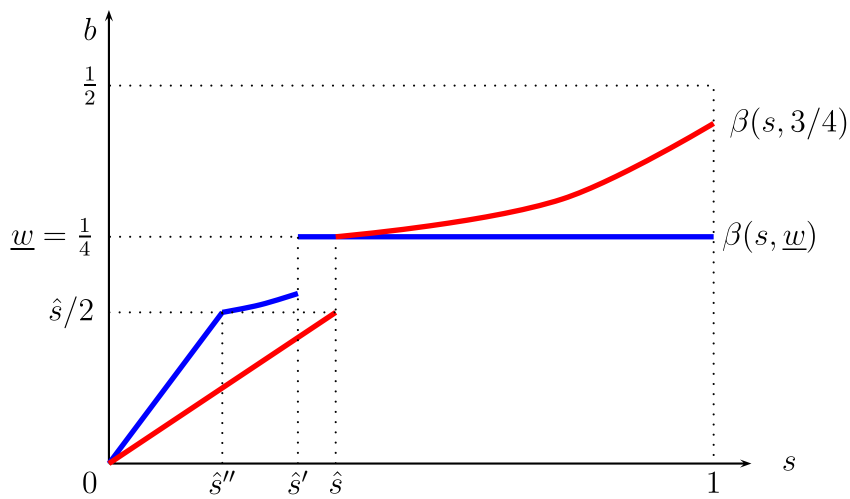


Figure A.1: Equilibrium strategy in Example 6. The large-budget (unconstrained) bidder increases her bid discontinuously at $\hat{s}$. Low-valuation, low-budget bidders bid relatively more because of risk aversion. (Figure not to scale.)

when $s \leq \hat{s}$. Therefore, this bidder has no profitable deviation into this range of bids. A completely analogous argument applies to a bidder of type $s > \hat{s}$ who considers deviating into the range of $\beta(s, 3/4)$.

We need only confirm that bids in the range of $\beta(s, 1/4)$ for $s \in (\hat{s}'', \hat{s}']$ are not profitable deviations. Consider first a bidder of type $s < \hat{s}$. Let

$$\Delta(x, s) = \frac{7s^2}{16} - (px + (1-p)\hat{s})(s - \beta(x, 1/4))$$

be the expected utility difference between a bid of $s/2$ and $\beta(x, 1/4)$ for a bidder of type s. (We partly simplified the express to the example's parameters.) It is sufficient to establish that $\Delta(x, s) \geq 0$ for all $s < \hat{s}$ and $x \in [\hat{s}'', \hat{s}']$. It is clear that $\Delta(\hat{s}'', \hat{s}) = 0$. Next, observe that when $x = \hat{s}''$,

$$\frac{\partial \Delta(\hat{s}'', s)}{\partial s} = \frac{7}{8}(s - \hat{s}) \leq 0.$$

Thus, it is sufficient to confirm that $\frac{\partial \Delta(x,s)}{\partial x} \geq 0$ for all $s < \hat{s}$ and $x \in [\hat{s}'', \hat{s}']$. Again, a direct calculation gives,

$$\frac{\partial\Delta(x,s)}{\partial x} = \frac{1}{48}\left(8\left(\sqrt{11}-3-3s\right)+32x-\frac{49\left(19\sqrt{11}-63\right)}{\left(x+\sqrt{11}-3\right)^2}\right)$$
$$\geq \frac{1}{48}\left(8\left(\sqrt{11}-3-3\hat{s}\right)+32\hat{s}''-\frac{49\left(19\sqrt{11}-63\right)}{\left(\hat{s}''+\sqrt{11}-3\right)^2}\right) = 0$$

where we used the fact that $19\sqrt{11}-63>0$. Therefore, a bidder of type $s<\hat{s}$ has no incentive to deviate into this range of bids.

Finally, consider a bidder of type $s>\hat{s}$ and as before define

$$\Delta(x,s) = (p+(1-p)s)(s-\beta(s,3/4)) - (px+(1-p)\hat{s})(s-\beta(x,1/4)).$$

Again it is sufficient to confirm that $\Delta(x,s)\geq 0$ for $s>\hat{s}$ and $x\in[\hat{s}'',\hat{s}']$. It is clear that $\Delta(\hat{s}'',\hat{s})=0$. Next, observe that when $x=\hat{s}''$,

$$\frac{\partial\Delta(x,\hat{s})}{\partial x} = 1-\frac{\sqrt{11}}{3}+\frac{2x}{3}-\frac{49\left(19\sqrt{11}-63\right)}{48\left(x+\sqrt{11}-3\right)^2}$$
$$\geq 1-\frac{\sqrt{11}}{3}+\frac{2\hat{s}''}{3}-\frac{49\left(19\sqrt{11}-63\right)}{48\left(\hat{s}''+\sqrt{11}-3\right)^2} = 0$$

Thus, it is sufficient to confirm that $\partial\Delta/\partial s\geq 0$ for all $s>\hat{s}$ and $x\in[\hat{s}'',\hat{s}']$. A direct calculation gives

$$\frac{\partial\Delta(x,s)}{\partial s} = \frac{1}{8}\left(4s-4\left(x+\sqrt{11}-3\right)+4\right)$$
$$\geq \frac{1}{8}\left(4\hat{s}-4\left(1+\sqrt{11}-3\right)+4\right) = 0.$$

Therefore, a unconstrained bidder of type $s>\hat{s}$ has no incentive to deviate into this range of bids and $\beta(s,3/4)$ is a best response.

Constrained Bidders ($w=1/4$) Consider a bidder of type $s<\hat{s}$. Among bids in the range $[0,\hat{s}/2]$, determining the optimal bid for a constrained bidder is equivalent to solving:

$$\max_{x\leq\hat{s}''}\left(px+(1-p)\cdot\frac{4}{3}x\right)\left(s-\frac{2x}{3}\right)^{\frac{1}{2}}$$

Clearly, $x = s$ is the solution. A similar argument shows that $\beta(s, 1/4)$ is the optimal bid for a bidder of type $s \in [\hat{s}'', \hat{s}')$. The constant $\hat{s}'$ is the unique solution to the equation

$$(p\hat{s}' + (1-p)\hat{s})(\hat{s} - b(\hat{s}))^{\frac{1}{2}} = \left(p\frac{\hat{s}'+1}{2} + (1-p)\hat{s}\right)(s - 1/4)^{\frac{1}{2}}$$

where $b(s) = \frac{16s^3+24\left(\sqrt{11}-3\right)s^2+33\left(19\sqrt{11}-63\right)}{24\left(s+\sqrt{11}-3\right)^2}$. We need only verify that a constrained bidder of type s does not wish to deviate to some alternative "segment" of $\beta(s, 1/4)$. There are three cases.

(i) Consider a bidder of type $s < \hat{s}''$. As $\hat{s}'' = \frac{3}{4}(\sqrt{11} - 3) \approx 0.237 < 1/4$, a bid of $\underline{w} = 1/4$ is dominated. If such a bidder bids $\beta(x, 1/4)$ for $x \in (\hat{s}'', \hat{s}')$, the bidder's expected payoff is

$$U(x, s) = (px + (1-p)\hat{s})(s - \beta(x, 1/4))^{\frac{1}{2}}$$

Differentiating this expression with respect to x gives

$$\frac{\partial U}{\partial x} = \frac{\sqrt{6}(s-x)}{\sqrt{8\left(3s+\sqrt{11}-3\right) - 16x - \frac{49\left(19\sqrt{11}-63\right)}{\left(x+\sqrt{11}-3\right)^2}}} \leq 0$$

implying that there is profitable deviation to announcement in excess of $\hat{s}''$.

(ii) Consider a bidder of type $s \in (\hat{s}'', \hat{s}')$. An analogous argument that that above established that such a bidder would not wish to deviate to a bid below $\hat{s}'$. We thus need only verify that no such bidder wishes to bid $\underline{w} = 1/4$. Let

$$\begin{aligned}
U_\beta(s) &= (ps + (1-p)\hat{s})(s - \beta(s, 1/4))^{\frac{1}{2}} \\
U_{\underline{w}}(s) &= \left(p\frac{\hat{s}'+1}{2} + (1-p)\hat{s}\right)(s - \underline{w})^{\frac{1}{2}} \\
\Delta(s) &= U_\beta(s) - U_{\underline{w}}(s)
\end{aligned}$$

By definition $U_\beta(\hat{s}') = U_{\underline{w}}(\hat{s}')$. It is sufficient to establish that $U'_\beta \leq U'_{\underline{w}}$ for all $s \in (\hat{s}'', \hat{s}')$. Consider

$$\begin{aligned} U'_{\underline{w}}(s) &= \frac{\hat{s}' + 2\sqrt{11} - 5}{4\sqrt{4s-1}} \\ &\geq \frac{\hat{s}'' + 2\sqrt{11} - 5}{4\sqrt{4\hat{s}-1}} = \frac{11\sqrt{11} - 29}{16\sqrt{4\left(\sqrt{11}-3\right)-1}} \approx 0.901 \end{aligned}$$

and

$$\begin{aligned} U'_\beta(s) &= \frac{\sqrt{\frac{3}{2}}\left(s + \sqrt{11} - 3\right)}{\sqrt{8s - \frac{49\left(19\sqrt{11}-63\right)}{\left(s+\sqrt{11}-3\right)^2} + 8\left(\sqrt{11}-3\right)}} \\ &\leq \frac{\sqrt{\frac{3}{2}}\left(\hat{s} + \sqrt{11} - 3\right)}{\sqrt{8\hat{s}'' - \frac{49\left(19\sqrt{11}-63\right)}{\left(\hat{s}''+\sqrt{11}-3\right)^2} + 8\left(\sqrt{11}-3\right)}} \\ &= \frac{\left(\sqrt{11}-3\right)^2}{\sqrt{38\sqrt{11}-126}} \approx 0.563 \end{aligned}$$

Therefore $U'_\beta < U'_{\underline{w}}$ for all $s \in (\hat{s}'', \hat{s}')$ and thus $\Delta'(s) \leq 0$ confirming that $U_\beta \geq U_{\underline{w}}$.

(iii) Consider a bidder of type $s \geq \hat{s}'$. With argument like the preceding, it is straightforward to establish that the optimal deviation for such a bidder is to bid $\beta(\hat{s}', 1/4)$. As before

$$U'_{\underline{w}}(s) = \frac{\hat{s}' + 2\sqrt{11} - 5}{4\sqrt{4s-1}}$$

And at a bid of $\beta(\hat{s}', 1/4)$, expected utility is

$$U_{\beta(\hat{s}')}(s) \equiv (p\hat{s}' + (1-p)\hat{s})(s - \beta(\hat{s}', 1/4))^{\frac{1}{2}}$$

As $U_{\underline{w}}(\hat{s}') = U_{\beta(\hat{s}')}(\hat{s}')$ it is sufficient to confirm that $U'_{\underline{w}}(s) \geq U'_{\beta(\hat{s}')}(s)$

for $s > \hat{s}'$. Consider therefore,

$$\begin{aligned}\frac{U'_{\underline{w}}(s)}{U'_{\beta(\hat{s}')}(s)} &= \frac{\left(\hat{s}' + 2\sqrt{11} - 5\right)\sqrt{8\left(3s + \sqrt{11} - 3\right) - 16\hat{s}' - \frac{49\left(19\sqrt{11}-63\right)}{\left(\hat{s}'+\sqrt{11}-3\right)^2}}}{2\sqrt{24s-6}\left(\hat{s}' + \sqrt{11} - 3\right)} \\ &\geq \frac{\left(\hat{s}'' + 2\sqrt{11} - 5\right)\sqrt{8\left(3s + \sqrt{11} - 3\right) - 16\hat{s} - \frac{49\left(19\sqrt{11}-63\right)}{\left(\hat{s}''+\sqrt{11}-3\right)^2}}}{2\sqrt{24s-6}\left(\hat{s} + \sqrt{11} - 3\right)} \\ &= \frac{\sqrt{\left(333 + 68\sqrt{11}\right)s - 86\sqrt{11} + \frac{502}{3}}}{8\sqrt{4s-1}}\end{aligned}$$

This final expression is decreasing in s and at $s = 1$ we can conclude that

$$\frac{U'_{\underline{w}}(s)}{U'_{\beta(\hat{s}')}(s)} \geq \frac{1}{24}\sqrt{1501 - 54\sqrt{11}} \approx 1.51 \geq 1$$

which is the desired result.

□

A.2 Proof of Propositions 1–4

A.2.1 Preliminaries and Notation

Penalty Function and Restriction to Feasible Strategies As a technical device, we will define the following *penalty function* for bidder i as

$$\varrho_i(w_i, b_i) = -\mathbf{1}(b_i \notin \mathcal{B}_i(w_i))K$$

Where K is an large constant such that $b_i > w_i \implies \sup(u_i) + \varrho_i(w_i, b_i) < \inf_{\mathbf{s},w_i} u_i(\mathbf{s}, w_i, l)$. As utility is bounded, such as K exists. It is clear that when a bidder's objective function is $u_i(\mathbf{s}, w_i, b_i) + \varrho(w_i, b_i)$ all bids in outside a bidder's feasible bid set are strictly dominated by the bid l and all best reply strategies will reside in $\mathscr{S}_i$—the set of strategies that respect the set of feasible bids. (Note that the penalty is imposed irrespective of auction outcome.) Whenever we consider an auction with budget constrained bidders we will implicitly think of an auction game where bidder's utilities are augmented with the above penalty function; without loss of generality we focus on strategies and bids respecting the budget constraint which set $\varrho(w_i, b_i) = 0$.

Utility and Allocations In keeping with the main text, $f(\theta)$ is the joint density of bidders' types. When budgets are mutually independent and independent of value signals we will write $f(\theta) = h(\mathbf{s})g(\mathbf{w})$ where $g(\mathbf{w}) = \prod g_i(w_i)$. We will work with the following allocation rule.

