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On Conformal Superspace and the One-loop Effective Action in Supergravity

by

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Abstract

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We outline a program for the calculation of the one-loop effective action for generic supergravity theories in superspace. The first step involves the construction of a conformal superspace (with the conformal algebra as the structure group) to facilitate the algebraic manipulations necessary to deal with the underlying conformal coupling of chiral matter to supergravity. Next we show how to expand actions to second order in the fundamental quantum variables to allow one-loop computations. Finally, we describe how the chiral loops may be handled by explicitly calculating their divergences and anomalies.

To the parents who shaped my past and the woman who will shape my future

Contents

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Chapter 1

Introduction and motivation

Over the last thirty years, supersymmetry has become nearly a cornerstone of modern research in particle theory. Its many theoretical successes (e.g. a possible resolution of the hierarchy problem, better gauge coupling unification, some insight into the origin of dark matter) lead us to confidently expect some superpartners to be detected by the LHC in the near future. On the other hand, the combination of supersymmetry with gravity has not solved the latter's problems. The fundamental divergences and concommitant non-renormalizability are softened by the presence of supersymmetry but not removed entirely.

Nevertheless quantum effects within supergravity may still give us some insight if we take the point of view that supergravity is some low energy approximation of an ultimately finite theory – the prime candidate being string theory. We expect then that the divergences in low energy loops involving supergravity, matter, and gauge fields should be cancelled by heavy string modes. Certainly, the form of any anomalies of the low energy theory should be extremely limited, with an effective four dimensional version of the Green-Schwarz anomaly cancellation mechanism playing some role. These features can be explored quite directly using standard field theoretic techniques. Indeed, this has been the guiding principle of the work of Gaillard and collaborators, who over the years have examined these very features in general $\mathcal{N} = 1$ supergravity theories using standard techniques [1, 2]. Because of the sheer complexity of the interactions involved, the standard approach has been to work in the background field formulation with background fermions turned off, breaking the background supersymmetry of the theory. There were two reasons for going about it this way: first, it allowed existing field theory techniques to be applied directly to a supersymmetric theory while remaining tractable; and second, there were simply no manifestly supersymmetric methods which could easily handle supergravity coupled to matter.

The goal of this thesis work has been to make supergravity calculations more tractable while maintaining manifest supersymmetry. The latter requirement is best handled by working in superspace, where the four dimensional manifold of spacetime is extended to a supermanifold with extra Grassmannian coordinates obeying fermionic statistics (i.e. they possess an odd grading under multiplication). The algebraic structure of supergravity on such a space is rather involved; for example, in the standard "old minimal" way of formulating supergravity coupled to matter, the Einstein-Hilbert and Rarita-Schwinger actions describing supergravity are mixed with the matter action under supersymmetry.

This thesis is divided into two parts. In part I, we describe the construction of conformal superspace and demonstrate how to expand rather generic Poincaré invariant supergravity theories in terms of the unconstrained superfields describing the underlying conformal structure. The material in this part comes from two previous papers by the author [3, 4]. In part II, we review how the one-loop effective action for chiral loops may be constructed and describe methods for calculating its divergences and anomalies within superspace. The material here also largely comes from a previous published work [5].

We use throughout the superspace notations and conventions of Binetruy, Girardi, and Grimm [6] (which are a slight modification of those of Wess and Bagger [7]) – with our own slight modification: we choose the superspace $U(1)$ connection to be Hermitian. That is, our connection A_M here is equivalent to $-iA_M$ of [6]; similarly, our corresponding generator A is equivalent to their iA. (The unfortunate coincidence of the generator and connection names will, we hope, not overly confuse the reader.)

Part I

Conformal superspace and its variational structure

Chapter 2

The conformal structure of superspace

The use of conformal techniques to address supergravity has a long history. Not all that long after Wess and Zumino discovered the superspace formulation of supergravity [8], Kaku, Townsend, and van Nieuwenhuizen, along with Ferrara and Grisaru, worked out the conformal structure of component supergravity and demonstrated that Poincare supergravity was a gauge-fixed version of conformal supergravity [9]. Howe first proposed superspace formulations of four-dimensional $\mathcal{N} \leq 4$ conformal supergravities by explicitly gauging $\mathrm{SL}(2,\mathcal{C}) \times \mathrm{U}(\mathcal{N})$ [10]. Work continued on conformal supergravity over the next few years (an excellent review [11] on the topic was written by Fradkin and Tseytlin) eventually culminating in the work of Kugo and Uehara, who not only popularized the conformal compensator approach to supergravity and matter systems [12] but also made a comprehensive analysis of the component transformation rules and spinorial derivative structure of $\mathcal{N} = 1$ conformal supergravity [13].

In large part, the results presented in this chapter are a superspace response to this last work. Here we will take a complementary approach, treating superspace as an honest supermanifold with a conformal structure. Unlike Howe, we will seek to gauge the *entire* superconformal algebra. Prior experience with superspace hints that this approach would be a foolish one – that the constraints required with a larger structure group would be more numerous and their evaluation more cumbersome. What we find is the opposite: the covariant derivatives of conformal supergravity have a Yang-Mills structure, with the algebra

$$\{\nabla_\alpha, \nabla_\beta\} = 0, \quad \{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\} = 0$$
$$\{\nabla_\alpha, \nabla_{\dot\alpha}\} = -2i\nabla_{\alpha\dot\alpha}$$
$$\{\nabla_\beta, \nabla_{\alpha\dot\alpha}\} = -2i\epsilon_{\beta\alpha}\mathcal{W}_{\dot\alpha}, \quad \{\nabla_{\dot\beta}, \nabla_{\alpha\dot\alpha}\} = -2i\epsilon_{\dot\beta\dot\alpha}\mathcal{W}_\alpha$$

where $\mathcal{W}_\alpha$ are the "gaugino superfields" for the superconformal group. The constraints of conformal superspace involve setting most of the $\mathcal{W}_\alpha$ to zero, and the evaluation of these constraints is no more difficult than in a conventional Yang-Mills theory, leading the non-vanishing $\mathcal{W}_\alpha$ to be expressed in terms of the single superfield $W_{\alpha\beta\gamma}$. When the theory is "degauged" to a $U(1)$ Poincaré supergravity, the extra gauge superfields can be reinterpreted as the familiar superfields R, G_c, and X_α. This is the main result of this work.

It is well known that the various equivalent formalisms of superspace supergravity – the minimal Poincaré [14], the minimal Kähler [6], and even the new minimal Poincaré [15] – are all derivable from a conformal superspace under different gauge-fixing constraints. We review one way of seeing how this occurs in our approach.

This chapter is divided into three sections. In the first, we present an elementary review of the structure of global and local symmetry groups as well as the structure of actions over both the full manifold and submanifolds of such theories. There is no pretense to completeness or even rigor, but standardizing notation and justifying what exactly a gauged special conformal transformation is are reasonable justifications for its inclusion. In the second, we discuss conformal representations of superfields on superspace and construct the constraints necessary for the existence of such a space. We also give the explicit form of all the curvatures from solving the Bianchi identities. In the third, we demonstrate how the auxiliary structure of $U(1)$ superspace is identical to a certain gauge-fixed version of conformal superspace. In addition, we explicitly construct the superspace of minimal supergravity, Kähler supergravity, and new minimal supergravity.

Although the theory discussed here ought to be properly denoted "superconformal superspace," this is an awkward term that we would like to avoid. Instead we use "conformal" when the subject is superspace. (Similarly, supertranslations on superspace are simply called translations.) When the component theory is under consideration, we restore the "super."

2.1 Geometric preliminaries

2.1.1 The structure of global symmetries

The global structure of the conformal symmetry groups of arbitrary manifolds (with or without torsion and Grassmann coordinates) benefits from first discussing a simple example: the conformal group on four dimensional Minkowski (or Euclidean) space.

The conformal group

The flat metric, $ds^2 = dx^m dx^n \eta_{nm}$, is preserved up to a conformal factor by the differential generators[1]

$$\begin{aligned}
p_a &= \partial_a, & (1 + \xi \cdot p)\, x^m &= x^m + \xi^m \\
m_{ab} &= -x_a \partial_b + x_b \partial_a, & \left(1 + \frac{1}{2}\omega^{ba} m_{ab}\right) x^m &= x^m - \omega^{mn} x_n \\
d &= x \cdot \partial, & (1 + \lambda d)\, x^m &= x^m + \lambda x^m \\
k_a &= 2 x_a\, x \cdot \partial - x^2 \partial_a, & (1 + \epsilon \cdot k)\, x^m &= x^m + 2(\epsilon \cdot x) x^m - x^2 \epsilon^m
\end{aligned} \tag{2.1.1}$$

[1]The convention used here for the generators eliminates factors of i in group elements while making most of the generators anti-Hermitian.

The special conformal generator k_a can also be thought of as a spatial inversion, followed by a translation and then another spatial inversion.

These generators are represented on fields by the operators P_a, M_{ab}, D, and K_a with the following algebra:

$$[M_{ab}, P_c] = P_a\eta_{bc} - P_b\eta_{ac}, \quad [M_{ab}, K_c] = K_a\eta_{bc} - K_b\eta_{ac} \tag{2.1.2}$$

$$[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} \tag{2.1.3}$$

$$[D, P_a] = P_a, \quad [D, K_a] = -K_a \tag{2.1.4}$$

$$[K_a, P_b] = 2\eta_{ab}D - 2M_{ab} \tag{2.1.5}$$

where all other commutators vanish. The action of such generators on fields is defined by their action at the origin. One usually takes for conformally primary fields Φ,

$$P_a\Phi(0) = \partial_a\Phi(0), \quad M_{ab}\Phi(0) = \mathcal{S}_{ab}\Phi(0), \quad D\Phi(0) = \Delta\Phi(0), \quad K_a\Phi(0) = 0 \tag{2.1.6}$$

Here $\mathcal{S}_{ab}$ is a differential rotation matrix appropriate for whatever representation of the rotation group Φ belongs to, Δ is the conformal scaling dimension, and the vanishing of K_a is called the *primary* condition. In order to discern the transformation rules at points beyond the origin, one must make use of the translation operator $e^{x\cdot P}$ to translate from the origin. This is formally a Taylor expansion:

$$\begin{aligned}\Phi(x) = e^{x\cdot P}\Phi(0) &= \Phi(0) + x^a P_a\Phi(0) + \frac{1}{2}x^a x^b P_a P_b\Phi(0) + \dots \\ &= \Phi(0) + x^a\partial_a\Phi(0) + \frac{1}{2}x^a x^b\partial_a\partial_b\Phi(0) + \dots\end{aligned}$$

The operator P_a acts only on the field Φ, returning its derivative, and has no action on the coordinate x, which is here just a parameter. The same is true for the other operators.

If g is any generator of the conformal algebra, the action of g on $\Phi(x)$ can be calculated easily by making use of the translation operator:

$$g\Phi(x) = e^{x\cdot P}e^{-x\cdot P}ge^{x\cdot P}\Phi(0) \equiv e^{x\cdot P}\tilde{g}(x)\Phi(0) \tag{2.1.7}$$

where $\tilde{g}(x) \equiv e^{-x\cdot P}ge^{x\cdot P}$ is an abbreviated notation for the translated g. It follows that

$$\begin{aligned}&\tilde{P}_a(x) = P_a, \quad \tilde{D}(x) = D + x^a P_a, \quad \tilde{M}_{ab}(x) = M_{ab} - x_{[a}P_{b]} \\ &\tilde{K}_a(x) = K_a + 2x_a D - 2x_b M_{ab} + 2x_a x_b P_b - x^2 P_a\end{aligned} \tag{2.1.8}$$

If these operators are taken to act on a pure function, they reproduce the derivative representations (2.1.1). It should be noted that the algebra of the derivative representations differs by a sign from the algebra of the field representations; the former can be thought of as a left action on the group manifold with the latter corresponding to a right action which yields an opposite sign in the commutator.

On a more general field these expansions involve extra terms appropriate for Φ's representation. For a primary field,

$$\begin{aligned}&D\Phi(x) = \Delta\Phi + x^a\partial_a\Phi, \quad M_{ab}\Phi(x) = \mathcal{S}_{ab}\Phi(x) - x_{[a}\partial_{b]}\Phi(x) \\ &K_a\Phi(x) = \left(2x_a\Delta - 2x_b\mathcal{S}_{ab} + 2x_a x_b\partial_b - x^2\partial_a\right)\Phi(x)\end{aligned} \tag{2.1.9}$$

The algebraic relations are simply applied. For example,

$$DP_a\Phi(x) = [D, P_a]\Phi(x) + P_a\left(\Delta + x^b P_b\right)\Phi(x) = (\Delta+1)P_a\Phi(x) + x^b P_b P_a\Phi(x)$$

from which one can define the intrinsic scaling dimension of $\partial_a\Phi(x)$ as $\Delta+1$. Similarly can one determine the behavior of the Lorentz rotation and special conformal generators:

$$\begin{aligned}
M_{bc}P_a\Phi(x) =& \left(\mathcal{S}_{bc}\delta_a^d + \eta_{a[c}\delta_{b]}^d\right)\partial_d\Phi(x) - x_{[b}\partial_{c]}\partial_a\Phi(x) \\
=& \mathcal{S}'_{bc}\partial_a\Phi(x) - x_{[b}\partial_{c]}\partial_a\Phi(x) \qquad (2.1.10)\\
K_b P_a\Phi(x) =& \left(2\eta_{ba}\Delta - 2\mathcal{S}_{ba}\right)\Phi(x) + 2x_b(\Delta+1)\partial_a\Phi(x) \\
& - 2x_c\left(\mathcal{S}_{bc}\delta_a^d + \eta_{a[c}\delta_{b]}^d\right)\partial_d\Phi(x) + \left(2x_b x_c\partial_c - x^2\partial_b\right)\partial_a\Phi(x) \\
=& \kappa_{ba}\Phi(x) + \left(2x_b\Delta' - 2x_c\mathcal{S}'{}_{bc} + 2x_b x_c\partial_c - x^2\partial_b\right)\partial_a\Phi(x) \qquad (2.1.11)
\end{aligned}$$

Both have precisely the forms expected, where Δ' and $S'{}_{bc}$ are the conformal dimension and rotation matrix appropriate for $\partial_a\Phi(x)$. The only interesting feature is that the special conformal generator *removes* the derivative; at the origin, $K_bP_a\Phi(0) = \kappa_{ba}\Phi(0) = (2\eta_{ba}\Delta - 2\mathcal{S}_{ba})\Phi(0)$. This same feature is found in the local theory.

The conformal group action we've discussed above involves transformations only on the fields, leaving the coordinate invariant. That is, the action of a differential generator g is

$$x \to x, \quad \Phi \to \Phi'(x) = \Phi(x) + g\Phi(x) \qquad (2.1.12)$$

If we begin with the action $S = \int d^4x\ \mathcal{L}$ (with the Lagrangian a function of fields and perhaps also the coordinate), the action of g is only on the fields:

$$\delta_g S = \int d^4x\ \left(\frac{\delta\mathcal{L}}{\delta\Phi}g\Phi + \frac{\delta\mathcal{L}}{\delta\partial_a\Phi}g\partial_a\Phi\right) \qquad (2.1.13)$$

For the case where $g = \xi\cdot P$, one finds $g\Phi = \xi\cdot\partial\Phi$ and $g\partial_a\Phi = \xi\cdot\partial\partial_a\Phi$. The term in parentheses is then equivalent to $\frac{d\mathcal{L}}{dx} - \frac{\partial\mathcal{L}}{\partial x}$. The first term vanishes as a total derivative; the second must also vanish, which tells that the Lagrangian cannot contain an explicit dependence on the coordinate. For the other choices of g, the obvious results are recovered: the Lagrangian must have $\Delta = 4$, it must be a Lorentz scalar, and it must be conformally primary. The simplest conformal action involving a single primary scalar field of dimension one is $\mathcal{L} = \phi\partial^2\phi/2 - a\phi^4$. (The only non-trivial check is to ensure the kinetic term vanishes at the origin under the action of the special conformal generator.)

The approach outlined above has the feature that it places all the transformation into the fields themselves. One often finds reference to a formalism where both the coordinates and the fields transform:

$$x \to x', \quad \Phi(x) \to \Phi'(x') \qquad (2.1.14)$$

For example, under translations and finite scalings, one would have

$$x \to x' = x - a, \quad \Phi(x) \to \Phi'(x') = \Phi(x) \qquad (2.1.15)$$

$$x \to x' = e^{-\lambda}x, \quad \Phi(x) \to \Phi'(x') = e^{\Delta\lambda}\Phi(x) \qquad (2.1.16)$$

The part of g which acts as a coordinate shift has been moved off the fields and onto the coordinate explicitly; the remaining action of g can be thought of as a generalized rotation operation, which vanishes if the field Φ is a pure function. The main reason this approach is employed is that it allows conformal transformations on scalar fields (but *only* scalar fields) to be compactly written

$$x \to x', \quad \phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/4} \phi(x). \tag{2.1.17}$$

where Δ is the conformal scaling dimension of ϕ. Invariance of the action can then be checked in one step for all the elements of the conformal group. The ϕ^4 term, for example, transforms as $\int d^4x\ \phi(x)^4 \to \int d^4x'\phi'(x')^4 = \int d^4xJJ^{-\Delta}\phi(x)^4$ where $J = |\partial x'/\partial x|$. Invariance is found for $\Delta = 1$.

Constant torsion

We will ultimately be concerned with a theory containing torsion, so it is useful to review the effects torsion induces. Assume the manifold possesses translation generators P_a with nontrivial (but constant) torsion: $[P_a, P_b] = -C_{ab}{}^c P_c$. All other points x relative to the priveleged origin are defined by the condition $f(x) = e^{x\cdot P} f(0)$ for pure functions f.[2] By Taylor's theorem, the P_a in the exponent is playing the same role as ∂_a and so they are equivalent when evaluated on the function at the origin. However, since the P_a do not commute, the operator $e^{x\cdot P}$ acting on a function $f(y)$ does *not* return $f(x+y)$ since $e^{x\cdot P}e^{y\cdot P} \neq e^{(x+y)\cdot P}$.

Now let Φ be a *field* valued on the manifold. All covariant fields Φ are simple representations of the translational isometries, obeying $\Phi(x) = e^{x\cdot P}\Phi(0)$. There are three reasonable but inequivalent notions of differentiation, which we denote the normal, left, and right differentiation:

$$\partial_a \Phi(x) \equiv \frac{\partial}{\partial x^a}\left[e^{x\cdot P}\Phi(0)\right] \tag{2.1.18}$$

$$D_a^{(L)}\Phi(x) \equiv P_a e^{x\cdot P}\Phi(0) \tag{2.1.19}$$

$$D_a^{(R)}\Phi(x) \equiv e^{x\cdot P} P_a \Phi(0) \tag{2.1.20}$$

In each of these definitions, the operation on the left is some sort of derivative on the group translation element $e^{x\cdot P}$ of the general form

$$D_a^{(L)} = e^{(L)}{}_a{}^m(x)\partial_m, \quad D_a^{(R)} = e^{(R)}{}_a{}^m(x)\partial_m, \tag{2.1.21}$$

where ∂_m is to be understood as a derivative on the group parameters x^m and $e^{(L)}{}_a{}^m(x)$ and $e^{(R)}{}_a{}^m(x)$ are functions of x chosen so that the definitions are satisfied. They are found most easily by differentiating with respect to x and moving all the P's to the left or to the right:

$$\partial_m e^{x\cdot P} = e^{(L)}{}_m{}^a(x) P_a e^{x\cdot P}, \quad \partial_m e^{x\cdot P} = e^{x\cdot P} e^{(R)}{}_m{}^a(x) P_a$$

[2]The index contraction $x \cdot P$ should be understood as $x^m \delta_m{}^a P_a$. We will shortly discover a nontrivial vierbein arising from the torsion, but it does not appear in the translation group element.

It is interesting to note the group commutation rules of these various derivative operations, which follow directly from their definitions. The normal differentiation has trivial commutator, $[\partial_a, \partial_b] = 0$, since these operations are simply derivatives of their parameter. Left differentiation is not so straightforward. First consider the product of two such operations:

$$D_a^{(L)} D_b^{(L)} \Phi(x) = D_a^{(L)} P_b e^{xP} \Phi = P_b D_a^{(L)} e^{xP} \Phi = P_b P_a e^{xP} \Phi \tag{2.1.22}$$

Since $D_a^{(L)}$ acts only on the translation generator as a series of derivatives on its parameters, it passes through the group generators. Here the order of operations has reversed, which reverses the sign of the commutator:

$$[D_a^{(L)}, D_b^{(L)}]\Phi(x) = [P_b, P_a] e^{xP} \Phi = +C_{ab}{}^c D_c^{(L)} \Phi(x) \tag{2.1.23}$$

A similar calculation with the right differentiation operators shows that they preserve the order, and we find

$$[D_a^{(R)}, D_b^{(R)}]\Phi(x) = -C_{ab}{}^c D_c^{(R)} \Phi(x) \tag{2.1.24}$$

The left and right derivatives formally commute with each other since they naturally place their corresponding P_a generators on opposite sides of the translation group element:

$$D_a^{(L)} D_b^{(R)} e^{x \cdot P} \Phi = D_a^{(L)} e^{x \cdot P} P_b \Phi = P_a e^{x \cdot P} P_b \Phi = D_b^{(R)} D_a^{(L)} e^{x \cdot P} \Phi \tag{2.1.25}$$

While each of these is interesting, only the right derivative is translationally covariant:

$$e^{x \cdot P} D_a^{(R)} \Phi(x_0) = e^{x \cdot P} e^{x_0 \cdot P} P_a \Phi = D_a^{(R)} \Phi(e^{x \cdot P} x_0). \tag{2.1.26}$$

(It is a straightforward exercise to show that the other derivative operations do not obey this rule unless torsion vanishes.) Therefore we may identify $D_a^{(R)} \equiv D_a$ as the covariant derivative, and $e^{(R)}{}_a{}^m \equiv e_a{}^m$ as the physical vierbein. It can be easily calculated by noting

$$e_m{}^a P_a \equiv e^{-x \cdot P} \partial_m e^{x \cdot P}$$

The result is[3]

$$e_m{}^a = \delta_m{}^a - \frac{1}{2} x^b C_{mb}{}^a + \frac{1}{3!} x^b x^c C_{mb}{}^d C_{dc}{}^a + \ldots \tag{2.1.27}$$

where the C's are understood to all possess Lorentz indices. (That is, the only vierbein in the expression is on the left hand side, and so this is an explicit, if unclosed, expression for the vierbein.) The above expansion can be written in a matrix form. Define the function $f(u) = (e^u - 1)/u$; then $e = f(xC)$ where $(xC)_a{}^b \equiv x^c C_{ca}{}^b$. It follows that the inverse vierbein can be expanded using the reciprocal:

$$e_a{}^m = (1/f(xC))_a{}^m = \delta_a{}^m + \frac{1}{2} x^b C_{ab}{}^m + \frac{1}{12} x^b x^c C_{ab}{}^d C_{dc}{}^m + \ldots \tag{2.1.28}$$

[3] This result can be generalized in the presence of local curvatures; see Appendix 2.1.2.

This relation for the vierbein can be shown to obey $\partial_{[n}e_{m]}{}^a = e_n{}^c e_m{}^b C_{cb}{}^a$ which shows that the torsion $T_{nm}{}^a$, in this flat case, is given in the Lorentz frame by the coefficients $C_{cb}{}^a$.

The above formalism is necessary in order to describe global supersymmetry in superspace. Begin with a Grassmann manifold with four bosonic dimensions x^a and four fermionic dimensions θ^α and $\bar{\theta}_{\dot{\alpha}}$. The translation isometries consist of the bosonic translations P_a and the fermionic ones Q_α and $\bar{Q}^{\dot{\alpha}}$, with a torsion term $\{Q_\alpha, Q_{\dot{\alpha}}\} = -2i\sigma_{\alpha\dot{\alpha}}{}^a P_a$. The torsion term here is found in the *anticommutator* of the fermionic Q's. It is useful to think of this anticommutator as just a normal commutator but with fermionic objects; whenever fermionic objects pass through each other, a relative sign is introduced, creating the anticommutator from a commutator.

A superfield $\Phi(x,\theta,\bar{\theta})$ is defined by the action at the origin:

$$\Phi(x,\theta,\bar{\theta}) = e^{x\cdot P+\theta Q+\bar{\theta}\bar{Q}}\Phi$$

Since P commutes with Q and $\bar{Q}$, this can be written as $\Phi(x,\theta,\bar{\theta}) = e^{\theta Q+\bar{\theta}\bar{Q}}\Phi(x)$. If we apply a theta derivative to this superfield, there are two avenues for simplification. One is to move the Q that is brought down all the way to the left, and the other is to move it all the way to the right. These two calculations are straightforward and yield

$$\begin{aligned}\partial_\alpha\Phi(x,\theta,\bar{\theta}) &= \partial_\alpha e^{\theta Q+\bar{\theta}\bar{Q}}\Phi(x) = \left(Q_\alpha + i\sigma^a_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}P_a\right)e^{\theta Q+\bar{\theta}\bar{Q}}\Phi(x)\\ &= \left(D^{(L)}_\alpha + i\sigma^a_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}P_a\right)\Phi(x,\theta,\bar{\theta})\end{aligned}$$

and

$$\begin{aligned}\partial_\alpha\Phi(x,\theta,\bar{\theta}) &= \partial_\alpha e^{\theta Q+\bar{\theta}\bar{Q}}\Phi(x) = e^{\theta Q+\bar{\theta}\bar{Q}}\left(Q_\alpha - i\sigma^a_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}P_a\right)\phi\\ &= \left(D^{(R)}_\alpha - i\sigma^a_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}P_a\right)\Phi(x,\theta,\bar{\theta})\end{aligned}$$

From these we see immediately that the various derivatives have the form

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}, \quad D^{(L)}_\alpha \equiv \partial_\alpha - i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m, \quad D^{(R)}_\alpha \equiv \partial_\alpha + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m \tag{2.1.29}$$

Note that in the literature [7], it is the right derivaive which is D_α, the supersymmetry-covariant derivative. The left derivative is often denoted Q_α and represents the supersymmetry isometry (it preserves the form of the vierbein), which is different from the supersymmetry-covariant derivative. We will discuss this further in the general context 2.1.2.

General case

Let $\mathcal{G}$ consist of the full set of symmetry transformations acting on fields on the manifold and $\mathcal{H}$ denote the subgroup spanned by all the elements aside from translations.[4]

[4] When the operators are defined by their action on the coordinates, one often finds $\mathcal{H}$ defined as the subgroup which leaves the origin invariant. The manifold M can therefore be viewed as the coset space $\mathcal{G}/\mathcal{H}$, which is the starting point of the group manifold approach to this same topic.

In practice, these normally consist of rotational, conformal, and any Yang-Mills transformations.

The instrinsic action of $G = \exp g$ on Φ is defined by $G\Phi(0)$, its action at the origin. The action of G elsewhere can always be reconstructed using the translations:

$$G\Phi(x) \equiv Ge^{x\cdot P}\Phi(0) = e^{x\cdot P}\tilde{G}(x)\Phi(0)$$

where $\tilde{G}(x) \equiv e^{-x\cdot P}Ge^{x\cdot P}$. The product group element $e^{x\cdot P}\tilde{G}$ can be rearranged into a part depending on P and an element of $\mathcal{H}$:

$$G\Phi(x) = Ge^{x\cdot P}\Phi(0) = e^{\tilde{x}\cdot P}H_G(x)\Phi(0) \tag{2.1.30}$$

where $H_G(x) \in \mathcal{H}$. All of the translations have been absorbed in a redefinition of $x \to \tilde{x}$. On a pure function $f(x)$ this would give $Gf(x) = f(\tilde{x})$, and so $\tilde{x}$ can be thought of as the action of G induced on x.

The differential version of (2.1.30) can be compactly written

$$g\Phi(x) = e^{x\cdot P}\tilde{g}(x)\Phi(0) = e^{x\cdot P}\left(\xi_g^a(x)P_a + h_g(x)\right)\Phi(0)$$

where we have separated $\tilde{g}(x)$ into a part ξ_g consisting only of translation generators and a part $h_g(x)$ consisting only of generators from $\mathcal{H}$. This formula can be further simplified by noting the first term involves the covariant derivative:

$$g\Phi(x) = \xi_g^a D_a\Phi(x) + e^{x\cdot P}h_g\Phi(0) = \xi_g^a e_a{}^m\partial_m\Phi(x) + e^{x\cdot P}h_g\Phi(0)$$

The action of g thus induces a shift in the coordinate from x^m to $\tilde{x}^m = x^m + \xi_g^a(x)e_a{}^m(x)$.

2.1.2 The structure of local symmetries

In the preceding sections we have discussed the construction of representations of spacetime symmetry groups which act on fields. There were several unsatisfying elements to this treatment: we had to choose a preferred point, the origin; there existed two alternative methods of describing the transformations, either as just transforming the fields or transforming the fields and the coordinates; and there was no clear way to generalize to local transformations.

Each of these objections can be answered by proceeding to a local formulation for the manifold. Again let $\Phi(x)$ denote the field Φ at the point x on the manifold. Let the symmetry group $\mathcal{G}$ consist of generators X_A. The action of such symmetry transformations on a field Φ is local; they transform the field into other fields at the same spacetime point. That is, $\delta_g\Phi(x) = g^A(x)X_A\Phi(x)$, where $g^A(x)$ is the position-dependent transformation. Here we view X_A as an operator and the product $X_A\Phi$ as a single object. If instead we view Φ as a column vector in its appropriate representation, then $X_A\Phi$ can be identified as $t_A\Phi$ where t_A is a matrix appropriate to that representation. The latter objects t_A are what are normally considered in treatments of Yang-Mills. It should be noted that their multiplication rule is backwards from that of the operators. That is, $X_AX_B\Phi = X_A(t_B\Phi) = t_BX_A\Phi = t_Bt_A\Phi$ since the operator X_A passes through the matrix t_B. It follows that if the algebra of the operators is

$$[X_A, X_B] = -f_{AB}{}^C X_C$$

then the algebra of the matrices is $[t_A, t_B] = +f_{AB}{}^C t_C$.

The generators can be decomposed into the translation generators P_a (more precisely, the generators of parallel transport) and the others $X_{\underline{a}}$. The existence of purely scalar, non-constant fields annihilated by $X_{\underline{a}}$ implies that the commutator of two such generators cannot give a P. In other words, $f_{\underline{cb}}{}^a = 0$ by assumption. (Supersymmetry in normal space violates this assumption since two internal symmetries Q anticommute to give a translation P. This is one advantage of using superspace instead.)

Associated with each generator is a gauge connection $W_m{}^A$, which can be similarly decomposed into the vierbein $e_m{}^a$ and the others $h_m{}^{\underline{a}}$. This decomposition can be written

$$W_m{}^A X_A = e_m{}^a P_a + h_m{}^{\underline{a}} X_{\underline{a}} \tag{2.1.31}$$

The nature of the connection is defined by its action on fields:

$$\Phi(x + dx) = (1 + dx^m W_m{}^A(x) X_A)\Phi(x) \tag{2.1.32}$$

where Φ is a scalar on the manifold but possibly nontrivial in the tangent space. (That is, it may possess Lorentz indices but no Einstein ones.) This equation is equivalent to

$$\partial_m \Phi(x) = W_m{}^A X_A \Phi(x) = e_m{}^a P_a \Phi(x) + h_m{}^{\underline{a}} X_{\underline{a}} \Phi(x) \tag{2.1.33}$$

which can be read as defining the action of P_a as that of the covariant derivative:[5]

$$e_m{}^a P_a \Phi(x) = \nabla_m \Phi(x) = \left(\partial_m - h_m{}^{\underline{a}} X_{\underline{a}}\right) \Phi(x) \tag{2.1.34}$$

Since the vierbein is generally invertible, $P_a\Phi(x) = e_a{}^m \nabla_m \Phi(x) = \nabla_a \Phi(x)$. Since P_a is equivalent to the covariant derivative, the algebra of the P_a's generally develops additional local elements corresponding to the various curvatures associated with the manifold. That is, the statement

$$[\nabla_c, \nabla_b]\Phi = -R_{cb}{}^A X_A \Phi$$

becomes a property of the algebra itself, $[P_c, P_b] = -R_{cb}{}^A X_A$. This alteration of the algebra is the only formal consequence when passing from a global to a local theory. In the language of the algebra, $f_{cb}{}^A = R_{cb}{}^A$ become structure *functions* in a local theory and depend on the value of the connections. We will see shortly how this comes about.

Under a gauge transformation, $\partial_m(\delta_g \Phi) = (\delta_g W_m{}^A) X_A \Phi + W_m{}^A \delta_g X_A \Phi$, where $X_A \Phi$ is considered a single object, leading to the gauge transformation of the connections,

$$\delta_g W_m{}^A = \partial_m g^A + W_m{}^B g^C f_{CB}{}^A. \tag{2.1.35}$$

A finite gauge transformation is found by exponentiating an element of the algebra. That is, for an element $G = \exp(g)$, $\Phi(x) \to \Phi'(x) = G(x)\Phi(x)$. Here G is understood as a power series expansion in $g = g^A t_A$ where the matrices t_A act only on the fields Φ. The relation (2.1.33) can also be straightforwardly integrated using a path-ordered exponential in the matrix language:

$$\Phi(x) = \mathcal{P} \exp\left(\int_{x_0}^{x} W^A t_A\right) \Phi(x_0). \tag{2.1.36}$$

[5] P_a is the operator which was frequently denoted Π_a in older literature, the *kinematic* momentum, as opposed to the canonical momentum.

This equation is strongly reminiscent of a Wilson line, but extended to the full symmetry group of the tangent space. It can be compactly written $\Phi(x) = U(x, x_0)\Phi(x_0)$ where $U(x, x_0)$ is the path-ordered exponential. A derivative yields $\partial_m \Phi(x) = W_m{}^A t_A \Phi(x) = W_m{}^A (X_A \Phi)(x)$. Under a gauge transformation,

$$\Phi(x) \to G(x)\Phi(x), \quad U(x, x_0) \to U'(x, x_0) = G(x)U(x, x_0)G(x_0)^{-1} \tag{2.1.37}$$

The integrated rule for the connections can be found by considering x vanishingly near to x_0:

$$W(x) \to W'(x) = -GdG^{-1} + GWG^{-1} \tag{2.1.38}$$

In order for the relation (2.1.36) to be path-independent, any path beginning and ending on the same point must vanish, $U(x, x) = 0$. This is equivalent to the condition that the formal gauge curvature $\mathcal{F}^A = dW^A - W^B W^C f_{CB}{}^A$ vanishes. It serves not as a restriction but as a definition of the covariant curvatures R. An explicit calculation of $\mathcal{F}$ using $[P_c, P_b] = -R_{cb}{}^A X_A$ yields

$$R^A = dW^A - e^b h^{\underline{c}} f_{\underline{c}b}{}^A - \frac{1}{2} h^{\underline{b}} h^{\underline{c}} f_{\underline{cb}}{}^A \tag{2.1.39}$$

as the relation between the covariant curvature (what we normally mean when we say the "curvature") and the gauge fields.[6]

Under a P-gauge transformation, the vierbein varies as a covariant Lie derivative:

$$\begin{aligned}
\delta_P(\xi) e_m{}^a &= \partial_m \xi^a + \xi^b R_{bm}{}^a - \xi^b h_m{}^{\underline{c}} f_{\underline{c}b}{}^a \\
&= \xi^n \nabla_n e_m{}^a + \partial_m \xi^n e_n{}^a
\end{aligned} \tag{2.1.40}$$

where $\xi^m \equiv \xi^a e_a{}^m$. One recovers the normal Lie derivative by making corresponding gauge transformations involving the gauge connections:

$$\mathcal{L}_\xi e_m{}^a = \left\{ \delta_P(\xi^m e_m{}^a) + \delta_H(\xi^m h_m{}^{\underline{a}}) \right\} e_m{}^a = \delta_{GC}(\xi) e_m{}^a = \xi^n \partial_n e_m{}^a + \partial_m \xi^n e_n{}^a \tag{2.1.41}$$

This rule can be generalized to any function with Einstein indices. Thus a gauge transformation with gauge parameter $\xi^m W_m$ is equivalent to a Lie derivative on the field in question. This is precisely the behavior expected of a diffeomorphism.

Jacobi and Bianchi identities

The generators X_A must obey the Jacobi identity:

$$0 = [X_C, [X_B, X_A]] + [X_A, [X_C, X_B]] + [X_B, [X_A, X_C]] \tag{2.1.42}$$

Assuming this is obeyed for the global theory, the consequences for the local theory are simple to derive. Only terms involving the curvatures will differ, so only two classes of

[6] This is the reverse of the usual approach, where one simply defines the covariant derivative and then calculates the curvatures. The condition $\mathcal{F} = 0$ is then nothing more profound than the commuting of the *coordinate* derivatives.

Jacobi identity must be checked: those with two P's and a generator of $\mathcal{H}$ and those with three P's. Taking

$$0 = [X_{\underline{d}}, [P_c, P_b]] + [P_b, [X_{\underline{d}}, P_c]] + [P_c, [P_b, X_{\underline{d}}]] \tag{2.1.43}$$

one finds

$$X_{\underline{d}} R_{cb}{}^A = -R_{cb}{}^F f_{F\underline{d}}{}^A - f_{\underline{d}[c}{}^f R_{fb]}{}^A - f_{\underline{d}[c}{}^{\underline{f}} f_{\underline{f}b]}{}^A \tag{2.1.44}$$

The term involving two f's can be eliminated using the global Jacobi identity, giving[7]

$$X_{\underline{d}} R_{cb}{}^A = -\Delta R_{cb}{}^F f_{F\underline{d}}{}^A - f_{\underline{d}[c}{}^f \Delta R_{fb]}{}^A \tag{2.1.45}$$

where ΔR^A represents the difference between the curvature in the local theory and in the global theory; in the cases we've discussed, the only curvature in the global theory is the constant torsion tensor C, so $\Delta R_{cb}{}^{\underline{a}} = R_{cb}{}^{\underline{a}}$, but $\Delta R_{cb}{}^a = T_{cb}{}^a - C_{cb}{}^a$.

The case of the three P's is also interesting. The rules found there correspond to the Bianchi identities for the covariant derivative. They read

$$0 = \sum_{[dcb]} \left(\nabla_d T_{cb}{}^a + T_{dc}{}^f T_{fb}{}^a + R_{dc}{}^{\underline{f}} f_{\underline{f}b}{}^a \right) \tag{2.1.46}$$

$$0 = \sum_{[dcb]} \left(\nabla_d R_{cb}{}^{\underline{a}} + T_{dc}{}^f R_{fb}{}^{\underline{a}} + R_{dc}{}^{\underline{f}} f_{\underline{f}b}{}^{\underline{a}} \right) \tag{2.1.47}$$

Gauge invariant actions over the manifold

An action S in four dimensions is the integral of a Lagrangian density $\mathcal{L}(x)$ over the manifold using the general coordinate invariant measure $d^4x\, e$. The invariance of the action under a non-translational symmetry $g^{\underline{b}}$ relates the transformation rule of $\mathcal{L}$ to that of e:

$$\delta_g S = \int d^4x\, e\, (\delta_g \mathcal{L} + \delta_g e_m{}^a e_a{}^m \mathcal{L}) = \int d^4x\, e\, (g^{\underline{b}} X_{\underline{b}} \mathcal{L} + g^{\underline{b}} f_{\underline{b}a}{}^a \mathcal{L}) \tag{2.1.48}$$

One concludes $X_{\underline{b}} \mathcal{L} = -f_{\underline{b}a}{}^a \mathcal{L}$ as a condition for invariance. One can now check invariance under a translational symmetry $g^a = \xi^a$, using $\xi^a \nabla_a = \xi^m \nabla_m$:

$$\delta_P S = \int d^4x\; e \left(e_b{}^n \xi^m \nabla_m e_n{}^b \mathcal{L} + \partial_m \xi^m \mathcal{L} + \xi^m \nabla_m \mathcal{L} \right) = \int d^4x \partial_m (\xi^m e \mathcal{L}) = 0 \tag{2.1.49}$$

This is nothing more than the statement that δ_P is equivalent to a general coordinate transformation followed by gauge transformations, under which the action is inert.

A good example of the local approach is again offered by the conformal group in four dimensions. The non-vanishing part of the conformal algebra is

$$\begin{gathered}
[M_{ab}, P_c] = P_a \eta_{bc} - P_b \eta_{ac}, \quad [M_{ab}, K_c] = K_a \eta_{bc} - K_b \eta_{ac} \\
[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc} \\
[D, P_a] = P_a, \quad [D, K_a] = -K_a \\
[K_a, P_b] = 2\eta_{ab} D - 2M_{ab}
\end{gathered} \tag{2.1.50}$$

[7] This transformation rule can also be derived from the definition of the R's in terms of the gauge connections, but the above is the more straightforward path.

Coupled to each of these generators is a gauge field,

$$W_m = e_m{}^a P_a + \frac{1}{2}\omega_m{}^{ba} M_{ab} + b_m D + f_m{}^a K_a \tag{2.1.51}$$

such that the action of P_a on physical fields is the covariant derivative; the other generators are defined by their intrinsic behavior:

$$P_a \Phi = \nabla_a \Phi, \quad M_{ab}\Phi = \mathcal{S}_{ab}\Phi, \quad D\Phi = \Delta\Phi, \quad K_a \Phi = 0 \tag{2.1.52}$$

(If Φ possesses any Einstein indices, we separate them out with the vierbein and treat only the Lorentz-indexed field as the actual Φ.) The difference between this and the approach discussed in the global theory is that these are the behaviors of the generators at *all* points on the manifold. The algebra of the generators allows one to calculate the transformation behavior of any covariant derivative of Φ by using the algebra. For example,

$$D\nabla_a \Phi = DP_a \Phi = (\Delta + 1)\nabla_a \Phi \tag{2.1.53}$$

$$K_b \nabla_a \Phi = K_b P_a \Phi = (2\eta_{ba}\Delta - 2\mathcal{S}_{ba})\,\Phi \tag{2.1.54}$$

$$M_{bc}\nabla_a \Phi = M_{bc} P_a \Phi = \left(\mathcal{S}_{bc}\delta_a^d + \eta_{a[c}\delta_{b]}^d\right)\nabla_d \Phi \tag{2.1.55}$$

Each of these generators acts locally with no derivative of its parameter.

The above relations can also be checked using the explicit definition of the covariant derivative. For that calculation, one would need the transformation of the gauge connections. For completeness, consider the arbitrary gauge parameter

$$\Lambda^A X_A = \xi^a P_a + \frac{1}{2}\theta^{ba} M_{ab} + \lambda D + \epsilon^a K_a \tag{2.1.56}$$

Under a gauge transformation with such a parameter, the gauge connections transform as

$$\delta_G(\Lambda) e_m{}^a = \partial_m \xi^a + \xi^b \omega_{mb}{}^a + \xi^a b_m + \theta^{ab} e_{mb} - \lambda e_m{}^a \tag{2.1.57}$$

$$\delta_G(\Lambda)\omega_m{}^{ba} = \partial_m \theta^{ba} + \theta^{[bc}\omega_{mc}{}^{a]} - 2\xi^{[b} f_m{}^{a]} - 2\epsilon^{[b} e_m{}^{a]} \tag{2.1.58}$$

$$\delta_G(\Lambda) b_m = \partial_m \lambda + 2\xi^a f_{ma} - 2\epsilon^a e_{ma} \tag{2.1.59}$$

$$\delta_G(\Lambda) f_m{}^a = \partial_m \epsilon^a + \epsilon^b \omega_{mb}{}^a - \epsilon^a b_m + \theta^{ab} f_{mb} + \lambda f_m{}^a \tag{2.1.60}$$

Using these definitions, one can check, for example, that $\delta_K(\epsilon)\nabla_a \Phi = \left(2\epsilon_a \Delta - 2\epsilon^b \mathcal{S}_{ba}\right)\Phi$ which agrees with the result from the algebra.

If an action S in conformally invariant, the Lagrangian must obey (using $X_{\underline{b}}\mathcal{L} = -f_{\underline{b}a}{}^a\mathcal{L}$)

$$D\mathcal{L} = 4\mathcal{L}, \quad M_{ab}\mathcal{L} = 0, \quad K_a \mathcal{L} = 0 \tag{2.1.61}$$

just as in the global case. Take as an example the standard ϕ^4 theory. It is interesting to note that the conventional way of writing the kinetic term, $\nabla_a \phi \nabla_a \phi$, is not actually inert under the special conformal transformations. Rather, one needs to use the covariant d'Alembertian ($\nabla^a \nabla_a$) to give a gauge-invariant action:

$$S = \int d^4x\, e \left(\frac{1}{2}\phi \nabla^a \nabla_a \phi - a\phi^4\right) \tag{2.1.62}$$

It is straightforward to check that this action is inert under all the gauge transformations. A more interesting question is to ask how the kinetic action differs from the conventional form. A convenient starting point is the identity

$$\partial_m(ee_a{}^m\phi\nabla^a\phi) = \nabla_m\left(ee_a{}^m\phi\nabla^a\phi\right) + f_m{}^b K_b\left(ee_a{}^m\phi\nabla^a\phi\right) \tag{2.1.63}$$

which follows since the expression in the parentheses is invariant under every gauge transformation except the special conformal one. The above expression can be easily evaluated to give

$$\partial_m(ee_a{}^m\phi\nabla^a\phi) = e\left(\nabla_a(\phi\nabla^a\phi) + T_{ba}{}^a\phi\nabla^b\phi + 2f_a{}^a\phi^2\right) \tag{2.1.64}$$

This allows one to integrate the action by parts:

$$S = \int d^4x\; e\;\left(\frac{1}{2}\phi\nabla^a\nabla_a\phi - a\phi^4\right) = \int d^4x\; e\;\left(-\frac{1}{2}\nabla^a\phi\nabla_a\phi - \frac{1}{2}T_{ba}{}^a\phi\nabla^b\phi - f_a{}^a\phi^2 - a\phi^4\right) \tag{2.1.65}$$

The trace of the torsion tensor usually vanishes in physically interesting theories, but the term involving the K-gauge field $f_m{}^a$ is physically of interest. In common theories of conformal gravity, it is related to the Ricci tensor and its trace is proportional to the Ricci scalar. In such theories, the Lagrangian above can be gauge fixed to yield the Einstein-Hilbert Lagrangian. (The quartic, if present, would give a cosmological constant.)

Global representations from local ones

We have discussed two ways of implementing the spacetime symmetry group on the fields. The first involved a selection of a privileged point, the origin, at which we defined the intrinsic behavior of the fields; the behavior elsewhere was then calculated by composing the group element with the translation element. The action of group elements was taken not only on the fields but also on the translation element, leading to non-trivial transformation rules for the fields away from the origin. The second way involved defining gauge connection 1-forms everywhere; no privileged point was needed, nor was there any discussion of moving points on the manifold. The advantage of this latter formulation was that it was trivial to implement *local* group transformations. The global structure should be represented by the local one when restricted to global gauge transformations.

Begin with a vanishing $\mathcal{H}$-connection and a P-connection as defined in (2.1.27) relative to some origin point 0. Construct a gauge transformation $\tilde{g}(x)$ which takes the value g at the origin but elsewhere is such as to keep the connections invariant. That is, $\tilde{g}(x)$ obeys

$$0 = \delta_{\tilde{g}} W_m{}^A = \partial_m \tilde{g}^A + e_m{}^b \tilde{g}^C f_{Cb}{}^A \tag{2.1.66}$$

This equation can be integrated to give $\tilde{g}(x) = e^{-x\cdot P} g e^{+x\cdot P}$ where $x\cdot P \equiv x^m \delta_m{}^a P_a$. To prove this is correct, recall that to first order in ξ, $1 + \xi^m e_m{}^a P_a = e^{-x\cdot P}e^{(x+\xi)\cdot P}$. It follows then that

$$\begin{aligned}
-\xi^m e_m{}^b \tilde{g}^C f_{Cb}{}^A = [\tilde{g}, \xi^m e_m{}^b P_b] &= e^{-x\cdot P} g e^{(x+\xi)\cdot P} - e^{-x\cdot P} e^{(x+\xi)\cdot P} e^{-x\cdot P} g e^{x\cdot P} \\
&= e^{-x\cdot P} g e^{(x+\xi)\cdot P} + e^{-(x+\xi)\cdot P} g e^{x\cdot P} - 2 \\
&= \xi^m \partial_m \tilde{g}^A
\end{aligned}$$

where the last two equalities hold only to first order in ξ. This gauge transformation, $\tilde{g}(x)$, is the transformation discussed in the global approach.

The general form of the locally invariant action $S = \int d^4x\, e\, \mathcal{L}$ obeying $X_{\underline{b}}\mathcal{L} = -f_{\underline{b}a}{}^a\mathcal{L}$ implies that the globally invariant form must also have that form. In particular the global measure must be $d^4x\, e$ where e is nontrivial in the case of a sufficiently complicated (but constant) torsion. (This is not normally an issue since even global supersymmetry has $E = 1$.) To prove this requirement, consider the global action $S = \int d^4x\, e\, \mathcal{L}$. Under a global gauge transformation $\tilde{g}$, the measure is invariant and the Lagrangian changes as $\delta\mathcal{L} = \tilde{g}^{\underline{b}}X_{\underline{b}}\mathcal{L} + \tilde{g}^b P_b\mathcal{L}$. We can first replace $\tilde{g}^{\underline{b}}X_{\underline{b}} \to -\tilde{g}^{\underline{b}}f_{\underline{b}a}{}^a$ and then equate that quantity to $e_a{}^m\partial_m\tilde{g}^a + \tilde{g}^b f_{ba}{}^a$ using the differential equation for $\tilde{g}$. Finally note that $\tilde{g}^b P_b\mathcal{L} = \tilde{g}^b e_b{}^m\partial_m\mathcal{L}$ and we find

$$\delta\mathcal{L} = e_a{}^m(\partial_m\tilde{g}^a)\mathcal{L} + \tilde{g}^b f_{ba}{}^a\mathcal{L} + \tilde{g}^b e_b{}^m\partial_m\mathcal{L} = e_b{}^m\partial_m(\tilde{g}^b\mathcal{L}) + \tilde{g}^b f_{ba}{}^a\mathcal{L} \tag{2.1.67}$$

Here by $f_{ba}{}^a$ we mean the trace of the torsion tensor, equivalently written $C_{ba}{}^a$ or $T_{ba}{}^a$ (these are identical in the global theory). The first term can be integrated by parts (if the measure is e) to cancel the second, rendering the action invariant.

The $\tilde{g}$'s discussed here represent the *isometries* of the flat space – the transformations which leave invariant the form of the connections. Of particular interest is the case where $g = g^a P_a$. There we find that $\tilde{g} = \tilde{g}^a P_a$ (no $\mathcal{H}$ bits are generated since the commutator of two P's is another P in the flat, ungauged space), with the interesting property that $\tilde{g}^a$ preserves the form of the vierbein. These are precisely the translation isometries of the space; that is, they are the diffeomorphisms which preserve the vierbein. We may write them as a coordinate transformation:

$$x^m \to x^m + \tilde{g}^a e_a{}^m, \quad e^a \to e^a, \quad D_a \to D_a \tag{2.1.68}$$

Recall that the vierbein used here was the one associated with right differentiation. The action of left differentiation was an isometry which preserved the form of the vierbein 1-form e^a and the right derivative operator D_a. We have recovered this isometry above; it represents the general form of the translation isometry of a flat space with torsion.

Normal gauge

In general relativity, there exists a preferred gauge for the metric, the choice of Riemann normal coordinates, which expands the metric in terms of the curvature and derivatives thereof. Similarly in Yang-Mills theories, there exists a preferred gauge, the Fock-Schwinger gauge, which gives the gauge connection in terms of the gauge curvature and derivatives thereof. It is possible to generalize both of these conditions to the sort of theory discussed here.

Recall that a field at a point x is related to the field at a fixed point x_0 by a Taylor expansion:

$$\begin{aligned}\phi(x) &= \exp\left((x - x_0)\cdot\partial\right)\phi(x_0)\\ &= \phi(x_0) + (x - x_0)^m\partial_m\phi(x_0) + \frac{1}{2}(x - x_0)^m(x - x_0)^n\partial_n\partial_m\phi(x_0) + \dots\end{aligned} \tag{2.1.69}$$

On the other hand, the parallel transport of the field from x_0 with parameter y is

$$\begin{aligned}\phi(x_0;y) &= \exp\left(y^a P_a\right)\phi(x_0)\\ &= \phi(x_0) + y^a \nabla_a \phi(x_0) + \frac{1}{2} y^a y^b \nabla_b \nabla_a \phi(x_0) + \ldots\end{aligned} \tag{2.1.70}$$

One can choose a gauge such that these coincide for $x = y + x_0$ for scalar fields; this generalizes Riemann normal coordinates for non-Riemannian geometries (for example, those with torsion). The further choice that these should coincide for *all* fields leads to a generalization of Fock-Schwinger gauge.

In principle, one can equate these series term-by-term to determine the gauge fields. A slightly simpler method is to note that $e_a{}^m \partial_m \phi - h_a{}^{\underline{b}} X_{\underline{b}} \phi$ is the covariant derivative; therefore one may equate

$$e_a{}^m(y) \frac{\partial}{\partial y^m} \exp\left(y^a P_a\right)\phi(x_0) - h_a{}^{\underline{b}}(y) \exp\left(y^a P_a\right) X_{\underline{b}} \phi(x_0) = \exp\left(y^a P_a\right) P_a \phi(x_0) \tag{2.1.71}$$

This can be rearranged to

$$\begin{aligned}\frac{\partial}{\partial y^m} e^{y\cdot P}\phi(x_0) &= e_m{}^a e^{y\cdot P} P_a \phi(x_0) + h_m{}^{\underline{b}}(y) e^{y\cdot P} X_{\underline{b}}\phi(x_0)\\ &= e^{y\cdot P}\tilde{e}_m{}^a P_a \phi(x_0) + e^{y\cdot P}\tilde{h}_m{}^{\underline{b}}(y) X_{\underline{b}}\phi(x_0)\end{aligned} \tag{2.1.72}$$

where we have defined $\tilde{e}_m{}^a$ and $\tilde{h}_m{}^{\underline{b}}$ by conjugation with $e^{y\cdot P}$. Multiplying by an overall factor gives

$$e^{-y\cdot P}\frac{\partial}{\partial y^m} e^{y\cdot P}\phi(x_0) = \tilde{e}_m{}^a P_a \phi(x_0) + \tilde{h}_m{}^{\underline{b}}(y) X_{\underline{b}}\phi(x_0) \tag{2.1.73}$$

The term on the left can be straightforwardly evaluated term by term:

$$\begin{aligned}e^{-y\cdot P}\partial_m e^{y\cdot P} &= \partial_m + P_m + \frac{1}{2}[P_m, y^a P_a] + \frac{1}{3!} L^2_{y\cdot P} P_m - \frac{1}{4!} L^3_{y\cdot P} P_m + \ldots\\ &= \partial_m + P_m + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} Q_m(j)\end{aligned} \tag{2.1.74}$$

where $L_{y\cdot P} f \equiv [y^a P_a, f] = y^a [P_a, f]$ and $Q_m(j) \equiv L^j_{y\cdot P} P_m$. In this expansion the y^a are to be treated as group parameters, inert under the action of the generators, and the explicit derivative ∂_m is with respect to the y only. One may formally solve for the gauge fields by defining

$$\begin{aligned}\tilde{e}_m{}^a &= \delta^a_m + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} Q_m{}^a(j)\\ \tilde{h}_m{}^{\underline{b}} &= \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} Q_m{}^{\underline{b}}(j)\end{aligned} \tag{2.1.75}$$

where we have expanded $Q_m = Q_m{}^a P_a + Q_m{}^{\underline{b}} X_{\underline{b}}$. Then conjugating by the group element $\exp(y \cdot P)$ generates the actual gauge fields:[8]

$$e_m{}^a = \delta_m^a + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \sum_{k=0}^{\infty} \frac{1}{k!} L_{y \cdot P}^k Q_m{}^a(j)$$

$$h_m{}^{\underline{b}} = \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \sum_{k=0}^{\infty} \frac{1}{k!} L_{y \cdot P}^k Q_m{}^{\underline{b}}(j) \tag{2.1.76}$$

Note that the conjugation generates covariant derivatives of the listed terms; for example, $L_{y \cdot P} Q_m{}^a(j) = y^b \nabla_b Q_m{}^a(j)$. All indices on the right hand side of these equations should be understood as Lorentz indices.

Since curvatures transform covariantly, the factor of $\sum_{k=0}^{\infty} \frac{1}{k!} L_{y \cdot P}^k$ in both of the above expressions serves only to replace the curvatures by their power series expansion in y. Therefore, we instead can write

$$e_m{}^a = \delta_m{}^a + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \tilde{Q}_m{}^a(j) \tag{2.1.77}$$

$$h_m{}^{\underline{b}} = \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \tilde{Q}_m{}^{\underline{b}}(j) \tag{2.1.78}$$

where $\tilde{Q}$ contain y-dependence both explicitly and implicitly. Assuming that torsion vanishes and the only curvatures are Lorentz and Yang-Mills, we find

$$\begin{aligned}
Q_m(1) &= -\mathcal{F}_{ym} \\
Q_m(2) &= -\nabla_y \mathcal{F}_{ym} + R_{ymy}{}^a P_a \\
Q_m(3) &= -\nabla_y^2 \mathcal{F}_{ym} + 2\nabla_y R_{ymy}{}^a P_a + R_{ymy}{}^b \mathcal{F}_{by} \\
Q_m(4) &= -\nabla_y^3 \mathcal{F}_{ym} + 3\nabla_y^2 R_{ymy}{}^a P_a + 3\nabla_y R_{ymy}{}^b \mathcal{F}_{by} + R_{ymy}{}^b \nabla_y \mathcal{F}_{by} - R_{ymy}{}^b R_{byy}{}^a P_a
\end{aligned}$$

These are sufficient to determine all of the connections to fourth order in y. It is easy to see that this gauge obeys

$$y^a \nabla_a = y^m \partial_m. \tag{2.1.79}$$

We note that this definition of normal coordinates generalizes both Riemann normal coordinates and Fock-Schwinger gauge for an abelian gauge theory. It is the simplest Lorentz invariant gauge one may define where the connections are power series in the curvatures. Non-Lorentz invariant gauges can be derived by rearranging the exponential in (2.1.70). A generalized temporal gauge ($h_0 = 0, e_0{}^a = \delta_0{}^a$) would correspond to defining

$$\Phi(y) = \exp(y^i P_i) \exp(y^0 P_0) \Phi(0)$$

In this gauge the temporal components are trivial, but the spatial components are rather more complicated.

For a complementary (and more rigorous) treatment of normal coordinates, we refer the reader to the recent papers [16, 17] and the references therein.

[8] In the case where there are no curvatures except for constant torsion, the above reduce to $h_m{}^{\underline{b}} = 0$ and $e_m{}^a$ given by (2.1.27). Normal gauge is the appropriate generalization.

Gauge invariant actions over submanifolds

In the case of global supersymmetry, we know that it is natural to consider not only integrals over the entire superspace of coordinates $(x, \theta, \bar{\theta})$ but also integrals over a chiral superspace of coordinates (y, θ) where $y = x + i\theta\sigma\bar{\theta}$. It is natural to think of the chiral superspace as lying on a submanifold characterized by a constant value of $\bar{\theta}$. Then change in coordinates from x to y is naturally understood, since in those coordinates $D^{\dot{\alpha}} = \partial^{\dot{\alpha}}$ and so chiral superfields (those annihilated by $D^{\dot{\alpha}}$) naturally live on such a submanifold.

Let us take this point of view seriously and derive some useful results about actions on submanifolds. We will assume that the space under consideration is purely bosonic so that our geometric intuition can be trusted. Let the full manifold M be D-dimensional on which we may define the parallel transport operators P_A, where $A = 1, \ldots, D$. Let P be decomposed as $P_A = (P_{\mathfrak{a}}, P_{\dot{\alpha}})$ where $\mathfrak{a} = 1, \ldots, \mathfrak{D}$ and $\dot{\alpha} = \mathfrak{D} + 1, \ldots, D$. We will use Gothic indices $\mathfrak{a}$ to denote the submanifold tangent space indices. Our object of interest is a submanifold $\mathfrak{M}$ of dimension $\mathfrak{D}$ defined so that $P_{\dot{\alpha}}$ annihilates the functions naturally integrated over $\mathfrak{M}$.

This can be made more concrete by choosing coordinates $z^M = (\mathfrak{z}^{\mathfrak{m}}, \bar{\theta}^{\dot{\mu}})$ so that $\mathfrak{M}$ is parametrized by $\mathfrak{z}^{\mathfrak{m}}$ with constant $\bar{\theta}^{\dot{\mu}}$; we will assume $\bar{\theta}^{\dot{\mu}} = 0$ for definiteness, but any constant will do. In this way the coordinates on M can be related nicely to the coordinates on $\mathfrak{M}$.[9] Then the condition that $P_{\dot{\alpha}}$ annihilates the natural integrands on $\mathfrak{M}$ means $P_{\dot{\alpha}} = \partial/\partial\bar{\theta}^{\dot{\alpha}}$ when acting on pure functions, or, equivalently, that $\mathfrak{M}$ lies at a constant slice of $\bar{\theta}^{\dot{\mu}}$. This choice of coordinates has the benefit of simplifying calculations while unfortunately forcing a breakdown in manifest general coordinate invariance on M; equivalently, this forces one to choose a certain P-gauge. We will therefore avoid making this explicit assumption until it is absolutely necessary.

Recall that an invariant integral on the whole manifold M is

$$S = \int_M E^1 \wedge E^2 \wedge \ldots \wedge E^D \, V = \int d^D z \, E \, V \tag{2.1.80}$$

where $E = \det(E_M{}^A)$ and V is an appropriate integrand to make the action gauge invariant. We have already shown that invariance under the non-translation symmetries $\mathcal{H}$ requires $\delta_g V = -g^{\underline{b}} f_{\underline{b}A}{}^A$, while invariance under P follows from general coordinate invariance. An invariant integral over $\mathfrak{M}$ can be very similarly defined:

$$\mathfrak{S} = \int_{\mathfrak{M}} E^1 \wedge E^2 \wedge \ldots \wedge E^{\mathfrak{D}} \, W = \int_{\mathfrak{M}} \mathcal{E}^1 \wedge \mathcal{E}^2 \wedge \ldots \wedge \mathcal{E}^{\mathfrak{D}} \, W = \int d^{\mathfrak{D}} \mathfrak{z} \, \mathcal{E} \, W, \tag{2.1.81}$$

where $\mathcal{E} = \det(\mathcal{E}_{\mathfrak{m}}{}^{\mathfrak{a}})$ is the volume measure and W is an appropriate integrand. The subvierbein form $\mathcal{E}^{\mathfrak{a}}$ is taken to be identical to $E^{\mathfrak{a}}$ when restricted to the manifold $\mathfrak{M}$.[10] Invariance of this integral under the action of $\mathcal{H}$ requires $\delta_g W = -g^{\underline{\mathfrak{b}}} f_{\underline{\mathfrak{b}}\mathfrak{a}}{}^{\mathfrak{a}} W$. (Note the trace of the structure constant is over the submanifold's Lorentz indices.) However, since the integral is over a submanifold, it is not obviously taken into itself under P-gauge transformations.

[9] Of course $\bar{\theta}$ here is to be understood as a bosonic coordinate at the moment.

[10] In the special coordinates where $\mathfrak{M}$ corresponds to $\bar{\theta} = 0$, the vierbein obeys $E_{\dot{\mu}}{}^{\mathfrak{a}}|_{\mathfrak{M}} = 0$. This condition is equivalent to the conditions $\mathcal{E}^{\mathfrak{a}} = E^{\mathfrak{a}}|_{\mathfrak{M}} = d\mathfrak{z}^{\mathfrak{m}} \mathcal{E}_{\mathfrak{m}}{}^{\mathfrak{a}}$.

We check first the requirement of $P_{\dot\alpha}$ invariance, which means essentially that such actions should not depend on the constant value of $\bar\theta$ used to define $\mathfrak{M}$. The action varies as

$$0 = \delta_\xi \mathfrak{S} = \int d^{\mathfrak{D}}\mathfrak{z}\, \mathcal{E} \left(\xi^{\dot\alpha} T_{\dot\alpha\mathfrak{m}}{}^{\mathfrak{b}} \mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}} W + \xi^{\dot\alpha}\nabla_{\dot\alpha} W\right). \tag{2.1.82}$$

(The term $\mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}}$ represents the inverse of the *sub*vierbein. It does not necessarily correspond to $E_{\mathfrak{b}}{}^{\mathfrak{m}}$, since the inverse of a submatrix is not necessarily the submatrix of the inverse unless certain requirements are placed on the coordinates $\mathfrak{z}$ being used for the submanifold, or equivalently, the gauge choice for the vierbein.) Each term should vanish separately. Requiring the second term to vanish enforces the covariant constancy of W in the direction of $P_{\dot\alpha}$. Requiring consistency of $\nabla_{\dot\alpha} W = 0$ with the algebra gives several additional constraints:

$$0 = [\nabla_{\dot\alpha}, \nabla_{\dot\beta}] W = -T_{\dot\alpha\dot\beta}{}^{\mathfrak{c}} \nabla_{\mathfrak{c}} W + R_{\dot\alpha\dot\beta}{}^{\underline{c}} f_{\underline{c}\mathfrak{d}}{}^{\mathfrak{d}} W \tag{2.1.83}$$

$$0 = [X_{\underline{a}}, \nabla_{\dot\beta}] W = -f_{\underline{a}\dot\beta}{}^{\mathfrak{c}} \nabla_{\mathfrak{c}} W + f_{\underline{a}\dot\beta}{}^{\underline{c}} f_{\underline{c}\mathfrak{d}}{}^{\mathfrak{d}} W \tag{2.1.84}$$

(The second commutator vanishes since $\nabla_{\dot\beta} X_{\underline{a}} W = -\nabla_{\dot\beta} f_{\underline{a}\mathfrak{b}}{}^{\mathfrak{b}} W = 0$.) From this simple result we learn $T_{\dot\alpha\dot\beta}{}^{\mathfrak{c}} = f_{\underline{a}\dot\beta}{}^{\mathfrak{c}} = 0$ as well as $R_{\dot\alpha\dot\beta}{}^{\underline{c}} f_{\underline{c}\mathfrak{d}}{}^{\mathfrak{d}} = f_{\underline{a}\dot\beta}{}^{\underline{c}} f_{\underline{c}\mathfrak{d}}{}^{\mathfrak{d}} = 0$. The other term in the variation of the subaction gives two new terms which must vanish:

$$T_{\dot\alpha\mathfrak{m}}{}^{\mathfrak{b}} \mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}} = T_{\dot\alpha\dot\gamma}{}^{\mathfrak{b}} E_{\mathfrak{m}}{}^{\dot\gamma} \mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}} + T_{\dot\alpha\mathfrak{b}}{}^{\mathfrak{b}}$$

The first of these, $T_{\dot\alpha\dot\gamma}{}^{\mathfrak{b}} E_{\mathfrak{m}}{}^{\dot\gamma} \mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}} = 0$, is already a condition for the existence of a covariantly constant W. The second, $T_{\dot\alpha\mathfrak{b}}{}^{\mathfrak{b}} = 0$, amounts to an additional constraint on the space.[11]

Next we check $P_{\mathfrak{a}}$ invariance of the subaction. One finds

$$0 = \delta_\xi \mathfrak{S} = \int d^{\mathfrak{D}}\mathfrak{z}\, \mathcal{E} \left(\nabla_{\mathfrak{m}} \xi^{\mathfrak{a}} \mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{m}} W + \xi^{\mathfrak{a}} T_{\mathfrak{a}\mathfrak{m}}{}^{\mathfrak{b}} \mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}} W + \xi^{\mathfrak{a}} \nabla_{\mathfrak{a}} W\right). \tag{2.1.85}$$

Integrating the first term by parts gives

$$0 = \delta_g \mathfrak{S} = \int d^{\mathfrak{D}}\mathfrak{z}\, \mathcal{E} \left(-\xi^{\mathfrak{a}} \mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{m}} \nabla_{\mathfrak{m}} W + \xi^{\mathfrak{a}} \nabla_{\mathfrak{a}} W - \xi^{\mathfrak{a}} \mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{n}} T_{\mathfrak{n}\mathfrak{m}}{}^{\mathfrak{b}} \mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}} W + \xi^{\mathfrak{a}} T_{\mathfrak{a}\mathfrak{m}}{}^{\mathfrak{b}} \mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}} W\right) \tag{2.1.86}$$

Invariance holds under the same set of conditions. For example,

$$\mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{m}} \nabla_{\mathfrak{m}} W = \mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{m}} E_{\mathfrak{m}}{}^{B} \nabla_B W = \mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{m}} E_{\mathfrak{m}}{}^{\mathfrak{b}} \nabla_{\mathfrak{b}} W = \nabla_{\mathfrak{a}} W$$

since W is covariantly constant with respect to $P_{\dot\alpha}$ and $E_{\mathfrak{m}}{}^{\mathfrak{b}}$ is equivalent to $\mathcal{E}_{\mathfrak{m}}{}^{\mathfrak{b}}$. A similar argument demontrates the cancellation of the torsion terms.

