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**The Marketing/Operations Management Interface:  
Toward a Science of Delivering Value**

by

Shan Li

A dissertation submitted in partial satisfaction of the  
requirements for the degree of  
Doctor of Philosophy

in

Engineering - Industrial Engineering and Operations Research

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

Committee in charge:  
Professor Zuo-Jun Max Shen, Chair  
Professor Teck-Hua Ho  
Professor George Shanthikumar  
Professor Ying-Ju Chen

Spring 2010

**The Marketing/Operations Management Interface:  
Toward a Science of Delivering Value**

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by

Shan Li

## Abstract

The Marketing/Operations Management Interface:  
Toward a Science of Delivering Value

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Shan Li

Doctor of Philosophy in Engineering - Industrial Engineering and Operations Research

University of California, Berkeley

Professor Zuo-Jun Max Shen, Chair

My thesis research explicitly emphasizes integrating marketing and operations management (hereafter OM) perspectives in the formulation of strategy. Shortened product life cycles, technological advancements in products and processes, globalization of markets, consumerism, and the rapidity of change have only exacerbated the perceived need to link Marketing/OM strategies. My primary research interest follows the trend and explores the science of delivering value to customers from an integrated view of marketing and OM with special emphasis on the timing of introducing product lines (Chapter 1) and the role of social contagion in forming customer lifetime value (Chapters 2 and 3).

In Chapter 1, we study the problem of when to introduce a line extension of a product with an existing version in an integrated inventory (supply) and diffusion (demand) framework. The launch of a new product with successive (and differentiated) versions always commands a large commitment of resources in production and marketing, thus the introduction strategy often requires careful planning. A key element in the introduction strategy is the introduction time. There is yet a formal model to quantify the impact of inventory cost on product line introduction timing decisions considering the demand dynamics in product life cycle and substitution among versions. This paper takes a first step towards filling this gap. On the demand side, we consider the demand dynamics of both versions during product life cycle, in marketplaces where repeated industry practices are observable to customers. Based on the Bass model, we propose a splitting Bass-like diffusion model to describe the adoption processes for the two successive (and differentiated) versions of one product, taking into account the role of customer expectation in shaping purchase choices. On the supply side, we model the impact of inventory holding cost that arises from a simple ordering policy. We show there exists a unique optimal time to introduce the line extension in the planning horizon. We quantify the optimal launch-time and both versions' sales trajectories. In contrary to the existing optimal policy in the literature (i.e., "Now or Never"), we find that the

optimal introduction can happen anytime from “Now” to “Never”, depending upon the characteristics of different products. We show that when inventory holding cost is small and the ordering cycle is short, the optimal introduction time is indeed “Now ” or “Never”. However, as inventory holding becomes substantial, the firm might choose to delay the introduction when the line extension is more profitable than the existing version, or to accelerate the introduction when the existing version generates more profit. Our integrated model sheds light on the necessity of coordinating marketing and operations management decisions.

In Chapters 2 and 3, we incorporate social contagion into customer lifetime value analysis. Prior research has assumed that a customer’s lifetime value (LV) only depends on her own purchase history. The rise of Internet and viral marketing casts doubt on this assumption. In the Web 2.0 economy, social contagion is so integral to customer’s shopping process that purchase behaviors are frequently interdependent. We investigate how social contagion might influence a customer’s lifetime value beyond her own purchases. We posit that a customer’s total lifetime value (LV) is a sum of her total purchase value (PV) (accounting for others’ influence on her purchases) and her total influence value (IV). Specifically we have:  $LV = PV + IV$ . Consequently a customer can still have a high lifetime value even if she has a low PV as long as she has a high IV.

Chapter 2 presents a model with homogeneous population. Building on the classical Bass diffusion model, we show that PV, IV, and LV decrease in the convex manner with adoption time. Hence a customer who adopts earlier is much more valuable than a customer who adopts later. While PV increases with the innovation parameter, IV decreases with it. Early adopters have their LV decrease with innovation parameter while later adopters have their LV increase with it. Interestingly, PV decreases with the imitation parameter and IV increases with it for early adopters and decreases with it for late adopters. LV increases with the imitation parameter if the timing of adoption is below a cutoff value and decreases with it if it is above the cutoff. We then examine how a firm might improve its overall customer LV by accelerating purchase made possible by offering introductory price discounts to a subset of customers. We characterize the optimal size of the targeted customers in terms of level of discount, innovation as well as imitation parameters and demonstrate that the firm can significantly increase its total customer LV by purchase acceleration. We also analyze the impact of purchase deceleration in a make-to-stock supply chain environment and a make-to-order supply chain environment respectively. We show that an out-of-stock phenomenon that occurs earlier in a product’s life cycle always leads to a significantly greater loss in total customer LV. We also demonstrate even a small lead time leads to a big loss in total customer LV.

Chapter 3 presents a model with heterogeneous population. We propose a four-segment model which considers the ex ante heterogeneity among customers in the tendency to be in tune with new developments and the tendency to influence (or be influenced by) others. Specifically, we segment customers into four types: type 1 customers are both innovators

and global influencers, type 2 customers are both innovators and local influencers, type 3 customers are both imitators and global influencers, and type 4 customers are both imitators and local influencers. We characterize the closed-form expressions for adoption rate of each customer type. Based on them, we derive closed-form expressions for the customer PV, IV and LV as a function of product adoption time. We also investigate how PV, IV and LV vary with the adoption time and the innovation parameters. Finally, we analyze the impact of purchase acceleration on customer LV, and propose an algorithm based on the LV of marginal customers to optimally allocate free samples among customers.

To my parents,  
for letting me pursue my dream,  
for so long  
so far away from home  
& To my husband,  
for giving me  
new dreams to pursue.

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# Chapter 1

## Timing Product Line Introductions Considering Inventory Cost

### 1.1 Introduction

Many firms introduce new products that are variants of the existing products in a given category to target different customer segments and satisfy customers' different desires (Krishnan and Ulrich 2001, Ramdas 2003). Because the launch of a new product always commands a large commitment of resources in production and marketing, the introduction strategy requires careful planning (Dobson and Kalish, 1988). A key element in the introduction strategy is the introduction time. Depending upon the product category, firms choose to time the introductions of product line extensions differently. We describe three examples below:

- In the automobile industry, “Volvo of North America released its 6-cylinder 760 model in Oct 1983 and the 4-cylinder 740 model 17 months later even though both cars share the same chassis and the 4-cylinder engine was available earlier.” (Moorthy and Png, 1992)
- In the fashion industry, fashion houses such as Armani first introduce new top-of-the-line designs at very high price points and only several months later do they introduce their lower-priced lines (Pesendorfer, 1995).
- In the publishing industry, hardcover books are introduced to the market first, while paperback versions are released about one year later (McDowell 1989, Shapiro and Varian 1999).

In all of these examples, firms chose different times for launching line extensions even though no technology constraints prevented them from making simultaneous releases. On one hand, as the versions are substitutes, delaying the introduction of one version leads to

less cannibalization of the existing version. On the other hand, a large body of empirical marketing research (e.g., Bass, 1969) suggests that demand diffusion begins slowly, speeds up and slows down after maturity, so if the firm waits too long, sales may have slowed considerably as the product has already diffused through the market (Druehl *et al.*, 2009). According to Wilson and Norton (1989), “the timing of the introduction affects sales, the timing of the sales and profits from both versions,” so the decision of when to introduce a new variant of an existing product is a critical tactical decision. In this paper, we use the terms “new variant”, “new version” and “line extension” interchangeably.

Several papers in the marketing literature have addressed the strategy for timing the release of two successive (and somewhat differentiated) versions of the same product when both versions could be offered. Yet, most of these papers compare the simultaneous introduction strategy with the sequential one for a two-period market window, ignoring the inherent demand dynamics over the product life cycle (e.g. Moorthy and Png 1992, Bhattacharya *et al.* 2003). To the best of our knowledge, so far Wilson and Norton (1989) is the only paper that addresses demand dynamics over the product life cycle in this context, and the optimal time to introduce the second version is shown to be “now or never” (i.e., the new version is introduced either immediately or never). However, this result is not consistent with the industry practices that were cited above. One possible reason for this discrepancy might be that determinants other than diffusion and substitution, should be considered in the decision model.

Inventory cost is one of the missing factors in this stream of literature. In practice, firms tend to manufacture or order products in large batches to achieve efficiency and minimize cost. In the publishing industry example, new books are often produced in large quantities, partly due to economies of scale in printing. In industries with relatively short product life cycles, such as apparel and consumer electronics, where rapidly-changing consumer preferences and frequent innovations have reduced product life cycles from years to months, a capacity-constrained business that offers many product variants will produce each variant only once in the planning horizon to avoid large setup costs associated with changeovers (Kurawarwala and Matsuo 1996, Bitran *et al.* 1986). However, there will be non-negligible inventory holding cost associated with this practice. Firms have to weigh the instantaneous profit from the new variant against the inventory holding cost resulting from a slowed demand rate of the older version (Bayus and Putsis Jr, 1999). Thus, the decision of when to release the new version is further complicated by consideration of inventory holding costs, and this will be the focus of this paper. It is challenging, however, to introduce the inventory aspect into an integrated model, given that most models accounting for only diffusion and substitution are very difficult to analyze (Wilson and Norton, 1989).

Another interesting feature of those industry practices that were cited above is customer expectation arising from repeated observations of firms introducing different versions of the same product (Prasad *et al.* 2004). Customers were aware that it was only a matter of time until Volvo released the 4-cylinder model, that they have to wait for paperback versions of hardcover books, and that they can purchase fashion goods at lower prices if they wait long enough (Pesendorfer, 1995). In these cases, customers form expectation about future

versions, and thus some of them may be inclined to wait for future releases, balking at the idea of an immediate purchase of the existing one. However, it is all a matter of time and patience that drives their actual purchases. Would knowing this fact affect a firm's decision on the timing of entries? To answer this, we propose an extension to the Bass model (1969), the splitting Bass-like diffusion model, which considers the feature of word-of-mouth effects due to customer expectation, to study the role of customer expectation in shaping purchase choices, and thereby, the optimal introduction timing for the new version.

To account for all the factors discussed above, we propose an integrated model that considers the S-curve market penetration of new products, substitution between versions, inventory cost and customer expectation in order to answer the question of when a new version should be launched to maximize total profits. To the best of our knowledge, this paper is the first attempt to develop an analytical model that determines the optimal time to introduce a new product and its line extension accounting for both supply and demand sides. Our paper belongs to the research stream that tries to coordinate the decisions of operations management and marketing science (Eliashberg and Steinberg 1987, Ho *et al.* 2002, Malhotra and Sharma 2002, Hausman *et al.* 2002, Chopra *et al.* 2004, Jerath *et al.* 2007). The motivation for this paper is to take a first step to explore the impact of inventory cost on the choice of the release time of a new variant of an existing product from a joint analysis of marketing science and operations management.

Our contributions to the marketing and operations management literature are three-fold. First, we propose the splitting Bass-like diffusion model to describe the adoption process of two successive versions of the same product, which also captures the role of customer expectation in shaping purchase choices. Second, we bring an operations management perspective to the introduction timing decision through a focus on inventory holding cost that arises from a simple ordering policy, and show that the optimal solution in the literature (e.g., Wilson and Norton, 1989) does not give the best outcome if inventory cost is accommodated. Our results illuminate how inventory cost influences the introduction timing decision, and how a better outcome can be achieved by taking inventory cost into consideration. Third, by developing an integrated model accounting for both demand and supply sides, we show that the decisions of marketing and operations management should be coordinated not only at the operational level, such as the match between demand and supply (Ho and Tang 2004), but at the tactical level as well, for example the introduction timing decision.

The rest of the chapter is organized as follows. Section 1.2 reviews the relevant literature. Section 1.3 presents the splitting Bass-like diffusion model. Based on that, section 1.4 presents three models in succession to discuss the effects of substitution and inventory holding cost from a simple ordering policy. In section 1.5, we present two extensions to the integrated model and conclude with a summary of key insights and suggestions for future research. To improve readability, all proofs and mathematical details are relegated to Appendix A.



## 1.2 Literature Review

In this section, we first review the literature that centers on the research of introduction timing of product line extensions, and then review some related work that lies at the interface between marketing and operations management.

There have been many studies about product line management (e.g., Quelch and Kenny 1994, Dobson and Kalish 1988, Krishnan and Ulrich 2001), but not enough attention has been given to considering time dynamics in this process (Ramdas, 2003). We broadly classify the existing literature on introduction timing into two categories: (1) continuous time models in the diffusion of innovation context, and (2) two-period models for comparing simultaneous and sequential strategies.

Research in the continuous-time category often relates to the seminal Bass diffusion model (Bass, 1969), which initiates the stream of examining demand diffusion for a single new product. Many studies have extended the Bass model into multi-product diffusion literature (e.g., Peterson and Mahajan 1978, Bayus *et al.* 2000). A subset of this group of work concentrates on modeling the diffusion paths of successive product generations (Norton and Bass 1987), where most entry timing research arises. Mahajane and Muller (1996) conclude the optimality of “now or at maturity” rule governing introduction of successive product generations, where the new generation product is introduced immediately or when the present generation product has reached sufficient sales. As technology improvement is a key ingredient of this branch of research, many researchers have addressed dynamic technology improvement in these kinds of problems. Krankel *et al.* (2006) incorporate technology improvement into the multi-generation diffusion demand context and provide a state-dependent threshold policy governing introduction timing decisions. Krishnan and Ramachandran (2008) study the trade-offs in timing product launches when the core technology available is improving rapidly. Druehl *et al.* (2009) analyze the impact of product development cost, the rate of margin decline and the cannibalization across generations on a firm’s time-pacing decision. However, the progression of product technology is not the demand driver in our model setting. As such, our model is more related to the next subset of papers centering on the case of releasing two successive (and somewhat differentiated) versions of the same product in the absence of development constraints.

A model of particular relevance to our work is that of Wilson and Norton (1989). Proceeding under a demand diffusion framework, they conclude a “now or never” rule that governs the introduction of the second version, provided both versions are available at all time from time 0. They show that the timing of introduction depends on the profit margins and the degree of substitutability between the two versions. However, though they mention, “almost all paperback books are released about a year later than the hardcover,” the gap between their theoretical findings and industry practices is left unexplained. Subsequently, Prasad *et al.* (2004) study the role of customer expectation on the timing of sequential entries of line extensions, and show a better outcome can be achieved if customer expectation is taken into account. However, their model is not directly based upon the diffusion literature, and the modeling of underlying word-of-mouth communication is not addressed. Besides

considering inventory cost, our model differs from the above two papers in that we encompass customer expectation in the diffusion setting by directly modeling its impact on word-of-mouth communication.

Research of the two-period model category is mainly to address the comparison of sequential and simultaneous introduction strategies. Moorthy and Png (1992) analyze the introduction strategy of a high-end product and its low-end variant. Their results suggest that if the firm can commit in advance to the subsequent prices and product designs, the introduction of low-end product should be delayed to alleviate cannibalization. In contrast, Bhattacharya *et al.* (2003) show that the strategy of introducing a low-end product before its high-end variant might be optimal if technological improvement is taken into account. None of the papers we have reviewed consider the impact of inventory on the introduction timing decisions.

Another relevant stream of literature studies the interface between marketing and operations management. In the literature of operations management, the classic approach often ignores the nonstationarity in demand inherent in new product diffusion (Shen and Su, 2007). On the other hand, marketing researchers typically focus on developing accurate characterizations of the demand process, and they seldom take supply side factors into consideration. Only recently have we seen some attempts to bridge the two areas. For example, Kurawarwala and Matsto (1998) present a model of procurement in which the demand process follows a Bass-type diffusion. Their model corresponds to an extension of a conventional newsvendor model and provides an example of how procurement policy can be influenced by new product diffusion dynamics. Ho *et al.* (2002) provide a joint analysis of demand and sales dynamics in a constrained new product diffusion context. Their analysis generalizes the Bass model to include backordering and customer losses, and determines the diffusion dynamics when the firm actively makes supply-related decisions to influence the diffusion process. Savin and Terwiesch (2005) present a model describing the demand dynamics of two new products competing for a limited target market, in which the demand trajectories of the two products are driven by a market saturation effect and an imitation effect reflecting the product experience of previous adopters. Schmidt and Druehl (2005) explore the influence of progressive improvements in product attributes and continual cost reduction on the new product diffusion process. Hopp and Xu (2005) analyze the cost and revenue trade-off of choosing optimal product line length and pricing decisions.

As inventory cost has been largely ignored in the introduction timing research, we are interested in finding out more managerial insights on the stream of introduction timing research from an integrated model that considers issues from both operations management and marketing sides.

## 1.3 The Splitting Bass-like Diffusion Model

### 1.3.1 Word-of-Mouth Communication under Customer Expectation

We consider a monopoly that plans on introducing two versions of a durable product. The two versions are differentiated along one dimension, for example, engine power in the case of automobiles, cover type in the case of books, brand franchise in the case of fashions. We assume the two versions share the same product life cycle, in other words, version 1 will not be phased out before version 2 is released. Customers belong to one of two segments: High or Low. High-type customers are more interested in version 1, whereas low-type customers prefer version 2. A customer obtains at most one product, never both. The firm introduces version 1 ahead of version 2 as otherwise cannibalization would be aggravated. Without loss of generality, version 1 is assumed to be introduced at time 0, so the key decision is on timing the introduction of version 2.

Our demand model is motivated by the classical diffusion model proposed by Bass (1969), a well-known parametric approach to estimating the demand trajectory of a single new product over time. We first give an overview of the Bass model. If  $f(t)$  is defined as the probability of adoption at time  $t$ , the fundamental premise is that the likelihood of adoption at time  $t$  given that one has not yet occurred is:

$$f(t) = (p + qF(t))(1 - F(t)) \quad (1.1)$$

The parameter  $p$  is called the coefficient of innovation and  $q$  the coefficient of imitation. In the Bass model, customers can be classified into “innovators” and “imitators”. Innovators adopt an innovation independently of the decisions of other individuals in a social system. Imitators, unlike innovators, are influenced in their adoption timing by previous buyers through word-of-mouth communication. A similar product line extension model studied by Wilson and Norton (1989) relies on the same assumption about word-of-mouth communication.

However, in a marketplace where firms repeatedly introduce different versions of the same product, customers form expectation of the future version prior to it being released, which subsequently shapes the underlying word-of-mouth communication differently from situations in which no expectations are formed about the release of the new variants. To see this, let us look at a situation often seen in daily life: After reading a hardcover copy of a new book, Annie recommends it to two friends, Betty and Cindy. Betty then buys one, but Cindy, a college student, wants to wait for the paperback version. During a dinner with David and Eva, Cindy mentions the endorsement she heard about that book. As a result, David buys the current hardcover version, while Eva decides to wait for the paperback one. Figure 1.1 illustrates the above typical interpersonal communication chain.

Shapiro and Varian (1999 p.55) state, “publishers design different versions to emphasize customer differences. Here, high-type customers are impatient to get the book, while lower-value customers can more easily wait. The main difference here involves patience. Thus,

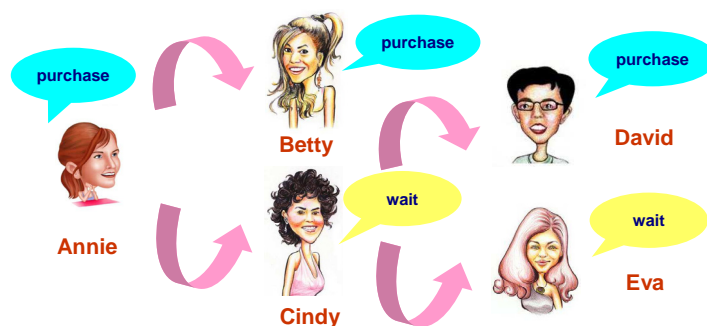


Figure 1.1: A Typical Interpersonal Communication Chain

the key to versioning books is to delay offering less expensive versions. This is precisely what publishers do.” Thus, a customer’s choice between versions is closely tied to her own valuation for the product and her inherent patience, regardless of her inclination for spreading word-of-mouth, and thereby it is likely that a low-type customer have influenced several purchases of version 1 from high-type customers whereas herself is waiting for version 2 patiently. Hence we posit the following assumption about word of mouth: *In the adoption of two successive (and differentiated) versions of the same product, due to customer expectation of the new version before it being released, word of mouth not only comes from the actual buyers of the existing version but also from those who intend to purchase the new one.*

We use the term “customer” to include anyone that contemplates a purchase. An individual who is waiting for version 2 is a customer yet a non-buyer. Based on the above assumption about word of mouth, an individual can be influenced by all previous customers, rather than just by previous buyers. In the anecdote illustrated before, Cindy contributes to the diffusion process by exerting a positive influence on David and Eva despite that herself is a non-buyer. One may argue that the population of actual buyers is a more credible source than people who are waiting to purchase but have not yet done so. In fact, imitation is jointly fueled by observations and communications (Zhang 2009). An imitator may become interested in the product either from observing predecessors’ buying actions, or through communications with anyone that endorses this product, including those whose own purchases have not yet realized.

### 1.3.2 The Demand Model

We continue with our demand model as follows: After version 1 is introduced, at any given time  $t$ , a customer who was previously not ready to buy might decide to purchase. If she decides to do so, with probability  $s$ , she is a high-type customer and will buy version 1 immediately. Note that  $s$  can also be interpreted as the market segmentation parameter and its value captures the sizes of the two customer segments (Moothy and Png, 1992). With probability  $1 - s$  she is of low type and prefers version 2: she will buy if version 2

has been released, and will wait otherwise. As customers are not fully aware of the exact introduction timing before it actually happens, we assume if one decides to wait for version 2, she would wait for at most  $l$  units of time (We will address customer heterogeneity in waiting time in Section 1.5.1). If version 2 becomes available before she loses patience, her purchase will be made at the time of it being introduced; Otherwise, she will reconsider the decision after waiting for  $l$  units of time, as a consequence, she loses patience switching to version 1 with probability  $\theta$ , or decides not to buy anything otherwise. We assume the latter choice does not mitigate her propensity to spread word of mouth, which is reasonable as we do not account for outside competition in this model. In fact, her decision of giving up the purchase is not necessarily a result of lack of interest, but may simply reflect a change of her budget situation, or other external factors. Figure 1.2 describes customer choices prior to version 2 being released.

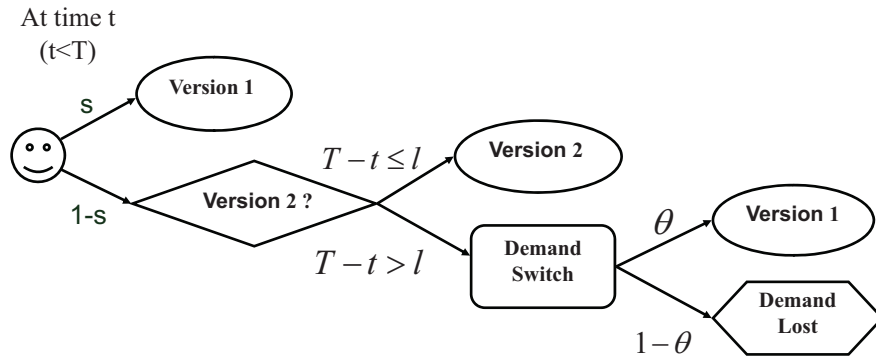


Figure 1.2: Customer Choices Prior to Version 2 Being Released

From Section 1.3.1 we know that an individual can be influenced by all previous customers, before or after version 2 is released. In other words, a unified information flow of both versions spreads over the entire planning horizon, independent of the introduction time. This allows us to directly analyze the unified social influence of the product (both versions), which is assumed to follow the classical Bass diffusion pattern.

According to the empirical justifications in Norton and Bass (1987), the innovation and imitation coefficients stay *the same* regardless of whether the new variant has been introduced or not. Consequently, similar to prior studies in this field (Wilson and Norton 1989, Joshi *et al.* 2008), in our model we do not distinguish between the diffusion parameters before and after the introduction, and use  $p$  and  $q$  as the respective innovation and imitation parameters of both versions. As customers' preferences are segmented according to  $s$ , we can think of the evolution of demand for the two versions as being “split.” We use the term “splitting Bass-like diffusion model” to refer to this diffusion process.

In the following,  $T$  denotes the introduction time of version 2,  $m$  denotes the total potential customers, and  $D_i(t; T)$  ( $i=1,2$ ) denotes the cumulative demand for version  $i$  until time  $t$  if version 2 is introduced at  $T$ . As version 2 is not available before  $T$ , when  $t < T$ ,

$D_2(t; T)$  is the potential yet not realized demand for version 2. We let  $W(t; T)$  be the population that spreads word-of-mouth at time  $t$  if version 2 is introduced at  $T$ . Note that  $W(t; T) \geq D_1(t; T) + D_2(t; T)$ , as we will illustrate later. To distinguish the specifications of actual sales from those of potential demand, we use  $S_i(t; T)$  ( $i=1,2$ ) to denote the cumulative sales of version  $i$  by time  $t$  if version 2 is introduced at  $T$ .

Next we discuss the demand specifications of the two versions. We use  $\tau$  to denote the length of the planning horizon. As version 2 may be introduced at any time  $T \geq 0$ , we distinguish between the following two cases.

1) If the introduction time  $T \leq l$ :

The demand/sales trajectory of version  $i$  ( $i=1,2$ ) can be written as

$$\frac{dW(t; T)}{dt} = [1 - W(t; T)][p + qW(t; T)] \quad (1.2)$$

$$D_1(t; T) = S_1(t; T) = sW(t; T) \quad (1.3)$$

$$D_2(t; T) = (1 - s)W(t; T) \quad (1.4)$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1 - s)W(t; T) & \text{if } T \leq t \leq \tau. \end{cases} \quad (1.5)$$

In (1.2) we analyze the entire population interested in the product (both versions), which follows the Bass dynamics. As version 2 is introduced sooner than any of its customers loses patience, the demand of the two versions splits, with a fraction  $s$  of the instantaneous demand going to high-type customers (version 1), remaining demand going to low-type customers (version 2). At any time  $t$ , if a version is available, all of its demand at that time turns into sales. Otherwise, its sales volume remains 0 whereas its potential demand is being accumulated.

In this case,  $W(t; T) = D_1(t; T) + D_2(t; T)$  as no demand is lost. The assumption that the version-level demand diffusion model sums up to the Bass product-level model may look restrictive, but it gives our model a well-founded behavioral basis and indirect validity because numerous empirical evidence has been found to support the Bass (1969) model.

2) If the introduction time  $T \geq l$ :

The demand/sales trajectory of version  $i$  ( $i=1,2$ ) can be written as

$$\frac{dW(t; T)}{dt} = [1 - W(t; T)][p + qW(t; T)] \quad (1.6)$$

$$D_1(t; T) = S_1(t; T) = \begin{cases} sW(t; T) & \text{if } 0 \leq t \leq l; \\ sW(t; T) + \theta(1 - s)W(t - l; T) & \text{if } l < t < T; \\ sW(t; T) + \theta(1 - s)W(T - l; T) & \text{if } T \leq t \leq \tau. \end{cases} \quad (1.7)$$

$$D_2(t; T) = \begin{cases} (1 - s)W(t; T) & \text{if } 0 \leq t \leq l; \\ (1 - s)(W(t; T) - W(t - l; T)) & \text{if } l < t < T; \\ (1 - s)(W(t; T) - W(T - l; T)) & \text{if } T \leq t \leq \tau. \end{cases} \quad (1.8)$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1 - s)(W(t; T) - W(T - l; T)) & \text{if } T \leq t \leq \tau. \end{cases} \quad (1.9)$$

Similar to the previous case, the population interested in the product (both versions) follows the Bass diffusion dynamics in (1.6). Obviously, demand switches between the two versions do not take place before  $l$ , so  $D_1(t; T)$  ( $t \leq l$ ) stays the same as in (1.3). By  $l < t < T$ , those low-type customers who waited for no less than  $l$  units of time stopped waiting, as a consequence, they either have turned to version 1 instead, or wouldn't buy anything. Simply put, demand for version 1 comes from two parts: high-type customers who would buy version 1 at the first place ( $sW(t; T)$ ), as well as a  $\theta$  fraction of those low-type customers who waited for version 2 for no less than  $l$  units of time ( $\theta(1-s)W(t-l; T)$ ). After version 2 becomes available at  $T$ , neither demand switch nor demand lost occurs, and thus by  $T \leq t \leq \tau$ , the amount of switchers is bounded by the number of willing-to-switch low-type customers up to  $T$ , stated as  $\theta(1-s)W(T-l; T)$ .

Similarly,  $D_2(t; T)$  ( $t \leq l$ ) stays the same as in (1.4). By  $l < t < T$ ,  $D_2(t; T)$  comes from those low-type customers being informed about the product after  $t-l$ , stated as  $(1-s)(W(t; T) - W(t-l; T))$ , because all low-type customers that started to wait before  $t-l$  either switched to version 1 or would not buy anything after having spent  $l$  units of time waiting. After  $T$ , neither demand switch nor demand lost occurs any more. As product 2 is not available before  $T$ ,  $S_2(t; T) = 0$  if  $t < T$ , and  $S_2(t; T) = D_2(t; T)$  otherwise.

In contrast to the previous case, we note that  $W(t; T) > D_1(t; T) + D_2(t; T)$  when the introduction happens later than  $l$ . The difference is due to the demand lost from low-type customers who have lost patience due to a long waiting and would rather go with outside option. When  $l$  and  $\theta$  are both small, the cost of a later introduction is substantially high.

To simplify notation, we define

$$F(t) = m \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}} \quad (1.10)$$

which is the solution of (3.10), known as the cumulative demand specification in the Bass model. And the Bass instantaneous demand expression

$$f(t) = m \frac{(p+q)^2 e^{-(p+q)t}}{p(\frac{q}{p}e^{-(p+q)t} + 1)^2} \quad (1.11)$$

Based on (1.2)-(1.10), we can now establish the demand/sales trajectories of the two versions in the following proposition.

**Proposition 1.** *1). If the introduction time  $T \leq l$ , (In fact, in this case, the demand diffusion specifications are not functions of the introduction time  $T$ , but we retain  $D_i(t; T)(S_i(t; T))$ , ( $i = 1, 2$ ) for notational consistency.)*

$$D_1(t; T) = S_1(t; T) = sF(t) \quad (1.12)$$

$$D_2(t; T) = (1-s)F(t) \quad (1.13)$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1-s)F(t) & \text{if } T \leq t \leq \tau. \end{cases} \quad (1.14)$$

2). If the introduction time  $T > l$ ,

$$D_1(t; T) = S_1(t; T) = \begin{cases} sF(t) & \text{if } 0 \leq t \leq l; \\ sF(t) + \theta(1-s)F(t-l) & \text{if } l < t < T; \\ sF(t) + \theta(1-s)F(T-l) & \text{if } T \leq t \leq \tau. \end{cases} \quad (1.15)$$

$$D_2(t; T) = \begin{cases} (1-s)F(t) & \text{if } 0 \leq t \leq l; \\ (1-s)(F(t) - F(t-l)) & \text{if } l < t \leq T; \\ (1-s)(F(t) - F(T-l)) & \text{if } T < t \leq \tau. \end{cases} \quad (1.16)$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1-s)(F(t) - F(T-l)) & \text{if } T \leq t \leq \tau. \end{cases} \quad (1.17)$$

### 1.3.3 Discussions on the Demand Model

To relate our demand model to the literature, we compare it with the demand model discussed in Wilson and Norton (1989), abbreviated as W/N. To the best of our knowledge, so far W/N is the only paper that addresses demand dynamics over the product life cycle in the context of introducing a new product with two versions.

In W/N, diffusion dynamics is modeled over the information function  $I(t; T)$ , which denotes the fraction of the population informed about the existence of the available versions at time  $t$ , i.e. about the first version alone if  $0 \leq t \leq T$ , and about two versions if  $t > T$ . Before version 2 is introduced ( $t \leq T$ ), W/N models the differential equation for  $I(t; T)$  as

$$\frac{dI(t; T)}{dt} = [1 - I(t; T)][p + qm_1I(t; T)] \quad (1.18)$$

$$D_1(t; T) = m_1I(t; T) \quad (1.19)$$

A fraction  $m_1$  of those who become informed about the product at any time  $t$  are high-type customers, who decide to purchase the current high-end version, and the remaining  $1 - m_1$  decide not to buy anything. W/N assumes that only actual buyers can get involved in communicating product information, so customers spreading WOM at any instant of time  $t$  are exactly those high-type buyers up to  $t$ , written as  $m_1I(t; T)$  in (1.18).

In contrast, we model the diffusion dynamics over the desire to buy the product (either version). As we demonstrated in Section 1.3.1, due to repeated observations of firms introducing different versions of the same product, a low-type customer who is interested in the product at  $t$ , will choose to wait for the low-end version if it has not been introduced to market. However, her propensity of spreading WOM starts from  $t$  even though she has not made the purchase at that time. This leads to the distinction of demand specifications in our model from those in W/N. In W/N, only those high-type individuals who have made the purchase will communicate, whereas our model allows for WOM communication from all customers (both high-type and low-type customers) who desire to buy the basic product, including those low-type customers that are expecting version 2's future release. We find that (1.2), (1.6) are identical to (1.18) except that all interested customers are engaged in



spreading WOM. If  $\theta = 0$ , (1.3), (1.7) are the same as (1.19) since  $s$  and  $m_1$  share the same interpretation in the two models.

When  $t > T$ , in W/N the diffusion dynamics is given by

$$\frac{dI(t;T)}{dt} = [1 - I(t;T)][p + q(m_1I(T;T) + m_2(I(t;T) - I(T;T)) + m_3(I(t;T) - I(T;T)))] \quad (1.20)$$

$$D_1(t;T) = m_1I(T;T) + m_2[I(t;T) - I(T;T)] \quad (1.21)$$

$$D_2(t;T) = m_3[I(t;T) - I(T;T)] \quad (1.22)$$

$m_2$  is the fraction of population who becomes informed at any instant of time after version 2's introduction who decides to purchase version 1 (i.e., high-type customers), and  $m_3$  is the fraction of customers who prefer version 2 (i.e., low-type customers). After the low-end version is released, the WOM influence comes from three sources of customers: the high-type buyers before version 2's introduction ( $m_1I(T;T)$ ), the high-type customers after version 2's introduction ( $m_2(I(t;T) - I(T;T))$ ) and the low-type customers ( $m_3(I(t;T) - I(T;T))$ ) after the introduction.

If we retain the assumptions in the Bass model that all interested customers will make the purchase as long as the product is available (so that  $m_2 + m_3 = 1$ ), and customers of different types have consistent choices of preference ( $m_1 = m_2$ ) over time, which is often true if customers have formed complete information set from repeated observations of industry practices in releasing similar products, equations (1.20) - (1.22) can then be written into

$$\frac{dI(t;T)}{dt} = [1 - I(t;T)][p + q(I(t;T) - (1 - m_1)I(T;T))] \quad (1.23)$$

$$D_1(t;T) = m_1I(t;T) \quad (1.24)$$

$$D_2(t;T) = (1 - m_1)[I(t;T) - I(T;T)] \quad (1.25)$$

Now we compare (1.2) - (1.9) with (1.23) - (1.25) in the following steps. First, we note that  $m_1$  in W/N is in fact the market segmentation parameter  $s$  in our model, both capturing the fraction of the high-type customers in the potential population. Second, when  $\theta = 0$  and  $l = 0$ , (1.2) - (1.9) are the same as (1.23) - (1.25) except that we accommodate the WOM influence from all customers in (1.2) - (1.9). This is because we focus on the marketplaces where repeated industry practices are observable to customers, whereas W/N studies the case where such practices are not observable to customers so that social influence is only allowed from actual buyers ( $(1 - m_1)I(T;T)$  refers to those who were informed before  $T$  but did not purchase). Finally, we further model the impact of customer expectations on version-wise demand dynamics, incorporating customer patience and the cross-version substitution from impatience low-type customers.

Therefore, we can conclude that, our demand model studies the marketplaces where repeated industry practices are observable to customers, whereas W/N studies the case where such practices are not observable to customers. Specifically, due to the repeated industry practices, customers are aware of the low-end line extension before its introduction. As stated earlier, such information would impact customer expectation and thereby change their purchase behaviors.

## 1.4 Modeling Substitution and Inventory Holding Cost

In this section we successively present three models, all built upon the splitting Bass-like diffusion model of Section 1.3: the first model (Section 1.4.1) discusses the substitution effects when one of the two versions is in supply scarcity, the second one (Section 1.4.2) focuses on the impact of inventory holding cost without taking substitution into account, and finally in Section 1.4.4 we discuss a general model with both inventory holding cost and substitution considerations.

### 1.4.1 Model with Substitution Due to Supply Scarcity

Many studies in marketing find that customers do not always end up empty-handed when the specific version they wanted to buy is no longer available, in fact, some of them are willing to settle for a similar one currently in stock instead. In this section we study this kind of substitution due to supply scarcity and model its impact on the firm's choice of introduction timing.

We assume that if the supply of version 2 alone falls short of its demand, an  $\alpha$  fraction of low-type customers who wanted to buy version 2 would settle for version 1 instead, and neither version is purchased by the remaining  $1 - \alpha$  fraction of low-type customers. When version 1 alone stocks out, a  $\beta$  fraction of high-type customers who planned on purchasing version 1 turn to version 2, and the demand from those non-switchers would be lost. Note that  $\alpha = 0$  or  $\beta = 0$  if such substitution can only happen one direction-wise. To focus on decision makings at the tactical level, we do not address the stock-outs resulting from different ordering policies, instead we concentrate on the overall ordering quantity over the life-cycle period. Denoting  $r_i$  ( $i = 1, 2$ ) as the unit profit margin of version  $i$ ,  $c_i$  the unit ordering cost of version  $i$ , and thereby unit price  $w_i = r_i + c_i$ . We assume version 1 has a higher profit margin, i.e.,  $r_1 > r_2$ , yet our model is not restricted to it as solutions/arguments can be easily mirrored when  $r_1 \leq r_2$ . Last, as we generally consider books or products with short life cycles (i.e. apparel, toys, consumer electronics, personal computers), retail prices and ordering costs of both versions are fixed during product life cycle (Kurawarwala and Matsuo 1996, Bitran *et al.* 1986), and we will discuss an extension that allows prices to change over time during life-cycle period in Section 1.5.2. Figure 1.3 depicts customer choices after version 2 being released.