Definition 5. The *standard allocation rule with uniform tie breaking* is

$$\varphi_i(\mathbf{b}) = \begin{cases} \frac{1}{|\{j\colon b_j = \max(\mathbf{b})\}|} & \text{if } b_i > l \text{ and } b_i = \max(\mathbf{b}) \\ 0 & \text{otherwise} \end{cases}.$$

As bidder's bidding sets are mutually disjoint ties do not occur and $\varphi_i(\mathbf{b}) \in \{0, 1\}$. Moreover, $\varphi_i(\mathbf{b})$ is nondecreasing in b_i and non-increasing in $\mathbf{b}_{-i}$.

Let $\tilde{u}_i(\theta, b) \equiv u_i(\mathbf{s}, w_i, b) - u_i(\mathbf{0}, w_i, 0)$. Fix a strategy profile β_{-i}. The expected payoff to a type θ_i from the individually rational bid $b \in \mathcal{B}_i(w_i)$ is

$$
\begin{aligned}
&U_i(b, \beta_{-i}|\theta_i) \\
&= \int\limits_{\Theta_{-i}} \big[\varphi_i(b, \beta_{-i}) u_i(\mathbf{s}, w_i, b) + (1 - \varphi_i(b, \beta_{-i})) u_i(\mathbf{0}, w_i, 0)\big] f(\theta_{-i}|\theta_i) d\theta_{-i} \\
&= u_i(\mathbf{0}, w_i, 0) + \int_{\Theta_{-i}} \varphi_i(b, \beta_{-i}) \tilde{u}_i(\theta, b) f(\theta_{-i}|\theta_i) d\theta_{-i} \\
&= u_i(\mathbf{0}, w_i, 0) + \mathbb{E}[\varphi_i(b, \beta_{-i}) \tilde{u}_i(\theta, b)|\theta_i].
\end{aligned}
$$

When bidder i employs the strategy $\beta_i \in \mathscr{S}_i$, her (ex ante) expected utility is

$$
\begin{aligned}
U_i(\beta_i, \beta_{-i}) &= \mathbb{E}[U_i(\beta_i(\theta_i), \beta_{-i}|\theta_i)] \\
&= \int_{\Theta} \left[u_i(\mathbf{0}, w_i, 0) + \varphi_i(\beta) \tilde{u}_i(\theta, \beta_i(\theta_i))\right] f(\theta) d\theta
\end{aligned}
$$

Definition 6. A *Nash equilibrium* is a (feasible) strategy profile $\beta^* \in \mathscr{S}$ such that for all i $U_i(\beta_i^*, \beta_{-i}^*) \geq U_i(\beta_i, \beta_{-i}^*)$ for all $\beta_i \in \mathscr{S}_i$. If $\beta^* \in \mathscr{I}$, the equilibrium is in nondecreasing strategies.

The following lemma illustrates the utility of the $\geq_{\theta_i}$ ordering.

Lemma 15. *Fix θ_{-i} and $b \in \mathcal{B}_i(w_i)$. Suppose $\theta_i \geq_{\theta_i} \theta_i'$, then $\tilde{u}_i(\theta, b) \geq \tilde{u}_i(\theta', b)$.*

Proof. As b and $\mathbf{s}_{-i}$ are fixed, $u_i(\cdot, \mathbf{s}_{-i}, \cdot, b)\colon \Theta_i \to \mathbb{R}$, which by assumption is a smooth function. Then the following inequalities can be established:

$$
\begin{aligned}
&u_i(s_i, \mathbf{s}_{-i}, w_i, b) - u_i(s_i', \mathbf{s}_{-i}, w_i', b) \\
&= \left.\frac{\partial u_i}{\partial s_i}\right|_{\theta_i=\hat{\theta}_i} (s_i - s_i') + \left.\frac{\partial u_i}{\partial w_i}\right|_{\theta_i=\hat{\theta}_i} (w_i - w_i') \\
&\geq \left(\inf \frac{\partial u_i}{\partial s_i}\right) (s_i - s_i') + \left(\inf \frac{\partial u_i}{\partial w_i}\right) (w_i - w_i') \\
&\geq \left(\inf \frac{\partial u_i}{\partial s_i}\right) \alpha_i (w_i - w_i') + \left(\inf \frac{\partial u_i}{\partial w_i}\right) (w_i - w_i') \\
&\geq \left(\sup \frac{\partial u_i}{\partial w_i}\right) (w_i - w_i') \\
&\geq \left.\frac{\partial u_i}{\partial w_i}\right|_{\theta_i=\tilde{\theta}_i} (w_i - w_i') \\
&= u_i(\mathbf{0}, w_i, 0) - u_i(\mathbf{0}, w_i', 0)
\end{aligned}
$$

The two equalities follow from the mean-value theorem and the inequalities are a consequence of the $\geq_{\theta_i}$ partial order and a substitution for α_i. Rearranging terms gives the desired result. □

A.2.2 Proposition 1 and Proposition 2.

Together the following lemmas give a proof of Lemmas 1 and 2. They confirm that we can apply Reny (2009, Theorem 2.1) in this setting to show that there is an equilibrium in nondecreasing strategies.

Lemma 16. *Consider the setting of Lemma 1. Fix β_{-i} and suppose $\theta_i \geq_{\theta_i} \theta_i'$. Let $b > b'$ be feasible bids for a type θ_i'. If $U_i(b, \beta_{-i}|\theta_i') \geq U_i(l, \beta_{-i}|\theta_i')$, then*

$$U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \geq (>) 0 \implies U_i(b_i, \beta_{-i}|\theta_i) - U_i(b_i', \beta_{-i}|\theta_i) \geq (>) 0.$$

Proof. As bidder's types are independent, it is sufficient to verify that ex-post payoffs display increasing differences in θ_i for all possible auction outcomes.

Define the events

$$A = \{\theta_{-i} \colon \varphi_i(b, \beta_{-i}(\theta_{-i})) = 1\}, \qquad A' = \{\theta_{-i} \colon \varphi_i(b', \beta_{-i}(\theta_{-i})) = 1\}.$$

Clearly, $A' \subset A$ as the allocation rule is nondecreasing in a bidder's own action. The utility difference now becomes,

$$\begin{aligned}&U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i')\\&= \int_{A'} \left[\tilde{u}(\theta', b) - \tilde{u}_i(\theta', b')\right] f(\theta_{-i}|\theta_i') d\theta_{-i} + \int_{A\setminus A'} \tilde{u}(\theta', b) f(\theta_{-i}|\theta_i') d\theta_{-i}.\end{aligned}$$

There are two cases:

1. $\theta_{-i} \in A'$. $\tilde{u}_i(\theta', b) - \tilde{u}_i(\theta', b') = u_i(s_i', \mathbf{s}_{-i}, w_i', b) - u_i(s_i', \mathbf{s}_{-i}, w_i', b')$ is nondecreasing in both s_i' and w_i' by assumption; thus, it is nondecreasing according to $\geq_{\theta_i}$.
2. $\theta_{-i} \in A \setminus A'$. Then by Lemma 15, $\theta_i \geq_{\theta_i} \theta_i' \implies \tilde{u}_i(\theta, b) \geq \tilde{u}_i(\theta', b)$.

As θ_{-i} is independent of θ_i, $f(\theta_{-i}|\theta_i') = f(\theta_{-i}|\theta_i)$. Therefore,

$$0 \leq U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \leq U_i(b_i, \beta_{-i}|\theta_i) - U_i(b_i', \beta_{-i}|\theta_i)$$

as required. □

Lemma 17. *Consider the setting of Lemma 2. Fix β_{-i} and suppose $\theta_i \geq_{\theta_i} \theta_i'$. Let $b > b'$ be feasible bids for a type θ_i'. If $U_i(b, \beta_{-i}|\theta_i') \geq U_i(l, \beta_{-i}|\theta_i')$, then*

$$U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \geq (>) 0 \implies U_i(b_i, \beta_{-i}|\theta_i) - U_i(b_i', \beta_{-i}|\theta_i) \geq (>) 0. \tag{A.10}$$

Proof. Let

$$A = \{\theta_{-i} \colon \varphi_i(b, \beta_{-i}(\theta_{-i})) = 1\}, \qquad A' = \{\theta_{-i} \colon \varphi_i(b', \beta_{-i}(\theta_{-i})) = 1\},$$

and define define $\tilde{A} \equiv A \setminus A'$. We can express the first difference in (A.10) as

$$\begin{aligned}&\int_{\tilde{A}} [v_i(s_i', \mathbf{s}_{-i}) - b] f(\theta_{-i}|\theta_i) d\theta_{-i} + \int_{A'} [b' - b] f(\theta_{-i}|\theta_i') d\theta_{-i}\\&= \int_{\tilde{A}} (v_i(s_i', \mathbf{s}_{-i}) - b) f(\theta_{-i}|\theta_i') d\theta_{-i} + (b' - b)\Pr[A'|\theta_i']\end{aligned}$$

Because A' is a decreasing set and θ is affiliated, $\Pr[A'|\theta_i'] \geq \Pr[A'|\theta_i]$; thus $0 \geq (b' - b)\Pr[A'|\theta_i] \geq (b' - b)\Pr[A'|\theta_i']$. Therefore, it suffices to verify that $\int_{\tilde{A}} (v_i(s_i', \mathbf{s}_{-i}) - b) f(\theta_{-i}|\theta_i') d\theta_{-i} \leq \int_{\tilde{A}} (v_i(s_i, \mathbf{s}_{-i}) - b) f(\theta_{-i}|\theta_i) d\theta_{-i}$. There are two cases:

1. Suppose $\Pr[\tilde{A}|\theta_i'] \geq \Pr[\tilde{A}|\theta_i]$. Then,

$$\begin{aligned}
&\int_{\tilde{A}} [v_i(s_i', \mathbf{s}_{-i}) - b] f(\theta_{-i}|\theta_i') d\theta_{-i} \\
&\quad = \int_{\tilde{A}} v_i(s_i', \mathbf{s}_{-i}) \lambda(\mathbf{s}_{-i}|s_i')(1 - H(\mathbf{s}_{-i}|s_i')) g(\mathbf{w}_{-i}) d\theta_{-i} - \Pr[\tilde{A}|\theta_i'] b \\
&\quad \leq \int_{\tilde{A}} v_i(s_i, \mathbf{s}_{-i}) \lambda(\mathbf{s}_{-i}|s_i)(1 - H(\mathbf{s}_{-i}|s_i)) g(\mathbf{w}_{-i}) d\theta_{-i} - \Pr[\tilde{A}|\theta_i] b \\
&\quad = \int_{\tilde{A}} [v_i(s_i, \mathbf{s}_{-i}) - b] f(\theta_{-i}|\theta_i) d\theta_{-i}
\end{aligned}$$

   The inequality follows from the assumption that $v_i(\cdot, \mathbf{s}_{-i})\lambda(\mathbf{s}_{-i}|\cdot)$ is non-decreasing and that $H(\mathbf{s}_{-i}|\cdot)$ is decreasing (by affiliation).

2. Suppose $\Pr[\tilde{A}|\theta_i'] < \Pr[\tilde{A}|\theta_i]$. As

$$\int_{\tilde{A}} [v_i(s_i', \mathbf{s}_{-i}) - b] f(\theta_{-i}|\theta_i') d\theta_{-i} = \Pr[\tilde{A}|\theta_i'] \mathbb{E}[v_i(s_i', \mathbf{s}_{-i}) - b)|\tilde{A}, \theta_i'] > 0,$$

   it is sufficient to verify that $\mathbb{E}[v_i(s_i', \mathbf{s}_{-i}) - b)|\tilde{A}, \theta_i'] \leq \mathbb{E}[v_i(s_i, \mathbf{s}_{-i}) - b)|\tilde{A}, \theta_i]$. The conclusion would be immediate if $\tilde{A}$ was a sublattice of Θ_{-i}. This however may not always be the case under the usual coordinate ordering; however, $\tilde{A}$ is a sublattice under the following partial order:

$$\theta \tilde{\geq}_i \theta' \iff \begin{cases} \mathbf{s} \geq \mathbf{s}' \\ w_i \geq w_i' \\ \mathbf{w}_{-i} \leq \mathbf{w}'_{-i} \end{cases}$$

   It is clear that $f(\theta) = h(\mathbf{s})g(\mathbf{w})$ is log-supermodular under the $\tilde{\geq}_i$ ordering and $v_i(s_i, \mathbf{s}_{-i}) - b$ is also nondecreasing. An application of Milgrom & Weber (1982, Theorem 23) therefore gives $\mathbb{E}[v_i(s_i', \mathbf{s}_{-i}) - b)|\tilde{A}, \theta_i'] \leq \mathbb{E}[v_i(s_i, \mathbf{s}_{-i}) - b)|\tilde{A}, \theta_i]$, the desired result.[3]

This is exhaustive of all cases. □

[3]This conclusion depends crucially on independent budgets which allow f to be log-supermodular under various re-orderings of Θ. Such re-orderings need to maintain Θ as a sublattice of $\mathbb{R}^{2N}$.

Lemma 18. *Let $\beta_{-i} \in \mathscr{I}_{-i}$. Player i's set of (interim) best reply bids is nonempty and increasing in the strong set order.*[4]

Proof. As action sets are finite and non-empty (they always contain l), the set $\rho(\beta_{-i};\theta_i) \equiv \arg\max_{b\in\mathcal{B}_i(w_i)} U_i(b,\beta_{-i}|\theta_i) \neq \emptyset$. That $\rho_i(\beta_{-i},\theta_i)$ is non-decreasing in the strong set order is an immediate consequence of Lemma 16. □

As the interim best-reply correspondence is nondecreasing, player i has at least one nondecreasing best reply when others employ nondecreasing strategies. This follows from Topkis (1998, Theorem 2.8.3) and depends on $\mathcal{B}_i(w_i)$ varying monotonically with a player's type. Let $\rho_i^*\colon \mathscr{I}_{-i} \to \mathscr{I}_{-i}$ be player i's best reply map. For $\beta_{-i} \in \mathscr{I}_{-i}$, $\rho_i^*(\beta_{-i}) \neq \emptyset$ and is point-wise join-closed.