The constraints we have found are:

$$T_{\dot\alpha\dot\beta}{}^{\mathfrak{c}} = 0, \quad f_{\underline{a}\dot\beta}{}^{\mathfrak{c}} = 0$$
$$R_{\dot\alpha\dot\beta}{}^{\underline{c}} f_{\underline{c}\mathfrak{d}}{}^{\mathfrak{d}} = 0, \quad f_{\underline{a}\dot\beta}{}^{\underline{c}} f_{\underline{c}\mathfrak{d}}{}^{\mathfrak{d}} = 0$$
$$T_{\dot\alpha\mathfrak{b}}{}^{\mathfrak{b}} = 0$$

[11] These constraints are stricter than necessary. One could choose that $\nabla_{\dot\alpha} W = -T_{\dot\alpha\mathfrak{m}}{}^{\mathfrak{b}} \mathcal{E}_{\mathfrak{b}}{}^{\mathfrak{m}} W$, as opposed to requiring each term to separately vanish. We have chosen to separate them in the way we have since it makes sense that the conditions we want should be simple conditions on W, like chirality, and simple conditions on the geometry, like vanishing of certain torsions, as opposed to something more complicated relating the two.

The next question to consider is whether integrals over a manifold M can be related to integrals over the submanifold $\mathfrak{M}$, and vice-versa. We will deal with $M \to \mathfrak{M}$ first and then consider the reverse.

Case 1: $M \to \mathfrak{M}$

Consider the integration of a function V over the whole manifold: $\int_M d^D z\, E\, V$. We would like to decompose it into an integral of some other function W over the submanifold $\mathfrak{M}$. The most straightforward way to do this is to adopt the coordinates (equivalently, choose the P-gauge) so that $z^M = (\mathfrak{z}^{\mathfrak{m}}, \bar\theta^{\dot\mu})$ and $\mathfrak{M}$ corresponds to $\bar\theta = 0$. Note that it is rather trivial to choose $E_{\dot\mu}{}^{\mathfrak{a}}|_{\mathfrak{M}} = 0$; it can be shown that the conditions we derived for the invariance of the subactions over $\mathfrak{M}$ allow us to extend this condition over all of M.[12] We then can assume a gauge choice where $E_{\dot\mu}{}^{\mathfrak{a}} = 0$ everywhere, as well as the additional requirements $h_{\dot\mu}{}^{\underline{b}} f_{\underline{b}\mathfrak{a}}{}^{\mathfrak{a}} = 0$. These two conditions mean that $\nabla_{\dot\alpha} W = 0$ is equivalent to $\partial_{\dot\mu} W = 0$. Given these, one may easily show that $\mathcal{E}$ is itself independent of $\bar\theta$:

$$\partial_{\dot\mu}\mathcal{E} = \partial_{\dot\mu} E_{\mathfrak{n}}{}^{\mathfrak{a}} \mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{n}} = \nabla_{\dot\mu} E_{\mathfrak{n}}{}^{\mathfrak{a}} \mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{n}} = T_{\dot\mu \mathfrak{n}}{}^{\mathfrak{a}} \mathcal{E}_{\mathfrak{a}}{}^{\mathfrak{n}} = 0 \tag{2.1.87}$$

This is important since the gauge choice for the vierbein implies $E = \mathcal{E}\bar\Sigma$, where $\bar\Sigma \equiv \det(E_{\dot\mu}{}^{\dot\alpha})$. Then E separates into a part ($\mathcal{E}$) independent of $\bar\theta$ and another ($\bar\Sigma$) which is an appropriate density in $\bar\theta$.

Under these assumptions, we find

$$\int_M d^D z\, E\, V = \int_{\mathfrak{M}} d^{\mathfrak{D}} \mathfrak{z}\, \mathcal{E}\, \mathcal{P}[V] \tag{2.1.88}$$

where

$$\mathcal{P}[V] \equiv \int d^{\dot d}\bar\theta\, \bar\Sigma\, V. \tag{2.1.89}$$

Note that $\mathcal{P}[V]$ is covariantly constant with respect to $P_{\dot\alpha}$ for a quite trivial reason: by construction, $\mathcal{P}[V]$ is independent of $\bar\theta$ and so $\partial_{\dot\mu}\mathcal{P}[V] = 0$ in a gauge where $\partial_{\dot\alpha} = \nabla_\alpha$. This operation can be extended to any gauge by first evaluating it in the special gauge used here and then transforming to the desired gauge using $\delta_g \mathcal{P}[V] = -g^{\underline{b}} f_{\underline{b}\mathfrak{a}}{}^{\mathfrak{a}} \mathcal{P}[V]$.

Case 2: $\mathfrak{M} \to M$

In principle an integral over a submanifold $\mathfrak{M}$ can be defined by an integral over the whole manifold M using an appropriate delta function Δ_c. Then

$$\int_{\mathfrak{M}} d^{\mathfrak{D}} \mathfrak{z}\, \mathcal{E}\, W = \int_M d^D z\, E\, W \Delta_c \tag{2.1.90}$$

That both sides remain gauge invariant under $\mathcal{H}$ implies $\delta_g \Delta_c = -g^{\underline{b}} f_{\underline{b}\dot\alpha}{}^{\dot\alpha} \Delta_c$. The simplest way to describe the constraints is to choose the coordinates z to decompose

[12] The construction will be given when needed for the explicit case of $\mathcal{N} = 1$ superspace.

as $z^M = (\mathfrak{z}^{\mathfrak{m}}, \bar\theta^{\dot\mu})$ where the submanifold $\mathfrak{M}$ lives at $\bar\theta^{\dot\mu} = 0$. In this special gauge, Δ_c takes the simple form

$$\Delta_c = \frac{\delta^{\dot d}(\bar\theta)}{\bar\Sigma}. \tag{2.1.91}$$

This is not the only such Δ_c that will work; an entire family is permissible, of the form

$$\Delta_c = \frac{X}{\mathcal{P}[X]}. \tag{2.1.92}$$

The choice $X = \delta^{\dot d}(\bar\theta)$ reproduces the simplest example. If, however, $\mathcal{P}[1]$ is a simple enough object, the choice $X = 1$ becomes extremely attractive. [13]

That both of these results should hold implies

$$\int_M d^D z\, E\, V = \int_{\mathfrak{M}} d^{\mathfrak{D}} \mathfrak{z}\, \mathcal{E}\, \mathcal{P}[V] = \int_M d^D z\, E\, \mathcal{P}[V] \Delta_c \tag{2.1.93}$$

Since Δ_c can be placed in the form $X/\mathcal{P}[X]$, the equivalence of the first and third forms implies $\mathcal{P}$ is a self-adjoint operation under the full integration.

While it is self-adjoint, $\mathcal{P}$ is not actually a projector, as it is not idempotent (that is, $\mathcal{P}^2 \neq \mathcal{P}$). The true projector (in the special gauge) is Π, which is defined by

$$\Pi[V] \equiv \int d^{\dot d} \bar\theta\, \bar\Sigma\, V \Delta_c. \tag{2.1.94}$$

This formula is a very complicated way of saying a simple thing: $\Pi[V]$ is formally identical (in this gauge) to $V|_{\bar\theta=0}$ provided we use the simplest Δ_c. The advantage of the more cumbersome form $X/\mathcal{P}[X]$ is that it can be extended to any other gauge since the gauge transformation properties of the various objects are well-defined.[14]

2.2 Conformal superspace

In the ensuing section we describe the gauge structure, geometric constraints, and curvatures of conformal superspace, which is defined as an $\mathcal{N} = 1$ superspace with the structure group of the superconformal algebra. We discuss representations of that algebra, invariant actions and chiral submanifold actions. As usual, constraints must be imposed to eliminate unwanted fields; we will discuss their construction and solution. But the first place to start is at the component level, where conformal supergravity is well-known and its properties well-established.

[13]The above construction applies very nicely to Poincaré supergravity, where if one chooses $X = 1$, one finds $\Delta = 1/2R$.

[14]Applying this to the case of Poincaré supergravity, one finds $\mathcal{P} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)$ and $\Pi = -\frac{1}{8R}(\bar{\mathcal{D}}^2 - 8R)$.

2.2.1 Conformal supergravity at the component level

Conformal supergravity at the component level begins with the extension of the Poincaré to the super-Poincaré algebra by the addition of fermionic internal symmetries Q_α. These anticommute to give spacetime translations:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i\sigma^a_{\alpha\dot{\alpha}} P_a \tag{2.2.1}$$

The rest of the super-Poincaré algebra is

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} \\
[M_{ab}, P_c] &= P_a\eta_{bc} - P_b\eta_{ac} \\
[M_{ab}, Q_\gamma] &= (\sigma_{ab})_\gamma{}^\beta Q_\beta
\end{aligned} \tag{2.2.2}$$

The bosonic part of the algebra can be extended to include the conformal algebra. This requires the introduction of the conformal scaling[15] operator D and the special conformal operator K_a, which loosely speaking can be understood as a translation conjugated by inversions. A brief review of the conformal algebra is given in Section 2.1.1.

These two generators can be added to the super-Poincaré algebra provided one also includes two new operators, the fermionic special conformal operator S_α (and its conjugate $\bar{S}^{\dot{\alpha}}$) and the chiral rotation operator A. (This last generator is the $U(1)$ R-symmetry.) It should be noted that the special conformal generators possess the same Lorentz transformation properties as the corresponding translation and supersymmetry generators, but have opposite weights under scalings and chiral rotations:

$$\begin{aligned}
[D, P_a] = P_a, \quad [D, Q_\alpha] = \frac{1}{2}Q_\alpha, \quad [D, \bar{Q}^{\dot{\alpha}}] = \frac{1}{2}\bar{Q}^{\dot{\alpha}} \\
[D, K_a] = -K_a, \quad [D, S_\alpha] = -\frac{1}{2}S_\alpha, \quad [D, \bar{S}^{\dot{\alpha}}] = -\frac{1}{2}\bar{S}^{\dot{\alpha}} \\
[A, Q_\alpha] = -iQ_\alpha, \quad [A, \bar{Q}^{\dot{\alpha}}] = +i\bar{Q}^{\dot{\alpha}} \\
[A, S_\alpha] = +iS_\alpha, \quad [A, \bar{S}^{\dot{\alpha}}] = -i\bar{S}^{\dot{\alpha}} \\
[M_{ab}, K_c] = K_a\eta_{bc} - K_b\eta_{ac} \\
[M_{ab}, S_\gamma] = (\sigma_{ab})_\gamma{}^\beta S_\beta
\end{aligned} \tag{2.2.3}$$

The special conformal generators have an algebra among each other that is similar to the supersymmetry algebra:

$$\{S_\alpha, \bar{S}_{\dot{\alpha}}\} = +2i\sigma^a_{\alpha\dot{\alpha}} K_a \tag{2.2.4}$$

Finally, the commutators of the special conformal generators with the translation and su-

[15] This operation is often called "dilatation."

persymmetry generators are

$$\begin{aligned}
[K_a, P_b] &= 2\eta_{ab} D - 2M_{ab} \\
[K_a, Q_\alpha] &= i\sigma_{a\alpha\dot{\beta}} \bar{S}^{\dot{\beta}}, \quad [K_a, \bar{Q}^{\dot{\alpha}}] = i\bar{\sigma}_a^{\dot{\alpha}\beta} S_\beta \\
[S_\alpha, P_a] &= i\sigma_{a\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}}, \quad [\bar{S}^{\dot{\alpha}}, P_a] = i\bar{\sigma}_a^{\dot{\alpha}\beta} Q_\beta \\
\{S_\alpha, Q_\beta\} &= 2D\epsilon_{\alpha\beta} - 2M_{\alpha\beta} - 3iA\epsilon_{\alpha\beta} \\
\{\bar{S}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} &= 2D\epsilon^{\dot{\alpha}\dot{\beta}} - 2M^{\dot{\alpha}\dot{\beta}} + 3iA\epsilon^{\dot{\alpha}\dot{\beta}}
\end{aligned} \tag{2.2.5}$$

All other commutators vanish.

We have made use of the convenient shorthand $M_{\alpha\beta} = (\sigma^{ba}\epsilon)_{\alpha\beta} M_{ab}$ and $M^{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}^{ba}\epsilon)^{\dot{\alpha}\dot{\beta}} M_{ab}$. These are projections of the Lorentz generator; $M_{\alpha\beta}$ rotates undotted spinors while $M^{\dot{\alpha}\dot{\beta}}$ rotates dotted spinors. For example,

$$\begin{aligned}
[M_{\alpha\beta}, Q_\gamma] &= -Q_\alpha \epsilon_{\beta\gamma} - Q_\beta \epsilon_{\alpha\gamma} \\
[M_{\alpha\beta}, Q_{\dot{\gamma}}] &= 0 \\
[M_{\alpha\beta}, P_{(\gamma\dot{\gamma})}] &= -P_{\alpha\dot{\gamma}} \epsilon_{\beta\gamma} - P_{\beta\dot{\gamma}} \epsilon_{\alpha\gamma}
\end{aligned}$$

where $P_{(\gamma\dot{\gamma})} \equiv P_c \sigma^c_{\gamma\dot{\gamma}}$. The canonical decomposition of a vector into dotted and undotted spinors is accomplished via contraction with a sigma matrix.

Conformal supergravity in four dimensions is the gauge theory of the above algebra. The connection forms $W_m{}^A$ can be collected with their generators X_A into the total connection form

$$W_m = e_m{}^a P_a + \frac{1}{2}\psi_m{}^{\underline{\alpha}} Q_{\underline{\alpha}} + \frac{1}{2}\omega_m{}^{ba} M_{ab} + b_m D + A_m A + f_m{}^a K_a + f_m{}^{\underline{\alpha}} S_{\underline{\alpha}} \tag{2.2.6}$$

Here $\underline{\alpha}$ denotes both spinor chiralities (α and $\dot{\alpha}$) and the spinor summation convention is that of [7]. In the local theory, the generator P_a is identified as the covariant derivative when acting on a covariant field.[16] The algebra of the P_a among themselves is altered by the introduction of curvatures. One finds on a covariant field Φ

$$[P_a, P_b]\Phi \equiv [\nabla_a, \nabla_b]\Phi = -R_{ab}\Phi \tag{2.2.7}$$

where the curvatures are

$$R_{nm} = T_{nm}{}^a P_a + T_{nm}{}^{\underline{\alpha}} Q_{\underline{\alpha}} + \frac{1}{2} R_{nm}{}^{ba} M_{ab} + H_{nm} D + F_{nm} A + R(K)_{nm}{}^a K_a + R(S)_{nm}{}^{\underline{\alpha}} S_{\underline{\alpha}} \tag{2.2.8}$$

Here we are using $T_{nm}{}^{\underline{\alpha}}$ as the supersymmetry curvature (anticipating that in superspace this will become part of the torsion), H and F as the curvatures associated with scalings and chiral rotations, and $R(K)$ and $R(S)$ as the curvatures associated with special conformal and fermionic special conformal transformations. (The curvatures – with Lorentz form indices

[16] A covariant field Φ transforms as $\delta_g \Phi = g^A X_A \Phi$. This is linear in Φ and involves no derivatives of the parameter g^A.

– are also covariant fields in the sense that a curvature transforms into another curvature.) The construction of a local gauge theory from a generic global theory is detailed in Section 2.1.2.

Constraints are imposed on these curvatures in such a way as to eliminate the spin connection ${\omega_m}^{ba}$ and the special conformal connections ${f_m}^a$ and ${f_m}^{\underline{\alpha}}$ in terms of the other fields. This procedure is summarized in the review literature [11] but the details do not concern us here.

The transformation rules of the various gauge fields are straightforward to calculate and are given in [11]. For our purposes, the only ones which will matter are the supersymmetry transformations of the unconstrained fields:

$$\delta_Q {e_m}^a = i(\xi \sigma^a \bar{\psi}_m - \psi_m \sigma^a \bar{\xi}) \tag{2.2.9}$$

$$\delta_Q {\psi_m}^\alpha = 2\nabla_m \xi^\alpha \tag{2.2.10}$$

$$\delta_Q b_m = 2{f_m}^{\underline{\alpha}} \xi_{\underline{\alpha}} \tag{2.2.11}$$

$$\delta_Q A_m = -3i{f_m}^\alpha \xi_\alpha + 3i f_{m\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} \tag{2.2.12}$$

The derivative ∇_m is covariant with respect to spin, scalings, and chiral rotations and ξ^α is assumed to transform with *opposite* scaling and chiral weights as Q_α. The gravitino transformation rule is especially simple.

It is also useful to record the transformation rules of chiral matter coupled to conformal supergravity. For the chiral multiplet $\Phi = (\phi, \psi, F)$,

$$\delta_Q \phi = \sqrt{2}\xi\psi, \quad \delta_Q \psi = \sqrt{2}\xi F + i\sqrt{2}\sigma^a \bar{\xi} \nabla_a \phi, \quad \delta_Q F = i\sqrt{2}(\bar{\xi}\bar{\sigma}^a \nabla_a \psi) \tag{2.2.13}$$

which is identical to the supersymmetry algebra except for the replacement of the regular derivative with the covariant one.

These sets of component transformation rules must be reproduced at the superfield level once we move to superspace; this will help us to find the proper constraints on the curvatures in superspace.

2.2.2 Conformal superspace and representations of the algebra

$\mathcal{N} = 1$ superspace is a manifold combining four-dimensional Minkowski coordinates x^m with four Grassmann coordinates $\theta^\alpha, \bar{\theta}_{\dot{\alpha}}$ into a single supermanifold with coordinate $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$. The superconformal algebra can be represented as a set of transformations on these coordinates. In differential form they read [18]

$$\begin{gathered}
p_a = \partial_a, \quad q_\alpha = \partial_\alpha - i(\bar{\theta}\bar{\sigma}^a \epsilon)_\alpha \partial_a, \quad \bar{q}^{\dot{\alpha}} = \partial^{\dot{\alpha}} - i(\theta\sigma^a\epsilon)^{\dot{\alpha}} \partial_a \\
m_{ab} = -x_{[a}\partial_{b]} + \theta\sigma_{ab}\partial_\theta + \bar{\theta}\bar{\sigma}_{ab}\partial_{\bar{\theta}} \\
d = x^m \partial_m + \frac{1}{2}\theta\partial_\theta + \frac{1}{2}\bar{\theta}\partial_{\bar{\theta}}, \quad a = -i\theta\partial_\theta + i\bar{\theta}\partial_{\bar{\theta}} \\
s_\alpha = -2\theta^2 \partial_\alpha + i(x_b - i\zeta_b)(\sigma_b \partial_{\bar{\theta}})_\alpha - (x_b + i\zeta_b)(\theta\sigma_c\bar{\sigma}_b\epsilon)_\alpha \partial_c \\
s^{\dot{\alpha}} = -2\bar{\theta}^2 \partial^{\dot{\alpha}} + i(x_b + i\zeta_b)(\bar{\sigma}_b \partial_\theta)^{\dot{\alpha}} - (x_b - i\zeta_b)(\bar{\theta}\bar{\sigma}_c\sigma_b\epsilon)^{\dot{\alpha}} \partial_c \\
k_a = 2x_a(x\cdot\partial) - x^2\partial_a - 2\zeta_a(\zeta\cdot\partial) + \zeta^2\partial_a - (x_b + i\zeta_b)(\theta\sigma_a\bar{\sigma}_b\partial_\theta) - (x_b - i\zeta_b)(\bar{\theta}\bar{\sigma}_a\sigma_b\partial_{\bar{\theta}})
\end{gathered} \tag{2.2.14}$$

where $\zeta^a \equiv \theta\sigma^a\bar{\theta}$. These operators can be found in several ways. The most straightforward is to write down the supersymmetry line element $ds^2 = \left(dx^a + i\theta\sigma^a d\bar{\theta} + i\bar{\theta}\bar{\sigma}^a d\theta\right)^2$ and require that it be preserved up to a conformal factor. This yields the coordinate representations we have given above. The elements p_a, q_α, $\bar{q}^{\dot{\alpha}}$, m_{ab} and a preserve the line element exactly; the others, d, k_a, s_α and $\bar{s}^{\dot{\alpha}}$ preserve it only up to a conformal factor.

The field representation possesses the same algebra as the coordinate representation but with the opposite sign. We will be most interested in the field representation, which is the only sensible approach when the symmetry is made a local one.

As it will be useful to collect terms in a way which makes manifest the supersymmetry, we will denote by P_A the set of generators P_a, Q_α, and $\bar{Q}^{\dot{\alpha}}$; P_A represents the super-translation generator on superspace. Similarly, the special conformal generators may be collected into a single K_A. The algebra as in Section 2.1 can then be written

$$\begin{gathered}
[D, P_A] = \lambda(A) P_A, \quad [A, P_A] = -i\omega(A) P_A \\
[D, K_A] = -\lambda(A) K_A, \quad [A, K_A] = +i\omega(A) K_A \\
[P_A, P_B] = -C_{AB}{}^C P_C, \quad [K_A, K_B] = C_{AB}{}^C K_C \\
[K_A, P_B] = +2\tilde{\eta}_{AB} D - 2M_{AB} + 3iA\eta_{AB}\omega(A) - \frac{1}{2}K^C C_{CBA} - \frac{1}{2}P^C C_{CAB}
\end{gathered} \tag{2.2.15}$$

The commutators and other objects are to be understood as carrying an implicit grading, which we briefly discuss in Appendix C.

The various objects defined above are

$$\begin{gathered}
P_A = (P_a, Q_\alpha, Q^{\dot{\alpha}}), \quad K_A = (K_a, S_\alpha, S^{\dot{\alpha}}) \\
M_{AB} = \left(M_{ab}, M_{\alpha\beta}, M^{\dot{\alpha}\dot{\beta}}\right) \\
\eta_{AB} = (\eta_{ab}, -\epsilon_{\alpha\beta}, -\epsilon^{\dot{\alpha}\dot{\beta}}), \quad \tilde{\eta}_{AB} = (\eta_{ab}, +\epsilon_{\alpha\beta}, +\epsilon^{\dot{\alpha}\dot{\beta}})
\end{gathered} \tag{2.2.16}$$

where mixed objects such as $M_{a\beta}$ and $\eta_{a\beta}$ are defined to be zero. Note that $\tilde{\eta}_{AB} = (-)^A \eta_{AB}$; this will be the origin of graded signs $(-)^A$ in subsequent formulae.

We also have the flat-space torsion tensor

$$C_{AB}{}^C = -C_{BA}{}^C = \begin{cases} -2i(\sigma^c\epsilon)_\alpha{}^{\dot{\beta}} & \text{if } A = \alpha, B = \dot{\beta}, C = c \\ 0 & \text{otherwise} \end{cases} \tag{2.2.17}$$

as well as the numerical coefficients

$$\begin{gathered}
\lambda(A) = \begin{cases} 1 & \text{if } A = a \\ \frac{1}{2} & \text{if } A = \alpha, \dot{\alpha} \end{cases} \\
\omega(A) = \begin{cases} 0 & \text{if } A = a \\ +1 & \text{if } A = \alpha \\ -1 & \text{if } A = \dot{\alpha} \end{cases}
\end{gathered} \tag{2.2.18}$$

The tensor C, like all explicitly supersymmetric objects, possesses an implicit grading.[17] The matrix η_{AB} is used to raise and lower indices; $\tilde{\eta}_{AB}$ is its transpose, and is equivalent to $\eta_{AB}(-)^{ab}$, the sign coming from the implicit grading.

[17] That is, we interpret its antisymmetry condition to mean $C_{ab}{}^C = -C_{ba}{}^C$ but $C_{\alpha\beta}{}^C = +C_{\beta\alpha}{}^C$. The implicit grading works by appending an extra sign whenever two fermionic objects (fields, indices, etc.) are permuted past each other.

The main limitation of this form is that the would-be super-rotation generator M_{AB} is highly constrained: only M_{ab} is independent. $M_{\alpha b}$ vanishes, and $M_{\alpha\beta}$ is just a left-handed projection of M_{ab}. Nevertheless, we may write its commutator with P_A in the elegant form

$$[M_{AB}, P_C] = P_A \eta_{BC} - P_B \eta_{AC} \tag{2.2.19}$$

where it is to be understood that the A, B, and C are all of the same type and the implicit grading is understood.

The representation theory of fields under the conformal group is discussed in [19] as well as in Section 2.1.1 and is rather straightforward. The only difference from Poincaré representations is that we require primary fields Φ to have constant conformal weight under D and to be annihilated by the special conformal generator K_a.

We may extend that discussion to the superconformal group with little effort. Let Φ be a primary superfield. It may or may not possess Lorentz indices, but we will suppress them for notational elegance. The action of the superconformal group is

$$\begin{gathered} P_A \Phi = \nabla_A \Phi, \quad M_{ab}\Phi = \mathcal{S}_{ab}\Phi \\ D\Phi = \Delta\Phi, \quad A\Phi = iw\Phi \\ K_A \Phi = 0 \end{gathered} \tag{2.2.20}$$

The action of P_A is the statement that the translation generator acts as the covariant derivative. The matrix $\mathcal{S}_{ab}$ is appropriate for whatever representation of the Lorentz algebra Φ belongs to. Δ and w represent its conformal and chiral weights.

Primary chiral superfields

The superconformal algebra by itself does not itself tell us anything more about an arbitrary superfield than the conformal algebra tells us in spacetime. The advantage comes when restrictions are imposed. For example, the most important theoretical and phenomenological superfields are chiral ones. These obey a constraint $\bar\nabla^{\dot\alpha}\Phi = 0$, where again we are suppressing Lorentz indices which Φ may possess. Requiring this to be superconformally invariant leads to a nontrivial condition on Φ:

$$0 = \{S^{\dot\alpha}, \bar\nabla^{\dot\beta}\}\Phi = \epsilon^{\dot\alpha\dot\beta}(2D + 3iA)\Phi - 2M^{\dot\alpha\dot\beta}\Phi = \epsilon^{\dot\alpha\dot\beta}(2\Delta - 3w)\Phi - 2M^{\dot\alpha\dot\beta}\Phi \tag{2.2.21}$$

The first term is antisymmetric in the indices, the second is symmetric. Therefore each must vanish separately. The first tells us $2\Delta = 3w$, that is, the $U(1)$ weight and scaling dimension of the field Φ have a fixed ratio. The second term tells us that Φ, when decomposed into irreducible spinors, contains no dotted indices, since $M^{\dot\alpha\dot\beta}$ acts only on these. Thus, Φ_α, $\Phi_{\alpha\beta}$, and $\Phi_{\alpha\beta\gamma}$ are valid chiral superfields, but $\Phi_{(\alpha\dot\beta)} = \sigma_{\alpha\dot\beta}{}^c\Phi_c$ is not. (We will use the notation $(\alpha\dot\alpha)$ to denote a vector index contracted with a sigma matrix.)

Primary gauge vector superfields

The next most important superfield is the gauge vector superfield V. It is related to the chiral superfield W_α by $W_\alpha = \mathcal{P}[\nabla_\alpha V]$ where $\mathcal{P}$ is the chiral projection operator.[18]

[18]It is convention in literature to call this object the "projection" operator even though it is not truly a projection operator, since $\mathcal{P}^2 \neq \mathcal{P}$. We denote Π as the *actual* projection operator where it matters.

In flat supersymmetry this condition reads $W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$ where D_A is the flat space covariant derivative; we will assume without (yet) a proof that this expression holds in the case of a nontrivial geometry simply by replacing D_A with ∇_A. If we demand that W_α be primary in addition to V being primary, we can deduce a nontrivial condition on V. The primary condition is actually three: the vanishing of K, S, and $\bar{S}$ on W_α. Since the anti-commutator of S and $\bar{S}$ yields K, we only need to check that S and $\bar{S}$ vanish. Consider S first:

$$0 = -4S_\beta W_\alpha = S_\beta \bar{\nabla}^2 \nabla_\alpha V = \bar{\nabla}^2 S_\beta \nabla_\alpha V = \bar{\nabla}^2 \left(2D\epsilon_{\beta\alpha} - 2M_{\beta\alpha} - 3iA\epsilon_{\beta\alpha}\right) V$$

Since V is real, its chiral weight vanishes. Furthermore, it is a scalar so M annihilates it. We are left with the condition $DV = 0$, that is V must have conformal dimension zero. This forces W_α to have conformal dimension $3/2$, which is sensible since it must possess the gaugino as its lowest component. The check that $\bar{S}^{\dot{\beta}}$ also annihilates W_α is straightforward; no further constraints are required. It therefore follows that W_α is conformally primary precisely when V has conformal dimension zero.

Primary F-term superfields

The last special case we will discuss is where V is a superfield and we demand that its chiral projection $U = \mathcal{P}[V]$ is primary. (This is of interest since if V is a D-term then U is the corresponding F-term.) Generalizing the flat space result gives $U = -\frac{1}{4}\bar{\nabla}^2 V$ (which we will show is the case later). We assume that V is primary with conformal weight Δ and chiral weight w. Then the primariness of U reduces to checking that $\bar{S}$ annihilates U, since it is clear that S annihilates U. This is a simple exercise:

$$\begin{aligned}
-4\bar{S}^{\dot{\beta}} U &= -\{\bar{S}^{\dot{\beta}}, \bar{\nabla}^{\dot{\alpha}}\}\bar{\nabla}_{\dot{\alpha}} V - \bar{\nabla}_{\dot{\alpha}}\{\bar{S}^{\dot{\beta}}, \bar{\nabla}^{\dot{\alpha}}\} V \\
&= -\left(2D\epsilon^{\dot{\beta}\dot{\alpha}} - 2M^{\dot{\beta}\dot{\alpha}} + 3iA\epsilon^{\dot{\beta}\dot{\alpha}}\right)\bar{\nabla}_{\dot{\alpha}} V - \bar{\nabla}_{\dot{\alpha}}\left(2D\epsilon^{\dot{\beta}\dot{\alpha}} - 2M^{\dot{\beta}\dot{\alpha}} + 3iA\epsilon^{\dot{\beta}\dot{\alpha}}\right) V \\
&= (8 - 4\Delta + 6w)\bar{\nabla}^{\dot{\beta}} V
\end{aligned}$$

It follows that $2\Delta - 3w = 4$ is the condition on V so that U is primary. If we also require that the antichiral projection of V be primary, then we would find $2\Delta + 3w = 4$, concluding that $w = 0$ and $\Delta = 2$. This latter case is most important since we will see if V is a D-term action, then U is the F-term action equivalent to V.

2.2.3 Local superconformal symmetry

A space of local symmetries includes a rule for parallel transport, which allows one to compare group elements at neighboring points. One demands that the supertranslation generators P_A act as parallel transport operators with the supervierbein $E_M{}^A$ as their corresponding gauge field. For each of the other generators X_A, one also introduces a gauge field $W_M{}^A$:

$$W_M{}^A X_A = E_M{}^A P_A + \frac{1}{2}\phi_M{}^{ba} M_{ab} + B_M D + A_M A + f_M{}^A K_A \tag{2.2.22}$$

In practice, it is useful to decompose the algebra into the generators of parallel transport and the other generators, which annihilate pure functions (i.e. scalar primary fields with vanishing chiral and scaling weights). We denote the latter group as $\mathcal{H}$, its generators by $X_{\underline{a}}$, and its gauge fields by $h_M{}^{\underline{a}}$. In this manner, the total gauge connection is simply

$$W_M{}^A X_A = E_M{}^A P_A + h_M{}^{\underline{a}} X_{\underline{a}} \tag{2.2.23}$$

The action of the generators on fields is defined by

$$\delta_G(\xi^M W_M{}^A X_A)\Phi \equiv \mathcal{L}_\xi \Phi. \tag{2.2.24}$$

For fields lacking Einstein indices, this reduces to the simpler

$$\xi^M W_M{}^A X_A \Phi = \xi^M \partial_M \Phi \tag{2.2.25}$$

Since the action of the non-translation generators is defined already, this defines the action of P_A. One finds the standard definition of the covariant derivative

$$\xi^A P_A \Phi = \xi^M \nabla_M \Phi = \xi^M \left(\partial_M - h_M{}^{\underline{a}} X_{\underline{a}}\right) \Phi \tag{2.2.26}$$

If Φ possesses an Einstein index, then an additional transformation rule for that index is required. For example, on the vierbien,

$$\delta_P(\xi) E_M{}^A = \xi^N \nabla_N E_M{}^A + \partial_M \xi^N E_N{}^A; \tag{2.2.27}$$

this rule can be used to define δ_P on any other Einstein tensor.

The algebraic structure of conformal superspace is identical to the flat case except for the introduction of curvatures to the $[P, P]$ commutator. We include here the results quoted in the most supersymmetric language:[19]

$$\begin{aligned}
[P_A, P_B] &= -T_{AB}{}^C P_C - \frac{1}{2} R_{AB}{}^{dc} M_{cd} - H_{AB} D - F_{AB} A - R(K)_{AB}{}^C K_C \\
[M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc} \\
[D, P_A] &= +\lambda(A) P_A \\
[A, P_A] &= -i\omega(A) P_A \\
[K_A, K_B] &= +C_{AB}{}^C K_C \\
[D, K_A] &= -\lambda(A) K_A \\
[A, K_A] &= +i\omega(A) K_A \\
[K_A, P_B] &= +2\tilde{\eta}_{AB} D - 2M_{AB} + 3iA\eta_{AB}\omega(A) - \frac{1}{2} K^C C_{CBA} - \frac{1}{2} P^C C_{CAB}
\end{aligned} \tag{2.2.28}$$

[19]We have adopted the notation $R(K)_{AB}{}^C$ for the special conformal curvature. One could similarly write $R_{AB}{}^{dc}$ as $R(M)_{AB}{}^{dc}$ but we have chosen to use the conventional name for the Lorentz curvature.

2.2.4 Invariant superconformal actions

Superspace actions fall into two types. The first is the D-type Lagrangian, involving an integration over the full Grassmannian manifold. The local action reads

$$S_D = \int d^4x\, e\, \mathcal{L}_D = \int d^4x\, d^4\theta\; E\; V \tag{2.2.29}$$

Here $e = \det(e_m{}^a)$ is the normal four dimensional measure factor, while $E = \det(E_M{}^A)$ is the appropriate generalization for a D-term.[20] (Setting $E = e = 1$ retrieves the global action.) Invariance requires $X_{\underline{b}}V = -(-)^A f_{\underline{b}A}{}^A V$, which amounts to

$$\begin{gathered} D\,V = 2V, \quad A\,V = 0, \quad M_{ab}V = 0, \\ K_aV = 0, \quad S_\alpha V = 0, \quad \bar{S}^{\dot\alpha}V = 0 \end{gathered}$$

V must have scaling dimension two; its chiral weight must vanish; it must be a Lorentz scalar; it must be superconformally primary. One should also in general check the action of P_a, Q_α, and $\bar{Q}^{\dot\alpha}$ to ensure that it is translation invariant and supersymmetric. Each of these gives a set of derivative operations; since the entire space is integrated over, each of these vanishes. (A review of this material was presented in Section 2.1.2.)

The second Lagrangian of concern is the F-type, which involves an integration over the chiral submanifold $\mathfrak{M}$ corresponding to $\bar\theta = 0$ (or to any other constant $\bar\theta$ slice):

$$S_F = \int d^4x\; e\; \mathcal{L}_F = \int d^4x\, d^2\theta\, \mathcal{E}\, W \tag{2.2.30}$$

(We will for brevity's sake write only the chiral part; but in physical theories one must of course include the antichiral part.) The chiral measure $\mathcal{E}$ is to be understood as $\det(E_{\mathfrak{m}}{}^{\mathfrak{a}})$ where $\mathfrak{m}$ is the Einstein index over the chiral coordinates y and θ and $\mathfrak{a} = (a, \alpha)$. This is a loose definition since the chiral y and θ need to be better defined. (Section 2.1.2 contains a brief discussion of this.)

For this action to be invariant under the non-translation part of the gauge group, W must obey

$$\begin{gathered} D\,W = 3W, \quad A\,W = 2iW, \quad M_{ab}W = 0 \\ K_aW = 0, \quad S_\alpha W = 0, \quad \bar{S}^{\dot\alpha}W = 0 \end{gathered}$$

For invariance under P, Q, and $\bar{Q}$, W must be chiral, $\nabla_{\dot\alpha}W = 0$. In addition, consistency of this result (i.e. $\{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\}W = 0$) leads to the following conditions on torsions and curvatures:

$$T_{\dot\alpha\dot\beta}{}^c = T_{\dot\alpha\dot\beta}{}^\gamma = 0, \quad H_{\dot\alpha\dot\beta} + \frac{2i}{3}F_{\dot\alpha\dot\beta} = 0 \tag{2.2.31}$$

These constraints are invariant under the action of $\mathcal{H}$, as is expected, and should be understood as the *minimal* set of constraints for a conformal superspace.

[20]This determinant becomes a super-determinant when the implicit grading is taken into account

2.2.5 Constraints

Since every superfield contains 16 independent components, the number of degrees of freedom represented by unconstrained gauge fields is staggering. The vierbein $E_M{}^A$ alone consists of 64 superfields, each possessing 16 independent components for a total of 1024 component fields. This problem can be circumvented by the imposition of certain constraints in superspace, followed by solving the Bianchi identities subject to these constraints. Conformal superspace must reduce to a Poincaré superspace upon the breaking of the conformal symmetry, so the constraints on its geometry ought to be more severe than those normally imposed. We will guess the constraints necessary by identifying the properties we would like to have. If this overconstrains the theory, so be it; the Bianchi identities will tell us if this occurs.

Perhaps the most fundamental constraint is the existence of chiral primary superfields, the absence of which would render any superspace theory practically useless. The chiral requirement, $\nabla_{\dot\alpha}\Phi = 0$, implies that the anticommutator $\{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\}\Phi$ vanishes. (We have suppressed any Lorentz indices which Φ may possess.) This commutator in turn gives the following constraints:

$$T_{\dot\alpha\dot\beta}{}^c = T_{\dot\alpha\dot\beta}{}^\gamma = 0, \quad H_{\dot\alpha\dot\beta} + \frac{2i}{3}F_{\dot\alpha\dot\beta} = 0, \quad R_{\dot\alpha\dot\beta}{}^{\gamma\delta} = 0 \tag{2.2.32}$$

(The complex conjugates are implied for the existence of anti-chiral superfields.) All of these conditions except the last we already knew; the last is required if chiral superfields with undotted spinor indices (such as W_α and $W_{\alpha\beta\gamma}$) should exist.

If we consider the component level behavior, more constraints may be deduced. The component conformal supergravity multiplet for a chiral matter scalar, ϕ, possesses the same transformation laws as in flat supersymmetry, only with the regular derivative replaced by a covariant one:

$$\delta_Q\phi = \sqrt{2}\xi\psi, \quad \delta_Q\psi = \sqrt{2}\xi F + i\sqrt{2}\sigma^a\bar\xi\nabla_a\phi, \quad \delta_Q F = i\sqrt{2}(\bar\xi\bar\sigma^a\nabla_a\psi) \tag{2.2.33}$$

These equations can be interpreted as superspace formulae with the superfields $\psi_\alpha \equiv \frac{1}{\sqrt{2}}\nabla_\alpha\phi$ and $F \equiv -\frac{1}{4}\nabla^2\phi$, and the formal definition of $\delta_Q \equiv \xi^\alpha\nabla_\alpha + \bar\xi_{\dot\alpha}\nabla^{\dot\alpha}$. Requiring that this variation δ_Q act on each of the superfields as indicated by the component transformation rules leads to a number of further constraints on the superspace curvatures:

$$T_{\alpha\beta}{}^\gamma = T_{\alpha\dot\beta}{}^{\underline{\gamma}} = T_{\underline{\alpha}b}{}^c = 0, \quad T_{\alpha\dot\beta}{}^c = 2i\sigma^c_{\alpha\dot\beta} \tag{2.2.34}$$

Other more complicated conditions are also implied, but they end up being satisfied automatically by the Bianchi identities, so we do not bother listing them here in detail.

We can further restrict the superspace structure by requiring the component transformation laws for the gravitino, $U(1)$ gauge field, and scaling gauge field to behave as in component conformal supergravity. For example, the gravitino ought to transform under supersymmetry into a covariant derivative of the supersymmetry parameter, $\delta_Q\psi_m = 2\nabla_m\xi$, without the need for any explicit auxiliary fields as in (2.2.10). Since we already know the transformation law for the gravitino is

$$\delta_Q E_m{}^\alpha = \nabla_m\xi^\alpha + E_m{}^c\xi^\beta T_{\beta c}{}^\alpha + E_m{}^c\xi_{\dot\beta}T^{\dot\beta}{}_c{}^\alpha \tag{2.2.35}$$

we are left to conclude $T_{\underline{\beta}c}{}^{\alpha} = 0$. (These are the torsion components which in the minimal multiplet give the superfields R and G_c whose lowest components are the supergravity auxiliaries M and b_m.) A similar analysis using the $U(1)$ and scaling gauge fields using (2.2.11) and (2.2.12) tells us $F_{\beta c} = H_{\beta c} = 0$.

One can continue in this manner to bootstrap constraints which seem reasonable. The ones discussed above are nearly sufficient to completely determine a minimal superspace formulation of conformal supergravity. It turns out only one additional constraint is needed: $R(K)_{\alpha\beta}{}^{C} = 0$ and its conjugate.

We summarize here the constraints we take. For torsion we have

$$\begin{aligned} T_{\gamma\beta}{}^{A} &= T_{\dot\gamma\dot\beta}{}^{A} = 0 \\ T_{\gamma\dot\beta}{}^{a} &= 2i\sigma^{a}_{\gamma\dot\beta} \\ T_{c\beta}{}^{A} &= T_{c\dot\beta}{}^{A} = 0 \\ T_{cb}{}^{a} &= 0 \end{aligned} \tag{2.2.36}$$

These define all torsion except for $T_{cb}{}^{\alpha}$ and $T_{cb}{}^{\dot\alpha}$ which remain undetermined. For the Lorentz curvature, we have

$$R_{\alpha\beta}{}^{cd} = R_{\alpha\dot\beta}{}^{cd} = R_{\dot\alpha\dot\beta}{}^{cd} = 0 \tag{2.2.37}$$

For the chiral curvature,

$$\begin{aligned} F_{\alpha\beta} = F_{\alpha\beta} &= F_{\dot\alpha\dot\beta} = 0 \\ F_{\alpha b} &= F_{\dot\alpha b} = 0 \end{aligned} \tag{2.2.38}$$

Similarly for the scaling curvature:

$$\begin{aligned} H_{\alpha\beta} = H_{\alpha\beta} &= H_{\dot\alpha\dot\beta} = 0 \\ H_{\alpha b} &= H_{\dot\alpha b} = 0 \end{aligned} \tag{2.2.39}$$

For the special conformal curvature, we take

$$R(K)_{\alpha\beta}{}^{C} = R(K)_{\dot\alpha\dot\beta}{}^{C} = R(K)_{\alpha\dot\beta}{}^{C} = 0 \tag{2.2.40}$$

This set of conditions for the curvatures is especially interesting for one particular reason: it includes the condition $R_{\underline{\alpha\beta}} = 0$ for all curvatures *except* for torsion, where we choose the flat result $T_{\alpha\dot\beta}{}^{c} = 2i\sigma^{c}_{\alpha\dot\beta}$. This is consistent with making the following demands on the fermionic covariant derivatives:[21]

$$\{\nabla_\alpha, \nabla_\beta\} = \{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\} = 0 \tag{2.2.41}$$

$$\{\nabla_\alpha, \nabla_{\dot\beta}\} = -2i\nabla_{\alpha\dot\beta} \tag{2.2.42}$$

The first of these implies the existence of a gauge choice where $\nabla_\alpha = \partial_\alpha$ and the second implies the conjugate; the third implies that no gauge exists where *both* these conditions hold. Moreover, in flat supersymmetry, the chiral projector $\mathcal{P}$ is proportional to $\bar{D}^2$. The condition that it should be $\bar{\nabla}^2$ in conformal supergravity is equivalent to the constraints $\{\nabla_\alpha, \nabla_\beta\} = \{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\} = 0$.

These constraints may at first glance seem exceedingly restrictive, certainly more so than those assumed in deriving Poincaré supergravity. It helps to recall that each of these objects, the torsion and the other curvatures, are internally more complicated than their non-conformal cousins due to the presence of the extra gauge fields. We will find that it is these fields, in particular those associated with the special conformal generators, which allow us to reconstruct normal Poincaré supergravity with its relaxed constraints after gauge fixing.

2.2.6 Jacobi and Bianchi identities

The discussion of the Jacobi and Bianchi identities in an arbitrary theory is given in 2.1.2 and merely needs to be specialized here. The Jacobi identity serves to define the gauge transformation properties of the curvatures:

$$\begin{aligned}
D\, T_{CB}{}^A &= (\Delta(C) + \Delta(B) - \Delta(A))\, T_{CB}{}^A \\
D\, R(K)_{CB}{}^A &= (\Delta(C) + \Delta(B) + \Delta(A))\, R(K)_{CB}{}^A \\
D\, R_{DC}{}^{BA} &= (\Delta(D) + \Delta(C))\, R_{DC}{}^{BA} \\
D\, F_{BA} &= (\Delta(B) + \Delta(A))\, F_{BA} \\
D\, H_{BA} &= (\Delta(B) + \Delta(A))\, H_{BA}
\end{aligned} \tag{2.2.43}$$

(With the exception of the K-curvature, these are entirely straightforward.) The $U(1)$ transformations are similarly simple:

$$\begin{aligned}
A\, T_{CB}{}^A &= -i\,(w(C) + w(B) - w(A))\, T_{CB}{}^A \\
A\, R(K)_{CB}{}^A &= -i\,(w(C) + w(B) + w(A))\, R(K)_{CB}{}^A \\
A\, R_{DC}{}^{BA} &= -i\,(w(D) + w(C))\, R_{DC}{}^{BA} \\
A\, F_{BA} &= -i\,(w(B) + w(A))\, F_{BA} \\
A\, H_{BA} &= -i\,(w(B) + w(A))\, H_{BA}
\end{aligned} \tag{2.2.44}$$

[21] These conditions alone are probably sufficient to define a conformal superspace with dynamical spin connection and torsion as well as their superpartners; we conjecture that the extra constraints are to eliminate the spin connection and its associated multiplet but as yet are unaware of any direct evidence for this.

The transformations under K_A are, however, less than obvious:[22]

$$
\begin{aligned}
K_D\ \ T_{CB}{}^A &= \frac{1}{2}\Delta T_{CB}{}^F C^A{}_{DF} + \frac{1}{2}C^F{}_{D[C}\Delta T_{B]F}{}^A \\
K_D\ \ H_{CB} &= -(-)^D 2\Delta T_{CBD} + \frac{1}{2}C^F{}_{D[C}H_{B]F} \\
K_D\ \ F_{CB} &= -3iw(D)\Delta T_{CBD} + \frac{1}{2}C^F{}_{D[C}F_{B]F} \\
K_D\ \ R(K)_{CB}{}^A &= R(K)_{CB}{}^F C_{FD}{}^A + \frac{1}{2}C^F{}_{D[C}R(K)_{B]F}{}^A - \frac{1}{2}\Delta T_{CB}{}^F C_F{}^A{}_D \\
&\quad - \lambda(D)H_{CB}\delta_D{}^A + iw(D)F_{CB}\delta_D{}^A + R_{CBD}{}^A \\
\frac{1}{2}\left(K_D\ \ R_{CB}{}^{a'a}\right)M_{aa'} &= 2\Delta T_{CB}{}^A M_{AD} - \frac{1}{4}C^F{}_{D[C}R_{B]F}{}^{a'a}M_{aa'}
\end{aligned}
\tag{2.2.45}
$$

The notation $[CB]$ in the above denotes graded antisymmetrization of the respective indices. The rule for the Lorentz curvature has been left in a form with the explicit Lorentz generators since they are not independent of each other. Since K_A is in a sense the inverse of P_A, these rules are like inverted Bianchi identities, and they provide a nontrivial check of consistency once the curvatures are specified.

We do not list explicitly the Lorentz transformation rules for the curvatures since each transforms as its indices indicate.

Invariance under parallel transports is equivalent to checking the Bianchi identities. These are significantly more complicated:

$$
\begin{aligned}
0 &= \sum_{[DCB]} \nabla_D F_{CB} + T_{DC}{}^F F_{FB} - 3iR(K)_{DCB}w(B) \\
0 &= \sum_{[DCB]} \nabla_D H_{CB} + T_{DC}{}^F H_{FB} - 2R(K)_{DCB}(-)^B \\
0 &= \sum_{[DCB]} \nabla_D T_{CB}{}^A + T_{DC}{}^F T_{FB}{}^A - R_{DCB}{}^A + \lambda(A)H_{DC}\delta_B{}^A + iw(A)F_{DC}\delta_B{}^A - \frac{1}{2}R(K)_{DC}{}^F C_F{}^A{}_B \\
0 &= \sum_{[DCB]} \nabla_D R(K)_{CBA} + T_{DC}{}^F R(K)_{FBA} - \frac{1}{2}R(K)_{DC}{}^F C_{BAF} \\
0 &= \sum_{[FDC]} \nabla_F R_{DC\beta\alpha} + T_{FD}{}^H R_{HC\beta\alpha} - \frac{1}{2}R(K)_{FD\{\beta\dot\phi}(\bar\sigma_c\epsilon)^{\dot\phi}{}_{\alpha\}} + 2R(K)_{FD\{\beta}\,\epsilon_{\alpha\}\delta}
\end{aligned}
\tag{2.2.46}
$$

The sum over $[DCB]$ denotes a sum over graded cyclic permutations of those indices. Also, the notation $\{\}$ on indices denotes symmetrization; for example, $X_{\{\alpha}Y_{\beta\}} \equiv X_\alpha Y_\beta + Y_\beta X_\alpha$. (The last identity involving the Lorentz curvature has been projected to the left-handed part of the Lorentz group. The right-handed part is found by complex conjugation.)

[22] Note that gradings arising from the order of the indices have been left off for simplicity of notation. To replace them, note the order of the indices on the left side of the equation and add appropriate gradings to arrive at the same order. Also, contracted indices must be placed next to each other with the raised index on the left. For example, in the first line, the order of indices on the left is $DCBA$. If we replace the gradings, we would have $K_D\ T_{CB}{}^A = \frac{1}{2}\Delta T_{CB}{}^F C^A{}_{DF}(-)^{AF+D(A+F+B+C)} + \frac{1}{2}C^F{}_{DC}\Delta T_{BF}{}^A(-)^{F(D+C+B)}$.

As in [7] the constraints we have chosen restrict our gauge potentials; we must ensure that the Bianchi identities are satisfied in order for these constraints to be valid. Though our constraints are stronger than in [7], our curvatures and Bianchi identities are more numerous. We avoid recounting the derivation in detail here (see Appendix A for that) and merely quote the result: every curvature either vanishes or is expressed in terms of a single chiral superfield $W_{\alpha\beta\gamma}$. It obeys

$$D\,W_{\alpha\beta\gamma} = \frac{3}{2}W_{\alpha\beta\gamma}, \quad A\,W_{\alpha\beta\gamma} = iW_{\alpha\beta\gamma}, \quad K_A W_{\alpha\beta\gamma} = 0 \tag{2.2.47}$$

That is, $W_{\alpha\beta\gamma}$ possesses the same scaling and $U(1)$ weights as it does in Poincaré supergravity and is conformally primary. Furthermore, it is constrained by its own Bianchi identity

$$\nabla^{\gamma}{}_{\dot\beta}\nabla^{\phi}W_{\phi\gamma\beta} = -\nabla_{\beta}{}^{\dot\gamma}\nabla^{\dot\phi}W_{\dot\phi\dot\gamma\dot\beta} \tag{2.2.48}$$

The results for the curvatures follow below.

Torsion

The conformal torsion two-form is defined in terms of the gauge connections:

$$T^A = dE^A + \lambda(A)E^A B - iw(A)E^A A + E^B \phi_B{}^A - \frac{1}{2}C^{ACB}E_B f_C \tag{2.2.49}$$

We group the various components in terms of their scaling dimension.

- Dimension 0

$$T_{\gamma\beta}{}^a = 0, \quad T^{\dot\gamma\dot\beta a} = 0 \tag{2.2.50}$$
$$T_{\gamma}{}^{\dot\beta a} = -2i(\sigma^a\epsilon)_{\gamma}{}^{\dot\beta} \tag{2.2.51}$$

- Dimension 1/2

$$T_{\underline{\gamma\beta}}{}^{\underline{\alpha}} = 0, \quad T_{\underline{\gamma} b}{}^a = 0 \tag{2.2.52}$$

- Dimension 1

$$T_{\gamma b}{}^{\alpha} = 0, \quad T^{\dot\gamma}{}_{b\dot\alpha} = 0 \tag{2.2.53}$$
$$T_{\gamma b\dot\alpha} = 0, \quad T^{\dot\gamma}{}_b{}^{\alpha} = 0 \tag{2.2.54}$$
$$T_{cb}{}^a = 0 \tag{2.2.55}$$

- Dimension 3/2

$$T_{cb}{}^{\alpha} \quad \rightsquigarrow \quad T_{(\gamma\dot\gamma)(\beta\dot\beta)\alpha} = +2\epsilon_{\dot\gamma\dot\beta}W_{\gamma\beta\alpha} \tag{2.2.56}$$
$$T_{cb\dot\alpha} \quad \rightsquigarrow \quad T_{(\gamma\dot\gamma)(\beta\dot\beta)\dot\alpha} = -2\epsilon_{\gamma\beta}\bar W_{\dot\gamma\dot\beta\dot\alpha} \tag{2.2.57}$$

Lorentz curvature

The conformal Lorentz curvature two-form is

$$R^{ba} = d\phi^{ba} + \phi^{bc}\phi_c{}^a - 2E^{[b}f^{a]} - 4E^\beta f^\alpha(\sigma^{ba}\epsilon)_{\alpha\beta} - 4E_{\dot\beta}f_{\dot\alpha}(\bar\sigma^{ba}\epsilon)^{\dot\alpha\dot\beta} \tag{2.2.58}$$

The notation $[b...a]$ denotes antisymmetrization of those indices; for example, $X_{[b}Y_{a]} \equiv X_bY_a - X_aY_b$.

Because the form is valued in the Lorentz group, it may be canonically decomposed:

$$R_{DC}{}^{ba} \quad \rightsquigarrow \quad R_{DC(\beta\dot\beta)(\alpha\dot\alpha)} = 2\epsilon_{\dot\beta\dot\alpha}R_{DC\beta\alpha} - 2\epsilon_{\beta\alpha}R_{DC\dot\beta\dot\alpha} \tag{2.2.59}$$

It is simplest to express the curvature results in terms of these components.

- Dimension 1

$$R_{\delta\gamma\,\beta\alpha} = 0, \quad R_{\delta\gamma\,\dot\beta\dot\alpha} = 0 \tag{2.2.60}$$
$$R_{\dot\delta\dot\gamma\,\beta\alpha} = 0, \quad R_{\dot\delta\dot\gamma\,\dot\beta\dot\alpha} = 0 \tag{2.2.61}$$
$$R_{\delta\dot\gamma\,\beta\alpha} = 0, \quad R_{\delta\dot\gamma\,\dot\beta\dot\alpha} = 0 \tag{2.2.62}$$

- Dimension 3/2

$$R_{\delta(\gamma\dot\gamma)\,\beta\alpha} = 0, \quad R_{\delta(\gamma\dot\gamma)\,\dot\beta\dot\alpha} = +4i\epsilon_{\delta\gamma}\bar W_{\dot\gamma\dot\beta\dot\alpha} \tag{2.2.63}$$
$$R_{\dot\delta(\gamma\dot\gamma)\,\dot\beta\dot\alpha} = 0, \quad R_{\dot\delta(\gamma\dot\gamma)\,\beta\alpha} = -4i\epsilon_{\dot\delta\dot\gamma}W_{\gamma\beta\alpha} \tag{2.2.64}$$

- Dimension 2

$$\begin{aligned} R_{(\delta\dot\delta)(\gamma\dot\gamma)\beta\alpha} &= +2\epsilon_{\dot\delta\dot\gamma}\chi_{\delta\underset{\smile}{\gamma\beta}\alpha} - \frac{1}{4}\epsilon_{\dot\delta\dot\gamma}\sum_{(\delta\gamma)}\sum_{(\beta\alpha)}\epsilon_{\delta\beta}\nabla^\phi W_{\phi\gamma\alpha} \\ &= +\epsilon_{\dot\delta\dot\gamma}\nabla_{\{\beta}W_{\alpha\}\delta\gamma} \end{aligned} \tag{2.2.65}$$

$$\begin{aligned} R_{(\delta\dot\delta)(\gamma\dot\gamma)\dot\beta\dot\alpha} &= -2\epsilon_{\delta\gamma}\chi_{\dot\delta\underset{\smile}{\dot\gamma\dot\beta}\dot\alpha} + \frac{1}{4}\epsilon_{\delta\gamma}\sum_{(\dot\delta\dot\gamma)}\sum_{(\dot\beta\dot\alpha)}\epsilon_{\dot\delta\dot\beta}\nabla^{\dot\phi}\bar W_{\dot\phi\dot\gamma\dot\alpha} \\ &= -\epsilon_{\delta\gamma}\nabla_{\{\dot\beta}W_{\dot\alpha\}\dot\delta\dot\gamma} \end{aligned} \tag{2.2.66}$$

The totally symmetric symbol χ is itself the spinorial curl of the superfield W:

$$\chi_{\delta\underset{\smile}{\gamma\beta}\alpha} = \frac{1}{4}(\nabla_\delta W_{\gamma\beta\alpha} + \nabla_\gamma W_{\delta\beta\alpha} + \nabla_\beta W_{\gamma\delta\alpha} + \nabla_\alpha W_{\gamma\beta\delta}) \tag{2.2.67}$$
$$\chi_{\dot\delta\underset{\smile}{\dot\gamma\dot\beta}\dot\alpha} = \frac{1}{4}(\nabla_{\dot\delta}\bar W_{\dot\gamma\dot\beta\dot\alpha} + \nabla_{\dot\gamma}\bar W_{\dot\delta\dot\beta\dot\alpha} + \nabla_{\dot\beta}\bar W_{\dot\gamma\dot\delta\dot\alpha} + \nabla_{\dot\alpha}\bar W_{\dot\gamma\dot\beta\dot\delta}) \tag{2.2.68}$$

Scaling and $U(1)$ curvatures

The conformal field strengths for scalings and chiral rotations are

$$H = dB + 2E^A F_A(-)^a \tag{2.2.69}$$
$$F = dA + 3iE^A F_A w(A) \tag{2.2.70}$$

- Dimension 1

$$H_{\delta\gamma} = F_{\delta\gamma} = 0 \tag{2.2.71}$$
$$H_{\delta\dot{\gamma}} = F_{\delta\dot{\gamma}} = 0 \tag{2.2.72}$$
$$H_{\dot{\delta}\dot{\gamma}} = F_{\dot{\delta}\dot{\gamma}} = 0 \tag{2.2.73}$$

- Dimension 3/2

$$H_{\mu(\gamma\dot{\gamma})} = F_{\mu(\gamma\dot{\gamma})} = 0 \tag{2.2.74}$$
$$H_{\dot{\mu}(\gamma\dot{\gamma})} = F_{\dot{\mu}(\gamma\dot{\gamma})} = 0 \tag{2.2.75}$$

- Dimension 2

$$H_{cb} \quad \rightsquigarrow \quad H_{(\gamma\dot{\gamma})(\beta\dot{\beta})} = 2\epsilon_{\dot{\gamma}\dot{\beta}}\tilde{H}_{\gamma\beta} - 2\epsilon_{\gamma\beta}\tilde{H}_{\dot{\gamma}\dot{\beta}} \tag{2.2.76}$$
$$F_{cb} \quad \rightsquigarrow \quad F_{(\gamma\dot{\gamma})(\beta\dot{\beta})} = 2\epsilon_{\dot{\gamma}\dot{\beta}}\tilde{F}_{\gamma\beta} - 2\epsilon_{\gamma\beta}\tilde{F}_{\dot{\gamma}\dot{\beta}} \tag{2.2.77}$$

The components $\tilde{H}$ and $\tilde{F}$ are themselves related to the spinorial divergence of the superfield W:

$$\nabla^\gamma W_{\gamma\beta\alpha} = \frac{4i}{3}\tilde{F}_{\beta\alpha} = +2\tilde{H}_{\beta\alpha} \tag{2.2.78}$$
$$\nabla^{\dot{\gamma}} W_{\dot{\gamma}\dot{\beta}\dot{\alpha}} = \frac{4i}{3}\tilde{F}_{\dot{\beta}\dot{\alpha}} = -2\tilde{H}_{\dot{\beta}\dot{\alpha}} \tag{2.2.79}$$

Special conformal curvature

The special conformal curvatures are

$$R(K)^a = df^A - \lambda(A)f^A B + iw(A)f^A A + f^B \phi_B{}^A + \frac{1}{2}C^{ACB} f_C E_B + \frac{1}{2} f^B f^C C_{CB}{}^A$$

We will group them by their form indices.

- Fermion/fermion

$$R(K)_{\gamma\beta\alpha} = 0, \quad R(K)_{\gamma\dot{\beta}\alpha} = 0, \quad R(K)_{\dot{\gamma}\dot{\beta}\alpha} = 0 \tag{2.2.80}$$
$$R(K)_{\gamma\beta\dot{\alpha}} = 0, \quad R(K)_{\gamma\dot{\beta}\dot{\alpha}} = 0, \quad R(K)_{\dot{\gamma}\dot{\beta}\dot{\alpha}} = 0 \tag{2.2.81}$$
$$R(K)_{\gamma\beta a} = 0, \quad R(K)_{\gamma\dot{\beta}a} = 0, \quad R(K)_{\dot{\gamma}\dot{\beta}a} = 0 \tag{2.2.82}$$

- Fermion/boson

$$R(K)_{\alpha(\beta\dot\beta)\gamma} = 0, \quad R(K)_{\dot\alpha(\beta\dot\beta)\dot\gamma} = 0 \tag{2.2.83}$$

$$R(K)_{\alpha(\beta\dot\beta)\dot\gamma} = +i\epsilon_{\alpha\beta}\nabla^{\dot\phi}W_{\dot\phi\dot\beta\dot\gamma}, \quad R(K)_{\dot\alpha(\beta\dot\beta)\gamma} = +i\epsilon_{\dot\alpha\dot\beta}\nabla^{\phi}W_{\phi\beta\gamma} \tag{2.2.84}$$

$$R(K)_{\alpha(\beta\dot\beta)(\gamma\dot\gamma)} = -2i\epsilon_{\alpha\beta}\nabla_{\gamma\dot\phi}W^{\dot\phi}{}_{\dot\beta\dot\gamma}, \quad R(K)_{\dot\alpha(\beta\dot\beta)(\gamma\dot\gamma)} = -2i\epsilon_{\dot\alpha\dot\beta}\nabla_{\phi\dot\gamma}W^{\phi}{}_{\beta\gamma} \tag{2.2.85}$$

- Boson/boson

$$R(K)_{cb\mu} = -\frac{i}{3}\nabla_\mu F_{cb}, \quad R(K)_{cb\dot\mu} = +\frac{i}{3}\nabla_{\dot\mu}F_{cb} \tag{2.2.86}$$

$$R(K)_{(\gamma\dot\gamma)(\beta\dot\beta)(\alpha\dot\alpha)} = -\epsilon_{\gamma\beta}\bar\nabla_{\dot\gamma}\nabla_\alpha{}^{\dot\phi}W_{\dot\phi\dot\beta\dot\alpha} - \epsilon_{\dot\gamma\dot\beta}\nabla_\gamma\nabla^{\phi}{}_{\dot\alpha}W_{\phi\beta\alpha} \tag{2.2.87}$$

where the chiral curvature F_{cb} has been used for notational simplicity.

2.2.7 Chiral projectors and component actions

One can use the details of Section 2.1.2, specifically equation (2.1.89) to construct an explicit form for the chiral projector in conformal superspace:

$$\mathcal{P}[V] = \int d^2\bar\theta\,\bar\Sigma\,V \tag{2.2.88}$$

where $\bar\Sigma$ is the superdeterminant constructed out of $E^{\dot\mu}{}_{\dot\alpha}$ in the gauge where $E_{m\dot\alpha}$ and $E_{\mu\dot\alpha}$ vanish. Let us explicitly construct the vierbein (and other connections) in this gauge.

Recall that the variation of the connections $W^{\dot\mu A}$ is

$$\delta_G W^{\dot\mu A} = \partial^{\dot\mu} g^A + W^{\dot\mu B} g^C f_{CB}{}^A. \tag{2.2.89}$$

The gauge parameter g^A is a superfield and so has a larger parameter space than what survives at the component level. In principle, every θ and $\bar\theta$-dependent part of g^A can be exhausted to put the connections in a desirable form without affecting the component Lagrangian. We will here use the $\bar\theta$-dependence of g^A to fix $W^{\dot\mu A}$ to a specific form. (This will correspond to a chiral version of Wess-Zumino gauge. Later on we shall fix the θ-dependence.)

Let $g^A = \bar\theta_{\dot\mu} g^{\dot\mu A} + \frac{1}{2}\bar\theta^2 g_2^A$ where the functions $g^{\dot\mu A}$ and g_2^A depend on x and θ but not $\bar\theta$. It is immediately clear by inspection of the gauge transformation law that $g^{\dot\mu A}$ can be chosen to fix the gauge $W^{\dot\mu A}|_{\bar\theta=0} = \delta^{\dot\mu A}$, meaning the vierbein is gauged to $\delta^{\dot\mu A}$ at lowest component and all other gauge fields set to zero. The gauge connection $\bar\theta$-expansion then becomes

$$W^{\dot\mu A} = \delta^{\dot\mu A} + \bar\theta_{\dot\nu} W^{\dot\nu\dot\mu A} + \frac{1}{2}\bar\theta^2 W_2^{\dot\mu A} \tag{2.2.90}$$

for fields $W^{\dot\nu\dot\mu A}$ and $W_2^{\dot\mu A}$ which depend on only x and θ. The remaining gauge parameter g_2^A can be used to eliminate the antisymmetric part of $W^{\dot\nu\dot\mu A}$, leaving $W^{\dot\nu\dot\mu A} = W^{\dot\mu\dot\nu A}$. This exhausts our $\bar\theta$-dependent gauge freedom. The curvatures then uniquely determine the remaining bits of the connection. By taking the definition of the curvature R and projecting to $\bar{\theta} = 0$, one finds $W^{\dot{\nu}\dot{\mu}A} = \frac{1}{2}R^{\dot{\nu}\dot{\mu}A}|_{\bar{\theta}=0}$. The remaining component of W is determined by taking the derivative of the curvature formula and projecting to $\bar{\theta} = 0$. One finds $W_2^{\dot{\mu}A} = -\frac{1}{3}\nabla_{\dot{\alpha}}R^{\dot{\alpha}\dot{\mu}A}|_{\bar{\theta}=0} - \frac{1}{6}R_{\dot{\alpha}}{}^{\dot{\mu}\underline{b}}f_{\underline{b}}{}^{\dot{\alpha}A}|_{\bar{\theta}=0}$. This gives the formula

$$W^{\dot{\mu}A} = \delta^{\dot{\mu}A} + \frac{1}{2}\bar{\theta}_{\dot{\alpha}}R^{\dot{\alpha}\dot{\mu}A}|_{\bar{\theta}=0} - \frac{1}{6}\bar{\theta}^2\nabla_{\dot{\alpha}}R^{\dot{\alpha}\dot{\mu}A}|_{\bar{\theta}=0} - \frac{1}{12}\bar{\theta}^2R_{\dot{\alpha}}{}^{\dot{\mu}\underline{b}}f_{\underline{b}}{}^{\dot{\alpha}A}|_{\bar{\theta}=0}. \tag{2.2.91}$$

Within conformal superspace, all of the $\bar{\theta}$-dependent terms vanish, giving

$$E^{\dot{\mu}A} = \delta^{\dot{\mu}A}, \quad h^{\dot{\mu}\underline{a}} = 0 \tag{2.2.92}$$

Therefore, the chiral projector is simply defined as

$$\mathcal{P}[V] = \int d^2\bar{\theta}V = -\frac{1}{4}\partial_{\dot{\mu}}\partial^{\dot{\mu}}V = -\frac{1}{4}\bar{\nabla}^2V \tag{2.2.93}$$

where the last equality follows due to the simplicity of the connections in this gauge. Since the left and right sides of this equation transform the same way under gauge transformations, their equality in this special gauge implies their equality in any.