Based on the demand specifications derived in Section 1.3.2, we formulate the problem of maximizing profit over the planning horizon as follows:

$$\begin{aligned} \text{Problem } (P^S): \max_{Q_1, Q_2, T} \quad & w_1 \min \{D_1(\tau; T), Q_1\} + w_2 \min \{D_2(\tau; T), Q_2\} - c_1 Q_1 - c_2 Q_2 \\ & + w_1 \min \{(D_2(\tau; T) - Q_2)^+ \alpha, (Q_1 - D_1(\tau; T))^+\} \\ & + w_2 \min \{(D_1(\tau; T) - Q_1)^+ \beta, (Q_2 - D_2(\tau; T))^+\} \end{aligned} \quad (1.26)$$

$$\text{s.t.} \quad Q_1 \geq 0, Q_2 \geq 0 \quad (1.27)$$

$$0 \leq T \leq \tau \quad (1.28)$$

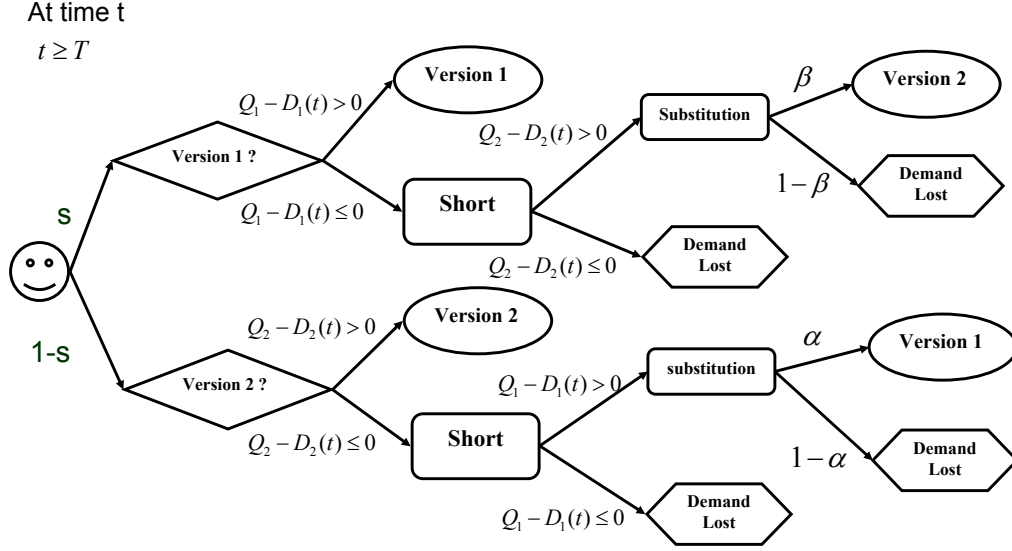


Figure 1.3: Customer Choices after Version 2 Being Released

In the objective,  $w_i \min \{ \bar{D}_i(T), Q_i \}$  ( $i = 1, 2$ ) is the total revenue from version  $i$  before substitution takes place.  $c_i Q_i$  is the total ordering cost of version  $i$ , ( $i = 1, 2$ ). The revenue from version 1 contributed by substitution is stated as  $w_1 \min \{ (D_2(\tau; T) - Q_2)^+ \alpha, (Q_1 - D_1(\tau; T))^+ \}$ , where  $(D_2(\tau; T) - Q_2)^+ = \max \{ D_2(\tau; T) - Q_2, 0 \}$  and  $(Q_1 - D_1(\tau; T))^+ = \max \{ Q_1 - D_1(\tau; T), 0 \}$ . The first term of the 'min' operator is the demand from those low-type customers who prefer version 2 but do not mind buying version 1 instead, and the second term is the excessive supply of version 1 over its own demand. Obviously, actual demand from substitution is the smaller one of those two terms. Similarly, the last term in the objective comes from substitution the other way around.

Before solving the above optimization problem, we first prove the following lemma, which will help us reformulate problem  $(P^S)$ .

**Lemma 1.** *Consider the following two problems:*

$$\begin{array}{ll}
 (PL_1) & z = \max_{x,y} f(x,y) \\
 s.t. & g_1(x) \leq 0 \\
 & g_2(x,y) \leq 0
 \end{array}
 \qquad
 \begin{array}{ll}
 (PL_2) & w = \max_x R(x) \\
 s.t. & g_1(x) \leq 0 \\
 \text{where} & R(x) = \max_y f(x,y) \\
 s.t. & g_2(x,y) \leq 0
 \end{array}$$

We claim:  $z = w$ .

Since constraint (1.28) is a function of  $T$  alone, fixing  $T$ , the feasible region of  $(P^L)$

can be partitioned into six disjoint areas according to the relations among  $D_i(\tau; T)$  and  $Q_i(i = 1, 2)$ , each of which can be characterized by functions of  $Q_1$ ,  $Q_2$  and  $T$ , and thus the objective function of  $(P^S)$  within each area has a specific form. By Lemma 1,  $(P^S)$  can be reformulated as:

$$\begin{aligned} \text{Problem } (\hat{P}^S): \quad & \max_{0 \leq T \leq \tau} R(T) \\ \text{where } R(T) = & \max_{j=1, \dots, 6} R_j(T) \\ \text{and } R_1(T) = & \max_{0 \leq Q_1 \leq D_1(\tau; T), 0 \leq Q_2 \leq D_2(\tau; T)} r_1 Q_1 + r_2 Q_2 \end{aligned}$$

Formulations of  $R_i(T)$ ,  $i = 2 \dots 6$ , are presented in Appendix A. The above bi-level optimization problem  $(\hat{P}^S)$  shares the same optimal solution as  $(P^S)$ . Therefore, we solve  $(\hat{P}^S)$  instead, and state results in the following propositions.

**Proposition 2.** *The optimal solution to problem  $R(T)$  can be characterized as follows: if  $\alpha \leq r_2/r_1$ ,  $Q_1^* = D_1(\tau; T)$ ,  $Q_2^* = D_2(\tau; T)$ ; Otherwise,  $Q_1^* = D_1(\tau; T) + D_2(\tau; T)\alpha$ ,  $Q_2^* = 0$ .*

The above results do not depend on  $\beta$ , because we have assumed a higher profit margin of version 1, which implicitly forces the firm to order version 1 more. Proposition 2 is very intuitive as basically it says that version 2 is better off being substituted if affluent consumers are willing to turn to version 1 after finding out version 2 stocks out. The substitute/no substitute decision is determined by the fraction of the willing-to-switch low-type customers and by the profit margin ratio, both independent of time.

**Proposition 3.** *If the planning horizon is comparable to the life-cycle period, when  $r_2/r_1 \geq \max\{\alpha, \theta\}$ , an immediate introduction is preferable, with  $Q_1^* = sF(\tau)$ ,  $Q_2^* = (1 - s)F(\tau)$ ; Otherwise, the second version should never be introduced, and the firm should commit the unavailability of version 2 in advance if  $\theta > \max\{r_2/r_1, \alpha\}$ .*

Proposition 3 reveals that when the planning horizon is comparable to the life-cycle period and low-type customers are relatively impatient, version 2 ought to be introduced either “now” or “never”, consistent with the findings in Wilson and Norton (1989). Besides that, if  $\alpha$  is sufficiently large, the firm is better off informing the customers in advance about the unavailability of version 2 to prevent their expectation of future releases (Moorthy and Png 1992). The demand curves are presented in Appendix A. In the following section, we will present a model that accounts for inventory holding cost, and will draw a comparison between the results of the two models to address the impact of inventory holding cost on the decision of introduction timing.

### 1.4.2 Model with Inventory Holding Cost

Now, we consider the impact of inventory holding cost on the decision of introduction timing. We ignore the substitution effects in this section by assuming  $\alpha = \beta = 0$ , and

the discussion of accommodating inventory holding and substitution in a unified model is deferred to Section 1.4.4.

To illustrate our idea, we consider a simple scheduled ordering policy: fixed interval ordering (Graves 1996, Cachon 1999). In reality it is often impossible to apply make-to-order and replenish inventory continuously, and thus the fixed interval ordering policy is motivated and widely used in practice. Delivery of orders is assumed to be instantaneous. We assume an exogenous ordering interval  $L$ , and to address the effect of inventory holding cost in general, we assume the inventory holding cost is  $h$  per unit item per unit time, same for both versions. To avoid trivial outcomes, profit margins are presumed to be higher than the cumulative holding cost in an ordering interval, that is,  $r_1 - h \min\{L, \tau\} > 0$  and  $r_2 - h \min\{L, \tau - l\} > 0$ . As in this section  $\alpha = \beta = 0$ , it is easy to verify that the total ordering quantity equals to the overall demand/sales over the life-cycle period ( $Q_i = D_i(\tau; T) = S_i(\tau; T)$   $i = 1, 2$ ).

On the demand side, with inventory holding cost included, the firm clearly would not release the second version before  $l$ . When  $T \geq l$  the demand and sales expressions can be found in (A.4)-(A.6):

$$D_1(t; T) = S_1(t; T) = \begin{cases} sF(t) & \text{if } 0 \leq t \leq l; \\ sF(t) + \theta(1-s)F(t-l) & \text{if } l < t \leq T; \\ sF(t) + \theta(1-s)F(T-l) & \text{if } T < t \leq \tau. \end{cases}$$

$$D_2(t; T) = \begin{cases} (1-s)F(t) & \text{if } 0 \leq t \leq l; \\ (1-s)(F(t) - F(t-l)) & \text{if } l < t \leq T; \\ (1-s)(F(t) - F(T-l)) & \text{if } T < t \leq \tau. \end{cases}$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1-s)(F(t) - F(T-l)) & \text{if } T < t \leq \tau. \end{cases}$$

With an exogenous ordering cycle  $L$ , the problem can then be modeled as follows:

Problem ( $P^H$ ):

$$\begin{aligned} \max_{l \leq T \leq \tau} \quad & r_1 D_1(\tau; T) + r_2 D_2(\tau; T) - h \sum_{i=1}^{\lfloor \tau/L \rfloor} \int_{(i-1)L}^{iL} (D_1(iL, T) - S_1(t, T)) dt \\ & - h \int_{\lfloor \tau/L \rfloor * L}^{\tau} (D_1(\tau; T) - S_1(t, T)) dt \\ & - h \sum_{j=1}^{\lfloor (\tau-T)/L \rfloor} \int_{T+(j-1)L}^{T+jL} (D_2(T+jL, T) - S_2(t, T)) dt \\ & - h \int_{\lfloor (\tau-T)/L \rfloor * L + T}^{\tau} (D_2(\tau; T) - S_2(t, T)) dt \end{aligned}$$

where  $D_1(iL, T)$  is the cumulative ordering quantity of version 1 in the  $i$ th ordering interval and thus  $h \int_{(i-1)L}^{iL} (D_1(iL, T) - S_1(t, T)) dt$  captures the inventory holding cost in that cycle, and if the planning horizon is not a multiple of  $L$ , the fourth term gives us the holding cost

in the last ordering interval. Similarly, holding cost of version 2 is captured by the last two terms, except that its order replenishment starts from  $T$  rather than 0.

The complex structure of the cumulative sales function  $S_i(t; T)$  ( $i = 1, 2$ ) along with the generality of ordering cycle length  $L$ , complicates the analysis substantially. In order to derive additional analytical insights, we now consider a special case of our model where each version can only be ordered at the time of it being released during the life-cycle period. In this special case, problem ( $P^H$ ) becomes

$$\begin{aligned} \text{Problem } (P_1^H): \quad \max_{l \leq T \leq \tau} \quad & r_1 D_1(\tau; T) - h \int_0^\tau (D_1(\tau; T) - S_1(t; T)) dt \\ & + r_2 D_2(\tau; T) - h \int_T^\tau (D_2(\tau; T) - S_2(t; T)) dt \end{aligned}$$

In order to characterize the solutions, we first define

$$A(T) = h \left( \frac{1 - e^{-(p+q)\tau}}{1 + q/p e^{-(p+q)\tau}} - \frac{1 - e^{-(p+q)T}}{1 + q/p e^{-(p+q)T}} \right) \quad (1.29)$$

$$B(T) = - \frac{(p+q)(1+q/p)e^{-(p+q)(T-l)}}{(1+q/p e^{-(p+q)(T-l)})^2} (\theta(r_1 - hT) - r_2) \quad (1.30)$$

$$h^*(\theta) = \frac{p(r_2 - \theta r_1)}{\frac{1 - e^{-(p+q)\tau}}{1 + q/p e^{-(p+q)\tau}} - \frac{1 - e^{-(p+q)l}}{1 + q/p e^{-(p+q)l}} - pl\theta} \quad (1.31)$$

$$\text{and } l^*(\theta) = \left\{ l : l + \frac{1 - e^{-(p+q)l}}{p\theta + q\theta e^{-(p+q)l}} = \frac{1 - e^{-(p+q)\tau}}{p\theta + q\theta e^{-(p+q)\tau}} \right\} \quad (1.32)$$

We note that  $l^*(\theta)$  is uniquely determined because  $l + \frac{1 - e^{-(p+q)l}}{p\theta + q\theta e^{-(p+q)l}}$  is strictly increasing in  $l$ . A set of general results characterizing the optimal introduction time  $T^*$  can be obtained as outlined below.

**Proposition 4.** (a) If  $\theta(r_1 - h\tau) > r_2$ ,  $T^* = \tau$ .

(b) Otherwise, let  $\bar{T}$  denote the value of  $T$  ( $T < \tau$ ) that satisfies  $A(T) = B(T)$ .

(i) If  $\bar{T} \geq l$ ,  $T^* = \bar{T}$ . (ii) Otherwise,  $T^* = l$ .

When the planning horizon is comparable to the product life cycle, and low-type customers are relatively impatient, in contrast to the prior introduction rule of “now or never” stated in Section 1.4.1, Proposition 4 shows that with inventory holding cost considerations, the optimal introduction could happen at any time “from now to never”. The introduction time depends upon the characteristics of different products (See Tables (1.4)-(1.7) in Appendix A).

Figure 1.4 plots a possible set of instantaneous demand and sales paths where the optimal introduction occurs later than  $l$  but sooner than  $\tau$ . One can observe from Figure 1.4(b) that when the second version is launched, a new left-truncated diffusion process starts.

Another interesting observation is that, a “spike” of total sales from both versions occurs at the time of version 2 being released, which has often been observed in practice. This happens because low-type customers who have been holding purchases back will buy when version 2 becomes available.

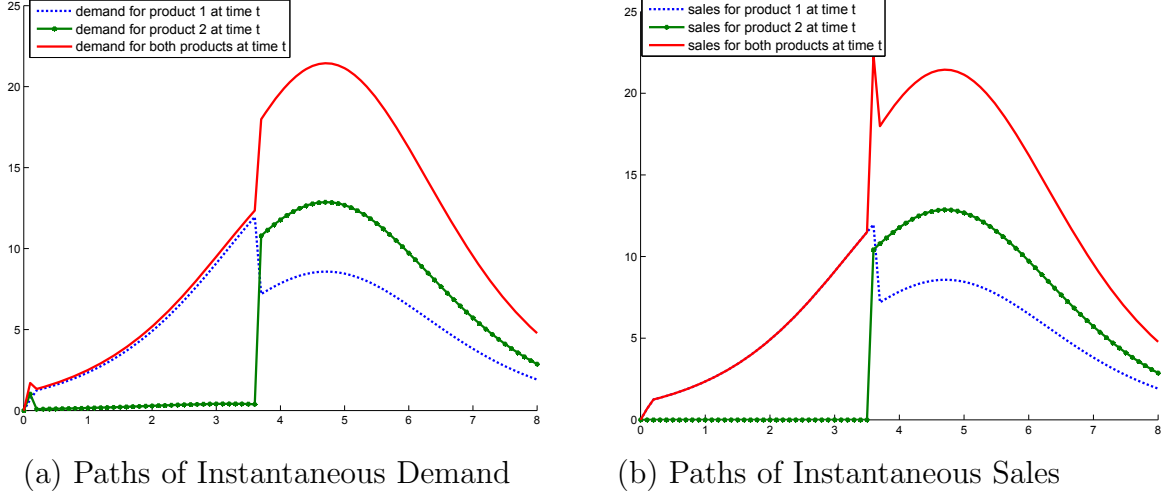


Figure 1.4: Instantaneous Demand and Sales Paths

( $m = 1000, s = 0.4, p = 0.0163221, q = 0.825044, l = 0.1, w_1 = 30, w_2 = 15, c_1 = 10, c_2 = 3, h = 1, \tau = 8$  months, and  $\theta = 0.5$ )

**Proposition 5.** (a)  $\bar{T}$  increases with the profit margin of version 1,  $r_1$ , and the proportion of low-type customers who would like to switch to version 1 after experiencing too long a wait,  $\theta$ .  $\bar{T}$  decreases with the profit margin of version 2,  $r_2$ .  
(b)  $T^* = \bar{T}$  ( $l < \bar{T} < \tau$ ) if  $\frac{1}{\tau} (r_1 - \frac{1}{\theta} r_2) < h < \frac{1}{l} (r_1 - \frac{1}{\theta} r_2)$ .

As expected, when the optimal introduction should happen *between* ‘now’ and ‘never’, it increases with version 1’s profit margin. A similar effect is observed when the proportion of willing-to-switch low-type customers, as measured by  $\theta$ , increases. Increases in profit margin of version 2,  $r_2$ , will lead to a sooner introduction of version 2.

Part (b) of Proposition 5 states sufficient conditions in terms of model primitives for the optimality of a introduction that lies between “now” and “never”. It is interesting to see that, this condition is independent of diffusion parameters  $p$  and  $q$ , though  $\bar{T}$  itself relates to the diffusion process. We note that this sufficient condition can never be satisfied if  $\theta \leq r_2/r_1$ , and we will elaborate this case in the following proposition.

**Proposition 6.** When  $\theta \leq r_2/r_1$ ,  
(a) If  $h < h^*(\theta)$ ,  $T^* = l$ . (b)  $T^* = \bar{T}$  otherwise.

Comparing Proposition 6 with Proposition 3, one may find the impact of inventory holding cost  $h$ . Under the same condition, Proposition 3 states that the introduction should

happen immediately, whereas with inventory holding cost consideration, the firm should either stick to an immediate introduction, or delay the introduction until  $\bar{T}$  if inventory holding cost exceeds the threshold. One can observe that the threshold is smaller for a larger  $r_1$ , and thereby, the firm is more likely to delay the introduction if version 1 is more profitable. In addition, as we know from Proposition 5 that, the more profitable version 1 is, the further the release of version 2 should be delayed.

**Proposition 7.** *When  $\theta > r_2/r_1$ ,*

- (a) *If  $h < \frac{1}{\tau} \left( r_1 - \frac{1}{\theta} r_2 \right)$ ,  $T^* = \tau$ .*
- (b) *If  $h \geq \frac{1}{\tau} \left( r_1 - \frac{1}{\theta} r_2 \right)$  and  $l \leq l^*(\theta)$ ,  $T^* = \bar{T}$ .*
- (c) *If  $h \geq \frac{1}{\tau} \left( r_1 - \frac{1}{\theta} r_2 \right)$  and  $l > l^*(\theta)$ ,*
  - (i) *If  $\frac{1}{\tau} \left( r_1 - \frac{1}{\theta} r_2 \right) < h < h^*(\theta)$ ,  $T^* = l$ .*
  - (ii) *Otherwise,  $T^* = \bar{T}$ .*

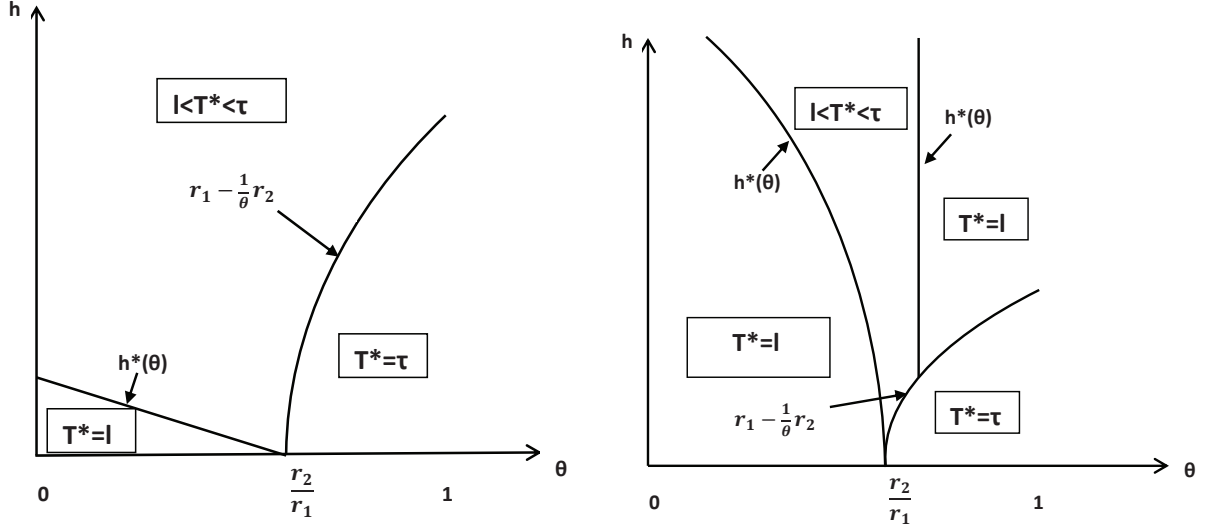
Let us also compare Proposition 7 with Proposition 3. Under the same condition, the firm should never release version 2 in Proposition 3. Yet the introduction rule is much more complicated when inventory is taken into account, as stated in Proposition 7. The decision of introduction timing relies on the cost of carrying inventory, as well as on patience of low-type customers.

The solutions in Propositions 6 and 7 are best explained using the graphic representation in Figure 1.5. It depicts the solution as a function of  $\theta$ , which is the willingness-to-switch of impatient low-type customers; and the inventory holding cost  $h$ . Although the figure is plotted for a specific set of problem parameters, all insights are not parameter specific because only the relative size of the regions changes and not the solution structure. The boundaries between the regions correspond to the closed-form thresholds.

More interesting is the impact of the inventory holding cost on the introduction timing when customers are impatient and when the planning horizon is comparable to product life cycle. In that case, we can interpret  $T^* = \tau$  as to introduce version 1 only,  $T^* = l$  as simultaneous introduction, and  $l < T^* < \tau$  as sequential introduction. Figure 1.6 depicts the graphic representation of the optimality of each introduction rules. When  $\theta < r_2/r_1$ , the firm should always introduce two versions, either simultaneously or sequentially, regardless of the cost of holding inventory. As  $\theta$  increases beyond  $r_2/r_1$ , a simultaneously introduction is no longer suitable for the firm, as a consequence, the firm should choose between the strategy of simultaneous introduction and that of introducing version 1 alone, depending upon the cost of holding inventory.

Two extreme situations are worth noting. First, when  $h > r_1 - r_2$ , which will happen when version 1 has slightly larger profit margin than that of version 2 while carrying inventory is quite costly, the firm should always adopt a sequential introduction strategy. Second, when  $\theta = r_1/r_2$ , this is the situation where the firm is indifferent between releasing version 2 and not releasing it if only considers the dimension of marginal profit, whereas for any positive inventory holding cost  $h$ , the firm is better off adopting a sequential introduction strategy.





(a) Low-type customers are impatient ( $l$  small) (b) Low-type customers are patient ( $l$  large)

Figure 1.5: The Impact of Inventory Holding on the Introduction Timing

### 1.4.3 Numerical Study

In this section, we present a systematic numerical analysis using the empirical results from Kurawarwala and Matsuo (1996) where the diffusion dynamics of four short life-cycle personal computers were estimated. We apply our integrated model discussed in the previous sections to assess the impact of inventory holding on the introduction timing of line extensions.

In the numerical study, we choose  $r_1 = 200$  and  $r_2 = 100$  for all products, such that the high-end version is twice profitable than the low-end version. We assume the 40% of potential customers are of high-type, and that low-type costumers can wait for at most 1 month to get the low-end version. Finally, we choose a comment life cycle of 18 months for all the four PC products M1 - M4. The coefficients of innovation ( $p$ ), imitation ( $q$ ) and market size ( $m$ ) of each product are obtained from empirical estimation in Kurawarwala and Matsuo (1996). In the following, we will first apply the analytical solutions for the optimal introduction time derived in Section 1.4.2 and present the solutions for the case where inventory of each version is replenished only once in the planning horizon, i.e. order at the time of its introduction. After that, we will proceed with the numerical study for the case where the firm can order each version multiple times.

**One Replenishment.** We choose  $h = 5$  and assume half of the impatient low-type customers would be willing to purchase version 1 instead ( $\theta = 0.5$ ). We then apply Proposition 4 and compute the optimal time to release the low-end version of each product given the existence of its high-end variant. We also compare the optimal profit  $\pi(T = T^*)$  with the one obtained using “Now or Never” policy ( $\max(\pi(T = l), \pi(T = \tau))$ ), and use *increase%* to

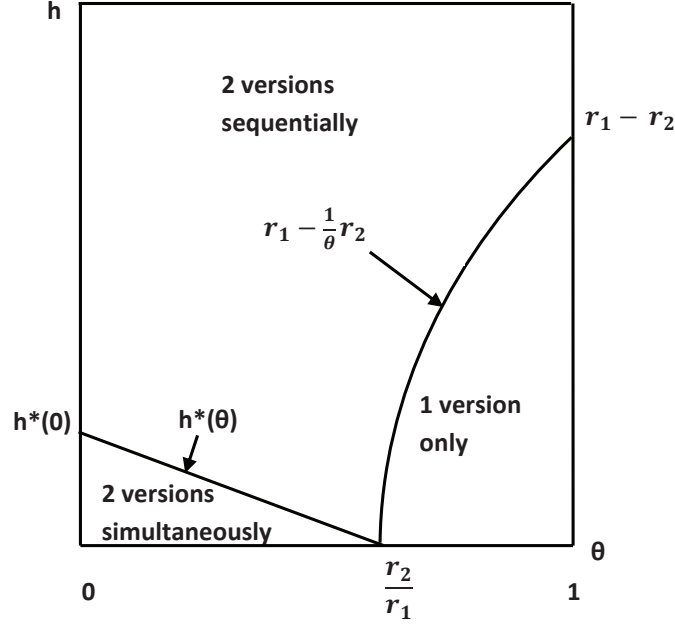


Figure 1.6: The Optimal Product Introduction Strategy

	(p, q, m)	$T^*$	$\pi(T = T^*)$	$\max(\pi(T = l), \pi(T = \tau))$	increase%
M1	(0.0728, 0.1230, 5158)	9.2	565260	549100	2.94%
M2	(0.0242, 0.4434, 5005)	7	598820	575100	4.12%
M3	(0.0152, 0.1937, 6364)	11.7	527570	498800	5.77%
M4	(0.0303, 0.3187, 5439)	8.2	625800	601110	4.11%

Table 1.1: Impact of Inventory Holding Cost on Introduction Timing for Products M1-M4 ( $r_1 = 200, r_2 = 100, h = 5, s = 0.4, \theta = 0.5, l = 1$  month and  $\tau = 18$  months)

evaluate the relative improved model performance. Table 1.1 presents the results for products M1-M4. As shown, the optimal introduction should happen at different times of the product life cycle, depending on different diffusion characteristics of each product. Clearly, the optimal timing policy derived from our model dominates the “Now or Never” policy, with a more than 4% profit increase on average over the four PC products.

**Multiple Replenishment.** For general ordering interval lengths  $O_i (i = 1, 2)$ , Problem ( $P^H$ ) is analytically untractable, the results in Table 1.2 shed light on the impact of ordering interval length  $O_i (i = 1, 2)$  on the optimal introduction time ( $T^*$ ) and the relative improved performance from the “Now or Never” policy (*increase%*). We vary  $O_i$  at values of 3, 6, 9 and 18. Note that version 2 will be ordered only at its release time if  $T^* + O_2 > \tau$ , so the last two columns have the same value.

We observe that  $T^*$  decreases in  $O_1$ , thus the firm would prefer a sooner introduction if

		$O_2$			
		3	6	9	18
$O_1$	3	(16.9, 0.11%)	(16.9, 0.11%)	(16.9, 0.11%)	(16.9, 0.11%)
	6	(14, 0.35%)	(15.8, 0.48%)	(15.8, 0.48%)	(15.8, 0.48%)
	9	(4.3, 0.12%)	(14.7, 1.17%)	(14.7, 1.17%)	(14.7, 1.17%)
	18	(4.3, 0.13%)	(8.2, 1.15%)	(11.7, 5.77%)	(11.7, 5.77%)

Table 1.2: ( $T^*$ , increase%) with Different Ordering Cycles ( $O_1, O_2$ ) for M3 ( $p = 0.0152, q = 0.1937, m = 6364, r_1 = 200, r_2 = 100, h = 5, s = 0.4, \theta = 0.5, l = 1, \tau = 18$  months)

version 1 has a longer ordering cycle. To the contrary,  $T^*$  increases in  $O_2$ , so a later release is preferred if version 2 can not be ordered frequently. Besides, we observe that with a fixed inventory cost, the difference of model performance appears to be higher if the ordering cycle becomes longer.

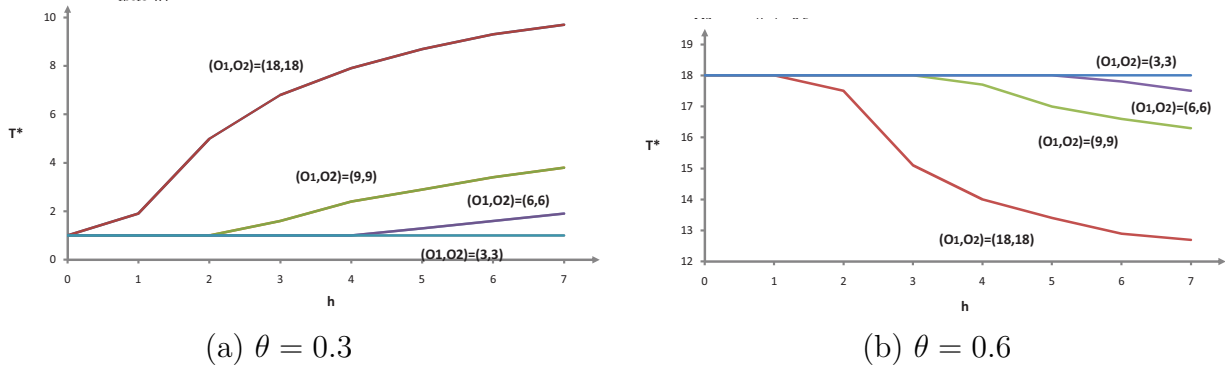


Figure 1.7: Impacts of Inventory Holding Cost and Ordering Cycles on the Introduction Timing

( $p = 0.0152, q = 0.1937, m = 6364, r_1 = 200, r_2 = 100, s = 0.4, l = 1, \tau = 18$  months).

Figure 1.7 shows the importance of re-assessing the rule of determining introduction timing. The ordering interval for each version  $O_i (i = 1, 2)$  and the inventory holding cost  $h$  are varied systematically. When  $\theta = 0.3$ , the "Now or Never" solution is "Now" (because  $\theta < r_1/r_2$ ), whereas in the presence of any non-zero inventory holding cost, the optimal introduction should happen at a time later than it. The delay is higher if the inventory is more costly to carry or if the orders have to be placed less frequently. When  $\theta = 0.6$ , the "Now or Never" solution is "Never" (because  $\theta > r_1/r_2$ ), whereas with inventory holding, the optimal introduction should happen at a time prior to it. The difference is again higher with a higher inventory holding cost and a longer ordering cycle. Thus, firms must adjust the introduction timing strategy according to its ordering schedule carefully especially in the situations when inventory is quite costly to carry or orders have to be placed less frequently.

Taken together, our results provide evidence for the necessity of coordinating the deci-

sions of operations management and marketing sciences and show that firms must accommodate both demand and supply sides in the decision of introduction timing. Furthermore, these two decisions should be synchronized not only on a day-to-day basis, such as the demand-supply match, but at the tactical level as well. As shown, the introduction timing must be carefully determined, especially when the product is costly to carry or when orders can not be placed frequently.

#### 1.4.4 Model with Substitution and Inventory Holding Cost

In this section, we propose a model accommodating the impacts of both substitution (discussed in Section 1.4.1) and inventory holding cost (discussed in Section 1.4.2). To better focus on the introduction timing decision without getting into too many operational details of how to choose a specific ordering policy, we assume each version is only ordered when it is introduced. Besides, we again assume version 1 is not phased out when version 2 being released

To proceed with the analysis, we use  $T_1(T_2)$  to denote the time version 1(2) sells out *before* substitution happens, and  $T'_1(T'_2)$  the time version 1(2) sells out *after* substitution. For any given  $Q_1$  and  $Q_2$ ,  $T_1$  can be written as the solution to

$$Q_1 = sF(T_1) + \theta(1 - s)F(T - l) \quad (1.33)$$

If (1.33) does not have a solution in  $[l, \tau]$ , we let  $T_1 = \tau$  to retain our focus inside the planning horizon  $[l, \tau]$ . Similarly, from (A.5),  $T_2$  is the solution to

$$Q_2 = (1 - s)(F(T_2) - F(T - l)) \quad (1.34)$$

If (1.34) does not have a solution in  $[l, \tau]$ , we let  $T_2 = \tau$ .

If  $T_1 < T_2$ , i.e., version 1 runs out before version 2 does, according to Section 1.4.1, a fraction  $\beta$  of high-type customers who wanted to buy version 1 would turn to version 2 until it sells out as well, and thereby in this case,  $T'_1 = T_1$  and  $T'_2$  is the solution to

$$Q_2 = (1 - s)(F(T'_2) - F(T - l)) + s\beta(F(T'_2) - F(T_1)),$$

where the first term on the right hand side is the sales from low-type customers, and the second term is the sales from high-type customers due to substitution. Consequently, the sales paths can be written as follows:

$$S_1(t; T) = \begin{cases} sF(t) & \text{if } 0 \leq t \leq l; \\ sF(t) + \theta(1 - s)F(t - l) & \text{if } l < t \leq T; \\ sF(t) + \theta(1 - s)F(T - l) & \text{if } T < t \leq T'_1; \\ 0 & \text{if } T'_1 < t \leq \tau. \end{cases}$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1 - s)(F(t) - F(T - l)) & \text{if } T \leq t \leq T'_1; \\ (1 - s)(F(t) - F(T - l)) + s\beta(F(t) - F(T_1)) & \text{if } T'_1 < t \leq T'_2; \\ 0 & \text{if } T'_2 < t \leq \tau. \end{cases}$$

If version 2 stocks out before version 1 does ( $T_1 \geq T_2$ ), a fraction  $\alpha$  of those low-type customers who wanted to buy version 2 would agree to purchase version 1 instead. As a consequence,  $T'_2 = T_2$  and  $T'_1$  is the solution to

$$Q_1 = sF(T'_1) + \theta(1-s)F(T-l) + (1-s)\alpha(F(T'_1) - F(T_2)),$$

where the first two terms refer to the sales from high-type customers, and the last term is the sales from low-type customers as a result of substitution. In this case, the sales paths can be written as follows:

$$S_1(t; T) = \begin{cases} sF(t) & \text{if } 0 \leq t \leq l; \\ sF(t) + \theta(1-s)F(t-l) & \text{if } l < t \leq T; \\ sF(t) + \theta(1-s)F(T-l) & \text{if } T < t \leq T'_2; \\ sF(t) + \theta(1-s)F(T-l) + (1-s)\alpha(F(t) - F(T'_2)) & \text{if } T'_2 < t \leq T'_1; \\ 0 & \text{if } T'_1 < t \leq \tau. \end{cases}$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } t < T; \\ (1-s)(F(t) - F(T-l)) & \text{if } T < t \leq T'_2; \\ 0 & \text{if } T'_2 < t \leq \tau. \end{cases}$$

By using the above expressions of  $S_i(t; T)$  ( $i = 1, 2$ ), the optimization model can then be formulated as follows:

$$\begin{aligned} \text{Problem } (P^{SH}): \quad & \max_{l \leq T \leq \tau, Q_1, Q_2} w_1 S_1(T'_1; T) + w_2 S_2(T'_2; T) - c_1 Q_1 - c_2 Q_2 \\ & - h \int_0^{\min\{\tau, T'_1\}} (Q_1 - S_1(t, T)) dt - h \int_T^{\min\{\tau, T'_2\}} (Q_2 - S_2(t, T)) dt \end{aligned}$$

To get a better understanding of the impacts of inventory holding cost and substitution on the introduction timing, we compare the solutions obtained from this model with those from Sections 1.4.1 and 1.4.2. For the ease of explanation, we index the model presented in Section 1.4.1 “model 1,” the one in Section 1.4.2 “model 2,” and the one Section 1.4.4 “model 3.” The comparisons of the three models are listed in Tables 1.4 - 1.7. Recall that in model 1, substitution is beneficial in cases 1(a) and 1(b) (The cases are summarized at the end of the proof of Proposition 3), but not favored in 2(a) and 2(b). As shown, under the parameter sets 2(a) and 2(b), model 3 is reduced to model 2 and the optimal solutions obtained in model 3 are the same as the ones in model 2 (The small differences are due to the computational rounding error), which shows that substitution is not attractive when inventory holding cost is considered, because the firm has to pay more inventory holding cost if more units of version 1 are being carried. Under the parameter set 1(b), substitution seems attractive in model 1, yet as the optimal introduction time turns out to be  $\tau$ , the end of the planning horizon, substitution does not actually happen. Therefore, it is not surprising to realize that models 2 and 3 share the same optimal solution. The reason why the introduction time shifts from 8 (model 1) to 6.3 (models 2 and 3) is because with holding cost included in the decision making, the firm would be more likely to order less in order to carry less inventory of version 1 compared with the situation where holding cost is not

h	L	$T^*$	$\pi(T = T^*)$	$\pi(T = l)$	$\pi(T = \tau)$	% improvement
1	8	3.6	11415	10131	10206	10.5874
1	4	2.6	12468	12445	11877	0.1851
1	2	0.5	13376	13371	12528	0.033
2	6	4	9790.8	8134.8	8872.1	9.3834
2	4	5.2	11036	10571	10561	4.2139
2	2	0.8	12446	12424	11862	0.1797
3	4	5.5	9765.7	8697.8	9245.2	5.3299
3	2	3	11570	11477	11197	0.806

Table 1.3: Optimal Introduction Timing and Optimal Profit for Different Values of  $h$  and  $L$

considered. As a result, we claim that with inventory holding cost included, the gain from substitution reduces.

Therefore, in the above three cases (cases 1(b), 2(a), 2(b)), the solutions to model 3 are exactly the same as those to model 2. By applying Proposition 4, we are able to quantify the optimal introduction time of the product line extension in terms of model primitives. It has not been addressed in the literature why the existing optimal policies (e.g., “now or never”, “now or at maturity”) are not consistent with the rules being applied in practice, and one can observe that by considering the inventory holding cost caused by a constraint ordering schedule (models 2 and 3), the existing policies are in fact no longer optimal. Our integrated model considers factors from both demand and supply sides, as a consequence, our result is closer to the industry practice.

We further observe that only in case 1(a) does model 3 achieve higher profit than model 2, where the cost from inventory holding fail to diminish all gains from substitution so that substitution is still attractive. Table 1.8 shows that the difference in profit between model 2 and model 3 decreases as per unit holding cost increases. And it should be noticed that both  $r_1 - h\tau > 0$  and  $r_2 - h\tau > 0$  are satisfied in the current parameter setting, so that the optimal solutions to model 2 are always feasible in model 3. But in fact, model 3 is not restricted to the cases where  $r_1 - h\tau > 0$  and  $r_2 - h\tau > 0$ . We use the setting only to make the results comparable to those from model 2.