Lemma 19. *There exists an equilibrium in nondecreasing strategies.*

Proof. The conclusion follows from Reny (2009, Theorem 4.1). Conditions G.1–G.6 are easy to verify. The partial order on Θ_i is measurable and by assumption $f(\cdot)$ is atomless. Condition G.3 follows from Reny (2009)'s analysis of the multi-unit auction with risk-averse bidders as the $\geq_\theta$ order is an adaptation of the analogous construction in that setting. The unrestricted action space is $\{l\}\cup([r_i,\bar{w}_i]\cap\mathcal{P}_i)$, satisfying G.4 and G.5. Finally, a bidder's utility is bounded, measurable, and continuous in $\mathbf{b}$. Continuity follows from the discretized action set.

When all others employ nondecreasing strategies, player i has a nondecreasing best response in $\mathscr{I}_i$ which is compact, nonempty, join-closed, piecewise-closed, and point-wise limit-closed. Arguments mirroring those in Reny (2009, Theorem 4.1) show that $\rho_i^*\colon \mathscr{I}_{-i} \to \mathscr{I}_i$ is upper-hemicontinuous and nonempty. Similarly, $\times\rho_i^*$ can be shown to be contractable-valued implying a fixed point (i.e. equilibrium) by Reny (2009, Theorem 2.1). □

A.2.3 Continuum Action Spaces

Proposition 3

Consider a private values setting where utility is $u_i(s_i,w_i,b_i)$. To extend the equilibrium to a continuum action space we offer an argument that mirrors

[4] Let X and Y be two sets. Then X is greater than Y in the strong set order if $\forall x \in X$ and $\forall y \in Y, x\vee y \in X$ and $x \wedge y \in Y$.

the proof of Reny & Zamir (2004, Theorem 2.1). We simply indicate the set-up and the notable modification. To fix ideas, define the set of allowable bids for player i as

$$\mathcal{B}_i(w_i)_k = \left(\left\{ \frac{m\,\bar{w}_i + \upsilon_i}{2^k} : m = 1, \ldots, 2^k \right\} \cap [r_i, w_i] \right) \cup \{l\}$$

where $\{\upsilon_i\}$ are small, fixed numbers chosen such that $\mathcal{B}_i(\bar{w})_k \cap \mathcal{B}_j(\bar{w})_k = \{l\}$ for all k. It is easy to verify that $\mathcal{B}_i(w_i)_k \cap \mathcal{B}_j(w_j)_k = \{l\}$.

Call an auction with such bidding sets Γ^k and let β_k be a nondecreasing equilibrium. Let $k \to \infty$ and consider the sequence of auction games $\{\Gamma^k\}$.

By Helly's selection theorem[5] for each i there is a nondecreasing function, β_i, such that $\beta_i^k(\theta_i) \xrightarrow{a.e.} \hat{\beta}_i(\theta_i)$ where we passed to a subsequence without relabeling.

As β_i^k is monotone and $\varphi_i(\beta^k(\cdot))$ is nondecreasing in θ_i and non-increasing in θ_{-i}. The corresponding limit, $\varphi_i(\beta^k(\cdot)) \xrightarrow{a.e.} \hat{\varphi}_i$ inherits these properties too. Without loss of generality, we may assume that β^k and $\varphi(\beta^k)$ converge on the same dense subset of Θ and are extended together to all of Θ. We will argue that $\hat{\beta}$ is an equilibrium of Γ, an auction with a continuum action space and uniform tie-breaking.

Consider the limiting strategy $\hat{\beta}$. For each i, $\hat{\beta}_i(\theta_i)$ has at most countably many mass points. Let $\mathscr{P} = \cup_{j=1}^N \left\{ b\colon \Pr[b = \hat{\beta}_j(\theta_j)] > 0 \right\}$. Let $\mathscr{T}_i = \{\theta_i\colon w_i \in \mathscr{P}\}$. $\mathscr{T}_i$ has zero measure. Henceforth consider only types $\theta_i \in \Theta_i \setminus \mathscr{T}_i$. Such bidders do not have a budget equal to a mass-point of the bid distribution; hence, should they ever choose to bid their budget versus $\hat{\beta}_{-i}$ they do not need to worry about it resulting in a tie with positive probability.

Given that all but a measure zero of bidder's types is capable of resolving relevant ties, i.e. those that occur with strictly positive probability, by bidding slightly more, we will establish that it is indeed in their interests to do so. Consider the limiting strategy $\hat{\beta}_{-i}$ and a bidder of type θ_i who places a feasible bid $b > l$. Let $A = \{\theta_{-i}\colon b \geq \max_{j \neq i} \hat{\beta}_j(\theta_j)\}$. And define the set $A(\mathbf{w}_{-i}) = \{\mathbf{s}_{-i}\colon (\mathbf{s}_{-i}, \mathbf{w}_{-i}) \in A\} \subset \mathcal{S}_{-i}$. $A(\mathbf{w}_{-i})$ is a slice of A fixing $\mathbf{w}_{-i}$. When bidders follow monotone strategies it is a sublattice of $\mathcal{S}_{-i}$.

A bid of b may win or may tie with competing bidders. As the bidder is not budget constrained in bidding b we can find a sequence of feasible bids $b' \downarrow b$ that tie with probability zero. That is $\mathbf{1}(\{\theta_{-i}\colon b' > \max_{j \neq i} \hat{\beta}_j(\theta_j)\}) \xrightarrow{a.e.} \mathbf{1}(A)$.

[5] See Reny (2009, Lemma A.10) for the appropriate generalization.

Then,

$$
\begin{aligned}
0 &\leq U_i(b, \hat{\beta}_{-i}|\theta_i) - u_i(\mathbf{0}, w_i, 0) \\
&= \mathbb{E}[\varphi_i(b; \hat{\beta}_{-i})\tilde{u}_i(\theta, b)|\theta_i] \\
&= \Pr[A|\theta_i]\mathbb{E}\left[\varphi(b; \hat{\beta}_{-i})\tilde{u}_i(\theta, b)|A, \theta_i\right] \\
&= \Pr[A|\theta_i]\mathbb{E}\left[\mathbb{E}[\varphi(b; \hat{\beta}_{-i})\tilde{u}_i(\theta, b)|A(\mathbf{w}_{-i}), \theta_i]|A, \theta_i\right] \\
&\leq \Pr[A|\theta_i]\mathbb{E}\big[\mathbb{E}[\varphi(b; \hat{\beta}_{-i})|A(\mathbf{w}_{-i}), \theta_i]\, \underbrace{\mathbb{E}[\tilde{u}_i(\theta, b)|A(\mathbf{w}_{-i}), \theta_i]}_{=\tilde{u}_i(\theta,b)\geq 0}|A, \theta_i\big] \qquad (\P) \\
&\leq \Pr[A|\theta_i]\mathbb{E}[\tilde{u}_i(\mathbf{s}, w_i, b)|\theta_i, A] \\
&= \lim_{b' \downarrow b} U_i(b', \hat{\beta}_{-i}|\theta_i) - u_i(\mathbf{0}, w_i, 0).
\end{aligned}
$$

The above inequalities show that under uniform tie breaking, a bidder always wishes to resolve a tie in her favor. As $\hat{\varphi}_i$ is decreasing in $\mathbf{s}_{-i}$ a similar argument establishes that a bidder wishes to improve upon payoffs obtained in the limit as $k \to \infty$. The remainder of the proof follows from the proof of Theorem 2.1 in Reny & Zamir (2004, pp. 1121–25).

Remark 1. At (¶) this argument fails when values are interdependent. Proceeding to the next inequality follows from setting $\mathbb{E}[\varphi(b; \hat{\beta}_{-i})|A(\mathbf{w}_{-i}), \theta_i] = 1$. With interdependent values however $\mathbb{E}[\tilde{u}_i(\theta, b)|A(\mathbf{w}_{-i}), \theta_i] \gtrless 0$.

Proposition 4

We will pass to a continuous action space via an endogenous tie-breaking / allocation rule. This rule will be defined as the limit of a sequence of allocations from equilibria of discretized versions of the auction game.

Relaxed Discrete Bidding Sets Modify the preceding framework as follows. For a (large) fixed k, a bidder of type $\theta_i = (s_i, w_i)$ can place a bid from the set

$$
\mathcal{B}_i(w_i)_k = \left(\left\{ \frac{m\ \bar{w}_i + v_i}{2^k} : m = 1, \ldots, 2^k \right\} \cap \left[r_i, w_i + \frac{1}{k} \right] \right) \cup \{l\}. \quad \text{(A.11)}
$$

The bidding set (A.11) allows bidders to place bids slightly above their budget limit for fixed k. (For k sufficiently large there are feasible bids on $[w_i, w_i +$

$1/k$].) Thus, bidders are able to approximate bidding their budget (w_i) with a decreasing sequence of bids for increasing k.

It is clear that for fixed k an auction with bidding sets defined by (A.11) continues to admit a monotone equilibrium $\beta^k(\theta)$ and as above we can consider a sequence of equilibria such that for all i, $\beta_i^k(\theta_i) \xrightarrow{a.e.} \hat{\beta}_i(\theta_i)$. It is also clear that the limiting strategy $\hat{\beta}_i(\theta_i)$ respects the (actual) budget limit: $\hat{\beta}_i(\theta_i) \leq w_i$ for all θ_i.

Limit Allocation Rules We will define an allocation rule that will map player's submitted bids and announced types, $\hat{\varphi}\colon \times_i \mathbb{R}_+ \cup \{l\} \times \Theta \to [0,1]^{N+1}$, into allocations.[6] Let $(\mathbf{b}, \theta)$ be a submitted bid profile and an announced type profile. For player i placing bid b_i and announcing type θ_i, define the sequence of bids $\tilde{\beta}_i^k(\theta_i)[b_i]$ as

$$\tilde{\beta}_i^k(\theta_i)[b_i] = \begin{cases} \beta_i^k(\theta_i) & \text{if } b_i = \lim \beta_i^k(\theta_i) \\ \tilde{b}_i^k & \text{if } b_i \neq \lim \beta_i^k(\theta_i) \end{cases} \tag{A.12}$$

where $\tilde{b}_i^k = \min\left\{b \in \mathcal{B}_i^k(w_i) \colon b \geq b_i\right\}$. If the bid b_i is the limit of a sequence of bids placed by a bidder of type θ_i, the sequence assumes those values. Otherwise, b_i is approached from above. Note that this sequence of bids is feasible within the relaxed budget set for large k. For notation, let $\tilde{\beta}^k(\theta)[\mathbf{b}] = (\tilde{\beta}_1^k(\theta_1)[b_1], \ldots, \tilde{\beta}_N^k(\theta_N)[b_N])$. For any profile $(\mathbf{b}, \theta)$ define the following allocation rule

$$\hat{\varphi}_i(\mathbf{b}, \theta) = \lim_k \varphi_i(\tilde{\beta}^k(\theta)[\mathbf{b}]). \tag{A.13}$$

The allocation rule (A.13) is standard in the sense that if the high bidder is unique, she wins the item. When bidders tie for the highest bid, this allocation rule breaks ties by relying on a limit of allocations from approximating games given the announced types of bidders. The following statement complete the proof of Lemma 4.

Lemma 20. *Consider a first-price auction where bidders of type θ_i can bid from $\mathcal{B}_i(w_i) = [r_i, w_i] \cup \{l\}$ and must announce their type. Suppose the allocation rule is defined by* (A.13) *and let $\hat{\sigma}_i(\theta_i) = \theta_i$. Then $(\hat{\beta}, \hat{\sigma})$ is an equilibrium.*

[6] $\hat{\varphi}_0 = 1 - \sum_{j \geq 1} \hat{\varphi}_j$ to account for cases where the item is unsold.

Proof. Suppose $(\hat{\beta}_i, \hat{\sigma}_i)$ is not an equilibrium and player i has a profitable deviation to $(\tilde{\beta}_i, \tilde{\sigma}_i)$.

Define the following sequence of strategies.

$$\breve{\beta}_i^k(\theta_i) = \tilde{\beta}_i^k(\tilde{\sigma}_i(\theta_i))[\tilde{\beta}_i(\theta_i)]$$

By construction $\breve{\beta}_i^k \to \tilde{\beta}_i$. Therefore the bidding strategy from the profitable deviation, $\tilde{\beta}_i$, can be approximated by a sequence of feasible strategies. As $\tilde{u}_i(\theta, b_i)$ is continuous in b_i, $\tilde{u}_i(\theta, \breve{\beta}_i^k) \to \tilde{u}_i(\theta, \tilde{\beta}_i)$. Similarly by construction, $\varphi_i(\tilde{\beta}^k(\theta)[\tilde{\beta}_i, \hat{\beta}_{-i}]) \to \hat{\varphi}_i(\tilde{\beta}_i, \hat{\beta}_{-i}, \tilde{\sigma}_i, \hat{\sigma}_{-i})$.