Since the result is suspiciously simple, we should check that this approach works for minimal supergravity where the chiral projector is known to be not so simple. There the vierbein should take the general form

$$E^{\dot{\mu}A} = \delta^{\dot{\mu}A} - \frac{1}{12}\bar{\theta}^2R_{\dot{\alpha}}{}^{\dot{\mu}\underline{b}}f_{\underline{b}}{}^{\dot{\alpha}A}|_{\bar{\theta}=0} \tag{2.2.94}$$

since the relevant torsion components vanish. The only curvature in Poincaré superspace is the Lorentz curvature, and it is straightforward to evaluate the term appearing here. One finds

$$E^{\dot{\mu}A} = \delta^{\dot{\mu}A} - \delta^{\dot{\mu}A}\bar{\theta}^2R \tag{2.2.95}$$

for the vierbein (as well as a non-vanishing spin connection which we will ignore since it turns out not to matter). The chiral projection formula becomes

$$\mathcal{P}[V] = \int d^2\bar{\theta}(1 + 2\bar{\theta}^2R)V = 2RV - \frac{1}{4}\partial_{\dot{\mu}}\partial^{\dot{\mu}}V = -\frac{1}{4}(\bar{\nabla}^2 - 8R)V \tag{2.2.96}$$

Here the spin connection is not zero but it contributes nothing when $\bar{\nabla}^2$ acts on a field without dotted indices, and so $\bar{\nabla}^2$ in this gauge is as simple in Poincaré superspace as it is in conformal superspace.

In either formalism, the conversion from a D to an F-term proceeds straightforwardly. Using (2.1.88), we find

$$\int d^4xd^4\theta E\,V = \int d^4xd^2\theta\mathcal{E}\,\mathcal{P}[V]. \tag{2.2.97}$$

where the second integration is understood to occur at $\bar{\theta} = 0$. Although the operations above were performed in a specific $\bar{\theta}$ gauge, the final results have been written in a gauge-invariant manner. In fact, since the gauge-fixing procedure undertaken had no effect on the fields at $\bar{\theta} = 0$, the right-hand side of the above equation *must* be independent of our gauge choices.

F-term integrations

We have shown that any D-term can be written as an F-term. It is still necessary to evaluate the component Lagrangian corresponding to an F-term. A chiral integral has the form

$$\int d^4x\, d^2\theta\, \mathcal{E}\, W, \tag{2.2.98}$$

an integral over the superspace slice where $\bar\theta = 0$. W is a chiral superfield transforming under the gauge group in order to leave the full action invariant.

We can evaluate this integral by the method of gauge-fixing, much like how we derived the D to F integral conversion formula. The first step is to use the θ-dependent part of the gauge transformations to fix the connections.[23] In a way entirely analogous to what we did in the previous section, we may choose[24]

$$W_\mu{}^A = \delta_\mu{}^A + \frac{1}{2}\theta^\alpha R_{\alpha\mu}{}^A|_{\theta=0} - \frac{1}{6}\theta^2 \nabla^\alpha R_{\alpha\mu}{}^A|_{\theta=0} + \frac{1}{12}\theta^2 R_\mu{}^{\alpha\underline{b}} f_{\underline{b}\alpha}{}^A|_{\theta=0}. \tag{2.2.99}$$

by exhausting the remaining θ-dependence of g^A. Here the projection to $\bar\theta = 0$ has also already been done, so we will avoid indicating it explicitly.

In conformal superspace, this expression is extremely simple. It gives

$$E_\mu{}^A = \delta_\mu{}^A, \quad h_\mu{}^{\underline{a}} = 0 \tag{2.2.100}$$

The F-term integration then becomes

$$\mathcal{L}_F = \int d^4x\, d^2\theta\, e\, W = -\frac{1}{4} e \partial^\mu \partial_\mu W - \frac{1}{2}\partial^\mu e \partial_\mu W - \frac{1}{4}(\partial^\mu \partial_\mu e) W \tag{2.2.101}$$

The first term is rather simple. In our gauge choice, it is easy to see that $\nabla^\alpha \nabla_\alpha W = \partial^\alpha \partial_\alpha W$ when $\theta = \bar\theta = 0$. The other terms are usually constructed in the literature from supersymmetric completion of this term; here we will evaluate them directly in this gauge. For example,

$$\partial_\mu e = e(\partial_\mu E_m{}^a) e_a{}^m = e(\partial_m E_\mu{}^a + T_{\mu m}{}^a) e_a{}^m = 0 + e T_{\mu \dot\beta}{}^a E_m{}^{\dot\beta} e_a{}^m = ie(\sigma^a \bar\psi_a)_\mu \tag{2.2.102}$$

where we have used $E_\mu{}^a| = 0$ as well as the torsion constraint $T_{\gamma\beta}{}^a = T_{\gamma b}{}^a = 0$. This allows us to evaluate the second term of $\mathcal{L}_F$; we find $ie(\bar\psi_a \bar\sigma_a)^\alpha \nabla_\alpha W/2$ (since $\partial_\alpha W = \nabla_\alpha W$ at $\theta = \bar\theta = 0$ in this gauge.)

The remaining third term is slightly more complicated. One begins with

$$\partial^\mu \partial_\mu e = \partial^\mu (e T_{\mu\dot\alpha}{}^a E_m{}^{\dot\alpha} e_a{}^m) \tag{2.2.103}$$

[23] It is useful to note that whether or not we gauge-fixed the $\bar\theta$-dependent part of the connections is irrelevant for evaluating an F-term as its integral occurs at $\bar\theta = 0$.

[24] This last gauge-fixing has an interesting effect on θ. Their Einstein index is now effectively a Lorentz index, since every Lorentz rotation which would alter the vierbein must be countered by a P-gauge (or coordinate) transformation. The θ's are therefore the same as the Θ variables of [7]. Their F-terms are written $\int d^2\Theta\mathcal{E}$ where Θ is equivalent to θ and their $\mathcal{E}$ is half of ours when we go to this gauge.

The outer spinorial derivative acts on each term in parentheses except the torsion (which is constant). From differentiating e, we find the term $e(\bar{\psi}_a\bar{\sigma}^a\sigma^b\bar{\psi}_b)$. From the inverse vierbein, one gets $-e(\bar{\psi}_a\bar{\sigma}^b\sigma^a\bar{\psi}_b)$. From the gravitino one finds no additional terms. This leads to

$$\partial^\mu\partial_\mu e = 4e(\bar{\psi}_a\sigma^{ab}\bar{\psi}_b) \tag{2.2.104}$$

which gives the chiral Lagrangian

$$\mathcal{L}_F = \int d^2\theta\,\mathcal{E}\,W = e\left(-\frac{1}{4}\nabla^\alpha\nabla_\alpha W + \frac{i}{2}(\bar{\psi}_a\bar{\sigma}_a)^\alpha\nabla_\alpha W - (\bar{\psi}_a\sigma^{ab}\bar{\psi}_b)W\right) \tag{2.2.105}$$

where the projection to $\theta = \bar{\theta} = 0$ is implicit.

Again, we may repeat this process for Poincaré superspace. One finds

$$E_\mu{}^A = \delta_\mu{}^A - \delta_\mu{}^A\theta^2\bar{R} \tag{2.2.106}$$

and for the F-term

$$\mathcal{L}_F = \int d^2\theta\,e\,(1+2\theta^2\bar{R})W = -\frac{1}{4}e\partial^\mu\partial_\mu W - \frac{1}{2}\partial^\mu e\partial_\mu W - \frac{1}{4}(\partial^\mu\partial_\mu e)W + 2\bar{R}W \tag{2.2.107}$$

The first and second terms are evaluated as before. The third gains an extra contribution of $-16e\bar{R}$ from (2.2.103) when the spinorial derivative hits the gravitino. This gives the chiral Lagrangian

$$\mathcal{L}_F = \int d^2\theta\,\mathcal{E}\,W = e\left(-\frac{1}{4}\nabla^\alpha\nabla_\alpha W + \frac{i}{2}(\bar{\psi}_a\bar{\sigma}_a)^\alpha\nabla_\alpha W - (\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b)W + 6\bar{R}W\right) \tag{2.2.108}$$

where the projection to $\theta = \bar{\theta} = 0$ is implicit.

D-term integrations

Within conformal superspace, the F-term component Lagrangian is

$$\mathcal{L}_F = \int d^2\theta\,\mathcal{E}\,W = e\left(F + \frac{i\sqrt{2}}{2}(\bar{\psi}_a\bar{\sigma}_a\rho) - (\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b)W\right) \tag{2.2.109}$$

where

$$F \equiv -\frac{1}{4}\nabla^2 W| \quad \text{and} \quad \rho_\alpha \equiv \frac{1}{\sqrt{2}}\nabla_\alpha W| \tag{2.2.110}$$

A D-term can be divided into two terms, one evaluated via a chiral integration and the other via an antichiral integration in order to give a manifestly Hermitean action:

$$\int d^4\theta\,E\,V = \frac{1}{2}\int d^2\theta\,\mathcal{E}\,U + \frac{1}{2}\int d^2\bar{\theta}\,\bar{\mathcal{E}}\,\bar{U} \tag{2.2.111}$$

where $U \equiv -\frac{1}{4}\bar{\nabla}^2 V$ and $\bar{U} \equiv -\frac{1}{4}\nabla^2 V$ are the chiral and antichiral projections of V. These two F-terms can then be evaluated using the F-term formula giving the general D-term formula

$$\mathcal{L}_D = \int d^4\theta\, E\, V = e\left(\frac{1}{2}(F+\bar{F}) + i\frac{\sqrt{2}}{4}\left(\bar{\psi}_a\bar{\sigma}^a\rho + \psi_a\sigma^a\bar{\rho}\right) - \frac{1}{2}(\bar{\psi}_a\bar{\sigma}_{ab}\bar{\psi}_b)U - \frac{1}{2}(\psi_a\sigma_{ab}\psi_b)\bar{U}\right) \tag{2.2.112}$$

where

$$U \equiv -\frac{1}{4}\bar{\nabla}^2 V|, \quad F \equiv \frac{1}{16}\nabla^2\bar{\nabla}^2 V|, \quad \text{and} \quad \rho_\alpha \equiv -\frac{1}{4\sqrt{2}}\nabla_\alpha\bar{\nabla}^2 V| \tag{2.2.113}$$

The fields F are actually not quite independent fields. In terms of the D-term of V, they are

$$F = D + \frac{1}{2}\nabla_c\nabla^c V + \frac{i}{2}\nabla_c V^c \tag{2.2.114}$$

$$\bar{F} = D + \frac{1}{2}\nabla_c\nabla^c V - \frac{i}{2}\nabla_c V^c \tag{2.2.115}$$

where[25]

$$D \equiv \frac{1}{16}\nabla^\alpha\bar{\nabla}^2\nabla_\alpha V = \frac{1}{16}\bar{\nabla}_{\dot{\alpha}}\nabla^2\bar{\nabla}^{\dot{\alpha}} V \tag{2.2.116}$$

$$V_c \equiv -\frac{1}{2}\bar{\sigma}_c^{\dot{\alpha}\alpha}[\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}]V \tag{2.2.117}$$

The imaginary part of the fields F and $\bar{F}$ is the divergence of the vector component of V. When evaluating a D-term integral, it is occasionally useful to use the fields D rather than F.

2.2.8 Kähler structure of conformal superspace of chiral superfields

It turns out that the conformal superspace of an arbitrary set of scalar chiral superfields possesses a simple Kähler structure due to its relation to the Kähler manifold $\mathbb{C}P^n$.

Suppose we are furnished with a set of chiral primary superfields Φ_I where $I = 0, 1, \ldots, n$. Our action consists in general of a D-term and an F-term which respectively take the forms

$$\mathcal{L}_D = -3\int d^4\theta\, E\, Z(\Phi_I, \bar{\Phi}_I), \quad \mathcal{L}_F = \int d^2\theta\, \mathcal{E}\, P(\Phi_I) \tag{2.2.118}$$

[25] As in normal superspace, one must be careful to note that ∇_c is covariant even with respect to supersymmetry. That is, $\nabla_c = e_c{}^m\left(\nabla_m - \frac{1}{2}\psi_m{}^{\underline{\alpha}}\nabla_{\underline{\alpha}}\right) = e_c{}^m\left(\partial_m - \frac{1}{2}\psi_m{}^{\underline{\alpha}}\nabla_{\underline{\alpha}} - h_m{}^{\underline{a}}X_{\underline{a}}\right)$ where $\underline{\alpha}$ denotes both spinor indices. In fact, were we to treat supersymmetry as a gauge theory in normal space with internal symmetry operators Q which happened to include translations in their algebra, we would denote $\frac{1}{2}\psi_m{}^{\underline{\alpha}}$ as the gauge field associated with the generator $Q_{\underline{\alpha}}$. Then the above formula is simply the covariant derivative. There is a further mild complication in conformal superspace: ∇_c will include the gauge action of $S_{\underline{\alpha}}$; therefore, a superconformal covariant derivative includes not only terms higher in the multiplet (due to Q), but also terms lower in the multiplet (due to S).

where Z is some real non-negative function of the fields with $\Delta(Z) = 2$ and P is some chiral function with $\Delta(P) = 3$ and $w(P) = 2$. (The assumption of non-negativity of Z is ultimately for stability of the underlying Einstein-Hilbert term. The factor of 3 is for convenience.) We can take the Φ_i as parametrizing some complex manifold. In order for Z to have a nonzero scaling weight, at least one of the Φ_i must have $\Delta_i \neq 0$. We will assume without loss of generality that this is Φ_0 (by renaming the fields if necessary) and that $\Delta_0 = 1$ (by redefining $\Phi_0 \to (\Phi_0)^{1/\Delta_0}$ if necessary).

It is then possible to trade the fields Φ_j with $j \geq 1$ for projective fields ξ_j which have zero weight. (The simplest way of doing this is by defining $\xi_j \equiv \Phi_j / \Phi_0^{\Delta_j}$.) Since the fields ξ_j have vanishing scaling weight, the fields Z and P in this parametrization are restricted in their form to

$$Z = \Phi_0 \bar{\Phi}_0 \exp\left(-K/3\right), \quad P = \Phi_0^3 W \tag{2.2.119}$$

where $K = K(\xi_j, \bar{\xi}_j)$ is some real function of the projective fields and $W = W(\xi_j)$ is some chiral function.[26] (The choice of this definition for real K is possible only if Z is assumed to be non-negative.) It is obvious that both Z and P, viewed as functions of the complex manifold spanned by the Φ_i, are independent of the projective representation chosen. A different representation is induced on the projective coordinates by the mapping

$$\Phi_0 \to \Phi_0 \exp(F/3), \quad K \to K + F + \bar{F}, \quad W \to e^{-F} W \tag{2.2.120}$$

where $F = F(\xi_j)$ is a holomorphic function of the projective parameters. (For example, trading Φ_0 for Φ_1 as the field to project with is accomplished by choosing $F = 3\log(\Phi_1/\Phi_0) = 3\log(\xi_1)$.) The above transformation law is simply a Kähler transformation, and the manifold under discussion is the complex projective space $\mathbb{C}P^n$, a simple example of a Kähler manifold.

The two actions then take the form

$$\mathcal{L}_D = -3 \int d^4\theta\, E\, \bar{\Phi}_0 e^{-K/3} \Phi_0, \quad \mathcal{L}_F = \int d^2\theta\, \mathcal{E}\, \Phi_0^3 W \tag{2.2.121}$$

where W is chiral and K is real. The factor of $e^{-K/3}$ is reminiscent of e^V for a theory with an internal $U(1)_K$ symmetry; this $U(1)_K$ is gauged not by an independent gauge multiplet but by the other chiral fields. We may make the $U(1)_K$ more manifest in the following manner. Define a new complex superfield Ψ_0 by

$$\Psi_0 \equiv e^{-K/6} \Phi_0, \quad \bar{\Psi}_0 \equiv e^{-K/6} \bar{\Phi}_0 \tag{2.2.122}$$

under which the actions become

$$-3 \int d^4\theta\, E\, \bar{\Psi}_0 \Psi_0, \quad \int d^2\theta\, \mathcal{E}\, \Psi_0^3 e^{K/2} W$$

[26]Since Φ_0 has scaling weight 1 and chiral weight 2/3 (their ratio is fixed at 3/2 for any primary chiral superfield) P has the correct scaling and chiral weights for an F-term.

The new field Ψ_0 and effective superpotential $e^{K/2}W$ are the only objects (besides K) which transform under Kähler transformations:

$$\Psi_0 \to \exp\left(+\frac{i}{3}\mathrm{Im}F\right)\,\Psi_0, \quad \bar{\Psi}_0 \to \exp\left(-\frac{i}{3}\mathrm{Im}F\right)\,\bar{\Psi}_0 \tag{2.2.123}$$

$$e^{K/2}W \to \exp\left(-i\mathrm{Im}F\right)\,e^{K/2}W, \quad e^{K/2}\bar{W} \to \exp\left(+i\mathrm{Im}F\right)\,e^{K/2}\bar{W} \tag{2.2.124}$$

We normalize the generator of Kähler transformations, $\mathbf{k}$, by requiring the above Kähler transformation to correspond to $\exp\left(-\mathrm{Im}F\,\mathbf{k}/2\right)$. In this way the Kähler weights of Ψ_0 and $e^{K/2}W$ are set to be $-2/3$ and 2, respectively:

$$\mathbf{k}\,\Psi_0 = -i\frac{2}{3}\Psi_0, \quad \mathbf{k}\,e^{K/2}W = +2ie^{K/2}W$$

(Note that $e^{K/2}W$ is chiral from the point of view of the Kähler covariant derivative, which carries a Kähler connection.) This normalization is purely a matter of convention; it is chosen so that $e^{K/2}W$ possesses the same Kähler and $U(1)$ weights.

The Kähler covariant derivative then takes the form

$$\nabla^{(K)} \equiv \nabla - \mathbb{A}\mathbf{k} \tag{2.2.125}$$

where $\mathbf{k}$ is the generator of the Kähler transformations. The Kähler connection $\mathbb{A}$ is defined in terms of the Kähler potential K:

$$\mathbb{A}_\alpha = +\frac{i}{4}\nabla_\alpha K, \quad \mathbb{A}_{\dot\alpha} = -\frac{i}{4}\nabla_{\dot\alpha}K$$

$$\mathbb{A}_{\alpha\dot\alpha} = \frac{i}{2}(\nabla_\alpha\mathbb{A}_{\dot\alpha} + \nabla_{\dot\alpha}\mathbb{A}_\alpha) = \frac{1}{8}[\nabla_\alpha, \nabla_{\dot\alpha}]K \tag{2.2.126}$$

(In these formulae, the function K is a primary scalar superfield and is therefore invariant under all the generators of the superconformal algebra.) The definition of $\mathbb{A}_{\alpha\dot\alpha}$ is conventional; it is chosen so that $\{\nabla^{(K)}_\alpha, \nabla^{(K)}_{\dot\alpha}\} = -2i\nabla^{(K)}_{\alpha\dot\alpha}$.

2.3 Degauging to Poincaré

Poincaré superspace lacks the explicit scaling and conformal symmetries enjoyed by conformal superspace. It may also, depending on the flavor of supergravity chosen, lack the $U(1)$ R-symmetry. Converting conformal supergravity to one of the flavors of Poincaré supergravity must then involve some measure of gauge-fixing. We will demonstrate how this is accomplished by first reducing the conformal multiplet to a theory with an explicit $U(1)$ symmetry and a nonlinearly realized conformal symmetry. To guide our path, we first review in broad strokes how it works without supersymmetry, the details of which can be found in [11].

2.3.1 Review: Conformal gravity and the Einstein-Hilbert Lagrangian

Conformal gravity consists of the following gauge connections:

$$W_m = e_m{}^a P_a + \frac{1}{2}\omega_m{}^{ba} M_{ab} + b_m D + f_m{}^a K_a \tag{2.3.1}$$

We will denote by $\breve{R}$ the curvatures of the conformal theory and by R the Poincaré curvatures. One usually takes the constraint of vanishing conformal torsion (which is equivalent to vanishing Poincaré torsion) to determine the spin connection $\omega_m{}^{ba}$ in terms of the vierbein and the scaling gauge field b_m. One also would like to express the special conformal gauge field $f_m{}^a$ in terms of other fields; this can be done by taking the conformal Ricci tensor to vanish, $\breve{R}_{mn}{}^{ba} e_b{}^n = 0$. Having done so, one finds

$$f_m{}^a = -\frac{1}{4}\left(\mathcal{R}_m{}^a - \frac{1}{6} e_m{}^a \mathcal{R}\right) \tag{2.3.2}$$

where $\mathcal{R}_m{}^a = R_{mn}{}^{ba} e_b{}^n$ is the Poincaré Ricci tensor and $\mathcal{R} = \mathcal{R}_m{}^a e_a{}^m$ the Poincaré Ricci scalar. One further, for simplicity, usually adopts the K-gauge choice $b_m = 0$. (This is possible since $\delta_K(\epsilon) b_m = -2 e_m{}^a \epsilon_a$ allows one to gauge b_m away.)

Having made these constraints and gauge choices, one then examines the simplest conformal action for a scalar field ϕ with $\Delta = 1$:

$$e^{-1}\mathcal{L} = \frac{1}{2}\phi \nabla^a \nabla_a \phi = -\frac{1}{2}\nabla^a \phi \nabla_a \phi - \frac{1}{2} T_{ba}{}^a \phi \nabla^b \phi - f_a{}^a \phi^2 \tag{2.3.3}$$

(We have integrated the covariant d'Alembertian by parts.) The torsion term vanishes by assumption. The term involving $\nabla_a \phi$ also vanishes if we fix the remaining D-gauge by gauging ϕ to the constant ϕ_0:

$$\nabla_a \phi_0 = e_a{}^m \partial_m \phi_0 = 0$$

(There is no scaling connection in the above expression since $b_m = 0$ has been chosen as our K-gauge.) This leaves for the Lagrangian

$$e^{-1}\mathcal{L} = \frac{1}{2}\phi_0 \nabla^a \nabla_a \phi_0 = -f_a{}^a \phi_0^2 = +\frac{1}{12}\mathcal{R}\phi_0^2 \tag{2.3.4}$$

This is almost the Einstein-Hilbert term $-\mathcal{R}/2$ (in units where the reduced Planck mass is unity). We need only start with the wrong sign for the kinetic term and then choose $\phi_0^2 = 6$.

If we had started with a complex gauge field ϕ, the Lagrangian would have been

$$e^{-1}\mathcal{L} = \phi^* \nabla^a \nabla_a \phi = -\nabla^a \phi^* \nabla_a \phi - 2 f_a{}^a \, |\phi|^2 \tag{2.3.5}$$

We may gauge $|\phi| = \phi_0$ but not the phase of ϕ, which we shall denote ω. This gives

$$e^{-1}\mathcal{L} = \phi^* \nabla^a \nabla_a \phi = -\phi_0^2 \partial^m \omega \partial_m \omega + \frac{1}{6} \mathcal{R} \phi_0^2 \tag{2.3.6}$$

Gauging $\phi_0^2 = 3$ and choosing to flip the sign of the Lagrangian gives back the Einstein-Hilbert term; unfortunately this also leaves an unstable kinetic term for ω. A model with an additional gauged $U(1)$ symmetry would be able to dispense with this phase. The superconformal algebra has such a symmetry, and we will find it is the supersymmetric version of this model with a complex ϕ which reproduces the minimal version of Poincaré supergravity.

2.3.2 $U(1)$ superspace

In conformal gravity, the scaling gauge field b_m was the only field that transformed under the special conformal symmetry; moreover, this symmetry was precisely enough to allow the choice $b_m = 0$. The latter property is also enjoyed in the superconformal case, even though not every other field is K-inert. It is here that we begin our gauge fixing procedure.

Recall that under the action of K_A with parameter ϵ^A, $\delta_K B_M = -2\epsilon^A E_{MA} (-)^a$. If we choose $\epsilon^A = \eta^M E_M{}^A (-)^a$, then we find $\delta_K B_M = -2\eta_M$ and we can freely choose the gauge $B = 0$. The generator D then drops out of the covariant derivative. We also have chosen a gauge for K_A and so we ought not to consider K_A a symmetry any longer. We denote this by the breakdown of the conformally covariant derivative ∇ to the Poincaré derivative $\mathcal{D}$.

Since K_A is no longer considered a symmetry, the fields $f_M{}^A$ are now auxiliary. In order to analyze the various properties of these objects, we must make use of the conformal curvatures. Most of these (torsion, Lorentz, and $U(1)$) can be viewed as the Poincaré versions with additional terms arising from the conformal gauge fields. The remaining ones (special conformal and scaling) have no Poincaré versions and so give pure constraints among the various fields $f_M{}^A$. After examining all the conformal constraints we will show that they reduce to the Poincaré constraints with precisely the auxiliary fields of $U(1)$ superspace.

For reference, we give here the relations among the various objects in the gauge where $B = 0$. For the conformal/Poincaré curvatures,

$$\breve{F}_{BA} = F_{BA} + 3i f_{BA} w(A) - 3i f_{AB} w(B) \tag{2.3.7}$$

$$\breve{T}_{CB}{}^A = T_{CB}{}^A + \frac{1}{2} f_{[C}{}^D C_D{}^A{}_{B]} \tag{2.3.8}$$

$$\breve{R}_{DC}{}^{\beta\alpha} = R_{DC}{}^{\beta\alpha} + 2\delta_{[D}{}^b f_{C]}{}^a (\epsilon\sigma_{ab})^{\beta\alpha} + 2\delta_{[D}{}^{\{\beta} f_{C]}{}^{\alpha\}} (-)^C \tag{2.3.9}$$

For the purely conformal curvatures,

$$\breve{H}_{BA} = 2f_{BA}(-)^a - 2f_{AB}(-)^b \tag{2.3.10}$$

$$\breve{R}(K)_{CB}{}^A = \mathcal{D}_{[C}f_{B]}{}^A + T_{CB}{}^D f_D{}^A + \frac{1}{2}f_{[CD}C_{B]}{}^{AD} - \frac{1}{2}f_{[C}{}^D f_{B]}{}^F C_{FD}{}^A \tag{2.3.11}$$

The covariant derivative appearing in $R(K)$ is Poincaré. $f_M{}^A$ is understood to transform as a Lorentz vector on the index A and to possess a scaling weight of $\lambda(A)$ and a $U(1)$ weight of $-w(A)$. (These latter two weights mean $f_M{}^A$ transforms oppositely under scalings and the $U(1)$ as $E_M{}^A$.) In the above and subsequent formulae, we will use the combination $f_B{}^A = E_B{}^M f_M{}^A$, which possesses scaling and $U(1)$ weights of $\lambda(A) + \lambda(B)$ and $-(w(A) + w(B))$, respectively.

Constraint analysis

We shall start with the scaling curvature:

$$\breve{H}_{BA} = (dB)_{BA} + 2f_{BA}(-)^a - 2f_{AB}(-)^b$$

Since B has been gauged away, the constraints on the H_{BA} give constraints on the fields $f_M{}^A$. These are:

$$\breve{H}_{\beta\alpha} = 0 \implies f_{\beta\alpha} = -f_{\alpha\beta} = -\epsilon_{\beta\alpha}\bar{R} \tag{2.3.12}$$

$$\breve{H}_{\dot\beta\dot\alpha} = 0 \implies f_{\dot\beta\dot\alpha} = -f_{\dot\alpha\dot\beta} = +\epsilon_{\dot\beta\dot\alpha}R \tag{2.3.13}$$

$$\breve{H}_{\beta\dot\alpha} = 0 \implies f_{\beta\dot\alpha} = -f_{\dot\alpha\beta} = -\frac{1}{2}G_{\beta\dot\alpha} \tag{2.3.14}$$

$$\breve{H}_{\underline{\beta}a} = 0 \implies f_{\underline{\beta}a} = -f_{a\underline{\beta}} \tag{2.3.15}$$

The above serve as definitions of the fields R and G_c. The complex conjugation properties of the above identities tell us $\bar{R} = R^\dagger$ and $G_c = (G_c)^\dagger$. The scaling weights of these objects are $\Delta(R) = \Delta(\bar{R}) = 2$ and $\Delta(G_c) = 2$; the $U(1)$ weights are $w(R) = -w(\bar{R}) = 2$ and $w(G_c) = 0$.

The next set of constraints to analyze are those of the $U(1)$ curvature. Recall

$$\breve{F}_{BA} = F_{BA} + 3if_{BA}w(A) - 3if_{AB}w(B)$$

which leads to

$$\breve{F}_{\beta\alpha} = 0 \implies F_{\beta\alpha} = 0 \tag{2.3.16}$$

$$\breve{F}_{\dot\beta\dot\alpha} = 0 \implies F_{\dot\beta\dot\alpha} = 0 \tag{2.3.17}$$

$$\breve{F}_{\beta\dot\alpha} = 0 \implies F_{\beta\dot\alpha} = 6if_{\beta\dot\alpha} = -3iG_{\beta\dot\alpha} \tag{2.3.18}$$

$$\breve{F}_{\beta a} = 0 \implies F_{\beta a} = -3if_{\beta a} \tag{2.3.19}$$

$$\breve{F}_{\dot\beta a} = 0 \implies F_{\dot\beta a} = +3if_{\dot\beta a} \tag{2.3.20}$$

Now consider the torsion. Noting that

$$\breve{T}_{CB}{}^A = T_{CB}{}^A + \frac{1}{2}F_{[C}{}^D C_D{}^A{}_{B]} \tag{2.3.21}$$

one can see the only torsions which differ between the conformal and Poincaré cases are those with A fermionic and either C or B (or both) bosonic. Thus the constraints on the conformal torsions pass unchanged for the fermion/fermion form indices:

$$\breve{T}_{\gamma\beta}{}^{A} = 0 \implies T_{\gamma\beta}{}^{A} = 0 \tag{2.3.22}$$

$$\breve{T}_{\dot\gamma\dot\beta}{}^{A} = 0 \implies T_{\dot\gamma\dot\beta}{}^{A} = 0 \tag{2.3.23}$$

$$\breve{T}_{\gamma\dot\beta}{}^{\underline{\alpha}} = 0 \implies T_{\gamma\dot\beta}{}^{\underline{\alpha}} = 0 \tag{2.3.24}$$

$$\breve{T}_{\gamma\dot\beta}{}^{a} = 2i\sigma^{a}_{\gamma\dot\beta} \implies T_{\gamma\dot\beta}{}^{\underline{\alpha}} = 2i\sigma^{a}_{\gamma\dot\beta} \tag{2.3.25}$$

For the fermion/boson form indices, it is only slightly more complicated:

$$\breve{T}_{\gamma b}{}^{\alpha} = 0 \implies T_{\gamma(\beta\dot\beta)\alpha} = +i\epsilon_{\beta\alpha}G_{\gamma\dot\beta} \tag{2.3.26}$$

$$\breve{T}_{\dot\gamma b}{}^{\alpha} = 0 \implies T_{\dot\gamma(\beta\dot\beta)\alpha} = -2i\epsilon_{\dot\gamma\dot\beta}\epsilon_{\beta\alpha}R \tag{2.3.27}$$

$$\breve{T}_{\gamma b}{}^{a} = 0 \implies T_{\gamma b}{}^{a} = 0 \tag{2.3.28}$$

$$\breve{T}_{\dot\gamma b}{}^{a} = 0 \implies T_{\dot\gamma b}{}^{a} = 0 \tag{2.3.29}$$

The only other torsion constraint was $\breve{T}_{cb}{}^{a} = 0$, which gives the same constraint on the Poincaré torsion

$$\breve{T}_{cb}{}^{a} = 0 \implies T_{cb}{}^{a} = 0. \tag{2.3.30}$$

The Lorentz curvature is quite simple to analyze:

$$\breve{R}_{\delta\gamma\beta\alpha} = 0 \implies R_{\delta\gamma\beta\alpha} = 4(\epsilon_{\delta\beta}\epsilon_{\gamma\alpha} + \epsilon_{\delta\alpha}\epsilon_{\gamma\beta})\bar{R} \tag{2.3.31}$$

$$\breve{R}_{\delta\gamma\dot\beta\dot\alpha} = 0 \implies R_{\delta\gamma\dot\beta\dot\alpha} = 0 \tag{2.3.32}$$

$$\breve{R}_{\dot\delta\dot\gamma\beta\alpha} = 0 \implies R_{\dot\delta\dot\gamma\beta\alpha} = 0 \tag{2.3.33}$$

$$\breve{R}_{\dot\delta\dot\gamma\dot\beta\dot\alpha} = 0 \implies R_{\dot\delta\dot\gamma\dot\beta\dot\alpha} = 4(\epsilon_{\dot\delta\dot\beta}\epsilon_{\dot\gamma\dot\alpha} + \epsilon_{\dot\delta\dot\alpha}\epsilon_{\dot\gamma\dot\beta})R \tag{2.3.34}$$

$$\breve{R}_{\delta\dot\gamma\beta\alpha} = 0 \implies R_{\delta\dot\gamma\beta\alpha} = -\epsilon_{\delta\beta}G_{\alpha\dot\gamma} - \epsilon_{\delta\alpha}G_{\beta\dot\gamma} \tag{2.3.35}$$

$$\breve{R}_{\delta\dot\gamma\dot\beta\dot\alpha} = 0 \implies R_{\delta\dot\gamma\dot\beta\dot\alpha} = -\epsilon_{\dot\gamma\dot\beta}G_{\delta\dot\alpha} - \epsilon_{\dot\gamma\dot\alpha}G_{\delta\dot\beta} \tag{2.3.36}$$

The remaining curvatures are:

$$\breve{R}(K)_{\gamma\beta\alpha} = 0 \implies \mathcal{D}_{\alpha}\bar{R} = 0 \tag{2.3.37}$$

$$\breve{R}(K)_{\gamma\beta\dot\alpha} = 0 \implies f_{\gamma(\beta\dot\alpha)} + f_{\beta(\gamma\dot\alpha)} = -\frac{i}{2}\mathcal{D}_{\{\gamma}G_{\beta\}\dot\alpha} \tag{2.3.38}$$

$$\breve{R}(K)_{\gamma\beta a} = 0 \implies \mathcal{D}_{\{\gamma}f_{\beta\}(\alpha\dot\alpha)} = +2iG_{\{\gamma\dot\alpha}\epsilon_{\beta\}\alpha}\bar{R} \tag{2.3.39}$$

$$\breve{R}(K)_{\gamma\dot\beta\alpha} = 0 \implies f_{\gamma(\alpha\dot\beta)} - 2f_{\alpha(\gamma\dot\beta)} = \frac{i}{2}\mathcal{D}_{\gamma}G_{\alpha\dot\beta} - i\epsilon_{\gamma\alpha}\mathcal{D}_{\dot\beta}\bar{R} \tag{2.3.40}$$

$$\breve{R}(K)_{\gamma\dot\beta a} = 0 \implies f_{(\beta\dot\beta)(\alpha\dot\alpha)} = \frac{i}{2}\mathcal{D}_{\{\beta}f_{\dot\beta\}(\alpha\dot\alpha)} + 2\epsilon_{\beta\alpha}\epsilon_{\dot\beta\dot\alpha}R\bar{R} + \frac{1}{2}G_{\alpha\dot\beta}G_{\beta\dot\alpha} \tag{2.3.41}$$

(We have used the spinor notation $f_{\gamma(\beta\dot{\alpha})} \equiv f_{\gamma c}\,\sigma^c_{\beta\dot{\alpha}}$ as well as $f_{(\beta\dot{\beta})(\alpha\dot{\alpha})} \equiv f_{ba}\,\sigma^a_{\alpha\dot{\alpha}}\,\sigma^b_{\beta\dot{\beta}}$.) The first condition indicates that $\bar{R}$ is an antichiral superfield; its complex conjugate tells that R is chiral. The second and fourth equations can be combined to yield

$$3if_{\beta(\alpha\dot{\alpha})} = +\frac{1}{2}\mathcal{D}_\beta G_{\alpha\dot{\alpha}} + \mathcal{D}_\alpha G_{\beta\dot{\alpha}} + \epsilon_{\beta\alpha}\mathcal{D}_{\dot{\alpha}}\bar{R} \tag{2.3.42}$$

as well as its conjugate

$$3if_{\dot{\beta}(\alpha\dot{\alpha})} = -\frac{1}{2}\mathcal{D}_{\dot{\beta}} G_{\alpha\dot{\alpha}} - \mathcal{D}_{\dot{\alpha}} G_{\alpha\dot{\beta}} - \epsilon_{\dot{\beta}\dot{\alpha}}\mathcal{D}_\alpha R. \tag{2.3.43}$$

This result can be substituted into the third equation, yielding

$$\mathcal{D}^2 G_c = 4i\mathcal{D}_c\bar{R}, \quad \bar{\mathcal{D}}^2 G_c = -4i\mathcal{D}_c R \tag{2.3.44}$$

The result given for $f_{\beta a}$ allows the determination of $F_{\beta a}$:

$$F_{\beta(\alpha\dot{\alpha})} = -3if_{\beta(\alpha\dot{\alpha})} = -\frac{3}{2}\mathcal{D}_\beta G_{\alpha\dot{\alpha}} - \epsilon_{\beta\alpha}\bar{X}_{\dot{\alpha}} \tag{2.3.45}$$

$$F_{\dot{\beta}(\alpha\dot{\alpha})} = +3if_{\dot{\beta}(\alpha\dot{\alpha})} = -\frac{3}{2}\mathcal{D}_{\dot{\beta}} G_{\alpha\dot{\alpha}} - \epsilon_{\dot{\beta}\dot{\alpha}}X_\alpha \tag{2.3.46}$$

where

$$X_\beta \equiv \mathcal{D}_\beta R - \mathcal{D}^{\dot{\beta}} G_{\beta\dot{\beta}}, \quad \bar{X}_{\dot{\beta}} \equiv \mathcal{D}_{\dot{\beta}}\bar{R} - \mathcal{D}^\beta G_{\beta\dot{\beta}} \tag{2.3.47}$$

just as in $U(1)$ superspace. Furthermore, (2.3.44) implies (after some algebra) the chirality of X_α:

$$\mathcal{D}_{\dot{\alpha}} X_\alpha = 0, \quad \mathcal{D}_\alpha X_{\dot{\alpha}} = 0 \tag{2.3.48}$$

Finally the fourth $R(K)$ constraint gives

$$\begin{aligned} f_{(\beta\dot{\beta})(\alpha\dot{\alpha})} =& \frac{i}{2}\mathcal{D}_{\{\beta} f_{\dot{\beta}\}(\alpha\dot{\alpha})} + 2\epsilon_{\beta\alpha}\epsilon_{\dot{\beta}\dot{\alpha}} R\bar{R} + \frac{1}{2}G_{\alpha\dot{\beta}}G_{\beta\dot{\alpha}} \\ =& -\frac{1}{12}[\mathcal{D}_\beta, \mathcal{D}_{\dot{\beta}}]G_{\alpha\dot{\alpha}} - \frac{1}{6}\mathcal{D}_\beta\mathcal{D}_{\dot{\alpha}}G_{\alpha\dot{\beta}} + \frac{1}{6}\mathcal{D}_{\dot{\beta}}\mathcal{D}_\alpha G_{\beta\dot{\alpha}} \\ & - \frac{1}{12}\epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{\beta\alpha}(\mathcal{D}^2 R + \bar{\mathcal{D}}^2\bar{R}) + 2\epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{\beta\alpha}R\bar{R} + \frac{1}{2}G_{\alpha\dot{\beta}}G_{\beta\dot{\alpha}} \end{aligned} \tag{2.3.49}$$

The special conformal gauge field ${f_B}^A$ is now entirely specified in terms of superfields R and G_c.

It is worth pausing a moment to take stock of our position. We have now checked that every constraint taken in conformal superspace reproduces (in the $B = 0$ gauge) a known result in $U(1)$ superspace; in particular, we have reproduced among our relations the constraint structure of $U(1)$ superspace. Since the $U(1)$ constraints uniquely specify $U(1)$ superspace, the gauge $B = 0$ of our constrained conformal superspace *must* correspond to the standard $U(1)$ superspace. All further checks are merely tests of consistency.

Some consistency checks

- Torsion
  The only torsion components we have not yet discussed are those which we did not constrain: $T_{cb}{}^{\underline{\alpha}}$. These also differ between conformal and Poincaré theories. Using

$$\breve{T}_{cb}{}^{\alpha} = T_{cb}{}^{\alpha} + i f_{[c\dot{\delta}}\bar{\sigma}_{b]}^{\dot{\delta}\alpha}$$

one finds

$$T_{(\gamma\dot{\gamma})(\beta\dot{\beta})\alpha} = +2\epsilon_{\dot{\gamma}\dot{\beta}}W_{\gamma\beta\alpha} + \epsilon_{\alpha\beta}\left(\mathcal{D}_{\dot{\beta}}G_{\gamma\dot{\gamma}} + \frac{2}{3}\epsilon_{\dot{\beta}\dot{\gamma}}X_{\gamma}\right) - \epsilon_{\alpha\gamma}\left(\mathcal{D}_{\dot{\gamma}}G_{\beta\dot{\beta}} + \frac{2}{3}\epsilon_{\dot{\gamma}\dot{\beta}}X_{\beta}\right) \tag{2.3.50}$$

$$T_{(\gamma\dot{\gamma})(\beta\dot{\beta})\dot{\alpha}} = -2\epsilon_{\gamma\beta}W_{\dot{\gamma}\dot{\beta}\dot{\alpha}} + \epsilon_{\dot{\alpha}\dot{\beta}}\left(\mathcal{D}_{\beta}G_{\gamma\dot{\gamma}} + \frac{2}{3}\epsilon_{\beta\gamma}\bar{X}_{\dot{\gamma}}\right) - \epsilon_{\dot{\alpha}\dot{\gamma}}\left(\mathcal{D}_{\gamma}G_{\beta\dot{\beta}} + \frac{2}{3}\epsilon_{\gamma\beta}\bar{X}_{\dot{\beta}}\right) \tag{2.3.51}$$

This is equivalent to the corresponding formulae in Eqs. (B-2.12)-(B-2.18) of [6]; therefore, the torsion of $U(1)$ supergravity is equivalent to the $B = 0$ gauge of conformal superspace.

- Lorentz curvatures
  The Lorentz curvatures in their canonically decomposed form are

$$\breve{R}_{DC}{}^{\beta\alpha} = R_{DC}{}^{\beta\alpha} + 2\delta_{[D}{}^{b}f_{C]}{}^{a}(\epsilon\sigma_{ab})^{\beta\alpha} + 2\delta_{[D}{}^{\{\beta}f_{C]}{}^{\alpha\}}(-)^{C} \tag{2.3.52}$$

The case of purely fermionic form indices has already been dealt with. Turn next to the fermion/boson case:

$$\breve{R}_{\delta(\gamma\dot{\gamma})\beta\alpha} = R_{\delta(\gamma\dot{\gamma})\beta\alpha} + \sum_{\beta\alpha}\left(-\epsilon_{\gamma\alpha}f_{\delta(\beta\dot{\gamma})} + 2\epsilon_{\delta\beta}f_{\alpha(\gamma\dot{\gamma})}\right) \tag{2.3.53}$$

Noting that $\breve{R}_{\delta(\gamma\dot{\gamma})\beta\alpha} = 0$ and inserting the explicit expression for $f_{\beta(\alpha\dot{\alpha})}$, one finds

$$R_{\delta(\gamma\dot{\gamma})\beta\alpha} = +i\sum_{\beta\alpha}\left(\frac{1}{2}\epsilon_{\delta\gamma}\mathcal{D}_{\beta}G_{\alpha\dot{\gamma}} + \frac{1}{2}\epsilon_{\delta\beta}\mathcal{D}_{\gamma}G_{\alpha\dot{\gamma}} - \epsilon_{\delta\beta}\epsilon_{\gamma\alpha}\bar{\mathcal{D}}_{\dot{\gamma}}\bar{R}\right) \tag{2.3.54}$$

as in $U(1)$ superspace [6]. The other Lorentz curvature term we need to calculate is

$$\begin{aligned}
R_{\dot{\delta}(\gamma\dot{\gamma})\beta\alpha} &= \breve{R}_{\dot{\delta}(\gamma\dot{\gamma})\beta\alpha} + \sum_{\beta\alpha} f_{\dot{\delta}(\beta\dot{\gamma})}\epsilon_{\gamma\alpha} \\
&= -4i\epsilon_{\dot{\delta}\dot{\gamma}}W_{\gamma\beta\alpha} + \sum_{\beta\alpha}\epsilon_{\gamma\alpha}\left(\frac{i}{6}\mathcal{D}_{\dot{\delta}}G_{\beta\dot{\gamma}} + \frac{i}{3}\mathcal{D}_{\dot{\gamma}}G_{\beta\dot{\delta}} + \frac{i}{3}\epsilon_{\dot{\delta}\dot{\gamma}}\mathcal{D}_{\beta}R\right) \\
&= -4i\epsilon_{\dot{\delta}\dot{\gamma}}W_{\gamma\beta\alpha} + i\sum_{\beta\alpha}\epsilon_{\gamma\alpha}\left(\frac{1}{2}\mathcal{D}_{\dot{\delta}}G_{\beta\dot{\gamma}} + \frac{1}{3}\epsilon_{\dot{\delta}\dot{\gamma}}X_{\beta}\right)
\end{aligned}$$

which is also as in $U(1)$ superspace [6].

At the dimension 2 level, results are a bit more interesting. Using (2.3.9), one finds

$$R_{(\delta\dot{\delta})(\gamma\dot{\gamma})\beta\alpha} = \breve{R}_{(\delta\dot{\delta})(\gamma\dot{\gamma})\beta\alpha} + \sum_{\beta\alpha}\Big(f_{(\delta\dot{\delta})(\beta\dot{\gamma})}\epsilon_{\gamma\alpha} - f_{(\gamma\dot{\gamma})(\beta\dot{\delta})}\epsilon_{\delta\alpha}\Big) \tag{2.3.55}$$

Recall that

$$\breve{R}_{(\delta\dot{\delta})(\gamma\dot{\gamma})\beta\alpha} = +2\epsilon_{\dot{\delta}\dot{\gamma}}\chi_{\delta\underset{\smile}{\gamma\beta}\alpha} - \frac{1}{4}\epsilon_{\dot{\delta}\dot{\gamma}}\sum_{(\delta\gamma)}\sum_{(\beta\alpha)}\epsilon_{\delta\beta}\mathcal{D}^{\phi}W_{\phi\gamma\alpha} \tag{2.3.56}$$

where

$$\chi_{\delta\underset{\smile}{\gamma\beta}\alpha} \equiv \frac{1}{4}(\mathcal{D}_\delta W_{\gamma\beta\alpha} + \mathcal{D}_\gamma W_{\delta\beta\alpha} + \mathcal{D}_\beta W_{\gamma\delta\alpha} + \mathcal{D}_\alpha W_{\gamma\beta\delta}).$$

We would like to show that (2.3.55) reduces to

$$R_{(\delta\dot{\delta})(\gamma\dot{\gamma})\beta\alpha} = +2\epsilon_{\dot{\delta}\dot{\gamma}}\chi_{\delta\gamma\beta\alpha} - 2\epsilon_{\delta\gamma}\epsilon_{\dot{\beta}\dot{\alpha}}\psi_{\dot{\delta}\dot{\gamma}\beta\alpha} \tag{2.3.57}$$

where

$$\chi_{\delta\gamma\beta\alpha} = \chi_{\delta\underset{\smile}{\gamma\beta}\alpha} + (\epsilon_{\delta\beta}\epsilon_{\gamma\alpha} + \epsilon_{\delta\alpha}\epsilon_{\gamma\beta})\chi \tag{2.3.58}$$

$$\psi_{\delta\gamma\dot{\beta}\dot{\alpha}} = \frac{1}{8}\sum_{\delta\gamma}\sum_{\dot{\beta}\dot{\alpha}}\left(G_{\delta\dot{\beta}}G_{\gamma\dot{\alpha}} - \frac{1}{2}[\mathcal{D}_\delta, \mathcal{D}_{\dot{\beta}}]G_{\gamma\dot{\alpha}}\right) \tag{2.3.59}$$

$$\chi = -\frac{1}{12}(\mathcal{D}^2 R + \bar{\mathcal{D}}^2\bar{R}) + \frac{1}{48}[\mathcal{D}^\alpha, \mathcal{D}^{\dot{\alpha}}]G_{\alpha\dot{\alpha}} - \frac{1}{8}G^{\alpha\dot{\alpha}}G_{\alpha\dot{\alpha}} + 2R\bar{R} \tag{2.3.60}$$

This is a straightforward (albeit tiresome) check. Some intermediate results help:

$$\sum_{\beta\alpha} f_{(\beta\dot{\phi})(\alpha}{}^{\dot{\phi})} = -\mathcal{D}^\phi W_{\phi\beta\alpha} \tag{2.3.61}$$

$$\sum_{\dot{\delta}\dot{\gamma}}\sum_{\beta\alpha} f_{(\beta\dot{\delta})(\alpha\dot{\gamma})} = 4\psi_{\dot{\delta}\dot{\gamma}\beta\alpha} \tag{2.3.62}$$

$$f_{(\phi\dot{\phi})}{}^{(\phi\dot{\phi})} = 4\chi \tag{2.3.63}$$

which allow the complete expression of the f terms from (2.3.55) in terms of the relevant Poincaré quantities. For example, (2.3.61) allows for the cancellation of the similar $\mathcal{D}^\phi W_{\phi\beta\alpha}$ terms in (2.3.56); the remaining terms involving ψ and χ combine with $\chi_{\delta\underset{\smile}{\gamma\beta}\alpha}$ to give the Poincaré Lorentz curvature.

- Scaling and $U(1)$ curvatures
  The only $U(1)$ curvature we haven't discussed yet is F_{ba}, but this is the same in both conformal and Poincaré theories. We have

$$F_{(\gamma\dot{\gamma})(\beta\dot{\beta})} = 2\epsilon_{\dot{\gamma}\dot{\beta}}\tilde{F}_{\gamma\beta} - 2\epsilon_{\gamma\beta}\tilde{F}_{\dot{\gamma}\dot{\beta}}$$

where $\mathcal{D}^\phi W_{\phi\beta\alpha} = \frac{4i}{3}\tilde{F}_{\beta\alpha}$. This is exactly as in [6] (aside from the extra factor of i).

For the scaling curvature,

$$H_{(\gamma\dot{\gamma})(\beta\dot{\beta})} = 2\epsilon_{\dot{\gamma}\dot{\beta}}\tilde{H}_{\gamma\beta} - 2\epsilon_{\gamma\beta}\tilde{H}_{\dot{\gamma}\dot{\beta}}$$

where $\mathcal{D}^{\phi}W_{\phi\beta\alpha} = +2\tilde{H}_{\beta\alpha}$. This is easily checked explicitly. Since $H_{(\gamma\dot{\gamma})(\beta\dot{\beta})} = 2f_{(\beta\dot{\beta})(\alpha\dot{\alpha})} - 2f_{(\alpha\dot{\alpha})(\beta\dot{\beta})}$, it follows that

$$\tilde{H}_{\beta\alpha} = -\frac{1}{2}\sum_{\beta\alpha} f_{(\beta\dot{\phi})(\alpha}{}^{\dot{\phi})} = +\frac{1}{2}\mathcal{D}^{\phi}W_{\phi\beta\alpha},$$

as needed.

- Special conformal curvatures
  These are by far the most complicated expressions remaining to check. The ones remaining for us to examine are $R(K)_{\gamma bA}$ and $R(K)_{cbA}$, which amount to five extra checks to perform. These give no extra insight or relations beyond those we already have, so we will avoid evaluating them explicitly here.

Conformal symmetry of $U(1)$ superspace

If $U(1)$ superspace is indeed a gauge-fixed version of a fully conformal superspace, then it must permit some form of scale transformation. This must be more than that of Howe and Tucker [20] since those authors were restricted to a chiral parameter in order to preserve the minimal torsion constraints. In fact, an unrestricted transformation *does* exist. Binetruy et al. [6] showed that the minimal matter coupling $e^{-K/3}$ could be absorbed into the frame of the vierbein provided the minimal superspace structure was enlarged to include a $U(1)$ superconnection. This can be understood as an unconstrained scale transformation.[27]

They postulated a transformation for the vierbein

$$E'_M{}^A = E_M{}^B X_B{}^A \tag{2.3.64}$$

with a parameter $X_B{}^A$ of the form

$$X_B{}^A = \begin{pmatrix} \delta_b{}^a X\bar{X} & X_b{}^\alpha & X_{b\,\dot{\alpha}} \\ 0 & \delta_\beta{}^\alpha X & 0 \\ 0 & 0 & \delta^{\dot{\beta}}{}_{\dot{\alpha}}\,\bar{X} \end{pmatrix} \tag{2.3.65}$$

where

$$X_b{}^\alpha \equiv \frac{i}{2}(\epsilon\sigma_b)^\alpha{}_{\dot{\alpha}}\bar{X}^{-1}\bar{\mathcal{D}}^{\dot{\alpha}}(X\bar{X}), \quad X_{b\dot{\alpha}} \equiv \frac{i}{2}(\epsilon\bar{\sigma}_b)_{\dot{\alpha}}{}^\alpha X^{-1}\mathcal{D}_\alpha(X\bar{X}) \tag{2.3.66}$$

is required to preserve torsion constraints. Otherwise, the factors X and $\bar{X}$ are totally unconstrained. By investigating the constraints of $U(1)$ superspace, they found the required

[27] Enlarging the structure group is not the only way to do this. Instead, one may choose fewer torsion constraints in Poincaré supergravity, which allow the superfield T_α in addition to R, $W_{\alpha\beta\gamma}$ and G_c. See for example [21] or [22].

transformation rules of the superfields

$$R' = (\bar{X})^{-2}\left(R - \frac{1}{8}(X\bar{X})^{-2}\bar{\mathcal{D}}^2(X\bar{X})^2\right) \tag{2.3.67}$$

$$G'_{\alpha\dot{\alpha}} = (X\bar{X})^{-1}\left(G_{\alpha\dot{\alpha}} - \frac{1}{2}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}]\log(X\bar{X}) + Y_\alpha \bar{Y}_{\dot{\alpha}}\right) \tag{2.3.68}$$

$$W'_{\alpha\beta\gamma} = (X\bar{X})^{-1}(\bar{X})^{-1}W_{\alpha\beta\gamma} \tag{2.3.69}$$

where $Y_A \equiv \mathcal{D}_A \log(X\bar{X})$. Although they restricted to the case where the $U(1)$ connection was initially zero, it is simple to extend to the case of a non-vanishing initial connection:

$$A'_M = A_M - i\frac{1}{2}Z_M - \frac{3i}{2}E_M{}^\alpha Y_\alpha + \frac{3i}{2}E_{M\dot{\alpha}}\bar{Y}^{\dot{\alpha}} + \frac{3}{4}E_M{}^{\alpha\dot{\alpha}}Y_\alpha \bar{Y}_{\dot{\alpha}} \tag{2.3.70}$$

where $Z_M \equiv \partial_M \log(X/\bar{X})$. Without loss of generality, the superfields X and $\bar{X}$ can be written

$$X = \exp(-\Lambda/2 + i\Omega), \quad \bar{X} = \exp(-\Lambda/2 - i\Omega), \tag{2.3.71}$$

for real superfields Ω and Λ. The infinitesimal transformation rules are

$$\delta E_m{}^a = -\Lambda E_m{}^a \tag{2.3.72}$$

$$\delta E_m{}^\alpha = \left(-\frac{1}{2}\Lambda + i\Omega\right)E_m{}^\alpha - \frac{i}{2}(\epsilon\sigma_m)^\alpha{}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\Lambda \tag{2.3.73}$$

$$\delta R = (\Lambda + 2i\Omega)R + \frac{1}{4}\bar{\mathcal{D}}^2\Lambda \tag{2.3.74}$$

$$\delta G_{\alpha\dot{\alpha}} = \Lambda G_{\alpha\dot{\alpha}} + \frac{1}{2}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}]\Lambda \tag{2.3.75}$$

$$\delta W_{\alpha\beta\gamma} = \left(\frac{3}{2}\Lambda + i\Omega\right)W_{\alpha\beta\gamma} \tag{2.3.76}$$

$$\delta A_M = \partial_M\Omega + \frac{3i}{2}E_M{}^\alpha\mathcal{D}_\alpha\Lambda - \frac{3i}{2}E_{M\dot{\alpha}}\mathcal{D}^{\dot{\alpha}}\Lambda \tag{2.3.77}$$

(Of the fields in the supervierbein, we have listed only those corresponding to the graviton and the gravitino. The other components of the supervierbein also transform, but they are unphysical so we'll ignore them for simplicity.) The above set of transformation rules is quite interesting. For the most part, they have the form of scale (Λ) and chiral (Ω) transformations, with A as the gauge field for the chiral transformations; however, for every term other than $E_m{}^a$, $W_{\alpha\beta\gamma}$, and $A_{\alpha\dot{\alpha}}$, there are modifications which depend on the derivative of the scale parameter Λ.

These extra modifications can be viewed as requirements forced by the torsion and curvature constraints of $U(1)$ superspace, but they can also be viewed as having a deeper geometrical origin. Our claim was that $U(1)$ superspace is a gauge-fixed version of conformal superspace. This is straightforward to see. The variation of the field B_M under D and K transformations is

$$\delta B_M = \partial_M\Lambda - 2E_M{}^A\epsilon_A(-)^a$$

where ϵ^A is the parameter for K transformations and Λ that of D. If we demand that $B_M = 0$ remain fixed, then every scale transformation must be accompanied by a K-transformation with $\epsilon^A = (-)^a \frac{1}{2} D_A \Lambda$. It is this corresponding K-transformation which generates the additional derivatives of Λ.

Consider first the vierbein. Under a K-transformation, $\delta_K E_M{}^A = \frac{1}{2} E_M{}^C \epsilon^B C^A{}_{BC}$, which corresponds to

$$\delta_K E_m{}^a = 0$$
$$\delta_K E_m{}^\alpha = -i\epsilon_{\dot\beta} \bar\sigma_m^{\dot\beta\alpha} = \frac{i}{2} \mathcal{D}_{\dot\beta} \Lambda \bar\sigma_m^{\dot\beta\dot\alpha}$$

for the graviton and gravitino, reproducing the additional terms exactly. Take the $U(1)$ connection next. Under a K-transformation, $\delta_K A_M = -3iw(A) E_M{}^A \epsilon_A$. Plugging in for ϵ we find

$$\delta_K A_M = \frac{3i}{2} w(A) E_M{}^A \mathcal{D}_A \Lambda$$

as expected.

The fields R and $G_{\alpha\dot\beta}$ are a bit more complicated. Recall that they are themselves related to the K-gauge fields by $f_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta} R$ and $f_{\alpha\dot\beta} = -G_{\alpha\dot\beta}/2$. The rule for the transformation of $f_{M\dot\beta}$ is $\delta_K f_{M\dot\beta} = \mathcal{D}_M \epsilon_{\dot\beta} - i E_M{}^\beta \epsilon_{\beta\dot\beta}$ which corresponds to

$$\delta_K G_{\alpha\dot\beta} = \mathcal{D}_\alpha \mathcal{D}_{\dot\beta} \Lambda + i \mathcal{D}_{\alpha\dot\beta} \Lambda = \frac{1}{2} [\mathcal{D}_\alpha, \mathcal{D}_{\dot\beta}] \Lambda.$$

For R, using $\delta_K f_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta} \bar{\mathcal{D}}^2 \Lambda/4$ gives

$$\delta_K R = \bar{\mathcal{D}}^2 \Lambda/4$$

These are precisely the extra terms enforced by the torsion constraints.

Finally, note that $W_{\alpha\beta\gamma}$ is a chiral primary superfield; thus it is inert under K and so has no extra terms.

2.3.3 Old minimal supergravity

We break the explicit scale invariance of the superspace theory by following as closely as possible the non-supersymmetric case. There a compensating matter field Φ was introduced with unit scaling weight. The D-gauge was then used to scale Φ to a constant, explicitly breaking the scale invariance and collapsing the kinetic Lagrangian into the Einstein-Hilbert term.

An analogous procedure can be undertaken in superspace. We must make use of a compensating superfield, and the simplest one is a chiral field. We denote it Φ_0, assume it to have a scaling weight of $\Delta(\Phi_0) = 1$ (and therefore a chiral weight of $\omega(\Phi_0) = 2/3$). The kinetic multiplet for Φ_0 is just the superspace D-term

$$-3 \int \breve{E} \; \bar\Phi_0 \Phi_0 \tag{2.3.78}$$

(Here and in the following we use $\breve{}$ over the measure to denote when we are in the conformal frame where the gauge is unfixed.) We would like to gauge $\Phi_0 = 1$. That converts the kinetic action into the supervolume, which reproduces the supersymmetrized Einstein-Hilbert term.

First let us note some things. After gauge-fixing Φ_0 to a constant, we are left with an issue of consistency, the equation of chirality for Φ_0:

$$0 = \nabla_{\dot{\alpha}}\Phi_0 = \left(D_{\dot{\alpha}} - B_{\dot{\alpha}} - \frac{2i}{3}A_{\dot{\alpha}}\right)\Phi_0 \tag{2.3.79}$$

We have explicitly used all of the K-gauge to fix $B = 0$. When Φ_0 is gauged to a constant, $A_{\dot{\alpha}} = 0$ follows. A corresponding analysis with $\bar{\Phi}_0$ leads us to conclude A_α vanishes as well. Using $F_{\alpha\dot{\alpha}} = (dA)_{\alpha\dot{\alpha}} = -3iG_{\alpha\dot{\alpha}}$, one can immediately deduce $A_{\alpha\dot{\alpha}} = -\frac{3}{2}G_{\alpha\dot{\alpha}}$. The $U(1)$ symmetry is broken; the bosonic component of A has become the auxiliary field G_c.

The superfield R also ultimately has an origin in the unfixed gauge. Recall that the F-term of the field Φ_0 was defined using the conformal superspace derivatives. We must convert these to Poincaré derivatives, giving, after gauge-fixing Φ_0 to a constant,

$$F = -\frac{1}{4}\nabla^2\Phi_0 = -\frac{1}{4}\left(\mathcal{D}^2 - 8\bar{R}\right)\Phi_0 = 2\bar{R}\Phi_0 \tag{2.3.80}$$

The anti-chiral superfield $\bar{R}$ is itself nothing more than the F-term of the chiral compensator, which is a well-known result. [28]

The chiral compensator and super-Weyl transformations

The normal approach to conformal supergravity [23] makes use of a chiral field Φ_0, introduced as a book-keeping device, whose bosonic component is used to fix the normalization of the Einstein-Hilbert term while the rest of the components are set to zero. This is completely analogous to the theory discussed above, except in those formulations the compensator is fixed at the component level. This theory also possesses a residual "super-Weyl" symmetry.

Begin with a model where the only field with scaling or chiral weight is the compensator Φ_0. It must therefore be employed to make the conformal $D-$ and F-terms invariant. These take the form

$$\mathcal{L}_D = \int d^4\theta\; \breve{E}\; \Phi_0\bar{\Phi}_0\; V, \quad \mathcal{L}_F = \int d^2\theta\; \breve{\mathcal{E}}\; \Phi_0^3\; W \tag{2.3.81}$$

Although V and W are generic real and chiral superfields of vanishing scaling and chiral weights, they possess a residual symmetry:

$$\Phi_0 \to \Phi_0 e^{2\Sigma}, \quad V \to e^{-2\Sigma-2\bar{\Sigma}}V, \quad W \to e^{-6\Sigma}W \tag{2.3.82}$$

where Σ is a chiral field of vanishing scaling and chiral weights. If we work in the gauge where $\Phi_0 = 1$, the above redefinition of the chiral compensator must be compensated by

[28] It can be shown (see for example [6]) that the theory above, with a remnant $U(1)$ field, can be converted to the theory of Wess and Bagger, where the $U(1)$ connection is entirely absent, by a simple modification of the torsion components.

an honest conformal transformation with a rescaling $\Lambda = -\Sigma - \bar{\Sigma}$ and a $U(1)$ rotation $\Omega = \frac{2i}{3}(\Sigma - \bar{\Sigma})$. This combined redefinition and conformal transformation is the super-Weyl transformation of Howe and Tucker [20] which preserves the form of the minimal Poincaré torsion constraints. V transforms as a real super-Weyl field with weight 2, W as a chiral super-Weyl field of weight 3, and the superdeterminant of the vierbein, E, as a real super-Weyl field with weight -2. (The transformation rules on the superfields R, G_c, the graviton, and gravitino can be derived from (2.3.72)-(2.3.76).)

The conformal transformations discussed in this article must be contrasted with these super-Weyl transformations. The former are unconstrained in superspace; the latter are *highly* constrained in superspace (the Σ must be chiral) but correspond to unconstrained superconformal transformations at the *component* level.

Integral relations between various formulations

We have several types of integrals (D- and F-terms, gauge fixed and unfixed) that describe the same physics, and we should demonstrate how they are related to each other.

The F-term action in conformal superspace can be rewritten

$$\int d^2\theta\ \breve{\mathcal{E}}\ \Phi_0^3\ W = -\frac{1}{4}\int d^2\theta\ \breve{\mathcal{E}}\ \bar{\nabla}^2\left(\frac{\bar{\Phi}_0\Phi_0^3 W}{\bar{F}}\right) \tag{2.3.83}$$

where $\bar{F} \equiv -\frac{1}{4}\bar{\nabla}^2\bar{\Phi}_0$. (The equivalency follows since the only non-chiral term in the parentheses is $\bar{\Phi}_0$, whose derivatives are cancelled by the denominator.) This is equivalent to a D-term:

$$-\frac{1}{4}\int d^2\theta\ \breve{\mathcal{E}}\ \bar{\nabla}^2\left(\frac{\bar{\Phi}_0\Phi_0^3 W}{\bar{F}}\right) = \int d^4\theta\ \breve{E}\ \frac{\bar{\Phi}_0\Phi_0^3 W}{\bar{F}} \tag{2.3.84}$$

Now we gauge Φ_0 to one. This leaves the inverse of the F-component of $\bar{\Phi}_0$, but this is nothing more than the chiral field R. Thus we find the following set of equalities:

$$\int d^2\theta\ \breve{\mathcal{E}}\ \Phi_0^3\ W = \int d^2\theta\ \mathcal{E}\ \ W = \frac{1}{2}\int d^4\theta\ \frac{E}{R}\ W \tag{2.3.85}$$

The term on the left is the expression for the chiral F-term in the presence of a conformal multiplet. The term in the middle is the chiral F-term after conformal gauge-fixing. The term on the right is the form of the chiral F-term used in [6]. Since the difference between the first and third terms is just a gauge-fixing, it should make no difference *when* we fix the gauge. Therefore if we were to evaluate the first term completely within conformal superspace and then gauge-fix, we would necessarily arrive at the same answer as the term on the right.[29]

To address the D-term, first note that in conformal superspace one can easily

[29] One may also note that the rather curious form of $1/2R$ as the term converting from an F to a D-term can be understood as a delta function. In particular, using the result of Apppendix 2.1.2, the chiral delta function is of a general form $\Delta_c = X/\mathcal{P}[X]$. For the case of $X = 1$, this gives $\Delta_c = -1/\frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)(1) = 1/2R$.

convert a D to an F-term:

$$\int d^4\theta\, \breve{E}\, \bar{\Phi}_0\Phi_0\, V = \int d^2\theta\, \breve{\mathcal{E}}\Phi_0 \left(\bar{F}V - \frac{1}{2}\nabla_{\dot{\alpha}}\bar{\Phi}_0\nabla^{\dot{\alpha}}V - \bar{\Phi}_0\frac{1}{4}\nabla^2 V\right)$$
$$= \int d^2\theta\, \breve{\mathcal{E}} \left(2R\bar{\Phi}_0\Phi_0 V - \frac{\Phi_0}{2}\nabla_{\dot{\alpha}}\bar{\Phi}_0\nabla^{\dot{\alpha}}V - \bar{\Phi}_0\Phi_0\frac{1}{4}\nabla^2 V\right) \qquad (2.3.86)$$

(Here V has zero scaling weight.) Now, let us gauge fix Φ_0 to unity and equate the first and final steps. We find

$$\int d^4\theta\, E\, V = -\frac{1}{4}\int d^2\theta\, \mathcal{E}(\bar{\mathcal{D}}^2 - 8R)V \qquad (2.3.87)$$

This tells us that the proper way in Poincaré superspace to convert a D to an F-term is through the use of the chiral Poincaré projector. This is actually quite intuitive if we use our other F to D-term conversion formula:

$$\int d^4\theta\, E\, V = -\frac{1}{4}\int d^2\theta\, \mathcal{E}\,(\bar{\mathcal{D}}^2 - 8R)V = -\frac{1}{8}\int d^4\theta\, \frac{E}{R}\,(\bar{\mathcal{D}}^2 - 8R)V \qquad (2.3.88)$$

The equality of the first and third terms follows by integration by parts in Poincaré superspace.[30]

2.3.4 Kähler supergravity

A general set of chiral fields coupled to conformal supergravity generically has D and F-terms

$$\mathcal{L}_D = -3\int d^4\theta\, \breve{E}\, \bar{\Phi}_0 e^{-K/3}\Phi_0, \quad \mathcal{L}_F = \int d^2\theta\, \breve{\mathcal{E}}\, \Phi_0^3\, W \qquad (2.3.89)$$

for chiral primary superfield Φ_0 with $\Delta = 1$ and $\omega = 2/3$. K is real and W is chiral, both with vanishing scale and chiral weights. The actions are invariant under a Kähler transformation

$$K \to K + F + \bar{F} \qquad (2.3.90)$$
$$\Phi_0 \to \Phi_0 e^{+F/3}, \;\; \bar{\Phi}_0 \to \bar{\Phi}_0 e^{+\bar{F}/3} \qquad (2.3.91)$$
$$W \to e^{-F}W, \;\; \bar{W} \to e^{-\bar{F}}W \qquad (2.3.92)$$

Here the superfields F and $\bar{F}$ are chiral/antichiral respectively. K is a real function of Kähler chiral matter fields ξ^i and $\bar{\xi}^i$ with vanishing conformal weight, and W is a function of only the chiral ones ξ^i. In the language of complex manifolds, W is a holomorphic function and K a real function. The transformation fields F and $\bar{F}$ are, respectively, holomorphic and antiholomorphic functions of the chiral and anti-chiral Kähler matter fields. Note that the Kähler transformation has no effect a priori on the supergravity sector.