## 1.5 Model Extensions and Future Research

### 1.5.1 Model Extension: Customer Heterogeneity in Waiting Time

In this section, we relax the assumption of constant waiting time for all low-type customers, and instead we capture customer heterogeneity in waiting time by assuming a stochastic waiting time  $l$ , which follows an exponential distribution with parameter  $\lambda = \frac{1}{l}$ . For each low-type customer who becomes interested in version 2 at time  $x$ , the probability

Model	obj	T	$Q_1$	$Q_1$	$D_1$	$D_2$
Model 1	15446	0	772.425	0	376.7927	565.189
Model 2	11415	3.6	454.4581	409.8582	454.4581	409.8582
Model 3	11965	0.1	771	2	376.7927	565.189

Table 1.4: Parameter Set 1(a):  $\theta = 0.5, \alpha = 0.7$ 

Model	obj	T	$Q_1$	$Q_1$	$D_1$	$D_2$
Model 1	16571	8	826.6570	0	826.65	2.8675
Model 2	12895	6.3	751.4277	96.8953	751.4277	96.8953
Model 3	12893	6.3	752	96	751.4277	96.8953

Table 1.5: Parameter Set 1(b):  $\theta = 0.8, \alpha = 0.7$ 

Model	obj	T	$Q_1$	$Q_1$	$D_1$	$D_2$
Model 1	14316	0	376.7927	565.189	376.7927	565.189
Model 2	11415	3.6	454.4581	409.8582	454.4581	409.8582
Model 3	11410	3.6	454	410	454.4581	409.8582

Table 1.6: Parameter Set 2(a):  $\theta = 0.5, \alpha = 0.2$ 

Model	obj	T	$Q_1$	$Q_1$	$D_1$	$D_2$
Model 1	15443	8	770.4178	2.8675	770.4178	2.8675
Model 2	12232	5.2	621.6091	215.4513	621.6091	215.4513
Model 3	12228	5.1	613	228	612.8363	227.9838

Table 1.7: Parameter Set 2(b):  $\theta = 0.7, \alpha = 0.5$ 

Model	h=0.1	h=0.3	h=0.5	h=1	h=1.5
Model 2(obj)	13924	13298	12736	11415	10135
Model 3(obj)	15094	14400	13704	11965	10226
Model 3(obj)-Model 2(obj)	1170	1102	968	550	91

Table 1.8: Impact of Inventory Holding Cost on Optimal Profit:  $\theta = 0.5, \alpha = 0.7$

that her demand gets lost by  $t$  can be written as:

$$\begin{aligned} 1 - e^{-\lambda(t-x)} & \quad \text{if } t \leq T \\ 1 - e^{-\lambda(T-x)} & \quad \text{if } t > T \end{aligned}$$

Based on the basic demand processes described in Section 1.3.2, it is easy to verify that the demand trajectories in this scenario can be defined as:

$$D_1(t; T) = \begin{cases} sF(t) + \theta(1-s) \int_0^t (1 - e^{-\lambda(t-x)}) f(x) dx & \text{if } t \leq T; \\ sF(t) + \theta(1-s) \int_0^T (1 - e^{-\lambda(T-x)}) f(x) dx & \text{if } t > T. \end{cases} \quad (1.35)$$

$$D_2(t; T) = \begin{cases} (1-s) \int_0^t e^{-\lambda(t-x)} f(x) dx & \text{if } t \leq T; \\ (1-s)(F(t) - F(T-l)) + (1-s) \int_0^T e^{-\lambda(T-x)} f(x) dx & \text{if } t > T. \end{cases} \quad (1.36)$$

where  $F(t)$  and  $f(t)$ , defined in (1.10) and (1.11), are the respective cumulative and instantaneous demand path in the Bass model. (1.35) and (1.36) are similar to (A.4) and (A.5), except that in the presence of heterogeneity in waiting time, we use the expected demand loss rather than the actual demand loss from impatient low-type customers.

## 1.5.2 Model Extension: Changes of Price and Holding Cost over Product Life Cycle

To model the impact of price changes on customer choices, we let  $w_i(t)$  ( $i = 1, 2$ ) denote the price at time  $t$  and  $h_i$  ( $i = 1, 2$ ) denote the holding cost at time  $t$  of each version  $i$ . We assume the market is dynamically segmented according to the price ratio of the two version,  $s(t) = w_2(t)/w_1(t)$ . Intuitively, if version 1 is charged a higher price, a smaller proportion of customers would be willing to buy it. In addition, we assume before introduction,  $w_2(t) = r$ ,  $\forall 0 \leq t < T$ , where  $r$  is the price of version 2 expected by low-value customers. Following the notation defined in Section 1.4.4, for any given  $Q_1$  and  $Q_2$ , we can express the time version 1 sells out before substitution happens,  $T_1$ , as the solution to

$$Q_1 = \int_0^{T_1} s(t) f(t) dt + \theta \int_0^{T-l} (1-s(t)) f(t) dt \quad (1.37)$$

where  $f(t)$  is defined in (1.11). Similarly, the time version 2 sells out before substitution happens,  $T_2$ , is the solution to

$$Q_2 = \int_{T-l}^{T_2} (1-s(t)) f(t) dt \quad (1.38)$$

If (1.37) (or (1.38)) does not have a solution in  $[l, \tau]$ , we let  $T_1 = \tau$  (or  $T_2 = \tau$ ) to retain our focus inside the planning horizon  $[l, \tau]$ .

If version 1 stocks out before version 2 does ( $T_1 < T_2$ ), then  $\beta$  proportion demand from high-type customers will be substituted by version 2 after version 1 runs out. Obviously,



$T'_1 = T_1$  and  $T'_2$  is the solution to

$$Q_2 = \int_{T-l}^{T'_2} (1 - s(t))f(t) dt + \beta \int_{T_1}^{T'_2} s(t)f(t) dt,$$

Consequently, the instantaneous sales paths of version  $i$  ( $i = 1, 2$ ) can be written as follows:

$$s_1(t; T) = \begin{cases} s(t)f(t) & \text{if } 0 \leq t \leq l; \\ s(t)f(t) + \theta(1 - s(t))f(t - l) & \text{if } l < t \leq T; \\ s(t)f(t) & \text{if } T < t \leq T'_1; \\ 0 & \text{if } T'_1 < t \leq \tau. \end{cases}$$

$$s_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1 - s(t))f(t) & \text{if } T \leq t \leq T'_1; \\ (1 - s(t))f(t) + s(t)\beta f(t) & \text{if } T'_1 < t \leq T'_2; \\ 0 & \text{if } T'_2 < t \leq \tau. \end{cases}$$

If version 2 stocks out before version 1 does ( $T_1 \geq T_2$ ), then  $\alpha$  proportion demand from low-type customers will be substituted by version 1 after version 2 runs out. As a consequence,  $T'_2 = T_2$  and  $T'_1$  is the solution to

$$Q_1 = \int_0^{T'_1} s(t)f(t) dt + \theta \int_0^{T-l} (1 - s(t))f(t) dt + \alpha \int_{T_2}^{T'_1} (1 - s(t))f(t) dt,$$

In this case, the sales paths can be written as follows:

$$s_1(t; T) = \begin{cases} s(t)f(t) & \text{if } 0 \leq t \leq l; \\ s(t)f(t) + \theta(1 - s(t))f(t - l) & \text{if } l < t \leq T; \\ s(t)f(t) & \text{if } T < t \leq T'_2; \\ s(t)f(t) + \alpha(1 - s(t))f(t) & \text{if } T'_2 < t \leq T'_1; \\ 0 & \text{if } T'_1 < t \leq \tau. \end{cases}$$

$$s_2(t; T) = \begin{cases} 0 & \text{if } t < T; \\ (1 - s(t))f(t) & \text{if } T < t \leq T'_2; \\ 0 & \text{if } T'_2 < t \leq \tau. \end{cases}$$

Then we can express the problem of profit maximization over the life-cycle period as

$$\text{Problem } (P^{SHP}): \max_{l \leq T \leq \tau, Q_1, Q_2} \int_0^{\min\{\tau, T'_1\}} w_1(t)s_1(t; T) dt + \int_0^{\min\{\tau, T'_2\}} w_2(t)s_2(t; T) dt - c_1Q_1 - c_2Q_2 \\ - \int_0^{\min\{\tau, T'_1\}} h_1(t)(Q_1 - S_1(t, T)) dt - h \int_T^{\min\{\tau, T'_2\}} h_2(t)(Q_2 - S_2(t, T)) dt$$

where  $S_i(t, T) = \int_0^t s_i(s; T) ds$  is the cumulative sales up to time  $t$ .

### 1.5.3 Summary and Future Research

This paper complements existing research of determining optimal introduction timing for a line extension product, which has often been studied within the marketing domain (e.g. Wilson and Norton 1989, Moorthy and Png 1992). However, as inventory holding is often unavoidable in most industry practices, this has led to a clear call in the literature to develop more comprehensive models addressing the timing decisions from both operations management and marketing science perspectives, with the hope to design methodologies to improve a firm's profit or enhance the supply chain's overall performance. As a result, we propose an integrated model that considers important factors from operation management area, including substitution in supply scarcity, ordering interval and inventory holding cost, and factors from marketing area, including diffusion, customer expectation and demand switch. On the demand side, based on the Bass model (Bass, 1969), we propose the splitting Bass-like diffusion model to describe the adoption process of two successive (and differentiated) versions of the same product, which also captures the role of customer expectation in shaping purchase choices. On the supply side, we address the effect of substitution due to supply scarcity and the impact of inventory holding cost from a simple ordering policy.

In contrast to the existing "now or never" policy in the literature (Wilson and Norton, 1989), we have shown that, with a constrained ordering schedule, firms should adjust the decision of introduction timing of line extensions by considering inventory holding of the products, as the optimal introduction can happen anytime from "now" to "never". Our result justifies that an interdisciplinary decision-making approach of both operations management and marketing science will help a firm achieve an improved profit. These two aspects of a firm should be synchronized not only at the operational level, but at the tactical level as well, so managers should understand both sides and then cook a recipe that is right for their company's particular situation.

The purpose of this paper is to take a first step towards understanding the implications of timing introductions of product lines by coordinating decisions of marketing and operations management. Our analysis opens up several opportunities for future research. For instance, we can study the relationship between the waiting time of customers and the unit price of the line extension. Customer's patience can often be endogenously determined, as the more benefit she will gain from waiting, the more patient she is willing to be. It would be interesting to quantify this trade-off and investigate how this relationship will impact the results in the paper. We also plan to find out the impact of customers' estimation of the introduction time on the firm's timing decision (Prasad *et al.*, 2004). With her own expectation of the introduction time, a rational customer compares the net present values between the two products and make the purchase that gives her more utility. We are also interested in analyzing a more complex operational cost structure, such as fixed production cost, non-linear ordering cost and multiple-period production (e.g., Ho *et al.*, 2002), and the asymmetry in the social influence process, i.e., a set of customers have more influential power (Van den Bulte and Joshi 2007, Joshi *et al.* 2008). We believe new interesting managerial insights can be obtained from these research directions.

## Chapter 2

# Incorporating Social Contagion into Customer Value Analysis: A Homogeneous Population

### 2.1 Introduction

Customers are assets. Indeed, the health of a customer base is crucial to a firm's growth and profitability. A common metric for assessing a customer's worth is her lifetime value (LV): the present value of all future profits generated by the customer excluding the cost of acquiring her (Gupta *et al.*, 2006). Customer LV is also frequently used to segment customers, which allows the firm to identify customers of high value and appropriately allocate scarce marketing resources to enhance its customer assets (Rust *et al.* 2004, Ho *et al.* 2006). Hence the importance of having an accurate metric for assessing customer value cannot be over-emphasized.

Prior models of customer LV assume that the value of a customer depends only on her own purchase history (e.g. Dwyer 1989, Gupta and Lehmann 2003, Ho *et al.* 2006, Gupta *et al.* 2006). That is, the behaviors and purchase histories of other customers will *not* influence a specific customer's LV. This assumption is valid for traditional purchase contexts where there is little interaction among customers and each customer may be treated as an independent buyer. The explosion of instant communication and viral Internet marketing however cast doubt on this assumption. In the Web 2.0 world, customers interact intensely with each other during their shopping process (Mayzlin, 2006). Hence, social contagion is likely to play an active role in shaping the adoption and purchase of new product. Consequently, a customer's LV not only derives from how much she buys, but also captures her influence on others' timing of adoption through post-purchase interpersonal communication (Gremler and Brown, 1998). This paper provides a formal approach to model this social phenomenon. By doing so, we hope to develop a revised metric for customer value that is better suited for online purchase contexts where social contagion is prevalent.

Specifically we posit that a customer's LV is a sum of her purchase value (PV) and influence value (IV)<sup>1</sup>. Formally we have:

$$LV = PV + IV.$$

Our central premise recognizes the potential and power of social contagion. In fact, some firms have begun to recognize a customer's IV by rewarding them in order to reinforce their positive behavior. For example, BMG Music Service sends free CDs to existing customers when they bring in new customers (Villanueva *et al.*, 2008). The San Francisco Symphony offers complimentary concert tickets to customers who refer new customers (Biyalogorsy *et al.*, 2001). Other examples include fashion house Gilt and online bank ING which reward customers that help them attract new customers. Although the effect of social contagion on LV has been emphasized and highlighted by many practitioners and academics (see, e.g., Gupta *et al.* 2006, Kumar *et al.* 2007, Villanueva *et al.* 2008, Ovchinnikov and Pfeifer 2009), there is yet a formal model to quantify its effect on LV and study its influence on marketing mix activities. This paper is a step towards filling this gap.

Positive social contagion has two benefits. First, it increases the total number of adopters for a new product. Customers who otherwise would not have bought the product may now change their minds because of the positive feedback. Second, positive feedback may dramatically reduce the timing of adoption. Uncertainty about a new product's benefits may be reduced by social contagion and as a consequence, potential buyers may speed up their adoption process. This uncertainty reduction process leads to purchase acceleration which in effect reduces the new product's life-cycle. In this paper we focus exclusively on the second benefit and analyze the effect of social contagion on timing of new product adoption and customer LV. We will characterize customer PV, IV, and LV in the context of the classical Bass diffusion model (in which the total number of adopters is fixed).

Under the Bass diffusion model (Bass, 1969), a new product diffusion is described by both the innovation and imitation processes. Innovation process measures the propensity of one's adoption independent of the decisions of other individuals in a social system. On the other hand, imitation process describes how an individual is influenced by previous buyers through social influence. We use the Bass model for two reasons: First, it receives wide empirical support, either in the diffusion of a brand new category (Bass, 1969) or in the diffusion of a new brand within an established category (e.g., Kurawarwala and Matsuo 1998, Sawhney and Eliashberg 1996, Mahajan *et al.* 1993); Second, it has been shown to be a nice building block for many generalizations including incorporating the effects of marketing-mix variables (Mahajan *et al.*, 1995). By building on the Bass diffusion model, we hope our results can either be directly applied or be extended to many practical situations. These results can be used to predict the dynamics of customer PV, IV and LV for new products that are well approximated by Bass diffusion curves.

One powerful way to accelerate product purchase is to increase positive social contagion

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<sup>1</sup>Unlike Kumar *et al.* (2007) and Ovchinnikov and Pfeifer (2009) where IV is treated as separate from LV, we conceptualize IV as a key component of LV. This is so because IV captures a separate stream of cash flow a customer brings in the form of others' purchases.

by offering introductory discounts (Van Ackere and Reyniers, 1995). Publishers offer introductory discounts for college textbooks in order to speed up the initial adoption. Similarly, when Hasbro launched a new handheld video game called POX in 2001, they chose 1,600 kids to be their agents of social contagion, each armed with a backpack filled with samples of the game to be handed out to their friends (Godes and Mayzlin, 2009). Other examples include the widespread practice of sending a limited quantity of free CDs when a new CD is released. These marketing strategies can be nicely analyzed and interpreted using our model framework. We explain these phenomena from a social influence standpoint and show how such marketing strategies can actually increase a firm’s total customer LV. We investigate how a firm should optimally determine the size of the promotion sample in order to maximize total customer LV.

Sometimes due to operational reasons, customers may not be able to get the product they ordered right away. Consequently, a customer does not generate social contagion until her order arrives. This may decelerate social contagion and slow down product diffusion. In this paper, we specifically examine how the duration of an out-of-stock phenomenon influences customer LV so that firms can quantify the benefit on a high inventory availability.

The benefit of reducing lead times have been well documented in both operations management and marketing literature. Most research in those areas typically focuses on internal pricing during lead time (e.g., Hill and Khosla 1992, Ray and Jewkes 2004) or the impact of lead-time commitment on customer satisfaction and demand (e.g. Kumar et al. 1997, Ho and Zheng 2004). The influence of lead time on customer LV has not been explicitly examined. We show how a lengthy lead time decelerates social contagion, slows down future sales and decreases customer LV. We explain that even a small lead time will make a big difference in the firm’s customer assets.

The paper makes the following contributions:

1. We investigate how social contagion influences the value of a customer beyond her purchase history. Our analysis suggests that it is crucial to account for social contagion in customer LV analysis. Ignoring social influence will greatly underestimate the value of early adopters and overstate the value of later adopters<sup>2</sup>. To the best of our knowledge, this is the first attempt to provide formal metrics for firms to explicitly apportion the direct value of a customer’s purchase and the indirect value of the customer’s social influence over time. We derive closed-form expressions for PV, IV, and LV.
2. We show that a customer who adopts earlier is more valuable than a customer who adopts later and LV decreases rapidly over time. In addition, we find that PV increases with the innovation parameter whereas IV decreases with it. Early adopters have their LV decrease with the innovation parameter while later adopters have their LV increase

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<sup>2</sup>Substantially, this paper augments the traditional CRM literature. To date, few studies have explicitly addressed the value of the social influence associated with a customer in the CRM literature. Hogan *et al.* (2003) suggest that the value of a lost customer goes beyond the loss of the revenue generated directly from her purchase, but should also include the lost value of her social network. Gupta *et al.* (2006) find strong direct and indirect network effects among buyers and sellers in a field dataset involving an auction house.

with it. Interestingly, PV always decreases with the imitation parameter whereas IV increases with it for early adopters but decreases with it for late adopters. LV increases with the imitation parameter if the timing of adoption is below a cutoff value and decreases with it if it is above the cutoff.

3. We determine the optimal size of the promotion sample and show how purchase acceleration in terms of introductory discounts can lead to significant improvements in total customer LV. Purchase acceleration works because we dramatically increase the IV of social contagion agents.
4. We show that an out-of-stock phenomenon that happens earlier in a product's life cycle always leads to a greater loss in total customer LV. We find that out-of-stock status can dramatically slow down social contagion and customer purchases. Consequently, it can significantly decrease a firm's customer LV.

The remainder of this chapter is organized as follows. In Section 2.2, we describe the modeling framework of customer PV, IV and LV. Section 2.3 analyzes the influence of purchase acceleration on LV. Section 2.4 analyzes the influence of purchase deceleration on LV. Section 2.5 summarizes the paper and discusses potential directions for future research. All proofs are presented in Appendix A.

## 2.2 The Model

Consider a firm that introduces a new durable product to a market of potential adopters. The customer adoption process is assumed to follow the Bass dynamics (Bass, 1969). At any time during the product life cycle, a potential adopter's decision is influenced by two factors: the external influence (e.g., advertising or mass media communication) and the internal influence (e.g. social contagion):

$$\frac{f(t)}{1 - F(t)} = p + qF(t) \quad (2.1)$$

where  $f(t)$  and  $F(t)$  are the instantaneous and cumulative proportions of adopters at time  $t$  respectively. Parameters  $p$  and  $q$  are the coefficients of innovation and imitation. They can also be interpreted as the coefficients of external influence and internal influence (Mahajan *et al.*, 1990, Van den Bulte 2000). Equation (3.10) means that an individual's likelihood of adopting at time  $t$  condition on no adoption in the past is determined by the individual's intrinsic motivation and the social influence at that time. If there are no pre-release purchases (i.e.,  $F(0) = 0$ ), the solutions to (3.10) can be written as:

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}} \quad (2.2)$$

$$f(t) = \frac{(p+q)^2/p e^{-(p+q)t}}{\left(1 + \frac{q}{p}e^{-(p+q)t}\right)^2} \quad (2.3)$$

In new products with substantial social contagion (e.g.,  $q > p$ ), (3.11) gives a bell-shaped curve with a single inflection point. The instantaneous adoption rate is small initially because only a few members of the potential pool adopt the innovation. After a building-up phase, the instantaneous adoption rate rises sharply until it reaches the peak of the bell-shaped curve. After that point, the instantaneous adoption rate decreases because the remaining pool of potential adopters has declined. Consequently, when (2.2) is plotted, one often observes an S-shaped curve.

### 2.2.1 The Social Influence Chain

Bass diffusion implies a social influence chain. Consider two potential adopters Amy and Betty. Suppose Betty buys at time  $t$ . Betty plays two roles in this social influence chain. On the one hand, her purchase might be influenced by a previous buyer, and thereby she is an *influencee*. On the other hand, after her purchase, she might exert social influence, and thus become an *influencer* of others' purchases. Consequently she might have her own influencees. We assume that an individual can have multiple influencees but can only be influenced by at most one influencer (i.e. she may adopt the product without others' influence).

Now we move ahead to take a further look at Betty's role as an influencer. Suppose Amy buys at  $s$  ( $s > t$ ). Her purchase was either driven by her external influence, such as advertisement or mass media, or due to the internal influence from previous buyers. From (3.10), the probability of her being an influencee can be written as:

$$\Pr[\text{Amy is an influencee}] = \frac{qF(s)}{p + qF(s)} \quad (2.4)$$

An implicit assumption in the Bass model is that "at any point in the process, all individuals who are yet to adopt have the *same* probability of adopting in a given time period, so that *differences* in individual adoption times are purely stochastic," (Chatterjee and Eliashberg, 1990). Therefore, at any moment in time each individual who is buying is equally likely to be an influencee of any previous buyer, and each previous buyer is equally likely to be the influencer of any individual that is buying. We observe that (2.4) is increasing in  $s$ , so later buyers are more likely to be influencees.

We use  $N$  to denote the size of the potential customers. Since each previous buyer was equally likely to influence Amy, the probability that she was influenced by Betty is given by

$$\Pr[\text{Amy is Betty's influencee}] = \frac{q/N}{p + qF(s)} \quad (2.5)$$

We have  $f(s)N$  independent buyers at time  $s$ , so the number of buyers at  $s$  who were influenced by Betty follows a Binomial distribution with parameters  $f(s)N$  and  $\frac{q/N}{p + qF(s)}$ . It follows that the expected number of customers buying at  $s$  that were influenced by Betty is given by

$$\mathbb{E}[\text{number of Betty's influencees at time } s] = \frac{qf(s)}{p + qF(s)} \quad (2.6)$$

Therefore, during the product life cycle, the expected total number of Betty's influencees is

$$\mathbb{E}[\text{total number of Betty's influencees}] = \int_t^\infty \frac{qf(s)}{p + qF(s)} ds \quad (2.7)$$

### 2.2.2 Customer Lifetime Value

Without loss of generality, normalize the product profit margin to 1. We characterize the LV of any customer currently making a purchase as the sum of her PV and IV. We model LV in the following way. Consider Betty, a buyer at  $t$ . She is an innovator with probability  $\frac{p}{p + qF(t)}$ . In this case, her PV is the present value of her profit discounted by  $e^{-rt}$  with rate  $r$ . Otherwise, she is influenced by people who buy before  $t$ , so she only earns  $1 - \delta$  fraction of the present value of the profit she produces. The remaining  $\delta$  fraction is credited back to her influencer. Therefore, Betty's PV is

$$PV(t) = e^{-rt} \left( \frac{p}{p + qF(t)} + \frac{qF(t)}{p + qF(t)}(1 - \delta) \right) \quad (2.8)$$

$$= \frac{p + q(1 - \delta) + q\delta e^{-(p+q)t}}{p + q} e^{-rt} \quad (2.9)$$

On the other hand, Betty might tell her friends about her purchase. She will earn  $\delta$  fraction of the present value of the resulting profit brought in by each new customer. Substituting (3.10) into (2.7)

$$\begin{aligned} IV(t) &= \delta \int_t^\infty \frac{e^{-rs} qf(s)}{p + qF(s)} ds \\ &= \delta q \int_t^\infty e^{-rs} (1 - F(s)) ds \end{aligned} \quad (2.10)$$

---

<sup>3</sup>As  $q < pN$  always holds empirically, (2.5) is well-defined.



Her LV is the sum of PV and IV:

$$LV(t) = PV(t) + IV(t) \quad (2.11)$$

We compare the PV, IV and LV with and without social contagion in Table 2.1. Without social contagion, a customer's LV is the present value of her own purchase independent of others. With social contagion, her LV depends on past and future purchases.

Table 2.1: Customer LV with/without Social Contagion

	Without Social Contagion	With Social Contagion
PV(t)	$e^{-rt}$	$\frac{p + q(1 - \delta) + q\delta e^{-(p+q)t}}{p + q} e^{-rt}$
IV(t)	0	$\delta q \int_t^\infty e^{-rs} (1 - F(s)) ds$
LV(t)	$e^{-rt}$	$\frac{p + q(1 - \delta) + q\delta e^{-(p+q)t}}{p + q} e^{-rt} + \delta q \int_t^\infty e^{-rs} (1 - F(s)) ds$

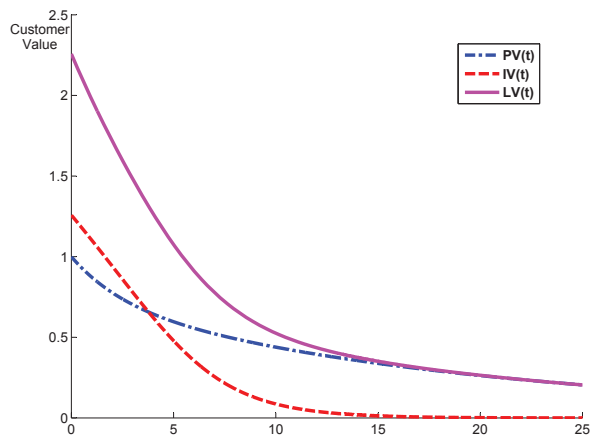


Figure 2.1: Customer PV, IV and LV  
 $(p = 0.0163221, q = 0.325044, r = 0.05, \delta = 0.3)$

Figure 2.1 plots the PV, IV and LV of a customer as a function of the time of purchase. We use the average coefficient of innovation described in Bass (1969) as the coefficient of innovation ( $p$ ). Similarly, the average coefficient of imitation from Bass (1969) is used as the coefficient of imitation ( $q$ ). The discounting factor  $r$  and the proportion of influential credit parameter  $\delta$  are set to 0.05 and 0.3 respectively. The high LV of early adopters is due to both their less-discounted PV and their post-purchase influence on later adopters. In fact, IV can be quite crucial for early adopters in a networked economy.

**Proposition 8.**  $\int_{t=0}^{\infty} LV(t) f(t) dt = \int_{t=0}^{\infty} e^{-rt} f(t) dt.$

Proposition 8 states that the total LV of the firm's customer base is the same in our framework as it is in traditional customer LV models. Note that in traditional models the value of a customer only comes from her own purchase. So social contagion is about value redistribution among the customer base rather than adding value to the customer base. Our approach enables the firm to directly model the value of interpersonal influence among customers. Moreover, we directly connect between the value of the firm and individual customer profitability avoiding double-counting of cash-flows when adding together the LV of the firm's customers can help measure the value of the firm (see, e.g., Rust *et al.* 2000, Gupta and Lehman 2003, Berger *et al.* 2006 and Gupta *et al.* 2006).

The following proposition establishes the existence of a cutoff time in any innovation diffusion process. All customers prior to the cutoff time are worth more to the firm than their own cash flow in a networked economy since they help to attract later buyers through social contagion. Purchases made after the cut-off time are worth less to the firm because part of the resulting profit is credited back to early adopters to reward their influence.

**Proposition 9.** *Let*

$$t^* = \left\{ t : \int_{s=t}^{\infty} e^{-r(s-t)} (1 - F(s)) ds = \frac{1 - e^{-(p+q)t}}{p+q} \right\}$$

When  $t \leq t^*$ ,  $LV(t) \geq e^{-rt}$ ; When  $t > t^*$ ,  $LV(t) < e^{-rt}$ .

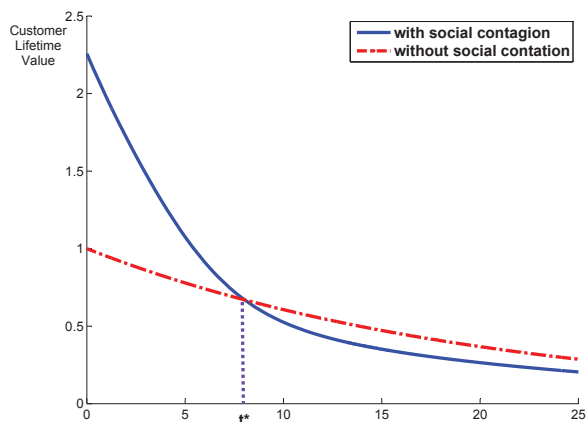


Figure 2.2: Customer LV with/without Social Contagion  
 $(p = 0.0163221, q = 0.325044, r = 0.05, \delta = 0.3)$

Figure 2.2 shows how social contagion redistributes value among customers. We shall say  $t^*$  is the cutoff time. Customers who adopt prior to  $t^*$  have higher value with social contagion than without it. In other words, ignoring social contagion may lead to an underestimate of the LV of early adopters and an overstatement of the LV of later adopters. Our model directly incorporates social influence and can help the firm form strategies to acquire customers over time.

Note, we assume a constant profit margin throughout the product life cycle. One might argue that the product profit margin typically decreases over time for most new durables. However, our qualitative results would not change. In this case, early customers are more valuable because they have higher cash flow value, greater influential power over other customers and they correspond to a greater profit margin.

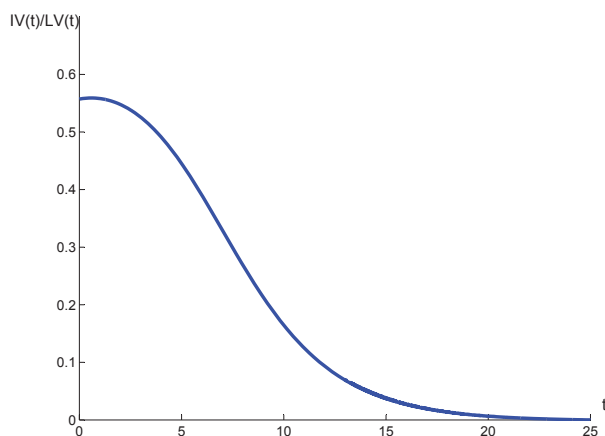


Figure 2.3: Importance of IV  
 $(p = 0.0163221, q = 0.325044, r = 0.05, \delta = 0.3)$

Figure 2.3 illustrates the significance of IV as a component of LV over time. We observe that IV is more significant for early adopters than late adopters. Here, the loss of an early adopter is more costly than the loss of a late adopter. Ho *et al.* (2002) showed that it may be beneficial to pre-produce some product before launching the new product to avoid losing early adopters under certain circumstances. Our result provides an LV perspective on this result. Also, noting the significant influence of early adopters, firms should increase post-purchase customer service early in the product life cycle. This strategy increases customer satisfaction and increases their willingness to spread positive social contagion.

It is worthwhile to discuss half-life customer value. This concept has been adopted across many marketing sub-fields. In the context of LV, the half-life of a customer is the length of time she can postpone her purchase before her value is halved. Figure 2.4 shows how a customer's LV decays by half as her purchase is delayed. We note that customer half-life is far less than half of the product life cycle. For a product with a 25-year life cycle, the LV of a customer who buys at product introduction is more than twice the LV of a customer who delays her adoption to year four. Further, an early adopter can be four times more valuable than a buyer in year fifteen.

### 2.2.3 Comparative Statics

In this section we consider comparative statics. In particular, we want to predict how the PV, IV and LV of customers change as parameters vary.

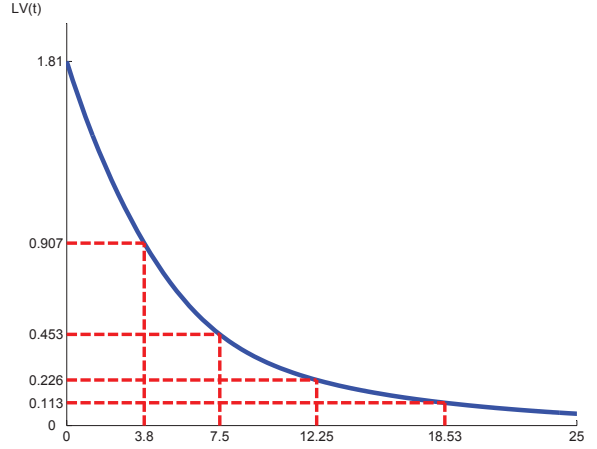


Figure 2.4: LV Curve  
 $p = 0.0163221, q = 0.325044, r = 0.1, \delta = 0.3$

**Proposition 10.**  $PV(t)$ ,  $IV(t)$  and  $LV(t)$  are all decreasing convex in  $t$ .

When a customer delays a purchase, her PV decreases because there is a more heavily discounted profit margin. At the same time, her IV decreases because the pool of potential adopters shrinks. Hence, a customer's value goes down rapidly as she waits to buy. Proposition 10 states that the rate of decrease is smaller as adoption diffuses.

**Proposition 11.** (1)  $PV(t)$  is increasing in  $p$ .  
 (2)  $IV(t)$  is decreasing in  $p$ .  
 (3) We define

$$t_1 = \left\{ t : e^{-rt} \frac{(1 - (pt + qt + 1)e^{-(p+q)t})}{(p+q)^2} = \int_t^\infty e^{-rs} \frac{\partial F(s)}{\partial p} ds \right\},$$

then  $LV(t)$  decreases with  $p$  for  $t < t_1$  and increases thereafter.

All other things being equal, Proposition 11 tells us that at any given time, stronger external influence increases the PV of customers making purchases. With stronger external influence, customers are more likely to be innovators who earn all the credit from their own purchases. However, stronger external influence speeds up the diffusion process, shrinks the pool of potential adopters, and thus reduces the social influence of all adopters. So, PV and IV move in opposite directions as external influence increases. As for LV, Proposition 11 proves the existence of an inflection point  $t_1$ . The LV of adopters prior to  $t_1$  decreases and the LV of buyers after  $t_1$  increases.

**Proposition 12.** (1)  $PV(t)$  is decreasing in  $q$ .  
 (2) We define

$$t_2 = \left\{ t : \int_t^\infty e^{-rs} \left( 1 - F(s) - q \frac{\partial F(s)}{\partial q} \right) ds = 0 \right\}$$

$IV(t)$  increases with  $q$  for  $t \leq t_2$  and decreases thereafter.

(3) We define

$$t_3 = \left\{ t : \int_t^\infty e^{-rs} \left( 1 - F(s) - q \frac{\partial F(s)}{\partial q} \right) ds = \frac{e^{-rt} p}{(p+q)^2} \left( 1 - \left( 1 - qt - \frac{q^2}{p} t \right) e^{-(p+q)t} \right) \right\}$$

$LV(t)$  increases with  $q$  for  $t \leq t_3$  and decreases thereafter.

Given fixed potential population and external influence, increased social contagion effects reduce the PV of customers currently making purchases at any given time. As social contagion speeds up diffusion, adopters are more likely to be imitators and only earn partial credit from purchases. However, two opposite forces drive IV. On one hand, a higher internal influence coefficient indicates that adopters are more likely to be imitators, leading to higher customer IV. On the other hand, a higher internal influence coefficient speeds up demand, so the market will be saturated sooner and customer IV will be lower. Part (2) of Proposition 12 addresses this situation. It states that as people become more inclined to spread social contagion, the IV of early customers (prior to  $t_2$ ) goes up. The IV of late customers (after  $t_2$ ) rapidly goes down. Part (3) of Proposition 12 specifies that LV changes as IV does, except LV changes around a different inflection point. It can be shown that  $t_3$  is always smaller than  $t_2$ .

These results are highly relevant in the era of new media marked by an increase in “buzz”. As mentioned earlier, ignoring social contagion leads to dramatic underestimation of the LV of early adopters. We summarize these comparative statics results in Table 2.2.

Table 2.2: Summary of Comparative Statics Results

	PV(t)	IV(t)	LV(t)
t	decreasing convex	decreasing convex	decreasing convex
p	increasing	decreasing	decreasing for $t < t_1$ increasing thereafter
q	decreasing	increasing for $t < t_2$ decreasing thereafter	increasing for $t < t_3$ decreasing thereafter

### 2.2.4 Mean Time until First Influence

In this section, we analyze customer value with a different metric – mean time until first influence. Mean time until first influence is the average time a customer takes to influence others for the first time after her purchase. We use this notion to measure how “fast” impacts are realized for customers that adopt at different times. Instead of analyzing LV directly, we consider the responsiveness of interpersonal influence. It has been widely acknowledged

that response time is an effective alternative measurement in value analysis. We proceed as follows.

Consider Betty who buys at time  $t$ . We use  $N_t^s$  to denote the number of Betty's influencees who buy at time  $s$  ( $s > t$ ). Recall (2.5). For each adopter at  $s$ , we know the probability of not being influenced by Betty is  $1 - \frac{q/N}{p + qF(s)}$ . Moreover,  $f(s)N$  independent customers will buy at  $s$  (assuming  $f(s)N$  is integer.) So, the probability that Betty has influencees at  $s$  is equal to the probability that not all imitators at  $s$  are influenced by others (except Betty.) This probability is

$$\begin{aligned} \Pr[\text{Betty has influencees at } s] &= \Pr(N_t^s \geq 1) \\ &= 1 - \left(1 - \frac{q/N}{p + qF(s)}\right)^{f(s)N} = \lambda(s) \end{aligned} \quad (2.12)$$

Note that (2.12) is independent of  $t$ , because each individual who buys before  $s$  is equally likely to influence anyone that is buying at  $s$ . We use  $\lambda(s)$  to represent (2.12).

**Proposition 13.**  $\lambda(s)$  is decreasing in  $s$ .

Compare this result with (2.4), which implies a later adopter is more likely to be an influencee. Proposition 13 shows that it becomes less likely for *anyone* to have influencees as the product diffuses in the market over time. The intuition is as follows: before demand peaks, there is a growing number of instantaneous adopters. As time goes by, each of these adopters becomes more likely to be an influencee. In the mean time, the number of previous buyers is increasing, so more buyers are competing to influence the growing number of instantaneous adopters. The number of previous buyers increases more rapidly than the number of instantaneous adopters. Thus, the competition effect always outweighs the effect of the growing number of instantaneous adopters anytime before the demand rate peaks. After the demand peaks, the number of instantaneous adopters declines as time passes. The likelihood that any of these adopters are influenced by Betty decreases over time because of the growing competition among previous buyers. Therefore Betty is less likely to have influencees after the demand peaks as adoption diffuses.

Let  $T_t$  be the duration between the time Betty buys  $t$  and the first time Betty influences others. Proposition 14 gives a closed-form expression for the mean time until first influence for a customer that purchases at any time  $t$  in the product life cycle.

**Proposition 14.**  $\mathbb{E}T_t = \int_0^\infty e^{-\int_t^{t+x} \lambda(s) ds} dx$

Our next proposition studies the mean duration between the time a customer adopts the product and the first time she influences others.

**Proposition 15.**  $\mathbb{E}T_t$  is increasing in  $t$ .

This result echoes those from previous sections. Late customers have lower IV because they have less time to influence others, but also because it takes longer for a later adopter

to influence others on average. A later customer is less valuable to the firm because she generates less profit (cash flow generated by herself and her influencees.) It also takes her more time before someone responds to her influence. These results supplement the customer value analysis.

## 2.2.5 Empirical Implications

In this section, we appeal to prior empirical studies on the Bass model to study the dynamics of customer LV for some specific products. Diffusion models have mainly been applied to consumer durable goods (e.g., Bass 1969, Easingwood *et al.* 1983, Gatignon *et al.* 1989, Sultan *et al.* 1990). Recently, researchers have found wide empirical evidence for the Bass model in the field of information technology innovation (e.g., Teng *et al.* 2002, Chu *et al.* 2009). So, we will examine the dynamics of customer LV in two product categories: consumer durable goods and information technology innovation.

### Consumer Durable Goods

We apply the empirical results from Bass (1969) to our LV framework with social contagion. We illustrate two examples: the diffusion of black & white television during 1946-1961 and the diffusion of the clothes dryer during 1948-1961. The coefficients of innovation ( $p$ ) and imitation ( $q$ ) are obtained empirically in Bass (1969). The annual discounting factor  $r$  and the proportion of influential credit parameter  $\delta$  are set to 0.1 and 0.3 respectively.