Define the following values:

$$\begin{aligned}
U_i(\beta^k) &= \int_\Theta \left[u_i(\mathbf{0}, w_i, 0) + \varphi_i(\beta^k)\tilde{u}_i(\theta, \beta_i^k)\right] f(\theta)d\theta \\
\hat{U}_i(\hat{\beta}, \hat{\sigma}) &= \int_\Theta \left[u_i(\mathbf{0}, w_i, 0) + \hat{\varphi}_i(\hat{\beta}, \hat{\sigma})\tilde{u}_i(\theta, \hat{\beta}_i)\right] f(\theta)d\theta \\
\hat{U}_i(\tilde{\beta}_i, \hat{\beta}_{-i}, \tilde{\sigma}_i, \hat{\sigma}_{-i}) &= \int_\Theta \left[u_i(\mathbf{0}, w_i, 0) + \hat{\varphi}_i(\tilde{\beta}_{-i}, \hat{\beta}_{-i}, \tilde{\sigma}_i, \hat{\sigma}_{-i})\tilde{u}_i(\theta, \tilde{\beta}_i)\right] f(\theta)d\theta \\
U_i(\tilde{\beta}_i^k, \beta_{-i}^k) &= \int_\Theta \left[u_i(\mathbf{0}, w_i, 0) + \varphi_i(\tilde{\beta}_i^k, \beta_{-i}^k)\tilde{u}_i(\theta, \tilde{\beta}_i^k)\right] f(\theta)d\theta
\end{aligned}$$

For player i, $U_i(\hat{\beta}^k) \to \hat{U}_i(\hat{\beta}, \hat{\sigma})$ while $U_i(\tilde{\beta}_i^k, \beta_{-i}^k) \to \hat{U}_i(\tilde{\beta}_i, \hat{\beta}_{-i}, \tilde{\sigma}_i, \hat{\sigma}_{-i})$ as all integrands are bounded. There exists an $\epsilon > 0$ sufficiently small such that

$$\hat{U}_i(\tilde{\beta}_i, \hat{\beta}_{-i}, \tilde{\sigma}_i, \hat{\sigma}_{-i}) > \hat{U}_i(\hat{\beta}, \hat{\sigma}) + \epsilon \tag{A.14}$$

But if β^k defines an equilibrium for the auction indexed by k, we have that $U_i(\beta^k) \geq U_i(\tilde{\beta}_i^k, \beta_{-i}^k)$ which implies a contradiction. □

A.3 Proofs

This appendix collects proofs and additional results referenced in the main text.

Lemma 21. *$\delta(b, x|s)$ is continuous and increasing in b and x and decreasing in s. $\eta(x|s)$ is continuous, differentiable, and increasing in both arguments. $\lambda(y|s)$ is decreasing in s.*

Proof. All of the constituent functions are continuous, thus, δ is also continuous. $g'(b) \leq 0$, therefore, δ is increasing in b. δ is increasing in x as $H(x|s)$ is increasing in x. Finally as S and Y are affiliated, if $s' > s$, $H(\cdot|s')$ first-order stochastically dominates $H(\cdot|s)$. Consequently, for all y $H(y|s') \leq H(y|s)$ and thus,

$$\frac{H(y|s')}{1 - H(y|s')} \leq \frac{H(y|s)}{1 - H(y|s)}.$$

as required. It is clear why $\eta(x|s)$ is increasing. $\lambda(y|s)$ is decreasing in s from affiliation. □

Proof of Lemma 1. This follows from the definition of $v(\cdot, \cdot)$. ■

Proof of Lemma 2. Clearly, $\dot{s}(s, b) = 0 \iff \eta(s|s) = \delta(b, s|s)$. For any s, $\delta(b, s|s)$ is strictly increasing in b. If a solution exists it is therefore unique. As $\delta(\underline{w}, s|s) \geq \eta(s, s)$, if a solution fails to exist at s it is because

$$\eta(s|s) < \underline{w} + \frac{H(s|s)}{g(\underline{w})(1 - H(s|s))}.$$

If $\psi(s) \leq \underline{w}$, then $\tilde{\psi}(s)$ is obviously continuous. Suppose instead that at some x, $\psi(x) > \underline{w}$. Then by the implicit function theorem, $\psi(\cdot)$ is continuous at x and there is an open set $\mathcal{O}_x$ containing x such that $s \in \mathcal{O}_x \implies \psi(s) > \underline{w}$. We can then write $\mathcal{X} = \cup_{\{x:\, \psi(x) > \underline{w}\}} \mathcal{O}_x$ as a countable union of disjoint open sets: $\mathcal{X} = [0, s_\psi^0) \cup \cup_{k=1}^{\infty} (s_\psi^{2k}, s_\psi^{2k+1})$.[7] $1 \notin \mathcal{X}$ as $\delta(b, 1|1)$ is not defined for all b. For $m \geq 0$, consider $\lim_{s \to s_\psi^m} \tilde{\psi}(s) > \underline{w}$ then $\psi(s_\psi^m) > \underline{w}$ which is a contradiction. Hence $\tilde{\psi}(s)$ is continuous. That $\lim_{s \to s_\psi} \tilde{\psi}(s) = \underline{w}$ follows from this observation. ■

[7] $[0, s_\psi^0)$ is open in the relative topology on $[0, 1]$. If $s_\psi^0 = 0$, $[0, s_\psi^0) = \emptyset$.

Proof of Lemma 3. The conclusion is immediate because $\delta(b, s|s)$ is strictly increasing in b. ■

Proof of Lemma 4. If $\underline{w} = 0$, then $\psi(0) \geq 0$ This follows from $\delta(0, 0|0) = 0$ and $0 \leq \eta(0|0) \leq 1$. As δ is strictly increasing in b, there will be a solution $\eta(0|0) = \delta(b, 0|0)$ for some $b \in [0, \eta(0|0)]$

If $\psi(0) = 0$, then $(0, 0)$ is a critical point. Alternatively, if $\psi(0) > 0$, then $\psi(s)$ is continuous on $[0, s_\psi^0)$ and $\lim_{s \uparrow s_\psi^0} \psi(s) = \underline{w}$. On the other hand, $\nu(s)$ is strictly increasing from 0 to $v(1, 1) \geq \eta(0|0) \geq 0$. Thus, $\exists s^* \in [0, s_\psi^0]$ at which $\nu(s^*) = \psi(s^*)$. ■

Proof of Lemma 5. It is sufficient to confirm that the eigenvalues of the Jacobian matrix evaluated at (s^*, b^*),

$$J = \left(\begin{array}{cc} \frac{\partial \dot{s}}{\partial s} & \frac{\partial \dot{s}}{\partial b} \\ \frac{\partial \dot{b}}{\partial s} & \frac{\partial \dot{b}}{\partial b} \end{array} \right) \Bigg|_{(s^*,b^*)},$$

are real-valued. The eigenvalues will be real if

$$\left(\frac{\partial \dot{s}}{\partial s} - \frac{\partial \dot{b}}{\partial b} \right)^2 + 4 \frac{\partial \dot{s}}{\partial b} \frac{\partial \dot{b}}{\partial s}$$

when evaluated at (s^*, b^*) is non-negative. Then,

$$\begin{aligned} \left.\frac{\partial \dot{s}}{\partial b}\right|_{(s^*,b^*)} &= \gamma'(b^*) \underbrace{(\eta(s^*|s^*) - \delta(b^*, s^*|s^*))}_{=0} + \gamma(b^*) \left(- \left.\frac{\partial \delta}{\partial b}\right|_{(s^*,b^*)} \right) \\ &= -\gamma(b^*) \left.\frac{\partial \delta}{\partial b}\right|_{(s^*,b^*)} < 0 \end{aligned}$$

where the conclusion follows from δ strictly increasing in b and $\gamma(b^*) \neq 0$. Also,

$$\begin{aligned} \left.\frac{\partial \dot{b}}{\partial s}\right|_{(s^*,b^*)} &= \left.\frac{\partial \lambda}{\partial s}\right|_{(s^*,b^*)} \underbrace{(b^* - v(s^*, s^*))}_{=0} + \lambda(s^*|s^*) \left(- \left.\frac{\partial v}{\partial s}\right|_{(s^*,b^*)} \right) \\ &= -\lambda(s^*|s^*) \left.\frac{\partial v}{\partial s}\right|_{(s^*,b^*)} < 0 \end{aligned}$$

Therefore, $4\frac{\partial \dot{s}}{\partial b}\frac{\partial \dot{b}}{\partial s} > 0$ as needed. ■

Proof of Proposition 5. We need to verify that the first-order characterization of the best response is optimal given that all others are following the strategy $\beta(s, w) = \min\{w, \bar{b}(s)\}$. Treating the auction as a revelation mechanism, the expected utility of a bidder of type (s, w) who bids as a type $x \in [\bar{b}^{-1}(0), \bar{b}^{-1}(w)]$ is

$$\begin{aligned} U(\bar{b}(x)|s, w) = &\int_0^{\bar{b}(x)} \int_0^1 \left(v(s, y) - \bar{b}(x)\right) h(y|s) g(z) dy dz \\ &+ \int_{\bar{b}(x)}^1 \int_0^x \left(v(s, y) - \bar{b}(x)\right) h(y|s) g(z) dy dz. \end{aligned}$$

Differentiating this expression with respect to x gives

$$\begin{aligned} \frac{\partial U(\bar{b}(x)|s, w)}{\partial x} = &(1 - G(x)) h(x|s) \left(v(s, x) - \bar{b}(x)\right) \\ &+ \bar{b}'(x) g(\bar{b}(x))(1 - H(x|s)) \left[\eta(x|s) - \delta(\bar{b}(x), x|s)\right]. \end{aligned}$$

Collecting hazard rate terms,

$$\begin{aligned} \frac{\partial U}{\partial x} = &g(\bar{b}(x))(1 - H(x|s)) \\ &\times \left[\frac{\lambda(x|s)}{\gamma(\bar{b}(x))} \left(v(s, x) - \bar{b}(x)\right) + \bar{b}'(x) \left(\eta(x|s) - \delta(\bar{b}(x), x|s)\right)\right]. \end{aligned}$$

To establish that $x = s$ is optimal it is sufficient to show that

$$\begin{aligned} x > s &\implies \frac{\partial U(\bar{b}(x)|s, w)}{\partial x} \leq 0 \\ x < s &\implies \frac{\partial U(\bar{b}(x)|s, w)}{\partial x} \geq 0 \end{aligned}$$

Suppose $x > s$. By affiliation, $\eta(x|x) \geq \eta(x|s)$ and $\delta(\bar{b}(x), x|x) \leq \delta(\bar{b}(x), x|s)$. Also due to affiliation $\lambda(x|x) \leq \lambda(x|s)$ and by assumption $v(s, x)\lambda(x|s) \leq v(x, x)\lambda(x|x)$. Thus for $x > s$,

$$\frac{\partial U(\bar{b}(x)|s, w)}{\partial x} \leq \frac{\partial U(\bar{b}(x)|x, w)}{\partial x} = 0. \qquad \text{(A.15)}$$

By a similar argument, if $x < s$, $\frac{\partial U(\bar{b}(x)|s,w)}{\partial x} \geq 0$. Thus, a bidder of type (s, w) has no incentive to deviate to a bid in the range of $\bar{b}(\cdot)$. The same argument applies to a bidder who bids $\beta(s, w) = w$; however, such a bidder can only consider mimicking a type $x < \bar{b}^{-1}(w)$ and is therefore at a constrained optimum (in the range of $\bar{b}$).

Bids in excess of $\bar{b}(1)$ are dominated. To rule out deviations below $\bar{b}(0)$, suppose a bidder bid $b < \bar{b}(0) \leq w$. This bid competes only against other bidders who are bidding exactly their budget. Hence, the bidder's expected utility is given by

$$U(b|s,w) = \int_0^b \int_0^1 (v(s,y) - b)\, h(y|s) g(w) dy dw = G(b)(\eta(0|s) - b)$$

This expression is concave in b and achieves a maximum at $\check{b}$ which solves $\eta(0|s) = \check{b} + G(\check{b})/g(\check{b})$. Recall that $\psi(0)$ satisfies $\eta(0|0) = \psi(0) + \frac{G(\psi(0))}{g(\psi(0))}$. Consequently, by affiliation, $\eta(0|s) \geq \eta(0|0) \implies \check{b} \geq \psi(0)$ because $b + G(b)/g(b)$ is increasing. However, for s sufficiently close to zero, $\bar{b}(s) \leq \psi(s)$, hence $\bar{b}(0) \leq \psi(0)$. Therefore the optimal bid (weakly) below $\bar{b}(0)$ is $\bar{b}(0)$, which was already recognized as inferior to $\beta(s, w) \geq \bar{b}(0)$.

Finally consider bidders with a budget of $w < \bar{b}(0)$. Given the strategy followed by all other bidders they solve $\max_{b \leq w} G(b)(\eta(0|s) - b)$. By the argument above, $\beta(s, w) = w$ is their optimal bid. ■

Proof of Proposition 6. Suppose $\tilde{s} < 1$ as otherwise there is little to prove. $\beta(s, w)$ is continuous and $\beta(s, \bar{w})$ is strictly increasing. We need only verify that no bidders wish to deviate within the range of $\beta(s, w)$. Treating the auction as a revelation mechanism suppose a bidder of type (s, w) instead bid as a type (x, w) while other bidders follow $\beta(s, w)$. Her expected utility from doing so is

$$U(\beta(x,w)|s,w) \tag{A.16}$$
$$= \begin{cases} \int_0^x (v(s,y) - b_f(s)) h(y|s) dy & \text{if } x \leq \tilde{s} \\ G(\tilde{b}(x)) \int_0^1 \left(v(s,y) - \tilde{b}(x)\right) h(y|s) dy \\ \qquad + (1 - G(\tilde{b}(x)) \int_0^x \left(v(s,y) - \tilde{b}(\hat{s})\right) h(y|s) dy & \text{if } x > \tilde{s} \end{cases}$$

There are two cases to consider:

Case 1. Suppose that $s \leq \tilde{s}$. Then $\beta(s,w) = b_f(s)$. As $b_f(s)$ is an equilibrium strategy profile in the first-price auction without budget constraints, the bidder has no profitable deviation imitating some type $x < \tilde{s}$. Therefore, we need only rule out deviations above $\underline{w}$. By an argument parallel to (A.15) we know that if $x > \tilde{s} \geq s$,

$$\frac{\partial U(\tilde{b}(x)|s,w)}{\partial x} \leq 0.$$

Thus, the best announcement for a type s bidder in excess of $\tilde{s}$ is $x = \tilde{s}$. However, this bid is in $b_f(s)$'s range and thus not optimal.