[30]Note the significance of these steps. Within conformal superspace as in flat supersymmetry, one can convert from a D to an F-term, but the reverse is not an easily defined operation. Upon gauge-fixing to minimal Poincaré superspace, we gain the field R which allows us to do so.

There are two straightforward ways to accomplish a conformal gauge fixing. The first is to gauge Φ_0 to one. As the Kähler transformations alter Φ_0, a corresponding conformal transformation must compensate every Kähler transforation. This is the well-known Howe-Tucker transformation [20], which when combined with the given Kähler transformations of K and W render the D and F-terms invariant. Unfortunately, the D-term action then yields a non-canonical Einstein-Hilbert term. There are two traditional methods for dealing with this. One may rescale fields at the component level in a quite complicated fashion; this is the path taken in [7]. One may also leave Φ_0 unscaled until the very end of the calculation; this is the chiral compensator approach popularized by Kugo and Uehara [23].

A newer method is that of Binetruy et al. [6]. They demonstrated that enlarging to $U(1)$ superspace from a minimal Poincaré superspace allowed an arbitrary super-Weyl transformation to absorb the factor $e^{-K/3}$ into E. From our point of view, their approach has a very simple interpretation. Rather than scale $\Phi_0 = 1$, choose the gauge $\Phi_0 = e^{K/6}$. The equation of chirality then reads $0 = \bar{\mathcal{D}}_{\dot\alpha}\Phi_0 = D_{\dot\alpha}\Phi_0 - \frac{2i}{3}A_{\dot\alpha}\Phi_0$ which implies $A_{\dot\alpha} = -\frac{i}{4}D_{\dot\alpha}K$. The antichirality of Φ_0 similarly implies $A_\alpha = \frac{i}{4}D_\alpha K$. The Poincaré constraint $F_{\alpha\dot\alpha} = -3iG_{\alpha\dot\alpha}$ then gives $A_{\alpha\dot\alpha}$. The entire connection is given in terms of K and $G_{\alpha\dot\alpha}$:

$$\begin{aligned} A_\alpha = +\frac{i}{4}D_\alpha K, \quad A_{\dot\alpha} = -\frac{i}{4}D_{\dot\alpha}K \\ A_{\alpha\dot\alpha} = -\frac{3}{2}G_{\alpha\dot\alpha} + \frac{1}{8}[\mathcal{D}_\alpha, \mathcal{D}_{\dot\alpha}]K \end{aligned} \tag{2.3.93}$$

The imaginary part of the Kähler transformation now plays the role of the $U(1)$ R-symmetry; the real part is equivalent to a super-Weyl transformation and corresponds to a rescaling of Φ_0.

Alternatively, one may absorb the Kähler potential into the fields Φ_0 to define Kähler-covariant fields Ψ_0 as in (2.2.122). Then the gauge choice $\Psi_0 = 1$ gives

$$0 = \nabla^{(K)}_{\dot\alpha}\Psi_0 = -\frac{2i}{3}A_{\dot\alpha} + \frac{2i}{3}\mathbb{A}_{\dot\alpha} \implies A_{\dot\alpha} = \mathbb{A}_{\dot\alpha} \tag{2.3.94}$$

where $A_{\dot\alpha}$ is the $U(1)$ connection and $\mathbb{A}_{\dot\alpha} = -\frac{i}{4}D_{\dot\alpha}K$ is the Kähler connection. We arrive at the same result as (2.3.93). The gauge $\Psi_0 = 1$ breaks one combination of the $U(1)$ and Kähler symmetries, leaving the combination where the $U(1)$ and Kähler transform together. Therefore, an effective transformation on the matter fields (the Kähler transformation) has been extended to the entire frame of superspace (by merging it with the $U(1)$ R-symmetry).

2.3.5 New minimal supergravity

In both of the prior cases, we have used the simplest superfield, a chiral one with eight components, to gauge fix to Poincaré supergravity. Needless to say this is not the only choice. Another minimal choice would be a linear multiplet, which also contains eight components. We begin with a real linear superfield L, obeying

$$\nabla^2 L = \bar\nabla^2 L = 0 \tag{2.3.95}$$

From the superconformal algebra, we know that L must possess a scaling weight of $\Delta(L) = 2$ and, by reality, a vanishing $U(1)$ weight. This latter feature will leave the $U(1)$ gauge symmetry unaffected by the gauge-fixing procedure.

Before fixing the gauge $L = 1$, one important feature of the linear multiplet must be discussed. Due to the linearity constraint, $[\nabla^2, \bar{\nabla}^2]L = 0$, which implies $\nabla^{\dot{\alpha}\alpha}[\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}]L = 0$ – the divergence of the vector component of L vanishes. In global supersymmetry, this implies the vector component is the dual of a three-form, but in supergravity this statement is modified by terms involving the gravitino. The simplest way to derive this behavior is to consider the two-form potential B_{MN}, whose three-form field strength $H = dB$ obeys a Bianchi identity, $dH = 0$. Following [21] and [6], one chooses H to obey the constraints

$$0 = H_{\underline{\gamma\beta\alpha}} = H_{\gamma\beta a} = H_{\dot{\gamma}\dot{\beta}a} \tag{2.3.96}$$

Then as a consequence of the Bianchi identities, one can show that

$$H_{\gamma\dot{\beta}}{}^a = 2i\sigma^a_{\gamma\dot{\beta}}L \tag{2.3.97}$$

$$H_{\gamma ba} = 2(\sigma_{ba})_\gamma{}^\phi \nabla_\phi L, \quad H^{\dot{\gamma}}{}_{ba} = 2(\bar{\sigma}_{ba})^{\dot{\gamma}}{}_{\dot{\phi}} \bar{\nabla}^{\dot{\phi}} L \tag{2.3.98}$$

$$H_{cba} = \epsilon_{cba}{}^d \Delta_d L \tag{2.3.99}$$

where L is a linear superfield and where we have defined

$$\Delta_{\alpha\dot{\alpha}}L \equiv -\frac{1}{2}[\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}]L. \tag{2.3.100}$$

It follows that the dual of the three form is

$$\begin{aligned}\frac{1}{3!}\epsilon^{pnml}H_{nml} &= e_a{}^p\Delta^a L - \frac{i}{2}\epsilon^{pnm\ell}(\psi_n\sigma_m\bar{\psi}_\ell)L + i(\psi_n\sigma^{np})^\phi\nabla_\phi L - i(\bar{\psi}_n\bar{\sigma}^{np})_{\dot{\phi}}\nabla^{\dot{\phi}}L \\ &= \frac{1}{2}\epsilon^{pnm\ell}\partial_n B_{m\ell}\end{aligned} \tag{2.3.101}$$

Let us now gauge fix $L = 1$. The equations of linearity become, in Poincaré superspace,

$$(\mathcal{D}^2 - 8\bar{R})L = (\bar{\mathcal{D}}^2 - 8R)L = 0 \tag{2.3.102}$$

Since L is a constant, the only way this can be satisfied is if $R = \bar{R} = 0$. From the relations relating R to G_c, this necessarily implies $\mathcal{D}_c G^c = 0$. Noting that

$$-2\Delta_{\alpha\dot{\alpha}}L = [\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}]L = [\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}]L - 4G_{\alpha\dot{\alpha}}L \tag{2.3.103}$$

and that both $\mathcal{D}_\alpha L$ and $\mathcal{D}_{\dot{\alpha}}L$ vanish (we have gauged B to zero, and the $U(1)$ connection appears in neither expression since L has no chiral weight), we derive that

$$\Delta_a L = 2G_a \tag{2.3.104}$$

in the gauge where $L = 1$. It follows that

$$e_a{}^p G^a| = \frac{1}{4}\epsilon^{pnm\ell}\partial_n b_{m\ell} + \frac{i}{4}\epsilon^{pnm\ell}(\psi_n\sigma_m\bar{\psi}_\ell) \tag{2.3.105}$$

where $b_{m\ell}$ denotes the bosonic lowest component $B_{m\ell}|$.

The bosonic two-form $b_{m\ell}$ corresponds to three real bosonic components (after accounting for its gauge invariance). The superfield R vanishes so no component field M is generated. However, the $U(1)$ symmetry has not been broken, and so we will have in our off-shell spectrum the bosonic field A_m which is the gauge field of the chiral gauge symmetry, giving three bosonic components. As in the (old) minimal model, we have introduced six extra bosonic degrees of freedom to close the supergravity algebra off-shell.

The immediate candidate for the simplest D-term action is

$$\int d^4\theta\, \breve{E}\; L \tag{2.3.106}$$

However, using the D to F conversion in conformal superspace, this becomes

$$\int d^4\theta\, \breve{E}\; L = -\frac{1}{4}\int d^2\theta\; \mathcal{E}\; \bar{\nabla}^2 L = 0. \tag{2.3.107}$$

The linearity condition tells us that this simple integral vanishes. This immediately implies (after gauging L to one) that in the new minimal Poincaré superspace the integral of the supervolume vanishes: $\int d^4\theta\;E = 0$. This is a well-known property of the new minimal model, and nothing more meaningful than the fact that $R = 0$ [24].

To derive the form of the new minimal supergravity action, we will use a duality transform (as discussed in [25]) to transform a chiral compensator to a linear one. The properly normalized Einstein-Hilbert action is derivable from

$$-3\int d^4\theta\, \breve{E}\; \Phi_0 \bar{\Phi}_0 \tag{2.3.108}$$

after fixing the gauge $\Phi_0 = 1$. This action can in turn be derived from the first-order action

$$-3\int d^4\theta\, \breve{E}\; \big(X - L\log(X/\Phi_0\bar{\Phi}_0)\big) \tag{2.3.109}$$

where L is a linear superfield, X is an arbitrary real superfield of scaling weight 2, and Φ_0 is some chiral superfield of scaling weight 1. (Although the theory seems to depend on Φ_0, this is illusory since the components of Φ_0 are modified by the redefinition $\Phi_0 \to \Phi_0 e^{F/3}$ for chiral F under which the first-order action is invariant.) Since a linear superfield L can be written as $L = \nabla^\alpha \bar{\nabla}^2 \Omega_\alpha + \text{h.c.}$ for Ω_α with $\Delta = 1/2$ and $w = -1$, an action of the form LZ has an L equation of motion which sets $Z = S + \bar{S}$ for chiral field S of vanishing conformal weight. Thus varying L gives $X = \Phi_0\bar{\Phi}_0$, up to chiral and antichiral fields which can be absorbed into a redefinition of Φ_0. This in turn restores the original action. On the other hand, we may vary X to conclude $X = L$, which gives the action

$$-3\int d^4\theta\, \breve{E}\; \big(L - L\log(L/\Phi_0\bar{\Phi}_0)\big) = \int d^4\theta\, \breve{E}\; L V_R \tag{2.3.110}$$

where we have defined $V_R \equiv 3\log(L/\Phi_0\bar{\Phi}_0)$ and dropped the term linear in L since a linear superfield has vanishing D-term. V_R is a scalar field with vanishing conformal and chiral weights, although it does possess a symmetry $V_R \to V_R - F - \bar{F}$ with chiral field F.

The prior gauge choice $\Phi_0 = \bar{\Phi}_0 = 1$ which gave a properly normalized Einstein-Hilbert term here corresponds to $L = 1$. Choosing this gauge gives the simple action $\int d^4\theta\, E\; V_R$ where $V_R = -3\log(\Phi_0\bar{\Phi}_0)$. It is fairly simple to see now what sort of object V_R is. Since we have gauge-fixed the scale symmetry in addition to fixing $B = 0$, the structure group of our space differs only from Poincaré supergravity by the presence of a $U(1)$ R-symmetry. These fields Φ_0 and $\bar{\Phi}_0$ are covariantly chiral with respect to a derivative containing the corresponding $U(1)$ connection. Any $U(1)$ theory of covariantly chiral superfields Φ ($\mathcal{D}_{\dot\alpha}\Phi = 0$) may be related to a theory with Einstein chiral superfields ϕ ($D_{\dot\alpha}\phi = E_{\dot\alpha}{}^M\partial_M\phi$) and a $U(1)$ prepotential V,

$$\bar{\Phi}\Phi \to \bar{\phi}e^{-V/3}\phi$$

By choosing F appropriately, one may eliminate ϕ, arriving at $V_R = V$.

While this is the simplest explanation for what V_R is, it is somewhat unsatisfying since throughout this chapter we have avoided discussing prepotentials. To arrive at the some point by a rather more circuitous route, one begins by partially fixing the $U(1)$ gauge which at the moment is still a full superfield symmetry. We choose $\Phi_0 = \bar{\Phi}_0$; that is, set their relative phase to zero. The symmetry $\Phi_0 \to \Phi_0 e^{F/3}$ must be compensated with a chiral rotation with parameter $\Omega = \frac{i}{4}(F - \bar{F})$. We have now fixed the unconstrained $U(1)$ parameter to the imaginary part of a chiral parameter, and we see immediately that V_R transforms suspiciously as if it were the prepotential of such a chiral version of R-symmetry. If we evaluate the spinorial derivatives of V_R, we find this is exactly so. Begin with

$$\mathcal{D}_\alpha V_R = -3\frac{1}{\Phi_0}\mathcal{D}_\alpha\Phi_0 = -3\frac{D_\alpha\Phi_0}{\Phi_0} + 2iA_\alpha$$

and then note that since as functions $\Phi_0 = \bar{\Phi}_0$,

$$D_\alpha\Phi_0 = D_\alpha\bar{\Phi}_0 = -\frac{2i}{3}A_\alpha\bar{\Phi}_0 = -\frac{2i}{3}A_\alpha\Phi_0$$

where we have used the chirality condition of $\bar{\Phi}_0$. It follows that

$$\mathcal{D}_\alpha V_R = 4iA_\alpha, \quad \mathcal{D}_{\dot\alpha} V_R = -4iA_{\dot\alpha}. \tag{2.3.111}$$

V_R plays here the role of the $U(1)$ R-symmetry prepotential, and so the term $\int d^4\theta\, E\; V_R$ is nothing more than the $U(1)$ Fayet-Iliopoulos term.

From our point of view, evaluating the D-term of V_R is particularly easy. One considers V_R in its original form involving Φ. One can evaluate the D-term component Lagrangian directly. After integrating a number of terms by parts, one arrives at[31]

$$\begin{aligned} e^{-1}\int d^4\theta\, E\; V_R =& \frac{1}{2}\mathcal{D}^\alpha X_\alpha - \frac{i}{2}(\psi_m\sigma^m)_{\dot\alpha}X^{\dot\alpha} - \frac{i}{2}(\bar\psi_m\bar\sigma^m)^\alpha X_\alpha \\ &+ (A_p + \frac{3}{2}e_p{}^c G_c) \times \left(-4G^b e_b{}^p + i\epsilon^{pnm\ell}(\psi_n\sigma_m\bar\psi_\ell)\right) \end{aligned} \tag{2.3.112}$$

The combination $A_p + \frac{3}{2}e_p{}^c G_c$ can be thought of as the $U(1)$ connection if one chooses to define the bosonic derivative so that $F_{\alpha\dot\alpha}$ vanishes. (Recall that $F_{\alpha\dot\alpha} = -3iG_{\alpha\dot\alpha}$ in our convention.)

[31]The calculation of this total expression can be simplified by noting that any terms which shift under the chiral transformation of Φ, such as $\mathcal{D}_\alpha \log\Phi$, must have vanishing coefficients.

Using the definition for the lowest component of G_b, one finds

$$e^{-1}\int d^4\theta\, E\, V_R = \frac{1}{2}\mathcal{D}^\alpha X_\alpha - \frac{i}{2}(\psi_m\sigma^m)_{\dot\alpha}X^{\dot\alpha} - \frac{i}{2}(\bar\psi_m\bar\sigma^m)^\alpha X_\alpha$$
$$- \epsilon^{pnm\ell}(A_p + \frac{3}{2}e_p{}^c G_c)\partial_n b_{m\ell} \tag{2.3.113}$$

The Einstein-Hilbert action will be contained within $\mathcal{D}^\alpha X_\alpha$ and the Rarita-Schwinger action within the terms involving X_α and $X^{\dot\alpha}$. The remaining term, while involving the gauge potential A_p directly, is gauge invariant when integrated by parts.

Recall that $\mathcal{D}^\alpha X_\alpha$ is as defined in $U(1)$ superspace [6] and obeys the equality

$$\mathcal{D}^2 R + \bar{\mathcal{D}}^2\bar R = -\frac{2}{3}R_{ba}{}^{ba} - \frac{2}{3}\mathcal{D}^\alpha X_\alpha + 4G^a G_a + 32R\bar R$$

Since $R = 0$, this equation serves to define

$$\frac{1}{2}\mathcal{D}^\alpha X_\alpha \equiv -\frac{1}{2}R_{ba}{}^{ba} + 3G^a G_a$$
$$= -\frac{1}{2}\mathcal{R} - i(\psi_b\sigma_a T^{ab}) - i(\bar\psi_b\bar\sigma_a T^{ab}) - \frac{i}{2}\epsilon^{k\ell mn}G_k\psi_\ell\sigma_m\bar\psi_n + 3G^a G_a$$

Using $(\psi_m\sigma^m\bar X) = -2(\psi_m\sigma^m\bar\sigma^{cb}T_{cb})$ and its conjugate, it is straightforward to derive

$$e^{-1}\int d^4\theta\, E\, V_R = -\frac{1}{2}\mathcal{R} + \frac{1}{2}\epsilon^{k\ell mn}(\bar\psi_k\bar\sigma_\ell\mathcal{D}'_m\psi_n) - \frac{1}{2}\epsilon^{k\ell mn}(\psi_k\sigma_\ell\mathcal{D}'_m\bar\psi_n) - \epsilon^{pnm\ell}A'_p\partial_n b_{m\ell}$$

where

$$A'_m \equiv A_m + \frac{3}{4}e_m{}^a G_m$$

and $\mathcal{D}'$ is defined with A' as its $U(1)$ connection. (This latter definition corresponds to choosing $F_{\alpha\dot\alpha} = -\frac{3i}{2}G_{\alpha\dot\alpha}$ in defining the bosonic derivative.)

In pure new minimal supergravity, the equation of motion of the two-form enforces the A' connection to (at least locally) be pure gauge, $A' = d\lambda$. The A' equation of motion on the other hand gives

$$0 = \epsilon^{k\ell mn}\left(\partial_\ell b_{mn} + i\psi_\ell\sigma_m\bar\psi_n\right)$$

Aside from the coupling of the gravitino to the field A', the auxiliary sector is that of a simple abelian BF model with topological action $\int b\wedge dA'$ and no propagating degrees of freedom.

New minimal supergravity coupled to matter

For reference, we include here the simplest couplings of new minimal supergravity to chiral matter of vanishing $U(1)_R$ charge. (This last condition forbids a superpotential, so these models are quite simple ones.) One can derive these by performing a duality transformation from the Kähler multiplet, where Ψ_0 is covariantly chiral with respect to a $U(1)_K$. The modification consists simply of exchanging Φ_0 with Ψ_0 in the definition of

V_R, which essentially shifts V_R to $V_R + K$. The kinetic matter coupling of new minimal supergravity is then

$$\int d^4\theta \, E \; K \tag{2.3.114}$$

as in global supersymmetry. Evaluating this is straightforward. One simply replaces X_α and A_m associated with V_R with X^K_α and A^K_m. Provided we make the definitions

$$X^K_\alpha = -\frac{1}{8}\bar{\mathcal{D}}^2 \mathcal{D}_\alpha K, \quad X^K_{\dot\alpha} = -\frac{1}{8}\mathcal{D}^2 \bar{\mathcal{D}}_{\dot\alpha} K \tag{2.3.115}$$

and

$$A^K_m = -\frac{1}{2}e_m{}^a \Delta_a K + \frac{i}{4}\psi_m{}^\alpha \mathcal{D}_\alpha K - \frac{i}{4}\psi_{m\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha} K \tag{2.3.116}$$

one finds

$$\int d^4\theta \, E \; K = -\frac{1}{2}\mathcal{D}^\alpha X^K_\alpha + \frac{i}{2}(\psi\sigma\bar{X}^K) + \frac{i}{2}(\bar\psi\bar\sigma X^K) + \frac{1}{2}\epsilon^{k\ell mn} A^K_k \partial_\ell b_{mn} \tag{2.3.117}$$

Unlike in old minimal supergravity, the presence of a Kähler potential does not lead to extra additions to the Einstein-Hilbert term. This is known to be altered when the chiral matter carries a $U(1)_R$ charge (see for example [26]).

Chapter 3

The variational structure of conformal superspace

3.1 Introduction

The background approach to quantization has a long pedigree in superspace approaches to supergravity. The foundational work of Grisaru and Siegel [27] (extended later by Grisaru and Zanon [28] to include off-shell background fields) showed how to expand old minimal Poincaré supergravity in terms of fundamental quantum variations about a classical background, but they restricted their consideration to old minimal supergravity alone. This is difficult enough to do given the constrained supergeometry, and its quantization requires the introduction of not only Fadeev-Popov ghosts but also ghosts for ghosts, Nielsen-Kallosh ghosts [29], and "hidden" ghosts [30] which a casual application of the Fadeev-Popov procedure might miss. The on-shell one-loop gauge-fixed quantum Lagrangian was found which allows certain simple calculations as well as the construction of covariant Feynman rules to handle more general theories perturbatively. This story is by now textbook material [25].

However, the calculation of even one-loop effects involving not only supergravity but also chiral matter and gauge fields has to our knowledge never been comprehensively undertaken in superspace. Part of this is undoubtedly the difficulty in dealing with not only the constrained structure of supergravity in superspace but also the Brans-Dicke coupling of chiral matter to the superspace Einstein-Hilbert term. In a purely Poincaré approach, this last feature requires either a component space Weyl rescaling [7] or the introduction of $U(1)$ superspace and a superfield Weyl rescaling [6]. In this respect, it is almost more straightforward to work at the component level and then to extract superspace results from the component ones. A conformal approach at the superfield level seems a more feasible method, and that is the approach we take here.

In order to deal ultimately with the conformal coupling of the canonical Kähler potential in the Einstein-Hilbert term, we have shown how, in the previous chapter, to extend the structure group of Poincaré superspace to include the superconformal group. The new conformally covariant derivatives possess an algebra which is identical to that of gauge theories: their curvatures are expressed in terms of "gaugino" superfields $\mathcal{W}_\alpha$ and $\mathcal{W}^{\dot\alpha}$ valued in the superconformal group, which obey a generalized chirality condition (3.2.2) as well as a Bianchi identity (3.2.3). The selection of a number of curvature constraints eliminate most of the these superfields, and the ones which remain may all be described by the single chiral superfield $W_{\alpha\beta\gamma}$, the chiral spinor field strength of conformal supergravity. The conformally covariant derivatives and their curvatures all transform covariantly under the superconformal algebra, which simplifies the calculation of superscale transformations considerably.

Were it not for the constraints on the $\mathcal{W}_\alpha$, the structure of the theory would be quite easy to solve. In analogy with Yang-Mills, one would expect unconstrained prepotentials V^A, one for each member of the superconformal algebra. The constraints on the curvatures clearly must eliminate most of these prepotentials since a large volume of literature (see for example the textbooks [25, 22] as well as the original work [31]) shows that the fundamental quanta of old minimal Poincaré supergravity are the superfields $H^M = (H^m, H^\mu, H_{\dot\mu})$ and a chiral compensator σ, with a gauge invariance allowing one to algebraically eliminate H^μ and $H_{\dot\mu}$. We will not attempt to solve the constraints on the full prepotentials here. Rather, as our interest is in performing one-loop calculations in a classical background, we will focus on calculating the allowed deformations of the prepotentials which preserve the curvature constraints. The degrees of freedom must, of course, be the same in either approach.

This chapter is composed of three sections. In the first, we establish that the theory, like Yang-Mills, is defined in terms of prepotentials. We study arbitrary first order deformations of the prepotentials and solve for the form that leave the constraints invariant to first order. In the second section, we consider two physical actions, one involving the arbitrary coupling of chiral superfields to supergravity and the other involving the minimal linear compensator model with a Kähler potential. We construct their first order variations in terms of their fundamental quanta about a classical background and demonstrate that they possess a common structure. In the third section, we proceed to second order and present the second order variation of the action for both models, which is sufficient (after gauge fixing) for one-loop computations.

3.2 Prepotential formulation of conformal superspace

The algebra of the conformally covariant derivatives are

$$\begin{gathered}\{\nabla_\alpha, \nabla_\beta\} = 0, \quad \{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\} = 0 \\ \{\nabla_\alpha, \nabla_{\dot\alpha}\} = -2i\nabla_{\alpha\dot\alpha} \\ \{\nabla_\beta, \nabla_{\alpha\dot\alpha}\} = -2i\epsilon_{\beta\alpha}\mathcal{W}_{\dot\alpha}, \quad \{\nabla_{\dot\beta}, \nabla_{\alpha\dot\alpha}\} = -2i\epsilon_{\dot\beta\dot\alpha}\mathcal{W}_\alpha\end{gathered} \tag{3.2.1}$$

where $\mathcal{W}_\alpha$ are the "gaugino superfields" for the superconformal group. These superfields are covariantly chiral in the sense that

$$\{\nabla_{\dot\alpha}, \mathcal{W}_\alpha\} = 0, \quad \{\nabla_\alpha, \mathcal{W}_{\dot\alpha}\} = 0 \tag{3.2.2}$$

and obey the Bianchi identity

$$\{\nabla^\alpha, \mathcal{W}_\alpha\} = \{\nabla_{\dot\alpha}, \mathcal{W}^{\dot\alpha}\} \tag{3.2.3}$$

The structure is clearly reminiscent of Yang-Mills, except for two differences: the gauge generators X_B do not commute with the covariant derivatives ($[X_B, \nabla_A] \neq 0$), and most of the $\mathcal{W}_\alpha$ are constrained to vanish. The combination of the constraints and the Bianchi identities then allow one to solve for the non-vanishing $\mathcal{W}_\alpha$ all in terms of the single chiral superfield $W_{\alpha\beta\gamma}$.

The structure of the covariant derivatives of conformal supergravity allows a solution in terms of prepotentials that is identical in its structure to that of gauge theories. For example, (3.2.1) implies the existence of a chiral (+) and an antichiral (-) gauge where

$$\nabla^{\dot{\alpha}(+)} = \partial^{\dot{\alpha}} = T\nabla^{\dot{\alpha}}T^{-1}, \quad \nabla_\alpha^{(-)} = \partial_\alpha = \bar{T}\nabla_\alpha\bar{T}^{-1} \tag{3.2.4}$$

where T and $\bar{T}$ represent the superconformal gauge transformations taking us from an arbitrary gauge to the two special ones. Inverting these formulae gives

$$\nabla_\alpha = \bar{T}^{-1}\partial_\alpha\bar{T}, \quad \nabla_{\dot{\alpha}} = T^{-1}\partial_{\dot{\alpha}}T \tag{3.2.5}$$

which serve to encode the details of the connections in an arbitrary gauge in terms of a complex gauge prepotential T.

It is clear that the special gauges T and $\bar{T}$ are ill-defined up to transformations of the form

$$T \to CT, \quad \bar{T} \to \bar{C}\bar{T} \tag{3.2.6}$$

where C is chiral ($[\partial_{\dot{\alpha}}, C] = 0$) and $\bar{C}$ is antichiral ($[\partial_\alpha, \bar{C}] = 0$). In addition, they transform under gauge transformations as

$$T \to TG^{-1}, \quad \bar{T} \to \bar{T}G^{-1} \tag{3.2.7}$$

Putting these two transformations together gives a combined gauge/chiral transformation of the form

$$T \to CTG^{-1}, \quad \bar{T} \to \bar{C}\bar{T}G^{-1} \tag{3.2.8}$$

It is convenient to define the object $U \equiv \bar{T}T^{-1}$, which represents the gauge transformation from the chiral to the antichiral gauge. That is, $\nabla_A^{(-)} = U\nabla_A^{(+)}U^{-1}$. Applying this formula and its inverse in the cases where the covariant derivative is simple leads to

$$\begin{aligned} &\nabla_\alpha^{(-)} = \partial_\alpha, \quad \nabla_{\dot{\alpha}}^{(-)} = U\partial_{\dot{\alpha}}U^{-1} \\ &\nabla_\alpha^{(+)} = U^{-1}\partial_\alpha U, \quad \nabla_{\dot{\alpha}}^{(+)} = \partial_{\dot{\alpha}} \end{aligned} \tag{3.2.9}$$

U is invariant under the full gauge transformations but transforms under chiral gauge transformations as

$$U \to \bar{C}UC^{-1}. \tag{3.2.10}$$

A (covariantly) chiral superfield Φ is a superfield constrained to obey $\nabla_{\dot{\alpha}}\Phi = 0$. This is not in practice a difficult constraint to satisfy. In the chiral gauge, we define the conventionally chiral superfield ϕ by $\phi \equiv \Phi^{(+)}$. The chirality condition is then simply the analytic statement that $\phi = \phi(x,\theta)$ is independent of $\bar{\theta}$. In any other gauge, we have

$$\Phi = T^{-1}\Phi^{(+)} = T^{-1}\phi \tag{3.2.11}$$

While Φ transforms under a gauge transformation as $\Phi \to G\Phi$, the conventionally chiral ϕ transforms as $\phi \to C\phi$ where C is the chiral gauge transformation parameter. One may make an analogous statement about antichiral superfields:

$$\Phi^\dagger = \bar{T}^{-1}\Phi^{\dagger(-)} = \bar{T}^{-1}\bar{\phi} \tag{3.2.12}$$

Under a gauge transformation, Φ and $\Phi^\dagger$ transform covariantly while ϕ and $\bar{\phi}$ transform as

$$\phi \to C\phi, \quad \bar{\phi} \to \bar{C}\bar{\phi} \tag{3.2.13}$$

The canonical kinetic action for Φ can be rewritten in terms of the conventionally chiral superfields

$$\int E\,\Phi^\dagger\Phi = \int E\,(\bar{T}^{-1}\bar{\phi})(T^{-1}\phi) \tag{3.2.14}$$

Since the action is gauge-invariant (provided Φ is of scaling dimension $\Delta = 1$), we may perform a gauge transformation with parameter $G = \bar{T}$; this gives

$$\int E\,\bar{\phi}(\bar{T}T^{-1}\phi) = \int E\,\bar{\phi}(U\phi) \tag{3.2.15}$$

The equality of the above two statements is formally equivalent to $\bar{T}^T = \bar{T}^{-1}$ where transposition is understood as moving the gauge generator off one term and onto another. (An integration by parts, of course, has the same property.) One may use this to adopt a notation where the kinetic term is written as

$$\Phi^\dagger\Phi = \bar{\phi}U\phi \tag{3.2.16}$$

where U may be understood as acting either to the right (as U) or to the left (as U^{-1}).

It is often useful to work in a Hermitian gauge. We denote such a gauge by (0); it is easily found by interpolating between the chiral and antichiral gauges:

$$\nabla_\alpha^{(0)} = U^{-1/2}\partial_\alpha U^{1/2}, \quad \nabla_{\dot{\alpha}}^{(0)} = U^{1/2}\partial_{\dot{\alpha}}U^{-1/2} \tag{3.2.17}$$

We note that it is often useful to represent U in an exponential form. We choose to define the superfield V^A by

$$U = \exp(-2iV^A X_A) \tag{3.2.18}$$

Under this definition, V^A is Hermitian and represents the superconformal analogue of the gauge prepotential. If the constraints (3.2.1) were the sole constraints on the geometry, the prepotentials V^A would be unconstrained. However, certain of the gaugino superfields $\mathcal{W}_\alpha$ are constrained to vanish, which serves to implicitly define some of the V^A in terms of the others. Experience in Poincaré geometry tells us that V^a is the unconstrained object out of which the others are defined.[1] We will not be concerned, however, with presenting a full solution of the constraints. Rather, as we are more concerned with one loop calculations around a classical background, we will seek to construct the V^A associated with the quantum deformations themselves.

3.2.1 Quantum deformations of conformal geometry

The standard recipe for quantum calculations in supergravity involves splitting the geometry into a background geometry and quantum fluctuations about that background. Since the gauge connections are encoded in T and $\bar{T}$ (and thereby in U), splitting the former into a background and quantum contribution is accomplished by doing the same with the latter. The method of splitting we will adopt is

$$T \to T T_Q, \quad \bar{T} \to \bar{T}\bar{T}_Q \tag{3.2.19}$$

which corresponds to

$$\nabla_\alpha \to \bar{T}_Q^{-1} \nabla_\alpha \bar{T}_Q, \quad \nabla_{\dot{\alpha}} \to T_Q^{-1} \nabla_{\dot{\alpha}} T_Q. \tag{3.2.20}$$

The new covariant derivatives can then be constructed perturbatively out of the old ones. Similarly, chiral superfields transform under these variations as

$$\Phi \to T_Q^{-1}\Phi, \quad \bar{\Phi} \to \bar{T}_Q^{-1}\bar{\Phi} \tag{3.2.21}$$

The prepotentials transform under the combined chiral and supergauge transformations as

$$T T_Q \to C T T_Q G^{-1}, \quad \bar{T}\bar{T}_Q \to \bar{C}\bar{T}\bar{T}_Q G^{-1}. \tag{3.2.22}$$

Just as in the component case, the gauge transformation can be interpreted as either a background or a quantum transformation. As a background transformation, we take T and $\bar{T}$ to transform as

$$T \to C T G^{-1}, \quad \bar{T} \to \bar{C}\bar{T} G^{-1}. \tag{3.2.23}$$

and the quantum prepotentials to transform homogeneously

$$T_Q \to G T_Q G^{-1}, \quad \bar{T}_Q \to G \bar{T}_Q G^{-1} \tag{3.2.24}$$

In practice, we will leave the background gauge unspecified; indeed, we will attempt to maintain background gauge covariance at all times.

As a quantum transformation, T is invariant and T_Q transforms as

$$T_Q \to C_Q T_Q G_Q^{-1}, \quad \bar{T}_Q \to \bar{C}_Q \bar{T}_Q G_Q^{-1} \tag{3.2.25}$$

[1]In the literature, V^a is usually replaced with H^m and would be defined from the above with the coordinate derivative ∂_M replacing the covariant ∇_A in the set of generators.

where $C_Q \equiv T^{-1}CT$ and $\bar{C}_Q \equiv \bar{T}^{-1}\bar{C}\bar{T}$ are chiral and antichiral operators, obeying respectively

$$0 = [\nabla_\alpha, \bar{C}_Q] = [\nabla^{\dot{\alpha}}, C_Q] \tag{3.2.26}$$

Henceforth, we will be concerned only with quantum transformations. The supergauge freedom of G_Q can be eliminated by choosing to work in quantum chiral, antichiral, or Hermitian gauge.

We prefer to work in a gauge which maintains manifest Hermiticity at all times, though it may occasionally be more cumbersome, so we choose the last of these gauges. To go to quantum Hermitian gauge, one takes $G_Q = \bar{T}_Q^{-1}U_Q^{1/2} = T_Q^{-1}U_Q^{-1/2}$ where $U_Q \equiv \bar{T}_Q T_Q^{-1}$. This yields $T_Q = U_Q^{-1/2}$, $\bar{T}_Q = U_Q^{1/2}$, giving

$$\nabla'_\alpha = U_Q^{-1/2}\nabla_\alpha U_Q^{+1/2}, \quad \nabla'_{\dot{\alpha}} = U_Q^{+1/2}\nabla_{\dot{\alpha}} U_Q^{-1/2} \tag{3.2.27}$$

for the covariant derivatives and

$$\Phi' = U_Q^{1/2}\Phi, \quad \bar{\Phi}' = U_Q^{-1/2}\bar{\Phi} \tag{3.2.28}$$

for the chiral and antichiral superfields. The residual gauge transformation acts on U_Q as

$$U_Q \to \bar{C}_Q U_Q C_Q^{-1} \tag{3.2.29}$$

Quantum chiral gauge consists of making the quantum gauge choice $T_Q = 1$, $\bar{T}_Q = U_Q$. In this approach, $\nabla_{\dot{\alpha}}$ remains unchanged under quantum deformations of the geometry and so chiral superfields remain unchanged. Quantum antichiral gauge is analogously constructued.

It is worth noting the relation between U_Q and U' in background Hermitian gauge:

$$U' = \bar{T}'T'^{-1} = \bar{T}\bar{T}_Q T_Q^{-1}T^{-1} = \bar{T}U_Q T^{-1} = U^{1/2}U_Q U^{1/2} \tag{3.2.30}$$

3.2.2 Conformally covariant quantum prepotentials

The perturbative quantum prepotentials are the Hermitian superfields V defined by[2]

$$U_Q = \exp\left(-2iV^B\nabla_B - 2iV^{\underline{b}}X_{\underline{b}}\right) \tag{3.2.31}$$

To maintain general covariance, we have chosen to parametrize the quantum prepotentials in terms of the background covariant derivatives ∇_B rather than the coordinate derivatives. The factor of -2 is conventional and the i is so that the superfields V^B have the obvious Hermiticity conditions – for example,

$$(V^b)^\dagger = V^b, \quad (V^\alpha)^\dagger = V_{\dot{\alpha}} \tag{3.2.32}$$

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[2]Notational consistency would demand that the V's be subscripted with Q's to denote that they are quantum prepotentials. Since we will never again mention the background prepotentials, it is easier to suppress the Q for a less cluttered notation.

These superfields are chosen to transform under the action of the group generators as

$$X_{\underline{b}} V^A = -V^C f_{C\underline{b}}{}^A \tag{3.2.33}$$

where A and C run over all indices and $f_{CB}{}^A$ are the structure constants as defined previously. We thus have a conformally covariant set of quantum prepotentials.

For the generators D and A, the V's transform contravariantly as their index indicates. Thus V^a (like $e_m{}^a$) has scaling and $U(1)_R$ weights $(\Delta, w) = (-1, 0)$, V^α (like $\psi_m{}^\alpha$) has weights $(-1/2, +1)$, but $V(K)^\alpha$ has weights $(+1/2, -1)$. For the Lorentz generators, the V's transform as their indices indicate. Only special conformal transformation properties are not obvious. Recall the action of K on a group element $g = (\xi, \omega, \Lambda, w, \epsilon)$ is

$$\begin{gathered} K_B \xi^A = -\frac{1}{2} C_B{}^A{}_c \xi^c, \quad \frac{1}{2}(K_B \omega^{dc}) M_{cd} = -2\xi^C M_{CB} \\ K_B \Lambda = -2(-)^B \xi_B, \quad K_B w = -3i \xi_B w(B) \\ K_B \epsilon^A = -\lambda(A) \Lambda \delta_B{}^A + i w(A) w \delta_B{}^A + \omega_B{}^A + \epsilon^C C_{CB}{}^A - \frac{1}{2} \xi^C C_C{}^A{}_B (-)^{BA} \end{gathered} \tag{3.2.34}$$

Since the prepotentials are group elements, they must have these same transformation properties, and since the special conformal generator acts quite like an antiderivative, these formulae encapsulate a good deal of information. By inspection, one can easily see that only V^a is conformally primary. This isn't too great of a surprise, since the prepotential of conformal supergravity is a real superfield H^m, and V^a is its obvious quantum variation. All other objects should in principle be given as derivatives of V^a or otherwise be pure gauge artifacts. Using the special conformal transformation rules, it is possible to rewrite each of the prepotentials as derivatives of V^a plus some remaining conformally primary object.

As an example, note that V^α obeys

$$S^{\dot\beta} V^\alpha = -i V^{\dot\beta \alpha}, \quad S_\beta V^\alpha = K_b V^\alpha = 0$$

This is easily solved by

$$V^\alpha = -\frac{i}{8} \nabla_{\dot\phi} V^{\dot\phi \alpha} + \tilde{V}^\alpha$$

where $\tilde{V}^\alpha$ is some conformally primary superfield. The other conditions are not all nearly so easy to solve, but the answer is straightforward to check. One finds

$$V^\alpha = -\frac{i}{8}\nabla_{\dot\phi} V^{\alpha\dot\phi} + \tilde V^\alpha \tag{3.2.35}$$

$$V_{\dot\alpha} = -\frac{i}{8}\nabla^{\phi} V_{\phi\dot\alpha} + \tilde V^{\dot\alpha} \tag{3.2.36}$$

$$\begin{aligned} V(D) &= \frac{1}{2}\nabla_c V^c + \frac{1}{2}\nabla^\alpha V_\alpha + \frac{1}{2}\nabla_{\dot\alpha} V^{\dot\alpha} + \tilde V(D) \\ &= \frac{1}{4}\nabla_c V^c + \frac{1}{2}\nabla^\alpha \tilde V_\alpha + \frac{1}{2}\nabla_{\dot\alpha} \tilde V^{\dot\alpha} + \tilde V(D) \end{aligned} \tag{3.2.37}$$

$$\begin{aligned} V(A) &= -\frac{1}{4}\Delta_c V^c - \frac{3i}{4}(\nabla^\alpha V_\alpha - \nabla_{\dot\alpha} V^{\dot\alpha}) + \tilde V(A) \\ &= +\frac{1}{8}\Delta_c V^c - \frac{3i}{4}(\nabla^\alpha \tilde V_\alpha - \nabla_{\dot\alpha} \tilde V^{\dot\alpha}) + \tilde V(A) \end{aligned} \tag{3.2.38}$$

$$\begin{aligned} V(M)_{\beta\alpha} &= +\frac{1}{2}\nabla_{\{\beta} V_{\alpha\}} + \frac{i}{8}\nabla^{\dot\phi}\nabla_{\{\beta} V_{\alpha\}\dot\phi} + \tilde V(M)_{\beta\alpha} \\ &= +\frac{1}{2}\nabla_{\{\beta} \tilde V_{\alpha\}} + \frac{i}{16}\nabla^{\dot\phi}\nabla_{\{\beta} V_{\alpha\}\dot\phi} - \frac{1}{8}\nabla_{\{\beta\dot\phi} V^{\dot\phi}{}_{\alpha\}} + \tilde V(M)_{\beta\alpha} \end{aligned} \tag{3.2.39}$$

$$\begin{aligned} V(M)_{\dot\beta\dot\alpha} &= +\frac{1}{2}\nabla_{\{\dot\beta} V_{\dot\alpha\}} - \frac{i}{8}\nabla^{\phi}\nabla_{\{\dot\beta} V_{\dot\alpha\}\phi} + \tilde V(M)_{\dot\beta\dot\alpha} \\ &= +\frac{1}{2}\nabla_{\{\dot\beta} \tilde V_{\dot\alpha\}} - \frac{i}{16}\nabla^{\phi}\nabla_{\{\dot\beta} V_{\dot\alpha\}\phi} + \frac{1}{8}\nabla_{\{\dot\beta\phi} V^{\phi}{}_{\dot\alpha\}} + \tilde V(M)_{\dot\beta\dot\alpha} \end{aligned} \tag{3.2.40}$$

where we have defined

$$[\nabla_\alpha, \nabla_{\dot\alpha}] \equiv -2\Delta_{\alpha\dot\alpha} \tag{3.2.41}$$

These prepotential formulae will be the most useful to us. We have given them both in terms of the conformally non-primary V^α and the primary $\tilde V^\alpha$. The other tilded objects are similarly primary.

For completeness, we include also the special conformal prepotentials, which are a little messier and which we will not have a great deal of use for in what follows:

$$\begin{aligned} V(K)_\alpha &= +\frac{1}{8}\nabla^2 V_\alpha - \frac{1}{4}\nabla^{\dot\phi}\nabla_\alpha V_{\dot\phi} + \frac{i}{96}\nabla^2\nabla_{\dot\phi} V_\alpha{}^{\dot\phi} + \frac{1}{24}\nabla_\alpha\nabla_{\beta\dot\beta} V^{\beta\dot\beta} + \tilde V(K)_\alpha \\ &= +\frac{1}{8}\nabla^2 \tilde V_\alpha - \frac{1}{4}\nabla^{\dot\phi}\nabla_\alpha \tilde V_{\dot\phi} + \frac{i}{96}\nabla_{\dot\phi}\nabla^2 V_\alpha{}^{\dot\phi} + \frac{1}{48}\nabla_{\{\beta}\nabla_{\alpha\}\dot\beta} V^{\beta\dot\beta} + \tilde V(K)_\alpha \end{aligned} \tag{3.2.42}$$

$$\begin{aligned} V(K)_{\dot\alpha} &= +\frac{1}{8}\bar\nabla^2 V_{\dot\alpha} - \frac{1}{4}\nabla_{\phi}\nabla_{\dot\alpha} V^{\phi} + \frac{i}{96}\bar\nabla^2\nabla^{\phi} V_{\phi\dot\alpha} + \frac{1}{24}\nabla_{\dot\alpha}\nabla_{\beta\dot\beta} V^{\beta\dot\beta} + \tilde V(K)_{\dot\alpha} \\ &= +\frac{1}{8}\bar\nabla^2 \tilde V_{\dot\alpha} - \frac{1}{4}\nabla_{\phi}\nabla_{\dot\alpha} \tilde V^{\phi} + \frac{i}{96}\nabla^{\phi}\bar\nabla^2 V_{\phi\dot\alpha} + \frac{1}{48}\nabla_{\{\dot\beta}\nabla_{\dot\alpha\}\beta} V^{\beta\dot\beta} + \tilde V(K)_{\dot\alpha} \end{aligned} \tag{3.2.43}$$

The objects $\tilde V(K)_\alpha$ are not themselves fully primary, but are related to $\tilde V(D)$, $\tilde V(A)$, and $\tilde V(M)_{\beta\alpha}$ by the action of S_β. When these latter objects vanish, $\tilde V(K)_\alpha$ is itself primary.

In addition, when we consider Yang-Mills theories, we will also need the prepotential Σ^r, the Yang-Mills prepotential associated with the Yang-Mills generator X_r. It is naturally conformally primary.

We emphasize that the separation we have made above is entirely dictated by conformality concerns; the tilded objects we have introduced are *defined* by the above equations. We will very quickly find that they are constrained to be pure gauge artifacts. To demonstrate this, we require two new pieces of information: the form of the chiral gauge transformations and the first-order solution of the supergravity constraints.

3.2.3 Chiral gauge transformations

In choosing to work in quantum Hermitian gauge, we have exhausted the full supergroup gauge transformation, but the chiral transformations remain. Recall they are given by

$$U_Q \to \bar{C}_Q U_Q C_Q^{-1} \tag{3.2.44}$$

where C_Q obeys a chirality condition, $[\nabla^{\dot{\alpha}}, C_Q] = 0$. If we define $U_Q \equiv \exp(-2iV)$, $C_Q^{-1} \equiv \exp(-2i\Lambda)$, and $\bar{C}_Q \equiv \exp(-2i\bar{\Lambda})$, then the above transformation rule is equivalent (for infinitesimal Λ) to

$$\delta V = \Lambda + \bar{\Lambda} - i[V, \Lambda - \bar{\Lambda}] + \mathcal{O}(V^2) \tag{3.2.45}$$

Writing $\Lambda = \xi^A \nabla_A + \frac{1}{2}\omega^{ba} M_{ab} + \Lambda D + wA + \epsilon^B K_B$, we can solve for the conditions that these various parameters must obey:

$$\begin{gathered}
\xi_{\alpha\dot{\alpha}} = -\nabla_{\dot{\alpha}} L_\alpha, \quad \xi_\alpha = \frac{i}{8}\bar{\nabla}^2 L_\alpha, \quad \xi_{\dot{\alpha}} = \text{arbitrary} \\
\Lambda = -\frac{1}{2}\nabla^{\dot{\alpha}}\xi_{\dot{\alpha}} + \phi(D), \quad w = -\frac{3i}{4}\nabla^{\dot{\alpha}}\xi_{\dot{\alpha}} + \frac{i}{2}\phi(D) \\
\omega_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\nabla_{\{\dot{\alpha}}\xi_{\dot{\beta}\}}, \quad \omega_{\alpha\beta} = -2iL^\gamma W_{\gamma\alpha\beta} + \phi(M)_{\alpha\beta} \\
\epsilon_{\dot{\alpha}} = \frac{1}{8}\bar{\nabla}^2\xi_{\dot{\alpha}}, \quad \epsilon_\alpha = +\frac{i}{2}L^\phi \nabla^\gamma W_{\gamma\phi\alpha} + \psi(K)_\alpha, \quad \epsilon_{(\alpha\dot{\alpha})} = +iL^\phi \nabla_{\dot{\alpha}}{}^\gamma W_{\gamma\phi\alpha} + i\nabla_{\dot{\alpha}}\psi(K)_\alpha
\end{gathered} \tag{3.2.46}$$

In the above formulae $\{\dot{\alpha}\dot{\beta}\}$ denotes the (unnormalized) symmetric sum $\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\alpha}$. The superfields $\phi(D)$ and $\phi(M)_{\alpha\beta}$ are chiral, $\psi(K)_\alpha$ is complex linear, $\xi_{\dot{\alpha}}$ is arbitrary, but none of these four is primary. L_α is both primary and arbitrary. As with the prepotentials, we may rewrite the non-primary operators as derivatives of primary ones plus some new primary object. Doing so gives

$$\begin{gathered}
\xi_{\alpha\dot{\alpha}} = -\nabla_{\dot{\alpha}} L_\alpha, \quad \xi_\alpha = \frac{i}{8}\bar{\nabla}^2 L_\alpha, \quad \xi_{\dot{\alpha}} = -\frac{i}{8}\nabla_\beta \nabla_{\dot{\alpha}} L^\beta + \tilde{\xi}_{\dot{\alpha}} \\
\Lambda = -\frac{1}{2}\nabla^{\dot{\alpha}}\xi_{\dot{\alpha}} - \frac{i}{16}\bar{\nabla}^2\nabla_\beta L^\beta + \tilde{\phi}(D), \quad w = -\frac{3i}{4}\nabla^{\dot{\alpha}}\xi_{\dot{\alpha}} + \frac{1}{32}\bar{\nabla}^2\nabla_\beta L^\beta + \frac{i}{2}\tilde{\phi}(D) \\
\omega_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\nabla_{\{\dot{\alpha}}\xi_{\dot{\beta}\}}, \quad \omega_{\alpha\beta} = -2iL^\gamma W_{\gamma\alpha\beta} - \frac{i}{16}\bar{\nabla}^2\nabla_{\{\alpha}L_{\beta\}} + \tilde{\phi}(M)_{\alpha\beta}
\end{gathered} \tag{3.2.47}$$

We have not included the terms corresponding to $\epsilon(K)$ since they are fairly messy and we don't actually have much use for these specific formulae in what follows.

The useful part of the above formulae is to note the correspondence between the tilded gauge objects and the tilded prepotentials. For example, if we could show that $\tilde{V}(K)_\alpha$

were constrained to be complex linear, then it is a pure gauge artifact, cancelling against $\psi(K)_\alpha$. Similarly, if we could show that $\tilde{V}(M)_{\alpha\beta}$ were chiral, we could cancel it against $\tilde{\phi}(M)_{\alpha\beta}$. Clearly $\tilde{V}_{\dot{\alpha}}$ already corresponds to $\tilde{\xi}_{\dot{\alpha}}$. To eliminate $\tilde{V}(D)$ and $\tilde{V}(A)$, we would need to show that they can be related to the appropriate sum (or difference) of a chiral and an antichiral field – in this case, $\tilde{\phi}(D)$ and its conjugate. Provided these constraints can be enforced, the theory becomes one entirely of V^a.

We should check that the number of degrees of freedom work out. V^a itself consists of 32 bosonic and 32 fermionic degrees of freedom. The gauge degree of freedom L_α, however, *also* seems to have 32+32 components. The solution to this puzzle is that L_α has weight $(-3/2, -1)$ which has precisely the ratio necessary to accomodate a primary chiral superfield. We will find in physical models, in fact, that L_α itself possesses a gauge symmetry of $L_\alpha \to L_\alpha + \phi_\alpha$, where ϕ_α has 8+8 components. Since it is a second order gauge degree of freedom (i.e. a gauge degree of freedom for a gauge degree of freedom), these components contribute positively to the counting. Put more simply,

$$32 + 32 - (32 + 32 - (8 + 8)) = 8 + 8$$

which is the right number for conformal supergravity. It is interesting that the physical degrees of freedom of conformal supergravity coincide with those of a chiral spinor.

For completeness, we also include the Yang-Mills variation:

$$\Lambda^r = iL^\beta W_\beta{}^r + \tilde{\Lambda}^r \tag{3.2.48}$$

where $\tilde{\Lambda}^r$ is chiral. Note that because we have included Σ^r with the supergravity prepotentials, its chiral gauge variation includes a term coming from supergravity, in addition to the usual chiral superfield.

3.2.4 First-order constraint solution

We next turn to the task of solving the supergravity constraints to first order. Because conformal supergravity is characterized by conventional constraints as in super Yang-Mills, the curvatures are entirely described by "gaugino" superfields $\mathcal{W}_\alpha$ which are given by the commutators

$$[\nabla_\alpha, \nabla_{\beta\dot{\beta}}] = -2i\epsilon_{\alpha\beta}\mathcal{W}_{\dot{\beta}}, \quad [\nabla_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] = -2i\epsilon_{\dot{\alpha}\dot{\beta}}\mathcal{W}_\beta \tag{3.2.49}$$

These are superfields which obey a chirality condition, $\{\nabla_{\dot{\alpha}}, \mathcal{W}_\beta\} = 0$. The constraints of conformal supergravity involve imposing $\mathcal{W}_\alpha(P)^B = \mathcal{W}_\alpha(D) = \mathcal{W}_\alpha(A) = 0$. From these it follows that $\mathcal{W}_\alpha(M)^{\dot{\beta}\dot{\gamma}} = 0$ and $\mathcal{W}_\alpha(K)_{\dot{\alpha}} = 0$ and that all the remaining curvatures can be expressed in terms of the single chiral superfield $W_{\alpha\beta\gamma}$.

The chiral superfield $\mathcal{W}_\alpha$ can be defined by

$$8\mathcal{W}_\alpha = [\nabla_{\dot{\alpha}}, \{\nabla^{\dot{\alpha}}, \nabla_\alpha\}] = +2i[\nabla^{\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}}] \tag{3.2.50}$$

Varying this object to first order involves varying each of the covariant derivatives on the right side. The easiest way to handle this is to adopt a chiral quantum gauge where we force all of the quantum variation onto ∇_α and leave $\nabla_{\dot{\alpha}}$ unchanged. If the gaugino superfield vanishes in this gauge, it vanishes in any gauge, including quantum Hermitian gauge. (This is equivalent to doing the variation in Hermitian gauge and then performing a quantum prepotential-dependent gauge transformation.)

Thus,

$$\delta_c \nabla_\alpha = [2iV, \nabla_\alpha], \quad \delta_c \nabla_{\dot\alpha} = 0 \tag{3.2.51}$$

where the subscript c denotes that the quantum gauge is chiral.

Note first that the Hermitian quantum variation of ∇_α is

$$\delta \nabla_\alpha = [iV, \nabla_\alpha] \equiv -H_\alpha{}^B X_B = -H_\alpha{}^B \nabla_B - \Omega_\alpha(M) - \Lambda_\alpha D - \omega_\alpha A - J_\alpha{}^B K_B \tag{3.2.52}$$

where

$$\begin{aligned}
H_\alpha{}^\beta &= +i\nabla_\alpha V^\beta - iV(M)_\alpha{}^\beta - \frac{i}{2} V(D)\delta_\beta{}^\alpha - V(A)\delta_\beta{}^\alpha && (3.2.53)\\
H_{\alpha\dot\beta} &= +i\nabla_\alpha V_{\dot\beta} && (3.2.54)\\
H_{\alpha(\beta\dot\beta)} &= +i\nabla_\alpha V_{(\beta\dot\beta)} + 4\epsilon_{\alpha\beta} V_{\dot\beta} && (3.2.55)\\
\Omega_\alpha(M) &= +iV^b R_{b\alpha}(M) + i\nabla_\alpha V(M) + 2iV(K)^\beta M_{\beta\alpha} && (3.2.56)\\
\Lambda_\alpha &= +i\nabla_\alpha V(D) + 2iV(K)_\alpha && (3.2.57)\\
\omega_\alpha &= +i\nabla_\alpha V(A) + 3V(K)_\alpha && (3.2.58)\\
J_\alpha{}^\beta &= +i\nabla_\alpha V(K)^\beta && (3.2.59)\\
J_{\alpha\dot\beta} &= +i\nabla_\alpha V(K)_{\dot\beta} + iV^c R_{c\alpha}(K)_{\dot\beta} + V(K)_{\alpha\dot\beta} && (3.2.60)\\
J_\alpha{}^b &= +i\nabla_\alpha V(K)^b + iV^c R_{c\alpha}(K)^b && (3.2.61)
\end{aligned}$$

In the chiral gauge we are using, the variation of ∇_α is simply twice this:

$$\delta_c \nabla_\alpha = -2H_\alpha{}^B \nabla_B - 2H_\alpha{}^{\underline{b}} X_{\underline{b}} \tag{3.2.62}$$

The variation of the bosonic derivative is rather easy to calculate in chiral gauge. One finds

$$\delta_c \nabla_{\alpha\dot\alpha} = -i\nabla_{\dot\alpha} H_\alpha{}^B X_B - i\nabla_{\dot\alpha} H_\alpha{}^{\underline{b}} X_{\underline{b}} - 2H_\alpha{}^\beta \nabla_{\beta\dot\alpha} + H_{\alpha(\beta\dot\alpha)} \mathcal{W}^\beta + iH_\alpha{}^{\underline{b}} f_{\underline{b}\dot\alpha}{}^D X_D \tag{3.2.63}$$

$\delta\mathcal{W}$ is then given by

$$\begin{aligned}
4\,\delta\mathcal{W}_\alpha = &- \bar\nabla^2 H_\alpha{}^B X_B + 4i\nabla_{\dot\alpha} H_\alpha{}^\beta \nabla_\beta{}^{\dot\alpha} \\
&+ \left(2i\nabla^{\dot\beta} H_{\alpha(\beta\dot\beta)} + 8H_{\alpha\beta}\right) \mathcal{W}^\beta \\
&+ \left(2\nabla_{\dot\alpha} H_\alpha{}^{\underline{b}} - H_\alpha{}^{\underline{c}} f_{\underline{c}\dot\alpha}{}^{\underline{b}}\right) f_{\underline{b}}{}^{\dot\alpha D} X_D
\end{aligned} \tag{3.2.64}$$

We begin the analysis by considering the constraints imposed on the prepotentials by $\mathcal{W}_\alpha(P) = 0$. These amount to two conditions, which we write as

$$\bar\nabla^2 H_{\alpha(\beta\dot\beta)} = 8i\nabla_{\dot\beta} H_{\alpha\beta} \tag{3.2.65}$$

$$8J_{\alpha\dot\alpha} = -\bar\nabla^2 H_{\alpha\dot\alpha} - \nabla_{\dot\alpha}\Lambda_\alpha - 2i\nabla_{\dot\alpha}\omega_\alpha + 2\nabla^{\dot\beta}\Omega_{\alpha\dot\beta\dot\alpha} \tag{3.2.66}$$

The second of these amounts to a definition of $V(K)_{\alpha\dot{\alpha}}$, on which $J_{\alpha\dot{\alpha}}$ linearly depends. (There is a third condition that we haven't listed which is a trivial consequence of the first.)

Choosing $\mathcal{W}_\alpha(D)$ and $\mathcal{W}_\alpha(A)$ to vanish amount to the condition

$$\bar{\nabla}^2\Lambda_\alpha = -\frac{2i}{3}\bar{\nabla}^2\omega_\alpha \tag{3.2.67}$$

All other conditions on the $\mathcal{W}_\alpha$'s follow from these three.

The third condition, (3.2.67), is the easiest to immediately evaluate. Using the above definitions for Λ_α and ω_α leads to

$$0 = \bar{\nabla}^2\left(i\nabla_\alpha V(D) - \frac{2}{3}\nabla_\alpha V(A) + 4iV(K)_\alpha\right)$$

Inserting the definitions of the V's in terms of the $\tilde{V}'s$, we discover a nice surprise. The above condition reduces to

$$0 = \bar{\nabla}^2\left(i\nabla_\alpha \tilde{V}(D) - \frac{2}{3}\nabla_\alpha \tilde{V}(A) + 4i\tilde{V}(K)_\alpha\right) \tag{3.2.68}$$

The first condition, (3.2.65), is the next easiest to check. Again using the $\tilde{V}$'s we can conclude that

$$0 = \nabla_{\dot{\beta}}\tilde{V}(M)_{\beta\alpha} \tag{3.2.69}$$

$$0 = \nabla_{\dot{\beta}}\left(\frac{i}{2}\tilde{V}(D) + \tilde{V}(A)\right) \tag{3.2.70}$$

The first of these implies that $\tilde{V}(M)_{\beta\alpha}$ is chiral and therefore pure gauge: it is in one-to-one correspondence with its chiral gauge parameter $\tilde{\phi}(M)_{\beta\alpha}$. We can therefore choose $\tilde{V}(M)$ to vanish. The second equation implies that

$$\tilde{V}(D) - 2i\tilde{V}(A) = 2\tilde{\phi}(D)$$

Together with its conjugate, this implies that $\tilde{V}(D)$ and $\tilde{V}(A)$ are the real and imaginary parts of a chiral superfield $\tilde{\phi}(D)$. Since this also precisely overlaps with their gauge degrees of freedom, we can similarly choose $\tilde{V}(D)$ and $\tilde{V}(A)$ to vanish.

This last point is an important one. In a theory with a conformal compensator Φ_0 of unit scaling dimension and matter fields Φ^i of vanishing scaling dimension, the quanta of Φ_0 are indistinguishable from the chiral degree of freedom $\tilde{\phi}(D)$. Both have an equally valid claim to be the chiral quanta which together with V^a make up the quanta of Poincaré supergravity, while the other is the pure gauge degree of freedom. From our point of view, it is almost always more sensible to remove $\tilde{\phi}(D)$ immediately. If desired, it can be restored by undoing the chiral scale transformation.

Whether or not we choose to eliminate $\tilde{\phi}(D)$, the condition that $\tilde{V}(D)$ and $\tilde{V}(A)$ are made up of a sum and a difference of a chiral and an antichiral superfield together with (3.2.68) implies that

$$\bar{\nabla}^2\tilde{V}(K)_\alpha = 0 \tag{3.2.71}$$

This means that $\tilde{V}(K)_\alpha$ is a complex linear superfield and so it too is in perfect correspondence with its gauge degree of freedom and so can be taken to vanish.

We return now to the second condition, (3.2.66). This boils down to

$$V(K)_{\alpha\dot{\alpha}} = -i\nabla_\alpha V(K)_{\dot{\alpha}} - i\nabla_{\dot{\alpha}} V(K)_\alpha + \frac{i}{8}\nabla_\alpha \bar{\nabla}^2 V_{\dot{\alpha}} + \frac{i}{8}\nabla_{\dot{\alpha}}\bar{\nabla}^2 V_\alpha + \frac{1}{32}\hat{\Delta}_D V_{\alpha\dot{\alpha}} \tag{3.2.72}$$

where we have defined

$$\hat{\Delta}_D V_{\alpha\dot{\alpha}} = \nabla^\beta \bar{\nabla}^2 \nabla_\beta V_{\alpha\dot{\alpha}} + 16\nabla_{\dot{\gamma}} W^{\dot{\gamma}\dot{\beta}}{}_\alpha V_{\alpha\dot{\beta}} + 16 W_\alpha{}^{\beta\gamma}\nabla_\gamma V_{\beta\dot{\alpha}} \tag{3.2.73}$$

One can show that $\hat{\Delta}_D V_a$ is Hermitian.

Before moving on, we note here the chiral variation of the conformal supergravity field strength in the chiral gauge where $\tilde{V}(D)$, $\tilde{V}(M)$, and $\tilde{V}(A)$ vanish:

$$\delta_c W_{\alpha\beta\gamma} = \sum_{(\alpha\beta\gamma)} \frac{i}{96}\bar{\nabla}^2 \nabla^{\dot{\phi}}{}_\alpha \nabla_\beta V_{\gamma\dot{\phi}} \tag{3.2.74}$$

We have discovered how to use the Yang-Mills-like features of the conformal supergravity algebra to extract the geometric quanta at first order. We turn next to some specific physical models.

3.3 Two physical models at first order

3.3.1 Linear compensator model

Although we will be most concerned with an arbitrary chiral model, we will first consider a simpler model. The minimally coupled linear compensator model with a Kähler potential consists of a D-term action of two terms

$$S = S_G + S_K. \tag{3.3.1}$$

The Einstein-Hilbert term is contained within the first term

$$S_G = \int ELV_R \equiv 3\int EL\log(L/\Phi_0\bar{\Phi}_0) \tag{3.3.2}$$

where L is the linear compensator and Φ_0 is a chiral superfield of scaling dimension 1, whose presence is almost solely to make the argument of the logarithm conformally invariant, as a redefinition

$$\Phi_0 \to e^\Lambda \Phi_0$$

for chiral Λ leaves the action invariant due to the linearity condition of L. In the gauge where $L = 1$, this has the form of a Fayet-Iliopoulos term for the supergravity $U(1)_R$.

The coupling of chiral matter to the theory is contained within the second term

$$S_K = \int ELK \tag{3.3.3}$$

where K is the Kähler potential, a dimension zero Hermitian function of chiral and antichiral superfields which possesses a symmetry

$$K \to K + F + \bar{F}, \tag{3.3.4}$$

also a consequence of the linearity of L.

We could also include Fayet-Iliopoulos terms for Yang-Mills fields by introducing them as $\int EL\mathrm{Tr}V$ where V is the gauge prepotential. In fact, one can likewise view S_K as essentially being the FI term for a $U(1)_K$ symmetry. One would then naturally combine all these to give the single term

$$-3\int E\, L \log\left(\Phi_0 e^{-(K+V)/3}\bar{\Phi}_0/L\right) \tag{3.3.5}$$

which can be understood as a sum of the FI terms for the Yang-Mills, Kähler, and $U(1)_R$ gauge sectors. We will exclude from our discussion Yang-Mills FI terms and treat the supergravity and Kähler sectors separately.