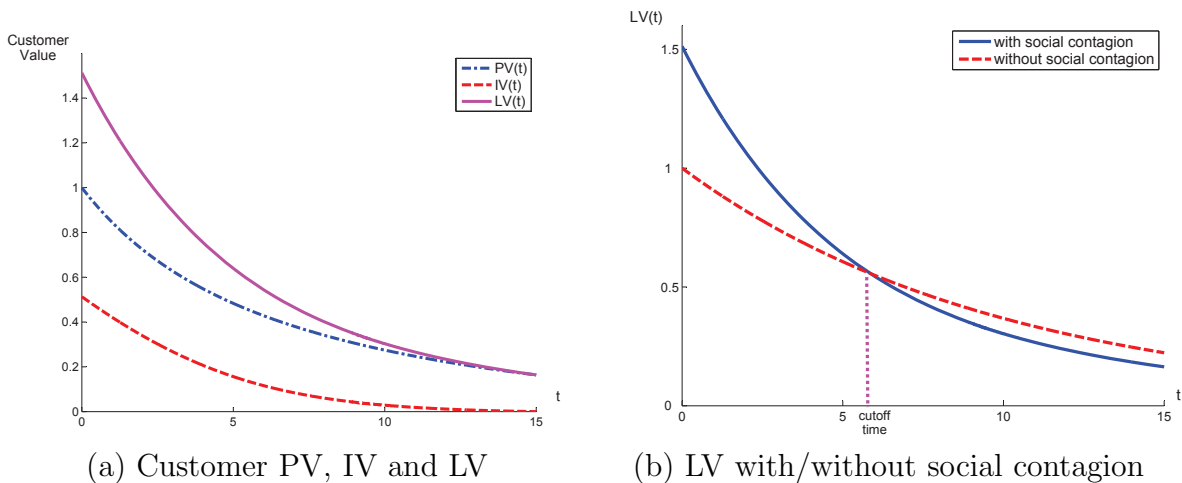


Figure 2.5: Black & White Television (1946 - 1961)  
 $(p = 0.027877, q = 0.25105, r = 0.1, \delta = 0.3)$

Figure 2.5(a) shows how customer PV, IV and LV change at different adoption times

for black & white TV during 1946-1961. We observe that for customers who bought black & white TV in 1946, 1/3 of the LV comes from influence on later buyers. Figure 2.5(b) tells us that customers who purchased before 1951 have a substantial effect on future customer acquisition through their ability to influence potential adopters. Also, their LV is higher than when social contagion is not considered.

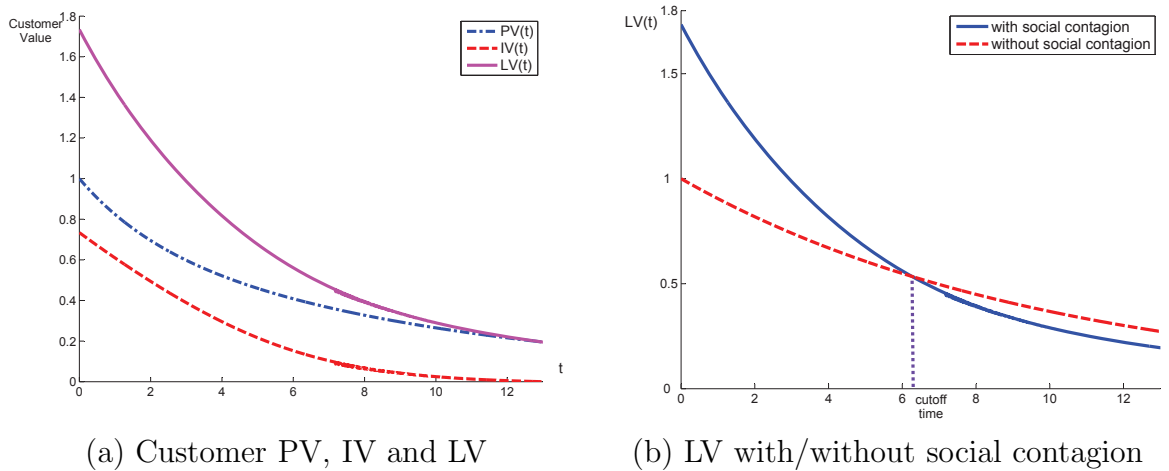


Figure 2.6: Clothes Dryer (1948 - 1961)  
 $(p = 0.017206, q = 0.35688, r = 0.1, \delta = 0.3)$

Figure 2.6 shows the customer's PV, IV and LV depend on when the purchase of a clothes dryer was made during 1948-1961. We observe that the LV of customers who adopted before 1955 is greater considering their social influence.

A comparison of Figure 2.5(a) and Figure 2.6(a) reveals that customers who bought a clothes dryer at introduction achieve higher IV than customers who adopted the black & white TV at introduction. The diffusion process of the clothes dryer has a lower coefficient of innovation and a higher coefficient of imitation compared to the diffusion coefficients of black & white TV. So, customers were subject to more social influence when adopting clothes dryers. The IV of customers who purchased at the earliest stages of the clothes dryer product life cycle is significantly higher.

## Information Technology Innovations

Nowadays, information technology innovation has become the main driving force of innovation generally. The value of customers in adoption of technology innovation is a pressing management issue. We examine empirical results of the diffusion dynamics of the mobile telephone in Taiwan (Chu *et al.*, 2009). We use our LV framework to examine the value of customers at different stages of adoption. Figure 2.7(a) plots the PV, IV and LV of mobile telephone customers in Taiwan over the past twenty years. We observe that a



customer who adopts at the earliest stage of the product life cycle has an LV 3.5 times of the profit resulting from her purchase. Her IV is so substantial that failure to include social contagion effects would lead to misallocation of scarce marketing resources during the critical early stages of a new market.

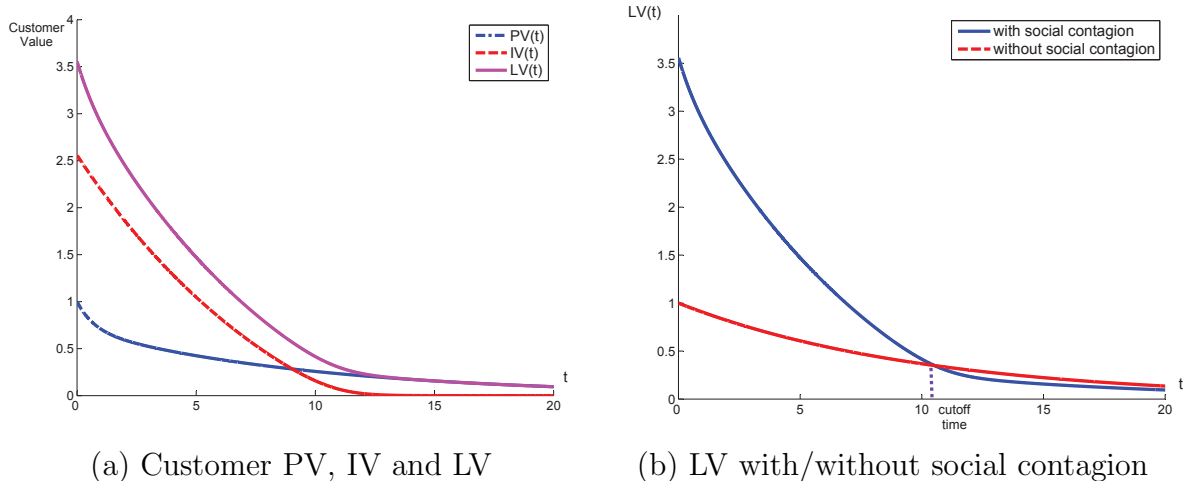


Figure 2.7: Mobile Telephone in Taiwan (1988 - 2008)  
 $(p = 8.93 \times 10^{-7}, q = 1.28, r = 0.1, \delta = 0.3)$

Figure 2.7(b) compares LV with and without social contagion. Social contagion leads to significant differences in LV. If one ignores the social influence among customers, then the value of early adopters is understated.

We apply our model based on the empirical estimation results from Teng *et al.* (2002). There, the diffusion pattern of twenty information technology innovations was examined based on a cross-sectional sample of 313 large American firms. Applying our LV framework<sup>4</sup>, we measure how much a customer contributes to the diffusion process of information technology innovations at different stages of buying. We see how much more she is actually worth considering her social influence versus the direct profit resulting from her purchase.

<sup>4</sup>The diffusion differential equation in Teng *et al.* (2002) is slightly different from the one in Bass (1969). We incorporate the demand saturation level  $m$ . So, we modify our LV calculation formula and apply the following expressions to calculate customer PV, IV and LV.

$$\begin{aligned}
 F(t) &= m \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}} \\
 PV(t) &= e^{-rt} \left( \frac{p}{p + qF(t)} + \frac{qF(t)}{p + qF(t)} (1 - \delta) \right) \\
 IV(t) &= \delta q \int_t^\infty e^{-rs} (m - F(s)) ds \\
 LV(t) &= PV(t) + IV(t)
 \end{aligned}$$

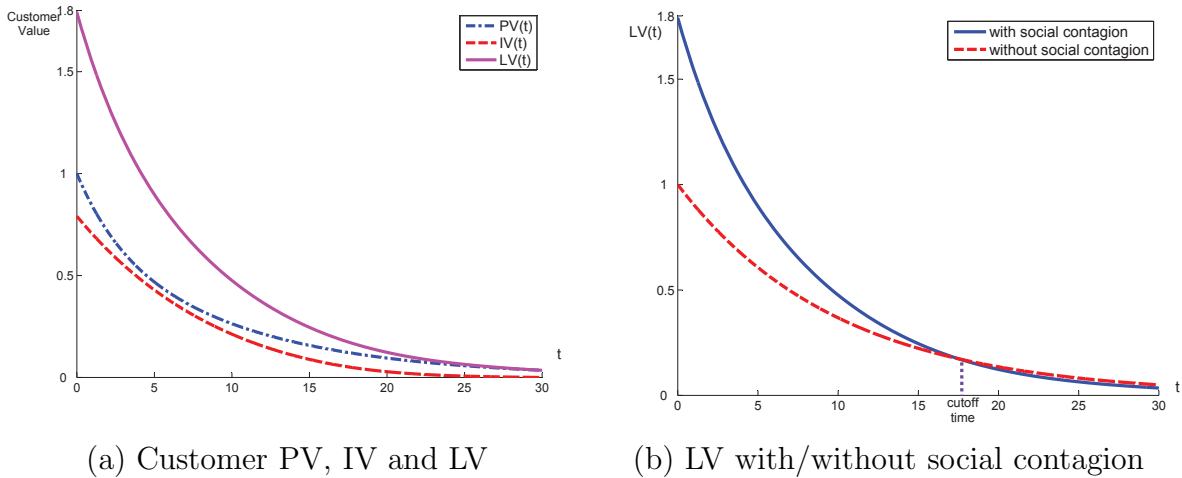


Figure 2.8: Large Sale Relational Database in U.S. (1971 - 2001)  
 $(p = 0.0006, q = 0.3074$  (saturation level = 87.39%),  $r = 0.1, \delta = 0.3)$

Figure 2.8 presents the diffusion of large sale relational databases in U.S. from 1971 to 2001. Figure 2.8(a) shows how customer PV, IV and LV all decrease as customers delay adoption. Figure 2.8(b) shows that customers who adopted large sale relational databases before 1989 are more valuable than the resulting profit. Adopters after 1989 are worth less than the present value of their own adoption.

## 2.3 Purchase Acceleration

In this section, we examine the effect of purchase acceleration on LV. The practice of giving introductory offers to accelerate purchase is widespread (e.g. Marks *et al.*, 1988, Dipak *et al.*, 1995). However, not all of these marketing activities can be easily rationalized by traditional LV models. Such practice however can be justified if one incorporates social contagion into customer value analysis.

To make the analysis tractable, we assume that every customer is equally likely to receive an introductory discount. In other words, we assume that the firm randomly samples a group of customers and offers them introductory discounts. Consequently, our uniform purchase acceleration scheme should not change the average propensity of internal and external influence on the potential population (i.e., the coefficients of innovation ( $p$ ) and imitation ( $q$ ) remain unchanged). In addition, we assume that the introductory discount is deep enough so that all potential customers are willing to adopt the product immediately.

The introductory discount will only be offered to a selected group of potential adopters at the time of product introduction. On one hand, introductory discounts stimulate more initial adoptions, and thus accelerate other potential adopters' purchase behavior. On the

other hand, introductory discounts reduce the PV of the selected customers. We will study how the firm should make the offering decisions considering these gains and losses. Let  $s$  be the unit selling price,  $c$  is the unit production cost, and  $d$  is the unit discount offered to the selected group of potential adopters whom we shall call *invited influencers*.

We begin by considering a fraction  $F_0$  of customers in the potential population. If no introductory discount is offered, the total PV of that fraction of customers is

$$(s - c) \left( \int_0^\infty PV(t; 0) f(t; 0) dt \right) F_0$$

where  $PV(t; 0)$  follows from (2.9). It denotes the normalized PV of a customer who buys at  $t$  with unit profit margin when a fraction of 0 customers are given the introductory discount at time 0 ( $F(0) = 0$ ).  $f(t; 0)$  is the demand rate at time  $t$  if  $F(0) = 0$  and follows from (3.11). Thus  $(s - c) \int_0^\infty PV(t; 0) f(t; 0) dt$  is the total PV of the entire potential population when the profit margin is  $s - c$ . The PV of the fraction  $F_0$  of customers is  $F_0$  times the total PV if the potential population is normalized to 1.

Similarly, the total IV of the fraction  $F_0$  of customers is

$$(s - c) \left( \int_0^\infty IV(t; 0) f(t; 0) dt \right) F_0$$

The total PV and IV of the remaining fraction  $1 - F_0$  of customers are  $(s - c) \left( \int_0^\infty PV(t; 0) f(t; 0) dt \right) (1 - F_0)$  and  $(s - c) \left( \int_0^\infty IV(t; 0) f(t; 0) dt \right) (1 - F_0)$  respectively.

Now suppose that at time 0, the fraction  $F_0$  of customers are selected to be invited influencers and are offered the introductory discount  $d$ . The firm earns  $s - c - d$  from each invited influencer, and  $s - c - d$  can be negative. When the introductory discount is offered at time 0, the PV of each invited influencer is the profit (or loss) she brings,  $s - c - d$ . When the population is normalized to 1, the total PV from all invited influencers (those  $F_0$  fraction of customers) is

$$(s - c - d) F_0 \tag{2.13}$$

The total IV from the fraction  $F_0$  of customers is

$$(s - c) IV(0; F_0) F_0 \tag{2.14}$$

where

$$IV(t; F_0) = \delta q \int_t^\infty e^{-rs} (1 - F(s; F_0)) ds$$

follows from (2.10) where  $F(s)$  is replaced with  $F(s; F_0)$ . Further

$$F(s; F_0) = \frac{p + qF_0 - p(1 - F_0)e^{-(p+q)s}}{p + qF_0 + q(1 - F_0)e^{-(p+q)s}} \tag{2.15}$$

is the cumulative sales solution to the Bass differential equation (3.10) with  $F(0) = F_0$ . The corresponding sales rate is

$$f(s; F_0) = \frac{(p + qF_0)(1 - F_0)(p + q)^2 e^{-(p+q)s}}{(p + qF_0 + q(1 - F_0)e^{-(p+q)s})^2} \quad (2.16)$$

The total PV of the remaining  $1 - F_0$  fraction of customers is

$$(s - c) \int_0^\infty PV(t; F_0) f(t; F_0) dt \quad (2.17)$$

where

$$PV(t; F_0) = e^{-rt} \left( \frac{p}{p + qF(t; F_0)} + \frac{qF(t; F_0)}{p + qF(t; F_0)} (1 - \delta) \right)$$

follows from (2.8) where  $F(t)$  is replaced with  $F(t; F_0)$  in (3.12). Similarly, the total IV of the non-invited customers is

$$(s - c) \int_0^\infty IV(t; F_0) f(t; F_0) dt \quad (2.18)$$

We summarize these results about PV and IV in Table 2.3.

Table 2.3: Total PV and IV with/without Introductory Discount

	Without Introductory Discount	Offer Discount to $F_0$ at time 0	
Invited Influencers	Total PV	$(s - c) \left( \int_0^\infty PV(t; 0) f(t; 0) dt \right) F_0$	$(s - c - d) F_0$
	Total IV	$(s - c) \left( \int_0^\infty IV(t; 0) f(t; 0) dt \right) F_0$	$(s - c) IV(0; F_0) F_0$
Non-Invited	Total PV	$(s - c) \left( \int_0^\infty PV(t; 0) f(t; 0) dt \right) (1 - F_0)$	$(s - c) \int_0^\infty PV(t; F_0) f(t; F_0) dt$
	Total IV	$(s - c) \left( \int_0^\infty IV(t; 0) f(t; 0) dt \right) (1 - F_0)$	$(s - c) \int_0^\infty IV(t; F_0) f(t; F_0) dt$

It is hard to see from Table 2.3 whether or not offering introductory discounts would benefit the firm. If discounts are beneficial, we want to know the optimal number of people to invite. So, we formulate the following optimization problem:

$$\begin{aligned} \pi_0^* = \max_{0 \leq F_0 \leq 1} & \quad (s - c) \int_0^\infty e^{-rt} f(t) dt + (s - c - d) F_0 \\ \text{s.t.} & \quad f(t) = (p + qF(t))(1 - F(t)) \\ & \quad F(0) = F_0 \end{aligned} \quad (2.19)$$

The objective of this program is the total profit from all customers (the summation of (2.13), (2.14), (2.17) and (2.18).) We maximize total profits subject to the Bass diffusion dynamics with fraction  $F_0$  of customers serving as invited influencers<sup>5</sup>. Proposition 16 characterizes the optimal solution.

<sup>5</sup>Dipak *et al.* (1995) studies a similar problem but does not include closed-form formulae to determine  $F_0$ .

**Proposition 16.** We let  $G(t, F_0) = \frac{(p+q)^2 e^{-(p+q+r)t}}{(p+qF_0+q(1-F_0)e^{-(p+q)t})^2}$ , then

(1) The optimal  $F_0^*$  is unique.

(2)  $F_0^* > 0$  if and only if  $\int_0^\infty G(t, 0) dt > \frac{d}{(s-c)r}$ . Otherwise,  $F_0^* = 0$ .

(3) If  $F_0^* > 0$ , then  $F_0^*$  satisfies  $\int_0^\infty G(t, F_0) dt = \frac{d}{(s-c)r}$ .

We conduct a numerical experiment to see what happens to the total PV and IV of invited influencers and non-invited customers when the optimal number of people are invited. Applying Table 2.3, the numerical results are presented in Table 2.4.

Table 2.4: Total PV and IV with/without Introductory Discount  
( $p=0.01$ ,  $q = 0.33$ ,  $r = 0.08$ ,  $s=100$ ,  $c=60$ ,  $d=60$ )

		Without Introductory Discount	Offer Discount to $F_0^*$
Invited Influencers	Total PV	0.8164	-1.2000
	Total IV	0.2819	1.3211
Non-Invited	Total PV	12.7899	15.5090
	Total IV	4.4162	4.5056
Total	Total LV	18.30	20.14

Table 2.4 reveals several insights. First, an invited influencer has a negative PV. Second, simple algebra will show that the LV of each invited influencer is always positive. Thus, the firm is trading off PV with IV. Note that each invited influencer always has a positive LV. Third, both the PV and IV of the non-invited group increase, because invited influencers accelerate product adoption within the non-invited group. Finally, the firm increases its overall customer LV through this practice. We summarize the losses and gains in the following 2-by-2 matrix: By offering introductory discounts to invited influencers, the firm loses some

		PV	IV
Invited Influencers		Loss	Gain
	Non-Invited Population	Gain	Gain

of their PV but increases their IV as well as the PV and IV from the non-invited population. In fact, the optimal  $F_0^*$  always maximizes the net gain. This result fills an important gap in the customer value analysis literature where typically only a customer's PV is used to judge whether or not to acquire a customer. Our model suggests that the acquisition of a customer should consider how this person is connected within the firm's customer social network.

Let's illustrate the impact of purchase acceleration on the value of a customer who adopts at the mean time of adoption. Use  $\hat{T}$  to denote the time when the product is adopted. We have

$$\mathbb{E}\hat{T} = \int_{t=0}^{\infty} t f(t; F_0) dt \quad (2.20)$$

$$= \frac{1}{q} \ln \frac{p+q}{p+qF_0} \quad (2.21)$$

where (2.21) follows from (2.20) by applying (3.13). Using the same parameter set as before ( $p = 0.01, q = 0.33, r = 0.08, s = 100, c = 60, d = 60$ ), we compare the PV, IV and LV at the mean time of adoption with and without purchase acceleration. Our result shows a boost in the LV of the customer who adopts at the mean adoption time if the firm adopts purchase acceleration.

Table 2.5: Comparisons of Customer Value at Mean Time of Purchase  
( $p=0.01, q = 0.33, r = 0.08, s=100, c=60, d=60$ )

	Original	Purchase Acceleration	Increase%
$F_0$	0	6.1%	–
Profit	14.64	18.90	29.1%
$E\hat{T}$	10.69	7.38	–
$PV(E\hat{T})$	12.19	15.88	30.3%
$IV(E\hat{T})$	1.91	2.21	15.8%
$LV(E\hat{T})$	14.10	18.09	28.3%

Finally, Proposition 17 explores sensitivity of the optimal number of invited influencers to changes in diffusion parameters (innovation and imitation), the level of introductory discounts offered, the profit margins and the time discount rate.

- Proposition 17.** (1)  $F_0^*$  decreases with  $p$ ;  
(2) If  $\int_0^{\infty} U(t, F_0^*) dt > 0$  ( $U(t, F_0^*)$  is defined in closed-form),  $F_0^*$  increases with  $q$ ; Otherwise,  $F_0^*$  decreases with  $q$ ;  
(3)  $F_0^*$  decreases with  $d$ .  
(4)  $F_0^*$  increases with the profit margin ( $s - c$ ).  
(5)  $F_0^*$  increases with  $r$ .

Part (1) of Proposition 17 suggests that the firm should always invite fewer influencers if the coefficient of innovation is high. In such cases, there are enough innovators to stimulate demand and any attempt to acquire more initial adopters may prove unprofitable. Part (2) gives conditions for which  $F_0^*$  is increasing (decreasing) in  $q$ . Our simulation results show that we have  $\int_0^{\infty} U(t, F_0^*) dt > 0$  for almost all products that follow Bass diffusion dynamics. Thus, the firm should invite more initial influencers when the coefficient of imitation ( $q$ ) increases. Part (3) states that fewer people should be invited if the discount is more costly. Part (4) implies that, for a given introductory discount value, the firm should invite more

influencers for a product with a higher profit margin. Finally, part (5) states that if the time discount rate is high, speeding up adoption becomes essential, and the firm should invite more influencers.

## 2.4 Purchase Deceleration

In this section, we examine the influence of purchase deceleration on LV. Due to operational constraints, customers may not be able to have their order fulfilled immediately. Consequently, they will not generate social contagion until they receive their product. This decrease in social contagion can lead to a significant drop in customer LV. In this section, we specifically examine two of such operational sources: product out-of-stock and lead time.

### 2.4.1 Product Out-of-Stock

Suppose an out-of-stock phenomenon occurs at time  $T$  and lasts for a duration  $L$ . Consequently, orders from potential adopters who purchase between  $[T, T + L)$  have to wait until  $T + L$  in order to be fulfilled. We assume all orders are backlogged.<sup>6</sup> After  $T + L$ , the demand will again follow the Bass diffusion dynamics.

The diffusion process goes through three distinct phases when an out-of-stock phenomenon occurs: 1) a pre-out-of-stock phase, 2) an out-of-stock phase, and 3) a post-out-of-stock phase. Below we provide a detailed analysis of each phase. Our main goal is to characterize the demand and sales trajectories at the three distinct phases. We use  $D(t)$  and  $S(t)$  to denote the cumulative demand and sales at  $t$  respectively, and the demand and sales rate at  $t$  are denoted by  $d(t)$  and  $s(t)$  respectively. The market size is normalized to 1.

During the pre-out-of-stock phase ( $t < T$ ), demand follows an unconstrained Bass diffusion pattern. The diffusion dynamics are described by

$$D(t) = S(t) = \frac{p(1 - e^{-(p+q)t})}{p + qe^{-(p+q)t}} \quad (t < T) \quad (2.22)$$

$$d(t) = s(t) = \frac{p(p + q)^2 e^{-(p+q)t}}{(p + qe^{-(p+q)t})^2} \quad (t < T) \quad (2.23)$$

Demand and sales are identical during this phase. This phase lasts until  $T$  when the out-of-stock period lasts for a duration  $L$ . During the out-of-stock phase ( $T \leq t < T + L$ ), those who place orders will not generate social contagion. Indeed, social contagion only comes from adopters who purchased the product before  $T$ . Thus, when  $T \leq t < T + L$ , the diffusion dynamics are

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<sup>6</sup>This is a standard assumption in the operations management literature. In practice, some customers may refuse to wait and abandon the adoption of the new product. Future research can look into generalizing the proposed model to incorporate this kind of customer loss explicitly.

$$\frac{dD(t)}{dt} = d(t) = (p + qD_1)(1 - D(t)) \quad (2.24)$$

$$s(t) = 0 \quad (2.25)$$

where

$$D_1 = \frac{p(1 - e^{-(p+q)T})}{p + qe^{-(p+q)T}}$$

is the cumulative number of adopters at time  $T$ . The solution to (2.24) subject to the initial condition  $D(T) = D_1$  is given by

$$D(t) = 1 - (1 - D_1)e^{-(p+qD_1)(t-T)} \quad (T \leq t < T + L) \quad (2.26)$$

$$d(t) = (p + qD_1)(1 - D_1)e^{-(p+qD_1)(t-T)} \quad (T \leq t < T + L) \quad (2.27)$$

The out-of-stock phase ends and all backorders are fulfilled at  $t = T + L$ . During the post-out-of-stock phase ( $t \geq T + L$ ), the diffusion continues to follow Bass diffusion dynamics as follow:

$$\begin{aligned} \frac{dD(t)}{dt} &= d(t) = (p + qD(t))(1 - D(t)) \\ S(t) &= D(t) \end{aligned} \quad (2.28)$$

The solution to (2.28) subject to the initial condition

$$D_2 = 1 - (1 - D_1)e^{-(p+qD_1)L} \quad (2.29)$$

is given by

$$D(t) = S(t) = \frac{p + qD_2 - p(1 - D_2)e^{-(p+q)(t-T-L)}}{p + qD_2 + q(1 - D_2)e^{-(p+q)(t-T-L)}} \quad (t \geq T + L) \quad (2.30)$$

$$d(t) = \frac{(p + q)^2(p + qD_2)(1 - D_2)e^{-(p+q)(t-T-L)}}{(p + qD_2 + q(1 - D_2)e^{-(p+q)(t-T-L)})^2} \quad (t \geq T + L) \quad (2.31)$$

Note that  $s(t)$  is identical to  $d(t)$  at all times except  $T + L$ , when all backorders turn into sales. The sales dynamics can be described as

$$\begin{aligned} s(T + L) &= d(T + L) + D_2 - D_1 \\ s(t) &= \frac{(p + q)^2(p + qD_2)(1 - D_2)e^{-(p+q)(t-T-L)}}{(p + qD_2 + q(1 - D_2)e^{-(p+q)(t-T-L)})^2} \quad (t > T + L) \end{aligned} \quad (2.32)$$

Formally, we characterize the demand process as follows:



**Proposition 18.** Denote by  $\tau_B = \frac{1}{p+q} \ln \frac{q}{p}$  the time of the maximum demand rate for Bass diffusion. Also, define  $\tau_1 = T + L + \frac{1}{p+q} \ln \frac{q(1-D_2)}{p+qD_2}$ ,  $d_1 = \frac{(p+q)^2}{4q}$ , and

$$\tau_2 = \left\{ t : (p+qd(t))L = \ln \frac{2q(1-d(t))}{p+q} \right\} \quad (2.33)$$

where  $d(t)$  is from (2.23).

The maximum demand rate occurs at

$$\tau_{max}^D = \begin{cases} \tau_1, & T < \tau_2; \\ \tau_B, & T > \tau_B; \\ \operatorname{argmax}\{t \mid d(t), t = T, T+L\}, & \text{o.w.} \end{cases}$$

and is equal to

$$d(\tau_{max}^D) = \begin{cases} d_1, & T < \tau_2 \text{ or } T > \tau_B; \\ \max\{d(t), t = T, T+L\}, & \text{o.w.} \end{cases}$$

It is interesting to see that if the out-of-stock phenomenon occurs after the original peak time (i.e.,  $\tau_B$ ) or the supply chain is recovered early in the product life cycle (before  $\tau_1$ ), the instantaneous peak demand rate always equals  $d_1$ . Note that the speed of diffusion during the post-out-of-stock phase is the same as the original unconstrained Bass diffusion. As a result, backorders do not influence the peak demand rate since the latter is determined only by the diffusion speed and is independent of the number of initial adopters. Note that the out-of-stock phenomenon drives the peak demand to a lower level only if it occurs close to the original peak time.

It can be shown that  $\tau_1 < \tau_B$ . Intuitively, a slowed-down social contagion takes a longer time for the demand to reach its peak even though the number of instantaneous adoptions at peak time stays the same. A larger number of initial adopters will make it faster to reach the peak, while a smaller number will make it slower.

Now we investigate the PV, IV and LV of customers at different adoption times. During the pre-out-of-stock phase, PV is identical to that in the Bass diffusion described in (2.9):

$$PV(t) = e^{-rt} \left( \frac{p}{p+qD(t)} + \frac{qD(t)}{p+qD(t)}(1-\delta) \right)$$

For a customer adopts at  $t$  ( $t < T$ ), the calculation of her IV is very complicated. To illustrate, we consider three distinct cases according to the adoption time of her influencees. We use  $IV_1(t)$  to denote her IV during the pre-out-of-stock phase,  $IV_2$  to denote her IV during the out-of-stock phase and  $IV_3$  to denote her IV during the post-out-of-stock phase. Thus, her IV can be expressed as

$$IV(t) = IV_1(t) + IV_2(t) + IV_3(t)$$

In the following, we will analyze  $IV_1(t)$ ,  $IV_2(t)$  and  $IV_3(t)$  respectively.

$IV_1(t)$  is identical to that defined in the Bass diffusion model described in (2.10).

$$IV_1(t) = \delta \int_t^T \frac{e^{-rs} q d(s)}{p + qD(s)} ds$$

Now consider her  $IV_2(t)$  during the out-of-stock phase.

$$IV_2(t) = \delta \int_T^{T+L} \frac{e^{-r(T+L)} q d(s)}{p + qD_1} ds$$

Finally,  $IV_3(t)$  is analogous to that defined in the Bass diffusion model:

$$IV_3(t) = \delta \int_{T+L}^{\infty} \frac{e^{-rs} q d(s)}{p + qD(s)} ds$$

During the out-of-stock phase, all orders during this phase are backlogged until  $T + L$ . So a customer's PV is the present value of her profit discounted by  $e^{-(T+L)}$  since her purchase won't be realized until  $T + L$ . Similarly, she will start to generate social contagion only if she has received the product. So we have

$$\begin{aligned} PV(t) &= e^{-r(T+L)} \left( \frac{p}{p + qD_1} + \frac{qD_1}{p + qD_1} (1 - \delta) \right) \\ IV(t) &= \delta \int_{T+L}^{\infty} \frac{e^{-rs} q d(s)}{p + qD(s)} ds \end{aligned}$$

For customers in the post-out-of-stock phase, the PV and IV are identical to those defined in the Bass demand dynamics.

$$\begin{aligned} PV(t) &= e^{-rt} \left( \frac{p}{p + qD(t)} + \frac{qD(t)}{p + qD(t)} (1 - \delta) \right) \\ IV(t) &= \delta \int_t^{\infty} \frac{e^{-rs} q d(s)}{p + qD(s)} ds \end{aligned}$$

And at any time,

$$LV(t) = PV(t) + IV(t)$$

In the following proposition, we state the difference of total customer LV with and without the out-of-stock phenomenon. To differentiate the diffusion dynamics in the two situations, we use  $d_{bass}(t)$  to denote the Bass diffusion dynamics (without out-of-stock phenomenon).

**Proposition 19.**  $\int_{t=0}^{\infty} LV(t) d(t) dt = \int_{t=0}^{\infty} e^{-rt} s(t) dt \leq \int_{t=0}^{\infty} e^{-rt} d_{bass}(t) dt.$

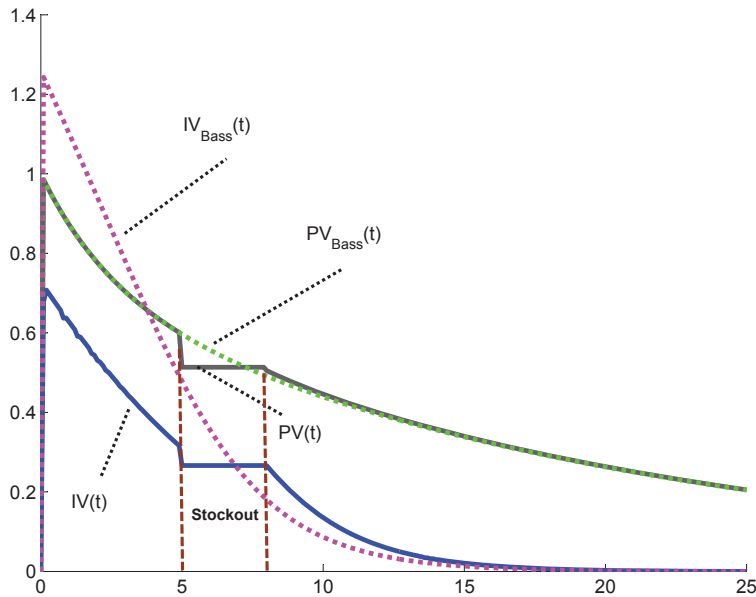


Figure 2.9: PV, IV under Product Out-of-Stock  
( $p = 0.0163221, q = 0.325044, r = 0.05, \delta = 0.3, T = 5, L = 3$ )

Proposition 19 states that the total LV of the firm's customer base is the same in our framework as it is in traditional customer LV models where the value of a customer comes only from her own purchase. It also states that the total customer LV is lower in the presence of the out-of-stock phenomenon.

Figure 2.9 presents the changes on PV and IV when there is an out-of-stock phenomenon. We label the PV and IV of the Bass model discussed in Section 2.2.2 as  $PV_{Bass}(t)$  and  $IV_{Bass}(t)$ , and use them as a benchmark to compare with the PV and IV obtained in this section, labeled as  $PV(t)$  and  $IV(t)$  respectively. Figure 2.9 shows that the IV of early adopters of the pre-out-of-stock phase drops dramatically. This is a striking result. Under social contagion, the LV of an early adopter greatly decreases if an out-of-stock phenomenon occurs.

**Proposition 20.** *For any fixed  $L$ , an out-of-stock phenomenon that happens earlier in the product life cycle always leads to a greater loss in total customer LV.*

Proposition 20 states that an unconstrained supply flow is crucial in the early stages of a product's life cycle. Put differently, if we divide the life cycle into four stages: Introduction, Growth, Maturity and Decline (Golder and Tellis 2004), an out-of-stock phenomena that occurs during the first 2 stages is the most detrimental to customer LV. Figure 2.10 shows the effect of purchase deceleration that occurs at different times across a product's life cycle.

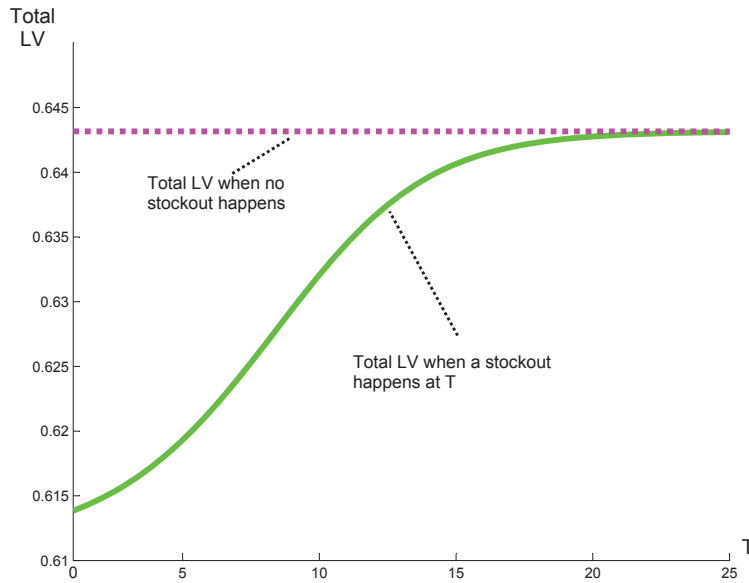


Figure 2.10: Total LV under Supply Disruption That Happens at Different Time  $T$   
( $p = 0.0163221$ ,  $q = 0.325044$ ,  $r = 0.05$ ,  $\delta = 0.3$ ,  $L = 3$ )

## 2.4.2 Lead Time

Firms have been increasingly improving their responsiveness to customers. We have seen many firms taking efforts to cut down production lead time in a make-to-order supply chain environment to reduce customer waiting time (So and Song 1998, Ho and Zheng 2004). In this section, we examine the influence of lead time on customer LV incorporating social contagion effects.

Assume the lead time exists all time, for all orders. We model the demand diffusion process with lead time using the delayed differential equation as follows.  $\bar{L}$  is denoted as the lead time. We use  $D(t)$  and  $S(t)$  to denote the cumulative demand and sales at  $t$  respectively, and use  $d(t)$  and  $s(t)$  to denote the demand and sales rate at  $t$  respectively. The market size is normalized to 1. We have

$$d(t) = (p + qS(t))(1 - D(t)) \quad (2.34)$$

$$S(t) = \begin{cases} 0 & \text{if } t < \bar{L}; \\ D(t - \bar{L}) & \text{if } t \geq \bar{L}. \end{cases} \quad (2.35)$$

$$s(t) = \begin{cases} 0 & \text{if } t < \bar{L}; \\ d(t - \bar{L}) & \text{if } t \geq \bar{L}. \end{cases}$$

At time  $t$ , only those orders placed prior to  $t - \bar{L}$  have been received. Thus only customers who have received the product, written as  $S(t)$ , can influence those who have not yet adopted. While in the Bass diffusion process, all customers who have placed the orders, regardless of whether the produce is received or not, can influence potential customers. Note that (2.34)

can be solved in a stepwise fashion. We first solve the initial value problem in the interval of  $[0, \bar{L}]$  by applying (2.35), then continue for the successive intervals by using the solution to the previous interval.