Case 2. Suppose $s > \tilde{s}$. The same argument as in Lemma 5 establishes that this bidder has no profitable deviation to bids above $\underline{w}$. Consider a deviation to some type $x < \tilde{s}$. Differentiating (A.16) in this domain gives

$$\begin{aligned}\frac{\partial U(b_f(x)|s,w)}{\partial x} &= H(x|s)\left[(v(s,x) - b_f(x))\frac{h(x|s)}{H(x|s)} - b_f'(x)\right] \\ &\geq H(x|s)\left[(v(x,x) - b_f(x))\frac{h(x|x)}{H(x|x)} - b_f'(x)\right] = 0\end{aligned}$$

where the inequality follows from affiliation. Therefore the best response of a type $s > \tilde{s}$ bidder in the range $[0, \underline{w}]$ is to bid $\underline{w}$. However, this was already shown to be suboptimal.

Finally, for bidders who are constrained, $\left.\frac{\partial U(\beta(x,w)|s,w)}{\partial x}\right|_{x=s} \geq 0$; therefore, they wish to bid more but cannot. Downward deviations are ruled out by the same argument as for an unconstrained bidder. ■

Proof of Corollary 2. It is sufficient to establish that $b_f'(\tilde{s}) < \lim_{s \to \tilde{s}^+} \tilde{b}'(s)$. At $\tilde{s}$, $\tilde{b}(\tilde{s}) = b_f(\tilde{s}) = \underline{w}$ and $G(\underline{w}) = 0$. Let $\tilde{H} \equiv H(\tilde{s}|\tilde{s})$, $\tilde{g} \equiv g(\tilde{b}(\tilde{s}))$ and $\tilde{\eta} \equiv \eta(\tilde{s}|\tilde{s})$. By concavity, $\tilde{s} > 0 \implies \tilde{g} < \infty$. Then,

$$\begin{aligned}\lim_{s \to \tilde{s}^+} \tilde{b}'(s) &= \frac{\lambda(\tilde{s}|\tilde{s})\left(\tilde{b}(\tilde{s}) - v(\tilde{s},\tilde{s})\right)}{\gamma(\tilde{b}(\tilde{s}))\left(\eta(\tilde{s}|\tilde{s}) - \delta(\tilde{b}(\tilde{s}),\tilde{s}|\tilde{s})\right)} \\ &= \frac{\frac{\tilde{H}}{1-\tilde{H}} \cdot \frac{1}{\tilde{g}}}{\frac{\tilde{H}}{1-\tilde{H}} \cdot \frac{1}{\tilde{g}} + \underline{w} - \tilde{\eta}} b_f'(\tilde{s})\end{aligned}$$

The equality follows from the substitution

$$b_f'(\tilde{s}) = [v(\tilde{s},\tilde{s}) - b_f(\tilde{s})]h(\tilde{s}|\tilde{s})/H(\tilde{s}|\tilde{s}).$$

As $\underline{w} - \tilde{\eta} < 0$ we can conclude that $\lim_{s\to\tilde{s}^+} \tilde{b}'(s) > b_f'(\tilde{s})$. ■

Proof of Lemma 6. At $s = s_0$, (3.13) becomes

$$\int_0^{s_0} (v(s_0,y) - \mu(s_0))\, h(y|s_0)dy + G(\mu(s_0)) \int_{s_0}^1 (v(s_0,y) - \mu(s_0))\, h(y|s_0)dy \\ \leq \int_0^{s_0} (v(s_0,y) - b_f(s_0))\, h(y|s_0)dy.$$

There are two cases. If $s_0 = 0$, this expression reduces to

$$G(\bar{b}(s_0)) \int_0^1 \big(v(0,y) - \bar{b}(0)\big)\, h(y|0)dy = G(\bar{b}(s_0))\, \big(\eta(0|0) - \bar{b}(0)\big) \leq 0,$$

which is true. If instead $s_0 > 0$, then $\mu(s_0) = \underline{w}$ and so,

$$\int_0^{s_0} (v(s_0,y) - \underline{w})\, h(y|s_0)dy \leq \int_0^{s_0} (v(s_0,y) - b_f(s_0))\, h(y|s_0)dy.$$

This expression is also true as $\underline{w} > b_f(s_0)$. Thus, $\mathcal{Z} \neq \emptyset$. At $s = \tilde{s} \equiv \bar{b}^{-1}(\underline{w})$,

$$\int_0^{\tilde{s}} (v(\tilde{s},y) - \mu(\tilde{s}))\, h(y|\tilde{s})dy + G(\mu(\tilde{s})) \int_{\tilde{s}}^1 (v(\tilde{s},y) - \mu(\tilde{s}))\, h(y|\tilde{s})dy \\ > \int_0^{\tilde{s}} (v(\tilde{s},y) - b_f(\tilde{s}))\, h(y|\tilde{s})dy.$$

This follows from the definition of $\mu(s)$. Thus $\hat{s} < \tilde{s}$ by continuity. ■

Proof of Proposition 7. The proof is divided into two parts. Part 1 confirms the existence of functions ϕ and b_1 with the stated properties. Their existence is not immediate as they are introduced in a self-referential manner. Part 2 verifies that the proposed strategy is an equilibrium. As the verification of equilibrium follows the standard argument, this part is abbreviated. Throughout the discussion below, it is convenient to extend $G(s)$ to all of $\mathbb{R}$, i.e. $G(w) = 0$ if $w \leq \underline{w}$ and analogously $G(w) = 1$ for $w > \bar{w}$.

Part 1: The functions b_1 and ϕ. This section derives some implied relationships between b_1 and ϕ and uses these relationships to construct a map Λ whose fixed point(s) will be used to define b_1 and ϕ. If ϕ and b_1 exist, they need to be consistent with three facts:

i) $\phi(s)$ describes an *indifference condition*; otherwise $\beta(s, w)$ could not be discontinuous along $\{(s, \phi(s))\}_{s>\hat{s}}$. Types $(s, \phi(s))$ are indifferent between bids of $b_1(s)$ and $\phi(s)$. The expected utility from bidding $\phi(s)$ is

$$U(\phi(s)|s, \phi(s)) = \int_0^{\hat{s}} (v(s, y) - \phi(s))h(y|s)dy \tag{A.17}$$

$$+ \int_{\hat{s}}^{s} (v(s, y) - \phi(s))G(\phi(y))h(y|s)dy \tag{A.18}$$

$$+ G(\phi(s)) \int_s^1 (v(s, y) - \phi(s))h(y|s)dy. \tag{A.19}$$

(A.17) is the contribution to utility from defeating all bidders of type $y < \hat{s}$, who are bidding according to $b_f(\cdot)$. (A.18) is the contribution of utility from defeating bidders of type $y \in [\hat{s}, s)$ but who have $w < \phi(y)$. Such bidders are bidding according to $b_1(\cdot)$. (A.19) is the contribution to utility from defeating all bidders of type $y \geq s$ who have a budget less than $\phi(s)$. The expected utility of bidding $b_1(x)$ is

$$U(b_1(x)|s, w) = \int_0^{\hat{s}} (v(s, y) - b_1(x))h(y|s)dy$$

$$+ \int_{\hat{s}}^{x} (v(s, y) - b_1(x))G(\phi(y))h(y|s)dy. \tag{A.20}$$

The first term is the contribution to utility from defeating all bidders who bid according to $b_f(\cdot)$. The second term is the contribution to utility from defeating all bidders of type $y > \hat{s}$ who bid less than $b_1(x)$. The bounds of integration follow from $b_1(\cdot)$ being strictly increasing. Setting $U(b_1(s)|s, \phi(s)) = U(\phi(s)|s, \phi(s))$ gives (3.17) which is reproduced here:

$$\frac{\phi(s) - b_1(s)}{\eta(s|s) - \phi(s)} = \frac{G(\phi(s))(1 - H(s|s))}{H(\hat{s}|s) + \int_{\hat{s}}^{s} G(\phi(y))h(y|s)dy}.$$

ii) $b_1(\cdot)$ must be optimal for all types who do not increase their bid discontinuously. Treating the auction as a revelation mechanism and differentiating (A.20) with respect a player's announced type gives the differential equation (3.16) which is reproduced here:

$$\left.\frac{\partial U(b_1(x)|s,w)}{\partial x}\right|_{x=s} = 0 \implies {b_1}'(s) = \frac{[v(s,s)-b_1(s)]h(s|s)G(\phi(s))}{H(\hat{s}|s)+\int_{\hat{s}}^{s} G(\phi(y))h(y|s)dy} \tag{A.21}$$

iii) If a bidder of type (s,w) increases her bid discontinuously, the large increase must be a good idea. That is, it must be the constrained optimal bid in the range $[\underline{w},w]$. The expected utility of a bid $b \in [\underline{w},\phi(\hat{s})]$ is

$$\begin{aligned} U(b|s,\phi(s)) = &\int_0^{\hat{s}} (v(s,y)-b)h(y|s)dy \\ &+ \int_{\hat{s}}^{\phi^{-1}(b)} (v(s,y)-b)G(\phi(y))h(y|s)dy \\ &+ G(b)\int_{\phi^{-1}(b)}^{1} (v(s,y)-b)h(y|s)dy. \end{aligned}$$

Each term's motivation mirrors that of the corresponding term (A.17)–(A.19). At $b=\phi(s)$,

$$\left.\frac{\partial U(b|s,\phi(s))}{\partial b}\right|_{b=\phi(s)} \geq 0. \tag{A.22}$$

We are concerned with showing that there exist b_1 and ϕ that meet (A.21), (A.21), and (A.22), along with the collection of boundary and monotonicity conditions from (3.14). Let $C[\hat{s},1]$ be the set of continuous functions on the interval $[\hat{s},1]$ equipped with the usual metric, $d(c_1,c_2)=\sup_{s\in[\hat{s},1]}|c_1(s)-c_2(s)|$, and define the following subsets:

- $\mathscr{B} \subset C[\hat{s},1]$.

  If $b \in \mathscr{B}$ then $b(s)$ is nondecreasing, $b(\hat{s})=b_f(\hat{s})$, $b(s) \leq \min\{v(s,s),\underline{w}\}$ and if $s,s' \in [\hat{s},1]$, then

$$|b(s)-b(s')| \leq \left[\sup_{s\in[\hat{s},1]} \frac{v(s,s)h(s|s)}{H(\hat{s}|1)}\right]|s-s'|.$$

- $\mathscr{Z} \subset C[\hat{s}, 1]$.

  If $\zeta \in \mathcal{Z}$ then $\zeta(\hat{s}) = H(\hat{s}|\hat{s})$, $0 < H(\hat{s}|1) \leq \zeta(s) \leq 1$ and if $s, s' \in [\hat{s}, 1]$, then

$$|\zeta(s) - \zeta(s')| \leq \left| \sup_{s \in [\hat{s},1]} \left(h(s|s) + \left. \frac{\partial H(\hat{s}|x)}{\partial x} \right|_{x=s} \right) \right| |s - s'|.$$

With $(b, \zeta) \in \mathscr{B} \times \mathscr{Z}$ consider the following function (suppressing its dependence on $b(\cdot)$ and $\zeta(\cdot)$):

$$\Phi(s) \equiv \min \left\{ \arg \min_{\hat{\phi} \in [\underline{w}, \eta(s|s)]} \left| \max \left\{ 0, \frac{\hat{\phi} - b(s)}{\eta(s|s) - \hat{\phi}} \right\} - \frac{G(\hat{\phi})(1 - H(s|s))}{\zeta(s)} \right| \right\}. \tag{A.23}$$

For a given b and ζ, the equation

$$\frac{\hat{\phi} - b(s)}{\eta(s|s) - \hat{\phi}} = \frac{G(\hat{\phi})(1 - H(s|s))}{\zeta(s)} \tag{A.24}$$

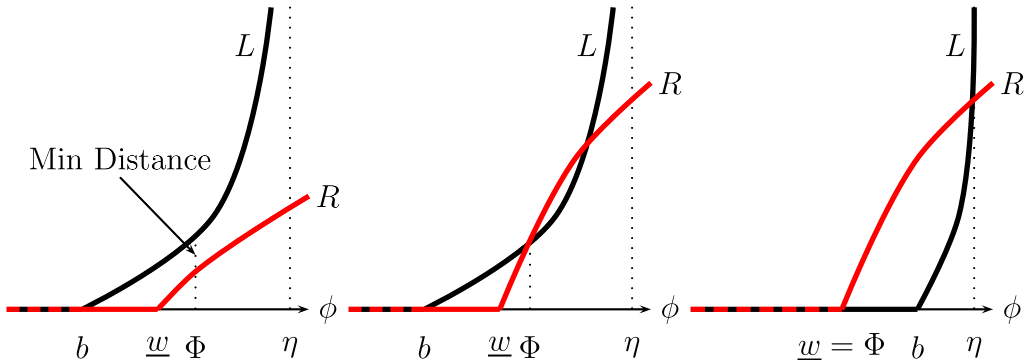
may not admit a solution—$\hat{\phi}$—greater than $\underline{w}$. $\Phi(s)$ returns an approximate solution, if no solution exists. If a solution does exist it selects the smallest solution. To develop an intuition for $\Phi(s)$, Figure A.2 presents three typical scenarios. Noting that both $b(\cdot)$ and $\zeta(\cdot)$ are continuous, $\Phi(s)$ is also continuous in $b(\cdot), \zeta(\cdot)$ and s.