In order to proceed, we need to determine the transformation of the various quantities. We will work in the gauge where $\tilde{V}(D) = \tilde{V}(A) = \tilde{V}(M) = \tilde{V}(K) = 0$. The non-primary object V^α we will leave for the moment unfixed and specify a gauge for it later.

The first order variation of E is

$$\begin{aligned}\delta E &= H^\alpha{}_\alpha + H_{\dot\alpha}{}^{\dot\alpha} + H^a{}_a \\ &= -3i\nabla^\alpha V_\alpha + 3i\bar{\nabla}_{\dot\alpha}V^{\dot\alpha} - \Delta_b V^b - 4V(A) = 0\end{aligned} \tag{3.3.6}$$

This is an initially surprising result, but it is owed to our working in a conformal theory. For example, in a component four dimensional theory, the first order variation of $\sqrt{g}$ is the trace of the graviton perturbation, which is the conformal mode of the graviton. We could set the scaling gauge in such a theory by forcing the conformal mode to vanish. This is something of a shell game, however, since the conformal mode of the graviton is essentially the same object as the conformal compensator in such a theory. In the current theory, the role of the "conformal mode" of the graviton will be taken up by the linear compensator (and later the chiral compensator) and so $\delta E = 0$ here.

The first order variation of a chiral superfield Φ of scaling dimension Δ and $U(1)_R$ weight $2\Delta/3$ is given in Hermitian gauge by

$$\begin{aligned}\delta\Phi &= -iV^B X_B\Phi + \delta_c\Phi \\ &= -iV^\beta\Phi - iV^b\nabla_b\Phi - i\left(V(D) + \frac{2i}{3}V(A)\right)\Delta\Phi - i\Sigma^r X_r\Phi + \eta\end{aligned} \tag{3.3.7}$$

where we define $\delta_c\Phi \equiv \eta$ as the variation in chiral gauge.

We next note that L may be written

$$L = \nabla^\alpha\Phi_\alpha + \nabla_{\dot\alpha}\Phi^{\dot\alpha} \tag{3.3.8}$$

in terms of chiral primary superfields Φ_α of weight $(3/2, 1)$. The variation of $\nabla^\alpha \Phi_\alpha$ is given by

$$\begin{aligned}\delta(\nabla^\alpha\Phi_\alpha) = & -i\nabla^\beta(V_\beta\nabla^\alpha\Phi_\alpha) + i\nabla_{\dot\beta}(V^{\dot\beta}\nabla^\alpha\Phi_\alpha) - \Delta_b(V^b\nabla^\alpha\Phi_\alpha) + 2V^{\dot\alpha\alpha}\bar W_{\dot\alpha}\Phi_\alpha \\ & + \frac{1}{4}\nabla_{\dot\alpha}\nabla^2(V^{\dot\alpha\alpha}\Phi_\alpha) + \nabla^\alpha(\delta_c\Phi_\alpha) \\ & - i\Sigma\nabla^\alpha\Phi_\alpha - 2i(\nabla^\alpha\Sigma^r)X_r\Phi_\alpha \end{aligned} \tag{3.3.9}$$

Assuming Φ_α to be a gauge singlet, we can write the variation of L as

$$\delta L = \mathcal{L} - i\nabla^\beta(V_\beta L) + i\nabla_{\dot\beta}(V^{\dot\beta}L) - \Delta_b(V^b L) \tag{3.3.10}$$

where

$$\mathcal{L} \equiv \nabla^\alpha\left(\delta_c\Phi_\alpha - \frac{1}{4}\bar\nabla^2(V^{\dot\alpha\alpha}\bar\Phi_{\dot\alpha})\right) + \text{h.c.} \equiv \nabla^\alpha\eta_\alpha + \text{h.c.} \tag{3.3.11}$$

η_α is a weight $(3/2, 1)$ chiral primary superfield, which we have defined to depend on both $\delta_c\Phi_\alpha$ and its conjugate in order to simplify the formula.

After several integrations by parts, one can show that

$$\delta S_G = \int E\left(\mathcal{L}V_R - 2V^b\Delta_b L + \frac{3}{2L}V^{\alpha\dot\alpha}\nabla_\alpha L\nabla_{\dot\alpha}L\right) \tag{3.3.12}$$

We may define a new weight (0,0) primary superfield G_b by

$$G_b \equiv \frac{1}{2}L^{-1}\Delta_b L - \frac{3}{8L^2}\nabla_\alpha L\nabla_{\dot\alpha}L = -L^{1/2}\Delta_b L^{-1/2} \tag{3.3.13}$$

So that

$$\delta S_G = \int E\left(\mathcal{L}V_R - 4LV^bG_b\right) \tag{3.3.14}$$

One can similarly work out the structure of S_K. Skipping details (the most difficult of which is an integration by parts) one finds

$$\delta S_K = \int EL\left(K_i\eta^i + K_{\bar j}\eta^{\bar j} + V^bK_b + \Sigma^rK_r\right) + \int E\mathcal{L}K \tag{3.3.15}$$

where

$$K_{\alpha\dot\alpha} \equiv K_{i\bar j}\nabla_\alpha\Phi^i\nabla_{\dot\alpha}\bar\Phi^{\bar j} \tag{3.3.16}$$

$$K_r \equiv -iK_iX_r\Phi^i + iK_{\bar j}X_r\bar\Phi^{\bar j} \tag{3.3.17}$$

Both K_a and K_r are conformally primary.

Combining these two variations gives

$$\delta S = \int E\left[LV^b\left(-4G_b + K_b\right) + L\Sigma^rK_r + LK_i\eta^i + LK_{\bar j}\eta^{\bar j} + \mathcal{L}(V_R + K)\right] \tag{3.3.18}$$

This is a surprisingly compact expression. When L is gauged to 1, G_b becomes the Poincaré superfield of the same name and represents the pure supergravity contribution to the energy-momentum tensor. K_b represents the matter contribution to the energy-momentum tensor, and K_r is the matter contribution to the gauge current.

Gauge invariance of the linear compensator model

The first feature we should observe about our linear compensator model is that at first order it is independent of V^α and $V_{\dot\alpha}$. This is certainly sensible since these are gauge degrees of freedom and should certainly not have any equations of motion associated with themselves.

The dynamical theory would seem to consist of V^a and Σ^r – the Hermitian superfields associated with the graviton and gauge multiplets – as well as the matter superfield η^i and $\bar\eta^{\bar j}$ and the linear compensator variation $\mathcal{L}$. We recall that V^a transforms under the quantum chiral gauge transformation as

$$\delta V_{\alpha\dot\alpha} = \nabla_\alpha L_{\dot\alpha} - \nabla_{\dot\alpha} L_\alpha \tag{3.3.19}$$

Under the L_α transformation, a chiral superfield transforms as

$$\Phi' = C_Q \Phi \tag{3.3.20}$$

Differentially, this reads

$$\delta\eta = 2i\Lambda\Phi = 2i\xi^a\nabla_a\Phi + 2i\xi^\alpha\nabla_\alpha\Phi + 2i\Lambda\Delta\Phi - \frac{4}{3}\omega\Delta\Phi + 2i\Lambda^r X_r\Phi \tag{3.3.21}$$

where Δ is the scaling dimension of Φ. Plugging in the values for superfields, we find

$$\delta\eta = -\frac{1}{4}\bar\nabla^2\left(L^\alpha\nabla_\alpha\Phi\right) - \frac{\Delta}{12}(\bar\nabla^2\nabla^\beta L_\beta)\Phi + 2i\tilde\Lambda^r X_r\Phi \tag{3.3.22}$$

The gauge superfield Σ^r transforms as

$$\delta\Sigma^r = \tilde\Lambda^r + \bar{\tilde\Lambda}^r + iL^\beta W^r_\beta + iL_{\dot\beta}\bar W^{\dot\beta r} \tag{3.3.23}$$

The quantum linear compensator varies as

$$\delta\mathcal{L} = \frac{1}{4}\nabla^\alpha\bar\nabla^2(L_\alpha L) + \text{h.c.} \tag{3.3.24}$$

Note that this last expression depends on Φ_α only implicitly via L.

One can check that the first-order action is invariant under this first-order shift in the quantum superfields, as it must be by construction.

3.3.2 Arbitrary chiral model

The minimal linear compensator model is notable for the clean decoupling of the gravitational and matter terms of the action, which gives a corresonding decoupling of their contributions to the gravitational current. The arbitrary chiral model will not be so immediately simple to evaluate, but we will find its first order variation shares the same features.

The chiral model classically dual to the minimal linear compensator model with a Kähler potential K is

$$S = -3\int E\,\Phi_0\bar\Phi_0\,e^{-K/3} \tag{3.3.25}$$

This action encapsulates not only the pure gravity effects (denoted S_G in the linear model) but also kinetic matter terms (denoted S_K). Here Φ_0 is a weight $(1, 2/3)$ conformally primary chiral superfield and K is as before a Hermitian function of weight $(0,0)$ chiral and antichiral superfields. A canonically normalized Einstein-Hilbert term is found in the gauge $\Phi_0\bar{\Phi}_0 = e^{K/3}$.

The above D-term is a special case of a more general theory involving an arbitrary set of chiral superfields of arbitrary weights,

$$S = -3\int EZ \equiv -3\,[Z]_D \tag{3.3.26}$$

We have introduced the shorthand that $[\quad]_D$ denotes integration of its argument over the full superspace. We can similarly define $[\quad]_F$ as integration over the chiral submanifold of superspace. In this expression, Z is a gauge invariant Hermitian superfield of scale dimension two construced from the chiral superfields Φ^i and their conjugates. The factor of -3 is necessary so that the gauge $Z = 1$ gives a canonical Einstein-Hilbert term. The proof of this is straightforward. Using the scaling and $U(1)_R$ weights of Z,

$$\begin{aligned}
DZ = 2Z &= Z_i\Delta_i\Phi^i + Z_{\bar{j}}\Delta_{\bar{j}}\bar{\Phi}^{\bar{j}}\\
-\frac{3i}{2}AZ = 0 &= Z_i\Delta_i\Phi^i - Z_{\bar{j}}\Delta_{\bar{j}}\bar{\Phi}^{\bar{j}}
\end{aligned}$$

and that the Einstein-Hilbert term is contained within

$$-3[Z]_D = -3\left[Z_{\bar{j}}\mathcal{P}\Phi^{\bar{j}} + \ldots\right]_F = -3Z_{\bar{j}}\bar{\mathcal{P}}\mathcal{P}\Phi^{\bar{j}} + \ldots = -3Z_{\bar{j}}\Box\Phi^{\bar{j}} + \ldots$$

where $\mathcal{P} = -\bar{\nabla}^2/4$, $\bar{\mathcal{P}} = -\nabla^2/4$ and $\Box$ are superconformal. That $\Box$ is superconformal means it contains $\mathcal{R}/6$ weighted by the scaling dimension of the field on which it acts, and so it is easy to see that the Einstein-Hilbert term is

$$-3[Z]_D \ni -\frac{1}{2}\mathcal{R}Z_{\bar{j}}\Delta_{\bar{j}}\Phi^{\bar{j}} = -\frac{Z}{2}\mathcal{R}$$

The gauge $Z = 1$ then corresponds to a canonical Einstein-Hilbert term.

Since $\delta E = 0$, we concern ourselves only with the first order variation of Z:

$$\begin{aligned}
\delta Z =& Z_i(\eta^i - iV\Phi^i) + Z_{\bar{j}}(\bar{\eta}^{\bar{j}} + iV\bar{\Phi}^{\bar{j}})\\
=& Z_i\eta^i + Z_{\bar{j}}\bar{\eta}^{\bar{j}} - iZ_i\Sigma^r X_r\Phi^i + iZ_{\bar{j}}\Sigma^r X_r\bar{\Phi}^{\bar{j}} - iZ_iV^b\nabla_b\Phi^i + iZ_{\bar{j}}V^b\nabla_b\bar{\Phi}^{\bar{j}}\\
&- iV^\alpha\nabla_\alpha Z + iV_{\dot{\alpha}}\bar{\nabla}^{\dot{\alpha}}Z + \frac{4}{3}V(A)Z
\end{aligned} \tag{3.3.27}$$

Plugging in the value of $V(A)$ gives

$$\begin{aligned}
\delta Z =& Z_i\eta^i + Z_{\bar{j}}\bar{\eta}^{\bar{j}} - iZ_i\Sigma^r X_r\Phi^i + iZ_{\bar{j}}\Sigma^r X_r\bar{\Phi}^{\bar{j}} - iZ_iV^b\nabla_b\Phi^i + iZ_{\bar{j}}V^b\nabla_b\bar{\Phi}^{\bar{j}}\\
&+ i\nabla_\alpha(V^\alpha Z) - i\bar{\nabla}^{\dot{\alpha}}(V_{\dot{\alpha}}Z) - \frac{1}{3}\Delta_bV^bZ
\end{aligned} \tag{3.3.28}$$

The two terms in the last line which appear to vanish as total derivatives actually do not. To see why, note that the actual statement of a vanishing total derivative involves only the coordinate derivative:

$$0 = \partial_M (E E_\alpha{}^M V^\alpha Z) = \nabla_M (E E_\alpha{}^M V^\alpha Z) + h_M{}^{\underline{b}} X_{\underline{b}} (E E_\alpha{}^M V^\alpha Z)$$

The term involving the connection usually vanishes by gauge invariance; however, in this case V^α is not conformally invariant (though the other terms in the parentheses are), and so the second term yields

$$E f_{\alpha\dot\alpha} \bar S^{\dot\alpha} (V^\alpha Z) = E \left(-i f_{\alpha\dot\alpha} V^{\alpha\dot\alpha} Z\right)$$

Evaluating the first term yields

$$E \left(\nabla_\alpha (V^\alpha Z) + T_{\alpha B}{}^B V^\alpha Z\right)$$

The trace of the torsion tensor vanishes, which leads to the identity

$$i \nabla_\alpha (V^\alpha Z) = -f_{\alpha\dot\alpha} V^{\alpha\dot\alpha} Z + \text{t.d.}$$

Integrating by parts on the $\Delta_b V^b$ term gives the same explicit connections but with the opposite sign, yielding

$$\delta S = -\,3 Z_i \eta^i - 3 Z_{\bar j} \bar\eta^{\bar j} + 3i Z_i \Sigma^r X_r \Phi^i - 3i Z_{\bar j} \Sigma^r X_r \bar\Phi^{\bar j} + V^b \left(\Delta_b Z + 3i Z_i \nabla_b \Phi^i - 3i Z_{\bar j} \nabla_b \bar\Phi^{\bar j}\right) \tag{3.3.29}$$

There are several annoying features of this expression. One is that the terms involving V^b are not individually conformally invariant. Another is that in the linear compensator model, we had a clear factor of L out front of all the terms which we could gauge to one. Here we would like to gauge $Z = 1$ to arrive at the supergravity of Binetruy, Girardi, and Grimm [6], but none of the terms possess an explicit Z out front. We can deal with both of these issues by the following field redefinition:

$$\mathcal{K} \equiv -3 \log Z \tag{3.3.30}$$

$\mathcal{K}$ is a superfield which transforms non-linearly under a conformal transformation. If we choose $Z = \Phi_0 \bar\Phi_0 e^{-K/3}$, we see that this $\mathcal{K}$ is essentially the same object as the canonical Kähler potential:

$$\mathcal{K} = K - 3 \log(\Phi_0 \bar\Phi_0)$$

The advantage of this definition is that we may now rewrite δS as

$$\delta S = Z \left(\mathcal{K}_i \eta^i + \mathcal{K}_{\bar j} \bar\eta^{\bar j} + \Sigma^r \mathcal{K}_r + V^b \left(-4 G_b + \mathcal{K}_b\right)\right) \tag{3.3.31}$$

where we have defined

$$G_b \equiv -Z^{1/2} \Delta_b Z^{-1/2} \tag{3.3.32}$$

$$\mathcal{K}_{\alpha\dot\alpha} \equiv \mathcal{K}_{i\bar j} \nabla_\alpha \Phi^i \nabla_{\dot\alpha} \bar\Phi^{\bar j} \tag{3.3.33}$$

$$\mathcal{K}_r \equiv -i \mathcal{K}_i X_r \Phi^i + i \mathcal{K}_{\bar j} X_r \bar\Phi^{\bar j} \tag{3.3.34}$$

If we choose $Z = \Phi_0 \bar{\Phi}_0 e^{-K/3}$, then we find

$$\delta S = Z \left(K_i \eta^i + K_{\bar{j}} \bar{\eta}^{\bar{j}} + \Sigma^r K_r + V^b \left(-4G_b + K_b\right) - \frac{3\eta_0}{\Phi_0} - \frac{3\bar{\eta}_0}{\bar{\Phi}_0} \right) \tag{3.3.35}$$

and the chiral first-order action is superficially the same as the linear one except for the exchange of the $\mathcal{L}$ sector for the η_0 sector and the exchange of the L compensator for Z.

The importance of this observation is that it simplifies the task of finding the second-order action for both of these theories. Rather than treating each individually, we can focus on their common features and only worry about where they specifically differ.

Let us consider several other terms that we might like to include in both of these models.

3.3.3 Superpotential terms

A superpotential term is a chiral action S_P defined as

$$S_P = \int \mathcal{E}\, P + \text{h.c.} \tag{3.3.36}$$

where P is some chiral superfield of weight $(3, 2)$. For the simplest chiral compensator model, $P = \Phi_0^3 W$ where W is the object one normally calls the superpotential. Because we're interested in linear compensator models as well as the general chiral model, we will use the more generic name P to denote this F-term superfield Lagrangian.

Since the superpotential terms involve purely chiral and antichiral actions, we can use the quantum chiral and antichiral gauges to describe them. We note that

$$\delta_c \mathcal{E} = H^\alpha{}_\alpha + H^a{}_a = 0 \tag{3.3.37}$$

in quantum chiral gauge, so only the chiral variation of the integrand remains. The variation of the superpotential term is then simply

$$\delta_c S_P = \int \mathcal{E}\, P_i \eta^i + \text{h.c.} \tag{3.3.38}$$

implying that the superpotential plays no rule in the pure conformal supergravity equations of motion. (That it plays a role in Poincaré supergravity arises because of the presence of the chiral compensator.)

3.3.4 Yang-Mills terms

The Yang-Mills term we will consider is

$$S_{YM} = \frac{1}{4} \int \mathcal{E}\, f_{rs} W^{\alpha r} W_\alpha{}^s + \text{h.c.} \tag{3.3.39}$$

where f_{rs} is a holomorphic covariant gauge coupling. In the simplest of cases, $f_{rs} = \delta_{rs}$, but we will for the moment allow for a more generic holomorphic coupling.

As before, one finds quantum chiral gauge the simplest for the chiral action. Using

$$\delta_c W^r_\alpha = -\frac{i}{4}\bar{\nabla}^2\nabla_\alpha\Sigma^r - \frac{1}{4}\bar{\nabla}^2\left(V_{\alpha\dot{\beta}}\bar{W}^{\dot{\beta}r}\right) \tag{3.3.40}$$

as well as

$$\delta_c f_{rs} = f_{rs,i}\eta^i \tag{3.3.41}$$

one immediately finds

$$\begin{aligned}\delta S_{YM} &= \int\mathcal{E}\left(\frac{1}{4}f_{rs,i}\eta^i W^{\alpha r}W_\alpha{}^s - \frac{i}{8}f_{rs}W^{\alpha r}\bar{\nabla}^2\nabla_\alpha\Sigma^s - \frac{1}{8}f_{rs}W^{\alpha r}\bar{\nabla}^2(V_{\alpha\dot{\beta}}\bar{W}^{\dot{\beta}s})\right) + \text{h.c.}\\ &= \int\mathcal{E}\left(\frac{1}{4}f_{rs,i}\eta^i W^{\alpha r}W_\alpha{}^s\right) + \int E\left(\frac{i}{2}f_{rs}W^{\alpha r}\nabla_\alpha\Sigma^s + \frac{1}{2}f_{rs}V_{\alpha\dot{\alpha}}W^{\alpha r}\bar{W}^{\dot{\alpha}s})\right) + \text{h.c.}\end{aligned} \tag{3.3.42}$$

There is the possibility of introducing the Yang-Mills interactions by requiring the linear compensator L to obey the modified linearity conditions

$$\bar{\nabla}^2 L = 2k\text{Tr}(W^\alpha W_\alpha), \quad \nabla^2 L = 2k\text{Tr}(\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}})$$

Then Yang-Mills interactions can be made part of the structure of superspace when the compensator is gauged to 1. This tends to introduce non-holomorphic gauge couplings. We will avoid this possibility for now and restrain ourselves to the normal holomorphic Yang-Mills terms.

3.3.5 Generic first-order structure

We summarize the generic structure that the arbitrary chiral model and the minimal linear compensator models possess. The common part of the first order action consists of a sum of four terms. They are:

$$(\delta S)_G = \left[-4XV^bG_b\right]_D \tag{3.3.43}$$

$$(\delta S)_K = \left[X(V^b\mathcal{K}_b + \Sigma^r\mathcal{K}_r + \eta^i\mathcal{K}_i + \bar{\eta}^{\bar{j}}\mathcal{K}_{\bar{j}})\right]_D \tag{3.3.44}$$

$$\delta S_P = \left[\eta^i P_i\right]_F + \text{h.c.} \tag{3.3.45}$$

$$\delta S_{YM} = [V^a\mathcal{Y}_a + \Sigma^r\mathcal{Y}_r]_D + \left[\eta^i\mathcal{Y}_i\right]_F + \left[\bar{\eta}^{\bar{j}}\bar{\mathcal{Y}}_{\bar{j}}\right]_{\bar{F}} \tag{3.3.46}$$

where X is the compensator (L or Z) and

$$G_b \equiv -X^{1/2}\Delta_b X^{-1/2} \tag{3.3.47}$$

$$\mathcal{K}_{\alpha\dot{\alpha}} \equiv \mathcal{K}_{i\bar{j}}\nabla_\alpha\Phi^i\nabla_{\dot{\alpha}}\bar{\Phi}^{\bar{j}} \tag{3.3.48}$$

$$\mathcal{K}_r \equiv -i\mathcal{K}_i X_r\Phi^i + i\mathcal{K}_{\bar{j}}X_r\bar{\Phi}^{\bar{j}} \tag{3.3.49}$$

$$\mathcal{Y}_i \equiv \frac{1}{4}f_{rs,i}W^{\alpha r}W_\alpha{}^s \tag{3.3.50}$$

$$\mathcal{Y}_{\alpha\dot{\alpha}} \equiv -(f_{rs} + \bar{f}_{rs})W_\alpha{}^r\bar{W}_{\dot{\alpha}}{}^s \tag{3.3.51}$$

$$\mathcal{Y}_r \equiv -\frac{i}{2}\nabla^\alpha\left(f_{rs}W_\alpha{}^s\right) + \text{h.c.} \tag{3.3.52}$$

We will find use to denote $G_{rs} \equiv f_{rs} + \bar{f}_{rs}$. Then the last two equations above may be written

$$\mathcal{Y}_{\alpha\dot{\alpha}} \equiv -G_{rs} W_\alpha{}^r \bar{W}_{\dot{\alpha}}{}^s$$
$$\mathcal{Y}_r \equiv -\frac{i}{2}(\nabla^\alpha G_{rs}) W_\alpha{}^s - \frac{i}{2}(\nabla_{\dot{\alpha}} G_{rs}) \bar{W}^{\dot{\alpha} s} - \frac{i}{2} G_{rs} \nabla^\alpha W_\alpha{}^s$$

using $\nabla^\alpha W_\alpha{}^r = \nabla_{\dot{\alpha}} \bar{W}^{\dot{\alpha} r}$.

The equations of motion amount to

$$0 = -4XG_b + X\mathcal{K}_b + \mathcal{Y}_b \tag{3.3.53}$$
$$0 = \mathcal{K}_r + \mathcal{Y}_r \tag{3.3.54}$$
$$0 = -\frac{1}{4}\bar{\nabla}^2(X\mathcal{K}_i) + P_i + \mathcal{Y}_i \tag{3.3.55}$$

For the linear compensator model, there is the additional term

$$\delta S_L = [\mathcal{L}(V_R + K)]_D \tag{3.3.56}$$

along with that model's equation of motion

$$0 = \bar{\nabla}^2 \nabla_\alpha (V_R + K) = \nabla^2 \bar{\nabla}^{\dot{\alpha}} (V_R + K) \tag{3.3.57}$$

which implies that $V_R = -K$ up to the real part of a chiral superfield.

The structure we have identified here is actually more general than this treatment indicates. The same features persist in arbitrary models involving any number of linear and chiral superfields. A brief discussion of the first order variation of an *arbitrarily* coupled linear superfield is given in Appendix E.

3.4 Going to second order

In order to construct a one-loop effective action, we require the action to second order in the quantum deformations. The simplest way to do this is a sort of bootstrap: vary our first order expression again to first order.

However, doing so immediately tends to produce a nasty set of terms involving many derivatives of the compensator X for the graviton's action. The reason is easy to see: the action for the graviton is hidden within the action for the compensator. In addition to a term $XV^a \Box V_a$, there would be a host of terms involving derivatives of X needed in order to make this expression invariant under special conformal transformations. One way to simplify this would be to eliminate many of these terms by choosing a gauge where X is constant and then degauging to Poincaré derivatives. Unfortunately this sacrifices the conformal invariance of the classical action before quantization has even taken place. A better approach would be to introduce conformally *invariant* derivatives, with respect to which X is covariantly constant. These would compactly encode the many terms involving derivatives of X in conformally invariant combinations. It is to this construction that we now turn.

3.4.1 A brief interlude: conformally invariant (or compensated) derivatives

Definition

In the preceding discussion, we introduced the conformally primary superfield G_b which was defined in terms of the dimension 2 compensator X. When X is gauged to unity and the conformally covariant derivatives are themselves "degauged", the object $-X^{1/2}\Delta_b X^{-1/2}$ reduces simply to the Poincaré superfield G_b, but the existence of this conformally primary combination means we may identify the equivalent of G_b even in the conformal theory. We may similarly identify other Poincaré equivalents and thereby perform something very much like a degauging while still maintaining the underlying conformal invariance.

We begin with X, a primary Hermitian superfield with $\Delta = 2$ and $w = 0$. Define $U = \log X$ so that under scalings, U transforms nonlinearly into a constant, here $DU = 2$. Then we define the compensator-associated derivatives as

$$\mathcal{D}_\alpha \equiv \nabla_\alpha - \frac{1}{2}\nabla_\alpha U D - \frac{1}{2}\nabla^\beta U M_{\beta\alpha} + \frac{3i}{4}\nabla_\alpha U A \tag{3.4.1}$$

$$\mathcal{D}^{\dot\alpha} \equiv \nabla^{\dot\alpha} - \frac{1}{2}\nabla^{\dot\alpha} U D - \frac{1}{2}\nabla_{\dot\beta} U M^{\dot\beta\dot\alpha} - \frac{3i}{4}\nabla^{\dot\alpha} U A \tag{3.4.2}$$

These new derivatives are constructed so that when they act on a conformally primary object, the result is conformally primary.

We are not the first to construct these objects. Kugo and Uehara, in their treatment of conformal supergravity [13], constructed these operators almost immediately out of the covariant derivatives, dubbing these the **u**-associated derivatives, where **u** denoted the compensator being used. Their motivation seemed to be the desire for operators that would act on conformally primary superfields to generate more conformally primary superfields. In that sense, these new operators are special conformal *invariant* rather than *covariant*.

The purely undotted objects have a new algebra

$$\{\mathcal{D}_\beta, \mathcal{D}_\alpha\} = \frac{1}{2}\left(\nabla^2 U + \nabla^\gamma U \nabla_\gamma U\right) M_{\beta\alpha} = \frac{1}{2}\frac{1}{X}\nabla^2 X M_{\beta\alpha} \equiv -4\bar{R} M_{\beta\alpha} \tag{3.4.3}$$

Similarly,

$$\{\mathcal{D}^{\dot\beta}, \mathcal{D}^{\dot\alpha}\} = -4R M^{\dot\beta\dot\alpha} \tag{3.4.4}$$

where we have defined

$$R \equiv -\frac{1}{8X}\bar\nabla^2 X, \quad \bar{R} \equiv -\frac{1}{8X}\nabla^2 X \tag{3.4.5}$$

From these definitions, R possesses scaling and $U(1)_R$ weights $(\Delta, w) = (1, +2)$ and $\bar R$ the weights $(1, -2)$. It is straightforward to show that in the limit where we gauge fix X to unity, these R's become the R's of Poincare supergravity. However, these versions are more useful since they are also conformally invariant by nature of the fact that the new covariant derivatives are themselves conformally invariant. Furthermore, one may show that they are chiral with respect to the new derivatives:

$$\mathcal{D}^{\dot{\alpha}} R = 0, \quad \mathcal{D}_\alpha \bar{R} = 0. \tag{3.4.6}$$

It is straightforward to guess the form of the analogues of G_c and X_α. Demanding that the definition of G_c match when X is fixed to unity (and also be conformally invariant) gives

$$G_{\alpha\dot{\alpha}} = -\frac{1}{4}[\nabla_\alpha, \nabla_{\dot{\alpha}}]U + \frac{1}{4}\nabla_\alpha U \nabla_{\dot{\alpha}} U = \frac{1}{2} X^{1/2} [\nabla_\alpha, \nabla_{\dot{\alpha}}] X^{-1/2} \tag{3.4.7}$$

which is as we have defined it before. Defining X_α as $\mathcal{D}_\alpha R - \mathcal{D}^{\dot{\alpha}} G_{\alpha\dot{\alpha}}$ leads to

$$X_\alpha = \frac{3}{8}\bar{\nabla}^2 \nabla_\alpha U, \quad X^{\dot{\alpha}} = \frac{3}{8}\nabla^2 \nabla^{\dot{\alpha}} U \tag{3.4.8}$$

which is conformally invariant automatically.

We briefly pause to note the following features. If $X = \Phi_0 \bar{\Phi}_0 e^{-K/3}$,

$$X_\alpha = -\frac{1}{8}\bar{\nabla}^2 \nabla_\alpha K = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha K$$

as in Kähler $U(1)$ supergravity. Similarly, if $X = L$, then $R = 0$ as in new minimal supergravity.

We next define the bosonic derivative $\mathcal{D}_{\alpha\dot{\alpha}}$ by the anti-commutator

$$\{\mathcal{D}_\alpha, \mathcal{D}^{\dot{\alpha}}\} \equiv -2i\mathcal{D}_\alpha{}^{\dot{\alpha}} - \lambda G^{\beta\dot{\alpha}} M_{\beta\alpha} + \lambda G_{\alpha\dot{\beta}} M^{\dot{\beta}\dot{\alpha}} + 3i\lambda G_\alpha{}^{\dot{\alpha}} A \tag{3.4.9}$$

We have introduced into this definition a parameter λ which parametrizes how much of the various bosonic connections of $\mathcal{D}_a$ is stored in the additional "curvatures" on the right hand side. $\lambda = 1$ corresponds to the standard $U(1)$ supergravity of Binetruy, Girardi, and Grimm [6] and what is achieved by straightforwardly degauging from conformal to Poincare supergravity (as in Section 2.3). $\lambda = 0$ corresponds to a redefinition of that theory so that the $\alpha\dot{\alpha}$ curvatures are trivial. (This is the choice made in [25] and [22].) The latter has the simplest-looking curvatures overall, but it introduces a nonzero torsion T_{cba} proportional to the dual of G_a, which leads to a bosonic Riemann curvature tensor lacking the common symmetries and with an auxiliary superfield hiding within the spin connection. For this reason $\lambda = 0$ seems to be ill-suited for component calculations; however, for the pure superfield manipulations we perform here, it leads to a simpler algebra for the covariant derivatives. The two definitions are completely equivalent, of course, and differ only in the definition of the bosonic connections.

These definitions lead to

$$\begin{aligned}\mathcal{D}_\alpha{}^{\dot{\alpha}} \equiv& \nabla_\alpha{}^{\dot{\alpha}} - \frac{i}{2}\nabla_\alpha U \mathcal{D}^{\dot{\alpha}} - \frac{i}{2}\nabla^{\dot{\alpha}} U \mathcal{D}_\alpha - \frac{1}{2}\nabla_\alpha{}^{\dot{\alpha}} U D + \left(+\frac{3}{8}[\nabla_\alpha, \nabla^{\dot{\alpha}}]U + \frac{3\lambda}{2} G_\alpha{}^{\dot{\alpha}}\right) A \\ &+ \left(-\frac{i}{4}\nabla_\alpha \nabla_{\dot{\beta}} U - \frac{i\lambda}{2} G_{\alpha\dot{\beta}}\right) M^{\dot{\beta}\dot{\alpha}} + \left(-\frac{i}{4}\nabla^{\dot{\alpha}} \nabla^\beta U + \frac{i\lambda}{2} G^{\beta\dot{\alpha}}\right) M_{\beta\alpha}\end{aligned} \tag{3.4.10}$$

The newly-defined curvatures are straightforward to work out. For the bosonic-fermionic curvatures,

$$T_{\gamma(\beta\dot\beta)\dot\alpha} = -2i\epsilon_{\gamma\beta}\epsilon_{\dot\beta\dot\alpha}\bar R \tag{3.4.11}$$

$$T_{\gamma(\beta\dot\beta)\alpha} = i\lambda G_{\gamma\dot\beta}\epsilon_{\beta\alpha} - 2i(1-\lambda)G_{\alpha\dot\beta}\epsilon_{\gamma\beta} \tag{3.4.12}$$

$$F_{\beta(\alpha\dot\alpha)} = -\frac{3\lambda}{2}\mathcal{D}_\beta G_{\alpha\dot\alpha} - \epsilon_{\beta\alpha}X_{\dot\alpha} \tag{3.4.13}$$

$$R_{\delta(\gamma\dot\gamma)\beta\alpha} = \sum_{\beta\alpha}\left[i\epsilon_{\delta\gamma}\mathcal{D}_\beta G_{\alpha\dot\gamma} + \frac{i\lambda}{2}\mathcal{D}_\delta G_{\beta\dot\gamma}\epsilon_{\gamma\alpha} - i\epsilon_{\delta\beta}\epsilon_{\gamma\alpha}\mathcal{D}_{\dot\gamma}\bar R\right] \tag{3.4.14}$$

$$R_{\delta(\gamma\dot\gamma)\dot\beta\dot\alpha} = 4i\epsilon_{\delta\gamma}W_{\dot\gamma\dot\beta\dot\alpha} + \sum_{\dot\beta\dot\alpha}\epsilon_{\dot\gamma\dot\alpha}\left[\frac{i}{3}\epsilon_{\delta\gamma}\bar X_{\dot\beta} + \frac{i\lambda}{2}\mathcal{D}_\delta G_{\gamma\dot\beta}\right] \tag{3.4.15}$$

Note that these curvatures simplify a fair amount by choosing $\lambda = 0$.

The bosonic torsions are

$$T_{(\beta\dot\beta)(\alpha\dot\alpha)}{}^{\gamma}\mathcal{D}_\gamma = -2\epsilon_{\dot\beta\dot\alpha}W_{\beta\alpha\gamma}\mathcal{D}^\gamma - \frac{1}{2}\epsilon_{\dot\beta\dot\alpha}\mathcal{D}_{\{\beta}R\,\mathcal{D}_{\alpha\}} - \frac{1}{6}\epsilon_{\dot\beta\dot\alpha}X_{\{\beta}\mathcal{D}_{\alpha\}} - \frac{1}{2}\epsilon_{\beta\alpha}\mathcal{D}_{\{\dot\beta}G_{\dot\alpha\}\gamma}\mathcal{D}^\gamma \tag{3.4.16}$$

$$T_{(\beta\dot\beta)(\alpha\dot\alpha)\dot\gamma}\mathcal{D}^{\dot\gamma} = -2\epsilon_{\beta\alpha}W_{\dot\beta\dot\alpha\dot\gamma}\mathcal{D}^{\dot\gamma} + \frac{1}{2}\epsilon_{\beta\alpha}\mathcal{D}_{\{\dot\beta}\bar R\,\mathcal{D}_{\dot\alpha\}} + \frac{1}{6}\epsilon_{\beta\alpha}\bar X_{\{\dot\beta}\mathcal{D}_{\dot\alpha\}} + \frac{1}{2}\epsilon_{\dot\beta\dot\alpha}\mathcal{D}_{\{\beta}G_{\alpha\}\dot\gamma}\mathcal{D}^{\dot\gamma} \tag{3.4.17}$$

$$T_{(\beta\dot\beta)(\alpha\dot\alpha)}{}^{c}\mathcal{D}_c = -2i(1-\lambda)G_{\beta\dot\alpha}\mathcal{D}_{\alpha\dot\beta} + 2i(1-\lambda)G_{\alpha\dot\beta}\mathcal{D}_{\beta\dot\alpha} \tag{3.4.18}$$

Note the last torsion vanishes for $\lambda = 1$.

The part of the Riemann tensor acting on spinor indices is

$$\begin{aligned}\frac{1}{2}R_{(\beta\dot\beta)(\alpha\dot\alpha)\gamma\phi}M^{\phi\gamma} =& \epsilon_{\dot\beta\dot\alpha}\sum_{\beta\alpha}\left(\frac{1}{2}\mathcal{D}_\beta W_{\alpha\phi\gamma}M^{\gamma\phi} + \frac{1}{12}\mathcal{D}_\beta X^\gamma M_{\gamma\alpha} - \frac{1}{8}\bar{\mathcal{D}}^2 R M_{\beta\alpha} + 2R\bar R M_{\alpha\beta}\right)\\ &- \frac{1}{4}\epsilon_{\beta\alpha}\mathcal{D}_{\{\dot\beta}\mathcal{D}_\gamma G_{\phi\dot\alpha\}}M^{\phi\gamma} - \frac{i\lambda}{2}\mathcal{D}_{\beta\dot\beta}G^{\phi}{}_{\dot\alpha}M_{\phi\alpha} + \frac{i\lambda}{2}\mathcal{D}_{\alpha\dot\alpha}G^{\phi}{}_{\dot\beta}M_{\phi\beta}\\ &- \frac{\lambda^2}{2}G_{\beta\dot\alpha}G^{\phi}{}_{\dot\beta}M_{\phi\alpha} + \frac{\lambda^2}{2}G_{\alpha\dot\beta}G^{\phi}{}_{\dot\alpha}M_{\phi\beta}\\ &+ \frac{1}{2}(\lambda^2-\lambda)\epsilon_{\dot\beta\dot\alpha}G_{\phi\dot\phi}G^{\phi\dot\phi}M_{\beta\alpha}\end{aligned} \tag{3.4.19}$$

The other half can be found by Hermitian conjugation.

The remaining $U(1)$ curvature is

$$F_{(\beta\dot\beta)(\alpha\dot\alpha)} = -\frac{3\lambda}{2}\mathcal{D}_{[(\beta\dot\beta)}G_{(\alpha\dot\alpha)]} - \frac{i}{4}\epsilon_{\beta\alpha}\mathcal{D}_{\{\dot\beta}X_{\dot\alpha\}} - \frac{i}{4}\epsilon_{\dot\beta\dot\alpha}\mathcal{D}_{\{\beta}X_{\alpha\}} \tag{3.4.20}$$

Again note the simplifications which occur for the choice $\lambda = 0$.

Deformation

The compensated derivatives (for $\lambda = 0$) can be compactly written as

$$\mathcal{D}_\alpha \equiv \nabla_\alpha + \frac{1}{4}(\nabla^\beta U)\{S_\beta, Q_\alpha\}, \quad \mathcal{D}^{\dot\alpha} \equiv \nabla^{\dot\alpha} + \frac{1}{4}(\nabla_{\dot\beta} U)\{\bar{S}^{\dot\beta}, \bar{Q}^{\dot\alpha}\}$$
$$\mathcal{D}_{\alpha\dot\alpha} \equiv \frac{i}{2}\{\mathcal{D}_\alpha, \mathcal{D}_{\dot\alpha}\}$$

provided we restrict them to only act on conformally primary objects. It is in this form that it is easiest to demonstrate that if Ψ is primary, so is $\mathcal{D}_\alpha \Psi$ where Ψ possesses arbitrary weights and Lorentz indices.

We have previously argued that to first order the spinor derivatives vary (in Hermitian quantum gauge) as $\delta\nabla_\alpha = [iV, \nabla_\alpha]$ and $\delta\nabla_{\dot\alpha} = [-iV, \nabla_{\dot\alpha}]$, where we had expanded

$$V \equiv V^A \nabla_A + V^{\underline{b}} X_{\underline{b}}$$

It follows then that the compensated spinor derivatives should vary as

$$\begin{aligned}\delta\mathcal{D}_\alpha &= [iV, \nabla_\alpha] + \frac{1}{4}\left([iV, \nabla_\beta]U + \nabla_\beta \delta U\right)\{S_\beta, Q_\alpha\} \\ &= [iV, \mathcal{D}_\alpha] + \frac{1}{4}\nabla^\beta(-iVU + \delta U)\{S_\beta, Q_\alpha\} \end{aligned} \tag{3.4.21}$$

where we have substituted $\mathcal{D}$ for ∇ in the commutator. Note that $(-iVU + \delta U)$ is conformally primary of dimension zero, and so we may replace the ∇^β acting on it with $\mathcal{D}^\beta$. Further simplifications arise if we choose to expand V in terms of the compensated derivative rather than the covariant derivative:

$$V = V^A \nabla_A + V^{\underline{b}} X_{\underline{b}} = V^{A'} \mathcal{D}_A + V^{\underline{b}'} X_{\underline{b}}$$

One may check that the V''s are now conformally primary objects. In particular, it is easy to show (by considering the variation of a chiral superfield of vanishing weight for example) that

$$V'^a = V^a, \quad \tilde{V}'^\alpha = \tilde{V}^\alpha \tag{3.4.22}$$

where $V'^\alpha \equiv -\frac{i}{8}\mathcal{D}_{\dot\phi} V'^{\dot\phi\alpha} + \tilde{V}'^\alpha$. Then provided we define a theory entirely in terms of V^a and $\tilde{V}^\alpha$, we can make use of these conformally invariant derivatives when we calculate deformations of the quantum theory.

Henceforth we suppress the primes and trade the conformally covariant prepotentials for the conformally invariant (or compensated) ones. One can show that

$$V(D) = \frac{1}{2}\mathcal{D}_b V^b + \frac{1}{2}\mathcal{D}^\alpha V_\alpha + \frac{1}{2}\mathcal{D}_{\dot\alpha} V^{\dot\alpha} + \tilde{V}(D) \tag{3.4.23}$$

$$V(A) = -\frac{1}{4}\Delta_b V^b + V^b G_b - \frac{3i}{4}\mathcal{D}^\alpha V_\alpha + \frac{3i}{4}\mathcal{D}_{\dot\alpha} V^{\dot\alpha} + \tilde{V}(A) \tag{3.4.24}$$

$$V(M)_{\beta\alpha} = +\frac{1}{2}\mathcal{D}_{\{\beta} V_{\alpha\}} + \frac{i}{8}\mathcal{D}^{\dot\phi}\mathcal{D}_{\{\beta} V_{\alpha\}\dot\phi} + \frac{i}{2} V_{\{\alpha\dot\phi} G_{\beta\}}{}^{\dot\phi} + \tilde{V}(M)_{\beta\alpha} \tag{3.4.25}$$

$$V(M)_{\dot\beta\dot\alpha} = +\frac{1}{2}\mathcal{D}_{\{\dot\beta} V_{\dot\alpha\}} - \frac{i}{8}\mathcal{D}^{\phi}\mathcal{D}_{\{\dot\beta} V_{\dot\alpha\}\phi} - \frac{i}{2} V_{\{\dot\alpha}{}^{\phi} G_{\dot\beta\}\phi} + \tilde{V}(M)_{\dot\beta\dot\alpha} \tag{3.4.26}$$

Note the forms are quite similar to what we had in (3.2.35), except for the appearance of the new superfield G_b. We have also introduced the conformally invariant operator $\Delta_{\alpha\dot\alpha} = -\frac{1}{2}[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}]$.

Since U obeys $\mathcal{D}_A U = 0$, it follows that

$$\delta\mathcal{D}_\alpha = [iV, \mathcal{D}_\alpha] + \frac{1}{4}\nabla^\beta(-2iV(D) + \delta U)\{S_\beta, Q_\alpha\} \tag{3.4.27}$$

from which we may derive the variations of each of the spinor connections. We find

$$\begin{aligned}
H_{\alpha\beta} &= i\mathcal{D}_\alpha V_\beta - iV^c T_{\alpha c\beta} - iV(M)_{\alpha\beta} + \frac{i}{2}V(D)\epsilon_{\alpha\beta} + V(A)\epsilon_{\alpha\beta} \\
H_{\alpha\dot\beta} &= i\mathcal{D}_\alpha V_{\dot\beta} - iV^c T_{\alpha c\dot\beta} \\
H_{\alpha(\beta\dot\beta)} &= i\mathcal{D}_\alpha V_{\beta\dot\beta} + 4V_{\dot\beta}\epsilon_{\alpha\beta} \\
\Lambda_\alpha &= \frac{1}{2}\mathcal{D}_\alpha(\delta U) \\
\omega_\alpha &= i\mathcal{D}_\alpha V(A) - iV^b F_{\alpha b} - \frac{3}{2}\mathcal{D}_\alpha V(D) - \frac{3i}{4}\mathcal{D}_\alpha \delta U \\
\Omega_\alpha(M) &= i\mathcal{D}_\alpha V(M) + 4i\bar{R}V^\beta M_{\beta\alpha} - iV^b R_{\alpha b}(M) - i\mathcal{D}^\beta V(D) M_{\beta\alpha} + \frac{1}{2}\mathcal{D}^\beta \delta U M_{\beta\alpha}
\end{aligned} \tag{3.4.28}$$

and for their conjugates

$$\begin{aligned}
H_{\dot\alpha\beta} &= -i\mathcal{D}_{\dot\alpha} V_\beta + iV^c T_{\dot\alpha c\beta} \\
H_{\dot\alpha\dot\beta} &= -i\mathcal{D}_{\dot\alpha} V_{\dot\beta} + iV^c T_{\dot\alpha c\dot\beta} + iV(M)_{\dot\alpha\dot\beta} + \frac{i}{2}V(D)\epsilon_{\dot\alpha\dot\beta} - V(A)\epsilon_{\dot\alpha\dot\beta} \\
H_{\dot\alpha(\beta\dot\beta)} &= -i\mathcal{D}_{\dot\alpha} V_{\beta\dot\beta} + 4V_\beta \epsilon_{\dot\alpha\dot\beta} \\
\Lambda_{\dot\alpha} &= \frac{1}{2}\mathcal{D}_{\dot\alpha}(\delta U) \\
\omega_{\dot\alpha} &= -i\mathcal{D}_{\dot\alpha} V(A) + iV^b F_{\dot\alpha b} - \frac{3}{2}\mathcal{D}_{\dot\alpha} V(D) + \frac{3i}{4}\mathcal{D}_{\dot\alpha}\delta U \\
\Omega^{\dot\alpha}(M) &= -i\mathcal{D}^{\dot\alpha} V(M) - 4iRV_{\dot\beta} M^{\dot\beta\dot\alpha} + iV_b R^{\dot\alpha b}(M) + i\mathcal{D}_{\dot\beta} V(D) M^{\dot\beta\dot\alpha} + \frac{1}{2}\mathcal{D}_{\dot\beta}\delta U M^{\dot\beta\dot\alpha}
\end{aligned} \tag{3.4.29}$$

The variation of the bosonic derivatives is straightforward to work out from the above results. Using these, one may for example work out the variations of the superfields $G_{\alpha\dot\alpha}$ and R in the language of these compensated derivatives. For R, it is actually easier to work in the original theory at first. Recall the chiral variation of an arbitrary superfield Ψ can be defined by

$$\delta_c\Psi = \delta\Psi + iV\Psi \tag{3.4.30}$$

which generalizes the case where Ψ is itself chiral. Then the chiral variation of R is

$$\delta_c R = -\frac{1}{8X}\bar\nabla^2\left(X\delta_c U\right) + \frac{1}{8X}\delta_c U\bar\nabla^2 X = -\frac{1}{8}\bar{\mathcal{D}}^2\delta_c U \tag{3.4.31}$$

Similarly, the chiral variation of X_α is

$$\begin{aligned}\delta_c X_\alpha &= \frac{3}{8}\bar{\nabla}^2 \left(\nabla_\alpha \delta U + 2iV\nabla_\alpha U - i\nabla_\alpha(VU)\right) \\ &= \frac{3}{8}(\bar{\mathcal{D}}^2 - 8R)\left(\mathcal{D}_\alpha \delta U + 2iZ_\alpha - 2i\mathcal{D}_\alpha V(D)\right)\end{aligned} \tag{3.4.32}$$

where

$$\begin{aligned}Z_\alpha &\equiv V\nabla_\alpha U \\ &= -\frac{1}{2}(\mathcal{D}^2 - 12\bar{R})V_\alpha + \frac{1}{2}\mathcal{D}^{\dot\beta}\mathcal{D}_\alpha V_{\dot\beta} \\ &\quad - \frac{1}{6}\mathcal{D}_\alpha\mathcal{D}_{\beta\dot\beta}V^{\dot\beta\beta} + \frac{i}{3}\mathcal{D}_\alpha(G_{\beta\dot\beta}V^{\dot\beta\beta}) + \frac{i}{24}(\mathcal{D}^2 - 12\bar{R})\mathcal{D}^{\dot\beta}V_{\dot\beta\alpha} \\ &\quad + \mathcal{D}^{\dot\beta}(\bar{R}V_{\alpha\dot\beta}) + \frac{i}{3}X^{\dot\beta}V_{\alpha\dot\beta}\end{aligned} \tag{3.4.33}$$

Calculating $\delta G_{\alpha\dot\alpha}$ is a bit more difficult since its definition in terms of X necessarily involves both dotted and undotted spinor derivatives in a symmetric fashion. The most straightforward way to proceed seems to be to work out its variation by calculating the variation of the torsion component $\delta T_{\gamma b\alpha}$. This gives the following rather complicated expression:

$$\begin{aligned}\delta G_{\alpha\dot\alpha} &= -\frac{1}{4}[\mathcal{D}_\alpha, \mathcal{D}_{\dot\alpha}]\tilde{\delta}U - H_{\alpha\dot\alpha}{}^b G_b - iV^\beta\mathcal{D}_\beta G_{\alpha\dot\alpha} - iV^{\dot\beta}\mathcal{D}_{\dot\beta}G_{\alpha\dot\alpha} \\ &\quad - \frac{1}{2}\Delta_{\alpha\dot\alpha}\Delta_b V^b - \frac{1}{2}\mathcal{D}_{\alpha\dot\alpha}\mathcal{D}_b V^b - \frac{1}{32}(\mathcal{D}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D} + \text{h.c.})V_{\alpha\dot\alpha} \\ &\quad + \frac{1}{2}(\mathcal{D}^\gamma V_{\dot\alpha}{}^\beta)W_{\gamma\beta\alpha} + \frac{1}{2}(\mathcal{D}^{\dot\gamma} V_\alpha{}^{\dot\beta})W_{\dot\gamma\dot\beta\dot\alpha} - \frac{1}{2}\Delta_{\alpha\dot\alpha}(V^b G_b) \\ &\quad + \frac{1}{8}\mathcal{D}^\beta V_{\beta\dot\alpha}\mathcal{D}_\alpha R + \frac{1}{8}\mathcal{D}_\alpha V_{\beta\dot\alpha}\mathcal{D}^\beta R + \frac{1}{6}\mathcal{D}_{\{\beta}V_{\alpha\}\dot\alpha}X^\beta \\ &\quad - \frac{1}{8}\mathcal{D}^{\dot\beta} V_{\dot\beta\alpha}\mathcal{D}_{\dot\alpha} R - \frac{1}{8}\mathcal{D}_{\dot\alpha}V_{\dot\beta\alpha}\mathcal{D}^{\dot\beta}R - \frac{1}{6}\mathcal{D}_{\{\dot\beta}V_{\dot\alpha\}\alpha}X^{\dot\beta} \\ &\quad - \bar{R}RV_{\alpha\dot\alpha} - \frac{1}{4}V_{\alpha\dot\alpha}\mathcal{D}^\beta X_\beta \\ &\quad - \frac{1}{2}(\Delta_{\alpha\dot\alpha}V^b)G_b - \frac{1}{2}\Delta_b(V^b G_{\alpha\dot\alpha}) + \frac{1}{2}(\Delta_b V^b)G_{\alpha\dot\alpha} \\ &\quad + \frac{i}{4}V_{\beta\dot\alpha}\mathcal{D}^{\dot\beta\beta}G_{\alpha\dot\beta} - \frac{i}{4}V_{\alpha\dot\beta}\mathcal{D}^{\dot\beta\beta}G_{\beta\dot\alpha}\end{aligned}$$

where we have defined

$$\tilde{\delta}U \equiv \delta U + i\mathcal{D}^\beta V_\beta - i\mathcal{D}_{\dot\beta}V^{\dot\beta} + \Delta_b V^b. \tag{3.4.34}$$

For the linear compensator model, $\tilde{\delta}U = L^{-1}\mathcal{L}$, but for the generic chiral model

$$\tilde{\delta}U = -\frac{1}{3}\left(\mathcal{K}_i\eta^i + \mathcal{K}_{\bar{j}}\bar{\eta}^{\bar{j}} - 2\Delta_b V^b - 4V^b G_b + V^b\mathcal{K}_b\right) \tag{3.4.35}$$

The expression for $\delta G_{\alpha\dot{\alpha}}$ involves a combination of the supergravity potentials that has been succinctly combined into $H_a{}^b$, which is the deformation of the bosonic vierbein. It can be calculated from

$$\delta\mathcal{D}_a = -H_a{}^B\mathcal{D}_B - H_a{}^{\underline{b}}X_{\underline{b}},$$

the left hand side of which can itself be calculated easily from $\delta\mathcal{D}_\alpha$ and $\delta\mathcal{D}_{\dot{\alpha}}$. The reason for collecting these terms in this way is that we will eventually find they cancel out.

Rearranging a number of terms leads to

$$\begin{aligned}
\delta G_{\alpha\dot{\alpha}} =& \frac{1}{2}\Delta_{\alpha\dot{\alpha}}\tilde{\delta}U - H_{\alpha\dot{\alpha}}{}^b G_b - iV^\beta\mathcal{D}_\beta G_{\alpha\dot{\alpha}} - iV^{\dot{\beta}}\mathcal{D}_{\dot{\beta}}G_{\alpha\dot{\alpha}} \\
&- \frac{1}{2}\Delta_{\alpha\dot{\alpha}}\Delta_b V^b - \frac{1}{2}\mathcal{D}_{\alpha\dot{\alpha}}\mathcal{D}_b V^b - \frac{1}{32}\{\mathcal{D}^2, \bar{\mathcal{D}}^2\}V_{\alpha\dot{\alpha}} + \frac{1}{2}\Box V_{\alpha\dot{\alpha}} \\
&+ \frac{1}{2}(R\mathcal{D}^2 + \bar{R}\bar{\mathcal{D}}^2)V_{\alpha\dot{\alpha}} + (\mathcal{D}^\gamma V_{\dot{\alpha}}{}^\beta)W_{\gamma\beta\alpha} + (\mathcal{D}^{\dot{\gamma}}V_\alpha{}^{\dot{\beta}})\bar{W}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} \\
&- G^b\Delta_b V_{\alpha\dot{\alpha}} - (\Delta_{\alpha\dot{\alpha}}V^b)G_b - \Delta_b(V^b G_{\alpha\dot{\alpha}}) + (\Delta_b V^b)G_{\alpha\dot{\alpha}} + \frac{1}{2}V^b\Delta_{[b}G_{(\alpha\dot{\alpha})]} \\
&+ \frac{1}{2}\mathcal{D}^\beta V_{\beta\dot{\alpha}}(\mathcal{D}_\alpha R - \frac{1}{3}X_\alpha) + \frac{1}{12}\mathcal{D}^\beta V_{\alpha\dot{\alpha}}X_\beta \\
&- \frac{1}{2}\mathcal{D}^{\dot{\beta}}V_{\dot{\beta}\alpha}(\mathcal{D}_{\dot{\alpha}}\bar{R} - \frac{1}{3}X_{\dot{\alpha}}) + \frac{1}{12}\mathcal{D}_{\dot{\beta}}V_{\alpha\dot{\alpha}}X^{\dot{\beta}} \\
&- \bar{R}RV_{\alpha\dot{\alpha}} - \frac{1}{8}V_{\alpha\dot{\alpha}}(\mathcal{D}^\beta X_\beta + \text{h.c.}) + \frac{i}{4}V_{\beta\dot{\alpha}}\mathcal{D}^{\dot{\beta}\beta}G_{\alpha\dot{\beta}} - \frac{i}{4}V_{\alpha\dot{\beta}}\mathcal{D}^{\dot{\beta}\beta}G_{\beta\dot{\alpha}}
\end{aligned} \tag{3.4.36}$$

3.4.2 Proceeding to second order

We would like to proceed to second order so that we can perform one-loop calculations. The immediate difficulty we face is that we solved our constraints only to first order. For example, $\tilde{V}(A)$ might also involve some second order object of the form $V^a\mathcal{O}_a{}^bV_b$ where $\mathcal{O}_a{}^b$ is some conformally invariant operator. Then in analyzing the variations of the $\mathcal{W}$'s, we should have worked to second order in V^a to find out if any such object exists.

There are two approaches one could take at this point. One would be to return to the original analysis and redo it to second order and determine what modifications are necessary. The second approach is to use our ability to take first order variations and to vary to first order the first order action that we already have – thereby bootstrapping to second order. This is possible since our first order solution was not dependent on any specific origin point on the constraint surface of conformal supergravity; it merely required that we remain *somewhere* on that surface.

This latter approach is the one we will take. The main difficulty is figuring out how to vary the quantum superfields V^a and Σ^r. On the one hand, varying these only shifts the action by a term proportional to the equations of motion, so it's not an immediate issue if we choose to work on shell. On the other, if there is some sort of natural variation of these objects, then we can possibly simplify the second-order action without the need to apply the equations of motion.

We begin by considering a primary chiral superfield of vanishing weights. In this way its variation can be defined solely in terms of V^a and $\tilde{V}^\alpha$. Then varying Φ in the most natural way amounts to

$$\Phi' = e^{-iV}(\Phi + \eta) = \Phi - iV\Phi + \eta - \frac{1}{2}V^2\Phi - iV\eta + \mathcal{O}(V^3) \tag{3.4.37}$$

where we have stopped the expansion at second order. Demanding that the second order terms agree with the first order variation of the first order terms gives

$$\delta\eta = -iV\eta, \quad \delta(V\Phi) = -iV^2\Phi \tag{3.4.38}$$

(In the calculation one must include an additional factor of 2 since the second variation is generated from half of the first order variation squared.) The first is a perfectly sensible definition (it amounts to $\delta_c\eta = 0$) and the second implies for the variations of V^a and Σ^r

$$\begin{aligned}
\delta V_{\alpha\dot\alpha} &= -8V_\alpha V_{\dot\alpha} + iV^\beta \mathcal{D}_\beta V_{\alpha\dot\alpha} - iV_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta} V_{\alpha\dot\alpha} + V^b H_{b(\alpha\dot\alpha)} \\
\delta\Sigma &= iV^\alpha \mathcal{D}_\alpha \Sigma - iV_{\dot\alpha}\mathcal{D}^{\dot\alpha}\Sigma + 2iV^\alpha V^b F_{b\alpha} + 2iV^{\dot\alpha} V^b F_{b\dot\alpha} - V^a \Delta_a \Sigma \\
&\quad + V^{\dot\alpha\alpha}\left(-\frac{1}{2}\mathcal{D}_\alpha V^b F_{b\dot\alpha} + \frac{1}{2}\mathcal{D}_{\dot\alpha} V^b F_{b\alpha} - \frac{1}{4}V^b \mathcal{D}_\alpha F_{b\dot\alpha} + \frac{1}{4}V^b \mathcal{D}_{\dot\alpha} F_{b\alpha}\right)
\end{aligned} \tag{3.4.39}$$

In the last equation we have suppressed the r index to simplify notation.

Note that $\delta V_a \ni V^b H_b{}^a$ and $\delta G_a = -H_a{}^b G_b$ and so there will be no $H_b{}^a$ in terms like $\delta(V^a G_a)$. We will similarly identify the combination $H_a{}^b$ in the variation of $\mathcal{K}_a$ and $\mathcal{Y}_a$ so that this cancellation occurs for these terms as well.

Variation of the η term

Beginning with

$$\delta_\eta S = \int \mathcal{E}\,\eta^i \left(X\mathcal{P}\mathcal{K}_i + P_i + \mathcal{Y}_i\right) + \text{h.c.} \tag{3.4.40}$$

we consider the effect of a second variation. Given the presence of η^i, it is most sensible to work in quantum chiral gauge where η^i has no further variation.

Taking the superpotential term, one finds simply

$$\delta\delta_\eta S \ni \int \mathcal{E}\,\eta^i\eta^j P_{ij} \tag{3.4.41}$$

The gauge field term is a bit more complicated:

$$\delta\delta_\eta S \ni \int \mathcal{E}\,\eta^i \left(\frac{1}{4}\eta^j f_{rs,ij} W^{\alpha r} W^s_\alpha + \frac{1}{2} f_{rs,i} W^{\alpha r}\delta_c W^s_\alpha\right) \tag{3.4.42}$$

Plugging in $\delta_c W^s_\alpha$ gives

$$\delta\delta_\eta S \ni \frac{1}{4}\int \mathcal{E}\,\eta^i\eta^j f_{rs,ij} W^{\alpha r} W^s_\alpha + \frac{1}{2}\int E\,\eta^i f_{rs,i} W^{\alpha r}\left(i\nabla_\alpha \Sigma^s + V_{\alpha\dot\alpha}\bar{W}^{\dot\alpha s}\right) \tag{3.4.43}$$

The term involving X and $\mathcal{K}_i$ is the most difficult to deal with. We rewrite it as a full superspace integral and then take the chiral quantum variation[3]

$$\delta\delta_\eta S \ni \delta_c X \eta^i \mathcal{K}_i + X\eta^i \mathcal{K}_{i\bar{j}} \left(\bar{\eta}^{\bar{j}} + 2iV^B X_B \bar{\Phi}^{\bar{j}}\right) + X\eta^i \mathcal{K}_{ij}\eta^j \tag{3.4.44}$$

The last term we will consider in tandem with P_{ij}. The second term can be simplified by noting that when $X_B = D$ or A, the result simplifies. First note

$$D\mathcal{K}_i = -\Delta_i \mathcal{K}_i = +\mathcal{K}_{ij}\Delta_j \Phi^j + \mathcal{K}_{i\bar{j}}\Delta_{\bar{j}}\bar{\Phi}^{\bar{j}} \tag{3.4.45}$$

$$\frac{3i}{2} A\mathcal{K}_i = \Delta_i \mathcal{K}_i = -\mathcal{K}_{ij}\Delta_j \Phi^j + \mathcal{K}_{i\bar{j}}\Delta_{\bar{j}}\bar{\Phi}^{\bar{j}} \tag{3.4.46}$$

which together imply

$$0 = \mathcal{K}_{i\bar{j}}\Delta_{\bar{j}}\bar{\Phi}^{\bar{j}}. \tag{3.4.47}$$

This gives

$$\mathcal{K}_{i\bar{j}}\eta^i \bar{\eta}^{\bar{j}} + 2i\eta^i \mathcal{K}_{i\bar{j}} \left(V^b \mathcal{D}_b + \Sigma^r X_r\right) \bar{\Phi}^{\bar{j}} + 2iV_{\dot{\alpha}}\mathcal{D}^{\dot{\alpha}}(\eta^i \mathcal{K}_i) \tag{3.4.48}$$

Next we observe that $\delta_c X$ is equivalent to

$$\delta_c X = X\delta U + iVX = X\tilde{\delta}U - iX\mathcal{D}^\beta V_\beta + iX\mathcal{D}_{\dot{\beta}}V^{\dot{\beta}} - X\Delta_b V^b + 2iV(D)X \tag{3.4.49}$$

where we have used (3.4.34) again. Plugging this in and using several integrations by parts, we can show that the total variation of this term is

$$\delta\delta_\eta S \ni X\left(i\mathcal{D}_b V^b K_i \eta^i - \Delta_b V^b K_i \eta^i + 2i\eta^i K_{i\bar{j}}(V+\Sigma)\bar{\Phi}^{\bar{j}} + \tilde{\delta}U K_i \eta^i + K_{ij}\eta^i\eta^j + K_{i\bar{j}}\eta^i\bar{\eta}^{\bar{j}}\right) \tag{3.4.50}$$

The combination $(V+\Sigma)$ is shorthand for $(V^b\mathcal{D}_b + \Sigma^r X_r)$. Note that the terms involving V^α and $V_{\dot{\alpha}}$ have dropped out. We can simplify this expression by combining the first two terms and then integrating by parts. The result is

$$\delta\delta_\eta S \ni X\left(-\frac{1}{2}V^{\dot{\alpha}\alpha}\mathcal{D}_\alpha(K_{i\bar{j}}\eta^i)\mathcal{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}} + 2i\eta^i K_{i\bar{j}}\Sigma\bar{\Phi}^{\bar{j}} + \tilde{\delta}U K_i\eta^i + K_{ij}\eta^i\eta^j + K_{i\bar{j}}\eta^i\bar{\eta}^{\bar{j}}\right) \tag{3.4.51}$$

Combining this with everything else yields

$$\begin{aligned}\delta\delta_\eta S = &\left[\eta^i(\mathcal{P}XK_{ij} + P_{ij} + \mathcal{Y}_{ij})\eta^j\right]_F + \text{h.c.} \\ &+ \left[X\eta^i K_{i\bar{j}}\bar{\eta}^{\bar{j}} + X\tilde{\delta}U\eta^i K_i + \eta^i\left(XK_{i,r} + \mathcal{Y}_{i,r}\right)\Sigma^r + XV^a K_{a,i}\eta^i + V^a\mathcal{Y}_{a,i}\eta^i\right]_D + \text{h.c.}\end{aligned} \tag{3.4.52}$$

[3]We have written this and many subsequent D-terms without an overall $\int E$ or with the brackets $[\ ]_D$ to keep the formulae from growing cluttered.

where we have defined

$$\mathcal{K}_{i,r} \equiv 2i\mathcal{K}_{i\bar{j}} X_r \bar{\Phi}^{\bar{j}} \tag{3.4.53}$$

$$\mathcal{Y}_{i,r} \equiv \frac{i}{2} f_{rs,i} W^{\alpha s} \mathcal{D}_\alpha = \frac{i}{2} G_{rs,i} W^{\alpha s} \mathcal{D}_\alpha \tag{3.4.54}$$

$$\mathcal{K}_{\alpha\dot{\alpha},i}\eta^i \equiv -\mathcal{D}_{\dot{\alpha}} \bar{\Phi}^{\bar{j}} \mathcal{D}_\alpha \left(K_{i\bar{j}}\eta^i\right) \tag{3.4.55}$$

$$\mathcal{Y}_{\alpha\dot{\alpha},i} \equiv -f_{rs,i} W^r_\alpha \bar{W}^s_{\dot{\alpha}} = -G_{rs,i} W^r_\alpha \bar{W}^s_{\dot{\alpha}} \tag{3.4.56}$$

Variation of the Σ term

The Σ term is

$$\int E\, \Sigma^r (\mathcal{Y}_r + X K_r)$$

where we recall

$$\mathcal{Y}_r \equiv -\frac{i}{2} \nabla^\alpha \left(f_{rs} W_\alpha{}^s\right) + \text{h.c.}$$
$$K_r \equiv -iK_i X_r \Phi^i + iK_{\bar{j}} X_r \bar{\Phi}^{\bar{j}}$$

The variation of the first term is given by using the formula

$$\begin{aligned}\delta(\nabla^\alpha \Phi_\alpha) = &- i\nabla^\beta (V_\beta \nabla^\alpha \Phi_\alpha) + i\nabla_{\dot{\beta}} (V^{\dot{\beta}} \nabla^\alpha \Phi_\alpha) - \Delta_b (V^b \nabla^\alpha \Phi_\alpha) + 2V^{\dot{\alpha}\alpha} \bar{W}_{\dot{\alpha}} \Phi_\alpha \\ &- i\Sigma \nabla^\alpha \Phi_\alpha - 2i(\nabla^\alpha \Sigma^r) X_r \Phi_\alpha + \frac{1}{4} \nabla_{\dot{\alpha}} \nabla^2 (V^{\dot{\alpha}\alpha} \Phi_\alpha) + \nabla^\alpha (\delta_c \Phi_\alpha)\end{aligned} \tag{3.4.57}$$

where Φ_α is an arbitrary chiral spinor superfield. This is written in terms of the old V_β and $V^{\dot{\beta}}$. Exchanging for the new conformally invariant ones gives

$$\begin{aligned}\delta(\nabla^\alpha \Phi_\alpha) = &- i\mathcal{D}^\beta (V_\beta \nabla^\alpha \Phi_\alpha) + i\mathcal{D}_{\dot{\beta}} (V^{\dot{\beta}} \nabla^\alpha \Phi_\alpha) - \Delta_b (V^b \nabla^\alpha \Phi_\alpha) + 2V^{\dot{\alpha}\alpha} \bar{W}_{\dot{\alpha}} \Phi_\alpha \\ &- i\Sigma \nabla^\alpha \Phi_\alpha - 2i(\mathcal{D}^\alpha \Sigma^r) X_r \Phi_\alpha + \frac{1}{4} \nabla_{\dot{\alpha}} \nabla^2 (V^{\dot{\alpha}\alpha} \Phi_\alpha) + \mathcal{D}^\alpha (\delta_c \Phi_\alpha)\end{aligned} \tag{3.4.58}$$

In this formula, we have mixed conventions with ∇'s and $\mathcal{D}$'s appearing in the same expression. Every isolated ∇_α (or $\nabla_{\dot{\alpha}}$) here is equivalent to $\mathcal{D}_\alpha$ (or $\mathcal{D}_{\dot{\alpha}}$), while ∇^2 is equivalent to $\mathcal{D}^2 - 8\bar{R}$. Δ_b is in terms of $\mathcal{D}$ and this will remain the case for the rest of this work.