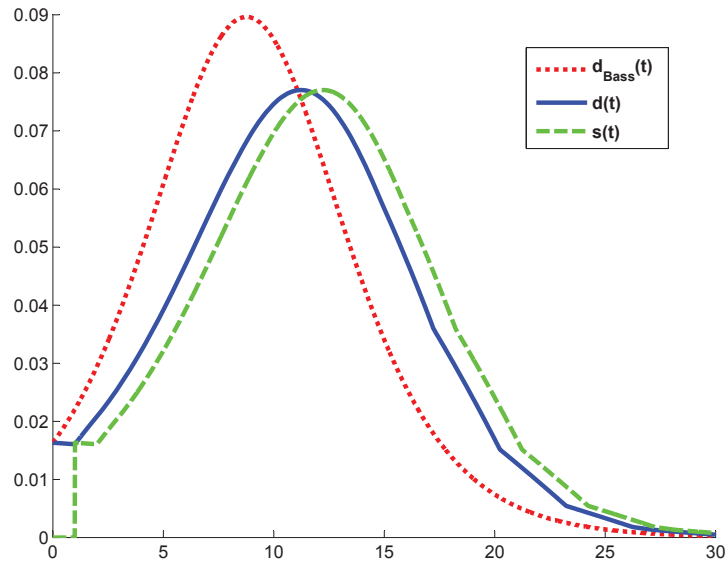


Figure 2.11: Rate of Demand and Sales with Lead Time  
 $(p = 0.0163221, q = 0.325044, r = 0.12, \delta = 0.3, \bar{L} = 1)$

Figure 2.11 shows the rate of demand and sales in the presence of lead time. We compare those with the demand rate in the Bass model, labeled as  $d_{Bass}(t)$ . We observe that the shape of  $s(t)$  is identical to that of  $d(t)$ , and  $s(t)$  can be obtained if we horizontally shift  $d(t)$  by  $\bar{L}$  to the right. Another interesting observation is that  $d(t)$  has a lower peak than  $d_{Bass}(t)$ . In the presence of lead time, customers that are waiting for the arrival of their orders do not generate social contagion, thus the social contagion delays. Even worse, the delay is successively passed (cascades) among customers. This leads to two outcomes. First, demand is slowed. Second, the peak is reduced. This is because the delayed information lessens the diffusion speed, and makes the diffusion process more spread out. Alternatively, we can see this as a smaller  $q$ . Since the peak demand rate is  $\frac{(p+q)^2}{4q}$  increases with  $q$ . A smaller  $q$  will lead to a smaller peak.

With the solutions of cumulative and instantaneous demand  $D(t)$  and  $d(t)$ , we now modify the LV formula in Section 2.2.2 considering lead time. For a customer who orders at  $t$ , her PV is given by

$$\begin{aligned} PV(t) &= e^{-rt} \left( \frac{p}{p+qS(t)} + \delta \frac{qS(t)}{p+qS(t)} \right) \\ &= e^{-rt} \frac{p + q\delta S(t)}{p + qS(t)} \end{aligned}$$

	No Lead Time	10-Day Lead Time	Decrease%	1-Month Lead Time	Decrease%
$\hat{ET}$	9.3294	9.3624	–	9.4836	–
Total PV	0.2884	0.2873	0.38%	0.2843	1.42%
Total IV	0.1371	0.1355	1.17%	0.1346	1.82%
Total LV	0.4255	0.4228	0.63%	0.4189	1.55%

Table 2.6: Comparisons of Customer Value with Different Lead Time  
( $p = 0.016322$ ,  $q = 0.325044$ ,  $r = 12\%$ )

Her IV is given by

$$\begin{aligned}
 IV(t) &= \delta \int_{t+L}^{\infty} \frac{e^{-rs} q d(s)}{p + qS(s)} ds \\
 &= \delta q \int_{t+\bar{L}}^{\infty} e^{-rs} (1 - D(s)) ds
 \end{aligned}$$

And her LV is the summation of her PV and IV:

$$LV(t) = PV(t) + IV(t)$$

To illustrate how lead time will affect customer LV for individuals at different adoption times, we use the mean value of the coefficients of innovation and imitation across 11 consumer durable products estimated in Bass (1969) ( $p_{avg} = 0.0163221$ ,  $q_{avg} = 0.325044$ ) and a 12% annual discount rate. Table 2.6 presents the comparison results of PV, IV and LV with different lead times.

We observe that even for a shipping time as short as 10 days, it will cost the firm more than 1% loss in its total customer influential value. A longer lead time will hurt the firm to a greater extent. As illustrated, a 1-month lead time implies that the firm has to give away more than 1.5% of its annual profit. Noting the essential effect of lead time on customer LV, we can conclude that even a small lead time will make a big difference.

To see how the impact of lead time varies with respect to different diffusion parameters, we classify the coefficients of innovation and imitation of 11 products estimated in Bass (1969) into 4 groups. To do so, we first classify the  $p$  values below  $p_{avg}$  into the  $p_L$  group, and then calculate the mean value of  $p$  within that group ( $p_L^{avg}$ ), which is being used to represent the  $p$  value in that  $p_L$  group. In this way, we can calculate  $p_L = 0.007019$ ,  $p_H = 0.021638$ ,  $q_L = 0.24668$  and  $q_H = 0.41908$ . Thus we identify the 2-by-2 classification as

$$\begin{aligned}
 (p_L, q_L) &= (0.007019, 0.24668); & (p_L, q_H) &= (0.007019, 0.41908); \\
 (p_H, q_L) &= (0.021638, 0.24668); & (p_H, q_H) &= (0.021638, 0.41908).
 \end{aligned}$$

Table 2.7 reports the comparison results of various kinds of customer value between no lead time versus 1-month lead time using different sets of diffusion parameters. As expected, when  $p$  is small and  $q$  is large, we observe a greater loss in all customer purchase, influential and lifetime value due to a lengthy shipping time. This is because in that case, social contagion is the main driving force for the product to take-off, and a lengthy time will

$(p_L, q_L)$	No Lead Time	1-Month Lead Time	Decrease%
Total PV	0.1746	0.1715	1.78%
Total IV	0.1022	0.1008	1.37%
Total LV	0.2768	0.2722	1.66%
$(p_L, q_H)$	No Lead Time	1-Month Lead Time	Decrease%
Total PV	0.2524	0.2465	2.34%
Total IV	0.1371	0.1335	2.63%
Total LV	0.3895	0.38	2.44%
$(p_H, q_L)$	No Lead Time	1-Month Lead Time	Decrease%
Total PV	0.2861	0.2833	0.98%
Total IV	0.1241	0.1224	1.37%
Total LV	0.4102	0.4056	1.12%
$(p_H, q_H)$	No Lead Time	1-Month Lead Time	Decrease%
Total PV	0.3515	0.3465	1.42%
Total IV	0.156	0.1527	2.12%
Total LV	0.5074	0.4992	1.62%

Table 2.7: Comparisons of Customer Value with 1-Month Lead Time under Different Diffusion Dynamics

significantly slow future purchases and thereby decrease the firm's customer assets. On the contrary, when  $p$  is large and  $q$  is small, lead time has the least impact on customer LV. This is because customers are more subject to external influence when adopting an innovation, and thus the adoption decision is not greatly affected by social contagion, even though it is still decelerated.

In this section, we have shown that a lengthy lead time decelerates social contagion, slows future purchases, and decreases total customer LV. Thus, our results suggest a new reason for firms to reduce lead time in order to accelerate social contagion and demand.

## 2.5 Conclusion

In this paper, we incorporate social contagion into customer value analysis. By doing so, we bridge the quantitative customer value analysis research with the social network research. This link is crucial because social contagion is a significant driver of customer LV, especially in the Web 2.0 world.

We build on the seminal work of Bass (1969) and derive closed-form expressions for the customer PV, IV and LV as a function of product adoption time. These formulae reveal that an early adopter is worth more than the cash flow she generates, while a late adopter is worth less than the profit she yields. We investigate how PV, IV and LV vary with the innovation and imitation parameters and illustrate the three value functions with two specific products.

In addition, we derive an expression for the mean time until first influence and show that it always takes longer for late adopters to pass on the first influence than it does for early adopters.

To show how a firm can increase its customer LV, we analyze the impact of purchase acceleration on LV. We determine the optimal number of customers to whom introductory discounts should be offered and show that the purchase acceleration can significantly improve total customer LV. We also investigate the sensitivity of the optimal size with respect to the innovation and imitation parameters, the level of introductory discount, the product profit margin, and the time discount rate. On the flip side, we also analyze the influence of purchase deceleration on LV. We show that an out-of-stock phenomenon that occurs earlier in the product life cycle always leads to a greater loss in total customer LV.

In the next chapter, we extend the model to include customer heterogeneity because potential adopters are not always equally affected by adopters (see Van den Bulte and Joshi, 2007).



## Chapter 3

# Incorporating Social Contagion into Customer Value Analysis: A Heterogeneous Population

### 3.1 Introduction

Interpersonal influence on purchase is critically important in consumer decision making and choices. More than 40% Americans actively seek the advices from families and friends when shop for doctors, lawyers or auto mechanics (Goldenberg et al. 2001). As a consequence, the value of a customer is more than the direct monetary value of her purchase, her influence on future customers through word of mouth of imitation is nonneglectable (Gremler and Brown, 1998). Recently, as the evolution of internet allows customers to overcome geographic boundaries and to communicate based on mutual interests, word-of-mouth has become a good substitute for the advertising for a product (Mayzlin, 2006). It has been acknowledged that if the firm attracts many “valuable” customers when introducing a new product, it can efficiently use social networks to increase new product awareness and adoption (e.g. Gladwell 2000). Therefore, there is a clear need that customer lifetime value should be aggregated to arrive at a strategic metric that can be useful for the firm’s strategic decision making.

There are at least two reasons why the customer lifetime value perspective has become important today. First, as discussed before, as online chat rooms and forums become more popular nowadays, social contagion starts to play a more important role than ever. Firms have set up different marketing plans to take advantages of the social network expansion, and some of them even started to establish its own social network to promote its products. For example, in the two years since Nike launched Nike+, a technology that tracks data of every run and connects runners around the world at a Web site, it has built millions of fans. The company is now using this social network to promote its basketball shoes. Second, the improvement of information technology have made it easier for firms to collect enormous

amount customer transaction data. This allows firms to use data on revealed consumer preferences. Many web-operated businesses have provided reference links for consumers to invite their friends, which makes it easier for firms to distinguish “evangelists” by collecting transaction data.

Due to the importance of social contagion in reshaping customer adoptions, we proposed a normative model in Chapter 2 for customer lifetime value, which consists the value of a customer’s own purchase (purchase value) and her value from social contagion (influence value). An important assumption in Chapter 2 is that customers are equal *ex ante*. But in real life, often times potential buyers are not *ex ante* equally affected by previous adopters, so it would be very interesting to look at the heterogeneity among consumers and analyze how that heterogeneity would affect the value redistribution among different customers. Besides, if we could take the impact of customer heterogeneity on customer lifetime value into account, a natural question is that how should businesses launch market activities to achieve more profit in the long run targeting different types of customers? This chapter aims to investigate these unknown area. Our study here will enrich the customer LV literature by highlighting the role of social contagion along the dimensions of adoption times and the intrinsic heterogeneity among customers.

To formulate the customer behavior in the adoption of new products, we draw on the literature of the Bass Model (Bass, 1969). On top of that, we consider the heterogeneity among customers in the tendency to be in tune with new developments and the tendency to influence (or be influenced by) others. In a social system, some individuals adopt an innovation independent of the decisions of others, others are influenced in the timing of adoption by the decisions of other individuals. Consistent with Bass (1969), we shall refer to the first group of customers as “innovators”, and the latter is “imitators”. At the same time, several theories and a large body of empirical research shows that some customers are more in touch with new developments than others, and often, their adoptions and options have a disproportionate influence on others’ adoptions (e.g. Gladwell 2000, Moore 1995, Rosen 2000, Slywotzky and Shapiro 1993, Katz and Lazarsfeld 1955, Rogers 2003, Weimann 1994). Considering the asymmetry in the influence process, Van den Bulte and Joshi (1997) propose a two-segment diffusion model, with discussions of five theories and frameworks that suggest the existence of *ex ante* global influencers and local influencers<sup>1</sup>. So we call customers who can reach a large amount of audience “global influencers”, and customer who get in touch with small number of people “local influencers”. To incorporate the customer heterogeneity along the two dimensions discussed above, we propose the four-segment diffusion model: type 1 customers are both innovators and global influencers, type 2 customers are both innovators and local influencers, type 3 customers are both imitators and global influencers, and type 4 customers are both imitators and local influencers. With the four types of customers, we believe that we can capture a good representation of reality.

The remainder of this chapter is organized as follows. In Section 3.2, we first set up our

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<sup>1</sup>To avoid confusion with the definition of “imitators” in Bass (1969), we label the two customer segments discussed in Van den Bulte and Joshi (2007) as “global influencers” and “local influencers” instead of “influentials” and “imitators”.

four-segment diffusion model and characterize the diffusion path. We then show our model is a generalization of the classic Bass diffusion model and the two-segment model described in Van den Bulte and Joshi (2007). In Section 3.3, we characterize the customer lifetime value in the four-segment model through a focus of each customer’s direct financial effect, as well as her influence value under social contagion. Section 3.4 shows how a firm can increase its total customer LV by purchase acceleration, in which we propose an algorithm to optimally allocate free samples among customers, which looks into the LV of marginal customers. Finally, we offer concluding remarks in Section 3.5. All proofs are presented in Appendix A.

## 3.2 Model Setup

We begin by looking at the heterogeneity among customers discussed in Bass (1969) and Van den Bulte and Joshi (2007). Bass (1969) considers customer heterogeneity in the tendency to be in tune with new products, adoptions either from self-motivated innovators, or from imitators as a result of social contagion, but the influence on imitators are assumed as homogeneous. On the other hand, Van den Bulte and Joshi (2007) study the heterogeneity in the influence process, distinguishing global influencers, whose influential powers are stronger, from local influencers, but innovators are not differentiated from imitators in their model. Figure 3.1 illustrates the customer segmentations addressed in the above two papers. To

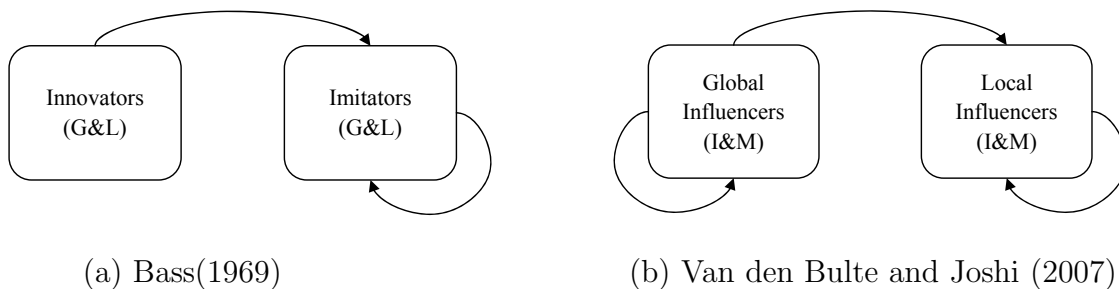


Figure 3.1: Customer Segmentation in Bass (1969) and Van den Bulte and Joshi (2007)

incorporate customer heterogeneity along the two dimensions discussed above, we consider a customer segmentation illustrated in Figure 3.2 and propose a four-segment diffusion model as follows.

The set of potential customers has a constant size  $N$  and consists of four a priori different types of consumers, and we use subscripts 1,...,4 to denote each type of customers according to the following table.

	Global Influencers	Local Influencers
Innovators	customer type 1	customer type 3
Imitators	customer type 2	customer type 4

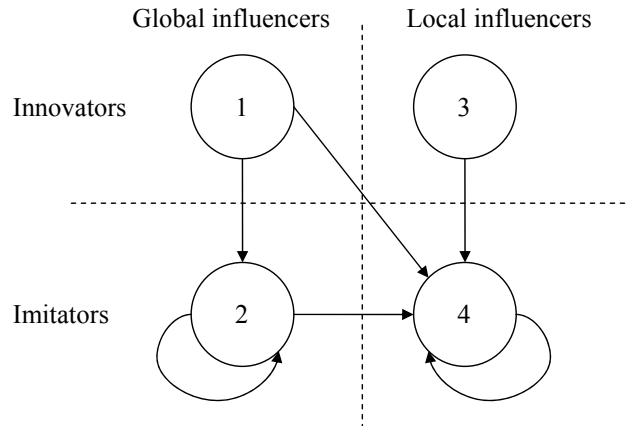


Figure 3.2: Customer Segmentation along two dimensions in Four-segment Diffusion Model

We use  $\theta_i$  to denote the proportion of type  $i$  customers in the population of eventual adopters ( $\theta_i \leq 1$  and  $\sum_{i=1}^4 \theta_i = 1$ ).  $F_i(t)$  stands for the cumulative adoption and  $f_i(t)$  the instantaneous adoption of type  $i$  customers. Following the Bass diffusion model (Bass 1969), Type  $i$  customers' demand density function is expressed as follows:

$$\text{Type 1: } f_1(t) = p_1(\theta_1 - F_1(t)) \quad (3.1)$$

$$\text{Type 2: } f_2(t) = q_1(F_1(t) + F_2(t))(\theta_2 - F_2(t)) \quad (3.2)$$

$$\text{Type 3: } f_3(t) = p_2(\theta_3 - F_3(t)) \quad (3.3)$$

$$\text{Type 4: } f_4(t) = q_2(w(F_1(t) + F_2(t)) + (1 - w)(F_3(t) + F_4(t)))(\theta_4 - F_4(t)) \quad (3.4)$$

Type 1 customers are both innovators and global influencers. They are self-motivated and the instantaneous growth rate of adopting a new product is described as a deterministic function of the innovation coefficient  $p_1$ , capturing a type 1 individual's intrinsic tendency to purchase, without being influenced by other customers.  $\theta_1 - F_1(t)$  describes the market potential of type 1 consumers at time  $t$ .

Type 2 customers are both imitators and global influencers. They will make the purchase decisions after being influenced by word of mouth of either type 1 or type 2, written as  $F_1(t) + F_2(t)$ . The imitation coefficient,  $q_1$ , captures a positive force of influence on a type 2 individual by previous adopters. As before,  $\theta_2 - F_2(t)$  describes the market potential of type 2 consumers at time  $t$ .

Type 3 customers are both innovators and local influencers. Their adoption behavior is similar to that of type 1 consumers, except that we use  $p_2$  to denote a type 3 individual's intrinsic tendency to purchase and  $\theta_3 - F_3(t)$  is the market potential of type 3 customers.

Type 4 customers are both imitators and local influencers. We use  $q_2$  to denote the imitation coefficient of type 4,  $\theta_4 - F_4(t)$  the market potential of type 4 customers. The word of mouth power influencing an individual of type 4 is described as  $w(F_1(t) + F_2(t)) +$

$(1-w)(F_3(t) + F_4(t))$ . Here,  $w$  denotes the relative importance that type 4 imitators attach to global influences' versus local influencer's behavior ( $0 \leq w \leq 1$ ) (Van den Bulte and Joshi (2007)).

We seek closed-form solution in the time domain for an innovation's diffusion path when the set of ultimate adopters consists of four types of consumers adopting according to Equations (3.1) to (3.4), as shown in Appendix A,

$$F_1(t) = \theta_1(1 - e^{-p_1 t}) \quad (3.5)$$

$$F_2(t) = \theta_2 + \exp\left(-q_1\theta_1 t - q_1\theta_2 t - \frac{q_1}{p_1}e^{-p_1 t}\theta_1\right) \quad (3.6)$$

$$\begin{aligned} & \left(\frac{q_1}{p_1}\left(\frac{q_1}{p_1}\theta_1\right)^{-\frac{q_1\theta_1+q_1\theta_2}{p_1}}\left(\Gamma\left(\frac{q_1\theta_1+q_1\theta_2}{p_1}, \frac{q_1}{p_1}\theta_1 e^{-p_1 t}\right)\right.\right. \\ & \left.\left.-\Gamma\left(\frac{q_1\theta_1+q_1\theta_2}{p_1}, \frac{q_1}{p_1}\theta_1\right)\right) - \frac{\exp\left(-\frac{q_1}{p_1}\theta_1\right)}{\theta_2}\right)^{-1} \\ F_3(t) = & \theta_3(1 - e^{-p_2 t}) \quad (3.7) \end{aligned}$$

$$F_4(t) = \theta_4 + \frac{R(t)}{q_2(1-w)\int_0^t R(s) ds - \frac{1}{\theta_4}} \quad (3.8)$$

where in (3.6)  $\Gamma(\eta, k)$  refers to the "upper" incomplete gamma function, that is,  $\Gamma(\eta, k) = \int_k^\infty v^{\eta-1} e^{-v} dv$ , and in (3.8)  $R(t) = \exp\left(\int_0^t -q_2(wF_1(s) + wF_2(s) + (1-w)F_3(s) + (1-w)\theta_4) ds\right)$  where  $F_1(s)$ ,  $F_2(s)$ ,  $F_3(s)$  can be obtained from Equations (3.5) to (3.7).

The expressions of (3.5) to (3.8) are plotted in Figure 3.3(a). We are interested in how our four-segment diffusion model is related to Bass (1969) and Van den Bulte and Joshi (2007) mathematically, and we present the comparison results as follows.

### Comparison with Bass (1969)

Consider the four-segment diffusion model with  $\theta_3 = \theta_4 = 0$ , in that case, only type 1 and type 2 customers exist in the market. Recall from (3.1) and (3.2) that the instantaneous adoption functions of type 1, 2 customers are :

Type 1:  $f_1(t) = p_1(\theta_1 - F_1(t))$

Type 2:  $f_2(t) = q_1(F_1(t) + F_2(t))(\theta_2 - F_2(t))$ , ( $\theta_1 + \theta_2 = 1$ )

Thus, the instantaneous adoption of the entire market is:

$$f_1(t) + f_2(t) = p_1(\theta_1 - F_1(t)) + q_1(F_1(t) + F_2(t))(\theta_2 - F_2(t)) \quad (3.9)$$

The instantaneous adoption in the Bass model (the mixed influence model) is:

$$f(t)/[1 - F(t)] = p + qF(t) \quad (3.10)$$

The solution to (3.10) is

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + q/pe^{-(p+q)t}} \quad (3.11)$$

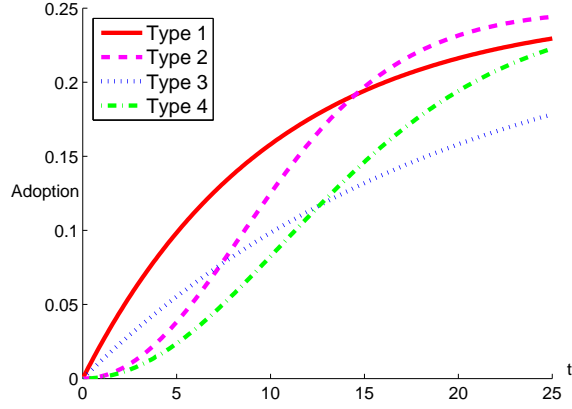
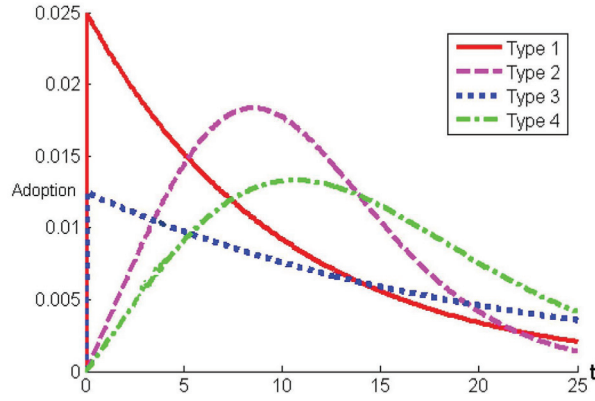
(1)  $F(t)$ (1)  $f(t)$ 

Figure 3.3: Cumulative Sale and Sales Rate of Different Customer Types  
 $(p_1 = 0.1, p_2 = 0.05, q_1 = 0.4, q_2 = 0.25, \theta_1 = \theta_2 = \theta_3 = \theta_4 = 0.25, \tau = 25, \text{ and } \delta = 0.3)$

To see how the Bass model relates to ours, we rewrite equation (3.10) into

$$f(t) = p(1 - F(t)) + qF(t)(1 - F(t)) \quad (3.12)$$

Substituting  $F(t) = F_1(t) + F_2(t)$  into (3.12), we have

$$f(t) = p(1 - F_1(t) - F_2(t)) + q(F_1(t) + F_2(t))(1 - F_1(t) - F_2(t)) \quad (3.13)$$

Comparing (3.13) with (3.9), we find that

$$p(t) = \frac{(\theta_1 - F_1(t))p_1}{1 - F_1(t) - F_2(t)} = \frac{p_1}{1 + \frac{\theta_2 - F_2(t)}{\theta_1 - F_1(t)}} \quad (3.14)$$

$$q(t) = \frac{(\theta_2 - F_2(t))q_1}{1 - F_1(t) - F_2(t)} = \frac{q_1}{1 + \frac{\theta_1 - F_1(t)}{\theta_2 - F_2(t)}} \quad (3.15)$$

Equations (3.14) and (3.15) show that our 2-type mixture model generalizes the mixed-influenced model by relaxing the constant restriction of the innovative and imitative coefficients and allowing them to be time dependent (Karmeshu and Goswami, 2001), so we shall use  $p(t)$  and  $q(t)$  to characterize their time dependent nature. Figure 3.4 shows how the innovative and imitative coefficients vary with respect to  $t$ . We note that the shape of  $p(t)$

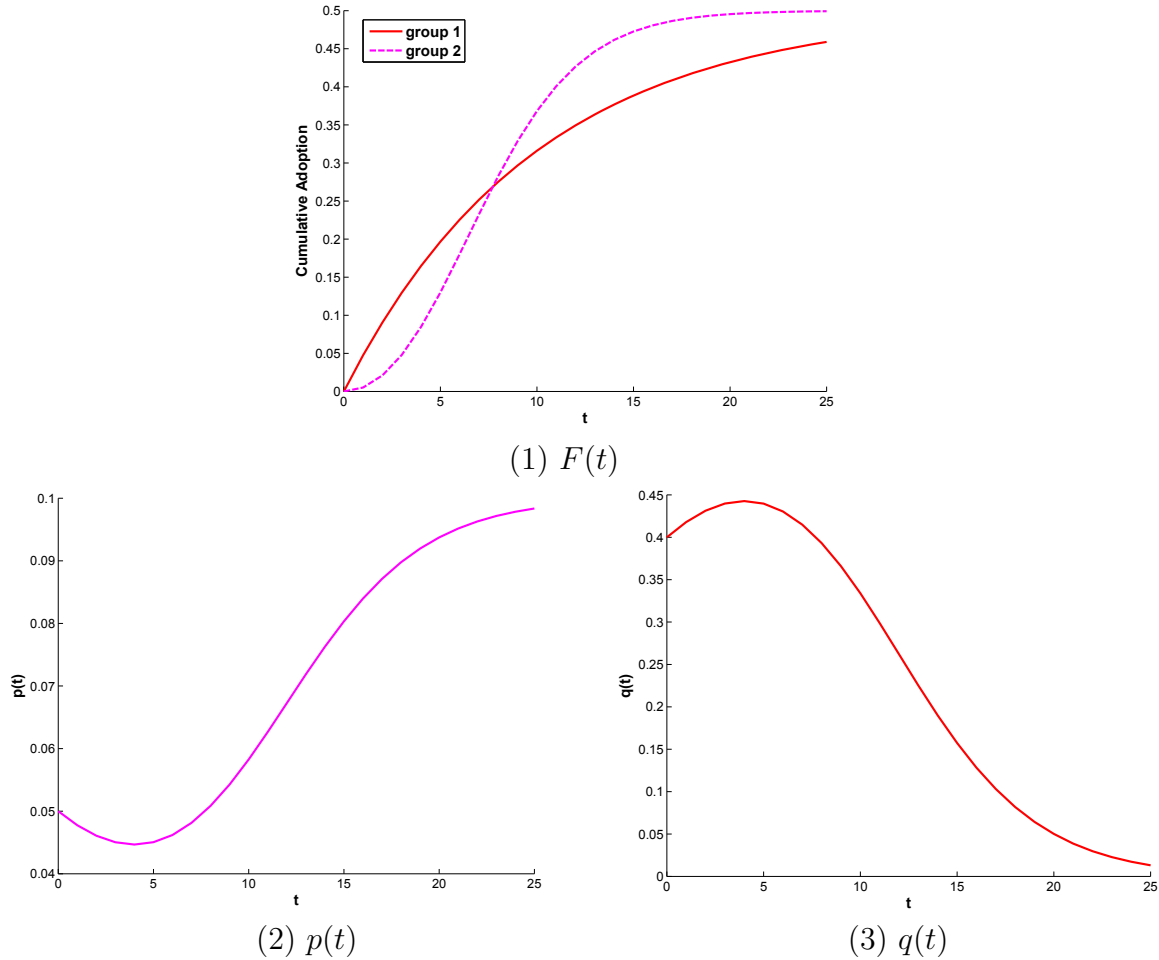


Figure 3.4: Curves of  $F_1(t)$ ,  $F_2(t)$ ,  $p(t)$  and  $q(t)$   
 $(p = 0.1, q = 0.4, \theta_1 = \theta_2 = 0.5, \tau = 25)$

and  $q(t)$  curves are parameter dependent. So the trend of the curve changes dramatically with different parameter sets. For example, we will get the curves with reversed trend at  $p = 0.1, q = 0.2, \theta_1 = 0.2, \theta_2 = 0.8, \tau = 25$  comparing with the ones illustrated in Figure 3.4.

### Comparison with Van den Bulte and Joshi (2007)

Van den Bulte and Joshi (2007) consider a 2-type mixture process with the 2 customer segments: global influencers and local influencers. We show how the four-segment diffusion model is related to theirs by comparing the model structure in Figure 3.1(b) and Figure 3.2.

It is clear that without distinguishing type 1 from type 2 customers, type 3 from type 4 customers, our model is reduced to Van den Bulte and Joshi (2007). To see this, we let  $\theta = \theta_1 + \theta_2$ , then  $1 - \theta = \theta_3 + \theta_4$ . We also define  $\hat{F}_1(t) = F_1(t) + F_2(t)$ , as well as  $\hat{F}_2(t) = F_3(t) + F_4(t)$ . Then our model becomes

$$\begin{aligned} \text{Type 1 \& 2 :} \quad & \hat{f}_1(t) = (p_1 + q_1 \hat{F}_1(t))(\theta - \hat{F}_1(t)) \\ \text{Type 3 \& 4 :} \quad & \hat{f}_2(t) = (p_2 + q_2(w\hat{F}_1(t) + (1-w)\hat{F}_2(t)))(1 - \theta - \hat{F}_2(t)) \end{aligned}$$

It follows that,

$$\begin{aligned} \hat{f}(t) &= \hat{f}_1(t) + \hat{f}_2(t) \\ &= (p_1 + q_1 \hat{F}_1(t))(\theta - \hat{F}_1(t)) + (p_2 + q_2(w\hat{F}_1(t) + (1-w)\hat{F}_2(t)))(1 - \theta - \hat{F}_2(t)) \end{aligned} \quad (3.16)$$

In the same time, the mixed adoption density in Van den Bulte and Joshi (2007) is written as,

$$\begin{aligned} f_m(t) &= \theta f_1(t) + (1 - \theta)f_2(t) \\ &= \theta(p_1 + q_1 F_1(t))(1 - F_1(t)) + (1 - \theta)(p_2 + q_2(wF_1(t) + (1-w)F_2(t)))(1 - F_2(t)) \end{aligned} \quad (3.17)$$

Comparing (3.16) with (3.17), it is not hard to see that  $\hat{F}_1(t) = \theta F_1(t)$  and  $\hat{F}_2(t) = (1 - \theta)F_2(t)$ . By expressing the fraction of customers not having adopted yet as  $\theta - \hat{F}_1(t)$  and  $(1 - \theta) - \hat{F}_2(t)$ , instead of  $1 - F_1(t)$  and  $1 - F_2(t)$ , our model resolves the misinterpretation in Van den Bulte and Joshi (2007), in which the sizes of each segment are ignored.

### 3.3 Customer Lifetime Value

In this section, we examine customer PV, IV and LV in the four-segment framework. Consider an individual of type 1, say Jimmy, adopting at  $t$  without having been influenced by anyone else, his purchase value is the present value of the profit (assuming the profit margin is 1) he generates discounted with rate  $r$ , so his PV equals to  $e^{-rt}$ . To calculate his IV, we first notice that he will influence customers of type 2 and type 4 whose purchases happen during  $[t, \tau]$ . For each customer he brings in, he earns  $\delta$  fraction of the present value of that purchase. Suppose a type 2 customer, Susan, purchases at time  $s$ ,  $t < s \leq \tau$ . The probability that she was influenced by Jimmy is  $\frac{1/N}{F_1(s) + F_2(s)}$  ( $N$  is the total potential customers), as every type 1 or type 2 customer who bought before  $s$  is equally likely to influence her. Furthermore, there are  $f_2(s)N$  independent type 2 buyers at  $s$ , so the number of Jimmy's influencees follows a Binomial distribution with parameters  $f_2(s)N$  and  $\frac{1/N}{F_1(s) + F_2(s)}$ . So the average number of Jimmy's type 2 influencees is  $\int_t^\tau \frac{1}{F_1(s) + F_2(s)} f_2(s) ds$ . Similarly,  $\int_t^\tau \frac{w}{w(F_1(s) + F_2(s)) + (1-w)(F_3(s) + F_4(s))} f_4(s) ds$  gives us Jimmy's type 4 influencees at



time  $s$ . Substituting  $f_i(t), i = 1, \dots, 4$  from (3.1) to (3.4), we have that,

$$IV_1(t) = IV_2(t) = q_1\delta \int_t^\tau e^{-rs}(\theta_2 - F_2(s)) ds + q_2\delta w \int_t^\tau e^{-rs}(\theta_4 - F_4(s)) ds \quad (3.18)$$

$$IV_3(t) = IV_4(t) = q_2\delta(1-w) \int_t^\tau e^{-rs}(\theta_4 - F_4(s)) ds \quad (3.19)$$

$$PV_1(t) = PV_3(t) = e^{-rt} \quad (3.20)$$

$$PV_2(t) = PV_4(t) = (1-\delta)e^{-rt} \quad (3.21)$$

The LV of each customer is the sum of her IV and PV, so  $LV_i(t) = PV_i(t) + IV_i(t), i = 1, \dots, 4, \forall t$ .

$$\begin{aligned} LV_1(t) &= IV_1(t) + PV_1(t) \\ &= q_1\delta \int_t^\tau e^{-rs}(\theta_2 - F_2(s)) ds + q_2\delta w \int_t^\tau e^{-rs}(\theta_4 - F_4(s)) ds + e^{-rt} \end{aligned} \quad (3.22)$$

$$\begin{aligned} LV_2(t) &= IV_2(t) + PV_2(t) \\ &= q_1\delta \int_t^\tau e^{-rs}(\theta_2 - F_2(s)) ds + q_2\delta w \int_t^\tau e^{-rs}(\theta_4 - F_4(s)) ds + (1-\delta)e^{-rt} \end{aligned} \quad (3.23)$$

$$\begin{aligned} LV_3(t) &= IV_3(t) + PV_3(t) \\ &= q_2\delta(1-w) \int_t^\tau e^{-rs}(\theta_4 - F_4(s)) ds + e^{-rt} \end{aligned} \quad (3.24)$$

$$\begin{aligned} LV_4(t) &= IV_4(t) + PV_4(t) \\ &= q_2\delta(1-w) \int_t^\tau e^{-rs}(\theta_4 - F_4(s)) ds + (1-\delta)e^{-rt} \end{aligned} \quad (3.25)$$

Figure 3.5 depicts the customer IV and LV curves. We observe that all values of all types decrease in a convex manner with adoption times.

**Proposition 21.**  $\sum_{i=1}^4 \int_{t=0}^\tau LV_i(t) f_i(t) dt = \sum_{i=1}^4 \int_{t=0}^\tau e^{-rt} f_i(t) dt$ .

Proposition 21 states that the total LV of the firm's customer base is identical with and without social contagion. This result echoes to the one we obtained in Chapter 2. So our methodology can be seen as a new accounting matrix, enabling the firm to directly capture the value of interpersonal influence among customers into customer lifetime value matrix.

**Proposition 22.**  $LV_i(t) i = 1, \dots, 4$  decreases with  $p_1$ ;  $LV_3(t)$  ( $IV_4(t)$ ) decreases with  $p_2$ .

All other things being equal, Proposition 22 tells us that at any given time, stronger external influence of the global influencers decreases the LV of all type customers making purchases. With stronger external influence, customers are more likely to be innovators who earn all credits from their own purchases. Similarly, stronger external influence of the local influencers decreases the LV of type 3 and type 4 customers making purchases.

**Proposition 23.**  $PV_i(t), IV_i(t)$  and  $LV_i(t) i = 1, \dots, 4$  are all decreasing convex in  $t$ .

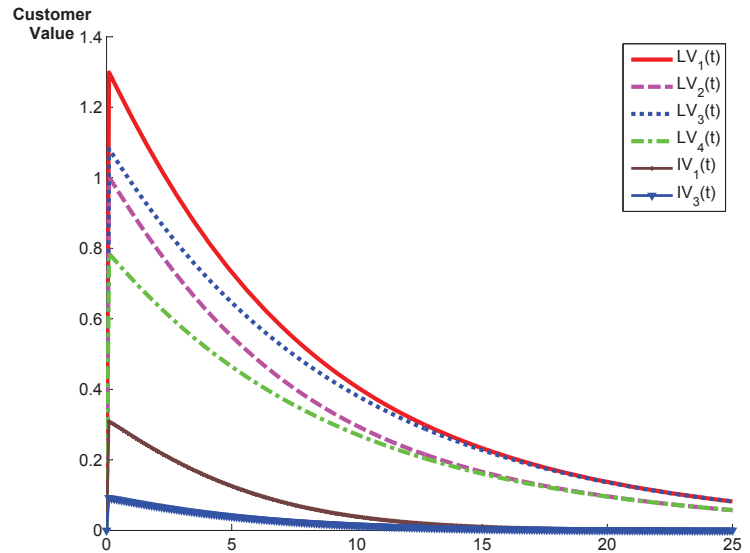


Figure 3.5: PV, IV and LV of Each Type's Customers

( $p_1 = 0.1, p_2 = 0.05, q_1 = 0.5, q_2 = 0.4, \theta_1 = \theta_2 = \theta_3 = \theta_4 = 0.25, \tau = 25, w = 0.5, \delta = 0.3$  and  $r = 0.1$ )

When a customer delays a purchase, her PV reduces due to a heavier discount, and her IV decreases because the pool of potential adopters shrinks. Hence, a customer's value goes down rapidly as she waits to buy. Proposition 23 states that the rate of decrease is smaller as adoption diffuses.

**Proposition 24.** *We have*

- 1)  $LV_1(t) > LV_2(t), LV_3(t) > LV_4(t), \forall t \in [0, \tau]$ .
- 2) If  $w \geq 1/2$ ,  $LV_1(t) \geq LV_3(t), LV_2(t) \geq LV_4(t), \forall t \in [0, \tau]$ .
- 3) **a.** If  $w \geq 1/2$ , there exists a cutoff time  $\hat{t}$ , s.t.,  $\forall t > \hat{t}, LV_3(t) > LV_2(t)$ .
- b.**  $\hat{t}$  decreases with  $p_1$  and  $p_2$ .

Proposition 24 states the relations among the LV of customer of different types. Part (1) tells us that at any purchase times, a type 1 customer is always more valuable than her type 2 peers, regardless of the behavior of product diffusion processes. At the same time, a type 3 customer is always worth more than her type 4 peers. Part (2) compares the LV of a type 1(2) customer with the LV of a type 3(4) customer who purchases at the same time. Provided that global influencers have more influential power on type 4 customers than local influencers, type 1(2) customers have greater LV than type 3(4) customers. Finally, part (3) tells us that if global influencers have more influential power on type 4 customers than local influencers, at any purchase times prior to a cutoff time, a type 3 customer is less valuable than a type 2 customer. All type 3 purchases after the cutoff time are worth more than the type 2 purchases. Moreover, the cutoff time is decreasing in both  $p_1$  and  $p_2$ .