Finally, define the map Λ as follows: $\Lambda(b, \zeta) = (\check{b}, \check{\zeta})$ such that

$$\check{b}(s) = b_f(\hat{s}) + \int_{\hat{s}}^{s} \frac{[v(x, x) - b(x)]G(\Phi(x))h(x|x)}{\zeta(x)} dx \tag{A.25}$$

$$\check{\zeta}(s) = H(\hat{s}|s) + \int_{\hat{s}}^{s} G(\Phi(x))h(x|s) dx \tag{A.26}$$

It is easy to see that both $\mathscr{B}$ and $\mathscr{Z}$ are equicontinuous subsets of $C[\hat{s}, 1]$. Moreover Λ is continuous and $\Lambda(\mathscr{B} \times \mathscr{Z}) \subset \mathscr{B} \times \mathscr{Z}$. By Schauder's fixed point theorem, Λ has a fixed point. We next verify that a fixed point of Λ satisfies several additional conditions, which are necessary for it to conform with the description of the strategy given above. (We do not make any statements concerning the uniqueness of the fixed point.)



(a) No Exact Solution (b) Two Solutions (c) Solution at $\underline{w}$.

Figure A.2: The value of Φ for a fixed $s, b(s)$ and $\zeta(s)$ in the three possible cases. Notation: $L(\phi) = \max\left\{0, \frac{\hat{\phi}-b(s)}{\eta(s|s)-\hat{\phi}}\right\}$, $R(\phi) = \frac{G(\hat{\phi})(1-H(s|s))}{\zeta(s)}$.

Lemma 22. *Let (b,ζ) be a fixed point of Λ. $b(s)$ satisfies the following properties:*

1. *$b(\hat{s}) = b_f(s)$ and $b(\cdot)$ is continuous and increasing whenever $\Phi(s) > \underline{w}$.*
2. *$b(s) \leq \underline{w}$.*

Proof. The first statement is trivial. To see that the second statement is true, suppose that $b(s) > \underline{w}$ for all $s > \tilde{s}$. By the definition of $\Phi(s)$ in (A.23) it follows that $\Phi(s) = \underline{w}$ for all $s > \tilde{s}$ and so $b'(s) = 0$ for all $s > \tilde{s}$. But if $\underline{w} = b(\tilde{s}) < b(1)$ then $b'(s) > 0$ for some $s \in (\tilde{s}, 1]$, a contradiction. ☐

Lemma 23. *Let (b,ζ) be a fixed point of Λ and let $\Phi(s)$ be defined as in* (A.23)*. Then for all $s \geq \hat{s}$,*

$$\frac{\Phi(s) - b(s)}{\eta(s|s) - \Phi(s)} = \frac{G(\Phi(s))(1 - H(s|s))}{\zeta(s)}. \tag{A.27}$$

Proof. If ever $b(s) = \underline{w}$, the (A.27) is easily seen to hold. Suppose $b(s) < \underline{w}$. We first establish that (A.27) holds at $s = \hat{s}$. When $s = \hat{s}$, (A.27) reduces to

$$\frac{\Phi - b_f(\hat{s})}{\eta(\hat{s}|\hat{s}) - \Phi} = \frac{G(\Phi)(1 - H(\hat{s}|\hat{s}))}{H(\hat{s}|\hat{s})}$$

From Lemma (6), we know that $\Phi = \mu(\hat{s}) > \underline{w}$ satisfied this equation.

To see that there is no $\phi < \mu(\hat{s})$ that could satisfy (A.23), suppose the contrary. Then any such ϕ must satisfy (suppressing the dependence on $\hat{s}$)

$$-b_f = (\eta - \phi)G(\phi)(1 - H)/H - \phi \tag{A.28}$$

Differentiating the right-hand side of (A.28) with respect to ϕ gives

$$\begin{aligned}
&\frac{d}{d\phi}\big((\eta - \phi)G(\phi)(1 - H)/H - \phi\big) \\
&= g(\phi)(\eta - \phi)\frac{1 - H}{H} - G(\phi)\frac{1 - H}{H} - 1 \\
&= g(\phi)\frac{1 - H}{H}\left(\eta - \phi - \frac{G(\phi)}{g(\phi)} - \frac{H}{g(\phi)(1 - H)}\right) \\
&= g(\phi)\frac{1 - H}{H}(\eta(\hat{s}|\hat{s}) - \delta(\phi, \hat{s}|\hat{s})).
\end{aligned} \tag{A.29}$$

When $\phi < \mu(\hat{s})$, $\eta(\hat{s}|\hat{s}) - \delta(\phi, \hat{s}|\hat{s}) > 0$. So the right-hand side of (A.28) is increasing as $\phi \uparrow \mu(\hat{s})$. Equality occurs only at $\phi = \psi(\hat{s}) \geq \mu(\hat{s})$. Therefore, there is no $\phi < \mu(\hat{s})$ that solves (A.28). If $\mu(\hat{s}) = \psi(\hat{s})$ then the solution to (A.28) is at a tangency point. The final term in (A.29) is zero at $\phi = \psi(\hat{s})$ and is positive (negative) for $\phi < (>) \psi(\hat{s})$. Thus, (A.28) has only one solution.

As $\Phi(s)$, $b(s)$, and $\zeta(s)$ are continuous, if (A.28) fails to be satisfied, it must fail on some interval, say $(\tilde{s}, \tilde{s}')$. It is clear that if (A.28) fails, it must be that for $s \in (\tilde{s}, \tilde{s}')$ at $\Phi(s)$,

$$\begin{aligned}
&\frac{\Phi(s) - b(s)}{\eta(s|s) - \Phi(s)} \geq \frac{G(\Phi(s))(1 - H(s|s))}{\zeta(s)} \\
\implies &(\Phi(s) - b(s))\zeta(s) \geq (\eta(s|s) - \Phi(s))G(\Phi(s))(1 - H(s|s))
\end{aligned} \tag{A.30}$$

As the inequality in (A.30) becomes strict for $s > \tilde{s}$, at $s = \tilde{s}$, the derivative of the left-hand side of (A.30) with respect to s (holding $\Phi(\tilde{s})$ fixed) must exceed the derivative of the right-hand side of (A.30) with respect to s. Intuitively, the right-hand side escapes downward from the left-hand side locally at $\tilde{s}$. Therefore, the following must be true,

$$\frac{d}{ds}\left[(\Phi(\tilde{s}) - b(s))\zeta(s)\right]|_{s=\tilde{s}} \geq \frac{d}{ds}\left[(\eta(s|s) - \Phi(\tilde{s}))G(\Phi(\tilde{s}))(1 - H(s|s))\right]|_{s=\tilde{s}}. \tag{A.31}$$

Following some algebra and substituting $b(\tilde{s}) = \frac{[v(\tilde{s},\tilde{s})-b(\tilde{s})]G(\Phi(\tilde{s}))h(\tilde{s}|\tilde{s})}{\zeta(\tilde{s})}$ and

$$\zeta'(\tilde{s}) = \left.\frac{\partial H(\hat{s}|s)}{\partial s}\right|_{s=\tilde{s}} + G(\Phi(\tilde{s}))h(\tilde{s}|\tilde{s}) + \int_{\hat{s}}^{\tilde{s}} G(\Phi(y)) \left.\frac{\partial h(y|s)}{\partial s}\right|_{s=\tilde{s}} dy$$

gives,

$$\begin{aligned}(\Phi(\tilde{s}) - b(\tilde{s})) &\left[\left.\frac{\partial H(\hat{s}|s)}{\partial s}\right|_{s=\tilde{s}} + \int_{\hat{s}}^{\tilde{s}} G(\Phi(y)) \left.\frac{\partial h(y|s)}{\partial s}\right|_{s=\tilde{s}} dy \right] \\ &+ \Phi(\tilde{s})G(\Phi(\tilde{s}) \left[h(\tilde{s}|\tilde{s}) - \left.\frac{dH(s|s)}{ds}\right|_{s=\tilde{s}} \right] \\ &\geq G(\Phi(\tilde{s}) \int_{\tilde{s}}^{1} \left.\frac{\partial}{\partial s} v(s,y)h(y|s)\right|_{s=\tilde{s}} dy\end{aligned}$$

By independence, the above reduces to

$$0 \geq \int_{\tilde{s}}^{1} \left.\frac{\partial}{\partial s} v(s,y)h(y|s)\right|_{s=\tilde{s}} dy > 0$$

Where the strict inequality follows from the assumption that $v(s,y)$ is strictly increasing in a player's own signal. As this final line is a contradiction we conclude that (A.27) indeed holds with equality for all $s \geq \hat{s}$. □

Lemma 24. *There exists $\hat{s}' \in (\hat{s}, 1]$ such that for all $s \geq \hat{s}'$, $b(s) = \Phi(s) = \underline{w}$.*

Proof. As (A.27) holds, at $s = 1$, we clearly have $\Phi(1) = b(1)$. Moreover, as $b(s) \leq \underline{w} \geq \Phi(s)$ we must have $b(1) = \Phi(1) = \underline{w}$. The lemma's conclusion follows. □

Lemma 25. *The following inequality holds:*

$$(1 - H(s|s)) \left(g(\Phi(s))(\eta(s|s) - \Phi(s)) - G(\Phi(s)\right) \geq \zeta(s). \qquad \text{(A.32)}$$

Proof. Recall that for each s, $\Phi(s)$ solves (suppressing the s),

$$\Phi - b = (\eta - \Phi)G(\Phi)(1 - H)/\zeta$$

For $\Phi > \underline{w}$, the righthand side of this expression is concave in Φ, therefore at the minimal solution to this equation, the slope of the right-hand side must exceed the slope of the left-hand side. Differentiation gives $[g(\Phi)(\eta - \Phi) - G(\Phi)](1 - H)/\zeta \geq 1$ which is (A.32). □

Lemma 26. $\Phi(s)$ *is non-increasing on* $[\hat{s}, 1]$.

Proof. Recall that $\Phi(s)$ solves $U(\Phi(s)|s, \Phi(s)) = U(b_1(s)|s, \Phi(s))$ for $s > \hat{s}$. Suppressing a bidder's budget-type in the following notation, differentiating the preceding expression with respect to s gives

$$\frac{\partial U(\Phi(s)|s)}{\partial \Phi(s)} \frac{d\Phi(s)}{ds} + \frac{\partial U(\Phi(s)|s)}{\partial s} = \frac{\partial U(b_1(s)|s)}{\partial b_1(s)} \frac{db_1(s)}{ds} + \frac{\partial U(b_1(s)|s)}{\partial s} \tag{A.33}$$

where

$$\frac{\partial U(\Phi(s)|s)}{\partial \Phi(s)} \frac{d\Phi(s)}{ds} \tag{A.34}$$

$$= \big[(1 - H(s|s))\,(g(\Phi(s))(\eta(s|s) - \Phi(s)) - G(\Phi(s)) - \zeta(s)\big]\Phi'(s) \tag{A.35}$$

$$\frac{\partial U(\Phi(s)|s)}{\partial s} \tag{A.36}$$

$$= \int_0^s \frac{d}{ds}\left[v(s,y)h(y|s)\right] G_\Phi(y)dy + G_\Phi(s) \int_s^1 \frac{d}{ds}\left[v(s,y)h(y|s)\right] dy$$
$$- \Phi(s)\left[\int_0^s \frac{d}{ds} h(y|s) G_\Phi(y)dy + G_\Phi(s)\int_s^1 \frac{d}{ds} h(y|s)dy\right] \tag{A.37}$$

$$\frac{\partial U(b_1(s)|s)}{\partial b_1(s)} \frac{db_1(s)}{ds} = 0 \tag{A.38}$$

$$\frac{\partial U(b_1(s)|s)}{\partial s} \tag{A.39}$$

$$= \int_0^s \frac{d}{ds}\left[v(s,y)h(y|s)\right] G_\Phi(y)dy - b_1(s)\int_0^s \frac{d}{ds} h(y|s) G_\Phi(y)dy \tag{A.40}$$

and for notation

$$G_\Phi(y) = \begin{cases} 1 & y < \hat{s} \\ G(\Phi(y)) & y \geq \hat{s} \end{cases}.$$

(A.38) is zero by the envelope theorem. The first terms of (A.35), $\frac{\partial U(\Phi(s)|s)}{\partial \Phi(s)}$, is nonnegative by Lemma 25. A rearrangement of (A.33) for $s \in (\hat{s}, \hat{s}')$ gives:

$$\frac{\partial U(\Phi(s)|s)}{\partial \Phi(s)} \frac{d\Phi(s)}{ds} = [\Phi(s) - b_1(s)] \int_0^s \frac{d}{ds} h(y|s) G_\Phi(y) dy \\ - G_\Phi(s) \underbrace{\int_s^1 \frac{d}{dx} \big[(v(x,y) - \Phi(s)) h(y|x)\big]_{x=s} dy}_{>0} < 0 \tag{A.41}$$

where the final inequality follows from independence. Therefore, $\Phi'(s) \leq 0$ as desired.

□

Remark 2. The inequality (A.32) implies that (A.22) holds. Therefore, bidders who increase their bids discontinuously and bid their entire budget are at a constrained optimum.

Choose a fixed point of the Λ mapping and define $b_1(s) \equiv b^*(s)$ and $\phi(s) = \Phi_0(s) = \Phi_1(s)$. From Lemma 23 we know that there is a $\hat{s}' \leq 1$ such that $s < \hat{s}' \implies, b_1'(s) > 0, \phi(s) > \underline{w}$ and $s < \hat{s}' \implies b_1(s) = \phi(s) = \underline{w}$. Therefore proposed the strategy exists.