Applying this formula to $\mathcal{Y}_r$ gives

$$\begin{aligned}\delta \mathcal{Y}_r = &- i\mathcal{D}^\beta (V_\beta \mathcal{Y}_r) + i\mathcal{D}_{\dot{\beta}} (V^{\dot{\beta}} \mathcal{Y}_r) - \Delta_b (V^b \mathcal{Y}_r) \\ &+ iV^{\dot{\alpha}\alpha} W_{\dot{\alpha}}{}^s W_\alpha{}^u f_{sr}{}^t f_{tu} + iV_{\alpha\dot{\alpha}} W^{\alpha s} \bar{W}^{\dot{\alpha}u} f_{sr}{}^t \bar{f}_{tu} \\ &+ \frac{i}{8} \nabla^\alpha \bar{\nabla}^2 (V_{\alpha\dot{\alpha}} G_{rs} \bar{W}^{\dot{\alpha}r}) - \frac{i}{8} \nabla_{\dot{\alpha}} \nabla^2 (V^{\dot{\alpha}\alpha} G_{rs} W_\alpha{}^s) \\ &+ i\Sigma^s f_{sr}{}^t \mathcal{Y}_t + \mathcal{D}^\alpha \Sigma^s f_{sr}{}^t W_\alpha{}^u f_{tu} - \bar{\mathcal{D}}_{\dot{\alpha}} \Sigma^s f_{sr}{}^t \bar{W}^{\dot{\alpha}u} \bar{f}_{tu} \\ &- \frac{1}{8} \nabla^\alpha \left(f_{rs} \bar{\nabla}^2 \nabla_\alpha \Sigma^s\right) - \frac{1}{8} \nabla_{\dot{\alpha}} \left(\bar{f}_{rs} \nabla^2 \nabla^{\dot{\alpha}} \Sigma^s\right) \\ &- \frac{i}{2} \mathcal{D}^\alpha (\eta^i f_{rs,i} W_\alpha{}^s) - \frac{i}{2} \mathcal{D}_{\dot{\alpha}} (\bar{\eta}^{\bar{j}} f_{rs,\bar{j}} \bar{W}^{\dot{\alpha}s})\end{aligned} \tag{3.4.59}$$

Including the variation of Σ and integrating by parts gives

$$\begin{aligned}\delta(\Sigma^r \mathcal{Y}_r) = &\left(2iV^\alpha \mathcal{D}_\alpha \Sigma^r - 2iV_{\dot\alpha}\mathcal{D}^{\dot\alpha}\Sigma^r + 2iV^\alpha V^b F_{b\alpha} + 2iV^{\dot\alpha} V^b F_{b\dot\alpha}\right)\mathcal{Y}_r \\ &+ V^{\dot\alpha\alpha}\left(-\frac{1}{2}\mathcal{D}_\alpha V^b F_{b\dot\alpha} + \frac{1}{2}\mathcal{D}_{\dot\alpha} V^b F_{b\alpha} - \frac{1}{4}V^b \mathcal{D}_\alpha F_{b\dot\alpha} + \frac{1}{4}V^b \mathcal{D}_{\dot\alpha} F_{b\alpha}\right)\mathcal{Y}_r \\ &- 2V^a(\Delta_a \Sigma^r)\mathcal{Y}_r \\ &+ \Sigma^r V^a \mathcal{Y}_{a,r} + \frac{i}{8}\Sigma^r \nabla^\alpha \bar\nabla^2 (V_{\alpha\dot\alpha} G_{rs} \bar W^{\dot\alpha r}) - \frac{i}{8}\Sigma^r \nabla_{\dot\alpha}\nabla^2(V^{\dot\alpha\alpha} G_{rs} W_\alpha{}^s) \\ &+ \Sigma^r \mathcal{D}^\alpha \Sigma^s f_{sr}{}^t W_\alpha{}^u f_{tu} - \Sigma^r \bar{\mathcal{D}}_{\dot\alpha}\Sigma^s f_{sr}{}^t \bar W^{\dot\alpha u} \bar f_{tu} \\ &- \frac{1}{8}\Sigma^r \nabla^\alpha \left(f_{rs}\bar\nabla^2 \nabla_\alpha \Sigma^s\right) - \frac{1}{8}\Sigma^r \nabla_{\dot\alpha}\left(\bar f_{rs}\nabla^2 \nabla^{\dot\alpha}\Sigma^s\right) \\ &+ \eta^i \mathcal{Y}_{ir}\Sigma^r + \bar\eta^{\bar j}\mathcal{Y}_{\bar j r}\Sigma^r \end{aligned} \tag{3.4.60}$$

where we have defined

$$\mathcal{Y}_{\alpha\dot\alpha,r} \equiv -2i\left(\bar W^s_{\dot\alpha} W^u_\alpha f_{sr}{}^t f_{tu} + W^s_\alpha \bar W^u_{\dot\alpha} f_{sr}{}^t \bar f_{tu}\right) \tag{3.4.61}$$

Varying $\mathcal{K}_r$ gives

$$\begin{aligned}\delta\mathcal{K}_r = &-iV^\beta \mathcal{D}_\beta \mathcal{K}_r + iV_{\dot\beta}\mathcal{D}^{\dot\beta}\mathcal{K}_r + \eta^i \mathcal{K}_{ir} + \bar\eta^{\bar j}\mathcal{K}_{\bar j r} \\ &+ 2\mathcal{K}_{i\bar j}(V+\Sigma)\Phi^i X_r \bar\Phi^{\bar j} + 2\mathcal{K}_{i\bar j} X_r \Phi^i (V+\Sigma)\bar\Phi^{\bar j}\end{aligned} \tag{3.4.62}$$

where again

$$V + \Sigma \equiv V^b \mathcal{D}_b + \Sigma^r X_r$$

Including the variation of X and Σ^r gives

$$\begin{aligned}X^{-1}\delta(\Sigma^r X \mathcal{K}_r) = &\left(2iV^\alpha \mathcal{D}_\alpha \Sigma^r - 2iV_{\dot\alpha}\mathcal{D}^{\dot\alpha}\Sigma^r + 2iV^\alpha V^b F_{b\alpha} + 2iV^{\dot\alpha} V^b F_{b\dot\alpha}\right)\mathcal{K}_r \\ &+ V^{\dot\alpha\alpha}\left(-\frac{1}{2}\mathcal{D}_\alpha V^b F_{b\dot\alpha} + \frac{1}{2}\mathcal{D}_{\dot\alpha} V^b F_{b\alpha} - \frac{1}{4}V^b \mathcal{D}_\alpha F_{b\dot\alpha} + \frac{1}{4}V^b \mathcal{D}_{\dot\alpha} F_{b\alpha}\right)\mathcal{K}_r \\ &- V^a \Delta_a \Sigma^r \mathcal{K}_r - (\Delta_b V^b)\Sigma^r \mathcal{K}_r + \tilde\delta U \Sigma^r \mathcal{K}_r \\ &+ 4\mathcal{K}_{i\bar j}\Sigma\Phi^i \Sigma\bar\Phi^{\bar j} + 2\mathcal{K}_{i\bar j} V\Phi^i \Sigma\bar\Phi^{\bar j} + 2\mathcal{K}_{i\bar j}\Sigma\Phi^i V\bar\Phi^{\bar j} \\ &+ \Sigma^r \mathcal{K}_{i,r}\eta^i + \Sigma^r \mathcal{K}_{\bar j,r}\bar\eta^{\bar j}\end{aligned} \tag{3.4.63}$$

Variation of the V^a term

The V^a term is

$$\left[V^b(-4XG_b + XK_b + \mathcal{Y}_b)\right]_D \tag{3.4.64}$$

We require the variations of $G_{\alpha\dot\alpha}$, $K_{\alpha\dot\alpha}$, and $\mathcal{Y}_{\alpha\dot\alpha}$ in order to continue.

The variation of $G_{\alpha\dot\alpha}$ contains the graviton kinetic term. We have already worked this out in (3.4.36), but we rewrite it here in the compact and useful form

$$\begin{aligned}
X^{-1}\delta(XG_{\alpha\dot\alpha}) =& \tilde\delta U G_{\alpha\dot\alpha} + \frac{1}{2}\Delta_{\alpha\dot\alpha}\tilde\delta U - H_{\alpha\dot\alpha}{}^{b}G_b - i\mathcal{D}^{\beta}(V_\beta G_{\alpha\dot\alpha}) + i\mathcal{D}_{\dot\beta}(V^{\dot\beta}G_{\alpha\dot\alpha}) \\
&- \frac{1}{2}\Delta_{\alpha\dot\alpha}\Delta_b V^b - \frac{1}{2}\mathcal{D}_{\alpha\dot\alpha}\mathcal{D}_b V^b - \frac{1}{32}\{\mathcal{D}^2, \bar{\mathcal{D}}^2\}V_{\alpha\dot\alpha} + \frac{1}{2}\Box_V V_{\alpha\dot\alpha} \\
&- G_{\alpha\dot\alpha}\Delta_b V^b - \Delta_{\alpha\dot\alpha}(V^b G_b) + \frac{1}{2}\mathcal{D}^{\beta}(R\mathcal{D}_\beta V_{\alpha\dot\alpha}) + \frac{1}{2}\mathcal{D}_{\dot\beta}(\bar R\mathcal{D}^{\dot\beta}V_{\alpha\dot\alpha}) \\
&- \frac{1}{2}\mathcal{D}_\alpha V_{\dot\alpha}{}^{\beta}X_\beta + \frac{1}{2}\mathcal{D}_{\dot\alpha}V_{\alpha}{}^{\dot\beta}X_{\dot\beta} - \frac{1}{2}V_b\mathcal{D}_c G_d\epsilon_{dcba}\sigma^a_{\alpha\dot\alpha} \\
&- \frac{1}{8}V^{\dot\alpha\alpha}(\mathcal{D}^2 R + \bar{\mathcal{D}}^2\bar R) - R\bar R V_{\alpha\dot\alpha} + \frac{1}{2}V^{\dot\alpha\alpha}\Delta_b G^b + \frac{1}{2}V^b\Delta_{[(\alpha\dot\alpha)}G_{b]}
\end{aligned} \tag{3.4.65}$$

where we have defined

$$\Box_V V_{\alpha\dot\alpha} \equiv \Box V_{\alpha\dot\alpha} - \frac{1}{2}\mathcal{D}^{[\beta}(G_{\beta\dot\beta}\mathcal{D}^{\dot\beta]}V_{\alpha\dot\alpha}) + \frac{1}{2}\mathcal{D}^{\gamma}V^{\dot\beta\beta}W_{\gamma(\beta\dot\beta)(\alpha\dot\alpha)} + \frac{1}{2}\mathcal{D}^{\dot\gamma}V^{\dot\beta\beta}W_{\dot\gamma(\beta\dot\beta)(\alpha\dot\alpha)} \tag{3.4.66}$$

$W_{\gamma ba}$ and its conjugate are defined by

$$\begin{aligned}
R_{\dot\delta(\gamma\dot\gamma)ba} &= 2i\epsilon_{\dot\delta\dot\gamma}W_{\gamma ba} \\
R_{\delta(\gamma\dot\gamma)ba} &= 2i\epsilon_{\delta\gamma}W_{\dot\gamma ba}
\end{aligned} \tag{3.4.67}$$

The variation we need is

$$\begin{aligned}
X^{-1}\delta(-4XV^aG_a) =& -8iV^\beta\mathcal{D}_\beta V^a G_a + 8iV_{\dot\beta}\mathcal{D}^{\dot\beta}V^a G_a - 16V^\alpha V^{\dot\alpha}G_{\alpha\dot\alpha} \\
&- 4\tilde\delta U(V^a G_a) - 2\Delta_a V^a\tilde\delta U \\
&+ 2(\Delta_b V^b)^2 - 2(\mathcal{D}_b V^b)^2 - \frac{1}{8}\mathcal{D}^2 V^{\dot\alpha\alpha}\bar{\mathcal{D}}^2 V_{\alpha\dot\alpha} + V^{\dot\alpha\alpha}\Box_V V_{\alpha\dot\alpha} \\
&+ 8V^a G_a\Delta_b V^b \\
&+ V^{\dot\alpha\alpha}\mathcal{D}^\beta(R\mathcal{D}_\beta V_{\alpha\dot\alpha}) + V^{\dot\alpha\alpha}\mathcal{D}_{\dot\beta}(\bar R\mathcal{D}^{\dot\beta}V_{\alpha\dot\alpha}) \\
&- V^{\dot\alpha\alpha}\mathcal{D}_\alpha V_{\dot\alpha}{}^{\beta}X_\beta + V^{\dot\alpha\alpha}\mathcal{D}_{\dot\alpha}V_\alpha{}^{\dot\beta}X_{\dot\beta} \\
&- \frac{1}{4}V^{\dot\alpha\alpha}V_{\alpha\dot\alpha}(\mathcal{D}^2 R + \bar{\mathcal{D}}^2\bar R) - 2R\bar R V^{\dot\alpha\alpha}V_{\alpha\dot\alpha} + V^{\dot\alpha\alpha}V_{\alpha\dot\alpha}\Delta_b G^b
\end{aligned} \tag{3.4.68}$$

Note that the combination $H_a{}^b$ cancels out of the expression.

Turning to the variation of the matter term, we begin by noting that $\mathcal{K}_{\alpha\dot\alpha}$ may be written a number of equivalent ways

$$\begin{aligned}
\mathcal{K}_{\alpha\dot\alpha} &= K_{i\bar j}\nabla_\alpha\Phi^i\nabla_{\dot\alpha}\bar\Phi^{\bar j} = \nabla_\alpha\Phi^i\nabla_{\dot\alpha}K_i = \nabla_\alpha K_{\bar j}\nabla_{\dot\alpha}\bar\Phi^{\bar j} \\
&= K_{i\bar j}\mathcal{D}_\alpha\Phi^i\mathcal{D}_{\dot\alpha}\bar\Phi^{\bar j} = \mathcal{D}_\alpha\Phi^i\mathcal{D}_{\dot\alpha}K_i = \mathcal{D}_\alpha K_{\bar j}\mathcal{D}_{\dot\alpha}\bar\Phi^{\bar j}
\end{aligned} \tag{3.4.69}$$

which can simplify its variation. We find after a lot of algebra

$$
\begin{aligned}
\delta\mathcal{K}_{\alpha\dot{\alpha}} = & - H_{\alpha\dot{\alpha}}{}^{b}\mathcal{K}_b - iV^{\beta}\mathcal{D}_{\beta}\mathcal{K}_{\alpha\dot{\alpha}} + iV_{\dot{\beta}}\mathcal{D}^{\dot{\beta}}\mathcal{K}_{\alpha\dot{\alpha}} - \Delta_{\alpha\dot{\alpha}}(V^b\mathcal{K}_b) \\
& + 2\mathcal{K}_{i\bar{j}}(V+\Sigma)\phi^i\mathcal{D}_{\alpha\dot{\alpha}}\bar{\phi}^{\bar{j}} + 2\mathcal{K}_{i\bar{j}}(V+\Sigma)\bar{\phi}^{\bar{j}}\mathcal{D}_{\alpha\dot{\alpha}}\phi^i \\
& - \Delta_{\alpha\dot{\alpha}}(\Sigma^r\mathcal{K}_r) + (\Delta_{\alpha\dot{\alpha}}\Sigma^r)\mathcal{K}_r \\
& - i\mathcal{D}_{\alpha}V_{\dot{\alpha}}{}^{\beta}W^r_{\beta}\mathcal{K}_r + i\mathcal{D}_{\dot{\alpha}}V_{\alpha}{}^{\dot{\beta}}W^r_{\dot{\beta}}\mathcal{K}_r \\
& - \frac{i}{2}V_{\dot{\alpha}}{}^{\beta}(\mathcal{D}_{\alpha}W^r_{\beta})\mathcal{K}_r + \frac{i}{2}V_{\alpha}{}^{\dot{\beta}}(\mathcal{D}_{\dot{\alpha}}W^r_{\dot{\beta}})\mathcal{K}_r \\
& - 2\mathcal{D}_{\alpha}V_{\dot{\alpha}}{}^{\beta}X^{(\mathcal{K})}_{\beta} + 2\mathcal{D}_{\dot{\alpha}}V_{\alpha}{}^{\dot{\beta}}X^{(\mathcal{K})}_{\dot{\beta}} \\
& - V_{\dot{\alpha}}{}^{\beta}\mathcal{D}_{\alpha}X^{(\mathcal{K})}_{\beta} + V_{\alpha}{}^{\dot{\beta}}\mathcal{D}_{\dot{\alpha}}X^{(\mathcal{K})}_{\dot{\beta}} \\
& + \mathcal{D}_{\alpha}\phi^i\mathcal{D}_{\dot{\alpha}}(\mathcal{K}_{i\bar{j}}\bar{\eta}^{\bar{j}}) - \mathcal{D}_{\dot{\alpha}}\bar{\phi}^{\bar{j}}\mathcal{D}_{\alpha}(\mathcal{K}_{i\bar{j}}\eta^i)
\end{aligned}
$$

where again we have collected a number of terms into the combination $H_a{}^b$. The object $X^{(\mathcal{K})}_{\beta}$ is defined as

$$
X^{(\mathcal{K})}_{\beta} \equiv -\frac{1}{8}\bar{\nabla}^2\nabla_{\beta}\mathcal{K} \tag{3.4.70}
$$

In the chiral model, this can further be identified as the $U(1)$ spinor field strength X_{β}.

Including the variation of the compensator and V^a gives

$$
\begin{aligned}
X^{-1}\delta(XV^a\mathcal{K}_a) = & + 2iV^{\beta}\mathcal{D}_{\beta}V^a\mathcal{K}_a - 2iV_{\dot{\beta}}\mathcal{D}^{\dot{\beta}}V^a\mathcal{K}_a + 4V^{\alpha}V^{\dot{\alpha}}\mathcal{K}_{\alpha\dot{\alpha}} \\
& + \tilde{\delta}UV^a\mathcal{K}_a - 2V^a\mathcal{K}_a\Delta_bV^b \\
& + 2\mathcal{K}_{i\bar{j}}(V+\Sigma)\Phi^iV\bar{\Phi}^{\bar{j}} + 2\mathcal{K}_{i\bar{j}}(V+\Sigma)\bar{\Phi}^{\bar{j}}V\Phi^i \\
& - \Delta_bV^b(\Sigma^r\mathcal{K}_r) + V^a(\Delta_a\Sigma^r)\mathcal{K}_r \\
& - V^{\dot{\alpha}\alpha}\left(-\frac{1}{2}\mathcal{D}_{\alpha}V^bF_{b\alpha} + \frac{1}{2}\mathcal{D}_{\dot{\alpha}}V^bF_{b\dot{\alpha}} - \frac{1}{4}V^b\mathcal{D}_{\alpha}F_{b\dot{\alpha}} + \frac{1}{4}V^b\mathcal{D}_{\dot{\alpha}}F_{b\alpha}\right)\mathcal{K}_r \\
& + V^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha}V_{\dot{\alpha}}{}^{\beta}X^{(\mathcal{K})}_{\beta} - V^{\dot{\alpha}\alpha}\mathcal{D}_{\dot{\alpha}}V_{\alpha}{}^{\dot{\beta}}X^{(\mathcal{K})}_{\dot{\beta}} \\
& + \frac{1}{4}V^{\dot{\alpha}\alpha}V_{\alpha\dot{\alpha}}\left(\mathcal{D}^{\beta}X^{(\mathcal{K})}_{\beta} + \mathcal{D}_{\dot{\beta}}X^{\dot{\beta}(\mathcal{K})}\right) \\
& - \frac{1}{2}V^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha}\Phi^i\mathcal{D}_{\dot{\alpha}}(\mathcal{K}_{i\bar{j}}\bar{\eta}^{\bar{j}}) + \frac{1}{2}V^{\dot{\alpha}\alpha}\mathcal{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}}\mathcal{D}_{\alpha}(\mathcal{K}_{i\bar{j}}\eta^i)
\end{aligned} \tag{3.4.71}
$$

The term arising from varying the Yang-Mills piece is fairly complicated. One finds

$$\begin{aligned}
\delta\mathcal{Y}_{\alpha\dot\alpha} =& - H_{\alpha\dot\alpha}{}^b\mathcal{Y}_b - (\Delta_{\alpha\dot\alpha}V^b)\mathcal{Y}_b - i\mathcal{D}^\beta(V_\beta\mathcal{Y}_{\alpha\dot\alpha}) + i\mathcal{D}_{\dot\beta}(V^{\dot\beta}\mathcal{Y}_{\alpha\dot\alpha}) \\
&+ \frac{1}{4}\mathcal{D}^\phi\mathcal{D}_{\{\dot\alpha}V_{\dot\beta\}\phi}\mathcal{Y}_\alpha^{\dot\beta} - \frac{1}{4}\mathcal{D}^{\dot\phi}\mathcal{D}_{\{\alpha}V_{\beta\}\dot\phi}\mathcal{Y}_{\dot\alpha}^\beta \\
&- V^{\dot\beta\beta}G_{\alpha\dot\beta}\mathcal{Y}_{\beta\dot\alpha} - V^{\dot\beta\beta}G_{\beta\dot\alpha}\mathcal{Y}_{\alpha\dot\beta} \\
&+ \frac{1}{4}G_{rs}\bar\nabla^2(V_{\alpha\dot\beta}W^{\dot\beta r})W_{\dot\alpha}^s - \frac{1}{4}G_{rs}\nabla^2(V_{\dot\alpha\beta}W^{\beta r})W_\alpha^s \\
&+ iV^b\mathcal{D}_b(f_{rs}W_\alpha{}^r)W_{\dot\alpha}^s + iV^b\mathcal{D}_bW_\alpha^rW_{\dot\alpha}{}^s\bar f_{rs} - iV^bW_\alpha{}^rf_{rs}\mathcal{D}_bW_{\dot\alpha}^s - iV^bW_\alpha^r\mathcal{D}_b(\bar f_{rs}W_{\dot\alpha}{}^s) \\
&- (G_{rs,i}\eta^i + G_{rs,\bar j}\bar\eta^{\bar j})W_\alpha^rW_{\dot\alpha}^s \\
&- 2if_{rs}W_\alpha^r\Sigma W_{\dot\alpha}^s + 2i\bar f_{rs}(\Sigma W_\alpha^r)W_{\dot\alpha}^s + \frac{i}{4}G_{rs}\bar\nabla^2\nabla_\alpha\Sigma^rW_{\dot\alpha}^s - \frac{i}{4}G_{rs}\nabla^2\nabla_{\dot\alpha}\Sigma^rW_\alpha^s
\end{aligned} \tag{3.4.72}$$

A number of somewhat complicated looking terms have been introduced in the first few lines, partly because the $H_a{}^b$ term is not generated here as readily as in δG_a and $\delta\mathcal{K}_a$. A more convenient arrangement of the above terms is given by

$$\begin{aligned}
\delta\mathcal{Y}_{\alpha\dot\alpha} =& - H_{\alpha\dot\alpha}{}^b\mathcal{Y}_b - i\mathcal{D}^\beta(V_\beta\mathcal{Y}_{\alpha\dot\alpha}) + i\mathcal{D}_{\dot\beta}(V^{\dot\beta}\mathcal{Y}_{\alpha\dot\alpha}) \\
&- 4(\mathcal{D}_bV_c)G_d^{\mathcal{Y}}\epsilon_{(\alpha\dot\alpha)bcd} + \mathcal{D}^\beta(G_{\alpha\dot\alpha}^{\mathcal{Y}}\mathcal{D}^{\dot\beta}V_{\alpha\dot\alpha}) - \mathcal{D}^{\dot\beta}(G_{\alpha\dot\alpha}^{\mathcal{Y}}\mathcal{D}^\beta V_{\alpha\dot\alpha}) \\
&+ 2\mathcal{D}^\beta(R^{\mathcal{Y}}\mathcal{D}_\beta V_{\alpha\dot\alpha}) + 2\mathcal{D}_{\dot\beta}(\bar R^{\mathcal{Y}}\mathcal{D}^{\dot\beta}V_{\alpha\dot\alpha}) - \mathcal{D}^\gamma V^{\dot\beta\beta}W^{\mathcal{Y}}_{\gamma(\beta\dot\beta)(\alpha\dot\alpha)} - \mathcal{D}^{\dot\gamma}V^{\dot\beta\beta}W^{\mathcal{Y}}_{\dot\gamma(\beta\dot\beta)(\alpha\dot\alpha)} \\
&+ 2\left(-\frac{1}{2}\mathcal{D}_\alpha V^bF_{b\alpha}{}^r + \frac{1}{2}\mathcal{D}_{\dot\alpha}V^bF_{b\dot\alpha}{}^r - \frac{1}{4}V^b\mathcal{D}_\alpha F_{b\dot\alpha}{}^r + \frac{1}{4}V^b\mathcal{D}_{\dot\alpha}F_{b\alpha}{}^r\right)\mathcal{Y}_r \\
&- 2V^{\dot\beta\beta}\mathcal{Y}_{(\alpha\dot\alpha)(\beta\dot\beta)} - (G_{rs,i}\eta^i + G_{rs,\bar j}\bar\eta^{\bar j})W_\alpha^r\bar W_{\dot\alpha}^s \\
&- 2if_{rs}W_\alpha^r\Sigma\bar W_{\dot\alpha}^s + 2i\bar f_{rs}(\Sigma W_\alpha^r)\bar W_{\dot\alpha}^s + \frac{i}{4}G_{rs}\bar\nabla^2\nabla_\alpha\Sigma^r\bar W_{\dot\alpha}^s - \frac{i}{4}G_{rs}\nabla^2\nabla_{\dot\alpha}\Sigma^rW_\alpha^s
\end{aligned}$$

where we have made a number of definitions. In particular,

$$R^{\mathcal{Y}} \equiv -\frac{1}{16}G_{rs}W^{\phi r}W_\phi{}^s \tag{3.4.73}$$

$$\bar R^{\mathcal{Y}} \equiv -\frac{1}{16}G_{rs}\bar W_{\dot\phi}{}^r\bar W^{\dot\phi s} \tag{3.4.74}$$

$$G_{\alpha\dot\alpha}^{\mathcal{Y}} \equiv \frac{1}{4}G_{rs}W_\alpha{}^r\bar W_{\dot\alpha}{}^s \tag{3.4.75}$$

These definitions should not be taken more seriously than just serving as convenient names. $R^{\mathcal{Y}}$, for example, is not chiral unless the gauge couplings are trivial. We have simply identified these combinations since they seem like they shall combine nicely with actual objects of those names in the graviton propagator.[4] In addition, we have written "curvature"

[4] It is plausible, although we haven't explored this possibility deeply yet, that if the linear compensator is coupled to the Chern-Simons term for the gauge sector, then the superfields R and G defined in terms of L will pick up contributions of the above form for the case $G_{rs} \propto \delta_{rs}$.

terms which will also combine with the similar term in $\Box_V$:

$$W^{\mathcal{Y}}_{\gamma(\beta\dot{\beta})(\alpha\dot{\alpha})} \equiv \frac{1}{4}\epsilon_{\dot{\beta}\dot{\alpha}} \sum_{\beta\alpha} \Big(\epsilon_{\alpha\gamma} W_\beta{}^r \mathcal{D}_{\dot{\phi}}(G_{rs}\bar{W}^{\dot{\phi}s}) + \epsilon_{\alpha\gamma} W_\beta{}^r \mathcal{D}^\phi (G_{rs} W_\phi{}^s) \\ - \epsilon_{\alpha\gamma} G_{rs} W_\beta{}^r (\mathcal{D}W)^s + G_{rs} W_\alpha{}^r \mathcal{D}_\gamma W_\beta{}^s \Big) \tag{3.4.76}$$

$$W^{\mathcal{Y}}_{\dot{\gamma}(\beta\dot{\beta})(\alpha\dot{\alpha})} \equiv -\frac{1}{4}\epsilon_{\beta\alpha} \sum_{\beta\alpha} \Big(\epsilon_{\dot{\alpha}\dot{\gamma}} \bar{W}_{\dot{\beta}}{}^r \mathcal{D}_{\dot{\phi}}(G_{rs}\bar{W}^{\dot{\phi}s}) + \epsilon_{\dot{\alpha}\dot{\gamma}} \bar{W}_{\dot{\beta}}{}^r \mathcal{D}^\phi (G_{rs} W_\phi{}^s) \\ - \epsilon_{\dot{\alpha}\dot{\gamma}} G_{rs} \bar{W}_{\dot{\beta}}{}^r (\mathcal{D}W)^s - G_{rs} \bar{W}_{\dot{\alpha}}{}^r \mathcal{D}_{\dot{\gamma}} \bar{W}_{\dot{\beta}}{}^s \Big) \tag{3.4.77}$$

as well as the "potential" term

$$\begin{aligned} V^{\dot{\beta}\beta}\mathcal{Y}_{(\alpha\dot{\alpha})(\beta\dot{\beta})} \equiv & - V^b G_b \mathcal{Y}_{\alpha\dot{\alpha}} - G_{\alpha\dot{\alpha}} V^b \mathcal{Y}_b + V_{\alpha\dot{\alpha}} G^b \mathcal{Y}_b \\ & + \frac{1}{4} V^{\dot{\beta}\beta} \left(\mathcal{D}_\beta W_{\alpha r} \mathcal{D}_{\dot{\beta}} W^r_{\dot{\alpha}} + \mathcal{D}_\beta W^r_\alpha \mathcal{D}_{\dot{\beta}} W_{\dot{\alpha} r} \right) - \frac{1}{2} V^b \Delta_b \mathcal{Y}_{\alpha\dot{\alpha}} \\ & - \frac{1}{8} V_{\alpha\dot{\beta}} (\bar{\mathcal{D}}^2 - 8R) \bar{W}^{\dot{\beta}r}\, G_{rs} W_{\dot{\alpha}}{}^s + \frac{1}{8} V_{\dot{\alpha}\beta} (\mathcal{D}^2 - 8\bar{R}) W^{\beta r}\, G_{rs} W_\alpha{}^s \\ & + \frac{1}{8} V_{\dot{\alpha}\beta} \mathcal{D}_\alpha W^{\beta r} \mathcal{D}^\gamma (f_{rs} W_\gamma{}^s) + \frac{1}{8} V_{\dot{\alpha}\beta} \mathcal{D}_\alpha W^{\beta r} \mathcal{D}_{\dot{\gamma}} (\bar{f}_{rs} \bar{W}^{\dot{\gamma}s}) \\ & - \frac{1}{8} V_{\alpha\dot{\beta}} \mathcal{D}_{\dot{\alpha}} \bar{W}^{\dot{\beta}r} \mathcal{D}^\gamma (f_{rs} W_\gamma{}^s) - \frac{1}{8} V_{\alpha\dot{\beta}} \mathcal{D}_{\dot{\alpha}} \bar{W}^{\dot{\beta}r} \mathcal{D}_{\dot{\gamma}} (\bar{f}_{rs} \bar{W}^{\dot{\gamma}s}) \end{aligned} \tag{3.4.78}$$

These look like they could be defined in terms of the new $R^{\mathcal{Y}}$ and $G^{\mathcal{Y}}$ objects we have mentioned before, but we will avoid doing so explicitly.

The combination we need is

$$\begin{aligned} \delta(V^a \mathcal{Y}_a) =& 2iV^\beta \mathcal{D}_\beta V^a \mathcal{Y}_a - 2iV_{\dot{\beta}} \mathcal{D}^{\dot{\beta}} V^a \mathcal{Y}_a + 4V^\alpha V^{\dot{\alpha}} \mathcal{Y}_{\alpha\dot{\alpha}} \\ & - 4V^a (\mathcal{D}_b V_c) G^{\mathcal{Y}}_d \epsilon_{abcd} \\ & - \frac{1}{2} V^{\dot{\alpha}\alpha} \mathcal{D}^\beta (G^{\mathcal{Y}}_{\beta\dot{\beta}} \mathcal{D}^{\dot{\beta}} V_{\alpha\dot{\alpha}}) + \frac{1}{2} V^{\dot{\alpha}\alpha} \mathcal{D}^{\dot{\beta}} (G^{\mathcal{Y}}_{\alpha\dot{\alpha}} \mathcal{D}^\beta V_{\alpha\dot{\alpha}}) \\ & - V^{\dot{\alpha}\alpha} \mathcal{D}^\beta (R^{\mathcal{Y}} \mathcal{D}_\beta V_{\alpha\dot{\alpha}}) - V^{\dot{\alpha}\alpha} \mathcal{D}_{\dot{\beta}} (\bar{R}^{\mathcal{Y}} \mathcal{D}^{\dot{\beta}} V_{\alpha\dot{\alpha}}) \\ & + \frac{1}{2} V^{\dot{\alpha}\alpha} \mathcal{D}^\gamma V^{\dot{\beta}\beta} W^{\mathcal{Y}}_{\gamma(\beta\dot{\beta})(\alpha\dot{\alpha})} + \frac{1}{2} V^{\dot{\alpha}\alpha} \mathcal{D}^{\dot{\gamma}} V^{\dot{\beta}\beta} W^{\mathcal{Y}}_{\dot{\gamma}(\beta\dot{\beta})(\alpha\dot{\alpha})} \\ & - V^{\dot{\alpha}\alpha} \left(-\frac{1}{2} \mathcal{D}_\alpha V^b F_{b\alpha} + \frac{1}{2} \mathcal{D}_{\dot{\alpha}} V^b F_{b\dot{\alpha}} - \frac{1}{4} V^b \mathcal{D}_\alpha F_{b\dot{\alpha}} + \frac{1}{4} V^b \mathcal{D}_{\dot{\alpha}} F_{b\alpha} \right) \mathcal{Y}_r \\ & + V^{\dot{\alpha}\alpha} V^{\dot{\beta}\beta} \mathcal{Y}_{(\alpha\dot{\alpha})(\beta\dot{\beta})} + \frac{1}{2} V^{\dot{\alpha}\alpha} (G_{rs,i}\eta^i + G_{rs,\bar{j}} \bar{\eta}^{\bar{j}}) W^r_\alpha \bar{W}^s_{\dot{\alpha}} \\ & + iV^{\dot{\alpha}\alpha} f_{rs} W^r_\alpha \Sigma \bar{W}^s_{\dot{\alpha}} - iV^{\dot{\alpha}\alpha} \bar{f}_{rs} (\Sigma W^r_\alpha) \bar{W}^s_{\dot{\alpha}} \\ & - \frac{i}{8} V^{\dot{\alpha}\alpha} G_{rs} \bar{\nabla}^2 \nabla_\alpha \Sigma^r \bar{W}^s_{\dot{\alpha}} + \frac{i}{8} V^{\dot{\alpha}\alpha} G_{rs} \nabla^2 \nabla_{\dot{\alpha}} \Sigma^r W^s_\alpha \end{aligned} \tag{3.4.79}$$

Variation of the $\mathcal{L}$ term

In the simple linear compensator model, there is one additional term – that involving $\mathcal{L}$. Beginning with

$$S_L = [\mathcal{L}(V_R + K)]_D \tag{3.4.80}$$

one varies it to find

$$\delta S_L = \mathcal{L}\left(3\frac{\mathcal{L}}{L} - 2\Delta_b V^b + V^b(K_b - 4G_b) + K_i\eta^i + K_{\bar{j}}\bar{\eta}^{\bar{j}} + \Sigma^r K_r\right) \tag{3.4.81}$$

3.4.3 Summary

We will break down our results into various sectors.

The terms involving just the chiral (and antichiral) quanta are

$$S_{\eta\eta} = \left[\eta^i X\mathcal{K}_{i\bar{j}}\bar{\eta}^{\bar{j}}\right]_D + \left[\eta^i(\mathcal{P}(X\mathcal{K}_{ij}) + P_{ij} + \mathcal{Y}_{ij})\eta^j\right]_F + \text{h.c.}$$

The terms involving chiral and gauge fields are

$$\begin{aligned} S_{\eta\Sigma} &= 4i\eta^i\mathcal{K}_{i\bar{j}}X_r\bar{\Phi}^{\bar{j}}\Sigma^r + i\eta^i f_{rs,i}W^{\alpha s}\nabla_\alpha\Sigma^r + \text{h.c.} \\ &= 2\eta^i\left(X\mathcal{K}_{ir} + \mathcal{Y}_{ir}\right)\Sigma^r + \text{h.c.} \end{aligned}$$

The terms involving chiral and gravity fields are

$$\begin{aligned} S_{\eta V} &= +V^{\dot{\alpha}\alpha}\mathcal{D}_{\dot{\alpha}}\bar{\Phi}^{\bar{j}}\mathcal{D}_\alpha(X\mathcal{K}_{i\bar{j}}\eta^i) + V^{\dot{\alpha}\alpha}W^r_\alpha\bar{W}^s_{\dot{\alpha}}f_{rs,i}\eta^i + \text{h.c.} \\ &= 2V^a\left(X\mathcal{K}_{a,i}\eta^i + \mathcal{Y}_{a,i}\right)\eta^i + \text{h.c.} \end{aligned}$$

The terms involving gravity and gauge fields are

$$\begin{aligned} S_{\Sigma V} &= \left(2iV^\alpha\mathcal{D}_\alpha\Sigma^r - 2iV_{\dot{\alpha}}\mathcal{D}^{\dot{\alpha}}\Sigma^r\right)\left(X\mathcal{K}_r + \mathcal{Y}_r\right) \\ &\quad - 2V^a(\Delta_a\Sigma^r)\mathcal{Y}_r - 2X(\Delta_b V^b)\Sigma^r\mathcal{K}_r \\ &\quad + \frac{i}{4}V_{\alpha\dot{\alpha}}G_{rs}\bar{W}^{\dot{\alpha}s}\bar{\nabla}^2\nabla^\alpha\Sigma^r - \frac{i}{4}V^{\dot{\alpha}\alpha}G_{rs}W_\alpha{}^s\nabla^2\nabla_{\dot{\alpha}}\Sigma^r \\ &\quad + 4X\mathcal{K}_{i\bar{j}}V\Phi^i\Sigma\bar{\Phi}^{\bar{j}} + 4X\mathcal{K}_{i\bar{j}}\Sigma\Phi^i V\bar{\Phi}^{\bar{j}} \\ &\quad + 2iV^{\dot{\alpha}\alpha}f_{rs}W^r_\alpha\Sigma\bar{W}^s_{\dot{\alpha}} - 2iV^{\dot{\alpha}\alpha}\bar{f}_{rs}(\Sigma W^r_\alpha)\bar{W}^s_{\dot{\alpha}} \end{aligned} \tag{3.4.82}$$

In the last two lines, we use a single Σ to denote $\Sigma^r X_r$ acting to the right. It seems reasonable to rearrange the second line of $S_{\Sigma V}$ so that it is proportional to the equation of motion.

$$\begin{aligned}S_{\Sigma V}=&\left(2iV^{\alpha}\mathcal{D}_{\alpha}\Sigma^{r}-2iV_{\dot{\alpha}}\mathcal{D}^{\dot{\alpha}}\Sigma^{r}\right)(X\mathcal{K}_{r}+\mathcal{Y}_{r})\\&-2(\Delta_{b}V^{b})\Sigma^{r}(X\mathcal{K}_{r}+\mathcal{Y}_{r})\\&+\frac{i}{4}V_{\alpha\dot{\alpha}}G_{rs}\bar{W}^{\dot{\alpha}s}\bar{\nabla}^{2}\nabla^{\alpha}\Sigma^{r}-\frac{i}{4}V^{\dot{\alpha}\alpha}G_{rs}W_{\alpha}{}^{s}\nabla^{2}\nabla_{\dot{\alpha}}\Sigma^{r}\\&-\Sigma^{r}\mathcal{D}^{\alpha}V_{\alpha\dot{\alpha}}\mathcal{D}^{\dot{\alpha}}\mathcal{Y}_{r}+\Sigma^{r}\mathcal{D}^{\dot{\alpha}}V_{\alpha\dot{\alpha}}\mathcal{D}^{\alpha}\mathcal{Y}_{r}\\&-2\Sigma^{r}V^{a}\Delta_{a}\mathcal{Y}_{r}\\&+4X\mathcal{K}_{i\bar{j}}V\Phi^{i}\Sigma\bar{\Phi}^{\bar{j}}+4X\mathcal{K}_{i\bar{j}}\Sigma\Phi^{i}V\bar{\Phi}^{\bar{j}}\\&+2iV^{\dot{\alpha}\alpha}f_{rs}W_{\alpha}^{r}\Sigma\bar{W}_{\dot{\alpha}}^{s}-2iV^{\dot{\alpha}\alpha}\bar{f}_{rs}(\Sigma W_{\alpha}^{r})\bar{W}_{\dot{\alpha}}^{s}\end{aligned}\tag{3.4.83}$$

The term with three spinor derivatives can be rearranged so that it is proportional to $\mathcal{D}^{\alpha}V_{\alpha\dot{\alpha}}(\bar{\mathcal{D}}^{2}-8R)\Sigma^{r}G_{rs}\bar{W}^{\dot{\alpha}s}$, which can be cancelled if we introduce a Gaussian smearing with the gauge fixing functions $\mathcal{D}^{\alpha}V_{\alpha\dot{\alpha}}$ for the gravity sector and $(\bar{\mathcal{D}}^{2}-8R)\Sigma^{r}$ for the gauge sector, which is the standard approach. [25]

Next we turn to the pure gauge sector. We find

$$\begin{aligned}S_{\Sigma\Sigma}=&4X\mathcal{K}_{i\bar{j}}\Sigma\Phi^{i}\Sigma\bar{\Phi}^{\bar{j}}+\Sigma^{r}\mathcal{D}^{\alpha}\Sigma^{s}f_{sr}{}^{t}W_{\alpha}{}^{t}f_{tu}-\Sigma^{r}\mathcal{D}_{\dot{\alpha}}\Sigma^{s}f_{sr}{}^{t}\bar{W}^{\dot{\alpha}t}\bar{f}_{tu}\\&-\frac{1}{8}\Sigma^{r}\nabla^{\alpha}(f_{rs}\bar{\nabla}^{2}\nabla_{\alpha}\Sigma^{s})-\frac{1}{8}\Sigma^{r}\nabla_{\dot{\alpha}}(\bar{f}_{rs}\nabla^{2}\nabla^{\dot{\alpha}}\Sigma^{s})\end{aligned}\tag{3.4.84}$$

It is conspicous that for arbitrary holomorphic f_{rs}, the last term yields the three spinor derivative term $\Sigma^{r}(\nabla^{\alpha}f_{rs})\bar{\nabla}^{2}\nabla_{\alpha}\Sigma^{s}$ which it does not seem possible to remove by a smeared gauge. It is not strictly speaking problematic to have a third order spinor derivative term (as it is still less divergent than the pure kinetic term and so can in principle be treated at least perturbatively), but it will lead to a more complicated one-loop analysis.

In any case, it is useful to rearrange the kinetic term into a form involving chiral projections of Σ. We use the identity

$$\begin{aligned}\frac{1}{8}\Sigma\nabla^{\alpha}(f\bar{\nabla}^{2}\nabla_{\alpha}\Sigma)+\text{h.c.}=&(\mathcal{D}_{a}\Sigma)G(\mathcal{D}_{a}\Sigma)+\frac{1}{8}(\bar{\mathcal{D}}^{2}-8R)\Sigma G(\mathcal{D}^{2}-8\bar{R})\Sigma\\&+\left(\frac{1}{8}\bar{\mathcal{D}}_{\dot{\alpha}}\Sigma\bar{\mathcal{D}}^{\dot{\alpha}}\bar{f}(\mathcal{D}^{2}-8\bar{R})\Sigma+\text{h.c.}\right)\\&+\frac{1}{2}\Sigma\mathcal{D}^{\alpha}f\Sigma\mathcal{D}_{\alpha}R+\frac{1}{2}\Sigma\mathcal{D}_{\dot{\alpha}}\bar{f}\Sigma\mathcal{D}^{\dot{\alpha}}\bar{R}\\&-8R\bar{R}\Sigma G\Sigma+\frac{1}{2}\Sigma G\Sigma(\mathcal{D}^{2}R+\bar{\mathcal{D}}^{2}\bar{R})\\&-\mathcal{D}_{\alpha}\Sigma G^{\dot{\alpha}\alpha}G\mathcal{D}_{\dot{\alpha}}\Sigma+\frac{i}{4}\mathcal{D}^{\alpha}\Sigma\mathcal{D}^{\dot{\alpha}}\Sigma(\mathcal{D}_{\alpha\dot{\alpha}}f-\mathcal{D}_{\alpha\dot{\alpha}}\bar{f})\\&+\mathcal{D}^{\alpha}\Sigma\bar{f}(W_{\alpha}\Sigma)-\mathcal{D}_{\dot{\alpha}}\Sigma f(\bar{W}^{\dot{\alpha}}\Sigma)\\&+\frac{i}{4}\bar{\mathcal{D}}^{\dot{\alpha}}\Sigma\mathcal{D}^{\alpha}f\mathcal{D}_{\alpha\dot{\alpha}}\Sigma+\frac{i}{4}\mathcal{D}_{\alpha}\Sigma\mathcal{D}_{\dot{\alpha}}\bar{f}\mathcal{D}^{\dot{\alpha}\alpha}\Sigma\end{aligned}\tag{3.4.85}$$

In the above, we have suppressed all gauge indices for the sake of a less cluttered notation. They should be contracted in the obvious way, taking care to note that $(W_{\alpha}\Sigma)^{r}\equiv$

$-W^s_\alpha \Sigma^t f_{ts}{}^r$. We have also chosen to integrate certain terms by parts so that the result is manifestly symmetric.

It is useful to define a generalized d'Alembertian for Σ based on the above formula. We choose

$$\Box^V_{rs}\Sigma^s \equiv \mathcal{D}^a(G_{rs}\mathcal{D}_a\Sigma^s) - \frac{1}{2}\mathcal{D}^{[\alpha}(G_{\alpha\dot{\alpha}}G_{rs}\mathcal{D}^{\dot{\alpha}]}\Sigma^s) + \mathcal{D}^\alpha\Sigma^s G_{su}W_\alpha{}^t f_{rt}{}^u - \mathcal{D}_{\dot{\alpha}}\Sigma^s G_{su}\bar{W}^{\dot{\alpha}t} f_{rt}{}^u \tag{3.4.86}$$

so that in compacted notation

$$\Sigma\Box_V\Sigma = \Sigma\mathcal{D}^a(G\mathcal{D}_a\Sigma) - \frac{1}{2}\Sigma\mathcal{D}^{[\alpha}(G_{\alpha\dot{\alpha}}G\mathcal{D}^{\dot{\alpha}]}\Sigma) - \mathcal{D}^\alpha\Sigma G W_\alpha\Sigma + \mathcal{D}_{\dot{\alpha}}\Sigma G\bar{W}^{\dot{\alpha}}\Sigma \tag{3.4.87}$$

This is a generalization of the scalar d'Alembertian $\Box_V$ discussed in [22], generalized to a superfield Σ with a nontrivial gauge sector with corresponding gaugino superfield W_α. The form of this operator also inspired the definition of $\Box_V V_{\alpha\dot{\alpha}}$ for the gravity sector.

We may then write

$$\begin{aligned}
S_{\Sigma\Sigma} =& \Sigma\Box_V\Sigma - \frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\Sigma G(\mathcal{D}^2 - 8\bar{R})\Sigma \\
& - \frac{1}{2}\Sigma G\Sigma(\mathcal{D}^2 - 8\bar{R})R - \frac{1}{2}\Sigma G\Sigma(\bar{\mathcal{D}}^2 - 8R)\bar{R} + 4X\mathcal{K}_{i\bar{j}}\Sigma\Phi^i\Sigma\bar{\Phi}^{\bar{j}} \\
& - \left(\frac{1}{8}\bar{\mathcal{D}}_{\dot{\alpha}}\Sigma\bar{\mathcal{D}}^{\dot{\alpha}}\bar{f}(\mathcal{D}^2 - 8\bar{R})\Sigma + \text{h.c.}\right) \\
& - \frac{1}{2}\Sigma\mathcal{D}^\alpha f\Sigma\mathcal{D}_\alpha R - \frac{1}{2}\Sigma\mathcal{D}_{\dot{\alpha}}\bar{f}\Sigma\mathcal{D}^{\dot{\alpha}}\bar{R} - \frac{i}{4}\mathcal{D}^\alpha\Sigma\mathcal{D}^{\dot{\alpha}}\Sigma(\mathcal{D}_{\alpha\dot{\alpha}}f - \mathcal{D}_{\alpha\dot{\alpha}}\bar{f}) \\
& - \frac{i}{4}\bar{\mathcal{D}}^{\dot{\alpha}}\Sigma\mathcal{D}^\alpha f\mathcal{D}_{\alpha\dot{\alpha}}\Sigma - \frac{i}{4}\mathcal{D}_\alpha\Sigma\mathcal{D}_{\dot{\alpha}}\bar{f}\mathcal{D}^{\dot{\alpha}\alpha}\Sigma \\
& + \Sigma^r\mathcal{D}^\alpha\Sigma^s W_\alpha{}^u(X_r f_{su}) - \Sigma^r\mathcal{D}_{\dot{\alpha}}\Sigma^s\bar{W}^{\dot{\alpha}u}(X_r\bar{f}_{su})
\end{aligned}$$

Note the last line involves the gauge generator acting on the holomorphic gauge couplings. If these are taken to be proportional to the identity, then the last line will vanish.

We turn finally to the pure gravity sector. The terms are quite numerous:

$$
\begin{aligned}
S_{VV} = &\left(2iV^{\alpha}V^{b}F_{b\alpha} + 2iV^{\dot\alpha}V^{b}F_{b\dot\alpha}\right)(X\mathcal{K}_r + \mathcal{Y}_r) \\
&\left(+2iV^{\beta}\mathcal{D}_{\beta}V^{a} - 2iV_{\dot\beta}\mathcal{D}^{\dot\beta}V^{a} + 4V^{\alpha}V^{\dot\alpha}\sigma^{a}_{\alpha\dot\alpha}\right)(-4XG_a + X\mathcal{K}_a + \mathcal{Y}_a) \\
&+ 2X(\Delta_b V^b)^2 - 2X(\mathcal{D}_b V^b)^2 - \frac{X}{8}\mathcal{D}^2 V^{\dot\alpha\alpha}\bar{\mathcal{D}}^2 V_{\alpha\dot\alpha} + XV^{\dot\alpha\alpha}\Box_V V_{\alpha\dot\alpha} \\
&- 2\Delta_b V^b\left(XV^a\mathcal{K}_a - 4XV^aG_a\right) \\
&+ XV^{\dot\alpha\alpha}\mathcal{D}^{\beta}(R\mathcal{D}_{\beta}V_{\alpha\dot\alpha}) + XV^{\dot\alpha\alpha}\mathcal{D}_{\dot\beta}(\bar{R}\mathcal{D}^{\dot\beta}V_{\alpha\dot\alpha}) \\
&- V^{\dot\alpha\alpha}\mathcal{D}^{\beta}(R^{\mathcal{Y}}\mathcal{D}_{\beta}V_{\alpha\dot\alpha}) - V^{\dot\alpha\alpha}\mathcal{D}_{\dot\beta}(\bar{R}^{\mathcal{Y}}\mathcal{D}^{\dot\beta}V_{\alpha\dot\alpha}) \\
&- \frac{1}{2}V^{\dot\alpha\alpha}\mathcal{D}^{\beta}(G^{\mathcal{Y}}_{\beta\dot\beta}\mathcal{D}^{\dot\beta}V_{\alpha\dot\alpha}) + \frac{1}{2}V^{\dot\alpha\alpha}\mathcal{D}^{\dot\beta}(G^{\mathcal{Y}}_{\alpha\dot\alpha}\mathcal{D}^{\beta}V_{\alpha\dot\alpha}) \\
&- 4V^a(\mathcal{D}_b V_c)G^{\mathcal{Y}}_{d}\epsilon_{abcd} \\
&+ XV^{\dot\alpha\alpha}\mathcal{D}_{(\alpha}V_{\dot\alpha}{}^{\beta}\hat{\mathcal{K}}_{\beta)} - XV^{\dot\alpha\alpha}\mathcal{D}_{(\dot\alpha}V_{\alpha}{}^{\dot\beta}\hat{\mathcal{K}}_{\dot\beta)} \\
&+ \frac{1}{2}V^{\dot\alpha\alpha}\mathcal{D}^{\gamma}V^{\dot\beta\beta}W^{\mathcal{Y}}_{\gamma(\beta\dot\beta)(\alpha\dot\alpha)} + \frac{1}{2}V^{\dot\alpha\alpha}\mathcal{D}^{\dot\gamma}V^{\dot\beta\beta}W^{\mathcal{Y}}_{\dot\gamma(\beta\dot\beta)(\alpha\dot\alpha)} \\
&- 2XR\bar{R}V^{\dot\alpha\alpha}V_{\alpha\dot\alpha} + 4X\mathcal{K}_{i\bar{j}}V\Phi^{i}V\bar{\Phi}^{\bar{j}} + V^{\dot\alpha\alpha}V^{\dot\beta\beta}\mathcal{Y}_{(\alpha\dot\alpha)(\beta\dot\beta)}
\end{aligned}
\tag{3.4.88}
$$

We have defined

$$
\hat{\mathcal{K}}_{\alpha} = \begin{cases} -\frac{1}{8}\bar{\nabla}^2\nabla_{\alpha}K - X_{\alpha} & \text{for the simple linear compensator model} \\ 0 & \text{for the arbitrary chiral model} \end{cases}
\tag{3.4.89}
$$

We have until now left the gauge for V^{α} unspecified. Inspection of its appearance in all the terms shows that it is always proportional to the equations of motion, so if we work with the background on-shell then the gauge of V^{α} (at least to one-loop order) is physically irrelevant. We will still choose the particular gauge $V^{\alpha} = 0$ for definiteness.

The above represent the common features of the linear and chiral models. They also each have a term involving $\tilde{\delta}U$:

$$
S_{\delta U,*} = \tilde{\delta}U(X\mathcal{K}_i\eta^i + X\mathcal{K}_{\bar{j}}\bar{\eta}^{\bar{j}} + X\mathcal{K}_r\Sigma^r - 4XV^bG_b + XV^b\mathcal{K}_b - 2X\Delta_b V^b)
$$

Depending on the model, the variation of the compensator may be quite different. The simple linear compensator model has

$$
\tilde{\delta}U = L^{-1}\mathcal{L}
$$

while the arbitrary chiral model possesses

$$
\tilde{\delta}U = -\frac{1}{3}\left(\mathcal{K}_i\eta^i + \mathcal{K}_{\bar{j}}\bar{\eta}^{\bar{j}} + \mathcal{K}_r\Sigma^r - 4V^bG_b + V^b\mathcal{K}_b - 2\Delta_b V^b\right)
$$

In addition, for the linear compensator model there are the terms arising from varying (3.4.81):

$$
S_{L,*} = \mathcal{L}\left(3\frac{\mathcal{L}}{L} - 2\Delta_b V^b + V^b(K_b - 4G_b) + K_i\eta^i + K_{\bar{j}}\bar{\eta}^{\bar{j}} + \Sigma^r K_r\right)
\tag{3.4.90}
$$

Combining these two effects gives the second order action for the linear compensator model

$$
\begin{aligned}
S^{(2)}_{\text{linear}} = & S_{VV} + S_{\Sigma V} + S_{\Sigma\Sigma} + S_{\eta V} + S_{\eta\Sigma} + S_{\eta\eta} \\
& + 3\frac{\mathcal{L}^2}{L} + 2\mathcal{L}\left(K_i\eta^i + K_{\bar{j}}\bar{\eta}^{\bar{j}} + \Sigma^r K_r + V^b(K_b - 4G_b) - 2\Delta_b V^b\right)
\end{aligned}
\tag{3.4.91}
$$

For the chiral model, we find

$$
\begin{aligned}
S^{(2)}_{\text{chiral}} = & S_{VV} + S_{\Sigma V} + S_{\Sigma\Sigma} + S_{\eta V} + S_{\eta\Sigma} + S_{\eta\eta} \\
& - \frac{X}{3}\left(\mathcal{K}_i\eta^i + \mathcal{K}_{\bar{j}}\bar{\eta}^{\bar{j}} + \mathcal{K}_r\Sigma^r + V^b(\mathcal{K}_b - 4G_b) - 2\Delta_b V^b\right)^2
\end{aligned}
\tag{3.4.92}
$$

For reference, we include here their first order variations as well:

$$
\begin{aligned}
S^{(1)}_{\text{chiral}} = & \left[V^a(X\mathcal{K}_a - 4XG_a + \mathcal{Y}_a) + \Sigma^r(X\mathcal{K}_r + \mathcal{Y}_r)\right]_D \\
& + \left[\eta^i(\mathcal{P}(X\mathcal{K}_i) + P_i + \mathcal{Y}_i)\right]_F + \left[\bar{\eta}^{\bar{j}}(\bar{\mathcal{P}}(X\mathcal{K}_{\bar{j}}) + \bar{P}_{\bar{j}} + \mathcal{Y}_{\bar{j}})\right]_F
\end{aligned}
\tag{3.4.93}
$$

$$
\begin{aligned}
S^{(1)}_{\text{linear}} = & \left[V^a(XK_a - 4XG_a + \mathcal{Y}_a) + \Sigma^r(XK_r + \mathcal{Y}_r)\right]_D \\
& + \left[\eta^i(\mathcal{P}(XK_i) + P_i + \mathcal{Y}_i)\right]_F + \left[\bar{\eta}^{\bar{j}}(\bar{\mathcal{P}}(XK_{\bar{j}}) + \bar{P}_{\bar{j}} + \mathcal{Y}_{\bar{j}})\right]_F \\
& + \left[\mathcal{L}(V_R + K)\right]_D
\end{aligned}
\tag{3.4.94}
$$

Their respective actions to second order in the quantum fields are then given by

$$
S_{\text{chiral}} = S^{(0)}_{\text{chiral}} + S^{(1)}_{\text{chiral}} + \frac{1}{2}S^{(2)}_{\text{chiral}} \tag{3.4.95}
$$

$$
S_{\text{linear}} = S^{(0)}_{\text{linear}} + S^{(1)}_{\text{linear}} + \frac{1}{2}S^{(2)}_{\text{linear}} \tag{3.4.96}
$$

When we consider that the linear compensator model is classically dual to a special case of the arbirary chiral model, it becomes perhaps unsurprising that their quantum actions should have so many features in common. This commonality is enough for us to ask whether the two theories might actually be equivalent at the one-loop level, at least on-shell. One can in fact make a rather straightforward argument, based on the existing proofs of equivalence for chiral spinors and chiral scalars [22, 32] that the two effective actions should be equivalent on-shell at one-loop.

3.5 Conclusion

The formulae listed above constitute the end of the algebraic manipulations necessary to produce a suitable action quadratic in the quantum superfields of supergravity, super Yang-Mills, and chiral matter. Further steps are necessary to produce one-loop results.

The first step is obviously to perform a gauge-fixing of the gravity and gauge sectors. Part of the procedure here will involve deciding just *how* to do it. Even if we choose a smeared gauge and aim for only $1/p^2$ propagators (as was the guiding principle in

[27]), we have the option of removing certain terms in $S_{\Sigma V}$ or S_{VV} involving operators of dimension less than two. Any choice must, of course, be physically equivalent to any other, but certain calculational simplifications may occur only one way.

The second is to actually perform the resulting path integrals. For background field calculations, one generally prefers a method which is non-perturbative, such as the Schwinger proper time method or the derivative expansion. Such a procedure here is a bit more difficult since while the gauge and gravity sectors involve generalized Laplacians, the chiral sector involves Dirac-like operators. If the couplings between these sectors do not vanish, some amount of perturbation seems necessary, since the determinant of an operator with a diagonal consisting of Laplace *and* Dirac operators is difficult to deal with without separating out the two sectors.

The first step toward fulfilling this program – the calculation of the effective action for chiral superfields in an arbitrary supergravity background – is the topic of Part II of this thesis.

Part II

Towards the effective action of Poincaré-invariant supergravity theories

Chapter 4

A brief review: The effective action for component field theories

4.1 Heat kernel analysis for Laplace operators

The one-loop contribution to the effective action for a generic quantum field theory usually boils down to the calculation of the regulated quantity $\mathrm{Tr}\log H$ where H is the second variation of the action around the quantum fields. After an appropriate Wick rotation, H usually becomes a differential operator with a positive spectrum – at least perturbatively.

For example, the Euclidean effective action for a complex bosonic field ϕ at one-loop generically amounts to performing the path integration

$$e^{-\Gamma_E} = \int \mathcal{D}\phi \exp\left(-\int d^4x\sqrt{g}\,\bar{\phi}\,(-\Box + Q)\,\phi\right) \tag{4.1.1}$$

where $\Box$ is some covariant Laplacian and Q is a generic matrix which may depend on background fields. To define the path integral requires specifying the measure. This is usually done implicitly by specifying the meaning of Gaussian integration. A sensible choice is

$$\int \mathcal{D}\phi \exp\left(-\int d^4x\sqrt{g}\,\bar{\phi}\phi\right) \equiv 1 \tag{4.1.2}$$

This defines $\tilde{\phi} = g^{1/4}\phi$ as the path integration variable and guarantees a manifestly diffeomorphism invariant measure.[1] For any internal symmetries it will often also be manifestly invariant since $\bar{\phi}$ is usually in the conjugate representation to ϕ. For classically Weyl invariant theories where ϕ has unit scaling dimension, one has $Q = -\frac{1}{6}\mathcal{R} + V$ where V is some conformal field of dimension 2. The Ricci scalar in Q combines with $\Box$ to give the conformally invariant Laplacian, $\Box + \mathcal{R}/6$. Unfortunately, the measure is *not* conformally invariant and this leads to the familiar conformal anomaly.[2]

[1] The measure is invariant because the "1" is invariant on the right side, the integrand is invariant on the left, and so the measure should be also.

[2] One could choose instead a different power of g in defining the measure to make it conformally invariant, but this would trade a conformal anomaly for a diffeomorphism anomaly.

Using the definition of Gaussian integration, the Euclidean effective action is given by

$$\Gamma_E = \mathrm{Tr}\log H = -\Gamma \tag{4.1.3}$$

where $H \equiv -\Box + Q$ and Γ is the Minkowski effective action. We would like to efficiently calculate properties of this object. One method to calculate $\mathrm{Tr}\log H$ is Schwinger's proper time technique. One makes use of the matrix equation[3]

$$\mathrm{Tr}\log H = -\mathrm{Tr}\int_0^\infty \frac{d\tau}{\tau}\exp(-\tau H). \tag{4.1.4}$$

which holds – up to an infinite constant – in the basis where H is diagonal. (To prove the equality, one differentiates both sides with respect to the eigenvalue of H.)

Usually H is afflicted with ultraviolet divergences. Then the above definition can be modified in several ways. One way, which is quite similar to dimensional regularization, is to add extra powers of τ in the definition of the trace:

$$[\mathrm{Tr}\log H]_s = -\mu^{2s}\mathrm{Tr}\int_0^\infty \frac{d\tau}{\tau^{1-s}}\exp(-\tau H). \tag{4.1.5}$$

The parameter μ has dimensions of mass and is added only to make the final result dimensionless. The integral then formally gives

$$[\mathrm{Tr}\log H]_s = -\mathrm{Tr}\left(\frac{H}{\mu^2}\right)^{-s}\Gamma(s) \tag{4.1.6}$$

Since the result is proportional to $\zeta_H(s)$, the zeta-function associated with H, this approach goes by the name of zeta-function regularization. Differentiating with respect to H gives

$$\left[\mathrm{Tr}\frac{1}{H}\right]_s = \mathrm{Tr}\left\{\left(\frac{H}{\mu^2}\right)^{-s}\frac{1}{H}\Gamma(s+1)\right\} \tag{4.1.7}$$

with the limit agreeing as s tends to zero.

Another method, which we shall adopt, is simply to introduce a small cutoff for the parameter τ:

$$[\mathrm{Tr}\log H]_\epsilon = -\mathrm{Tr}\int_\epsilon^\infty \frac{d\tau}{\tau}\exp(-\tau H). \tag{4.1.8}$$

Differentiating then gives

$$\left[\mathrm{Tr}\frac{1}{H}\right]_\epsilon = \mathrm{Tr}\left(e^{-\epsilon H}\frac{1}{H}\right) \tag{4.1.9}$$

The parameter ϵ has dimensions of length squared (or inverse energy squared).

[3]It is not necessary for the function in the integral to be an exponential. Any function f with certain boundary conditions – namely $f(0) = 1$ and $f(\infty) = 0$ sufficiently quickly – would work. The advantage of using the exponential is the ease of differentiating it.

In many problems, one can use either regulation scheme by working in a momentum basis, performing a derivative expansion, and then doing the resultant momentum integrals. But it is advantageous to have a formalism which does not require doing so directly. Such an approach is the heat kernel.[4]

The heat kernel is the formal operator $U(\tau) = \exp(-\tau H)$. Its two point function is given by

$$U(x, x'; \tau) = \langle x|e^{-\tau H}|x'\rangle. \tag{4.1.10}$$

and is subject to two conditions: the initial condition $U(x, x'; 0) = \delta(x, x')$ and the "heat equation"

$$\frac{dU}{d\tau} = -HU. \tag{4.1.11}$$

One is usually concerned with H's which are perturbatively related to the Laplacian $H_0 = -\partial^m \partial_m$ in flat space. This case is directly solvable via Fourier transform.[5] The result (written in four dimensions) is

$$U_0(x, x'; \tau) = \frac{1}{(4\pi\tau)^2} \exp\left(-|x - x'|^2/4\tau\right) \tag{4.1.12}$$

This can be generalized to $H = -\partial^m \partial_m + m^2$ for constant m^2 in d dimensions by[6]

$$U_0(x, x'; \tau) = \frac{1}{(4\pi\tau)^{d/2}} \exp\left(-|x - x'|^2/4\tau - \tau m^2\right) \tag{4.1.13}$$

but we will keep $d = 4$ in all our calculations.