Figure 3.6 shows the ratio between a customer's IV over her PV as a function of the

adoption times. Customers of type 1 or type 2, who buy at the same time, have identical IV, but type 1 customers always have a higher PV than their type 2 peers. So when looking at the ratio of IV over PV, type 1 customers have a lower IV/PV ratio than type 2 customer. Similar arguments can be applied to type 3 and type 4 customers. Type 3 customers always have a lower IV/PV ratio than their type 4 peers.

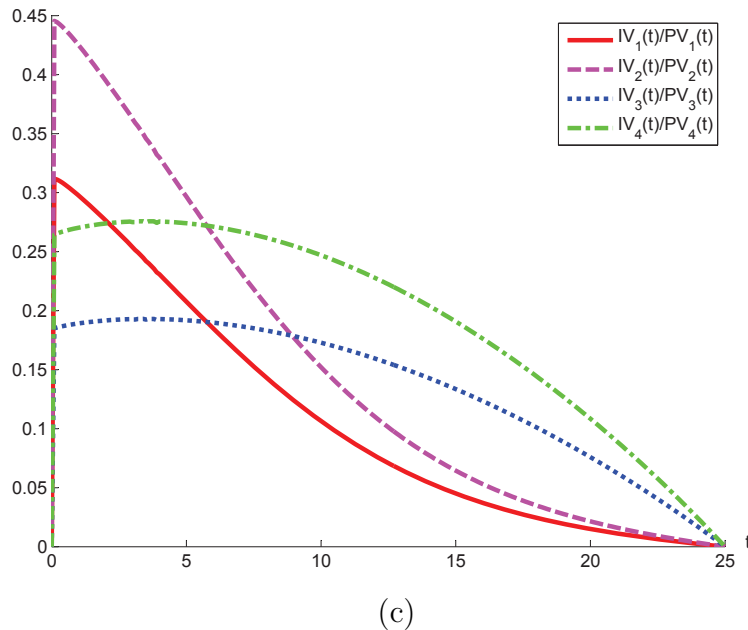


Figure 3.6:  $IV(t)/PV(t)$  of Different Customer Types

( $p_1 = 0.1, p_2 = 0.05, q_1 = 0.5, q_2 = 0.4, \theta_1 = \theta_2 = \theta_3 = \theta_4 = 0.25, \tau = 25, w = 0.5, \delta = 0.3$  and  $r = 0.1$ )

### 3.4 Purchase Acceleration

So far, the firm's marketing actions has been exogenously specified. Now, suppose the firm can take marketing actions to affect the diffusion path by sending free samples to customers at time 0, we are interested in exploring how the firm should make the sampling decision so as to accelerate purchase. We have studied the sample optimization problem in the Bass model in Chapter 3, and now let's draw attention to the four-segment model.

Specifically, we address two questions in this section: 1) What is the optimal sample size for the firm to balance the gains and losses from such practices? 2) Given the optimal sample size, how should the firm distribute samples among the four types of customers? To

answer these questions, we recap the diffusion density expressions from (3.1) to (3.4).

$$\text{Type 1: } f_1(t) = p_1(\theta_1 - F_1(t))$$

$$\text{Type 2: } f_2(t) = q_1(F_1(t) + F_2(t))(\theta_2 - F_2(t))$$

$$\text{Type 3: } f_3(t) = p_2(\theta_3 - F_3(t))$$

$$\text{Type 4: } f_4(t) = q_2(w(F_1(t) + F_2(t)) + (1 - w)(F_3(t) + F_4(t)))(\theta_4 - F_4(t))$$

We first look at the situation of sending free samples to type 1 customers. The distribution of free samples reduces the number of paid purchases from type 1 customers, which can be considered as a negative impact on the profit from type 1 consumers. On the other hand, sending free samples to type 1 customers accelerates the adoption of type 1 customers, so  $F_1(t)$  is greater than in the case without samples. As a consequence, a larger  $F_1(t)$  accelerates the purchase penetration rates of type 2 and type 4 customers as stated in Equations (3.2) and (3.4), so sending free samples to type 1 customers has a positive acceleration impact on type 2 and type 4 customers. In Table 3.1 we mark this with a positive sign.

When free samples are offered to type 2 consumers, the effects are two-fold. On one hand, the size of the remaining type 2 customers is reduced. On the other hand, with more earlier adopters, the diffusion rate among type 2 consumers speeds up. The consequence of the two contrary forces is regarded as a tradeoff between acceleration and cannibalization. So the total impact of a free sample promotion on sales may be positive or negative, depending on the strength of those two forces over time. At the same time, its effect on type 4 consumers is positive, followed by the same argument as of type 1 consumers.

Similar decision rules can be applied to the cases when free samples are offered to type 3 or type 4 consumers. We collect the analysis results in Table 3.1, where “-” indicates negative effects, “+” positive, “0” unchanged, and “?” undetermined.

Table 3.1: The Impact of Offering Samples to Type  $i$  Customers on Type  $j$  Customers

		Impact on Customer Type $j$			
		Type 1	Type 2	Type 3	Type 4
Offer Samples to Type $i$	Type 1	-	+	0	+
	Type 2	0	?	0	+
	Type 3	0	0	-	+
	Type 4	0	0	0	?

By obliterations from Equations (3.1) to (3.4), one immediate proposition is stated as follows:

**Proposition 25.** *If  $\theta_1 = \theta_3$ ,  $p_1 = p_2$ , it's preferable to distribute gifts to type 1 customers than to type 3 customers. If  $\theta_2 = \theta_4$ ,  $q_1 = q_2$ ,  $w \geq 1/2$ , it's preferable to distribute gifts to type 2 customers than to type 4 customers.*

The above proposition requires restrictive conditions. To generate more rigorous results with less restrictions, we approach this problem by looking at the LV of a marginal consumer. Specifically, the firm would have the incentive to offer a free sample to a customer if the marginal benefit of doing so exceeds the marginal cost. Assuming that the product has a marginal profit  $s$  and a marginal cost  $c$ . We propose the following algorithm to optimally allocate free samples among customers of different types in the steps outlined below:

*Step 1:* Denote  $F_0^i$  as the number of free samples that are distributed to type  $i$  customers at time 0. Initialize  $F_0^1 = F_0^2 = F_0^3 = F_0^4 = 0$ .

*Step 2:* Let

$$\Delta W_1 = IV_1(0; F_0^1 + 1, F_0^2, F_0^3, F_0^4) - \int_0^\tau LV_1(t; F_0^1, F_0^2, F_0^3, F_0^4) f_1(t; \cdot) dt \quad (3.26)$$

$$\Delta W_2 = IV_2(0; F_0^1, F_0^2 + 1, F_0^3, F_0^4) - \int_0^\tau LV_2(t; F_0^1, F_0^2, F_0^3, F_0^4) f_2(t; \cdot) dt \quad (3.27)$$

$$\Delta W_3 = IV_3(0; F_0^1, F_0^2, F_0^3 + 1, F_0^4) - \int_0^\tau LV_3(t; F_0^1, F_0^2, F_0^3, F_0^4) f_3(t; \cdot) dt \quad (3.28)$$

$$\Delta W_4 = IV_4(0; F_0^1, F_0^2, F_0^3, F_0^4 + 1) - \int_0^\tau LV_4(t; F_0^1, F_0^2, F_0^3, F_0^4) f_4(t; \cdot) dt \quad (3.29)$$

where  $(\cdot)$  stands for the argument of  $F_0^1, F_0^2, F_0^3, F_0^4$

$$IV_1(t; \cdot) = IV_2(t; \cdot) = q_1 \delta \int_t^\tau e^{-rs} (\theta_2 - F_2(s; \cdot)) ds + q_2 \delta w \int_t^\tau e^{-rs} (\theta_4 - F_4(s; \cdot)) ds \quad (3.30)$$

$$IV_3(t; \cdot) = IV_4(t; \cdot) = q_2 \delta (1 - w) \int_t^\tau e^{-rs} (\theta_4 - F_4(s; \cdot)) ds \quad (3.31)$$

and

$$LV_1(t; \cdot) = IV_1(t; \cdot) + e^{-rt} \quad (3.32)$$

$$LV_2(t; \cdot) = IV_2(t; \cdot) + (1 - \delta) e^{-rt} \quad (3.33)$$

$$LV_3(t; \cdot) = IV_3(t; \cdot) + e^{-rt} \quad (3.34)$$

$$LV_4(t; \cdot) = IV_4(t; \cdot) + (1 - \delta) e^{-rt} \quad (3.35)$$

Equations (3.30) to (3.35) are obtained from (3.22) and (3.25) by replacing  $F_i(t)$  with  $F_i(t; F_0^1, F_0^2, F_0^3, F_0^4)$ , whose closed-form expressions are presented in Appendix A.

*Step 3:* If  $\Delta W_i \leq \frac{c}{s}$ ,  $\forall i$ , stop. Otherwise, go to *Step 4*.

*Step 4:* Let  $\bar{i} = \operatorname{argmax}\{\Delta W_i, i = 1, 2, 3, 4\}$ ,  $F_0^{\bar{i}} \leftarrow F_0^{\bar{i}} + 1$ . Go to *Step 2*.

To see why our algorithm gives the optimal sample distribution levels of different customer types, we demonstrate this by illustrating one iteration. Suppose currently the firm plans to send  $F_0^i$  free samples to customers of type  $i$  ( $i = 1, \dots, 4$ ), the firm would like to send one more sample to a type 1 customer if the benefit of doing so exceeds the cost. The benefit to the firm is the IV afterwards from that type 1 customer who receives that sample, stated as  $sIV_1(0; F_0^1 + 1, F_0^2, F_0^3, F_0^4)$ . The cost for the firm comprises two parts: the marginal cost of the sample and the expected LV of that type 1 customer who receives that sample, written as

$$s \int_0^\tau LV_1(t; F_0^1, F_0^2, F_0^3, F_0^4) f_1(t, \cdot) dt + c$$

Cluster Parameters	$p_1$	$q_1$	$p_2$	$q_2$	$w$	$\theta$	$\theta_1$	$\theta_2$	$\theta_3$	$\theta_4$
Cluster 1 (K=2)	0.181	1.907	0.009	0.195	1.463	0.252	0.05	0.20	0.15	0.60
Cluster 2 (K=2)	0.038	0.386	0.013	1.596	0.166	0.730	0.15	0.58	0.05	0.22
Cluster 1 (K=3)	0.038	0.386	0.013	1.596	0.166	0.730	0.15	0.58	0.05	0.22
Cluster 2 (K=3)	0.092	7.431	0.000	0.051	1.000	0.147	0.03	0.12	0.17	0.68
Cluster 3 (K=3)	0.198	0.802	0.011	0.224	0.355	0.273	0.05	0.22	0.15	0.58

Table 3.2: Cluster Results

This is because if that customer did not receive the sample, she might pay for product at time  $t$  with probability  $f_1(t)$ . So by pushing her adoption to time 0, the firm loses both her PV and IV resulted from the paid purchase. Since the paid purchase might happen at anytime during the product life cycle, we use the expected LV to represent the loss from sending her the sample at time 0.

Consequently, given current sample distribution level  $F_0^1, F_0^2, F_0^3, F_0^4$ , the total impact of sending one more sample to a type 1 customer is

$$s \left( IV_1(0; F_0^1 + 1, F_0^2, F_0^3, F_0^4) - \int_0^\tau LV_1(t; F_0^1, F_0^2, F_0^3, F_0^4) f_1(t; \cdot) dt \right) - c$$

Similar expressions for the total impact hold if the firm chooses to send the  $(F_0^1 + F_0^2 + F_0^3 + F_0^4 + 1)$ th sample gives to a customer of type 2, 3, or 4. Therefore, the firm should compare the outcome of the four potential choices and then choose to send the  $(F_0^1 + F_0^2 + F_0^3 + F_0^4 + 1)$ th sample to the customer segment that has the maximum positive impact. Simulation results show that the term in the parentheses is generally very small. It tells us giving free samples can be an effective market strategy only if marginal cost is remarkably less than marginal profit. This result feature explains why detailing is more widely applied in pharmaceutical industry (where marginal profit is much higher than marginal cost) than elsewhere.

From the above algorithm, we can obtain some properties about sample distribution in special cases, described in the two propositions below.

**Proposition 26.** *If  $w = 1$ , the firm should never send samples to type 3 or type 4 customers.*

**Proposition 27.** *If  $w = 0, q_1 = 0$ , the firm should never send samples to type 2 customers.*

To illustrate how the sample should be distributed for real-world products, we use the empirical estimation of new product diffusion processes from Van den Bulte and Joshi (2007), in which 34 data series are analyzed, including the diffusion of the broad-spectrum antibiotic tetracycline, music CDs and high-technology products. Rather than examine the data series individually, we classify the 34 sets of results into K clusters and look at the cluster mean. Table 3.2 summarizes the empirical estimation results of  $p_1, q_1, p_2, q_2, w$  and  $\theta$  of cluster mean with K=2 and K=3, where  $\theta_i$  ( $i = 1, \dots, 4$ ) are determined according to  $\theta_1 = 1/4\theta_2, \theta_3 = 1/4\theta_4, \theta_1 + \theta_2 = \theta, \sum_{i=1}^4 \theta_i = 1$ .

Sample	K=2		K=3		
	Cluster 1	Cluster 2	Cluster 1	Cluster 2	Cluster 3
1	1	3	3	1	3
2	1	3	3	1	3
3	1	3	3	1	3
4	1	3	3	N/A	3
5	1	1	1	N/A	3
6	3	1	1	N/A	3
7	3	1	1	N/A	3
8	3	1	1	N/A	3
9	3	1	1	N/A	3
10	3	1	1	N/A	3

Table 3.3: Sample Assignments

Sample	K=2		K=3		
	Cluster 1	Cluster 2	Cluster 1	Cluster 2	Cluster 3
s/c=1/4	4	2	2	2	0
s/c=1/3	4	10	10	2	0
s/c=1/2	5	10	10	3	0
s/c=2/3	5	10	10	3	10
s/c=1	10	10	10	3	10
s/c=2	10	10	10	3	10

Table 3.4: Sample Sizes

In order to determine the sample assignment ordering, we rank the terms inside the brackets of equations (3.26) - (3.29) without imposing any assumptions on marginal profit and marginal cost. We set a relatively small number to  $N$  (total population) to emphasize on the marginal effect of each sample. The results of sample assignments are presented in Table 3.3 using  $\tau = 25$ ,  $N = 100$ ,  $\delta = 0.3$  and  $r = 0.1^2$ . Note that the ratio of the marginal profit over the marginal cost is crucial in deciding the optimal sample size, however it is independent of the order of sample distribution. For example, for different  $s/c$  ratios, Table 3.4 shows the optimal sample size when 10 samples are available.

The results are interesting. One might conjecture that, a type 1 customer should always be given a higher priority in the sample distribution ordering, because she is not only the driver of the diffusion, but also connects with more customers. Table 3.3 shows that this conjecture is not true. Free samples sent to the most influential person is not always the best strategy. With a fixed customer base, distributing free samples of a new product consists of two forces: an acceleration effect and a cannibalization effect (Bawa and Shoemaker, 2004). On one hand, sending more samples accelerates the spread of social contagion, and thus

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<sup>2</sup>N/A indicates the terms inside the brackets are all negative, thus no sample should be sent to any cluster.

customers are likely to make purchases earlier than they otherwise would do; On the other hand, sampling reduces the number of paid purchase, because some customers who have received the sample would be willing to pay for the product otherwise. Sending a sample to a type 1 customer might have greater acceleration impact, but its cost of cannibalization can be even higher. Only by balancing those two forces, can firms decide whether to send one more sample and whom it should be send to, thus can avoid misdirecting marketing efforts and can, instead, optimally target to the customer who contribute most with a limited amount of free samples. For example, using the empirical diffusion results of CD “Smoking Popes”, the optimal order of distributing 10 samples is type 2, type 3, type 3, type 3, type 3, type 3, type 4, type 2, type 4 and type 4.

### 3.5 Conclusion

As an extension to the model discussed in Chapter 2, in this chapter we incorporate social contagion into customer lifetime value analysis in a framework where customers are heterogeneous. Customer heterogeneity is important because potential adopters are not always equally affected by adopters (see Van den Bulte and Joshi, 2007).

Building upon the Bass model, we propose a four-segment model which considers the heterogeneity among customers in the tendency to be in tune with new developments and the tendency to influence (or be influenced by) others. Specifically, we segment customers into four types: type 1 customers are both innovators and global influencers, type 2 customers are both innovators and local influencers, type 3 customers are both imitators and global influencers, and type 4 customers are both imitators and local influencers. We characterize the closed-form expressions for adoption rate of each customer type as a function of the adoption time. Based on them, we derive closed-form expressions for the customer PV, IV and LV as a function of product adoption time. We investigate how customer PV, IV and LV vary with the innovation parameters.

To show how a firm can increase its total customer LV, we analyze the impact of purchase acceleration on LV. We propose an algorithm to optimally allocate free samples among customers, which looks into the LV of marginal customers. We illustrate the algorithm by using the empirical results from Van den Bulte and Joshi (2007), in which 34 data series were analyzed.

Our model paves the way for several new research avenues. First, it will be interesting to consider purchase deceleration in the four-segment model. Second, it will be worthwhile to extend our model to capture active rivalry (see for example Savin and Terwiesch, 2005). Finally, it would be interesting to look at the impact of social contagion in a service adoption context.



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# Appendix A

## Proofs

**Lemma 1:** Consider the following two problems:

$$\begin{array}{ll}
 (PL_1) & z = \max_{x,y} f(x,y) \\
 \text{s.t.} & g_1(x) \leq 0 \\
 & g_2(x,y) \leq 0
 \end{array}
 \qquad
 \begin{array}{ll}
 (PL_2) & w = \max_x R(x) \\
 \text{s.t.} & g_1(x) \leq 0 \\
 \text{where} & R(x) = \max_y f(x,y) \\
 \text{s.t.} & g_2(x,y) \leq 0
 \end{array}$$

We claim:  $z = w$ .

Proof:

Let  $(x^*, y^*)$  be the optimal solution for  $(PL_1)$ , and  $(\bar{x}, \bar{y})$  the optimal solution for  $(PL_2)$ . On one hand,  $w = f(\bar{x}, \bar{y}) \leq f(x^*, y^*) = z$ , as  $(\bar{x}, \bar{y})$  is a feasible solution for  $(PL_1)$ , and its corresponding objective value cannot be greater than the one achieved by the optimal solution  $(x^*, y^*)$ . On the other hand,  $w \geq R(x^*) \geq f(x^*, y^*) = z$ . The first inequality follows from the fact that  $(x^*, y^*)$  satisfies  $g_1(x^*) \leq 0$ , and the second inequality is because  $(x^*, y^*)$  satisfies  $g_2(x^*, y^*) \leq 0$ . (Note that the constraints are not restricted to be “ $\leq$ ”, and our proof will still apply when  $g_1(x) \geq 0$  or  $g_2(x, y) \geq 0$ .) Q. E. D.

**Proposition 1:**

1). If the introduction time  $T \leq l$ , (In fact, in this case, the demand diffusion specifications are not functions of the introduction time  $T$ , but we retain  $D_i(t; T)(S_i(t; T))$ , ( $i = 1, 2$ ) for notational consistency.)

$$D_1(t; T) = S_1(t; T) = sF(t) \tag{A.1}$$

$$D_2(t; T) = (1 - s)F(t) \tag{A.2}$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1 - s)F(t) & \text{if } T \leq t \leq \tau. \end{cases} \tag{A.3}$$

2). If the introduction time  $T > l$ ,

$$D_1(t; T) = S_1(t; T) = \begin{cases} sF(t) & \text{if } 0 \leq t \leq l; \\ sF(t) + \theta(1-s)F(t-l) & \text{if } l < t < T; \\ sF(t) + \theta(1-s)F(T-l) & \text{if } T \leq t \leq \tau. \end{cases} \quad (\text{A.4})$$

$$D_2(t; T) = \begin{cases} (1-s)F(t) & \text{if } 0 \leq t \leq l; \\ (1-s)(F(t) - F(t-l)) & \text{if } l < t \leq T; \\ (1-s)(F(t) - F(T-l)) & \text{if } T < t \leq \tau. \end{cases} \quad (\text{A.5})$$

$$S_2(t; T) = \begin{cases} 0 & \text{if } 0 \leq t < T; \\ (1-s)(F(t) - F(T-l)) & \text{if } T \leq t \leq \tau. \end{cases} \quad (\text{A.6})$$

Proof:

The results follows straightforward from (1.2) - (1.10).

**Proposition 2:**

The optimal solution to problem  $R(T)$  can be characterized as follows: if  $\alpha \leq r_2/r_1$ ,  $Q_1^* = D_1(\tau; T)$ ,  $Q_2^* = D_2(\tau; T)$ ; Otherwise,  $Q_1^* = D_1(\tau; T) + D_2(\tau; T)\alpha$ ,  $Q_2^* = 0$ .

Proof:

Formulations for  $R_i(T)$ :

$$\begin{aligned} R_1(T) &= \max_{0 \leq Q_1 \leq D_1(\tau, T), 0 \leq Q_2 \leq D_2(\tau, T)} r_1 Q_1 + r_2 Q_2 \\ R_2(T) &= \max_{Q_1 \geq D_1(\tau, T), 0 \leq Q_2 \leq D_2(\tau, T)} w_1 D_1(\tau, T) + w_2 Q_2 - c_1 Q_1 - c_2 Q_2 + w_1 \alpha (D_2(\tau, T) - Q_2) \\ &\text{s.t.:} \quad (D_2(\tau, T) - Q_2) \alpha \leq Q_1 - D_1(\tau, T) \\ R_3(T) &= \max_{Q_1 \geq D_1(\tau, T), 0 \leq Q_2 \leq D_2(\tau, T)} w_1 D_1(\tau, T) + w_2 Q_2 - c_1 Q_1 - c_2 Q_2 + w_1 (Q_1 - D_1(\tau, T)) \\ &\text{s.t.:} \quad (D_2(\tau, T) - Q_2) \alpha \geq Q_1 - D_1(\tau, T) \\ R_4(T) &= \max_{0 \leq Q_1 \leq D_1(\tau, T), Q_2 \geq D_2(\tau, T)} w_1 Q_1 + w_2 D_2(\tau, T) - c_1 Q_1 - c_2 Q_2 + w_2 \beta (D_1(\tau, T) - Q_1) \\ &\text{s.t.:} \quad (D_1(\tau, T) - Q_1) \beta \leq Q_2 - D_2(\tau, T) \\ R_5(T) &= \max_{0 \leq Q_1 \leq D_1(\tau, T), Q_2 \geq D_2(\tau, T)} w_1 Q_1 + w_2 D_2(\tau, T) - c_1 Q_1 - c_2 Q_2 + w_2 (Q_2 - D_2(\tau, T)) \\ &\text{s.t.:} \quad (D_1(\tau, T) - Q_1) \beta \geq Q_2 - D_2(\tau, T) \\ R_6(T) &= \max_{Q_1 \geq D_1(\tau, T), Q_2 \geq D_2(\tau, T)} w_1 D_1(\tau, T) + w_2 D_2(\tau, T) - c_1 Q_1 - c_2 Q_2 \end{aligned}$$

We solve the subcase problems  $R_1(T) - R_6(T)$  accordingly and we denote  $\pi_i(T)$  to be the objective of  $R_i(T)$ .

1. Problem  $R_1(T)$ :  $\pi_1(T) = r_1 Q_1 + r_2 Q_2$ , as  $r_1, r_2 > 0$ ,  $Q_1^* = D_1(\tau, T)$ ,  $Q_2^* = D_2(\tau, T)$ .
2. Problem  $R_2(T)$ :  $\pi_2(T) = (r_2 - w_1 \alpha) Q_2 - c_1 Q_1 + w_1 D_1(\tau, T) + w_1 \alpha D_2(\tau, T)$

$$\text{As } Q_1 \geq (D_2(\tau, T) - Q_2) \alpha + D_1(\tau, T), \pi_2(T) \leq (r_2 - r_1 \alpha) Q_2 + w_1 D_1(\tau, T) + w_1 \alpha D_2(\tau, T)$$

There are three sub-cases:

- (a) If  $r_2 - r_1 \alpha > 0$ , clearly,  $Q_2^* = D_2(\tau, T)$ ,  $Q_1^* = D_1(\tau, T)$ .
- (b) If  $r_2 - r_1 \alpha < 0$ , then  $Q_2^* = 0$ ,  $Q_1^* = D_1(\tau, T) + D_2(\tau, T) \alpha$ .



(c) If  $r_2 - r_1\alpha = 0$ ,  $\pi_2(T) = c_1(\alpha Q_2 - Q_1) + w_1 D_1(\tau, T) + w_1\alpha D_2(\tau, T)$ ,

and thereby it follows  $Q_2^* = D_2(\tau, T)$ ,  $Q_1^* = D_1(\tau, T)$ .

3. Problem  $R_3(T)$ : The results obtained are the same as those in problem  $R_2(T)$ .

4. Problem  $R_4(T)$ :  $\pi_4(T) = (r_1 - w_2\beta)Q_1 - c_2Q_2 + w_2D_2(\tau, T) + w_2\beta D_1(\tau, T)$

As  $Q_2 \geq (D_1(\tau, T) - Q_1)\beta + D_2(\tau, T)$ ,  $\pi_4(T) \leq (r_1 - r_2\beta)Q_1 + w_2D_2(\tau, T) + w_2\beta D_1(\tau, T)$

Since  $r_1 > r_2\beta$ ,  $Q_1^* = D_1(\tau, T)$ ,  $Q_2^* = D_2(\tau, T)$ .

5. Problem  $R_5(T)$ : The results obtained are the same as those in problem  $R_4(T)$ .

6. Problem  $R_6(T)$ :  $\pi_6(T) = w_1D_1(\tau, T) + w_2D_2(\tau, T) - c_1Q_1 - c_2Q_2$ , so  $Q_1^* = D_1(\tau, T)$ ,  $Q_2^* = D_2(\tau, T)$ .

To summarize the above 6 cases, we first notice that in  $R_1(T)$ ,  $R_4(T)$ ,  $R_5(T)$  and  $R_6(T)$ ,  $Q_1^* = D_1(\tau, T)$ ,  $Q_2^* = D_2(\tau, T)$ , but in  $R_2(T)$  and  $R_3(T)$ , the optimal solution varies. so if  $r_2 \geq \alpha r_1$ , the optimal solutions in all cases coincide,  $Q_1^* = D_1(\tau, T)$ ,  $Q_2^* = D_2(\tau, T)$ ; otherwise, in  $R_2(T)$  and  $R_3(T)$ ,  $Q_1^* = D_1(\tau, T) + D_2(\tau, T)\alpha$ ,  $Q_2^* = 0$ , but  $Q_1^* = D_1(\tau, T)$ ,  $Q_2^* = D_2(\tau, T)$  for others. However, since  $Q_1 = D_1(\tau, T)$ ,  $Q_2 = D_2(\tau, T)$  is a feasible solution for  $R_2(T)$  and  $R_3(T)$  when  $r_2 < \alpha r_1$ , which shows  $Q_1^* = D_1(\tau, T) + D_2(\tau, T)\alpha$ ,  $Q_2^* = 0$  outperforms  $Q_1 = D_1(\tau, T)$ ,  $Q_2 = D_2(\tau, T)$ . Hence, we claim that if  $r_2 < \alpha r_1$ ,  $Q_1^* = D_1(\tau, T) + D_2(\tau, T)\alpha$ ,  $Q_2^* = 0$ . Q.E.D.

### Proposition 3:

If the planning horizon is comparable to the life-cycle period, when  $r_2/r_1 \geq \max\{\alpha, \theta\}$ , an immediate introduction is preferable, with  $Q_1^* = sF(\tau)$ ,  $Q_2^* = (1 - s)F(\tau)$ ; Otherwise, the second version should never be introduced, and the firm should commit the unavailability of version 2 in advance if  $\theta > \max\{r_2/r_1, \alpha\}$ .

Proof:

We consider two cases as outlined below:

Case 1:  $T > l$ . The cumulative demand of each version is given by

$$\begin{aligned} D_1(\tau; T) &= sF(\tau) + (1 - s)\theta F(T - l) \\ D_2(\tau; T) &= (1 - s)(F(\tau) - F(T - l)) \end{aligned}$$

Case 2:  $T \leq l$ . The demand paths are given by

$$\begin{aligned} D_1(\tau; T) &= sF(\tau) \\ D_2(\tau; T) &= (1 - s)F(\tau) \end{aligned}$$

Recall from Proposition 2, it is easy to verify that

1. When  $\alpha > r_2/r_1$ ,  $Q_1^* = D_1(\tau; T) + D_2(\tau; T)\alpha$ ,  $Q_2^* = 0$ .

(a) If  $\theta \leq \alpha$ , the firm ought to inform customers that NO line extension will be introduced, as more low-value customers will buy version 1 if being informed version 2 will not be introduced at a later time.

- (b) Otherwise,  $T^* = \tau$ .
2. When  $\alpha \leq r_2/r_1$ ,  $Q_1^* = D_1(\tau; T)$ ,  $Q_2^* = D_2(\tau; T)$ .
- (a) If  $\theta \leq r_2/r_1$ , the firm ought to introduce immediately.
- (b) Otherwise,  $T^* = \tau$ .

**Proposition 4:**

- (a) If  $\theta(r_1 - h\tau) > r_2$ ,  $T^* = \tau$ .
- (b) Otherwise, let  $\bar{T}$  denote the value of  $T$  ( $T < \tau$ ) that satisfies  $A(T) = B(T)$ .
- (i) If  $\bar{T} \geq l$ ,  $T^* = \bar{T}$ . (ii) Otherwise,  $T^* = l$ .

Proof: Denoting by  $\pi(T)$  the profit given introduction time  $T$ . To get the optimal  $T^*$ , we take the first derivative of  $\pi(T)$  with respect to  $T$ :

$$\frac{d\pi(T)}{dT} = hm(1-s)\left(\frac{1 - e^{-a\tau}}{1 + be^{-a\tau}} - \frac{1 - e^{-aT}}{1 + be^{-aT}}\right) + \frac{a(1+b)e^{-a(T-l)}}{(1 + be^{-a(T-l)})^2}m(1-s)(\theta(r_1 - hT) - r_2) \quad (\text{A.7})$$

where  $a = p + q > 0$ ,  $b = q/p > 0$ .

(a) When  $\theta(r_1 - h\tau) - r_2 > 0$ , then  $\theta(r_1 - hT) - r_2 > 0$ ,  $\forall T \in [l, \tau]$ . At the same time,  $\frac{1 - e^{-a\tau}}{1 + be^{-a\tau}} - \frac{1 - e^{-aT}}{1 + be^{-aT}} \geq 0$ ,  $\forall T \in [l, \tau]$ . Thus,  $\frac{d\pi(T)}{dT} > 0$ ,  $\forall T \in [l, \tau]$ , and thereby  $T^* = \tau$ .

(b) When  $\theta(r_1 - h\tau) - r_2 \leq 0$ , we first look at the second order of  $\pi(T)$ :

$$\begin{aligned} \frac{d^2\pi(T)}{dT^2} &= hm(1-s)a(1+b)\left(-\frac{e^{-aT}}{(1 + be^{-aT})^2}\right) - h\theta m(1-s)\frac{a(1+b)e^{-a(T-l)}}{(1 + be^{-a(T-l)})^2} \\ &\quad + \frac{a^2(1+b)e^{-a(T-l)}(be^{-a(T-l)} - 1)}{(1 + be^{-a(T-l)})^3}m(1-s)(\theta(r_1 - hT) - r_2) \end{aligned} \quad (\text{A.8})$$

It is difficult to see from (A.8) whether the objective is concave in  $T$ . Alternatively, equate (A.7) to zero and solve for the corresponding  $T$ , denoted as  $\bar{T}$ , then  $\bar{T}$  satisfies:

$$-h\left(\frac{1 - e^{-a\tau}}{1 + be^{-a\tau}} - \frac{1 - e^{-a\bar{T}}}{1 + be^{-a\bar{T}}}\right) = \frac{a(1+b)e^{-a(\bar{T}-l)}}{(1 + be^{-a(\bar{T}-l)})^2}(\theta(r_1 - h\bar{T}) - r_2) \quad (\text{A.9})$$

Substituting the right hand side of (A.9) with its left hand side expression into (A.8), we have:

$$\begin{aligned} \frac{d^2\pi(T)}{dT^2}\Big|_{T=\bar{T}} &= hm(1-s)a(1+b)\left(-\frac{e^{-a\bar{T}}}{(1 + be^{-a\bar{T}})^2}\right) - h\theta m(1-s)\frac{a(1+b)e^{-a(\bar{T}-l)}}{(1 + be^{-a(\bar{T}-l)})^2} \\ &\quad + m(1-s)ha\frac{(be^{-a(\bar{T}-l)} - 1)}{1 + be^{-a(\bar{T}-l)}}\left(\frac{1 - e^{-a\bar{T}}}{1 + be^{-a\bar{T}}} - \frac{1 - e^{-a\tau}}{1 + be^{-a\tau}}\right) \end{aligned} \quad (\text{A.10})$$

Since

$$hm(1-s)a(1+b)\left(-\frac{e^{-a\bar{T}}}{(1 + be^{-a\bar{T}})^2}\right) < 0,$$

$$-h\theta m(1-s)\frac{a(1+b)e^{-a(\bar{T}-l)}}{(1+be^{-a(\bar{T}-l)})^2} < 0,$$

$$\frac{1-e^{-a\bar{T}}}{1+be^{-a\bar{T}}} - \frac{1-e^{-a\tau}}{1+be^{-a\tau}} < 0$$

• If  $be^{-a(\bar{T}-l)} - 1 \geq 0$ , then obviously,  $\frac{d^2\pi(T)}{dT^2}|_{T=\bar{T}}$  is negative.

• Otherwise, let us consider the function  $f(\bar{T}) = -\frac{e^{-a\bar{T}}}{(1+be^{-a\bar{T}})^2}$ .

Note that

$$\frac{df(\bar{T})}{d\bar{T}} = -\frac{ae^{-a\bar{T}}(b^2e^{-2a\bar{T}} - 1)}{(1+be^{-a\bar{T}})^4}$$

Because  $be^{-a\bar{T}} < be^{-a(\bar{T}-l)} < 1$ , we have  $b^2e^{-2a\bar{T}} - 1 < 0$ , so  $f(\bar{T})$  is increasing in  $\bar{T}$ . Therefore,

$$\begin{aligned} \frac{d^2\pi(T)}{dT^2}|_{T=\bar{T}} &\leq hm(1-s)a(1+b)\left(-\frac{e^{-a\bar{T}}}{(1+be^{-a\bar{T}})^2}\right) - h\theta m(1-s)\frac{a(1+b)e^{-a\bar{T}}}{(1+be^{-a\bar{T}})^2} \\ &\quad + m(1-s)ha\frac{(be^{-a(\bar{T}-l)} - 1)}{1+be^{-a(\bar{T}-l)}}\left(\frac{1-e^{-a\bar{T}}}{1+be^{-a\bar{T}}} - \frac{1-e^{-a\tau}}{1+be^{-a\tau}}\right) \\ &\leq hm(1-s)a(1+b)\left(-\frac{e^{-a\bar{T}}}{(1+be^{-a\bar{T}})^2}\right) - h\theta m(1-s)\frac{a(1+b)e^{-a\bar{T}}}{(1+be^{-a\bar{T}})^2} \\ &\quad + m(1-s)ha\frac{(be^{-a(\bar{T}-l)} - 1)}{1+be^{-a\bar{T}}}\left(\frac{1-e^{-a\bar{T}}}{1+be^{-a\bar{T}}} - \frac{1-e^{-a\tau}}{1+be^{-a\tau}}\right) \\ &\leq \frac{hm(1-s)a}{(1+be^{-a\bar{T}})^2}[-(1+b)e^{-a\bar{T}} - \theta(1+b)e^{-a\bar{T}} \\ &\quad + (be^{-a(\bar{T}-l)} - 1)(1 - e^{-a\bar{T}}) - (be^{-a(\bar{T}-l)} - 1)(1 + be^{-a\bar{T}})] \\ &= \frac{hm(1-s)a}{(1+be^{-a\bar{T}})^2}[-(1+\theta)(1+b)e^{-a\bar{T}} + (1-be^{-a(\bar{T}-l)})(1+b)(e^{-a\bar{T}})] \end{aligned}$$

The first inequality comes from  $e^{-a(\bar{T}-l)} > e^{-a\bar{T}}$ ; the second inequality is the result of  $\frac{1}{1+be^{-a(\bar{T}-l)}} < \frac{1}{1+be^{-a\bar{T}}}$ ; the last inequality is because  $\frac{1-e^{-a\tau}}{1+be^{-a\tau}} \leq 1$  and  $be^{-a(\bar{T}-l)} - 1 < 0$ .

Since  $1 - be^{-a(\bar{T}-l)} < 1 < 1 + \theta$ ,  $\frac{d^2\pi(T)}{dT^2}|_{T=\bar{T}} < 0$ .

Thus, it is always true that  $\frac{d^2\pi(T)}{dT^2}|_{T=\bar{T}} < 0$ , and thereby  $\bar{T}$  is guaranteed to be a local maximum. Therefore, (i) If  $\bar{T} \geq l$ ,  $T^* = \bar{T}$ ; (ii) Otherwise,  $T^* = l$ . Q. E. D.

**Proposition 5:**

(a)  $\bar{T}$  increases with the profit margin of version 1,  $r_1$ , and the proportion of low-type customers who would like to switch to version 1 after experiencing too long a wait,  $\theta$ .  $\bar{T}$  decreases with the profit margin of version 2,  $r_2$ .

(b)  $T^* = \bar{T}$  ( $l < \bar{T} < \tau$ ) if  $\frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2) < h < \frac{1}{l}(r_1 - \frac{1}{\theta}r_2)$ .

Proof: (a) As  $r_1$  increases,  $B(T)$  will decrease, and thereby we need a larger  $T$  to keep  $A(T) = B(T)$  as  $A(T)$  is decreasing in  $T$ . Thus,  $\bar{T}$  increases. One can show  $\bar{T}$  increases in  $\theta$  and decreases in  $r_2$  by a similar proof.

(b) Evaluating (A.7) at  $T = \tau$ , the first term is 0, and when  $h > \frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2)$ , the second term is negative, and thereby  $\frac{d\pi(T)}{dT}|_{T=\tau} < 0$ ; On the other hand, evaluating (A.7) at  $T = l$ , the first term is positive, and when  $h < \frac{1}{l}(r_1 - \frac{1}{\theta}r_2)$ , the second term is positive as well, and thereby  $\frac{d\pi(T)}{dT}|_{T=l} > 0$ . Consequently, when  $\frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2) < h < \frac{1}{l}(r_1 - \frac{1}{\theta}r_2)$ , we have  $\frac{d\pi(T)}{dT}|_{T=l} > 0$  and  $\frac{d\pi(T)}{dT}|_{T=\tau} < 0$ , so there exists a  $t \in [l, \tau]$ , such that  $\frac{d\pi(T)}{dT}|_{T=t} = 0$  due to continuity of  $\frac{d\pi(T)}{dT}$ . From Proposition 4 we know that  $t = \bar{T}$ . Q. E. D.

### Proposition 6:

When  $\theta \leq r_2/r_1$ ,

(a) If  $h < h^*(\theta)$ ,  $T^* = l$ . (b)  $T^* = \bar{T}$  otherwise.