Part 2. Verification of Equilibrium. Arguments analogous to the other cases confirm that $\beta(s, w)$ is a best response for bids above $\mu(\hat{s})$ and for bids below $b_f(\hat{s})$.

Bids placed according to $b_1(s)$ in its range are optimal by the routine argument (Krishna, 2002, Proposition 6.3). The following simple lemma, of independent interest, is useful in carrying out this step.

Lemma 27. *Let $f \colon \mathbb{R} \to \mathbb{R}_+$ be a non-increasing function, $f \neq 0$, and let X, Y be affiliated random variables with joint density $h(x, y)$. Then*

$$\sigma_f(y|x) = \frac{f(y)h(y|x)}{\int_0^y f(z)h(z|x)dx}$$

is nondecreasing in x.

Proof. If $f(y) = 0$, then the result is trivial. Suppose $f(y) > 0$ and let $x' > x$. Then

$$\begin{aligned}\sigma_f(y|x) &= \frac{f(y)h(y|x)}{\int_0^y f(z)h(z|x)dz} \\ &= \frac{f(y)\frac{h(y|x)}{H(y|x)}}{\int_0^y f(z)\frac{h(z|x)}{H(y|x)}dz} \\ &\leq \frac{f(y)\sigma(y|x')}{\mathbb{E}[f(Y)|X = x, Y \leq y]} \\ &\leq \frac{f(y)\sigma(y|x')}{\mathbb{E}[f(Y)|X = x', Y \leq y]} = \sigma_f(y|x')\end{aligned}$$

The first inequality follows from $\sigma(y|\cdot)$ being nondecreasing. The second inequality follows from affiliation and $f(\cdot)$ being non-increasing. Therefore, the numerator decreases in x. □

Bidders who increase their bid discontinuously only do so when they are indifferent between bidding $\phi(s)$ and $b_1(s)$; therefore, such an increase is optimal. The only unusual element in need of verification is that a bidder of type (s, w) does not bid $\phi(\check{s})$ for some $\check{s} > \phi^{-1}(w)$. (These are the only remaining feasible deviations for such a bidder. Bids of $\phi(\check{s})$ if $\check{s} < s$ are not feasible because ϕ is decreasing.) To see that such a deviation is not worthwhile fix $\check{s}$, let $\check{b} = b_1(\check{s})$ and $\check{\phi} = \phi(\check{s})$ and consider the expression

$$\begin{aligned}\Delta(s, \check{s}) &= U(\phi(\check{s})|s, w) - U(b_1(\check{s})|s, w) \\ &= \int_0^1 \underbrace{\begin{pmatrix} (\check{b} - \check{\phi}) \cdot \mathbf{1}(y < \hat{s}) \\ +(\check{b} - \check{\phi})G(\phi(y)) \cdot \mathbf{1}(\hat{s} \leq y < \check{s}) \\ +(v(s, y) - \check{\phi}) \cdot \mathbf{1}(\check{s} \leq y) \end{pmatrix}}_{\alpha(y,s)} h(y|s)dy \\ &= \mathbb{E}[\alpha(Y, s)|S = s].\end{aligned}$$

$\alpha(y, s)$ is a nondecreasing function of y and is increasing in s. By affiliation, therefore, $\Delta(s, \check{s})$ is increasing in s and is zero when $\check{s} = s$. Thus, when $s < \check{s}$,

$$U(\phi(\check{s})|s, w) \leq U(b_1(\check{s})|s, w) \leq U(b_1(s)|s, w).$$

The second inequality follows from $b_1(s)$ being the optimal bid for a bidder of type (s, w) in $b_1(\cdot)$'s range. This completes the proof of Proposition 7. ■

Proof of Lemma 7. If there are no critical points, the result is immediate as $\beta_1(s,w) \leq v(s,s) \leq \beta_2(s,w)$. Suppose instead that there is a critical point (s^*,b^*) and let $\bar{b}(s)$ be the solution (3.3) passing-through (s^*,b^*). It is sufficient to show that $\bar{b}(s) \leq b_2(s)$ whenever they are defined. From their definitions, it is easy to see that for all $s \in [s^*,1]$, $b_2(s) > v(s,s) \geq \bar{b}(s)$.

Suppose instead that $b_2(s) < \bar{b}(s)$ for some $s < s^*$. As both functions are continuous, there must exists a s such that $b_2(s) = \bar{b}(s)$. Noting that $\delta(b,s|s) > b$, $b_2'(s) < \bar{b}'(s)$. Therefore $b_2(\cdot)$ always crosses $\bar{b}(\cdot)$ from above when $s < s^*$. This contradicts $b_2(s^*) > v(s^*,s^*)$. Thus, $\bar{b}(s) < b_2(s)$ and so $\beta_1(s,w) \leq \beta_2(s,w)$.

The conclusion that $\hat{s}_2 \leq \hat{s}_1$ is a corollary to the preceding argument by noting that the initial discontinuous bid increases in the first-price auction (if it happens at all) is to some bid $\lim_{s\to\hat{s}_1^+} \beta_1(s,\bar{w}) \in (\underline{w}, \bar{b}(\hat{s}_1)]$ while the in the second-price auction it is to a bid of $\lim_{s\to\hat{s}_2^+} \beta_2(s,\bar{w}) = \underline{w}$. ■

Proof of Lemma 8. Let (s_k^*, b_k^*) be the critical point when $t = t_k$ and let $\psi_k(s)$ and $\nu_k(s)$ be the associated nullclines. As v is increasing in t, $s > 0 \implies \nu_1(s) > \nu_0(s)$ and $\psi_1(s) > \psi_0(s)$. Consequently, $b_1^* > b_0^*$. Let $\underline{s}^*$ be the solution to $\psi_0(s) = \nu_1(s)$ (or 0 if no solution exists). Let $\bar{s}^*$ be the solution to $\psi_1(s) = \nu_0(s)$. Then $\forall k, s_k^* \in [\underline{s}^*, \bar{s}^*]$ and so $s \in [\underline{s}^*, \bar{s}^*] \implies \bar{b}(s,t_0) < \bar{b}(s,t_1)$. Thus, in a neighborhood of the critical points value-enhancing information increases the unconstrained bidder's bid.

To prove that $\bar{b}(s,t_0) < \bar{b}(s,t_1)\ \forall s$, suppose the contrary. There are two cases:

1. Suppose $\bar{b}(\check{s},t_0) \geq \bar{b}(\check{s},t_1)$ for some $\check{s} \leq \underline{s}^*$.

   As $\bar{b}(\cdot,t_k)$ is continuous, without loss of generality suppose $\bar{b}(\check{s},t_0) = \bar{b}(\check{s},t_1) \equiv b$. As the reader can verify, at this $\check{s}$, $\bar{b}'(\check{s},t_1) > \bar{b}'(\check{s},t_0)$. Thus, $\bar{b}(s,t_1)$ must cross $\bar{b}(s,t_0)$ from below at $\check{s}$. For notation let

$$\lambda(\check{s}|\check{s},t_k) \equiv \check{\lambda}_k \qquad v(\check{s},\check{s}|t_k) \equiv \check{v}_k$$
$$\eta(\check{s}|\check{s},t_k) \equiv \check{\eta}_k \qquad \delta(b,\check{s}|\check{s},t_k) \equiv \check{\delta}_k$$

   Therefore,

$$\begin{aligned} \bar{b}'(\check{s},t_1) > \bar{b}'(\check{s},t_0) &\iff \frac{\check{\lambda}_1(b-\check{v}_1)}{\gamma(b)(\check{\eta}_1-\check{\delta}_1)} > \frac{\check{\lambda}_0(b-\check{v}_0)}{\gamma(b)(\check{\eta}_0-\check{\delta}_0)} \\ &\iff \check{\lambda}_1(b-\check{v}_1)(\check{\eta}_0-\check{\delta}_0) > \check{\lambda}_0(b-\check{v}_0)(\check{\eta}_1-\check{\delta}_1). \end{aligned}$$

As $\check{\lambda}_1 < \check{\lambda}_0$, $b \geq \check{v}_1 > \check{v}_0$, $\check{\eta}_1 > \check{\eta}_0$, and $\check{\delta}_1 < \check{\delta}_0$, this is a contradiction.

2. Suppose $\bar{b}(\check{s}, t_0) \geq \bar{b}(\check{s}, t_1)$ for some $\check{s} \geq \bar{s}^*$. Without loss of generality, suppose $\bar{b}(\check{s}, t_0) = \bar{b}(\check{s}, t_1) \equiv b$. With similar reasoning to case 1, $\bar{b}(s, t_1)$ crosses $\bar{b}(s, t_0)$ from above, i.e. $\bar{b}'(\check{s}, t_1) < \bar{b}'(\check{s}, t_0)$. Thus,

$$\begin{aligned} \bar{b}'(\check{s}, t_1) < \bar{b}'(\check{s}, t_0) &\iff \frac{\check{\lambda}_1(b - \check{v}_1)}{\gamma(b)(\check{\eta}_1 - \check{\delta}_1)} < \frac{\check{\lambda}_0(b - \check{v}_0)}{\gamma(b)(\check{\eta}_0 - \check{\delta}_0)} \\ &\iff \frac{\check{\lambda}_1(\check{v}_1 - b)}{\check{\delta}_1 - \check{\eta}_1} < \frac{\check{\lambda}_0(\check{v}_0 - b)}{\check{\delta}_0 - \check{\eta}_0} \\ &\iff \check{\lambda}_1(\check{v}_1 - b)(\check{\delta}_0 - \check{\eta}_0) < \check{\lambda}_0(\check{v}_0 - b)(\check{\delta}_1 - \check{\eta}_1). \end{aligned}$$

Observing that $\check{\delta}_0 - \check{\eta}_0 > \check{\delta}_1 - \check{\eta}_1$ and using the assumption that $v(s, y|t)\lambda(y|s, t)$ is nondecreasing in (s, t) yields the contradiction.

Therefore $t_0 < t_1 \implies \bar{b}(s, t_0) \leq \bar{b}(s, t_1)$ for all $s \in [0, 1]$. ■

The following are supplemental lemmas referenced in the main text.

Lemma 28. *Let* $\delta(b, x|s) = b + \frac{G(b)}{g(b)} + \frac{H(x|s)}{g(b)(1-H(x|s))}$. *For* $k \geq 0$, $kb + \delta(b, x|s)$ *is increasing in* b $\forall(x, s) \iff G''(b) \leq 0$.

Proof. ($\Longleftarrow$) Trivial. ($\Longrightarrow$) Suppose $\delta(b, x|s)$ is nondecreasing in b for all (x, s), then

$$\begin{aligned} &\frac{d}{db}\big(kb + \delta(b, x|s)\big) = 2 + k + \frac{[G(b) + H(x|s) - G(b)H(x|s)]g'(b)}{(H(x|s) - 1)g(b)^2} \geq 0 \\ &\implies [G(b) + H(x|s) - G(b)H(x|s)]g'(b) \leq (2 + k)(1 - H(x|s))g(b)^2 \end{aligned}$$

For $x = 1$, this inequality becomes $g'(b) = G''(b) \leq 0$. □

Lemma 29. *Suppose budgets and value-types are independent. Then* $G''(\cdot) \leq 0$ *if and only if Assumption 5 from Che & Gale (1998) is satisfied.*

Proof. Fix s and let

$$\tilde{G}(w, v) = 1 - \int_w^{\bar{w}} \int_v^1 g(\tilde{w})h(\tilde{v}|s)d\tilde{v}d\tilde{w}.$$

$\tilde{G}$ is the "$G(\cdot,\cdot)$" function from Che & Gale (1998, p. 10) adapted to section 3's information structure of affiliated value-signals and independent budgets on the type space $[0,1] \times [\underline{w}, \bar{w}]$. (There is typographic error in its original introduction in that article.) If $\tilde{G}_1$ is the derivative of $\tilde{G}$ with respect to the first argument, Assumption 5 from Che & Gale (1998) is:

> For all $v \in (0,1)$ and $N \geq 2$, $(N-1)w + \frac{\tilde{G}(w,v)}{\tilde{G}_1(w,v)}$ is strictly increasing in w.

Some algebra gives

$$\begin{aligned}(N-1)w + \frac{\tilde{G}(w,v)}{\tilde{G}_1(w,v)} &= (N-1)w + \frac{G(w)}{g(w)} + \frac{H(v|s)}{g(w)(1-H(v|s))} \\ &= (N-2)w + \delta(w,v|s).\end{aligned}$$

The equivalence follows from Lemma 28. □

A.4 The Symmetric Model: Extensions

This appendix considers caveats introduced in Section 3.

A.4.1 Multiple Critical Points

In general the set of critical points, $\mathcal{C}^*$, may not be a singleton. Multiple critical points can arise if $\psi(s)$ is not monotonic. This may happen if $G(\cdot)$ is not sufficiently concave and/or affiliation is relatively strong. When $\psi(s)$ is not monotone, the essence of the procedure is to identify a family of connected, upward sloping (i.e., $\dot{b}/\dot{s} > 0$) orbits from (3.5) and to construct $\bar{b}(s)$ by linking appropriate manifolds.

For example, Figure A.3 presents the case of three critical points and $\underline{w} = 0$. The central critical point is an unstable node while the others are saddle points. The thick arrows indicate the motion of (3.5) in the various regions and the dashed lines denote unstable manifolds (the central node has infinitely many such manifolds). Confining attention to the regions between $\nu(s)$ and $\psi(s)$, we are able to link adjacent nodes constructing a path for $\bar{b}(s)$ meeting Working Assumption 1.