When the model is modified with a potential or to include a Yang-Mills gauge field, one expects the corrections to U to come in a simple perturbative way. One takes

$$U(x, x'; \tau) = \frac{1}{(4\pi\tau)^2} \exp\left(-|x - x'|^2/4\tau\right) F(x, x'; \tau) \tag{4.1.14}$$

where $F(\tau)$ is assumed to be an analytic function in τ regular at $\tau = 0$ and obeying $F(x, x; 0) = 1$. Applying the heat equation to this ansatz for U gives

$$\frac{\partial F}{\partial \tau} + \frac{1}{\tau}(x^m - x'^m) D_m F = (-D^m D_m + Q) F \tag{4.1.15}$$

where we have taken $H = -\Box + Q$. [7] Taking $y = x - x'$, $\mathcal{O} = -H$, and writing $F = \sum_{n=0}^{\infty} a_n \tau^n / n!$, we find a set of recursion relations for the coefficients a_n

$$a_n + \frac{1}{n} y^m D_m a_n = \mathcal{O} a_{n-1} \tag{4.1.16}$$

---

[4]The heat kernel method has a long history, with much of its properties worked out originally by DeWitt [33]. A review of the heat kernel can be found in [34].

[5]This is the only location where a momentum basis calculation is used.

[6]Zeta function regularization essentially replaces d in this formula with $d - 2s$, which is why it is similar to dimensional regularization.

[7]If Q contains a constant mass term, one generally separates it out by positing F to have an overall factor $e^{-\tau m^2}$.

for $n \geq 1$, and

$$y^m D_m a_0 = 0. \tag{4.1.17}$$

for $n = 0$. These relations can be solved as power series in y for each coefficient, using the initial condition that $[a_0] = 1$, where the brakets denote taking the "coincident limit" of $y = x - x' \to 0$.

The inclusion of gravity requires one to reinterpret $|x - x'|^2 = y^2$ in a coordinate-invariant way. One makes the replacement $|x-x'|^2/2 \to \sigma$, where σ is a symmetric bi-scalar function (that is, a scalar function of both x and x'). The heat equation becomes

$$-\frac{2}{\tau}F + \frac{\sigma}{2\tau^2}F + \frac{\partial F}{\partial \tau} = \frac{1}{4\tau^2}\nabla^a\sigma\nabla_a\sigma F - \frac{\Box\sigma}{2\tau}F - \frac{1}{\tau}\nabla^a\sigma\nabla_a F - HF \tag{4.1.18}$$

In order for F to be analytic at $\tau = 0$, the term that goes as $1/\tau^2$ must be trivially satisfied, giving

$$2\sigma = \nabla^a\sigma\nabla_a\sigma. \tag{4.1.19}$$

This equation, together with $[\nabla_a\sigma] = 0$ and $[\nabla_a\nabla_b\sigma] = \eta_{ab}$ uniquely determines σ as $\sigma = \frac{1}{2}g_{mn}(x')(x-x')^m(x-x')^n + \mathcal{O}((x-x'))^3$. The remaining equation can be written in a form analogous to (4.1.15) provided we rescale F

$$F \to \Delta^{1/2}\tilde{F} \tag{4.1.20}$$

where Δ obeys

$$\nabla^a\sigma\nabla_a \log\Delta + \Box\sigma = 4 \tag{4.1.21}$$

with the initial condition $[\Delta] = 1$. The resultant equation reads

$$\frac{\partial\tilde{F}}{\partial\tau} + \frac{1}{\tau}\nabla^a\sigma\nabla_a\tilde{F} = \Delta^{-1/2}\mathcal{O}\Delta^{1/2}\tilde{F} \equiv \tilde{\mathcal{O}}\tilde{F} \tag{4.1.22}$$

where $\tilde{\mathcal{O}} = \Delta^{-1/2}\mathcal{O}\Delta^{1/2}$.

The bi-scalars σ and Δ are well-known from the study of geodesics. σ is the geodetic interval – half of the integral of ds^2 along the geodesic connecting x' to x. Δ is known as the Van Vleck-Morette determinant and represents the Jacobian between an arbitrary coordinate system and geodesic coordinates. The precise definitions of these objects will not concern us, since we will show that in a suitable coordinate system both σ and Δ take especially simple forms.

Expanding $\tilde{F}$ in a power series, we find the set of recursion relations

$$\tilde{a}_n + \frac{1}{n}\nabla^a\sigma\nabla_a\tilde{a}_n = \tilde{\mathcal{O}}\tilde{a}_{n-1} \tag{4.1.23}$$

for $n \geq 1$ and

$$\nabla^a\sigma\nabla_a\tilde{a}_0 = 0. \tag{4.1.24}$$

for $n = 0$. These relations were first written down by DeWitt [33] and solved recursively, using the $x \to x'$ limit of certain quantities to derive all of them.

The importance of these coefficients lies in recalling the definition of the regulated determinant:

$$[\mathrm{Tr}\log H]_\epsilon = -\mathrm{Tr}\int_\epsilon^\infty \frac{d\tau}{\tau}\exp(-\tau H) = -\int_\epsilon^\infty \frac{d\tau}{\tau}\mathrm{Tr}\langle x|U(\tau)|x\rangle \tag{4.1.25}$$

$$= -\int_\epsilon^\infty \frac{d\tau}{\tau}\frac{1}{(4\pi\tau)^2}\mathrm{Tr}\,\tilde{F}(x,x;\tau) \tag{4.1.26}$$

$$= -\int_\epsilon^\infty \frac{d\tau}{\tau}\frac{1}{(4\pi\tau)^2}\sum_{n=0}^\infty \frac{\tau^n}{n!}\mathrm{Tr}\,\tilde{[a_n]} \tag{4.1.27}$$

where we have used $[\sigma] = 0$ and $[\Delta] = 1$. The total effective action is given by the $x = x'$ limit of the coefficients a_n. In particular, the divergent terms in four dimensions are

$$[\mathrm{Tr}\log H]_\epsilon = -\frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{[a_0]}{2\epsilon^2} + \frac{[a_1]}{\epsilon} - \frac{[a_2]}{2}\log\epsilon + \text{finite}\right) \tag{4.1.28}$$

where the limit $x = x'$ has been taken.

Since the coincident limit of the heat kernel coefficients are by construction local, the divergences in the above expression can be removed by adding local counterterms. One can take

$$A_\epsilon^{ct} = +\frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{[a_0]}{2\epsilon^2} + \frac{[a_1]}{\epsilon} - \frac{[a_2]}{2}\log\epsilon\right) \tag{4.1.29}$$

and then the regulated trace can be defined as the limit where ϵ tends to zero

$$[\mathrm{Tr}\log H]_{reg} = \lim_{\epsilon\to 0}\left([\mathrm{Tr}\log H]_\epsilon + A_\epsilon^{ct}\right) \tag{4.1.30}$$

The result is explicitly ϵ-independent and corresponds to a minimal substraction scheme at one-loop.

This is not the only application of this method. In particular, any theory with a potential anomaly at one-loop can be understood by the nonzero symmetry transformation $\delta_g H$ where g is an element of the potentially anomalous symmetry group. (This can be seen to arise via the non-invariance of the path integral measure, which was Fujikawa's perspective [35].) Using the proper time regulation scheme, the transformation of the effective action is given by

$$\delta_g[\mathrm{Tr}\log H]_\epsilon = -\mathrm{Tr}\int_\epsilon^\infty \frac{d\tau}{\tau}\delta_g\exp(-\tau H) = \int_\epsilon^\infty d\tau\,\mathrm{Tr}\left(\delta_g H\exp(-\tau H)\right) \tag{4.1.31}$$

where we have used cyclicity of the trace. In most cases of interest, the anomaly has the form $\delta_g H = a\Lambda H + bH\Lambda$ for some numerical coefficients a and b and some quantity Λ which may or may not be local. Then using cyclicity of the trace, one finds

$$\delta_g[\mathrm{Tr}\log H]_\epsilon = (a+b)\int_\epsilon^\infty d\tau\,\mathrm{Tr}\left(\Lambda H\exp(-\tau H)\right) = (a+b)\mathrm{Tr}\left(\Lambda e^{-\epsilon H}\right)$$

$$= \frac{(a+b)}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{[\Lambda a_0]}{\epsilon^2} + \frac{[\Lambda a_1]}{\epsilon} + \frac{[\Lambda a_2]}{2} + \mathcal{O}(\epsilon)\right) \tag{4.1.32}$$

In the case of the conformal anomaly for a conformally invariant action, $a = 3$ and $b = -1$ (that is, $H' = e^{3\Lambda} H e^{-\Lambda}$) and Λ is a local function, one finds

$$\delta_c[\mathrm{Tr}\log H]_\epsilon = \frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{2\Lambda[a_0]}{\epsilon^2} + \frac{2\Lambda[a_1]}{\epsilon} + \Lambda[a_2] + \mathcal{O}(\epsilon)\right) \tag{4.1.33}$$

Usually (and we will demonstrate this) the coefficients $[a_n]$ are such that the conformal transformation of the counter terms cancels the effect of the two leading divergences. Then we may take $\epsilon \to 0$ for the finite regulated action and find the finite conformal anomaly depends only on $[a_2]$.

4.1.1 Analysis in normal coordinates

DeWitt's original analysis of the heat kernel coefficients was performed using the recursion relations and the differential equations for σ and Δ. This approach works reasonably well for the first few coefficients but quickly becomes unwieldy. A much more efficient method was developed by Avramidi [36], who was also the first to evaluate the coefficient $[a_4]$ in curved space. We will review how his approach works here using normal coordinates, which we reviewed in Section 2.1.2.

In normal coordinates, one would expect the geodetic interval to take the simple form

$$\sigma = \frac{y}{2} \tag{4.1.34}$$

where y is the normal coordinate for x centered at x'. In order for this choice to obey the required equation (4.1.19), one must have

$$\nabla_a \sigma = e_a{}^m y_m = \delta_a{}^m y_m = y_a \tag{4.1.35}$$

Normal coordinates possess the property that $y^m e_m{}^a = y^a$ as well as $y^a e_a{}^m = y^m$, but the condition we require is slightly different. It can be shown that if the stucture group is Riemannian plus some internal degrees of freedom, normal coordinates possess also this additional quality.[8]

The Van Vleck-Morette determinant is also quite simple in this coordinate system:

$$\Delta = \det(e_a{}^m) = \det(e_m{}^a)^{-1} \tag{4.1.36}$$

which is essentially the Jacobian between x and the normal coordinates y. It is straightforward to show this obeys (4.1.21).

The recursion relation for the coefficients now reads

$$\left(1 + \frac{D}{n}\right)\tilde{a}_n = \tilde{\mathcal{O}}\tilde{a}_{n-1} \tag{4.1.37}$$

[8] For Einstein-Cartan geometry with torsion, one can define normal coordinates using a Riemannian connection and then relate the results with Riemannian curvatures and derivatives to the torsioned quantities.

where $D \equiv \nabla^a\sigma\nabla_a = y^m\partial_m$ with the special case $D\tilde{a}_0 = 0$. These can be formally solved by taking $\tilde{a}_0 = 1$ and

$$\tilde{a}_n = \left(1 + \frac{D}{n}\right)^{-1} \tilde{\mathcal{O}} \left(1 + \frac{D}{n-1}\right)^{-1} \tilde{\mathcal{O}} \cdots (1 + D)^{-1} \tilde{\mathcal{O}}. \tag{4.1.38}$$

The operator $D \equiv y^m\partial_m$ can be thought of as the derivative along the Riemannian geodesic. It is formally a one-dimensional derivative and possesses eigenvalues $|n\rangle$, which are the totally symmetric n-tensors

$$|n\rangle = |b_1, \ldots, b_n\rangle \equiv \frac{1}{n!} y^{b_1} \cdots y^{b_n} \tag{4.1.39}$$

where $D\,|n\rangle = n\,|n\rangle$. Provided we are concerned only with quantities which are analytic in y (i.e. only those quantities which admit an analytic normal coordinate expansion) this set of eigenvalues forms a basis. Associated with these tensors are the dual tensors

$$\langle m| = \langle a_1, \ldots, a_m| = \partial_{a_1} \cdots \partial_{a_n}. \tag{4.1.40}$$

The inner product $\langle m|n\rangle$ is defined in the obvious way with $y = 0$ taken at the end:

$$\langle m|n\rangle = \delta_{mn}\delta^{b_1 \ldots b_n}_{a_1 \ldots a_n}. \tag{4.1.41}$$

We can therefore solve for $\tilde{a}_n$ as a power series in y. In the language of the bras and kets,

$$\tilde{a}_n = \sum_{k=0}^{\infty} |k\rangle\langle k|\tilde{a}_n\rangle \tag{4.1.42}$$

where

$$\begin{aligned}\langle k|\tilde{a}_n\rangle = \sum_{j_1,\ldots,j_{k-1}\geq 0} &\left(1 + \frac{k}{n}\right)^{-1} \left(1 + \frac{j_{n-1}}{n-1}\right)^{-1} \cdots (1 + j_1)^{-1} \times \\ &\langle k|\tilde{\mathcal{O}}|j_{n-1}\rangle\langle j_{n-1}|\tilde{\mathcal{O}}|j_{n-2}\rangle \cdots \langle j_1|\tilde{\mathcal{O}}|0\rangle\end{aligned} \tag{4.1.43}$$

The $y = 0$ limit of $\tilde{a}_n$ is given by $\langle 0|\tilde{a}_n\rangle = [\tilde{a}_n]$ and its kth order derivative given by the k-tensor $\langle k|\tilde{a}_n\rangle$.

The essence of (4.1.43) is that the heat kernel coefficients are given by matrix elements of the operator $\tilde{\mathcal{O}}$. To evaluate such elements, we first write $\tilde{\mathcal{O}}$ in terms of normal coordinates as

$$\tilde{\mathcal{O}} = X^{mn}\partial_m\partial_n + Y^m\partial_m + Z \tag{4.1.44}$$

For the case $\tilde{\mathcal{O}} = \Delta^{-1/2}(\nabla^a\nabla_a - Q)\Delta^{1/2}$, we find

$$\begin{aligned}X^{mn} &= g^{mn} \\ Y^m &= -2g^{mn}h_n + \partial_n g^{nm} \\ Z &= g^{mn}h_m h_n - \partial_n g^{nm}h_m - g^{mn}\partial_m h_n - Q + \Delta^{-1/2}\nabla^a\nabla_a\Delta^{1/2}\end{aligned} \tag{4.1.45}$$

where h_m is the connection found in $\nabla_m \equiv \partial_m - h_m$. Z can be rewritten as

$$Z = g^{mn} h_m h_n - \partial_n g^{nm} h_m - g^{mn} \partial_m h_n - Q$$
$$- \frac{1}{2} \partial_n g^{nm} \partial_m \log e - \frac{1}{2} g^{nm} \partial_m \partial_n \log e - \frac{1}{4} g^{nm} \partial_m \log e \partial_n \log e \tag{4.1.46}$$

which shows that the original operator $\tilde{\mathcal{O}}$ could have been written

$$\tilde{\mathcal{O}} = g^{-1/4} \nabla_m g^{mn} \sqrt{g} \nabla_n g^{-1/4} \tag{4.1.47}$$

This is an indicator that we have essentially used the scalar density $g^{1/4}\phi$ as the path integral variable. Moreover this operator is manifestly symmetric. We will encounter a similar structure when we deal with chiral superfields.

The divergences and anomalies are related to the $y = 0$ limits of the first three heat kernel coefficients. The zeroth coefficient is the simplest, $\langle 0|\tilde{a}_0\rangle = 1$, and gives the quartic divergence.

The quadratic divergence is given by the first coefficient

$$\langle 0|\tilde{a}_1\rangle = \langle 0|\tilde{\mathcal{O}}|0\rangle = [Z]$$

To evaluate $[Z]$, first note that

$$[Z] = -Q - \frac{1}{2}[\partial^m \partial_m \log e]$$

in normal coordinates as $y \to 0$. (Clearly $[\partial_m \log e]$ vanishes since there are no covariant vectors of the right dimension to correspond to it.) We need the expansion of $\log e$ to y^2. The vierbein in normal coordinates is given by

$$e_m{}^a = \delta_m{}^a + \frac{1}{6} R_{ymy}{}^a + \mathcal{O}(y^3)$$

where we have used the notation that a y in an index slot means a y is contracted with that index. Thus $\log e$ is given by

$$\log e = \frac{1}{6}\mathcal{R}_{yy} + \mathcal{O}(y^3)$$

where $\mathcal{R}_{ab}$ is the Ricci tensor. One easily finds

$$\langle 0|\tilde{a}_1\rangle = -Q - \frac{1}{6}\mathcal{R}. \tag{4.1.48}$$

The logarithmic divergences are given by the $y = 0$ limit of $\tilde{a}_2$:

$$\langle 0|\tilde{a}_2\rangle = \sum_{j_1=0} (1 + j_1)^{-1} \; \langle 0|\tilde{\mathcal{O}}|j_1\rangle \times \langle j_1|\tilde{\mathcal{O}}|0\rangle$$

Although the sum is over all values of j_1, the first matrix element vanishes for $j_1 \geq 3$. We easily find

$$\langle 0|\tilde{a}_2\rangle = [Z]^2 + \frac{1}{2}[Y^m][\partial_m Z] + \frac{1}{3}[X^{mn}][\partial_m \partial_n Z]$$

Using $[X^{mn}] = \eta^{mn}$ and $[Y^m] = 0$,

$$\langle 0|\tilde{a}_2\rangle = \left(Q + \frac{1}{6}\mathcal{R}\right)^2 + \frac{1}{3}[\partial^m\partial_m Z]$$

The remaining term is a little complicated to evaluate. Begin by expanding it out, using $[h_m] = 0$ and $[\partial_p g_{mn}] = 0$:

$$\begin{aligned}\frac{1}{3}[\partial^m\partial_m Z] = \frac{2}{3}[g^{pq}\partial^m h_p \partial_m h_q] - \frac{2}{3}[\partial^m \partial_p g^{pq}\partial^m h_q] - \frac{1}{3}[g^{pq}\partial^m\partial_m\partial_p h_q] \\ - \frac{1}{3}[\partial^m\partial_m \Delta^{-1/2}\nabla^a\nabla_a\Delta^{1/2}] - \frac{1}{3}[\partial^m\partial_m Q]\end{aligned}$$

where we have used that $[h_m] = 0$ and $[\partial_p g_{mn}] = 0$. Most of the terms can be evaluated by noting

$$g^{mn} = \eta^{mn} - \frac{1}{3}R_y{}^m{}_y{}^n + \mathcal{O}(y^3), \quad h_m = \frac{1}{2}\mathcal{F}_{ym} + \mathcal{O}(y^2)$$

These give

$$\frac{1}{3}[\partial^m\partial_m Z] = \frac{1}{6}\mathcal{F}^2 - \frac{1}{3}\Box Q - \frac{1}{3}[\partial^2\partial^m h_m] + \frac{1}{3}[\partial^2\Delta^{-1/2}\Box\Delta^{1/2}]$$

The gauge field h is given to cubic order by

$$h_n = \frac{1}{2}\mathcal{F}_{yn} + \frac{1}{3}\nabla_y\mathcal{F}_{yn} + \frac{1}{8}\nabla_y^2\mathcal{F}_{yn} - \frac{1}{4!}R_{yny}{}^b\mathcal{F}_{by} + \mathcal{O}(y^4)$$

and one easily finds $[\partial^m\partial_m\partial^n h_n] = 0$.

The remaining term is significantly more messy. After some work, we find

$$\begin{aligned}[\partial^m\partial_m\Delta^{-1/2}\nabla^a\nabla_a\Delta^{1/2}] &= -\frac{1}{5}\nabla^2\mathcal{R} - \frac{1}{30}\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{1}{45}R^{abcd}\left(R_{abcd} + R_{adcb}\right) \\ &= -\frac{1}{5}\nabla^2\mathcal{R} - \frac{1}{30}\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{1}{30}R^{abcd}R_{abcd}\end{aligned}$$

using the symmetry properties of the Riemann tensor. The second heat kernel coefficient (and the logarithmic divergences) is then given by

$$\langle 0|\tilde{a}_2\rangle = \left(Q + \frac{1}{6}\mathcal{R}\right)^2 + \frac{1}{6}\mathcal{F}^2 - \frac{1}{15}\Box\mathcal{R} - \frac{1}{90}\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{1}{90}R^{abcd}R_{abcd} - \frac{1}{3}\Box Q \tag{4.1.49}$$

It is useful to rewrite some of the quantities appearing here. The square of the conformal Weyl tensor can be written

$$C^{abcd}C_{abcd} = R^{abcd}R_{abcd} - 2\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{1}{3}\mathcal{R}^2 \tag{4.1.50}$$

This quantity (and C_{abcd} itself) transforms covariantly. The four dimensional Gauss-Bonnet term

$$\begin{aligned}L_\chi &= R^{abcd}R_{abcd} - 4\mathcal{R}^{ab}\mathcal{R}_{ab} + \mathcal{R}^2 \\ &= C^{abcd}C_{abcd} - 2\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{2}{3}\mathcal{R}^2\end{aligned} \tag{4.1.51}$$

is topological, its integral being invariant under arbitrary local (including conformal) deformations of the metric.

We can thereby rewrite $[a_2]$ as

$$\langle 0|\tilde{a}_2\rangle = \left(Q + \frac{1}{6}\mathcal{R}\right)^2 + \frac{1}{6}\mathcal{F}^2 - \frac{1}{3}\Box\left(Q + \frac{1}{6}\mathcal{R}\right) + \frac{1}{90}\left(\frac{3}{2}C^{abcd}C_{abcd} - \frac{1}{2}L_\chi - \Box\mathcal{R}\right) \tag{4.1.52}$$

It is worth noting that if we wanted H to transform covariantly under conformal transformations, we would choose $Q = -\frac{1}{6}\mathcal{R} + V$ where V transforms conformally. Then $[a_1]$ and $[a_2]$ would be

$$[a_1] = -V \tag{4.1.53}$$

$$[a_2] = V^2 + \frac{1}{6}\mathcal{F}^2 - \frac{1}{3}\Box V + \frac{1}{90}\left(\frac{3}{2}C^{abcd}C_{abcd} - \frac{1}{2}L_\chi - \Box\mathcal{R}\right) \tag{4.1.54}$$

and $[a_1]$ would be conformal (with dimension 2) and $[a_2]$ would be conformal (with dimension 4) up to total derivatives.

Thus if we calculate the conformal transformation of the counter-terms, we find

$$\delta_c A_\epsilon^{ct} = +\frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\left(4\Lambda\frac{[a_0]}{\epsilon^2} + 2\Lambda\frac{[a_1]}{\epsilon}\right) \tag{4.1.55}$$

and the regulated trace anomaly is finite and given by

$$\delta_c[\mathrm{Tr}\log H]_{reg} = \frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\,(\Lambda[a_2]) \tag{4.1.56}$$

4.2 Heat kernel analysis for Dirac operators

A common Dirac fermion model is

$$S = \int d^4x\sqrt{g}\,(\bar{\Psi}i\nabla\!\!\!/\Psi + \bar{\Psi}\mu\Psi) \tag{4.2.1}$$

where μ is a generic mass term and $\nabla\!\!\!/ = \gamma^a\nabla_a$ is a covariant derivative. Written in two-component notation, the Lagrangian is

$$\begin{pmatrix} \chi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} \mu\delta_\alpha{}^\beta & i\sigma^a_{\alpha\dot{\beta}}\nabla_a \\ i\bar{\sigma}_a^{\dot{\alpha}\beta}\nabla^a & \mu\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \begin{pmatrix} \psi_\beta \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix} \tag{4.2.2}$$

We assume Ψ and $\bar{\Psi}$ to transform in conjugate representations. This means that the Weyl fermion ψ is gauge conjugate not only to $\bar{\psi}$ but also to χ.

One can define the path integral of a Gaussian in the obvious way:

$$\int \mathcal{D}\Psi\exp\left(-\int d^4x\sqrt{g}\,\bar{\Psi}\Psi\right) \equiv 1 \tag{4.2.3}$$

This definition is clearly diffeomorphism, Lorentz, and gauge invariant and so we expect these symmetries to be non-anomalous. The (Euclidean) effective action is

$$\Gamma_E = -\mathrm{Tr}\log D \tag{4.2.4}$$

where

$$D = i\not{\nabla} + \mu \tag{4.2.5}$$

One normally proceeds using the standard fermion doubling trick, arguing that Γ_E cannot depend on the sign of μ. Equivalently, one could argue that Γ_E cannot depend on the convention for the gamma matrices. Either way, one can introduce a new operator with a relative sign flip between the kinetic and mass terms

$$\tilde{D} = -i\not{\nabla} + \mu \tag{4.2.6}$$

which should yield the same determinant as D. Then one may define

$$\Gamma_E = -\frac{1}{2}\mathrm{Tr}\log D - \frac{1}{2}\mathrm{Tr}\log\tilde{D} = -\frac{1}{2}\mathrm{Tr}\log(\tilde{D}D) \tag{4.2.7}$$

where[9]

$$\tilde{D}D = \mu^2 - i[\not{\nabla}, \mu] - \mathcal{F}_{ab}S^{ab} - \Box \tag{4.2.8}$$

A greater level of sophistication is required when the model of interest is chiral. Taking the above model with $\chi = \bar{\chi} = 0$ we find in two-component notation

$$S = \int d^4x\sqrt{g}\left(i\bar{\psi}_{\dot{\alpha}}\bar{\sigma}_b^{\dot{\alpha}\alpha}\nabla^b\psi_\alpha\right) \tag{4.2.9}$$

A Majorana mass term may be included:

$$S = \int d^4x\sqrt{g}\left(i\bar{\psi}_{\dot{\alpha}}\bar{\sigma}_b^{\dot{\alpha}\alpha}\nabla^b\psi_\alpha + \frac{1}{2}\psi^\alpha\mu\psi_\alpha + \frac{1}{2}\bar{\psi}_{\dot{\alpha}}\bar{\mu}\bar{\psi}^{\dot{\alpha}}\right) \tag{4.2.10}$$

The difficulty with this model arises because the simplest Lorentz invariant definition for the Gaussian path integration is

$$\int \mathcal{D}\psi\exp\left(-\frac{1}{2}\int d^4x\sqrt{g}\left(\psi^2 + \bar{\psi}^2\right)\right) \equiv 1 \tag{4.2.11}$$

For the massless case, the classical action is gauge invariant but the measure is not.[10]

[9]The same basic approach holds if we replace $\mu \to \mu + i\nu\gamma_5$. The only major modification is that one of the terms generated is linear in a derivative, $\nu\gamma_5\gamma^a\nabla_a$, which must be treated as a matrix connection. One absorbs it into a new definition of the derivative ∇' and again proceeds as before.

[10]The measure used here has the structure of a Majorana mass term, which in four dimensions joins objects of the same chirality. In $d = 2 + 4n$ dimensions, both the Majorana mass term and the Dirac mass term join objects of *opposite* chirality and so there is no Lorentz invariant way to define Gaussian integration. This is one way of explaining the celebrated gravitational (or Lorentz) anomaly found by Alvarez-Gaumé and Witten [37].

Explicit two-component notation can be avoided by combining ψ and $\bar{\psi}$ into a Majorana fermion Ψ_M where

$$\bar{\Psi}_M = \begin{pmatrix} \psi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix}, \quad \Psi_M = \begin{pmatrix} \psi_\beta \\ \bar{\psi}^{\dot{\beta}} \end{pmatrix}$$

Then the action reads

$$S = \frac{1}{2} \int d^4x \sqrt{g} \bar{\Psi}_M \left(i\hat{\nabla\!\!\!/} + \hat{\mu} \right) \Psi_M \tag{4.2.12}$$

with measure

$$\int \mathcal{D}\psi \exp\left(-\frac{1}{2} \int d^4x \sqrt{g} \bar{\Psi}_M \Psi_M \right) \equiv 1 \tag{4.2.13}$$

where $\hat{\mu} = \mathrm{Re}\,\mu + i\gamma_5\,\mathrm{Im}\,\mu$ is the Majorana mass and the Majorana derivative is

$$\hat{\nabla\!\!\!/} = \begin{pmatrix} 0 & \sigma^a_{\alpha\dot{\beta}} \tilde{\nabla}_a \\ \bar{\sigma}_a^{\dot{\alpha}\beta} \nabla^a & 0 \end{pmatrix} \tag{4.2.14}$$

where ∇_a is the derivative in the representation of ψ and $\tilde{\nabla}_a$ is the derivative in the conjugate representation of $\bar{\psi}$. This is problematic even in the massless case since the square of this object involves operators like $\tilde{\nabla}_a \nabla_b$ which do not transform covariantly and therefore make calculation especially difficult.

We restrict ourselves now to the case of vanishing Majorana mass. Defining $D \equiv i\hat{\nabla\!\!\!/}$, path integration yields a Pfaffian, which can be interpreted as the square root of a determinant:

$$\Gamma_E = -\log \mathrm{Pf}\, D = -\frac{1}{2} \mathrm{Tr} \log D \tag{4.2.15}$$

The properties of the effective action are then related to the properties of the determinant of the operator D. This operator can be thought of as a mapping

$$D : C_+(\mathbf{r}) \oplus C_-(\bar{\mathbf{r}}) \to C_+(\bar{\mathbf{r}}) \oplus C_-(\mathbf{r}) \tag{4.2.16}$$

where $\mathbf{r}$ is the representation of ψ, $\bar{\mathbf{r}}$ is that of $\bar{\psi}$, and $+$ and $-$ denote the positive and negative chirality sectors. As a formal operator, its determinant is ill-defined since the domain and range are different spaces; this is just another way of saying that its determinant does not transform in a gauge-invariant manner. One way of making sense of this object is to note that when the gauge coupling vanishes, D ceases to a problematic operator since there is no longer a distinction between a representation and its conjugate. Varying the trace with respect to the coupling, we find

$$\delta \mathrm{Tr} \log D = \mathrm{Tr} \left(D^{-1} \delta D \right) \tag{4.2.17}$$

If this expression can be suitably regulated and then integrated, we are left with a reasonable definition of the effective action. This approach was pioneered by Leutwyler [38] in the case of fermions and by McArthur and Osborn for the case of chiral superfields in background Yang-Mills [39].

Following Leutwyler, we regulate (4.2.17) by introducing the dual operator

$$\tilde{D} = \begin{pmatrix} 0 & -i\sigma^a_{\alpha\dot{\beta}}\nabla_a \\ -i\bar{\sigma}_a^{\dot{\alpha}\beta}\tilde{\nabla}^a & 0 \end{pmatrix} \tag{4.2.18}$$

so that

$$H = \tilde{D}D = \begin{pmatrix} -\Box - \mathcal{F}_{ab}\sigma^{ab} & 0 \\ 0 & -\tilde{\Box} - \tilde{\mathcal{F}}_{ab}\bar{\sigma}^{ab} \end{pmatrix} \tag{4.2.19}$$

$\mathcal{F}_{ab} = -[\nabla_a, \nabla_b]$ is the field strength associated with the covariant derivative and $\sigma^{ab} = \frac{1}{4}(\sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a)$ in the conventions of [6].

We define

$$L_\epsilon = \mathrm{Tr}\left(e^{-\epsilon H}D^{-1}\delta D\right) = \mathrm{Tr}\int_\epsilon^\infty d\tau \left(e^{-\tau H}\tilde{D}\delta D\right) \tag{4.2.20}$$

This operator can be separated into parts which are even and odd under parity: $L_\epsilon = L_\epsilon^+ + L_\epsilon^-$ where

$$L_\epsilon^+ = \frac{1}{2}\mathrm{Tr}\int_\epsilon^\infty d\tau \left(e^{-\tau H}\tilde{D}\delta D + e^{-\tau\tilde{H}}D\delta\tilde{D}\right) \tag{4.2.21}$$

$$L_\epsilon^- = \frac{1}{2}\mathrm{Tr}\int_\epsilon^\infty d\tau \left(e^{-\tau H}\tilde{D}\delta D - e^{-\tau\tilde{H}}D\delta\tilde{D}\right) \tag{4.2.22}$$

The operator $\tilde{H} = D\tilde{D}$ is the conjugate of H. Using cyclicity of the trace, one can immediately deduce that

$$L_\epsilon^+ = \frac{1}{2}\mathrm{Tr}\int_\epsilon^\infty d\tau \left(e^{-\tau H}\delta H\right) = \delta\left(-\frac{1}{2}\mathrm{Tr}\int_\epsilon^\infty \frac{d\tau}{\tau}e^{-\tau H}\right) = \frac{1}{2}\delta[\mathrm{Tr}\log H]_\epsilon \tag{4.2.23}$$

which is trivially integrable. In retrospect, the even part is certainly integrable since it corresponds to introducing a Weyl spinor $\bar{\chi}$ transforming as ψ; then one can simply combine ψ and $\bar{\chi}$ into a Dirac fermion. A straightforward calculation shows that

$$\frac{1}{2}[\mathrm{Tr}\log H]_\epsilon = -\frac{1}{32\pi^2}\left(\frac{\mathrm{Tr}[a_0^D]}{2\epsilon^2} + \frac{\mathrm{Tr}[a_1^D]}{\epsilon} - \frac{1}{2}\log\epsilon\,\mathrm{Tr}[a_2^D] + \text{finite}\right) \tag{4.2.24}$$

where

$$\mathrm{Tr}[a_0^D] = 4 \tag{4.2.25}$$

$$\mathrm{Tr}[a_1^D] = \frac{1}{3}\mathcal{R} \tag{4.2.26}$$

$$\mathrm{Tr}[a_2^D] = -\frac{4}{3}\mathrm{Tr}(F^{ab}F_{ab}) - \frac{1}{10}C^{abcd}C_{abcd} + \frac{11}{180}L_\chi + \frac{1}{15}\Box\mathcal{R} \tag{4.2.27}$$

The odd part is not generally integrable. If it were, then L_ϵ^- would be the variation of the odd part of the effective action. Interpreting the δ in L_ϵ^- as a differential operator, L_ϵ^- would be an exact form and would obey $\delta L_\epsilon^- = 0$. However, one can show that

$$C_\epsilon \equiv \delta L_\epsilon^- = \epsilon \int_0^1 d\lambda \, \mathrm{Tr}\left(\delta D e^{-\epsilon\lambda H} \delta\tilde{D} e^{-\epsilon\tilde{\lambda}\tilde{H}}\right) \tag{4.2.28}$$

(where $\tilde{\lambda} = 1 - \lambda$) does not vanish in the limit of vanishing ϵ due to singularities in the small ϵ limit of the heat kernel operators appearing in the expression. Since $\delta D = -\omega$ and $\delta\tilde{D} = -\tilde{\omega}$ are local operators, we can perform the trace with a single insertion of a complete set of states, giving

$$C_\epsilon = \epsilon \int d^4x\, d^4x' \sqrt{g}\sqrt{g'} \int_0^1 d\lambda \, \mathrm{Tr}\left(\omega(x) U(x, x'; \epsilon\lambda) \tilde{\omega}(x') \tilde{U}(x', x; \epsilon\tilde{\lambda})\right) \tag{4.2.29}$$

Since $\sigma(x, x') = \sigma(x', x)$ and $\Delta(x, x') = \Delta(x', x)$, the above can be written as

$$\begin{aligned} C_\epsilon = \frac{1}{(16\pi^2)^2\epsilon^3} \int_0^1 d\lambda \frac{1}{(\lambda\tilde{\lambda})^2} \int d^4x\, d^4x' \sqrt{g}\sqrt{g'} e^{-\sigma/2\epsilon\lambda\tilde{\lambda}} \Delta(x, x') \\ \mathrm{Tr}\left(\omega(x) F(x, x'; \epsilon\lambda) \tilde{\omega}(x') \tilde{F}(x', x; \epsilon\tilde{\lambda})\right) \end{aligned} \tag{4.2.30}$$

One chooses x' to be expanded in a normal coordinate system y' about x. Then rescaling $y' = y \times 2\sqrt{\epsilon\lambda\tilde{\lambda}}$

$$C_\epsilon = \frac{1}{16\pi^4\epsilon} \int d^4x \sqrt{g} \int_0^1 d\lambda \int d^4y \, e^{-y^2} \, \mathrm{Tr}\left(\omega(x) F(x, y'; \epsilon\lambda) \tilde{\omega}(y') \tilde{F}(y', x; \epsilon\tilde{\lambda})\right) \tag{4.2.31}$$

One generally finds that $\mathrm{Tr}(\omega\tilde{\omega})$ vanishes (it certainly does in this case) and the triviality of $[a_0]$ guarantees that the only contribution comes from the two a_1 coefficients:

$$C = \lim_{\epsilon\to 0} C_\epsilon = \frac{1}{32\pi^2} \int d^4x \sqrt{g} \, \mathrm{Tr}\left(\omega[a_1]\tilde{\omega} + \omega\tilde{\omega}[\tilde{a}_1]\right) = \frac{i}{8\pi^2} \int d^4x \sqrt{g} \, \mathrm{Tr}\left(\omega_a \omega_b \mathcal{F}_{cd}\right) \epsilon^{abcd} \tag{4.2.32}$$

where $\delta\mathcal{A}_b = \omega_b$. This vanishes precisely when the symmetrized trace of three generators vanishes. This is the standard anomaly cancellation condition and implies that the odd part of the effective action can indeed be defined.

Since C is by construction an exact local term, it can generally be represented as the variation of a local finite counterterm $-\ell$ (defined up to a closed form). Then one may add this counterterm to the L_ϵ^- and define (schematically)

$$\delta[\mathrm{Tr}\log D]_\epsilon \equiv \frac{1}{2}\delta[\mathrm{Tr}\log H]_\epsilon + \left(L_\epsilon^- + \ell\right) \tag{4.2.33}$$

$\mathrm{Tr}\log H$ is generally free of gauge (but not conformal) anomalies, and so the gauge anomaly is found in the two terms L_ϵ^- and ℓ by considering δD to have the form of a gauge transformation. Then L_ϵ^- gives the covariant gauge anomaly and ℓ a finite piece which ensures that the sum has the form of a consistent gauge anomaly. Since ℓ is defined only up to a closed form, the consistent gauge anomaly is defined only up to the gauge variation of some local term. The definition of $[\mathrm{Tr}\log D]_\epsilon$ so arrived at is not likely to coincide with what we would have found by naively squaring the operator, since the regulation method we have used here damps out the high energy spectrum of the gauge invariant operator H, whereas damping the high energy spectrum of D^2 does not have a gauge invariant meaning. The method used here is to be preferred since C is generally free of divergences and therefore the divergent part of $[\mathrm{Tr}\log D]_\epsilon$ is straightforwardly integrable. This procedure is quite analogous to the normal perturbative calculation, where one finds that the triangle diagram is not itself divergent but when regulated produces an ambiguity in the effective action which requires a prescription (which can be interpreted as the addition of a finite local counterterm) in order to be defined.

4.3 Mixed Laplace-Dirac operators

One generally does not encounter an isolated Dirac or Majorana fermion in models of physical interest; generally they mix with bosons. One therefore requires some general prescription for how to define the effective action in these situations.

For definiteness, we will give a specific schematic model:

$$S = \int d^4x\sqrt{g}\left(\frac{1}{2}\phi H\phi + \frac{1}{2}\bar{\Psi}D\Psi + \phi Q\Psi\right) \tag{4.3.1}$$

where Ψ is a Majorana fermion, ϕ is a boson, and Q is some interaction term. If Q were zero, one could proceed by combining results of the previous two sections, so we proceed in a way that generalizes most easily to the $Q = 0$ case. Naturally, such a method will be a perturbative one.

We begin by making note of the following Dyson-like expansion:

$$\begin{aligned} \mathrm{Tr}\log(H+V) &= -\int \frac{d\tau}{\tau}e^{-\tau(H+V)} \\ &= \mathrm{Tr}\log H + \int d\tau \mathrm{Tr}(e^{-\tau H}V) - \frac{1}{2}\int d\tau \int_0^\tau d\sigma \mathrm{Tr}(e^{-\sigma H}Ve^{-\tilde{\sigma} H}V) + \mathcal{O}(V^3) \end{aligned} \tag{4.3.2}$$

Here we have not specifically regulated the trace but one should assume some regularization scheme to be in play; the details do not specifically affect the calculation.

Our original action involves an operator $\mathcal{H}$ with the structure

$$\mathcal{H} = \begin{pmatrix} H & Q \\ Q^T & D \end{pmatrix}$$

when understood to act on the space $\Phi = (\phi, \Psi)$. We introduce the dual Dirac operator

$$\tilde{\mathcal{D}} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{D} \end{pmatrix}$$

Then we seek $\mathrm{Tr}\log\mathcal{H}$ via $\mathrm{Tr}\log\mathcal{H} = \mathrm{Tr}\log(\tilde{\mathcal{D}}\mathcal{H}) - \mathrm{Tr}\log(\tilde{\mathcal{D}})$. Writing

$$\tilde{\mathcal{D}}\mathcal{H} = \begin{pmatrix} H & Q \\ \tilde{D}Q^T & \tilde{D}D \end{pmatrix} = \mathcal{H}_0 + \mathcal{V}$$

where $\mathcal{H}_0$ is the diagonal and $\mathcal{V}$ the off-diagonal pieces, we then have the formal expression $\mathrm{Tr}\log\mathcal{H} = \mathrm{Tr}\log(\mathcal{H}_0 + \mathcal{V}) - \mathrm{Tr}\log\tilde{\mathcal{D}}$. Making use of the perturbative expasion in $\mathcal{V}$, we immediately arrive at

$$\begin{aligned} \mathrm{Tr}\log\mathcal{H} &\equiv \mathrm{Tr}\log H + \mathrm{Tr}\log D + \mathcal{C} \\ \mathcal{C} &= -\int d\tau \int_0^\tau d\sigma\, \mathrm{Tr}(e^{-\sigma H} Q e^{-\tilde{\sigma}\tilde{D}D}\tilde{D}Q^T) + \mathcal{O}(Q^4) \end{aligned} \tag{4.3.3}$$

where for $\mathrm{Tr}\log D$, we take the definition given in the previous section. Note that were it not for the need to regularize the τ integral, $\mathcal{C}$ would be formally independent of $\tilde{D}$ since the integration variables could be swapped from (τ, σ) to $(\sigma, \tilde{\sigma})$, with the $\tilde{\sigma}$ integration formally yielding $D^{-1}\tilde{D}^{-1}$.

Properly regulated, (4.3.3) serves as the *definition* of the effective action for a mixed Laplace-Dirac system. The advantage of this perturbative definition is that it is independent of certain choices made in the steps leading up to it. For example, if we were to have multiplied $\mathcal{H}$ on the right by $\tilde{\mathcal{D}}$, the perturbative result would be completely equivalent *even when regulated*.

Chapter 5

Effective action of chiral superfields

5.1 Physical motivation and previous work

As we have discussed, the most straightforward kinetic coupling of chiral superfields to (old) minimal supergravity involves an exponential factor involving the Kähler potential in the form

$$S = -\frac{3}{\kappa^2} \int d^8 z \, E \, e^{-\kappa^2 K/3} \tag{5.1.1}$$

Here we have restored the Planck length κ^2, which previously we have set to one. The limit of $\kappa^2 \to 0$ represents the decoupling of supergravity from the Kähler potential, and the globally supersymmetric Kähler term is restored with the familiar Kähler invariance of

$$K \to K + F + \bar{F} \tag{5.1.2}$$

In the locally supersymmetric case, the action is invariant under a certain combination of Kähler and super-Weyl transformations under which the determinant E of the supervierbein transforms counter to the Kähler potential. However, this coupling of K yields a noncanonical Einstein-Hilbert term which must be fixed either by a complicated component-level rescaling of the various supergravity fields [7], or via the reformulation of the geometry of superspace to the so-called Kähler superspace formulation [6].

In either formulation, calculating the effective action for chiral matter coupled to supergravity in superspace itself (thus maintaining manifest supersymmetry) is a difficult task. The Kähler formulation, while being more elegant for classical calculations, makes the origin of the supersymmetric form of the Kähler anomaly unclear [40], as it undoubtedly becomes intertwined with conformal transformations. On the other hand, calculating in the original formulation (as advocated in [40]) is clearly an inelegant task.

Here we advocate an alternative route. Having introduced the formulation of conformal superspace, the original action can be rewritten

$$S = -\frac{3}{\kappa^2} \int d^8 z \, E \, \Phi_0 \bar{\Phi}_0 e^{-\kappa^2 K/3} \tag{5.1.3}$$

where Φ_0 is the conformal compensator, originally introduced in [23] at the level of the tensor calculus. As is well known, the original Poincaré formulation is found by the gauge choice $\Phi_0 = 1$ while the Kähler formulation is found by the choice $\Phi_0 = e^{\kappa^2 K/6}$. The original Kähler symmetry in the conformal formulation is then a classical symmetry of the action provided we also transform

$$\Phi_0 \to \Phi_0 e^{\kappa^2 F/6} \tag{5.1.4}$$

We have subsequently shown how to expand generic actions coupling supergravity, super Yang-Mills, and chiral matter to quadratic order in quantum superfields in order to enable the calculation of one loop effects in arbitrary locally supersymmetric models in superspace.

As a first step toward that result, in this chapter we will formally construct the one loop effective action from all chiral loops[1]. Our approach to the calculation is not a new one, but constitutes a generalization and combination of two classic papers by McArthur [41] and one by Buchbinder and Kuzenko [42] calculating heat kernel coefficients in a Poincaré supergravity background[2]and another by McArthur and Osborn [39] about calculating anomalies in supersymmetric gauge theories.

This chapter is divided into two sections. In the first section, we consider the general case of chiral superfields coupled to arbitrary background supergravity and super Yang-Mills. The results are similar to those found in [41, 39], except for the change from Poincaré superspace to $U(1)$ superspace, which as we have shown can be understood as a gauge-fixed version of conformal superspace. In the second section, we apply the chiral loop calculation to the action (5.1.3) with the addition of a superpotential term. We find the covariant form of the reparametrization, Kähler, and gauge anomalies in a form which is non-perturbative in the Kähler potential, thus expanding the well-known results of [40] which restricted to a limited set of these anomalies. The remaining non-covariant part will be dependent on the precise choice of the definition of the effective action, and should presumably be fixed by details of the actual UV completion of the theory.

5.2 Setting up the problem

The standard textbook coupling of supergravity to chiral matter can be described by the conformal action[3]

$$\begin{aligned} S &= -3 \int d^4\theta E \bar{\Phi}_0 \Phi_0 e^{-K/3} + \left(\int d^2\theta \mathcal{E} \Phi_0^3 W + \text{h.c.} \right) \\ &= -3 \left[\bar{\Phi}_0 \Phi_0 e^{-K/3} \right]_D + \left(\left[\Phi_0^3 W \right]_F + \text{h.c.} \right) \end{aligned} \tag{5.2.1}$$

[1]We include the conformal compensator but exclude any chiral fields that may (and will) be introduced by the gauge-fixing procedure in the supergravity and super Yang-Mills sectors.

[2]McArthur worked in normal coordinates, which is the approach we will take in order to most easily apply Avramidi's non-recursive method. Buchbinder and Kuzenko worked in a generally covariant fashion and necessarily identified more of the interesting features of the supergeometry. See for example their followup paper [43] where the anomaly term was integrated.

In this expression, K is the Kähler potential, a Hermitian function of the chiral superfields Φ^i and their antichiral conjugates $\bar{\Phi}^{\bar{i}}$; W is the superpotential, a chiral function of only Φ^i; and Φ_0 is the conformal compensator, the only chiral superfield with non-vanishing conformal and $U(1)_R$ weights, which are 1 and 2/3, respectively. We denote the conformal and $U(1)_R$ weights of superfields by the ordered pair (Δ, w), so Φ_0 has weight $(1, 2/3)$ and $\bar{\Phi}_0$ has weight $(1, -2/3)$. The action is invariant to redefinitions of $\Phi_0 \to \Phi_0 e^{F/3}$ provided K and W transform as $K \to K + F + \bar{F}$ and $W \to e^{-F} W$. When Φ_0 is absorbed into the frame of superspace, its reparametrization becomes the super Weyl symmetry of Howe and Tucker [20] and the combined transformation is the Kähler transformation.

Because the conformal requirements of the action are satisfied by Φ_0, K and W are allowed to be arbitrary. To retrieve the original minimal supergravity formulation, one fixes the conformal gauge by taking $\Phi_0 = 1$. The formulation of Cremmer et al [44], found by taking $\Phi_0 = W^{-1/3}$, is strictly valid only when W nowhere vanishes. The formulation of Binetruy, Girardi and Grimm [6] corresponds to $\Phi_0 = e^{K/6}$. Yet in each of these formulations, the *quanta* of Φ_0 remain in the Poincaré supergravity sector. Therefore, we will avoid explicitly fixing the gauge of Φ_0 until after path integrals are taken.

This is not the only way to define a supergravity theory in superspace. Another possibility is to allow the fields Φ^i to have non-vanishing conformal dimension. One is immediately led to the more general form

$$S = [Z]_D + [P]_F + \left[\bar{P}\right]_{\bar{F}} \tag{5.2.2}$$

where Z is a weight $(2, 0)$ function of chiral superfields Φ^I and their conjugates, and P is a weight $(3, 2)$ purely chiral function. In the gauge where $Z = -3$, the Einstein-Hilbert term has the standard normalization. This more arbitrary choice is classically equivalent to the previous one by choosing to single out a particular chiral superfield of weight $(1, 2/3)$ and rescaling all of the other fields by it, turning them into projective variables. The Kähler symmetry is then a redefinition of the projective coordinates.

One may also choose to allow more general superfields than chiral ones. A linear superfield of weight $(2, 0)$ allows one to formulate new minimal supergravity, where the matter couplings can be described by

$$S \ni [LK]_D \tag{5.2.3}$$

Here K is a Hermitian function of chiral superfields Φ^i of vanishing weight. This theory is classically dual to (5.2.1) in the absence of a superpotential, which cannot be posed because Φ^i have vanishing $U(1)_R$ weight and so there is no way to formulate a function of them with the necessary dimension. Allowing non-vanishing dimension for the chiral superfields leads immediately to the more general form

$$S = [\mathcal{Z}]_D + [P]_F + \left[\bar{P}\right]_{\bar{F}} \tag{5.2.4}$$

where $\mathcal{Z}$ is weight $(2, 0)$ and P is $(3, 2)$. One can suppose $\mathcal{Z}$ to be linear in L, as $\mathcal{Z} = LK$, but there is no reason (beyond simplicity) to impose this constraint. (In fact, one may even introduce several linear superfields.)

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[3]For simplicity, we have neglected to include the possibility of a nontrivial holomorphic gauge coupling for the Yang-Mills sector.

These different conformal theories, even when classically dual, are not necessarily quantum mechanically equivalent. The major stumbling block is to formulate the Gaussian path integration for a quantum chiral superfield η of conformal dimension Δ. Only for $\Delta = 3/2$ (and therefore $U(1)_R$ weight $w = 1$) is the chiral Gaussian

$$\int \mathcal{D}\eta\mathcal{D}\bar{\eta}\exp\left(-\int d^2\theta\,\mathcal{E}\,\eta^T\eta + \text{h.c.}\right) \equiv 1 \tag{5.2.5}$$

conformal and $U(1)_R$ invariant. These last invariances are necessary for the chiral action to be supersymmetric. It is further evident that this definition of the measure is only gauge invariant if η is in a real representation of the gauge group.

For more general η, it is possible to construct a gauge invariant measure through the introduction of a field M[4]

$$\int \mathcal{D}\eta\mathcal{D}\bar{\eta}\exp\left(-\int d^2\theta\,\mathcal{E}\,\eta^T M\eta + \text{h.c.}\right) \equiv 1 \tag{5.2.6}$$

M here is assumed to have the appropriate transformation properties to render the measure gauge invariant. If an appropriate M is naturally furnished by the theory (as a function, perhaps, of the background fields) then it may be used, but more often no such object exists. Inserting a spurion field by hand does render a gauge invariant path integral, but this does not eliminate the anomaly. Instead of having an effective action which changes under a gauge transformation, one has an effective action which changes if a different M is chosen. These are, of course, the same thing.

For the original supergravity and chiral matter model (5.2.1), the conformal and $U(1)_R$ symmetries are effectively removed from the theory through the use of Φ_0 as a compensator field. All of the other fields Φ^i and their quanta η^i are chosen to have vanishing conformal and $U(1)_R$ weights, and Φ_0^3 is placed in all chiral superspace integrations. In this way, the chiral measure essentially becomes $\mathcal{E}\Phi_0^3$. These theories amount then to the choice $M = \Phi_0^3$. Any fields in complex representations of gauge groups must have their path integration defined using some other method, usually a perturbative method such as in [39].

This effectively converts the conformal theory with background Φ_0 into a Poincaré theory. The independent conformal and $U(1)_R$ symmetries of the original theory survive as Kähler transformations of the Poincaré theory. We note that if Φ_0 is used in this way, the choice $\Phi_0 = 1$ seems the simplest and most reasonable Gaussian path integration for the Poincaré theory, but the choice for the overall factor of the measure should presumably be equivalent to the choice of how precisely to regulate the theory.

We will be concerned with calculating anomalies and divergences involving chiral loops. Using the background field formalism, we split all chiral fields into a background piece Φ^i and a quantum variation η^i,

$$\Phi^i \to \Phi^i + \eta^i \tag{5.2.7}$$

[4]That the measure integral has the same structure as a mass term is not coincidental; one way to regulate the effective action involves using this measure field M in a way analogous to a Pauli-Villars field.

All of the above theories we have mentioned have a common structure for the part of the action quadratic in the quantum chiral superfield η^i:

$$S^{(2)} = \left[\bar{\eta}^{\bar{i}} Z_{\bar{i}j} \eta^j\right]_D + \frac{1}{2}\left(\left[\eta^i \mu_{ij} \eta^j\right]_F + \text{h.c.}\right) \tag{5.2.8}$$

Any D-terms of the form $\eta^i Z_{ij} \eta^j$ have been chirally projected and absorbed into μ_{ij}. In performing the splitting (5.2.7), we have broken any manifest reparametrization invariance. In many classical theories, chiral superfields parametrize a Kähler manifold with the reparametrization symmetry

$$\Phi^i \to \Lambda^i(\Phi) \tag{5.2.9}$$

This symmetry is manifested on the η as

$$\eta'^i = \frac{\partial \Lambda^i}{\partial \Phi^j} \eta^j + \mathcal{O}(\eta^2) = \Lambda^i{}_j \eta^j + \mathcal{O}(\eta^2) \tag{5.2.10}$$

In order to consistently truncate the expansion at the first term, one would need to introduce a chiral connection for the coordinates Φ [45]. Unfortunately, there is no natural object in the theory to play this role, (the Kähler affine connection being non-chiral). However, provided we work on shell, this will not be an issue.[5]

These concerns are not major ones at the moment. As far as we are concerned, the index i can be interpreted as a gauge index; hence we regard $S^{(2)}$ as simply

$$S^{(2)} = \left[\bar{\eta} Z \eta\right]_D + \frac{1}{2}\left(\left[\eta^T \mu \eta\right]_F + \text{h.c.}\right) \tag{5.2.11}$$

Writing this in Majorana form,

$$S^{(2)} = \frac{1}{2}\begin{pmatrix} \int \mathcal{E}\eta^T & \int \bar{\mathcal{E}}\bar{\eta} \end{pmatrix} \begin{pmatrix} \mu & \mathcal{P}Z^T \\ \bar{\mathcal{P}}Z & \bar{\mu} \end{pmatrix} \begin{pmatrix} \eta \\ \bar{\eta}^T \end{pmatrix} \tag{5.2.12}$$

The "column vector" on the right is an element of $C_+(\mathbf{r}) \oplus C_-(\bar{\mathbf{r}})$, where C_+ and C_- denote respectively the spaces of chiral and antichiral superfields and $\mathbf{r}$ and $\bar{\mathbf{r}}$ denote the representations. The matrix in the center can be thought of as an operator mapping $C_+(\mathbf{r}) \oplus C_-(\bar{\mathbf{r}})$ to the dual space $C_+(\mathbf{s}) \oplus C_-(\bar{\mathbf{s}})$. $\mathbf{r}$ and $\mathbf{s}$ are "dual" in the following way: their index structures are conjugate in the normal Yang-Mills sense, but their conformal and $U(1)_R$ charges are dual in the sense that they add to 3 and 2, respectively.

We can introduce some suitable measure by requiring that the path integral of

$$S_M = \frac{1}{2}\begin{pmatrix} \int \mathcal{E}\eta^T & \int \bar{\mathcal{E}}\bar{\eta} \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & \bar{M} \end{pmatrix} \begin{pmatrix} \eta \\ \bar{\eta}^T \end{pmatrix} \tag{5.2.13}$$

be unity. Then path integration of the action $S^{(2)}$ involves calculating the formal determinant of the operator

$$\begin{pmatrix} M & 0 \\ 0 & \bar{M} \end{pmatrix}^{-1} \begin{pmatrix} \mu & \mathcal{P}Z^T \\ \bar{\mathcal{P}}Z & \bar{\mu} \end{pmatrix} = \begin{pmatrix} M^{-1}\mu & M^{-1}\mathcal{P}Z^T \\ \bar{M}^{-1}\bar{\mathcal{P}}Z & \bar{M}^{-1}\bar{\mu} \end{pmatrix} \tag{5.2.14}$$

[5] Alternatively, one could choose to introduce a chiral metric by hand (which would presumably correspond to a "chiral measure metric" M_{ij}). But this only cloaks the anomaly in a different form.

on the space $C_+(\mathbf{r}) \oplus C_-(\bar{\mathbf{r}})$. This is an endomorphism by construction (i.e. its domain and range are the same space), so its determinant is at least formally sensible. Equivalently, one could also calculate

$$\begin{pmatrix} \mu & \mathcal{P}Z^T \\ \bar{\mathcal{P}}Z & \bar{\mu} \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & \bar{M} \end{pmatrix}^{-1} = \begin{pmatrix} \mu M^{-1} & \mathcal{P}Z^T\bar{M}^{-1} \\ \bar{\mathcal{P}}ZM^{-1} & \bar{\mu}\bar{M}^{-1} \end{pmatrix} \tag{5.2.15}$$

on the space $C_+(\mathbf{s}) \oplus C_-(\bar{\mathbf{s}})$.

The above structure can be clarified by the example of a chiral superfield in a background Yang-Mills field. We transform from the space of covariantly chiral superfields Φ (which obey $\nabla^{\dot{\alpha}}\Phi = 0$) to the space of conventionally chiral superfields ϕ (which obey $D^{\dot{\alpha}}\phi = 0$). The transformation to the conventionally chiral notation involves the introduction of the gauge prepotential V and the action reads

$$S = \left[\bar{\eta} e^V \eta\right]_D + \frac{1}{2}\left(\left[\eta^T \mu \eta\right]_F + \text{h.c.}\right) \tag{5.2.16}$$

where μ is some chiral Majorana mass term. The path integral measure can be defined by requiring the Gaussian integration of

$$S_M = \frac{1}{2}\left[\eta^T \eta\right]_F + \text{h.c.} \tag{5.2.17}$$

to yield unity. This amounts to choosing the spurionic measure field M to be unity in this particular gauge. The operator corresponding to $S^{(2)}$ is

$$\begin{pmatrix} \mu & -\frac{1}{4}\bar{D}^2 e^{V^T} \\ -\frac{1}{4}D^2 e^V & \bar{\mu} \end{pmatrix} \tag{5.2.18}$$

and maps the space $C_+ \oplus C_-$ to itself. By "degauging" the theory, we can define an operator whose determinant is at least sensible, however it it not particularly calculable. Its square yields operators like $\bar{D}^2 e^{V^T} D^2 e^V$ which are difficult to deal with unless in a real representation, and there is no clear reason that the action should be invariant under gauge transformations.[6]

In classical supergravity with a conformal compensator, the above action we considered would instead have the form

$$S^{(2)} = \left[\bar{\Phi}_0 \Phi_0 e^{-K/3}\, \bar{\eta} e^V \eta\right]_D + \frac{1}{2}\Big(\left[\Phi_0^3 \eta^T \mu \eta\right]_F + \text{h.c.}\Big) \tag{5.2.19}$$

with the measure

$$S_M = \frac{1}{2}\Big(\left[\Phi_0^3 \eta^T \eta\right]_F + \text{h.c.}\Big). \tag{5.2.20}$$

[6]Even if a series of η are chosen to have vanishing anomaly coefficients, the determinant defined above will *still* give an anomalous effective action. In this case, though, the anomaly will be cohomologically trivial: it can be removed by the addition of a local counterterm to the effective action.

This yields the operator

$$\begin{pmatrix} \mu & -\frac{1}{4}\Phi_0^{-3}\bar{\nabla}^2\bar{\Phi}_0\Phi_0 e^{-K/3}e^{V^T} \\ -\frac{1}{4}\bar{\Phi}_0^{-3}\nabla^2\bar{\Phi}_0\Phi_0 e^{-K/3}e^{V} & \bar{\mu} \end{pmatrix} \tag{5.2.21}$$

where ∇ is the conformally covariant derivative. Note this approach involves degauging the Yang-Mills structure but leaving the chiral superfields covariant with respect to the superconformal group. Thus the operator acts on the space $C_+(0) \oplus C_-(0)$ where 0 denotes the conformal weight of η. Different choices for the conformal gauge of Φ_0 give superficially different forms of the off-diagonal terms, but they are all conformally equivalent.

Another approach is to absorb a factor of $\Phi_0^{3/2}$ into η, or equivalently, split the measure factor onto both sides of the operator. This gives

$$\begin{pmatrix} \mu & -\frac{1}{4}\bar{\nabla}^2(\bar{\Phi}_0\Phi_0)^{-2}e^{-K/3}e^{V^T} \\ -\frac{1}{4}\nabla^2(\bar{\Phi}_0\Phi_0)^{-2}e^{-K/3}e^{V} & \bar{\mu} \end{pmatrix} \tag{5.2.22}$$

which acts on $C_+(3/2) \oplus C_-(3/2)$, but has the same determinant as (5.2.21). We will use this approach in what follows.

The structure of these operators is quite generic in conformal theories (or Poincaré theories with conformal compensators). One generally finds

$$\begin{pmatrix} \mu & \mathcal{P}e^{V^T}X^{-1/2} \\ \bar{\mathcal{P}}e^{V}X^{-1/2} & \bar{\mu} \end{pmatrix} \tag{5.2.23}$$

acting on the space $C_+(3/2) \oplus C_-(3/2)$. The projectors $\mathcal{P} = -\frac{1}{4}\bar{\nabla}^2$ and $\bar{\mathcal{P}} = \frac{1}{4}\nabla^2$ are conformally covariant, X is Hermitian function of conformal dimension two, and V is some generalized internal symmetry matrix. We will henceforth interpret V as a background gauge prepotential.

There is a classical invariance where a factor in $e^V/X^{1/2}$ may be considered either as a contribution to the $U(1)$ part of V or as a contribution to X. We will refer to this as the "$U(1)$ ambiguity." This classical symmetry is broken by our definition of the effective action, which treats e^V and X in an asymmetric way, and naturally an anomaly is introduced. It turns out that this anomaly term is cohomologically trivial – it is the variation of a local counterterm – and so the anomaly isn't truly physical.

In the operator (5.2.23), the dimension two object X could be eliminated by fixing the conformal gauge so that X is constant. There is an equivalent way of proceeding which does *not* explicitly fix the conformal symmetry. We may introduce conformally compensated derivatives $\mathcal{D}$ along with superfields R, G_c and X_α defined in terms of X so that X becomes covariantly constant and the derivatives become those of Poincaré $U(1)$ supergravity. Then $\mathcal{P} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)$ and $\bar{\mathcal{P}} = -\frac{1}{4}(\mathcal{D}^2 - 8\bar{R})$, where we use the supergravity conventions of [6]. This gives a structure that is formally identical to gauging X to be a constant, but because the conformal symmetry has only been hidden as opposed to fixed, it is a bit more aesthetically appeasing. Note that in this approach the $U(1)_R$ structure remains.[7]

[7] It is possible to remove even the $U(1)_R$ symmetry by introducing another compensator Y with weight $(0,1)$. The combination of X and Y can then be combined into a complex compensator Ψ of weight $(1,w)$ for arbitrary nonzero w. When $w = 2/3$, Ψ may be further restricted to be chiral, and the original Poincaré supergravity of [7] is recovered.

The similarity of the structure of (5.2.23) to the Dirac operator is compelling. We may define D as this operator in the massless limit

$$D \equiv \begin{pmatrix} 0 & \mathcal{P}e^{V^T}X^{-1/2} \\ \bar{\mathcal{P}}e^{V}X^{-1/2} & 0 \end{pmatrix} \tag{5.2.24}$$

and define its conjugate operator

$$\tilde{D} = \begin{pmatrix} 0 & -\mathcal{P}e^{-V}X^{-1/2} \\ -\bar{\mathcal{P}}e^{-V^T}X^{-1/2} & 0 \end{pmatrix} \tag{5.2.25}$$

In choosing $\tilde{D}$ to enable a Leutywler-like quantization, we have explicitly broken the classical $U(1)$ ambiguity since $e^{-V}/X^{1/2}$ is not invariant under the same exchange of $U(1)$ factors as its conjugate.

The Hermitian operator H is

$$H = \tilde{D}D = \begin{pmatrix} -\mathcal{P}e^{-V}X^{-1/2}\bar{\mathcal{P}}e^{V}X^{-1/2} & 0 \\ 0 & -\bar{\mathcal{P}}e^{-V^T}X^{-1/2}\mathcal{P}e^{V^T}X^{-1/2} \end{pmatrix} \tag{5.2.26}$$

Note that since $\tilde{D}$ is conjugate to D, the operators appearing in H are actually gauge covariant. We may absorb the various factors of e^V into gauge covariant derivatives (as well as commuting various factors of X past the derivatives) to yield

$$H = X^{-1}\begin{pmatrix} -\frac{1}{16}(\bar{\mathcal{D}}^2 - 8R)(\mathcal{D}^2 - 8\bar{R}) & 0 \\ 0 & -\frac{1}{16}(\mathcal{D}^2 - 8\bar{R})(\bar{\mathcal{D}}^2 - 8R) \end{pmatrix} \tag{5.2.27}$$

where we should properly interpret the space this acts on as $C_+(1, \mathbf{r}) \oplus C_-(-1, \mathbf{s})$, the 1 and -1 denoting just the $U(1)_R$ charges now, since the conformal structure has been hidden. (Before the conformal and $U(1)_R$ charges were related so we needed only specify the former.) Note that X appears only as an overall factor, compensating the conformal scale of the rest of the operator. In actual calculations, X can be presumed to be unity during calculations and then restored in the final results using dimensional analysis.

As we found in the case of the Dirac operator, the heat kernel expansion of this operator encodes a great deal of information, so we turn next to a derivation of that. Operators such as that above have been considered many times in the literature before [41, 22], but usually in the limit where the supergravity $U(1)_R$ was absent. This corresponds to the case where X is simply the product of a chiral and an antichiral superfield (i.e. $X = \Phi_0\bar{\Phi}_0$). As the $U(1)_R$ is quite necessary for our purposes, we will rederive similar results as those done before, but in the case where X is arbitrary and so the supergravity $U(1)_R$ field strength X_α does not necessarily vanish. Our results will therefore differ slightly from the literature by terms involving X_α.

5.3 Heat kernel for a generic chiral superfield

In deriving the heat kernel for a generic chiral superfield, we follow closely the setup of Buchbinder and Kuzenko from their classic paper [43] as summarized in their textbook

[22]. We refer the interested reader to their treatment of the subject. The major difference here is that we work in $U(1)$ supergravity and utilize normal coordinates in superspace in order to more easily apply Avramidi's non-recursive technique.

The first step in deriving anomalies and divergences of (5.2.24) is to analyze the heat kernel structure of (5.2.27). Recall that the heat kernel for a generic chiral superfield is the gauge and $U(1)_R$ covariant operator $e^{\tau \mathcal{O}_+}$ where

$$\mathcal{O}_+ \equiv \frac{1}{16}(\bar{\mathcal{D}}^2 - 8R)(\mathcal{D}^2 - 8\bar{R}) \tag{5.3.1}$$

acts on a chiral superfield of unit $U(1)_R$ weight. This generalizes the global supersymmetric $\frac{1}{16}\bar{D}^2 D^2$. Since the operator $\mathcal{O}_+$ acts only on chiral superfields, we may expand it out as

$$\begin{aligned}\mathcal{O}_+\phi =& \Box\phi + W^\alpha \mathcal{D}_\alpha \phi + \frac{1}{2}(\mathcal{D}^\alpha W_\alpha)\phi - iG^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}}\phi \\ &+ \frac{1}{2}\mathcal{D}^\alpha R \mathcal{D}_\alpha \phi + \frac{1}{2}R\mathcal{D}^2\phi - \frac{1}{2}\bar{\mathcal{D}}^2\bar{R}\phi + 4R\bar{R}\phi \\ &+ \frac{1}{2}(1-w)X^\alpha \mathcal{D}_\alpha\phi - \frac{1}{4}w(\mathcal{D}^\alpha X_\alpha)\phi\end{aligned} \tag{5.3.2}$$

where ϕ is assumed to be a chiral field of $U(1)_R$ weight w. Our concern will be the case $w = 1$, but we quote the general formula for reference. With the exception of the two terms involving W_α, which is specific to the gauge group of ϕ, all of the other terms in this expression are generic supergravity terms.