Proof: In the case when  $r_2/r_1 \geq \theta$ , it follows that  $\theta(r_1 - h\tau) - r_2 < 0$ , and thereby  $\frac{d\pi(T)}{dT}|_{T=\tau} < 0$ ;

In addition,

(a) If  $h < h^*(\theta)$ , we have  $\frac{1 - e^{-(p+q)\tau}}{1 + q/pe^{-(p+q)\tau}} - \frac{1 - e^{-(p+q)l}}{1 + q/pe^{-(p+q)l}} < \frac{p}{h}(r_2 - \theta(w_1 - c_2 - hl))$ , and it follows that  $\frac{d\pi(T)}{dT}|_{T=l} < 0$ .

We then claim  $\frac{d\pi(T)}{dT} < 0, \forall T \in [l, \tau]$ . We show this by contradiction. Suppose  $\frac{d\pi(T)}{dT}|_{T=t} > 0$ , for some  $t \in [l, \tau]$ , then due to continuity of  $\frac{d\pi(T)}{dT}$ , there exists a  $\hat{t} \in [l, t]$  such that  $\frac{d\pi(T)}{dT}|_{T=\hat{t}} = 0$  and  $\frac{d^2\pi(T)}{dT^2}|_{T=\hat{t}} > 0$ . This contradicts Proposition 4, in which we know all points satisfying first-order condition are local maximum.

Therefore, we have  $\frac{d\pi(T)}{dT} < 0, \forall T \in [l, \tau]$ , and thereby  $T^* = l$ .

(b) Otherwise, we have  $\frac{d\pi(T)}{dT}|_{T=l} > 0$ . Following a similar claim as above,  $T^* = \bar{T}$ . Q. E. D.

### Proposition 7:

When  $\theta > r_2/r_1$ ,

(a) If  $h < \frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2)$ ,  $T^* = \tau$ .

(b) If  $h \geq \frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2)$  and  $l \leq l^*(\theta)$ ,  $T^* = \bar{T}$ .

(c) If  $h \geq \frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2)$  and  $l > l^*(\theta)$ ,

(i) If  $\frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2) < h < h^*(\theta)$ ,  $T^* = l$ . (ii) Otherwise,  $T^* = \bar{T}$ .

Proof: In the case that  $r_2/r_1 < \theta$ ,

(a) If  $h < \frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2)$ , which indicates  $\frac{d\pi(T)}{dT} > 0, \forall T \in [l, \tau]$ , and thereby  $T^* = \tau$ ;

(b) Otherwise,

(i) If  $\frac{1}{\tau}(r_1 - \frac{1}{\theta}r_2) < h < h^*(\theta)$ ,  $\frac{d\pi(T)}{dT}|_{T=l} < 0, \frac{d\pi(T)}{dT}|_{T=\tau} < 0$ , and thus  $T^* = l$  following from the proof of Proposition 6.

(ii) Otherwise,  $T^* = \bar{T}$ .

In addition, When  $l \leq l^*(\theta)$ , we have

$$l - \left( \frac{1 - e^{-(p+q)\tau}}{p\theta + q\theta e^{-(p+q)\tau}} - \frac{1 - e^{-(p+q)l}}{p\theta + q\theta e^{-(p+q)l}} \right) < 0$$

and this guarantees

$$\frac{1}{\tau} > \frac{1}{l - \left( \frac{1 - e^{-(p+q)\tau}}{p\theta + q\theta e^{-(p+q)\tau}} - \frac{1 - e^{-(p+q)l}}{p\theta + q\theta e^{-(p+q)l}} \right)}$$

which in turn leads to  $\frac{1}{\tau} (r_1 - \frac{1}{\theta} r_2) > h^*(\theta)$ , so (i) of case (b) can not happen.

When  $l > l^*(\theta)$ , we have

$$l - \left( \frac{1 - e^{-(p+q)\tau}}{p\theta + q\theta e^{-(p+q)\tau}} - \frac{1 - e^{-(p+q)l}}{p\theta + q\theta e^{-(p+q)l}} \right) > 0$$

Besides, as  $\tau > l$ , we have

$$\tau > l - \left( \frac{1 - e^{-(p+q)\tau}}{p\theta + q\theta e^{-(p+q)\tau}} - \frac{1 - e^{-(p+q)l}}{p\theta + q\theta e^{-(p+q)l}} \right) > 0$$

and thus

$$\frac{1}{\tau} < \frac{1}{l - \left( \frac{1 - e^{-(p+q)\tau}}{p\theta + q\theta e^{-(p+q)\tau}} - \frac{1 - e^{-(p+q)l}}{p\theta + q\theta e^{-(p+q)l}} \right)}$$

which leads to  $\frac{1}{\tau} (r_1 - \frac{1}{\theta} r_2) < h^*(\theta)$ , so all cases can happen. Q. D. E.

**Proposition 8:**  $\int_{t=0}^{\infty} LV(t)f(t) dt = \int_{t=0}^{\infty} e^{-rt} f(t) dt$ .

Proof:

$$\begin{aligned} LHS &= \int_{t=0}^{\infty} PV(t)f(t) dt + \int_{t=0}^{\infty} IV(t)f(t) dt \\ &= \int_{t=0}^{\infty} e^{-rt} \left( \frac{p}{p + qF(t)} + \frac{qF(t)(1 - \delta)}{p + qF(t)} \right) (p + qF(t))(1 - F(t)) dt + \int_{t=0}^{\infty} IV(t)f(t) dt \\ &= \int_{t=0}^{\infty} e^{-rt} (p + qF(t)(1 - \delta))(1 - F(t)) dt + \delta q \int_{t=0}^{\infty} \int_{s=t}^{\infty} e^{-rs} (1 - F(s)) f(t) ds dt \\ &= \int_{t=0}^{\infty} e^{-rt} (p + qF(t)(1 - \delta))(1 - F(t)) dt + \delta q \int_{s=0}^{\infty} \int_{t=0}^s f(t) dt e^{-rs} (1 - F(s)) ds \\ &= \int_{t=0}^{\infty} e^{-rt} (p + qF(t)(1 - \delta))(1 - F(t)) dt + \int_{t=0}^{\infty} e^{-rt} (1 - F(t)) (q\delta F(t)) dt \\ &= \int_{t=0}^{\infty} e^{-rt} (1 - F(t)) (p + qF(t)) dt \\ &= RHS \end{aligned}$$

Q.E.D.

**Proposition 9:** Let

$$t^* = \left\{ t : \int_{s=t}^{\infty} e^{-r(s-t)} (1 - F(s)) ds = \frac{1 - e^{-(p+q)t}}{p+q} \right\}$$

When  $t \leq t^*$ ,  $LIV(t) \geq e^{-rt}$ ; When  $t > t^*$ ,  $LIV(t) < e^{-rt}$ .

Proof:

We define  $J(t) = LIV(t) - e^{-rt}$ . Take the derivative of  $J(t)$  with respect to  $t$  and set it equal to 0

$$q\delta(1 + e^{-(p+q)t} - F(t)) + rLIV(t) = r \quad (\text{A.11})$$

We note that the LHS of (A.11) is decreasing in  $t$ . Moreover, we know that  $LHS > r$  when  $t = 0$  and that  $LHS < r$  when  $t \rightarrow \infty$ . Hence, there exists a unique  $t^*$  such that (A.11) holds. Therefore  $J(t)$  decreases with  $t$  for  $t < t^*$  and increases thereafter.

$LIV(0) > 1$  so we know from Proposition 8 that there must exist a  $t'$  such that  $LIV(t') < e^{-rt'}$ . Since  $J(0) > 0$ ,  $J(t') < 0$  and  $J(t)$  decreases with  $t$  for  $t < t^*$  and increases thereafter. Thus, there exists a unique  $t^*$  such that  $LIV(t) > e^{-rt}$  for all  $t < t^*$  and  $LIV(t) < e^{-rt}$  thereafter. Solving  $LIV(t) = e^{-rt}$  gives us the expression for  $t^*$ . Q.E.D.

**Proposition 10:**  $PV(t)$ ,  $IV(t)$  and  $LIV(t)$  are all decreasing convex in  $t$ .

Proof:

We first show that  $PV(t)$  and  $IV(t)$  are decreasing convex functions of  $t$ . The claim for  $LIV(t)$  will follow since  $LIV(t) = PV(t) + IV(t)$ . To prove that  $PV(t)$  is decreasing convex in  $t$ , we need to show that  $PV'(t) < 0$  and  $PV''(t) > 0$ . We have

$$PV'(t) = -e^{-rt} (q\delta e^{-(p+q)t} + rPV(t)) < 0,$$

and

$$PV''(t) = re^{-rt} (q\delta e^{-(p+q)t} + rPV(t)) - e^{-rt} \left( -(p+q)q\delta e^{-(p+q)t} + r \frac{dPV(t)}{dt} \right) > 0,$$

Establishing that  $PV(t)$  is decreasing convex in  $t$ . Next, we will show that  $IV(t)$  is decreasing convex in  $t$ .

$$IV(t) = \delta q \int_t^{\infty} e^{-rs} (1 - F(s)) ds$$

The first derivative is

$$IV'(t) = -e^{-rt} \delta q (1 - F(t)) < 0$$

Also

$$IV''(t) = re^{-rt} \delta q (1 - F(t)) + e^{-rt} \delta q f(t) > 0$$

Thus,  $IV(t)$  is decreasing convex in  $t$ . Q.E.D.

- Proposition 11:** (1)  $PV(t)$  is increasing in  $p$ .  
(2)  $IV(t)$  is decreasing in  $p$ .  
(3) We define

$$t_1 = \left\{ t : e^{-rt} \frac{(1 - (pt + qt + 1)e^{-(p+q)t})}{(p+q)^2} = \int_t^\infty e^{-rs} \frac{\partial F(s)}{\partial p} ds \right\},$$

then  $LV(t)$  decreases with  $p$  for  $t < t_1$  and increases thereafter.

Proof:

- 1) To show that  $PV(t)$  increases with  $p$ , it suffices to show that  $\frac{\partial PV(t)}{\partial p} > 0$  for all  $t$ . We know  $\frac{\partial PV(t)}{\partial p} = e^{-rt} \frac{q\delta R(t)}{(p+q)^2}$ , where  $R(t) = 1 - (pt + qt + 1)e^{-(p+q)t}$ . Note that  $R(0) = 0$ . Moreover,  $R'(t) = t(p+q)^2 e^{-(p+q)t} > 0$  for  $t > 0$ . Therefore  $R(t) > 0, \forall t > 0$ . It follows that  $\frac{\partial PV(t)}{\partial p} > 0$ .

- 2) Similarly, we show that  $IV(t)$  is decreasing in  $p$

$$\frac{\partial IV(t)}{\partial p} = -\delta q \int_t^\infty e^{-rs} \frac{\partial F(s)}{\partial p} ds < 0, \quad (\text{A.12})$$

- 3) Let  $H(t) = \frac{\partial LV(t)}{\partial p} = \frac{\partial PV(t)}{\partial p} + \frac{\partial IV(t)}{\partial p}$ .  $\frac{\partial PV(t)}{\partial p}$  and  $\frac{\partial IV(t)}{\partial p}$  are derived in the proofs of part (1) and (2). By the definition of  $t_1$ , we must have that  $H(t_1) = 0$ . Now we will show that  $t_1$  is unique:

From the proofs of part (1) and (2), we know that whenever  $H(t) = 0$  we must have

$$e^{-rt} \frac{(1 - (pt + qt + 1)e^{-(p+q)t})}{(p+q)^2} = \int_t^\infty e^{-rs} \frac{\partial F(s)}{\partial p} ds \quad (\text{A.13})$$

Now, let  $R_1(t) = e^{-rt} \frac{(1 - (pt + qt + 1)e^{-(p+q)t})}{(p+q)^2}$ ,  $R_2(t) = \int_t^\infty e^{-rs} \frac{\partial F(s)}{\partial p} ds$ .

$$R_1'(t) = e^{-rt} q \left( te^{-(p+q)t} - r \frac{1 - (pt + qt + 1)e^{-(p+q)t}}{(p+q)^2} \right) \quad (\text{A.14})$$

Now we want to show that  $R_1(t)$  is unimodal in  $t$  and that there exists an interior maximizer. Set (A.14) to zero

$$(p+q)(p+q+r)te^{-(p+q)t} + re^{-(p+q)t} = r \quad (\text{A.15})$$

Use  $G(t)$  to denote the LHS of (A.15), i.e.  $G(t) = (p+q)(p+q+r)te^{-(p+q)t} + re^{-(p+q)t}$ . Then

$$G'(t) = (p+q)(p+q - t(p+q+r))e^{-(p+q)t} \quad (\text{A.16})$$

Now,  $G'(t) > 0$  for  $t < \frac{p+q}{p+q+r}$  and  $G'(t) \leq 0$  thereafter, so  $G(t)$  is unimodal in  $t$ . Moreover,  $G(t)|_{t=0} = r$  and  $\lim_{t \rightarrow \infty} G(t) < r$ . Thus (A.15) has exactly one interior solution of  $t$  and it is either a maximizer or a minimizer of  $R_1(t)$ . Since  $R_1'(t)|_{t=0} = 0$  and  $R_1''(t)|_{t=0} > 0$ , we must have that  $R_1'(t)|_{t=\epsilon} > 0$  where  $\epsilon$  is a small positive number. It follows that  $R_1(t)$  must have one interior maximizer.  $R_1(t)$  is increasing in  $t$  to the left of the maximizer and decreasing to the right.

Next, we will show that  $t_1$  is uniquely determined (i.e.  $R_1(t) = R_2(t)$  has a unique solution in  $t$ .) From above, we know that  $R_1$  is unimodal in  $t$  and has an interior maximizer.

Note that  $R_2$  is strictly decreasing in  $t$  since  $\frac{\partial F(s)}{\partial p} > 0$ . Moreover, because

$$\lim_{t \rightarrow \infty} \frac{(1 - (pt + qt + 1)e^{-(p+q)t})}{(p+q)^2} = \frac{1}{(p+q)^2} > 0,$$

$$\lim_{t \rightarrow \infty} \int_t^\infty \frac{\partial F(s)}{\partial p} ds = 0,$$

there must exist a  $\hat{t}$ , s.t. for all  $t \geq \hat{t}$ ,

$$\frac{(1 - (pt + qt + 1)e^{-(p+q)t})}{(p+q)^2} > \int_t^\infty \frac{\partial F(s)}{\partial p} ds$$

Hence,

$$R_1(\hat{t}) > e^{-r\hat{t}} \int_{\hat{t}}^\infty \frac{\partial F(s)}{\partial p} ds > R_2(\hat{t})$$

Moreover, since  $R_1(0) < R_2(0)$  and  $\lim_{t \rightarrow \infty} R_1(t) = \lim_{t \rightarrow \infty} R_2(t) = 0$ , it follows that  $t_1$  is uniquely determined by  $R_1(t) = R_2(t)$ .

Lastly, since  $H(t)|_{t=0} < 0$  and  $H(t)|_{t=\hat{t}} > 0$ , we must have that  $H(t) < 0$  for  $t < t_1$  and  $H(t) \geq 0$  thereafter. To the left of  $t_1$ ,  $\frac{\partial LV(t)}{\partial p}$  is decreasing and to the right,  $\frac{\partial LV(t)}{\partial p}$  is increasing. Q.E.D.

**Proposition 12:** 1)  $PV(t)$  is decreasing in  $q$ .

(2) We define

$$t_2 = \left\{ t : \int_t^\infty e^{-rs} \left( 1 - F(s) - q \frac{\partial F(s)}{\partial q} \right) ds = 0 \right\}$$

$IV(t)$  increases with  $q$  for  $t \leq t_2$  and decreases thereafter.

(3) We define

$$t_3 = \left\{ t : \int_t^\infty e^{-rs} \left( 1 - F(s) - q \frac{\partial F(s)}{\partial q} \right) ds = \frac{e^{-rt} p}{(p+q)^2} \left( 1 - \left( 1 - qt - \frac{q^2}{p} t \right) e^{-(p+q)t} \right) \right\}$$



$LV(t)$  increases with  $q$  for  $t \leq t_3$  and decreases thereafter.

Proof:

1) It suffices to show that  $\frac{\partial PV(t)}{\partial q} < 0 \forall t > 0$ .

$$\frac{\partial PV(t)}{\partial q} = e^{-rt} \frac{p\delta(R(t) - 1)}{(p + q)^2},$$

where  $R(t) = (1 - qt - \frac{q^2}{p}t)e^{-(p+q)t}$ . Note that  $\frac{\partial PV(t)}{\partial q} < 0$  if and only if  $R(t) < 1$ . Moreover, we have

$$R'(t) = \frac{1}{p}(qt - 1)(p + q)^2 e^{-(p+q)t}$$

Now observe that  $R'(t) < 0$  for  $t < \frac{1}{q}$  and  $R'(t) \geq 0$  thereafter. Thus  $R(t)$  decreases with  $t$  first and then increases, and it has a unique minimizer. Thus  $R(t)$  reaches its maximum at  $t = 0$  or  $t = \infty$ . Note also that  $R(0) = 1$  and that  $\lim_{t \rightarrow \infty} R(t) < 1$ . Hence,  $R(t) < 1$  for all  $t > 0$ . It follows that  $\frac{\partial PV(t)}{\partial q} < 0 \forall t > 0$ .

2) It suffices to show that  $\frac{\partial IV(t)}{\partial q} > 0$  for  $t < t_2$  and  $\frac{\partial IV(t)}{\partial q} < 0$  thereafter.

$$\frac{\partial IV(t)}{\partial q} = \delta \int_{s=t}^{\infty} e^{-rs} \left(1 - F(s) - q \frac{\partial F(s)}{\partial q}\right) ds \quad (\text{A.17})$$

Let  $H(s) = 1 - F(s) - q \frac{\partial F(s)}{\partial q}$ . Equating  $H(s)$  to 0 yields

$$\frac{q(p + q)}{p^2} e^{-2(p+q)s} + \left(\frac{q}{p} + 1\right) e^{-(p+q)s} = \frac{q^2}{p} s + qs \quad (\text{A.18})$$

Note that the LHS of (A.18) is decreasing in  $s$ , while the RHS is increasing in  $s$ . Moreover, when  $s = 0$ , the LHS is greater than the RHS ( $H(s) > 0$ ). When  $s$  is large, the RHS exceeds the LHS ( $H(s) < 0$ ). Thus  $H(s)$  hits 0 exactly once. In (A.17), the term inside the integral is positive for small  $s$  and is negative for large  $s$ . Therefore, in (A.17), the integral from  $t$  to  $\infty$  is positive for  $t < t_2$  and negative for  $t > t_2$ .  $t_2$  is uniquely determined by solving for the value of  $t$  such that  $\int_t^{\infty} e^{-rs} \left(1 - F(s) - q \frac{\partial F(s)}{\partial q}\right) ds = 0$ .

3) First, from parts (1) and (2), we know that  $\frac{\partial LV(t)}{\partial q} = \frac{\partial PV(t)}{\partial q} + \frac{\partial IV(t)}{\partial q} < 0$  for  $t > t_2$ .

Now we will show that  $\frac{\partial PV(t)}{\partial q}$  first decreases with  $t$  and then increases. Look at

$$\frac{\partial^2 PV(t)}{\partial q \partial t} = \frac{p\delta}{(p + q)^2} e^{-(r+p+q)t} G(t) \quad (\text{A.19})$$

Clearly  $G(t) > 0$  for all  $t > \frac{1}{q}$ . Moreover, since  $G'(t) > 0$  and  $G(0) < 0$ , we must have  $G(t) < 0$  for  $t$  less than some cutoff and  $G(t) > 0$  thereafter. Hence,  $\frac{\partial PV(t)}{\partial q}$  first decreases with  $t$  and then increases.

Next, we note that  $\frac{\partial IV(t)}{\partial q}$  is decreasing in  $t$  for  $t < t_2$ . Equivalently,  $-\frac{\partial IV(t)}{\partial q}$  is increasing in  $t$  for  $t < t_2$ .

We need to show that  $\frac{\partial PV(t)}{\partial q}$  has unique intersection with  $-\frac{\partial IV(t)}{\partial q}$  when  $t < t_2$ . When  $t < t_2$ ,  $\frac{\partial PV(t)}{\partial q}$  is either decreasing or unimodal in  $t$ , and  $-\frac{\partial IV(t)}{\partial q}$  is increasing in  $t$ . It suffices to compare the values between  $\frac{\partial PV(t)}{\partial q}$  and  $-\frac{\partial IV(t)}{\partial q}$  at  $t = 0$  and  $t = t_2$ .  $\frac{\partial PV(t)}{\partial q}|_{t=0} > -\frac{\partial IV(t)}{\partial q}|_{t=0}$  and  $\frac{\partial PV(t)}{\partial q}|_{t=t_2} < -\frac{\partial IV(t)}{\partial q}|_{t=t_2}$ . Hence there must exist a unique  $t_3$  such that  $\frac{\partial PV(t)}{\partial q} = -\frac{\partial IV(t)}{\partial q}$  ( $\frac{\partial LV(t)}{\partial q} = 0$ ).  $\frac{\partial LV(t)}{\partial q} > 0$  for  $t < t_3$  and  $\frac{\partial LV(t)}{\partial q} < 0$  thereafter. Q.E.D.

**Proposition 13:**  $\lambda(s)$  is decreasing in  $s$ .

Proof:

We first define  $\mu(s) = \ln(1 - \lambda(s))$ . Take the first derivative

$$\begin{aligned} \frac{d\mu(s)}{ds} &= \frac{N}{p}(p+q)^3 \ln \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} e^{-(p+q)s} \right) \right) \frac{\left( \frac{q}{p} e^{-2(p+q)s} - e^{-(p+q)s} \right)}{\left( 1 + \frac{q}{p} e^{-(p+q)s} \right)^3} \\ &\quad + \frac{q^2(p+q)^2}{p^2} \frac{e^{-2(p+q)s}}{\left( 1 + \frac{q}{p} e^{-(p+q)s} \right) \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} e^{-(p+q)s} \right) \right)} \end{aligned} \quad (\text{A.20})$$

Note that the second term is positive and decreasing in  $s$  everywhere. To examine the first term, define  $\mu_1(x) = \ln \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} x \right) \right) \frac{\left( \frac{q}{p} x^2 - x \right)}{\left( 1 + \frac{q}{p} x \right)^3}$ , where  $x = e^{-(p+q)s}$ . The sign of the first term in (A.20) is the same as  $\text{Sign}(\mu_1(x))$ . Note also that  $\ln \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} x \right) \right) < 0$ . There are two cases:

(1) If  $x \leq \frac{p}{q}$ ,  $\mu_1(x) \geq 0$ , then  $\frac{d\mu(s)}{ds} > 0$ .

(2) If  $\frac{p}{q} < x \leq 1$ , then  $\mu_1(x) \leq 0$ . To determine the sign of  $\frac{d\mu(s)}{ds}$ , take derivative of  $\mu_1(x)$  and write it as

$$\mu'_1(x) = \frac{1}{\left( 1 + \frac{q}{p} x \right)^4} (\mu_{11}(x) + \mu_{12}(x)),$$

where

$$\begin{aligned}\mu_{11}(x) &= \ln \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p}x \right) \right) \left( 3 - \left( \frac{q}{p}x - 2 \right)^2 \right), \\ \mu_{12}(x) &= - \left( \frac{q}{p}x^2 - x \right) \frac{q^2}{Np(p+q)} \frac{\left( 1 + \frac{q}{p}x \right)}{\left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p}x \right) \right)}.\end{aligned}$$

If  $\frac{p}{q} < x < \min\{1, \frac{p}{q}(\sqrt{3} + 2)\}$ , then  $\mu_{11}(x) < 0$  since  $3 - (\frac{q}{p}x - 2)^2 > 0$ . Moreover, because  $\frac{p}{q} < x \leq 1$ , we must have  $\mu_{12} < 0$  and  $\mu'_1(x) < 0$ .

If  $\min\{1, \frac{p}{q}(\sqrt{3} + 2)\} \leq x \leq 1$ , then  $\mu_{11}(x) \geq 0$ . Both  $3 - (\frac{q}{p}x - 2)^2$  and  $\ln \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p}x \right) \right)$  are nonpositive and decreasing in  $x$ , so we know that  $x = 1$  maximizes  $\mu_{11}(x)$ . Moreover, we see that  $\mu_{12} < 0$  and that  $x = 1$  minimizes  $\mu_{12}(x)$  since  $\mu'_{12}(x) < 0$ . Thus we will show that  $\mu'_1(x) < 0$  if we can show  $\mu_{11}(x) + \mu_{12}(x)|_{x=1} < 0$ .

$$\begin{aligned}& \mu_{11}(x) + \mu_{12}(x)|_{x=1} \\ &= \ln \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} \right) \right) \left( 3 - \left( \frac{q}{p} - 2 \right)^2 \right) - \left( \frac{q}{p} - 1 \right) \frac{q^2}{Np(p+q)} \frac{\left( 1 + \frac{q}{p} \right)}{\left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} \right) \right)} \\ &< - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} \right) \left( 3 - \left( \frac{q}{p} - 2 \right)^2 \right) - \left( \frac{q}{p} - 1 \right) \frac{q^2}{Np(p+q)} \frac{\left( 1 + \frac{q}{p} \right)}{\left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} \right) \right)} \quad (\text{A.21})\end{aligned}$$

$$\begin{aligned}&= \frac{q \left( 1 + \frac{q}{p} \right) \left( \left( \frac{q}{p} - 2 \right)^2 - 3 \right) \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p}x \right) \right) - q \left( \frac{q}{p} - 1 \right)}{N(p+q) \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p}x \right) \right)} \\ &< \frac{q \left( 1 + \frac{q}{p} \right) \left( \left( \frac{q}{p} - 2 \right)^2 - 3 \right) - q \left( \frac{q}{p} - 1 \right)}{N(p+q) \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p}x \right) \right)} \quad (\text{A.22})\end{aligned}$$

$$\begin{aligned}&= \frac{q \left( 1 + \frac{q}{p} \right) - \frac{q^2}{p^2} - \frac{q^2}{p} + \frac{4q}{p} + q - 1}{N(p+q) \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p}x \right) \right)} \\ &< 0 \quad (\text{A.23})\end{aligned}$$

(A.21) is because  $\ln(1 - y) < -y$  for  $0 < y < 1$ . (A.22) is due to  $(\frac{q}{p} - 2)^2 - 3 > 0$  and  $0 < 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p}x \right) < 1$ . (A.23) follows from  $-\frac{q^2}{p^2} + \frac{4q}{p} < 0$  and  $-\frac{q^2}{p} + q < 0$  as  $\frac{q}{p} > \sqrt{3} + 2$ . We have shown that  $\mu'_1(x) < 0$  for  $\frac{p}{q} < x < 1$ . Substituting  $x = e^{-(p+q)s}$  into the expression for  $\mu_1$ , we see that  $\mu_1(s)$  is increasing in  $s$ . Thus, in this case,  $s = 0$  minimizes  $\mu_1(s)$ .  $\mu_2(s)$  is always positive, so we will show  $\frac{d\mu(s)}{ds} > 0$  if we can show  $\frac{d\mu(s)}{ds}|_{s=0} > 0$ .

Plugging in  $s = 0$ , (A.20) becomes

$$\begin{aligned}\frac{d\mu(s)}{ds}|_{s=0} &= \frac{N}{p} (p+q)^3 \ln \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} \right) \right) \frac{\left( \frac{q}{p} - 1 \right)}{\left( 1 + \frac{q}{p} \right)^3} \\ &\quad + \frac{q^2 (p+q)^2}{p^2} \frac{1}{\left( 1 + \frac{q}{p} \right) \left( 1 - \frac{q}{N(p+q)} \left( 1 + \frac{q}{p} \right) \right)}\end{aligned} \quad (\text{A.24})$$

Because

$$\ln \left( 1 - \frac{q}{m(p+q)} \left( 1 + \frac{q}{p} \right) \right) = \ln \frac{Np - q}{Np} = -\ln \frac{Np}{Np - q} = -\ln \left( 1 + \frac{q}{Np - q} \right) > -\frac{q}{Np - q} \quad (\text{A.25})$$

Applying (A.25), (A.24) becomes

$$\left. \frac{d\mu(s)}{ds} \right|_{s=0} > \frac{Nq}{p(Np - q)} (p^3 + pq^2) > 0$$

Thus, in both cases  $\frac{d\mu(s)}{ds} > 0$  holds, implying  $\mu(s)$  increases with  $s$ . Therefore,  $\lambda(s)$  is decreasing in  $s$ . Q.E.D.

**Proposition 14:**  $\mathbb{E}T_t = \int_0^\infty e^{-\int_t^{t+x} \lambda(s) ds} dx$

Proof:

Let us first look at the discrete-time case. Use  $p_k$  to denote the probability that Betty influences someone who buys at  $k$  days after Betty.

The probability that Betty influences other people for the first time  $x$  or more days after her purchase is  $Pr(T_t > x) = \prod_{k=t+1}^{t+x} (1 - p_k)$ . Divide each day into  $n$  time slots. Also, cut each  $p_k$  in  $\frac{1}{n}$ . The probability that Betty's first influence happens after  $x$  days becomes:

$$Pr(T_t > x) = \prod_{k=t+1}^{t+x} \left( 1 - \frac{p_k}{n} \right)^n \quad (\text{A.26})$$

To see what happens to (A.26) as  $n$  approaches infinity, take the logarithm of the RHS of (A.26).

$$\log \left( \prod_{k=t+1}^{t+x} \left( 1 - \frac{p_k}{n} \right)^n \right) = \sum_{k=t+1}^{t+x} \left( n \log \left( 1 - \frac{p_k}{n} \right) \right) \quad (\text{A.27})$$

Using a Taylor expansion, we know that  $\log \left( 1 - \frac{p_k}{n} \right) = -\frac{p_k}{n}$  as  $n$  goes to infinity. Therefore,  $Pr(T_t > x) = e^{-\sum_{k=t+1}^{t+x} p_k}$  if  $n$  approaches infinity.

Now we want to interpolate to cover all points in time. Replace  $p_k$  with  $\lambda(s)$ . The distribution function is now  $Pr(T_t > x) = e^{-\int_t^{t+x} \lambda(s) ds}$ . This quantity is well defined for all real values of  $x \geq 0$ . Therefore,  $\mathbb{E}T_t = \int_0^\infty e^{-\int_t^{t+x} \lambda(s) ds} dx$ . Q.E.D.

**Proposition 15:**  $\mathbb{E}T_t$  is increasing in  $t$ .

Proof:

We have

$$\frac{\partial \mathbb{E}T_t}{\partial t} = \int_0^\infty \frac{\partial e^{-\int_t^{t+x} \lambda(s) ds}}{\partial t} dx \quad (\text{A.28})$$

Applying Proposition 13, we see that  $e^{-\int_t^{t+x} \lambda(s) ds}$  is increasing in  $t$  for any fixed  $x$ . So  $\frac{\partial \mathbb{E}T_t}{\partial t} > 0$ . Q.E.D.

**Proposition 16:** Let  $G(t, F_0) = \frac{(p+q)^2 e^{-(p+q+r)t}}{(p+qF_0 + q(1-F_0)e^{-(p+q)t})^2}$ , then

(1) The optimal  $F_0^*$  is unique.

(2)  $F_0^* > 0$  if and only if  $\int_0^\infty G(t, 0) dt > \frac{d}{(s-c)r}$ . Otherwise,  $F_0^* = 0$ .

(3) If  $F_0^* > 0$ ,  $F_0^*$  satisfies  $\int_0^\infty G(t, F_0) dt = \frac{d}{(s-c)r}$ .

Proof:

First rewrite (2.19) as

$$\begin{aligned} \pi_0^* &= \max_{0 \leq F_0 \leq 1} (s-c)(e^{-rt}F(t)|_{t=\infty} - e^{-rt}F(t)|_{t=0} + r \int_0^\infty e^{-rt}F(t) dt) + (s-c-d)F_0 \\ &= \max_{0 \leq F_0 \leq 1} (s-c)(r \int_0^\infty e^{-rt}F(t) dt - F_0) + (s-c-d)F_0 \end{aligned} \quad (\text{A.29})$$

where  $F(t) = \frac{p+qF_0 - p(1-F_0)e^{-(p+q)t}}{p+qF_0 + q(1-F_0)e^{-(p+q)t}}$

The first order condition (the second derivative is negative) is

$$\int_0^\infty \frac{(p+q)^2 e^{-(p+q+r)t}}{(p+qF_0 + q(1-F_0)e^{-(p+q)t})^2} dt = \frac{d}{(s-c)r} \quad (\text{A.30})$$

Let  $G(t, F_0) = \frac{(p+q)^2 e^{-(p+q+r)t}}{(p+qF_0 + q(1-F_0)e^{-(p+q)t})^2}$ , and then (A.30) becomes

$$\int_0^\infty G(t, F_0) dt = \frac{d}{(s-c)r} \quad (\text{A.31})$$

As  $G$  decreases with  $F_0$ , the LHS of (A.30) is decreasing in  $F_0$ . Thus there exists a unique optimal  $F_0^*$  which can be found by a simple bisection search. Part (1) and part (3) follow. Part (2) follows as  $G$  decreases with  $F_0$ . Q.E.D.

**Proposition 17:** (1)  $F_0^*$  decreases with  $p$ ;

(2) If  $\int_0^\infty U(t, F_0^*) dt > 0$  ( $U(t, F_0^*)$  is defined in closed-form),  $F_0^*$  increases with  $q$ ; Otherwise,  $F_0^*$  decreases with  $q$ ;

(3)  $F_0^*$  decreases with  $d$ .

(4)  $F_0^*$  increases with the profit margin  $(s-c)$ .

(5)  $F_0^*$  increases with  $r$ .

Proof:

1) First,  $\frac{\partial G(t, 0)}{\partial p} < 0$ , so for a larger value of  $p$  it is more likely to have

$$\int_0^\infty G(t, 0) dt < \frac{d}{(s-c)r},$$

leading to  $F_0^* = 0$ .

To show  $F_0^*$  decreases with  $p$ , we need  $\frac{dF_0^*}{dp} < 0$ .

By Proposition 16, we have

$$\int_0^\infty G(t, F_0^*) dt - \frac{d}{(s-c)r} = 0$$

Let  $H = \int_0^\infty G(t, F_0^*) dt - \frac{d}{(s-c)r}$ , because  $\frac{\partial G(t, F_0^*)}{\partial p} = \frac{q(F_0^* - 1)(1 - e^{-(p+q)t})}{(p + qF_0^* + q(1 - F_0^*)e^{-(p+q)t})^4} e^{-rt} < 0$ . Thus  $\frac{\partial H}{\partial p} < 0$ .

Similarly,  $\frac{\partial G(t, F_0^*)}{\partial F_0^*} < 0$ , so we have  $\frac{\partial H}{\partial F_0^*} < 0$ .

By the Implicit Function Theorem,  $\frac{dF_0^*}{dp} = -\frac{\frac{\partial H}{\partial p}}{\frac{\partial H}{\partial F_0^*}} < 0$ .

2) We need  $\frac{dF_0^*}{dq} > 0$  if  $\int_0^\infty U(t, F_0^*) dt > 0$  and  $\frac{dF_0^*}{dq} < 0$  otherwise.

Taking the derivative of  $G(t, F_0^*)$  with respect to  $q$  yields

$$\frac{\partial G(t, F_0^*)}{\partial q} = U(t, F_0^*)$$

where  $U(t, F_0^*) = \frac{(p+q)e^{-(r+p+q)t}}{(p+qF_0^*+q(1-F_0^*)e^{-(p+q)t})^3} (2p - pqt - 2pF_0^* - pqtF_0^* - q^2F_0^*t - p^2t) - e^{-(p+q)t} (2p - 2pt - 2pF_0^* + pqtF_0^* + q^2F_0^*t - q^2t)$ . Hence, if  $\int_0^\infty U(t, F_0^*) dt > 0$ ,  $\frac{\partial H}{\partial q} > 0$ , and thereby  $\frac{dF_0^*}{dq} > 0$ , then  $F_0^*$  is increasing in  $q$ . Otherwise,  $F_0^*$  is decreasing in  $q$ .

3) In (A.31), the RHS is increasing in  $d$ . The LHS is decreasing in  $F_0$ , thus  $F_0^*$  is decreasing in  $d$ .

4) In (A.31), the RHS is decreasing in  $s - c$ . The LHS is decreasing in  $F_0$ , thus  $F_0^*$  is increasing in  $s - c$ .

5) The RHS of (A.31) is decreasing in  $r$ . The LHS is decreasing in  $F_0$ . So  $F_0^*$  is increasing in  $r$ .

Q.E.D.

**Proposition 18:** Denote by  $\tau_B = \frac{1}{p+q} \ln \frac{q}{p}$  the time of the maximum demand rate for Bass diffusion. Also, define  $\tau_1 = T + L + \frac{1}{p+q} \ln \frac{q(1-D_2)}{p+qD_2}$ ,  $d_1 = \frac{(p+q)^2}{4q}$ , and

$$\tau_2 = \left\{ t : (p+qd(t))L = \ln \frac{2q(1-d(t))}{p+q} \right\} \quad (\text{A.32})$$

where  $d(t)$  is from (2.23).

The maximum demand rate occurs at

$$\tau_{max}^D = \begin{cases} \tau_1, & T < \tau_2; \\ \tau_B, & T > \tau_B; \\ \operatorname{argmax}\{t \mid d(t), t = T, T + L\}, & \text{o.w.} \end{cases}$$

and is equal to

$$d(\tau_{max}^D) = \begin{cases} d_1, & T < \tau_2 \text{ or } T > \tau_B; \\ \max\{d(t), t = T, T + L\}, & \text{o.w.} \end{cases}$$

Proof: Taking the first order condition of (2.23), we obtain  $t = \tau_B$ . Further, the first order condition of (2.31) gives us that  $t = \tau_1$ . Thus, if  $\tau_B$  is attainable ( $T > \tau_B$ ),  $\tau_{max}^D = \tau_B$ ; or if  $\tau_1$  is attainable ( $T + L < \tau_1$ ),  $\tau_{max}^D = \tau_1$ . The condition that  $T + L < \tau_1$  can be simplified as  $q(1 - D_2) > p + qD_2$ . Plugging in (2.29), we have

$$(p + qD_1)L < \ln \frac{2q(1 - D_1)}{p + q} \quad (\text{A.33})$$

The LHS of (A.33) increases with  $D_1$ , while the RHS decreases with it. Also note that  $LHS < RHS$  when  $D_1 = 0$  and  $LHS > RHS$  when  $D_1 = 1$ . We can claim that (A.32) gives a unique  $\tau_2$  as  $d(t)$  is strictly increasing in  $t$  from (2.23). So (A.33) holds if  $T < \tau_2$ .

Finally, if neither  $\tau_B$  nor  $\tau_1$  is attainable, the maximum demand rate is achieved during  $[T, T + L]$ . Because  $d(t)$  decreases with  $t$  when  $T \leq t < T + L$ , the maximum demand rate is achieved at either  $T$  or  $T + L$ . And the result follows. Q.E.D.

**Proposition 19:**  $\int_{t=0}^{\infty} LV(t)d(t) dt = \int_{t=0}^{\infty} e^{-rt}s(t) dt \leq \int_{t=0}^{\infty} e^{-rt}d_{bass}(t) dt$ .