If $\underline{w}$ is intermediate and there are multiple critical points, we can use a similar linking procedure to construct the appropriate solution for $\hat{b}(s)$. Consider, for example, the situation in Figure A.4. There are again three critical points. Suppose the equilibrium strategy is discontinuous at $\hat{s}$ and $\mu(s)$ is indicated as the dotted curve. As the critical point s_2^* is a node solutions for $\hat{b}(s)$ will be absorbed by the point. Beyond s_2^*, $\hat{b}(s)$ is extended by $\bar{b}(s) = \hat{b}(s)$ which ensures passage through the final critical point at s_3^*.

A.4.2 Relaxation of $G''(\cdot) \leq 0$

A non-concave distribution of budgets implies that budget constraints are relatively lax with large realizations of W_i relatively likely. From a strictly technical perspective, $G(\cdot)$'s concavity guarantees that $\psi(s)$ is single valued. This is no longer the case if $G(w)$, $w < v(1,1)$, is convex. If $\psi(s)$ is multivalued, there can be multiple critical points and, depending on model specifics, some critical points may need to be excluded when constructing $\bar{b}(\cdot)$. Although discontinuities in the equilibrium strategy are not ruled out, if we set this possibility aside then many previously developed tools continue to be useful as illustrated in the following example.

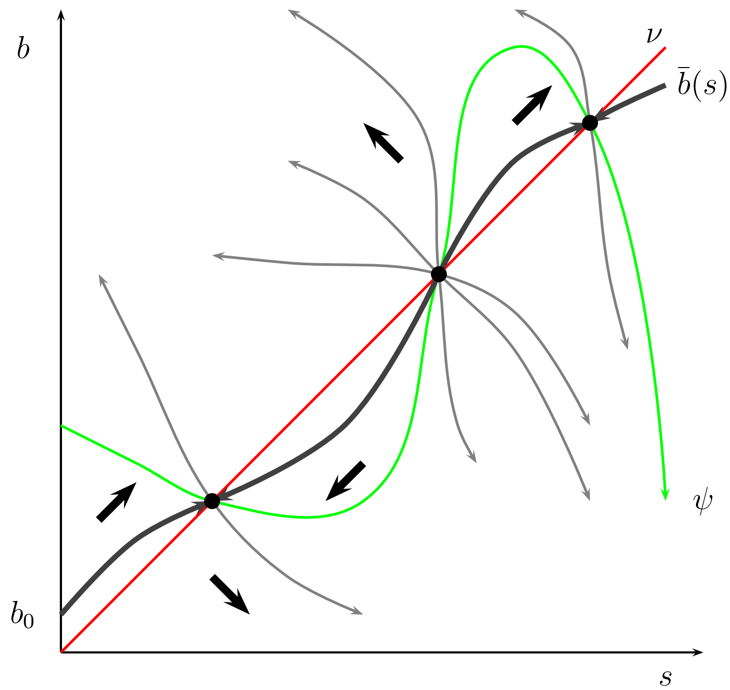


Figure A.3: Linking orbits to construct $\bar{b}(s)$. There are three critical points. The function $\bar{b}(s)$ can be constructed by linking together manifolds connecting adjacent critical points as shown.

Example 7. Consider Example 3 but suppose that the distribution of budgets is now $G(w) = (w/2)^2$. We can consider an equilibrium of the form $\beta(s,w) = \min\left\{\bar{b}(s), w\right\}$ where $\bar{b}(s)$ is a solution of (3.2).[8] The resulting differential equation is

$$\bar{b}'(s) = \frac{(2s - \bar{b}(s))\left(\bar{b}(s)^2 - 4\right)}{(1 + 2s - 3s^2)\,\bar{b}(s) + 3(s-1)\bar{b}(s)^2 - 4s}.$$

$\bar{b}(0) = 0$ is the only boundary condition that admits a strictly increasing solution for $s \in (0,1]$. Figure A.5 illustrates the situation. The ψ and ν

[8]The concavity of $G(\cdot)$ was not used to verify the equilibrium; therefore, the same argument applies as all equilibrium bids are in the range of $\bar{b}(s)$.

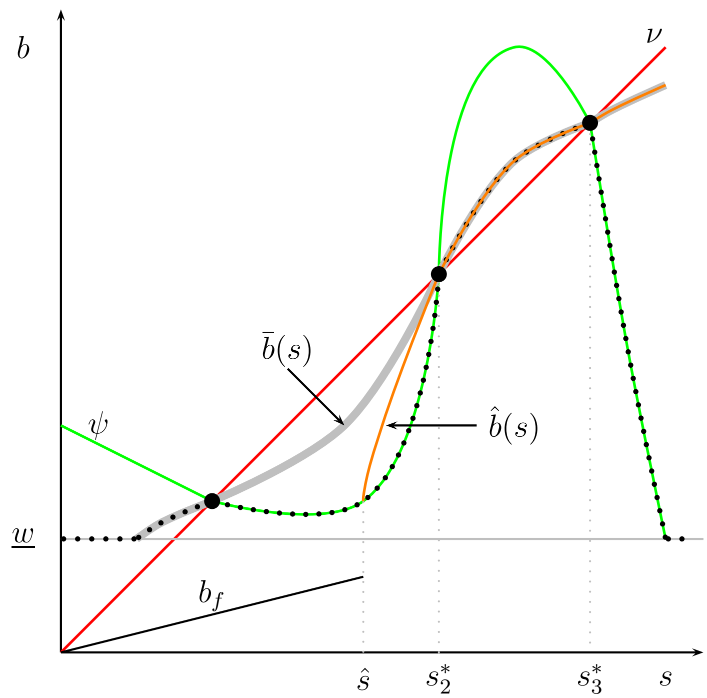


Figure A.4: Construction of $\hat{b}(s)$ when $\underline{w}$ is intermediate and there are multiple critical points. μs is denoted as the dotted curve. For $s > s_2^*$, $\hat{b}(s) = \bar{b}(s)$. $b_1(s)$ and $\phi(s)$ are not shown.

loci intersect only at $(s^*, b^*) = (0, 0)$. The inequality $\bar{b}(s) > b_f(s)$ for $s > 0$ sufficiently small is due to the interaction of interdependent values and budget constraints. As bids increase there is an increase in the rate at which bidders hit their budget limits. This ameliorates the winner's curse and encourages relatively more aggressive bidding. The convexity of the budget distribution implies that this effect only "kicks in" at relatively high bid levels.

To appreciate why, unlike Example 3, $\bar{b}(0) \not> 0$ consider the bidding decision faced by a bidder of type $(0, w)$. If such a bidder bids $b > 0$, the item will have strictly positive expected value as she will defeat all bidders who have a budget less than b. However given $G(\cdot)$'s convexity there are relatively few such types. Hence, b must be large to win with meaningful probability. However as b increases, given the strategy followed by the other

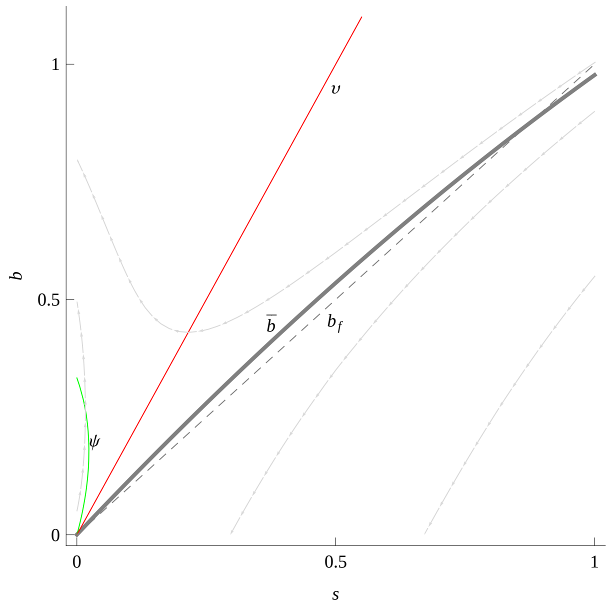


Figure A.5: Equilibrium characterization in Example 7. The $\psi(s)$ correspondence is multivalued and the critical point is at $(0, 0)$.

bidders, more types with *low* value-signals are defeated thus dampening the payoff conditional on winning. The increase in payoff does not compensate for the necessary increase in b.

Appendix B

Experimental Appendix

B.1 Experiment Instructions

Below are abridged instructions for the experiment. Full instructions included additional information about logging into the experimental platform.

In this experiment you will participate in several auctions. Please pay careful attention to the instructions as a considerable amount of money is at stake.

The entire experiment should be complete within 1.5 hours. At the end of the experiment you will be paid privately. You will receive $5 as a participation fee (for showing up on time).

You are seated at a computer terminal. Note the computer number carefully. This is your participant ID. It will be used to match you to your earnings at the end of the experiment.

In this session you will participate in several rounds of bidding. In each round, you will place bids in several separate auctions. All rounds and auctions are independent. Your performance in each auction has no bearing on any other rounds or auctions.

In each round you will be randomly matched with other bidders. This matching changes after each round. You will not know the identity of the other bidders and it will not be revealed to you afterwards. The number of other bidders will be indicated on your screen.

What is for sale?

In each auction a financial commodity is available for purchase. You will receive a signal regarding the value of this commodity. All signals are determined randomly and are private information. You will be informed about the distributions of your own signals and of all other bidders on your bidding screen; however, you will not be informed of others realized signals.

Throughout the experiment, the value of the commodity is expressed in Experimental Currency Units (ECU). In this session, the signals you receive determine the value of each commodity to you by the following formula:

Value of Commodity X to you in ECU = Your Signal about Commodity X

Note: Other bidders may value the same commodity more or less than you.

Your Budget

For each auction your are given a budget of ECUs that you can use to bid. All budgets are determined randomly and are private information. You will be informed about the distributions of your own budget and of all other bidders on your bidding screen. Note that other bidders may have different budgets than you.

Your budget is only useful for bidding in the auction for which it is issued. It cannot be saved for use in future auctions nor transferred to other auctions in the same round.

The Auction Rules

In each auction you can place one bid. The highest bidder will win the auction.

The price paid by the winning bidder for the commodity will be equal to his/her own bid.

The losing bidders will not pay anything.

If the winning bidder bids in excess of their budget, they forfeit all earnings in the auction and their budget. The commodity remains unallocated.[1]

[1]In this experiment the software prevented subjects from placing bids in excess of their budget. This fact was verbally communicated by the moderator at the start of the experiment. This point (and point 3 in the following subsection) were left in the instructions to allow the same set of instructions to be employed across a variety of treatments and auction formats.

If there is a tie for the winning bid, the computer will randomly choose a winner from the high bidders.

Your Earnings

Suppose that your valuation is V, your budget is M and the price paid by the winner is P.

1. If you win the auction your earnings will be: E = V + M - P

   You earn the value of the commodity. You make a payment from your budget equal to the price.

2. If you do not win the auction your earnings will be: E = M

   You make no payments and you keep your budget.

3. If the price is in excess of your budget and you win the auction your earnings will be: E = 0 (zero)

Your Payment

At the end of the experiment one round and one auction will be selected at random. Each round and auction will have the same probability of being selected.

If your earnings in the selected auction are E ECU your payment (including the $5 show-up fee) for the experiment will be: Payment = $ 2 × E + 5

FAQ

Q: Will I see the outcome of each auction where I place a bid?

A: No. You will only be informed of the outcome of the auction which is selected at random at the end of the experiment to determine your payment.

Q: The interface / program is not working correctly. What do I do?

A: Please raise your hand and a moderator will come to assist you.

Q: Can I use my mobile phone or use the Internet while waiting?

A: Out of courtesy for others, please refrain from activities not associated with the experiment.

The Interface

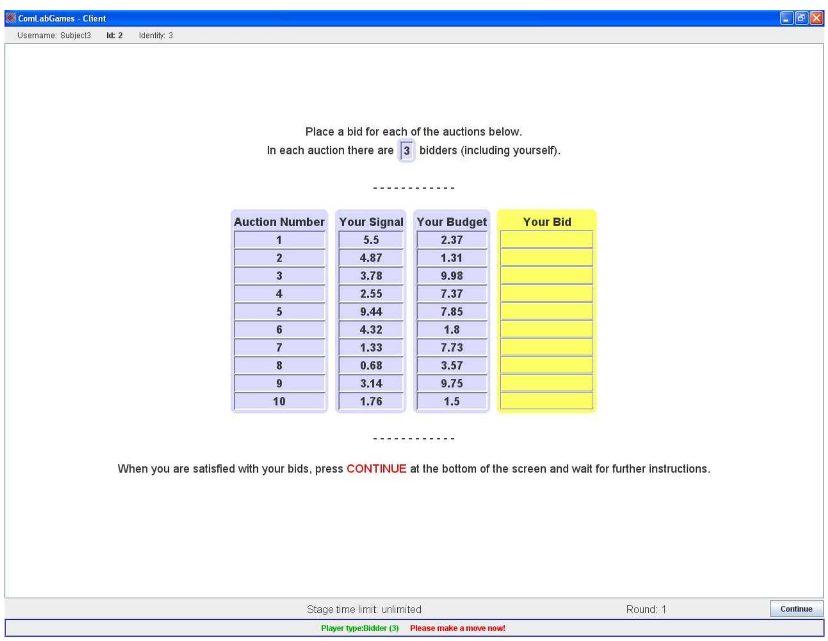
At the top of the bidding screen there is some text regarding the current auctions. Please read this information carefully. It may change from round-to-round but applies equally to all auctions within a round.

In the middle of the screen there are 4 columns. Auction Number identifies the given auction. Your Signal is the realization of your signal for the commodity available for purchase in the corresponding auction. Your Budget defines your budget limit for each corresponding auction.

You can use the keyboard to enter your bid in the Your Bid column for each corresponding auction.

At the bottom-right of the screen you will find the Continue button. Press this button when you are finished placing bids. You may have to wait for others to complete bidding after pressing Continue.

Figure B.1: A Typical Bidding Screen



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