Proof: We first show that  $\int_{t=0}^{\infty} LV(t)d(t) dt = \int_{t=0}^{\infty} e^{-rt}s(t) dt$ .

$$\begin{aligned} & \int_{t=0}^{\infty} LV(t)d(t) dt \\ = & \int_{t=0}^T PV(t)d(t) dt + \int_{t=0}^T IV(t)d(t) dt + \int_{t=T}^{T+L} PV(t)d(t) dt + \int_{t=T}^{T+L} IV(t)d(t) dt \\ & + \int_{t=T+L}^{\infty} PV(t)d(t) dt + \int_{t=T+L}^{\infty} IV(t)d(t) dt \\ = & \left( \int_{t=0}^T PV(t)d(t) dt + \int_{t=0}^T IV_1(t)d(t) dt \right) + \left( \int_{t=0}^T IV_2(t)d(t) dt + \int_{t=T}^{T+L} PV(t)d(t) dt \right) \\ & + \left( \int_{t=0}^T IV_3(t)d(t) dt + \int_{t=T}^{T+L} IV(t)d(t) dt + \int_{t=T+L}^{\infty} PV(t)d(t) dt + \int_{t=T+L}^{\infty} IV(t)d(t) dt \right) \end{aligned}$$

By some algebra, the terms in the first brackets are

$$\begin{aligned}
& \int_{t=0}^T PV(t)d(t) dt + \int_{t=0}^T IV_1(t)d(t) dt \\
&= \int_{t=0}^T (e^{-rt}(p + qD(t)(1 - \delta))(1 - D(t)) + \delta qe^{-rt}(1 - D(t))D(t)) dt \\
&= \int_{t=0}^T e^{-rt}d(t) dt
\end{aligned} \tag{A.34}$$

The terms in the second brackets are

$$\begin{aligned}
& \int_{t=0}^T IV_2(t)d(t) dt + \int_{t=T}^{T+L} PV(t)d(t) dt \\
&= \frac{\delta qe^{-r(T+L)}(D_2 - D_1)D_1}{p + qD_1} + e^{-r(T+L)}(D_2 - D_1)\left(\frac{p}{p + qD_1} + \frac{qD_1(1 - \delta)}{p + qD_1}\right) \\
&= e^{-r(T+L)}(D_2 - D_1)
\end{aligned} \tag{A.35}$$

The terms in the third brackets are

$$\begin{aligned}
& \int_{t=0}^T IV_3(t)d(t) dt + \int_{t=T}^{T+L} IV(t)d(t) dt + \int_{t=T+L}^{\infty} PV(t)d(t) dt + \int_{t=T+L}^{\infty} IV(t)d(t) dt \\
&= D_1 \int_{t=T+L}^{\infty} \delta qe^{-rt}(1 - D(t)) dt + (D_2 - D_1) \int_{t=T+L}^{\infty} \delta qe^{-rt}(1 - D(t)) dt \\
&+ \int_{t=T+L}^{\infty} e^{-rt}(1 - D(t))(p + qD(t)(1 - \delta)) dt \\
&+ \int_{t=T+L}^{\infty} e^{-rt}q\delta(D(t) - D_2)(1 - D(t)) dt \\
&= \int_{t=T+L}^{\infty} e^{-rt}(1 - D(t))(p + qD(t)) dt \\
&= \int_{t=T+L}^{\infty} e^{-rt}d(t) dt
\end{aligned} \tag{A.36}$$

Therefore,

$$\int_{t=0}^{\infty} LV(t)d(t) dt = \int_{t=0}^T e^{-rt}d(t) dt + e^{-r(T+L)}(D_2 - D_1) + \int_{t=T+L}^{\infty} e^{-rt}d(t) dt = \int_{t=0}^{\infty} e^{-rt}s(t) dt$$

We then show that  $\int_{t=0}^{\infty} e^{-rt}s(t) dt \leq \int_{t=0}^{\infty} e^{-rt}d_{bass}(t) dt$ . To see this, we first note that for any  $\tau < T$ ,  $S(\tau) = D_{bass}(\tau)$ . For  $T \leq \tau < T + L$ ,  $S(\tau) < D_{bass}(\tau)$ . Finally, for  $\tau > T + L$ ,  $S(\tau) \leq D_{bass}(\tau)$ , because with the same diffusion speed, a diffusion process with a higher initial coverage can not lead to a lower final coverage. Therefore, we have  $S(\tau) \leq D_{bass}(\tau)$ ,  $\forall \tau$ . This is equivalent to

$$\int_{t=0}^{\tau} s(t) dt \leq \int_{t=0}^{\tau} d_{bass}(t) dt$$



Multiplying  $e^{-r\tau}$  on both sides, we have

$$e^{-r\tau} \int_{t=0}^{\tau} s(t) dt \leq e^{-r\tau} \int_{t=0}^{\tau} d_{bass}(t) dt \quad (\text{A.37})$$

Since (A.37) is true for all  $\tau$ , we take integration on both sides,

$$\int_{\tau=0}^{\infty} e^{-r\tau} \int_{t=0}^{\tau} s(t) dt d\tau \leq \int_{\tau=0}^{\infty} e^{-r\tau} \int_{t=0}^{\tau} d_{bass}(t) dt d\tau$$

Interchanging integration,

$$\int_{t=0}^{\infty} s(t) \int_{\tau=t}^{\infty} e^{-r\tau} d\tau dt \leq \int_{t=0}^{\infty} d_{bass}(t) \int_{\tau=t}^{\infty} e^{-r\tau} d\tau dt$$

Simplifying the inner integral gives us

$$\int_{t=0}^{\infty} e^{-rt} s(t) dt \leq \int_{t=0}^{\infty} e^{-rt} d_{bass}(t) dt$$

This completes the proof. Q.E.D.

**Proposition 20:** For any fixed  $L$ , an out-of-stock phenomenon that happens earlier in the product life cycle always leads to a greater loss in total customer LV.

Proof: To show this, we use  $S(t, T)$  and  $s(t, T)$  to denote the cumulative sales and sales rate respectively at  $t$  when a product out-of-stock happens at  $T$ .

Suppose  $T_1 < T_2$ . For any  $\tau < T_1$ , we have  $S(\tau, T_1) = S(\tau, T_2)$ . For  $T_1 \leq \tau \leq T_2$ ,  $S(\tau, T_1) < S(\tau, T_2)$ . For  $\tau > T_2$ ,  $S(\tau, T_1) \leq S(\tau, T_2)$ , because with the same diffusion speed, a higher initial coverage can not lead to lower final coverage.

Therefore, with a fixed  $L$ , for  $T_1 < T_2$ , we must have  $S(t, T_1) \leq S(t, T_2)$  for all  $t$ .

$$\int_{t=0}^{\tau} s(t, T_1) dt \leq \int_{t=0}^{\tau} s(t, T_2) dt$$

Multiplying  $e^{-r\tau}$  on both sides, we have

$$e^{-r\tau} \int_{t=0}^{\tau} s(t, T_1) dt \leq e^{-r\tau} \int_{t=0}^{\tau} s(t, T_2) dt \quad (\text{A.38})$$

Since (A.38) is true for all  $\tau$ , we take integration on both sides,

$$\int_{\tau=0}^{\infty} e^{-r\tau} \int_{t=0}^{\tau} s(t, T_1) dt d\tau \leq \int_{\tau=0}^{\infty} e^{-r\tau} \int_{t=0}^{\tau} s(t, T_2) dt d\tau$$

Interchanging integration,

$$\int_{t=0}^{\infty} s(t, T_1) \int_{\tau=t}^{\infty} e^{-r\tau} d\tau dt \leq \int_{t=0}^{\infty} s(t, T_2) \int_{\tau=t}^{\infty} e^{-r\tau} d\tau dt$$

Simplifying the inner integral gives us

$$\int_{t=0}^{\infty} e^{-rt} s(t, T_1) dt \leq \int_{t=0}^{\infty} e^{-rt} s(t, T_2) dt$$

By Proposition 19, we then claim that the stockout at  $T_1$  leads to a lower total customer LV than the stockout at  $T_2$ . Q.E.D.

**Solutions for  $F_i(t, \cdot)$ :**

To simplify notation, we omit the time argument from functions and write  $F_i$  instead of  $F_i(t, F_0^1, F_0^2, F_0^3, F_0^4)$ ,  $i = 1, 2, 3, 4$ .

**Solution for  $F_1(t, F_0^1, F_0^2, F_0^3, F_0^4)$**

$f_1 = p_1(\theta_1 - F_1)$ , that is,  $\frac{dF_1}{\theta_1 - F_1} = p_1 dt$ , the solution for this differential equation is  $F_1 = \theta_1 - e^{-p_1(t+C)}$ . Since  $F_1 = \frac{F_0^1}{N}$ ,  $\frac{F_0^1}{N} = \theta_1 - e^{-p_1 C}$ , and  $C = -\frac{1}{p_1} \ln \theta_1$ . Substituting the value of  $C$ , we have  $F_1 = \theta_1(1 - e^{-p_1 t}) + \frac{F_0^1}{N} e^{-p_1 t}$ .

**Solution for  $F_2(t, F_0^1, F_0^2, F_0^3, F_0^4)$**

We have

$$\begin{aligned} \frac{dF_2}{dt} &= q_1(F_1 + F_2)(\theta_2 - F_2) \\ &= (q_1\theta_1 - q_1\theta_1 e^{-p_1 t} + q_1 \frac{F_0^1}{N} e^{-p_1 t} + q_1 F_2)(\theta_2 - F_2) \\ &= (q_1\theta_1 - q_1\theta_1 e^{-p_1 t} + q_1 \frac{F_0^1}{N} e^{-p_1 t})\theta_2 + (q_1\theta_2 - q_1\theta_1 + q_1\theta_1 e^{-p_1 t} - q_1 \frac{F_0^1}{N} e^{-p_1 t})F_2 \\ &\quad - q_1 F_2^2 \end{aligned} \tag{A.39}$$

Equation (A.39) is a Ricatti equation of the general form  $\frac{dF_2}{dt} = P(t) + Q(t)F_2 + R(t)F_2^2$ , with

$$\begin{aligned} P(t) &= \theta_2\theta_1 q_1(1 - e^{-p_1 t}) + q_1 \frac{F_0^1}{N} e^{-p_1 t} \\ Q(t) &= q_1\theta_2 - q_1\theta_1 + \theta_1 q_1 e^{-p_1 t} - q_1 \frac{F_0^1}{N} e^{-p_1 t} \\ R(t) &= -q_1 \end{aligned}$$

We observe that  $F_2 = \theta_2$  is a potential solution, and let  $z = \frac{1}{F_2 - \theta_2}$ , then  $F_2 = \theta_2 + \frac{1}{z}$ , and

$$\frac{dF_2}{dt} = -\frac{1}{z^2} \frac{dz}{dt}.$$

Equation (A.39) now becomes

$$\begin{aligned}
-\frac{1}{z^2} \frac{dz}{dt} &= P(t) + Q(t) \frac{z\theta_2 + 1}{z} - q_1 \frac{(z\theta_2 + 1)^2}{z^2} \\
\frac{dz}{dt} &= (-P(t) - Q(t)\theta_2 + q_1\theta_2^2)z^2 - Q(t)z + 2q_1\theta_2z + q_1 \\
\frac{dz}{dt} &= q_1 + q_1\theta_1z - \theta_1q_1e^{-p_1t}z + q_1\theta_2z + q_1 \frac{F_0^1}{N} e^{-p_1t}
\end{aligned} \tag{A.40}$$

Equation (A.40) is of the form  $\frac{dz}{dt} + P_1(t)z = Q_1(t)$ , with

$$\begin{aligned}
P_1(t) &= \theta_1q_1e^{-p_1t} - q_1\theta_1 - q_1\theta_2 - q_1 \frac{F_0^1}{N} e^{-p_1t} \\
Q_1(t) &= q_1
\end{aligned}$$

For  $F_2$  continuous in  $[0, \theta_2]$ ,  $z$  is continuous in  $(-\infty, -\frac{1}{\theta_2}]$ , thus the general solution for equation (A.40) is

$$z = \frac{\int R(s)Q_1(s) ds + C}{R(t)} \tag{A.41}$$

where  $R(t) = \exp(\int P_1(s) ds)$ .

$$\begin{aligned}
\int P_1(s) ds &= -q_1\theta_1t - q_1\theta_2t + (\theta_1 - \frac{F_0^1}{N}) \int q_1e^{-p_1s} ds \\
&= -q_1\theta_1t - q_1\theta_2t - \frac{(\theta_1 - \frac{F_0^1}{N})q_1}{p_1} e^{-p_1t}
\end{aligned}$$

We get

$$R(t) = \exp\left(-q_1\theta_1t - q_1\theta_2t - \frac{(\theta_1 - \frac{F_0^1}{N})q_1}{p_1} e^{-p_1t}\right)$$

And hence

$$\int R(s)Q_1(s) ds = \int_{-\infty}^t \exp\left(-q_1\theta_1s - q_1\theta_2s - \frac{(\theta_1 - \frac{F_0^1}{N})q_1}{p_1} e^{-p_1s}\right) q_1 ds \tag{A.42}$$

To solve (A.42), we do another transformation by letting  $a = e^{-p_1s}$ , thus  $s = -\frac{1}{p_1} \ln a$ , and  $ds = -\frac{1}{p_1 a} da$ . Equation (A.42) then becomes

$$\begin{aligned}
\int R(s)Q_1(s) ds &= q_1 \int_{e^{-p_1t}}^{\infty} \exp\left(\frac{q_1}{p_1}(\theta_1 - \frac{F_0^1}{N}) \ln a + \frac{q_1}{p_1} \theta_2 \ln a - \frac{q_1}{p_1} a \theta_1\right) \left(\frac{1}{p_1 a}\right) da \\
&= \frac{q_1}{p_1} \int_{e^{-p_1t}}^{\infty} a^{\frac{q_1\theta_1 + q_1\theta_2}{p_1} - 1} e^{-\frac{q_1(\theta_1 - \frac{F_0^1}{N})a}{p_1}} da \\
&= \frac{q_1}{p_1} \left(\frac{q_1}{p_1}(\theta_1 - \frac{F_0^1}{N})\right)^{-\frac{q_1\theta_1 + q_1\theta_2}{p_1}} \Gamma\left(\frac{q_1\theta_1 + q_1\theta_2}{p_1}, \frac{q_1}{p_1}(\theta_1 - \frac{F_0^1}{N})a\right)
\end{aligned} \tag{A.43}$$

Where in (A.43),  $\Gamma(\eta, k)$  is the ‘‘upper’’ incomplete gamma function,  $\Gamma(\eta, k) = \int_k^\infty v^{\eta-1} e^{-v} dv$ . Substituting  $a = e^{-p_1 t}$  in (A.43) and then back to (A.41), we obtain

$$z(t) = \frac{\frac{q_1}{p_1} \left( \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)^{-\frac{q_1 \theta_1 + q_1 \theta_2}{p_1}} \Gamma \left( \frac{q_1 \theta_1 + q_1 \theta_2}{p_1}, \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) e^{-p_1 t} \right) + C}{\exp \left( -q_1 \theta_1 t - q_1 \theta_2 t - \frac{q_1}{p_1} e^{-p_1 t} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)}$$

Transforming  $z$  back to  $F_2$ , we get

$$F_2 = \theta_2 + \frac{\exp \left( -q_1 \theta_1 t - q_1 \theta_2 t - \frac{q_1}{p_1} e^{-p_1 t} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)}{\frac{q_1}{p_1} \left( \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)^{-\frac{q_1 \theta_1 + q_1 \theta_2}{p_1}} \Gamma \left( \frac{q_1 \theta_1 + q_1 \theta_2}{p_1}, \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) e^{-p_1 t} \right) + C}$$

As  $F_2(0, F_0^1, F_0^2, F_0^3, F_0^4) = \frac{F_0^2}{N}$ , we have

$$\frac{F_0^2}{N} = \theta_2 + \frac{\exp \left( -\frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)}{\frac{q_1}{p_1} \left( \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)^{-\frac{q_1 \theta_1 + q_1 \theta_2}{p_1}} \Gamma \left( \frac{q_1 \theta_1 + q_1 \theta_2}{p_1}, \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right) + C}$$

Then

$$C = -\frac{\exp \left( -\frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)}{\theta_2 - \frac{F_0^2}{N}} - \frac{q_1}{p_1} \left( \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)^{-\frac{q_1 \theta_1 + q_1 \theta_2}{p_1}} \Gamma \left( \frac{q_1 \theta_1 + q_1 \theta_2}{p_1}, \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)$$

Thus,

$$\begin{aligned} F_2 = & \theta_2 + \exp \left( -q_1 \theta_1 t - q_1 \theta_2 t - \frac{q_1}{p_1} e^{-p_1 t} \left( \theta_1 - \frac{F_0^1}{N} \right) \right) \\ & \left( \frac{q_1}{p_1} \left( \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right) \right)^{-\frac{q_1 \theta_1 + q_1 \theta_2}{p_1}} \left( \Gamma \left( \frac{q_1 \theta_1 + q_1 \theta_2}{p_1}, \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) e^{-p_1 t} \right) \right. \\ & \left. - \Gamma \left( \frac{q_1 \theta_1 + q_1 \theta_2}{p_1}, \frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right) \right) - \frac{\exp \left( -\frac{q_1}{p_1} \left( \theta_1 - \frac{F_0^1}{N} \right) \right)}{\theta_2 - \frac{F_0^2}{N}} \end{aligned}$$

**Solution for  $F_3(t, F_0^1, F_0^2, F_0^3, F_0^4)$**

$f_3 = p_2(\theta_3 - F_3)$ , so we perform a similar solution procedure of  $F_1$ , and obtain  $F_3 = \theta_3(1 - e^{-p_2 t}) + \frac{F_0^3}{N} e^{-p_2 t}$ .

**Solution for**  $F_4(t, F_0^1, F_0^2, F_0^3, F_0^4)$

We have

$$\begin{aligned} \frac{dF_4}{dt} &= q_2(w(F_1 + F_2) + (1 - w)(F_3 + F_4))(\theta_4 - F_4) \\ &= q_2(wF_1 + wF_2 + (1 - w)F_3)\theta_4 + (q_2(1 - w)\theta_4 - q_2(wF_1 + wF_2 + (1 - w)F_3))F_4 \\ &\quad - q_2(1 - w)F_4^2 \end{aligned} \quad (\text{A.44})$$

Equation (A.44) is a Ricatti equation of the general form  $\frac{dF_4}{dt} = P(t) + Q(t)F_4 + R(t)F_4^2$ , with

$$\begin{aligned} P(t) &= q_2\theta_4(wF_1 + wF_2 + (1 - w)F_3) \\ Q(t) &= q_2(1 - w)\theta_4 - q_2(wF_1 + wF_2 + (1 - w)F_3) \\ R(t) &= -q_2(1 - w) \end{aligned}$$

We observe that  $F_4 = \theta_4$  is a potential solution, and let  $z = \frac{1}{F_4 - \theta_4}$ , then  $F_4 = \theta_4 + \frac{1}{z}$ , and

$$\frac{dF_4}{dt} = -\frac{1}{z^2} \frac{dz}{dt}.$$

Equation (A.44) now becomes

$$\begin{aligned} -\frac{1}{z^2} \frac{dz}{dt} &= P(t) + Q(t) \frac{z\theta_4 + 1}{z} - q_2(1 - w) \frac{(z\theta_4 + 1)^2}{z^2} \\ \frac{dz}{dt} &= (2q_2\theta_4(1 - w) - Q(t))z + q_2(1 - w) \end{aligned} \quad (\text{A.45})$$

$$= q_2(wF_1 + wF_2 + (1 - w)F_3 + (1 - w)\theta_4)z + q_2(1 - w) \quad (\text{A.46})$$

Equation (A.46) is of the form  $\frac{dz}{dt} + P_1(t)z = Q_1(t)$ , with

$$\begin{aligned} P_1(t) &= -q_2(wF_1 + wF_2 + (1 - w)F_3 + (1 - w)\theta_4) \\ Q_1(t) &= q_2(1 - w) \end{aligned}$$

For  $F_4$  continuous in  $[0, \theta_4]$ ,  $z$  is continuous in  $(-\infty, -\frac{1}{\theta_4}]$ , thus the general solution for equation (A.40) is

$$z = \frac{\int_{-\infty}^t R(s)Q_1(s) ds + C}{R(t)} \quad (\text{A.47})$$

where  $R(t) = \exp(\int P_1(s) ds)$ .

Transforming  $z$  back to  $F_4$ , we get

$$F_4 = \theta_4 + \frac{R(t)}{\int_{-\infty}^t R(s)Q_1(s) ds + C}$$

As  $F_4(0) = \frac{F_0^4}{N}$ , we have

$$\frac{F_0^4}{N} = \theta_4 + \frac{R(0)}{\int_{-\infty}^0 R(s)Q_1(s) ds + C}$$

Then

$$C = -\frac{1}{\theta_4 - \frac{F_0^4}{N}}$$

Thus,

$$F_4 = \theta_4 + \frac{R(t)}{q_2(1-w) \int_0^t R(s) ds - \frac{1}{\theta_4 - \frac{F_0^4}{N}}} \quad (\text{A.48})$$

where  $R(t) = \exp(\int_0^t -q_2(wF_1(s, \cdot) + wF_2(s, \cdot) + (1-w)F_3(s, \cdot) + (1-w)\theta_4) ds)$  and  $F_1(s, \cdot)$ ,  $F_2(s, \cdot)$ ,  $F_3(s, \cdot)$  can be obtained from previous results.

Equations (3.5) to (3.8) can be obtained as a special case where  $F_0^i = 0, \forall i$ .

**Proposition 21:**  $\sum_{i=1}^4 \int_{t=0}^{\tau} LV_i(t) f_i(t) dt = \sum_{i=1}^4 \int_{t=0}^{\tau} e^{-rt} f_i(t) dt$ .

Proof:

First, let us look at the aggregated customer LV of type 1 customers.

$$\begin{aligned} & \int_{t=0}^{\tau} LV_1(t) f_1(t) dt \\ = & \delta \left( \int_{t=0}^{\tau} \int_{s=t}^{\tau} e^{-rt} \left( \frac{f_2(s)}{F_1(s) + F_2(s)} + \frac{w f_4(s)}{w(F_1(s) + F_2(s)) + (1-w)(F_3(s) + F_4(s))} \right) ds f_1(t) dt \right) \\ & + \int_{t=0}^{\tau} e^{-rt} f_1(t) dt \\ = & \delta \left( \int_{s=0}^{\tau} \int_{t=0}^s f_1(t) dt e^{-rt} \left( \frac{f_2(s)}{F_1(s) + F_2(s)} + \frac{w f_4(s)}{w(F_1(s) + F_2(s)) + (1-w)(F_3(s) + F_4(s))} \right) ds \right) \\ & + \int_{t=0}^{\tau} e^{-rt} f_1(t) dt \\ = & \delta \left( \int_{s=0}^{\tau} \frac{e^{-rt} F_1(s)}{F_1(s) + F_2(s)} f_2(s) ds + \int_{s=0}^{\tau} \frac{e^{-rt} w F_1(s)}{w(F_1(s) + F_2(s)) + (1-w)(F_3(s) + F_4(s))} f_4(s) ds \right) \\ & + \int_{t=0}^{\tau} e^{-rt} f_1(t) dt \end{aligned}$$

The aggregated customer LV of other types can be written in similar expressions. Summing them up we conclude that  $\sum_{i=1}^4 \int_{t=0}^{\tau} LV_i(t) f_i(t) dt = \sum_{i=1}^4 \int_{t=0}^{\tau} e^{-rt} f_i(t) dt$ . Q. E. D.

**Proposition 22:**

$LV_i(t)$   $i = 1, \dots, 4$  decreases with  $p_1$ ;  $LV_3(t)$  and  $LV_4(t)$  decreases with  $p_2$ .

Proof: As  $p_1$  increases, we know from (3.5) that  $F_1(t)$  will increase. It follows from (3.2) and (3.4) that  $F_2(t)$  and  $F_4(t)$  will both increase as well. Therefore, all  $IV_i(t)$   $i = 1, \dots, 4$  will decrease from (3.18) and (3.19). As PVs are independent of  $p_1$ ,  $LV_i(t)$   $i = 1, \dots, 4$  decreases with  $p_1$ .

As  $p_2$  increases, we know from (3.7) that  $F_3(t)$  will increase. It follows from (3.4) that  $F_4(t)$  will increase as well. Therefore, we have that all  $IV_3(t)$  ( $IV_4(t)$ ) will decrease from (3.19).

As  $PV_3(t)$  and  $PV_4(t)$  are independent of  $p_2$ , we have  $LV_3(t)$  and  $LV_4(t)$  decreases with  $p_2$ . Q. E. D.

**Proposition 23:**

$PV_i(t), IV_i(t)$  and  $LV_i(t)$   $i = 1, \dots, 4$  are all decreasing convex in  $t$ .

Proof:

$$\begin{aligned} \frac{dIV_3(t)}{dt} &= -\delta q_2(1-w)(\theta_4 - F_4(t))e^{-rt} < 0 \\ \frac{d^2IV_3(t)}{dt^2} &= \delta q_2(1-w)(e^{-rt}f_4(t) + re^{-rt}(\theta_4 - F_4(t))) > 0 \end{aligned}$$

Thus,  $IV_3(t)$  ( $IV_4(t)$ ) is decreasing convex in  $t$ . Similarly,  $IV_1(t)$  ( $IV_2(t)$ ) is decreasing convex in  $t$ . Besides, as PVs are all decreasing convex in  $t$ , the above results hold for  $LV_i(t)$   $i = 1, \dots, 4$ . Q. E. D.

**Proposition 24:**

We have

- 1)  $LV_1(t) > LV_2(t)$ ,  $LV_3(t) > LV_4(t)$ ,  $\forall t \in [0, \tau]$ .
- 2) If  $w \geq 1/2$ ,  $LV_1(t) \geq LV_3(t)$ ,  $LV_2(t) \geq LV_4(t)$ ,  $\forall t \in [0, \tau]$ .
- 3) **a.** If  $w \geq 1/2$ , there exists a cutoff time  $\hat{t}$ , s.t.,  $\forall t > \hat{t}$ ,  $LV_3(t) > LV_2(t)$ .  
**b.**  $\hat{t}$  decreases with  $p_1$  and  $p_2$ .

Proof: Part (1) and (2) are straightforward from (3.18) and (3.19). To see part (3a), we first note that  $LV_2(t) > LV_3(t)$  if and only if

$$q_1 \int_t^\tau e^{-rt}(\theta_2 - F_2(s)) ds + q_2(2w - 1) \int_t^\tau e^{-rt}(\theta_4 - F_4(s)) ds > 1 \quad (\text{A.49})$$

Taking derivative (with respect to  $t$ ) to the LHS in (A.49), and when  $w \geq 1/2$  we have

$$e^{-rt}(-q_1(\theta_2 - F_2(t)) - q_2(2w - 1)(\theta_4 - F_4(t))) < 0$$

So the LHS of (A.49) decreases with  $t$ . Note also that (A.49) does not hold when  $t = \tau$ . Let  $\hat{t} = \{t : q_1 \int_t^\tau \theta_2 - F_2(s) ds + q_2(2w - 1) \int_t^\tau \theta_4 - F_4(s) ds = 1\}$ , then  $\forall t > \hat{t}$ ,  $LV_3(t) > LV_2(t)$ . Part (3b) directly follows from Proposition 22. Q. E. D.

**Proposition 25:**

If  $\theta_1 = \theta_3$ ,  $p_1 = p_2$ , it's preferable to distribute gifts to type 1 customers than to type 3 customers. If  $\theta_2 = \theta_4$ ,  $q_1 = q_2$ ,  $w \geq 1/2$ , it's preferable to distribute gifts to type 2 customers than to type 4 customers.

Proof: This is straightforward from Equations (3.1) to (3.4).

**Proposition 26:**

If  $w = 1$ , the firm should never send samples to type 3 or type 4 customers.

Proof: If  $w = 1$ ,  $IV_3(0, \cdot) = 0$ , as  $\int_0^\tau LV_3(t, F_0^1, F_0^2, F_0^3, F_0^4) f_3(t, \cdot) dt \geq 0$ ,  $IV_3(0, F_0^1, F_0^2, F_0^3 + 1, F_0^4) - \int_0^\tau LV_3(t, F_0^1, F_0^2, F_0^3, F_0^4) f_3(t, \cdot) dt \leq 0$ . This holds for  $i = 4$  as well. So the firm

should never send samples to type 3 or type 4 customers. Q. E. D.

**Proposition 27:**

If  $w = 0$ ,  $q_1 = 0$ , the firm should never send samples to type 2 customers.

Proof: If  $w = 0$ ,  $q_1 = 0$ ,  $IV_2(0, \cdot) = 0$ , as  $\int_0^\tau LV_2(t, F_0^1, F_0^2, F_0^3, F_0^4)f_2(t, \cdot) dt \geq 0$ ,  $IV_2(0, F_0^1, F_0^2, F_0^3+1, F_0^4) - \int_0^\tau LV_2(t, F_0^1, F_0^2, F_0^3, F_0^4)f_2(t, \cdot) dt \leq 0$ . So the firm should never send samples to type 2 customers. Q. E. D.



# Appendix B

## A Pharmaceutical Game

Education research shows that playing games is far more engaging than listening to lectures, as inspiring people to learn new things often requires challenges, curiosity, fantasy and control. Motivated by this, I have led to design and develop a teaching game on the topic of new drug development in the pharmaceutical industry. In the game, students working in teams, act as executives of a pharmaceutical firm. They need to figure out strategies of planning resource capacity, budgeting capital, managing R&D portfolio, and then apply them to a simulated pharmaceutical R&D environment to compete for final profit. The first version of the game was played in the course of *Service Operations Design and Analysis* in 2007 Fall, then an improved version was played in 2009 Spring. Students LOVE this game! They were so enjoyed and the winning team was thrilled when they received the prize - chocolate! Each year after the game, my email box was filled with excitement and appreciation from students. They were so amazed that the R&D management in pharmaceutical industry, which is considered as a very complicated and overwhelming decision process, can become so crystal-clear after a 1-hour game with a lot of fun. I gave a talk on this teaching game at INFORMS Annual Conference in 2008 and received many positive feedbacks.

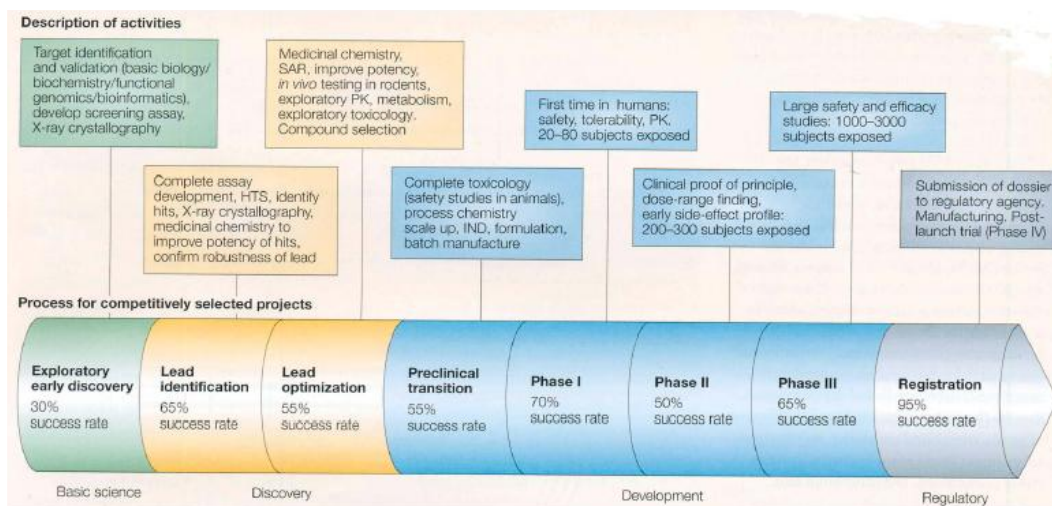
### **Game Background:**

Founded in 1868, Golden Bear Health Care has established itself as one of the biggest and most successful producers of pharmaceutical products in Bay area. With its consistent innovation in medicine, Golden Bear now has a steady stream of blockbuster drugs, including Aekit, Mavoir, Tloea and Nigxx, bringing approximately 20 million profit every year.

To continue its business success, in the next 30 years, Golden Bear is planning to invest in the development process of several drugs. You are hired as the executive of Golden Bear's R&D Planning Department to manage the new product pipeline as well as to develop a robust strategy to hedge against the risk, in order to sustain the firm's long term growth. Before starting the exciting journey, you first go over the inherently complicated structure of pharmaceutical industry and the challenges you are facing:

1. Process: New drug development in the pharmaceutical industry is regulated and thus, proceeds along a series of self-defined steps: After going through the basic research,

drugs are tested on animals at pre-clinical stage, then administered to healthy human volunteers at Phase I, to small-scale and large-scale patients with target disease or indication at Phase II and III, and finally released to market if FDA approves. Pharmaceutical firms make some of the biggest gambles of any industry: no financial benefit accrues until a drug is marketed, while costs are incurred from the moment when research starts, typically 10-15 years earlier. Figure B.1 shows the various activities involved in the development of a new drug product.



From Nature Reviews 2:919, 2003

Figure B.1: New Drug Development Process

2. **Patent Protection:** The patent protection system by FDA offers exclusive rights to sell a drug approved by FDA, creating a temporary monopoly position for the pharmaceutical firm in the marketplace with limited competitive pressure. Patents are usually protected for 20 years, and when they are due to expire, knockoffs will eat deeply into the market share. However, as patent clock starts ticking from pre-clinical, the longer a drug remains in the development pipeline, the shorter its patent covered sales period.
3. **Uncertainties:** High uncertainties at both technical and marketing sides are another big challenge to Pharmaceutical industry. Because of the low success rate, long and variable development durations, resource requirement, facility capital cost, internal dependencies, it makes the technical of pharmaceutical industry very much unpredictable. The broad uncertainty in sales estimates along with that in technical side enforces pharmaceutical firms face the fundamental tradeoffs between risk and revenues.
4. **Risk Management:**
  - (a) **Risk management for a drug:** This refers to the rule that the firm always has the freedom to invest on more back-up compounds with more pharmacologists

and patients to increase the success rate, since the more compounds go into a stage, the higher the probability that there is at least one that passes the stage. But unfortunately, no rule is perfect, even under extremely heavy investment, it won't rule out the possibility of losing the gamble.

- (b) Risk management across drugs (Resource management): With limited resources, such as scientists, infrastructure and capitals, all drugs can not be developed simultaneously. On one hand, resource constraints tend to delay the progress of these drugs to market introduction. On the other hand, the inherently low success rates associated with new drug development, along with the uncertain test duration at each stage, make the demand for resources at different stages of the development pipeline highly variable and hence very difficult to predict. Risk management across drugs refers to the investment decision on resources when a firm wants to develop more than one drug at the same time. If resource is under-investment, one may see potential drugs waiting for development resources, leading to more possible revenue lost. But if the firm invests too much, we might see development resources waiting for drugs, causing resource wastes. Golden Bear has the option to expand its resources every 5 years, but once expanded, they can not be reduced back.
5. Dependencies: On the cost side, due to resource sharing, the combined cost of development activities for drugs targeting the same disease is less than the sum of the individual costs. On the financial return side, cannibalization occurs when more than one drug succeeds in a category, competing each other in the marketplace, making the combined revenue of successful drugs targeting the same disease less than the sum of the individual revenues. While synergies occur if products are complement to each other, and will enhance the total sales.

Golden Bear is planning to invest in the development process of several drugs targeting three diseases: obesity, depression and diabetes. Specifically, there are 3 anti-obesity drugs specially designed for kids (Obesity-A), female adults (Obesity-B) and male adults (Obesity-C) respectively, 2 drugs with different potency to achieve adequate relief of depressive symptoms (Depression-A, Depression-B) and 3 drugs targeting diabetes of type 1 (Diabetes-A), type 1.5 (Diabetes-B) and type 2 (Diabetes-C) respectively. Information of each drug candidate is listed as follows:

1. Golden Bear's objective is to maximize its profit for the next 30 years. Each drug candidate has to go through preclinical, Phase I, Phase II, Phase III stages and subject to FDA approval.
2. Developing each drug requires 1 unit of resource, and Golden Bear has the option to expand resources at each stage every 5 years. Each year it chooses to expand resource for some stage  $k$ ,  $u^k$  units of resources are added to that stage one year later. Note that once the capacity gets expanded, it cannot be reduced during the horizon. Each

year at stage  $k$ , each unit of existing resource incurs  $\$V_f^k$  fixed cost, each unit of resource being occupied incurs  $\$V_m^k$  manufacturing cost, and each unit of resource being expanded incurs  $\$V_e^k$  expansion cost.

Stage $k$	$u^k$	$V_f^k$	$V_m^k$	$V_e^k$
Pre-Clinic	2	50	10	2000
Phase I	1	50	10	2000
Phase II	1	50	15	2000
Phase III	1	20	40	2000

3. The probability of drug  $i$  successfully passing stage  $k$  ( $p_k^i$ ) is measured by level “Low (L),” “MediumLow (ML),” “Medium (M),” “MediumHigh (MH)” or “High (H)”.

Probability Level	Prob Value
Low	0.17
MediumLow	0.33
Medium	0.50
MediumHigh	0.67
High	0.85
FDA	0.99

4. Patent clock starts ticking from pre-clinical, and patents are usually protected for 20 years. The yearly revenue for Drug  $i$  achieves  $r_1^i$  if a successful drug is under protection, and drops to  $r_2^i$  otherwise.

Drug	$r_1^i$	$r_2^i$	$p_{preclinical}^i$	$p_{phaseI}^i$	$p_{phaseII}^i$	$p_{phaseIII}^i$	$p_{FDA}^i$
Obesity-A	2000	400	M	M	M	M	FDA
Obesity-B	1500	350	M	ML	ML	ML	FDA
Obesity-C	1000	200	ML	L	L	ML	FDA
Depression-A	3000	400	ML	ML	ML	L	FDA
Depression-B	4000	500	M	L	L	L	FDA
Diabetes-A	4000	500	M	L	L	ML	FDA
Diabetes-B	2000	200	M	M	ML	ML	FDA
Diabetes-C	1000	50	ML	M	L	M	FDA

5. The development time for drug  $i$  at stage  $k$  follows a triangular distribution, with specified values for parameters of min, middle, max.

Stage	Obe-A	Obe-B	Obe-C	Dep-A	Dep-B	Dia-A	Dia-B	Dia-C
Preclinical(min)	1	1	1	1	1	1	1	1
Preclinical(middle)	2	3	2	2	2	2	2	2
Preclinical(max)	3	4	5	5	4	4	3	4
PhaseI(min)	1	1	1	1	1	1	1	1
PhaseI(middle)	4	4	3	2	2	2	3	2
PhaseI(max)	5	5	5	4	4	3	4	3
PhaseII(min)	1	1	1	1	1	1	1	1
PhaseII(middle)	2	2	2	3	3	3	3	3
PhaseII(max)	3	5	3	4	5	4	4	4
PhaseIII(min)	2	2	2	1	1	1	1	1
PhaseIII(middle)	4	3	3	2	2	2	2	3
PhaseIII(max)	5	5	5	4	4	3	3	4

- At any stage, Golden Bear can choose to invest on the pass rate for a drug by hiring more scientists or patients. The investment cost is  $N * 1000$  if the pass rate is invested to be  $N$  level up (i.e.  $N=1$ ,  $L \rightarrow ML$ ,  $ML \rightarrow M$ ,  $M \rightarrow MH$ ,  $MH \rightarrow H$ ;  $N=2$ ,  $L \rightarrow M, \dots$ ).
- The manufacturing dependency at each stage is captured by a cost coefficient (Total cost = Sum of individual costs \* Cost coefficient), depending on  $M$ , the number of drugs of the same category being developed in that stage. Similarly, the financial dependency is captured by a revenue coefficient, determined by the number of drugs of the same category that are being marketed.

Drug Category	M	Coefficient(cost)	Coefficient(revenue)
Obesity	2	0.8	0.8
Obesity	3	0.7	0.7
Depression	2	0.8	0.8
Diabetes	2	0.8	0.8
Diabetes	3	0.7	0.7

Now is your turn! Enjoy the journey and good luck!