One begins with the chiral heat kernel for the free theory

$$U_0(\mathfrak{z}, \mathfrak{z}'; \tau) = \frac{1}{(4\pi\tau)^2}\exp\left(-|y - y'|^2/4\tau\right)(\theta - \theta')^2 \tag{5.3.3}$$

in chiral coordinates $\mathfrak{z} = (y, \theta)$, where $\bar{\mathcal{D}}^{\dot{\alpha}} = \partial^{\dot{\alpha}}$. The additional factor of $(\theta - \theta')^2$ is to reproduce the chiral delta function: $U_0(\mathfrak{z}, \mathfrak{z}'; 0) = \delta^4(y - y')\delta^2(\theta - \theta') = \delta^4(y - y')(\theta - \theta')^2$. We generalize this to

$$U(\tau) = \frac{1}{(4\pi\tau)^2}\exp\left(-\Sigma/2\tau\right) F \tag{5.3.4}$$

where $U(\mathfrak{z}, \mathfrak{z}'; \tau)$ (and F) is formally a bi-tensor chiral field of $U(1)_R$ weight 1 at *both* of its spacetime points. That is, for operators acting on $\mathfrak{z}$, U is $U(1)_R$ weight 1. However, under a *global* $U(1)_R$ phase transformation, U transforms with a total weight of 2, just as U_0 does. The chiral bi-scalar Σ has no chiral weight.

We demand $U(\tau)$ obey the heat equation

$$\frac{\partial U}{\partial \tau} = \mathcal{O}_+ U \tag{5.3.5}$$

where $\mathcal{O}_+ = \frac{1}{16}(\bar{\mathcal{D}}^2 - 8R)(\mathcal{D}^2 - 8\bar{R})$.

Before proceeding further, it is helpful to work out various operators we will encounter. The first is $\Box_+$, which is the chiral generalization of the d'Alembertian:

$$\begin{aligned}\Box_+\phi \equiv& \frac{1}{16}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}^2\phi \\ =& \Box\phi + W^\alpha \mathcal{D}_\alpha\phi + \frac{1}{2}(\mathcal{D}^\alpha W_\alpha)\phi - iG^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}}\phi \\ &+ \frac{1}{2}\mathcal{D}^\alpha R\mathcal{D}_\alpha\phi + \frac{1}{2}R\mathcal{D}^2\phi + \frac{1}{2}(1-w)X^\alpha\mathcal{D}_\alpha\phi - \frac{w}{4}(\mathcal{D}^\alpha X_\alpha)\phi \end{aligned} \tag{5.3.6}$$

This is related to $\mathcal{O}_+$ by

$$\mathcal{O}_+ = \Box_+ - \frac{1}{2}(\bar{\mathcal{D}}^2 - 8R)\bar{R} \tag{5.3.7}$$

Note that $\Box_+$ vanishes on a covariantly constant ϕ, while $\mathcal{O}_+$ includes an extra supergravity "mass" term.

Also of use will be the chiral generalization of $\mathcal{D}^a\Sigma\mathcal{D}_a\phi$, which following Buchbinder and Kuzenko, we denote $\Sigma * \phi$:

$$\begin{aligned}\Sigma * \phi \equiv& \frac{1}{16}(\bar{\mathcal{D}}^2 - 8R)\left(\mathcal{D}^\alpha\Sigma\mathcal{D}_\alpha\phi\right) \\ =& \mathcal{D}^a\Sigma\mathcal{D}_a\phi + \frac{R}{2}\mathcal{D}^\alpha\Sigma\mathcal{D}_\alpha\phi - \frac{1}{4}w\mathcal{D}^\alpha\Sigma X_\alpha\phi + \frac{1}{2}\mathcal{D}^\alpha\Sigma W_\alpha\phi\end{aligned} \tag{5.3.8}$$

In terms of these new operations, the chiral heat equation takes the form

$$-\frac{2}{\tau}F + \frac{\Sigma}{2\tau^2}F + \frac{\partial F}{\partial\tau} = \mathcal{O}_+F - \frac{1}{2\tau}\Box_+\Sigma\, F + \frac{1}{4\tau^2}\left(\Sigma * \Sigma\right)\, F - \frac{1}{\tau}\Sigma * F \tag{5.3.9}$$

which should be compared to the corresponding bosonic equation (4.1.18). As before, we demand the $1/\tau^2$ term yield an identity

$$2\Sigma = \Sigma * \Sigma \tag{5.3.10}$$

This equation is consistent with the chirality requirement of Σ. The remaining term for F can be simplified if we rescale F by $F = \Delta^{1/2}\tilde{F}$ where Δ is some chiral determinant. The result is

$$-\frac{2}{\tau}\tilde{F} + \frac{\partial\tilde{F}}{\partial\tau} = \tilde{\mathcal{O}}_+\tilde{F} - \frac{1}{2\tau}\Box_+\Sigma\,\tilde{F} - \frac{1}{\tau}\Sigma * \tilde{F} - \frac{1}{2\tau}\left(\Sigma * \log\Delta\right)\tilde{F}$$

where $\tilde{\mathcal{O}}_+ = \Delta^{-1/2}\mathcal{O}_+\Delta^{1/2}$. We require Δ to obey the chiral equation

$$4 = \Box_+\Sigma + \Sigma * \log\Delta. \tag{5.3.11}$$

Provided there is no barrier to finding a chiral Σ and Δ which obey these properties, we find the simple chiral equation

$$\frac{\partial\tilde{F}}{\partial\tau} + \frac{1}{\tau}D\tilde{F} = \tilde{\mathcal{O}}_+\tilde{F} \tag{5.3.12}$$

where we have introduced the chiral operator $D\tilde{F} \equiv \Sigma * \tilde{F}$ to mimic the final form of the bosonic expression (4.1.22). Given the similarity between the above formulae and the bosonic formulae, we expect their solution to take roughly the same form. Aside from some complications and some simplifications, this will be the case.

Note that we have not yet specified the chiral weight of Δ and $\tilde{F}$. In the non-supersymmetric case, Δ was given in normal coordinates by e^{-1}; we expect the chiral Δ to be given in normal coordinates by $\mathcal{E}^{-1}$. Thus we shall take Δ to have chiral weight 2 on its $\mathfrak{z}$ coordinate and -2 on its $\mathfrak{z}'$ coordinate, and so $\tilde{F}$ has vanishing chiral weight on $\mathfrak{z}$ but weight 2 on $\mathfrak{z}'$.

5.3.1 Chiral normal coordinates

Before proceeding to a comprehensive analysis of the chiral heat kernel, we need to construct a useful set of normal coordinates as in the non-supersymmetric case. Here the procedure is a little more sophisticated, since we have coordinates associated with P, Q, and $\bar{Q}$ and so several ways one might define a normal coordinate system.

Recall that normal gauge in bosonic cooridnates was defined by requiring that the Taylor expansion $\phi(y) = e^{y\partial}\phi$ match the covariant Taylor expansion $\phi(y) = e^{y\cdot P}\phi$ where P was the formal parallel transport operator (i.e. the covariant derivative). In superspace, there are three distinct coordinates $(x, \theta, \bar{\theta})$ and – even in flat superspace – several different ways of constructing a normal coordinate system. Within global supersymmetry, Hermitian (or vector) superspace is defined by

$$\Psi(x, \theta, \bar{\theta}) = \exp(xP + \theta Q + \bar{\theta}\bar{Q})\Psi \tag{5.3.13}$$

whereas chiral superspace is defined by

$$\Psi(y, \theta, \bar{\theta}) = \exp(yP + \theta Q)\exp(\bar{\theta}\bar{Q})\Psi. \tag{5.3.14}$$

where Ψ is an arbitrary superfield. The advantage of chiral superspace is that the chirality condition reduces to independence of the coordinate $\bar{\theta}$ (since formally $\bar{Q}$ annihilates any chiral superfield). Thus $D^{\dot{\alpha}} = \partial^{\dot{\alpha}}$ and the antichiral vierbein $E^{\dot{\mu}A}$ and its inverse $E^{\dot{\alpha}M}$ are especially simple.

We require a chiral set of normal coordinates so we shall follow suit in placing $\exp(\bar{\theta}\bar{Q})$ to the far right. However, there are several ways in which one might define the remainder. The simple Lorentz invariant options are

$$\exp(yP + \eta Q), \quad \exp(yP)\exp(\eta Q), \quad \text{or} \quad \exp(\eta Q)\exp(yP)$$

where we introduce η to denote the normal coordinate difference between θ and θ'. Within global supersymmetry, these are equivalent since $[Q, P]$ vanishes, but not so in curved superspace. The first is the most symmetric and yields a normal mode expansion in y and η completely analogous to the bosonic case. The second is the one most useful when the spinor connections need to be simplified. In fact, in converting an F-term integral to a component x-space integral, one works in a coordinate system that amounts to having extracted $\exp(\eta Q)$ to the far right. That this is suitable for components is clear by noting that the expansion of $\phi(y,\eta)$ then looks like

$$\mathcal{D}_y \cdots \mathcal{D}_y \mathcal{D}_\eta \cdots \mathcal{D}_\eta \phi.$$

which is how one would naturally order these derivatives when projecting to lowest components.

However, both of these latter two coordinate systems turn out to lack the properties we will need. It turns out that the best system for our purposes is the third. We define therefore

$$G \equiv \exp(\eta Q) \exp(yP) \exp(\bar{\eta}\bar{Q}) \tag{5.3.15}$$

The connections are then found by first differentiating G,

$$G^{-1}\partial_M G = \tilde{E}_M{}^A P_A + \tilde{H}_M{}^{\underline{b}} X_{\underline{b}} \tag{5.3.16}$$

and then operating with G on the result:[8]

$$E_M{}^A \equiv G\tilde{E}_M{}^A, \quad H_M{}^{\underline{b}} \equiv G\tilde{H}_M{}^{\underline{b}}.$$

Here P_A represents the formal translation operator (which is represented on fields by the covariant derivative) and the set of $X_{\underline{b}}$ consists of Lorentz, $U(1)_R$, and Yang-Mills generators. $H_M{}^{\underline{b}}$ are the connections corresponding to the $X_{\underline{b}}$.

One immediately finds for $M = \dot{\mu}$ the connections take the rather simple form

$$E^{\dot{\mu}A} = \delta^{\dot{\mu}A}\left(1 - \bar{\eta}^2 R\right), \quad \omega^{\dot{\mu}}(M) = \frac{1}{2}\bar{\eta}_{\dot{\alpha}} R^{\dot{\alpha}\dot{\mu}}(M), \quad A^{\dot{\mu}} = 0, \quad \mathcal{A}^{\dot{\mu}} = 0 \tag{5.3.17}$$

Here we use an italicized $\mathcal{A}$ for the Yang-Mills connection to distinguish it from the supergravity $U(1)_R$ connection A. The inverse vierbein is easily found and allows us to write the connections with a Lorentz form index

$$E^{\dot{\alpha}M} = \delta^{\dot{\alpha}M}\left(1 + \bar{\eta}^2 R\right), \quad \omega^{\dot{\beta}}(M) = \frac{1}{2}\bar{\eta}_{\dot{\alpha}} R^{\dot{\alpha}\dot{\beta}}(M), \quad A^{\dot{\alpha}} = 0, \quad \mathcal{A}^{\dot{\alpha}} = 0 \tag{5.3.18}$$

from which it is straightforward to show that when acting on an arbitrary superfield Ψ without any dotted spinor indices,

$$(\bar{\mathcal{D}}^2 - 8R)\Psi = \partial_{\dot{\mu}}\partial^{\dot{\mu}}(1 + 2\bar{\eta}^2 R)\Psi \tag{5.3.19}$$

and so the result is explicitly independent of $\bar{\eta}$ and therefore chiral.

For $M = m$, the connections are given by

$$\tilde{W}_m = \exp(-\bar{\eta}\bar{Q}) e^{-yP} \partial_m e^{yP} \exp(\bar{\eta}\bar{Q}) \tag{5.3.20}$$

Defining

$$\hat{W}_m = e^{-yP}\partial_m e^{yP} \tag{5.3.21}$$

[8] One can simplify the last step by reinterpreting the tilded connections as having an extra implicit y dependence in all the covariant terms, replacing each with their covariant Taylor expansion.

we then have

$$\begin{aligned}\tilde{W}_m &= \exp(-\bar{\eta}\bar{Q})\hat{W}_m{}^A \exp(\bar{\eta}\bar{Q}) \times \exp(-\bar{\eta}\bar{Q}) X_A \exp(\bar{\eta}\bar{Q}) \\ &= \exp(-\bar{\eta}\bar{Q})\hat{W}_m{}^B \exp(\bar{\eta}\bar{Q}) \times X(\bar{\eta})_B{}^A X_A \end{aligned} \tag{5.3.22}$$

The final result is

$$W_m{}^A = \left(e^{\eta Q} e^{yP} \hat{W}_m{}^A\right) \times GX(\bar{\eta})_B{}^A \tag{5.3.23}$$

Note that $X(0)_B{}^A = \delta_B{}^A$.

For $M = \mu$, the connections are given by

$$\tilde{W}_\mu = \exp(-\bar{\eta}\bar{Q}) e^{-yP} e^{-\eta Q} \partial_\mu e^{\eta Q} e^{yP} \exp(\bar{\eta}\bar{Q}) \tag{5.3.24}$$

We first define

$$\hat{W}_\mu = e^{-\eta Q} \partial_\mu e^{\eta Q} \tag{5.3.25}$$

which is rather simple. One finds

$$\hat{E}_\mu{}^A = \delta_\mu{}^A \left(1 - \eta^2 \bar{R}\right), \quad \hat{\omega}_\mu(M) = \frac{1}{2}\eta^\alpha R_{\alpha\mu}(M), \quad \hat{A}_\mu = 0, \quad \hat{\mathcal{A}}_\mu = 0 \tag{5.3.26}$$

Defining $G_y = \exp(yP)$ and $G_{\bar{\eta}} = \exp(\bar{\eta}\bar{Q})$, we then have

$$\begin{aligned}\tilde{W}_\mu &= G_{\bar{\eta}}^{-1} G_y^{-1} \hat{W}_\mu{}^A G_y G_{\bar{\eta}} \times G_{\bar{\eta}}^{-1} G_y^{-1} X_A G_y G_{\bar{\eta}} \\ &= G_{\bar{\eta}}^{-1} G_y^{-1} \hat{W}_\mu{}^B G_y G_{\bar{\eta}} \times X(y, \bar{\eta})_B{}^A X_A \end{aligned} \tag{5.3.27}$$

which gives

$$W_\mu{}^A = \left(G_\eta \hat{W}_\mu{}^B\right) \times GX(y, \bar{\eta})_B{}^A \tag{5.3.28}$$

We are most interested in the case where $\bar{\eta} = 0$, since our heat kernel has $\bar{\theta}'$ equal to $\bar{\theta}$. Following the non-supersymmetric case, we would like to define $\Sigma = y^2/2$. For this to work requires $E_a{}^m y_m = y_a$ as well as $E_\alpha{}^m y_m = 0$ – both of which we take when $\bar{\eta}$ vanishes but for arbitrary y and η. Note that if we define $Y^M = (y^m, 0, 0)$, then the above conditions – along with $E^{\dot{\alpha}m} = 0$ which always holds in chiral coordinates – lead to

$$E_A{}^M Y_M = Y_A \iff Y_M = E_M{}^A Y_A$$

so we require $E_m{}^a y_a = y_m$ and $E_\mu{}^a y_a = 0$. The first is easy to see. It follows from $\hat{E}_m{}^a y_a = y_m$, which is true just as in the non-supersymmetric case. Any term generated in $\hat{E}_m{}^a$ past the leading term arose from commuting a P with a P or with an M. (No P can be generated by commuting a P with a Q or $\bar{Q}$.) Thus all the terms with a free index a will be of the form $T_{cy}{}^a$ or $R_{DCy}{}^a$. The latter vanishes by antisymmetry of the final two indices and the former vanishes since in the space we have, the bosonic torsion T_{cba} is totally antisymmetric. (It is proportional to $G^d \epsilon_{dcba}$.)

The condition for $E_\mu{}^a y_a = 0$ follows for essentially the same reason. One notes that since the only nonzero hatted connections are $\hat{E}_\mu{}^\alpha$ and $\hat{\omega}_\mu(M)$, we need only show that $X_\alpha{}^a y_a = 0$ and $X_{(M)}{}^a y_a = 0$. The Lorentz term vanishes since conjugating M_{cd} by e^{-yP} only gives a P from terms that look like $[M, yP]$ or $[P, yP]$ – these both vanish as in the non-supersymmetric case. The Q_α term vanishes since the only way to generate a P from commuting several yP's with the initial Q_α is to first generate an M, then commute $[M, yP]$. (This is because $[Q, P]$ by itself does not generate a P.)

Thus we are free to define $\Sigma = y^2/2$. This then obeys

$$2\Sigma = \Sigma * \Sigma = \mathcal{D}^a \Sigma \mathcal{D}_a \Sigma + 0 = y^a y_a \tag{5.3.29}$$

trivially. Note this result is consistently chiral.

Next we turn to our definition of Δ. We define $\Delta = \det(E_\mathcal{A}{}^\mathcal{M}) = \mathcal{E}^{-1}$ where we understand the indices $\mathcal{A}$ and $\mathcal{M}$ in $\mathcal{E}$ to be only over (a, α) and (m, μ). We require

$$4 = \Box_+ \Sigma + \Sigma * \log \Delta. \tag{5.3.30}$$

which amounts to

$$4 = \Box\Sigma - iG^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}}\Sigma + \frac{1}{2}\mathcal{D}^\alpha R \mathcal{D}_\alpha \Sigma + \frac{1}{2} R \mathcal{D}^2 \Sigma + \mathcal{D}^a \Sigma \mathcal{D}_a \log \Delta + \frac{R}{2}\mathcal{D}^\alpha \Sigma \mathcal{D}_\alpha \log \Delta$$

Proceeding in a way analogous to the non-supersymmetric case, we consider taking a derivative of $\log \Delta$:

$$\mathcal{D}_\mathcal{A} \log \Delta = \mathcal{D}_\mathcal{A} E_\mathcal{B}{}^\mathcal{M} E_\mathcal{M}{}^\mathcal{B} = E_\mathcal{M}{}^\mathcal{B} \mathcal{D}_\mathcal{B} E_\mathcal{A}{}^\mathcal{M} - T_{\mathcal{A}\mathcal{B}}{}^\mathcal{M} E_\mathcal{M}{}^\mathcal{B}$$

Here we are using an implicit grading for the indices. Since $E^{\dot{\mu}\mathcal{B}}$ vanishes, the last term becomes a trace of the torsion tensor in the chiral space. The remaining terms become

$$\begin{aligned}
\mathcal{D}_\mathcal{A} \log \Delta &= \mathcal{D}_\mathcal{M} E_\mathcal{A}{}^\mathcal{M} - E_{\mathcal{M}\dot{\beta}} \mathcal{D}^{\dot{\beta}} E_\mathcal{A}{}^\mathcal{M} - T_{\mathcal{A}\mathcal{B}}{}^\mathcal{B} \\
&= \mathcal{D}_\mathcal{M} E_\mathcal{A}{}^\mathcal{M} + E_{\mathcal{M}\dot{\beta}} T^{\dot{\beta}}{}_\mathcal{A}{}^\mathcal{M} - T_{\mathcal{A}\mathcal{B}}{}^\mathcal{B} \\
&= \mathcal{D}_\mathcal{M} E_\mathcal{A}{}^\mathcal{M} + \left(T_{\dot{\beta}\mathcal{A}}{}^{\dot{\beta}} - T_{\dot{\beta}\mathcal{A}}{}^D E_{D\dot{\mu}} E^{\dot{\mu}\dot{\beta}}\right) - T_{\mathcal{A}\mathcal{B}}{}^\mathcal{B} \\
&= \mathcal{D}_\mathcal{M} E_a{}^\mathcal{M} - T_{\mathcal{A}\mathcal{B}}{}^\mathcal{B} + T_{\mathcal{A}\dot{\beta}}{}^\mathcal{D} E_\mathcal{D}{}^{\dot{\beta}}
\end{aligned}$$

This gives (using $T_a{}^\beta{}_\beta = 2iG_a$)

$$4 = \mathcal{D}_\mathcal{M}\left(\mathcal{D}^a \Sigma E_a{}^\mathcal{M} + \frac{R}{2}\mathcal{D}^\alpha \Sigma E_\alpha{}^\mathcal{M}\right)$$

Since the result in the parentheses is invariant under all symmetry operations, we can replace the overall $\mathcal{D}_\mathcal{M}$ by $\partial_\mathcal{M}$. Since the derivative involves only y and η derivatives, we can cleanly set $\bar{\eta} = 0$ within the parentheses, which leave behind a single factor of y^m within, giving the result.

For the calculation of the chiral heat kernel, we will need the vierbein to second order in the coordinates y and η. Omitting the details, the result is

$$
\begin{aligned}
E_m{}^a &= \delta_m{}^a + \frac{1}{2}T_{ym}{}^a + \frac{1}{3}\mathcal{D}_y T_{ym}{}^a + \frac{1}{2}\mathcal{D}_\eta T_{ym}{}^a - \frac{1}{6}T_{ym}{}^b T_{by}{}^a + \frac{1}{6}R_{ymy}{}^a \\
E_m{}^\alpha &= \frac{1}{2}T_{ym}{}^\alpha + \frac{1}{3}\mathcal{D}_y T_{ym}{}^\alpha + \frac{1}{2}\mathcal{D}_\eta T_{ym}{}^\alpha - \frac{1}{6}T_{ym}{}^B T_{By}{}^\alpha \\
E_{m\dot\alpha} &= \frac{1}{2}T_{ym\dot\alpha} + \frac{1}{3}\mathcal{D}_y T_{ym\dot\alpha} + \frac{1}{2}\mathcal{D}_\eta T_{ym\dot\alpha} - \frac{1}{6}T_{ym}{}^B T_{By\dot\alpha} \\
E_\mu{}^\alpha &= \delta_\mu{}^\alpha + T_{y\mu}{}^\alpha + \frac{1}{2}\mathcal{D}_y T_{y\mu}{}^\alpha - \frac{1}{2}T_{y\mu}{}^\beta T_{\beta y}{}^\alpha - \frac{1}{2}T_{y\mu\dot\beta}T^{\dot\beta}{}_y{}^\alpha + \mathcal{D}_\eta T_{y\mu}{}^\alpha - \eta^2 \bar{R}\,\delta_\mu{}^\alpha \\
E_\mu{}^a &= \frac{1}{2}R_{y\mu y}{}^a + \frac{1}{2}R_{\eta\mu y}{}^a \\
E_{\mu\dot\alpha} &= T_{y\mu\dot\alpha} + \frac{1}{2}\mathcal{D}_y T_{y\mu\dot\alpha} - \frac{1}{2}T_{y\mu}{}^\beta T_{\beta y\dot\alpha} - \frac{1}{2}T_{y\mu\dot\beta}T^{\dot\beta}{}_{y\dot\alpha}
\end{aligned}
\tag{5.3.31}
$$

We will need the following inverses to second order:

$$
\begin{aligned}
E_a{}^m &= \delta_a{}^m - \frac{1}{2}T_{ya}{}^m - \frac{1}{3}\mathcal{D}_y T_{ya}{}^m - \frac{1}{2}\mathcal{D}_\eta T_{ya}{}^m - \frac{1}{12}T_{ya}{}^b T_{by}{}^m - \frac{1}{6}R_{yay}{}^m \\
E_a{}^\mu &= -\frac{1}{2}T_{ya}{}^\mu - \frac{1}{3}\mathcal{D}_y T_{ya}{}^\mu - \frac{1}{2}\mathcal{D}_\eta T_{ya}{}^\mu - \frac{1}{12}T_{ya}{}^b T_{by}{}^\mu - \frac{1}{3}T_{ya}{}^\beta T_{\beta y}{}^\mu + \frac{1}{6}T_{ya\dot\beta}T^{\dot\beta}{}_y{}^\mu \\
E_\alpha{}^\mu &= \delta_\alpha{}^\mu - T_{y\alpha}{}^\mu - \frac{1}{2}\mathcal{D}_y T_{y\alpha}{}^\mu - \frac{1}{2}T_{y\alpha}{}^\beta T_{\beta y}{}^\mu + \frac{1}{2}T_{y\alpha\dot\beta}T^{\dot\beta}{}_y{}^\mu - \mathcal{D}_\eta T_{y\alpha}{}^\mu + \eta^2 \bar{R}\,\delta_\alpha{}^\mu \\
E_\alpha{}^m &= -\frac{1}{2}R_{y\alpha y}{}^m - \frac{1}{2}R_{\eta\alpha y}{}^m
\end{aligned}
\tag{5.3.32}
$$

One specific combination which we will use a great deal is

$$
\begin{aligned}
X^\mu{}_\mu =& E^{a\mu}E_{a\mu} - \frac{1}{2}RE^{\alpha\mu}E_{\alpha\mu} \\
=& \frac{1}{4}T_y{}^{a\mu}T_{ya\mu} - R + RT_{y\alpha}{}^\alpha + \frac{1}{2}\mathcal{D}_y T_{y\alpha}{}^\alpha + R\mathcal{D}_\eta T_{y\alpha}{}^\alpha - 2\eta^2 R\bar{R} \\
& - \frac{R}{2}T_{y\alpha\dot\beta}T^{\dot\beta}{}_y{}^\alpha - \frac{R}{2}T_y{}^{\alpha\mu}T_{y\alpha\mu} + \frac{R}{2}T_y{}^{\alpha\beta}T_{y\beta\alpha}
\end{aligned}
\tag{5.3.33}
$$

The explicit R terms in the above are to be understood as $R(y,\eta)$ where

$$
R(y,\eta) = R + \mathcal{D}_y R + \mathcal{D}_\eta R + \frac{1}{2}\mathcal{D}_y\mathcal{D}_y R + \frac{1}{2}\mathcal{D}_\eta\mathcal{D}_\eta R + \mathcal{D}_\eta\mathcal{D}_y R + \dots \tag{5.3.34}
$$

5.3.2 Chiral heat kernel analysis

The remaining differential equation for our heat kernel reads

$$
\frac{\partial\tilde{F}}{\partial\tau} + \frac{D\tilde{F}}{\tau} = \tilde{\mathcal{O}}_+\tilde{F} \tag{5.3.35}
$$

for

$$
D \equiv \mathcal{D}^a\Sigma\mathcal{D}_a + \frac{R}{2}\mathcal{D}^\alpha\Sigma\mathcal{D}_\alpha + \frac{1}{2}\mathcal{D}^\alpha\Sigma W_\alpha \tag{5.3.36}
$$

(Recall that $\tilde{F}$ has $U(1)_R$ weight 0 on its $\mathfrak{z}$ coordinate, where D acts.) In normal coordinates at $\bar{\eta} = 0$, the above simplifies drastically. We end up with

$$D = y^a \mathcal{D}_a = y^m \partial_m \tag{5.3.37}$$

We assume F can be expanded as a power series in τ with $\tilde{F} = \sum_{n=0} A_n \tau^n / n!$, which gives recursion relations which we can solve just as before. (We neglect placing tildes on the coefficients A for notational simplicity.) We fix

$$A_0 = \eta^2 \tag{5.3.38}$$

to obey both the differential equation and the necessary $\tau = 0$ boundary condition. The rest of the coefficients follow via the formal solution of Avramidi [36]

$$A_n = \left(1 + \frac{D}{n}\right)^{-1} \tilde{\mathcal{O}}_+ \left(1 + \frac{D}{n-1}\right)^{-1} \tilde{\mathcal{O}}_+ \cdots (1+D)^{-1} \tilde{\mathcal{O}}_+ \eta^2 \tag{5.3.39}$$

As before we seek analytic power series solutions, except now the power series are in η as well as y, giving a generic ket $|n, \nu\rangle$. Since the η series terminates for $\nu \geq 3$, we have the the generic kets

$$|n, 0\rangle = |n\rangle, \quad |n, 1\rangle = |n\rangle \times \eta^{\beta_1}, \quad |n, 2\rangle = |n\rangle \times \eta^2 \tag{5.3.40}$$

where $|n\rangle$ is as defined in the non-supersymmetric case. We define the corresponding bras by

$$\langle n, 0| = \langle n|, \quad \langle n, 1| = \langle n| \times \partial_{\alpha_1}, \quad \langle n, 2| = -\frac{1}{4} \langle n| \times \partial^\alpha \partial_\alpha \tag{5.3.41}$$

It then follows easily as in the non-supersymmetric case

$$\begin{aligned} \langle k, \kappa | A_n \rangle = \sum_{j_1, \ldots, j_{k-1} \geq 0} \; \sum_{2 \geq \gamma_1, \ldots, \gamma_{k-1} \geq 0} \left(1 + \frac{k}{n}\right)^{-1} \left(1 + \frac{j_{n-1}}{n-1}\right)^{-1} \cdots (1 + j_1)^{-1} \times \\ \langle k, \kappa | \tilde{\mathcal{O}}_+ | j_{n-1}, \gamma_{n-1} \rangle \langle j_{n-1}, \gamma_{n-1} | \tilde{\mathcal{O}}_+ | j_{n-2}, \gamma_{n-2} \rangle \cdots \langle j_1, \gamma_1 | \tilde{\mathcal{O}}_+ | 0, 2 \rangle \end{aligned} \tag{5.3.42}$$

We turn now to the structure of $\tilde{\mathcal{O}}_+$. One finds after a great deal of work

$$\begin{aligned} \tilde{\mathcal{O}}_+ \tilde{F} = & \mathcal{D}_{\mathcal{M}} \left(E^{a\mathcal{M}} \mathcal{D}_a \tilde{F} + \frac{1}{2} R E^{\alpha\mathcal{M}} \mathcal{D}_\alpha \tilde{F} \right) + W^\alpha \mathcal{D}_\alpha \tilde{F} + \frac{1}{2} (\mathcal{D}^\alpha W_\alpha) \tilde{F} + \frac{1}{2} W^\alpha (\mathcal{D}_{\mathcal{M}} E_\alpha{}^{\mathcal{M}}) \tilde{F} \\ & + \left(\Delta^{-1/2} \mathcal{O}_+ \Delta^{1/2} \right) \tilde{F} \end{aligned}$$

This operator can be rewritten in the manifestly symmetric form

$$\tilde{\mathcal{O}}_+ = \mathcal{D}_{\mathcal{M}} X^{\mathcal{M}\mathcal{N}} \mathcal{D}_{\mathcal{N}} + \frac{1}{2} W^\alpha \mathcal{E}_\alpha{}^{\mathcal{M}} \mathcal{D}_{\mathcal{M}} + \frac{1}{2} \mathcal{D}_{\mathcal{M}} \mathcal{E}^{\alpha\mathcal{M}} W_\alpha + \left(\Delta^{-1/2} \mathcal{O}_+ \Delta^{1/2} \right) \tag{5.3.43}$$

We have used $\mathcal{E}_{\mathcal{A}}{}^{\mathcal{M}}$ in place of $E_{\mathcal{A}}{}^{\mathcal{M}}$ since all $\bar{\eta}$ derivatives have been removed and so we may take $\bar{\eta}$ to vanish without incident. The above form is particularly striking since the operator is clearly self-adjoint up to a change in the representation of the gauge field strength:

$$\tilde{\mathcal{O}}_+^T(W_\alpha) = \tilde{\mathcal{O}}_+(-W_\alpha^T) \tag{5.3.44}$$

This is sensible since $\mathcal{O}_+$ appears naturally acting between a chiral superfield Φ_1 and its conjugate Φ_2,

$$\int \mathcal{E}\, \Phi_2^T \mathcal{O}_+ \Phi_1 = \int \mathcal{E}\, (\mathcal{O}_+ \Phi_2)^T \Phi_1 = \int \mathcal{E}\, \Phi_1^T \mathcal{O}_+ \Phi_2 \tag{5.3.45}$$

which is a gauge invariant expression only if Φ_2 is in the representation conjugate to Φ_1.

We have introduced the "chiral metric"

$$X^{\mathcal{M}\mathcal{N}} = \mathcal{E}^{a\mathcal{M}} \mathcal{E}_a{}^{\mathcal{N}} + \frac{1}{2} R \mathcal{E}^{\alpha \mathcal{M}} \mathcal{E}_\alpha{}^{\mathcal{N}} \tag{5.3.46}$$

where $\mathcal{M}$ and $\mathcal{N}$ are only the chiral spinor and bosonic index. In all these formulae an implicit grading has been used.

In general $\tilde{\mathcal{O}}_+$ has the form

$$\tilde{\mathcal{O}}_+ = X^{\mathfrak{M}\mathcal{N}} \partial_{\mathcal{N}} \partial_{\mathfrak{M}} + Y^{\mathfrak{M}} \partial_{\mathfrak{M}} + Z \tag{5.3.47}$$

We have

$$\begin{aligned} Y^{\mathfrak{M}} =& -2X^{\mathfrak{M}\mathcal{N}} H_{\mathcal{N}} + \partial_{\mathcal{N}} X^{\mathcal{N}\mathfrak{M}} + W^\alpha E_\alpha{}^{\mathfrak{M}} \\ Z =& - X^{\mathfrak{M}\mathcal{N}} \partial_{\mathcal{N}} H_{\mathfrak{M}} + X^{\mathfrak{M}\mathcal{N}} H_{\mathcal{N}} H_{\mathfrak{M}} - (\partial_{\mathfrak{M}} X^{\mathfrak{M}\mathcal{N}}) H_{\mathcal{N}} \\ &+ \frac{1}{2} \mathcal{D}^\alpha W_\alpha + \frac{1}{2} (\partial_{\mathfrak{M}} \mathcal{E}^{\alpha \mathfrak{M}}) W_\alpha - W^\alpha E_\alpha{}^{\mathfrak{M}} H_{\mathfrak{M}} + \Delta^{-1/2} \mathcal{O}_+ \Delta^{1/2} \end{aligned} \tag{5.3.48}$$

Aside from the terms involving W^α, the above form is strikingly similar to the non-supersymmetric case, with $X^{\mathcal{M}\mathcal{N}}$ replacing g^{mn}. The connection H is really just the Yang-Mills connection $\mathcal{A}$; the heat kernel function $\tilde{F}$ has only a Yang-Mills structure since all its $U(1)_R$ weight is on the $\mathfrak{z}'$ coordinate, not the $\mathfrak{z}$ coordinate. If we were to generalize our approach to include chiral superfields with Lorentz indices, the Lorentz connection would appear here as well.

Before proceeding further, we should note the projections to $y = 0$ and $\eta = 0$ of the terms given above:

$$[X^{mn}] = \eta^{mn}, \quad [X^{m\nu}] = 0, \quad [X^{\mu\nu}] = \frac{1}{2} R \epsilon^{\mu\nu}$$
$$[Y^m] = 0, \quad [Y^\mu] = \frac{1}{2} \mathcal{D}^\mu R + W^\mu$$
$$[Z] = \frac{1}{2} \mathcal{D}^\alpha W_\alpha + [\Delta^{-1/2} \mathcal{O}_+ \Delta^{1/2}]$$

The quartic divergence is proportional to $\langle 0, 0 | A_0 \rangle$ which vanishes as required by supersymmetry.

The quadratic divergence is proportional to

$$\langle 0,0|A_1\rangle = \langle 0,0|\tilde{\mathcal{O}}_+|0,2\rangle = 2[X^\mu{}_\mu] = -2R \tag{5.3.49}$$

This is an F-term, so the corresponding D-term would simply be 1. In a sense, the quadratic divergence in superspace is most like the quartic divergence in normal space.

The logarithmic divergence is given by

$$\langle 0,0|A_2\rangle = \sum_{j_1,\gamma_1} (1+j_1)^{-1} \langle 0,0|\tilde{\mathcal{O}}_+|j_1,\gamma_1\rangle\langle j_1,\gamma_1|\tilde{\mathcal{O}}_+|0,2\rangle$$

The first matrix element vanishes trivially unless $\gamma_1 + j_1 \leq 2$. Those satisfying this requirement are

$$\begin{aligned}
\langle 0,0|\tilde{\mathcal{O}}_+|0,0\rangle &= [Z] \\
\langle 0,0|\tilde{\mathcal{O}}_+|0,1\rangle &= [Y^{\beta_1}] = W^{\beta_1} + \frac{1}{2}\mathcal{D}^{\beta_1}R \\
\langle 0,0|\tilde{\mathcal{O}}_+|1,0\rangle &= [Y^{b_1}] = 0 \\
\langle 0,0|\tilde{\mathcal{O}}_+|0,2\rangle &= [2X^\alpha{}_\alpha] = -2R \\
\langle 0,0|\tilde{\mathcal{O}}_+|1,1\rangle &= [X^{b_1\beta_1}] = 0 \\
\langle 0,0|\tilde{\mathcal{O}}_+|2,0\rangle &= [X^{b_1b_2}] = \eta^{b_1b_2}
\end{aligned}$$

We require the product of these with $\langle j_1,\gamma_1|\tilde{\mathcal{O}}_+|0,2\rangle$ for $(j_1,\gamma_1) = \{(0,0),(0,1),(0,2),(2,0)\}$. The first case we've already found. The second is

$$\langle 0,1|\tilde{\mathcal{O}}_+|0,2\rangle = [2\partial_{\alpha_1}X^\phi{}_\phi] + 2[Y_{\alpha_1}]$$

It is straightforward to show $[\partial_{\alpha_1}X^\phi{}_\phi] = -\mathcal{D}_\alpha R$, giving

$$\langle 0,1|\tilde{\mathcal{O}}_+|0,2\rangle = -\mathcal{D}_{\alpha_1}R + 2W_{\alpha_1}$$

The third term is

$$\begin{aligned}
\langle 0,2|\tilde{\mathcal{O}}_+|0,2\rangle &= [-\frac{1}{2}\partial^\alpha\partial_\alpha X^\beta{}_\beta] - [\partial^\alpha Y_\alpha] + [Z] \\
&= -\mathcal{D}^\alpha W_\alpha + [\partial_\nu\partial_m X^{m\nu}] + [Z]
\end{aligned}$$

but a straightforward calculation shows the middle term vanishes, leaving

$$\langle 0,2|\tilde{\mathcal{O}}_+|0,2\rangle = -\mathcal{D}^\alpha W_\alpha + [Z]$$

The fourth and final term is

$$\langle 2,0|\tilde{\mathcal{O}}_+|0,2\rangle = +2\partial_{a_1}\partial_{a_2}X^\beta{}_\beta$$

For simplicity, we note that only the contracted part of this is necessary, so we focus on

$$\begin{aligned}\partial^b\partial_b X^\alpha{}_\alpha =&\frac{1}{2}T^{cb\alpha}T_{cb\alpha} - \Box R + 2\mathcal{D}^b R T_{b\alpha}{}^\alpha + R\mathcal{D}^b T_{b\alpha}{}^\alpha \\ &- RT_{b\alpha\dot\beta}T^{\dot\beta b\alpha} - RT^{b\alpha\mu}T_{b\alpha\mu} + RT^{b\alpha\beta}T_{b\beta\alpha} \\ =&W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} + \frac{1}{4}\mathcal{D}^\alpha R\mathcal{D}_\alpha R + \frac{1}{2}X^\alpha\mathcal{D}_\alpha R - \frac{1}{12}X^\alpha X_\alpha + \frac{1}{2}\mathcal{D}_{\dot\alpha}G^b\mathcal{D}^{\dot\alpha}G_b \\ &- \Box R - 4i\mathcal{D}^b R G_b - 2iR\mathcal{D}_b G^b + 8R^2\bar R + 4RG^2\end{aligned}$$

Several terms can be collected into manifestly chiral terms, using

$$\Box_+ R =\Box R + 2iG^b\mathcal{D}_b R + \frac{1}{2}\mathcal{D}^\alpha R\mathcal{D}_\alpha R + \frac{1}{2}R\mathcal{D}^2 R - \frac{1}{2}X^\alpha\mathcal{D}_\alpha R - \frac{1}{2}(\mathcal{D}^\alpha X_\alpha)R$$

as well as

$$\frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)G^2 = \frac{1}{2}\mathcal{D}_{\dot\alpha}G_b\mathcal{D}^{\dot\alpha}G^b - 2iG^b\mathcal{D}_b R + 4G^2 R$$

to give

$$\begin{aligned}\partial^b\partial_b X^\alpha{}_\alpha =&W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} + \frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)G^2 - \Box_+ R - \frac{1}{12}X^\alpha X_\alpha \\ &+ \frac{3}{4}\mathcal{D}^\alpha R\mathcal{D}_\alpha R + 8R^2\bar R + \frac{1}{2}R\bar{\mathcal{D}}^2\bar R - \frac{1}{2}R\mathcal{D}^\alpha X_\alpha\end{aligned}$$

Putting all of this together gives

$$\begin{aligned}\langle 0,0|\tilde{\mathcal{O}}_+|0,0\rangle\langle 0,0|\tilde{\mathcal{O}}_+|0,2\rangle &= -2R[Z] \\ \langle 0,0|\tilde{\mathcal{O}}_+|0,1\rangle\langle 0,1|\tilde{\mathcal{O}}_+|0,2\rangle &= 2W^\alpha W_\alpha - \frac{1}{2}\mathcal{D}^\alpha R\mathcal{D}_\alpha R \\ \langle 0,0|\tilde{\mathcal{O}}_+|0,2\rangle\langle 0,2|\tilde{\mathcal{O}}_+|0,2\rangle &= -2R[Z] + 2R\mathcal{D}^\alpha W_\alpha \\ \frac{1}{3}\langle 0,0|\tilde{\mathcal{O}}_+|2,0\rangle\langle 2,0|\tilde{\mathcal{O}}_+|0,2\rangle &= \frac{2}{3}\partial^a\partial_a X^\beta{}_\beta \\ &= \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} + \frac{1}{6}(\bar{\mathcal{D}}^2 - 8R)G^2 - \frac{2}{3}\Box_+ R - \frac{1}{18}X^\alpha X_\alpha \\ &\quad + \frac{1}{2}\mathcal{D}^\alpha R\mathcal{D}_\alpha R + \frac{16}{3}R^2\bar R + \frac{1}{3}R\bar{\mathcal{D}}^2\bar R - \frac{1}{3}R\mathcal{D}^\alpha X_\alpha\end{aligned}$$

the sum of which is

$$\begin{aligned}[A_2] =&2W^\alpha W_\alpha + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} + \frac{1}{6}(\bar{\mathcal{D}}^2 - 8R)G^2 - \frac{2}{3}\Box_+ R - \frac{1}{18}X^\alpha X_\alpha \\ &- 4R[Z] + 2R\mathcal{D}^\alpha W_\alpha + \frac{16}{3}R^2\bar R + \frac{1}{3}R\bar{\mathcal{D}}^2\bar R - \frac{1}{3}R\mathcal{D}^\alpha X_\alpha\end{aligned}$$

We must still evaluate $[Z]$. Begin by noting

$$[Z] = \frac{1}{2}\mathcal{D}^\alpha W_\alpha + [\Delta^{-1/2}\mathcal{O}_+\Delta^{1/2}] = \frac{1}{2}\mathcal{D}^\alpha W_\alpha - \frac{1}{2}(\bar{\mathcal{D}}^2 - 8R)\bar R + [\Delta^{-1/2}\Box_+\Delta^{1/2}]$$

Evaluating the term involving $\Box_+$ is a somewhat laborious task. The most straightforward way of doing it is to expand out all the terms so that they involve $\log\Delta$ and then to work out the expansion of $\log\Delta$ to the necessary order. The expansion of $\log\Delta$ to second order is

$$\log\Delta = -2iG_y + \frac{1}{24}T_y{}^{ba}T_{yba} - \frac{1}{6}R_{ymy}{}^m - i\mathcal{D}_y G_y + \frac{1}{2}T_{\alpha y\dot\beta}T^{\dot\beta}{}_y{}^\alpha - 2\eta^2\bar R$$

Note that there are no terms linear in η. The $\Box_+$ term yields

$$\begin{aligned}
[\Delta^{-1/2}\Box_+\Delta^{1/2}] &= -\frac{1}{4}\mathcal{D}^\alpha X_\alpha + \frac{1}{4}R\partial^\alpha\partial_\alpha\log\Delta + iG^b\partial_b\log\Delta + \frac{1}{2}\partial^b\partial_b\log\Delta + \frac{1}{4}\partial^b\log\Delta\partial_b\log\Delta \\
&= -\frac{1}{4}\mathcal{D}^\alpha X_\alpha + 2R\bar R + \frac{1}{24}T^{cba}T_{cba} - \frac{1}{6}R_{ab}{}^{ab} - i\mathcal{D}_b G^b + \frac{1}{2}T_{\alpha c\dot\beta}T^{\dot\beta c\alpha} + G^2
\end{aligned}$$

Using

$$\begin{aligned}
R_{ab}{}^{ab} &= -\mathcal{D}^\beta X_\beta - \frac{3}{2}(\mathcal{D}^2 R + \bar{\mathcal{D}}^2\bar R) + 48R\bar R \\
T^{cba}T_{cba} &= -24G^2, \quad T_{\alpha c\dot\beta}T^{\dot\beta c\alpha} = 8R\bar R
\end{aligned}$$

we find that

$$[Z] = \frac{1}{2}\mathcal{D}^\alpha W_\alpha - \frac{1}{12}\mathcal{D}^\alpha X_\alpha + 2R\bar R$$

which gives the net result of

$$\begin{aligned}
[A_2] &= 2W^\alpha W_\alpha + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} + \frac{1}{6}(\bar{\mathcal{D}}^2 - 8R)G^2 - \frac{2}{3}\mathcal{O}_+ R - \frac{1}{18}X^\alpha X_\alpha \\
&= 2W^\alpha W_\alpha + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} - \frac{1}{18}X^\alpha X_\alpha - \frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)\left(-\frac{2}{3}G^2 + \frac{1}{6}(\mathcal{D}^2 - 8\bar R)R\right)
\end{aligned} \tag{5.3.50}$$

The divergences associated with the heat kernel of this operator are

$$[\mathrm{Tr}\log H]_\epsilon = -\frac{1}{16\pi^2}\int \mathcal{E}\,\mathrm{Tr}\left(\frac{[A_0]}{2\epsilon^2} + \frac{[A_1]}{\epsilon} - \frac{[A_2]}{2}\log\epsilon + \text{finite}\right) + \text{h.c.} \tag{5.3.51}$$

which we may write as

$$\begin{aligned}
[\mathrm{Tr}\log H]_\epsilon = &+ \frac{1}{16\pi^2\epsilon}\int E + \frac{\log\epsilon}{16\pi^2}\int \mathcal{E}\left(W^\alpha W_\alpha + \frac{1}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} - \frac{1}{36}X^\alpha X_\alpha\right) \\
&+ \frac{\log\epsilon}{16\pi^2}\int E\left(-\frac{1}{3}G^2 - \frac{2}{3}\bar R R\right) + \text{h.c.} + \text{finite}
\end{aligned} \tag{5.3.52}$$

where we have dropped a total derivative. This result for $U(1)$ supergravity agrees with the traditional calculation (up to factors of two in the definition of the supergravity superfields) in Poincaré supergravity when X^α vanishes [41].

In the non-supersymmetric calculation (provided only a classically conformal action was used) there was a striking feature where the logarithmic divergent term consisted solely of conformal or topological terms. Since we could have written our result here in terms of the heat kernel of a conformally coupled bosonic scalar and fermionic superpartner, it should have the same property.

Consider a small shift in the choice of compensator X of the form $\delta X = X\delta U$ where δU is a dimension zero superfield. First note that δE and $\delta\mathcal{E}$ both vanish if X is changed a small amount. This is because the choice of X while redefining $E_a{}^M$ does so only by shifting the spinor derivative part of the bosonic derivative. That is, $\delta E = -E\delta E_a{}^M E_M{}^a = -E\delta E_a{}^a$ vanishes. Similarly $\delta\mathcal{E}$ vanishes.

It is straightforward to work out that

$$\delta X_\alpha = \frac{3}{8}\bar\nabla^2\nabla_\alpha \delta U = \frac{3}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha \delta U \tag{5.3.53}$$

We similarly may calculate

$$\delta R = -\frac{1}{8}\bar\nabla^2\delta U - \frac{1}{4X}\nabla_{\dot\alpha}X\nabla^{\dot\alpha}\delta U = -\frac{1}{8}\bar{\mathcal{D}}^2\delta U \tag{5.3.54}$$

and

$$\delta G_{\alpha\dot\alpha} = -\frac{1}{4}[\mathcal{D}_\alpha, \mathcal{D}_{\dot\alpha}]\delta U \tag{5.3.55}$$

It is straightforward to check that the specific combination

$$\left[G^2 + 2R\bar R\right]_D + \frac{1}{6}\left[X^\alpha X_\alpha\right]_F \tag{5.3.56}$$

is invariant to *any* deformation of the compensator. It corresponds at the component level to the expression

$$-\frac{1}{6}F^{ab}F_{ab} - \frac{1}{8}\mathcal{R}^{ab}\mathcal{R}_{ab} + \frac{1}{24}\mathcal{R}^2 + \text{fermions}$$

where F_{ab} is the field strength of the $U(1)_R$. Noting that

$$\left[W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F = \frac{1}{6}F^{ab}F_{ab} + \frac{1}{16}C^{abcd}C_{abcd} + \text{fermions} \tag{5.3.57}$$

we find

$$\left[G^2 + 2R\bar R\right]_D + \frac{1}{6}\left[X^\alpha X_\alpha\right]_F + \left[W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F = \frac{1}{16}L_\chi + \text{fermions} \tag{5.3.58}$$

up to total derivatives, where L_χ is the topological Gauss-Bonnet term. Since $W^{\alpha\beta\gamma}$ is X-independent automatically, this combination must be independent under deformations of both the compensator X *and* the conformal supergravity structure. Showing this directly at the superspace level is straightforward, but requires solving the constraint structure of supergravity. This can be done using the formulae given in the previous chapters which we leave as an exercise to the interested reader.

This superfield topological combination will appear several times, so it is useful to introduce a label for the superfield expression. We choose to define the Hermitian combination

$$
\begin{aligned}
S_\chi \equiv \left[G^2 + \mathcal{P}\bar{R} + \bar{\mathcal{P}}R - 2R\bar{R}\right]_D + \frac{1}{12}\left[X^\alpha X_\alpha\right]_F + \frac{1}{12}\left[\bar{X}_{\dot\alpha}\bar{X}^{\dot\alpha}\right]_{\bar{F}} \\
+ \frac{1}{2}\left[W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F + \frac{1}{2}\left[\bar{W}_{\dot\gamma\dot\beta\dot\alpha}\bar{W}^{\dot\gamma\dot\beta\dot\alpha}\right]_{\bar{F}}
\end{aligned}
\tag{5.3.59}
$$

where, one should recall, $\mathcal{P} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)$. (We have chosen to reintroduce a total derivative which formerly dropped out previously since when we calculate the conformal anomaly this term will not in general vanish.) We can then write the divergences that we found as

$$
\begin{aligned}
[\mathrm{Tr}\log H]_\epsilon = &+ \frac{1}{8\pi^2\epsilon}\Big[X\Big]_D - \frac{\log\epsilon}{24\pi^2}S_\chi \\
&+ \frac{\log\epsilon}{16\pi^2}\left[W^\alpha W_\alpha + \frac{1}{36}X^\alpha X_\alpha + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F \\
&+ \frac{\log\epsilon}{16\pi^2}\left[\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha} + \frac{1}{36}\bar{X}_{\dot\alpha}\bar{X}^{\dot\alpha} + \frac{2}{3}\bar{W}_{\dot\gamma\dot\beta\dot\alpha}\bar{W}^{\dot\gamma\dot\beta\dot\alpha}\right]_{\bar{F}} \\
&+ \text{finite}
\end{aligned}
\tag{5.3.60}
$$

where we have reintroduced the compensator X. Its only explicit appearance is in the quadratically divergent D-term, where it provides the necessary conformal weight to render a conformally invariant expression. Although it is implicitly used to define S_χ, as we noted S_χ is independent of small deformations of X. The remaining presence in X_α is purely the part of X that can be regarded as a $U(1)$ prepotential, if say we were to decompose X as $\Phi_0\bar{\Phi}_0 e^V$ for some $U(1)$ prepotential V.

One also suspects it should combine with W^α in a way that removes the classical “U(1) ambiguity.” Indeed, noting that

$$
W_\alpha = \frac{1}{8}\bar{\nabla}^2 e^{-V}\nabla_\alpha e^V, \quad X_\alpha = \frac{3}{8}\bar{\nabla}^2\nabla_\alpha \log X \tag{5.3.61}
$$

the combination

$$
W_\alpha - \frac{1}{6}X_\alpha = \frac{1}{8}\bar{\nabla}^2\left(e^{-V+\log X/2}\nabla_\alpha e^{V-\log X/2}\right) \tag{5.3.62}
$$

corresponds to the way the factors of V and X appear in the original theory, and so we note that the divergent term seems to correspond to only the combination $(W_\alpha - \frac{1}{6}X_\alpha)^2$. We are missing, of course, the cross-term $W^\alpha X_\alpha$, but this is to be expected. The determinant of H corresponds to the part of the effective action even under charge conjugation. If this cross term exists, it should be found in the superfield version of the odd part of the effective action. We turn to that analysis next.

5.3.3 Integration of the odd part

Recall that D and $\tilde{D}$ are defined in the massless case by

$$D = \begin{pmatrix} 0 & \mathcal{P}e^{V^T}X^{-1/2} \\ \bar{\mathcal{P}}e^{V}X^{-1/2} & 0 \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} 0 & -\mathcal{P}e^{-V}X^{-1/2} \\ -\bar{\mathcal{P}}e^{-V^T}X^{-1/2} & 0 \end{pmatrix} \tag{5.3.63}$$

Defining $H = \tilde{D}D$ and $\tilde{H} = D\tilde{D}$, the effective action $\mathrm{Tr}\log D$ is divided into two terms

$$[\mathrm{Tr}\log D]_\epsilon = \frac{1}{2}\left[\mathrm{Tr}\log H\right]_\epsilon + \int \left(L_\epsilon^- + \ell\right) \tag{5.3.64}$$

the first of which we have already found. The objects L_ϵ^- and ℓ are one-forms in the space of all possible variations of the gauge prepotential, and ℓ is chosen so that $L_\epsilon^- + \ell$ is a closed form. It is therefore (at least locally) the variation of some other expression and can be integrated, which we have indicated with a schematic $\int$ symbol which shall be better defined later.

In analogy to the fermionic case, we define

$$L_\epsilon^- = \frac{1}{2}\mathrm{Tr}\int_\epsilon^\infty d\tau \left(e^{-\tau H}\tilde{D}\delta D - e^{-\tau\tilde{H}}D\delta\tilde{D}\right) \tag{5.3.65}$$

ℓ itself is defined by integrating the formula $\delta\ell = -C$ where

$$C_\epsilon = \delta L_\epsilon^- = \epsilon\int_0^1 d\lambda \mathrm{Tr}\left(\delta D e^{-\epsilon\lambda H}\delta\tilde{D}e^{-\epsilon\tilde{\lambda}\tilde{H}}\right) \tag{5.3.66}$$

where $\tilde{\lambda} = 1-\lambda$. Using cyclicity of the trace, we find

$$L_\epsilon^- = \frac{1}{2}\int_\epsilon^\infty d\tau \mathrm{Tr}\left(\delta D\tilde{D}e^{-\tau\tilde{H}} - \delta\tilde{D}De^{-\tau H}\right) \tag{5.3.67}$$

We denote

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{5.3.68}$$

and similarly for $\tilde{H}$.

The operator product $\delta D\tilde{D}$ is given by

$$\delta D\tilde{D} = \begin{pmatrix} -\mathcal{P}\delta e^{V^T}\bar{\mathcal{P}}e^{-V^T} & 0 \\ 0 & -\mathcal{P}\delta e^{V}\bar{\mathcal{P}}e^{-V} \end{pmatrix} = \begin{pmatrix} -\mathcal{P}\Delta V^T e^{V^T}\bar{\mathcal{P}}e^{-V^T} & 0 \\ 0 & -\bar{\mathcal{P}}\bar{\Delta}V e^{V}\bar{\mathcal{P}}e^{-V} \end{pmatrix}$$

and its conjugate $\delta\tilde{D}D$ by

$$\delta\tilde{D}D = \begin{pmatrix} -\mathcal{P}\delta e^{-V}\bar{\mathcal{P}}e^{V} & 0 \\ 0 & -\mathcal{P}\delta e^{-V^T}\bar{\mathcal{P}}e^{V^T} \end{pmatrix} = \begin{pmatrix} \mathcal{P}\Delta V e^{-V}\bar{\mathcal{P}}e^{V} & 0 \\ 0 & \bar{\mathcal{P}}\bar{\Delta}V^T e^{-V^T}\bar{\mathcal{P}}e^{V^T} \end{pmatrix}$$

where we have defined

$$\Delta V = e^{-V}\delta e^{V}, \quad \Delta V^T = (\delta e^{V^T})e^{-V^T}$$
$$\bar{\Delta} V = (\delta e^{V})e^{-V}, \quad \bar{\Delta} V^T = \delta e^{V^T} e^{-V^T} \tag{5.3.69}$$

The operators above are defined in a purely chiral or antichiral gauge, but it is clear that we can rewrite them in a general basis. The way to do this is to absorb the various factors of e^V in the operators above to define *covariant* chiral projectors $\mathcal{P}$ and $\bar{\mathcal{P}}$. In so doing, we would like to interpret ΔV and $\bar{\Delta} V$ (as well as their transposes) as covariant objects. To do this, we *define*

$$\omega \equiv \Delta V \text{ (chiral gauge).} \tag{5.3.70}$$

and extend ω into any other gauge by requiring it to transform covariantly. It follows that in antichiral gauge, $\omega = e^V \Delta V e^{-V} = \bar{\Delta} V$. We may now write L_ϵ^- in a covariant way:

$$L_\epsilon^- = -\frac{1}{2}\int_\epsilon^\infty d\tau \; \mathrm{Tr}_+ \left(\mathcal{P}\omega^T \bar{\mathcal{P}} e^{-\tau \tilde{H}_+} + \mathcal{P}\omega\bar{\mathcal{P}} e^{-\tau H_+}\right) + \text{h.c.}$$

where we have broken the trace up into the part over the separate chiral and antichiral spaces. Noing that the exponential term is the heat kernel, we find

$$\begin{aligned} L_\epsilon^- &= -\frac{1}{2}\int_\epsilon^\infty d\tau \int \mathcal{E} \left(\mathcal{P}[\omega^T \bar{\mathcal{P}} \tilde{U}_+(\tau)] + \mathcal{P}[\omega\bar{\mathcal{P}} U_+(\tau)]\right) + \text{h.c.} \\ &= -\frac{1}{2}\int_\epsilon^\infty d\tau \int E \left(\omega^T \bar{\mathcal{P}} \tilde{U}_+(\tau) + \omega\bar{\mathcal{P}} U_+(\tau)\right) + \text{h.c.} \end{aligned} \tag{5.3.71}$$

The heat kernel U_+ is

$$U_+(\tau) = \frac{1}{(4\pi\tau)^2} e^{-\Sigma/2\tau} \Delta^{1/2} F(\tau) \tag{5.3.72}$$

Noting that $[\Sigma] = 0$, $[\mathcal{D}_\alpha \Sigma] = 0$, and $[\mathcal{D}^2 \Sigma] = 0$, we find

$$L_\epsilon^- = -\frac{1}{2}\int_\epsilon^\infty \frac{d\tau}{(4\pi\tau)^2} \int E\,\mathrm{Tr}\left(\omega^T[\bar{\mathcal{P}}\Delta^{1/2}\tilde{F}] + \omega[\bar{\mathcal{P}}\Delta^{1/2}F]\right) + \text{h.c.}$$

Note that $\tilde{F}$ has the same form as F but in a conjugate representation. Next we note that $[\Delta^{1/2}] = 1$, $[\mathcal{D}_\alpha \Delta^{1/2}] = 0$, and $[\mathcal{D}^\alpha \mathcal{D}_\alpha \Delta^{1/2}] = 4\bar{R}$, giving

$$\begin{aligned} L_\epsilon^- &= -\frac{1}{2}\int_\epsilon^\infty \frac{d\tau}{(4\pi\tau)^2} \int E\,\mathrm{Tr}\left(\omega^T[\bar{\mathcal{P}}\tilde{F} - \bar{R}\tilde{F}] + \omega[\bar{\mathcal{P}}F - \bar{R}F]\right) + \text{h.c.} \\ &= -\frac{1}{2}\int_\epsilon^\infty \frac{d\tau}{(4\pi\tau)^2} \int E\,\mathrm{Tr}\left(\omega^T\left[-\frac{1}{4}\mathcal{D}^2\tilde{F} + \bar{R}\tilde{F}\right] + \omega\left[-\frac{1}{4}\mathcal{D}^2 F + \bar{R}F\right]\right) + \text{h.c.} \end{aligned}$$

Since $F(\lambda) = \sum_{n=0}^\infty A_n \lambda^n/n!$, only the terms involving A_0 and A_1 contribute to the divergences – the former to the quadratic and the latter to the logarithmic. Using $[A_0] = 0$ and $[\mathcal{D}^2 A_0] = -4$, we find for the quadratic divergences

$$L_\epsilon^- \ni -\frac{1}{32\pi^2}\frac{1}{\epsilon}\int E\,\mathrm{Tr}\left(\omega^T + \omega\right) + \text{h.c.} = -\frac{1}{16\pi^2}\frac{2}{\epsilon}\int E\,\delta\mathrm{Tr}\,(V) \tag{5.3.73}$$

which is a divergent contribution to the Fayet-Iliopoulos term.

For the logarithmic divergences, we note from our experience with the heat kernel, we immediately may conclude that $[A_1] = -2R$ and $[\mathcal{D}^2 A_1] = 2\mathcal{D}^\alpha W_\alpha + \frac{1}{3}\mathcal{D}^\alpha X_\alpha - 8R\bar{R}$ which give

$$L_\epsilon^- = +\frac{\log\epsilon}{32\pi^2}\int E\,\mathrm{Tr}\left(\omega^T\left[-\frac{1}{2}\mathcal{D}^\alpha\tilde{W}_\alpha - \frac{1}{12}\mathcal{D}^\alpha X_\alpha\right] + \omega\left[-\frac{1}{2}\mathcal{D}^\alpha W_\alpha - \frac{1}{12}\mathcal{D}^\alpha X_\alpha\right]\right) + \text{h.c.} \tag{5.3.74}$$

In chiral gauge, $W_\alpha = -\frac{1}{2}\mathcal{P}\left(e^{-V}\mathcal{D}_\alpha e^V\right)$ and $\tilde{W}_\alpha = -\frac{1}{2}\mathcal{P}\left(e^{V^T}\mathcal{D}_\alpha e^{-V^T}\right) = -W_\alpha^T$. Transposing cancels out the even term, leaving the odd term

$$L_\epsilon^- = -\frac{\log\epsilon}{16\pi^2}\int E\left(\omega\times\frac{1}{12}\mathcal{D}^\alpha X_\alpha\right) + \text{h.c.}$$

Noting that $\delta W_\alpha = -\frac{1}{2}\mathcal{P}\mathcal{D}_\alpha\omega$, this is equivalent to

$$L_\epsilon^- \ni -\frac{\log\epsilon}{16\pi^2}\times\frac{1}{6}\int\mathcal{E}\,\mathrm{Tr}\left(\delta W^\alpha X_\alpha\right) + \text{h.c.} \tag{5.3.75}$$

which is trivially integrable.

We summarize here our results: the quadratic divergences of the operator D are (restoring the compensator)

$$[\mathrm{Tr}\log D]_\epsilon \ni +\frac{1}{16\pi^2\epsilon}\Big[\mathrm{Tr}\left(1-2V\right)X\Big]_D \tag{5.3.76}$$

and the logarithmic divergences are

$$\begin{aligned}[\mathrm{Tr}\log D]_\epsilon \ni &-\frac{\log\epsilon}{48\pi^2}S_\chi + \frac{\log\epsilon}{32\pi^2}\left[\left(W^\alpha - \frac{1}{6}X^\alpha\right)^2 + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F \\ &+\frac{\log\epsilon}{32\pi^2}\left[\left(\bar{W}_\alpha - \frac{1}{6}\bar{X}_{\dot\alpha}\right)^2 + \frac{2}{3}\bar{W}_{\dot\gamma\dot\beta\dot\alpha}\bar{W}^{\dot\gamma\dot\beta\dot\alpha}\right]_{\bar{F}}\end{aligned} \tag{5.3.77}$$

Calculation of ℓ

The non-integrability of the finite part of L_ϵ is due to the non-vanishing of

$$\begin{aligned}C_\epsilon &= -\epsilon\int_0^1 d\lambda\mathrm{Tr}\left(\delta\tilde{D}e^{-\epsilon\tilde{\lambda}\tilde{H}}\delta De^{-\epsilon\lambda H}\right) \\ &= -\epsilon\int_0^1 d\lambda\mathrm{Tr}_+\left(\mathcal{P}\Delta Ve^{-\epsilon\tilde{\lambda}\tilde{H}_-}\bar{\mathcal{P}}\Delta Ve^{-\epsilon\lambda H_+}\right) - \text{conjugate rep} \\ &= -\epsilon\int_0^1 d\lambda\int E\int E'\;\mathrm{Tr}\left(\omega(z)U_-(z,z';\epsilon\tilde{\lambda})\,\omega(z')U_+(z',z;\epsilon\lambda)\right) - \text{conjugate rep}\end{aligned} \tag{5.3.78}$$

where we have written everything in a covariant notation as well as promoting ω to a 1-form in analogy to the fermionic case. The above expression includes the subtraction of the conjugate ($V \to -V^T$) representation; thus in a self-conjugate representation C_ϵ vanishes and L_ϵ is integrable by itself.

In the last line of the above formula we have taken a trace over chiral coordinates, introduced a complete set of antichiral coordinates in the center, and converted both systems into total superspace integrals using the explicit projectors.[9]

The evaluation of this expression is somewhat technical, so we relegate it to Appendix D where we explicitly evaluate the expression

$$Z(\omega_2,\omega_1;\epsilon,\lambda) = \int E \int E' \;\mathrm{Tr}\left(\omega_2(z)U_-(z,z',\epsilon\tilde{\lambda})\omega_1(z')U_+(z',z,\epsilon\lambda)\right) \tag{5.3.79}$$

where $\tilde{\lambda} = 1-\lambda$. We find

$$\begin{aligned} Z = \frac{1}{16\pi^2\epsilon^2}\int E\,\mathrm{Tr}\Bigg\{ &\omega_2\omega_1 - \frac{\epsilon\lambda}{2}R\mathcal{D}^\alpha\omega_2\mathcal{D}_\alpha\omega_1 - \frac{\epsilon\tilde{\lambda}}{2}\bar{R}\bar{\mathcal{D}}_{\dot\alpha}\omega_2\bar{\mathcal{D}}^{\dot\alpha}\omega_1 \\ &- \frac{\epsilon}{12}\mathcal{D}^\alpha X_\alpha\omega_2\omega_1 - \epsilon\lambda\tilde{\lambda}\mathcal{D}^a\omega_2\mathcal{D}_a\omega_1 \\ &+ \frac{\epsilon\lambda}{2}(\mathcal{D}^\alpha\omega_2\omega_1 W_\alpha - \omega_2\mathcal{D}^\alpha\omega_1 W_\alpha) \\ &+ \frac{\epsilon\tilde{\lambda}}{2}(\bar{\mathcal{D}}_{\dot\alpha}\omega_2 W^{\dot\alpha}\omega_1 - \omega_2 W_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\omega_1) + \mathcal{O}(\epsilon^2)\Bigg\} \end{aligned} \tag{5.3.80}$$

For the case of interest here,

$$\begin{aligned} C_\epsilon = -\frac{1}{16\pi^2}\int E\,\mathrm{Tr}\Bigg\{ &\frac{1}{\epsilon}\omega\omega - \frac{1}{6}\mathcal{D}^a\omega\mathcal{D}_a\omega - \frac{1}{4}R\mathcal{D}^\alpha\omega\mathcal{D}_\alpha\omega - \frac{1}{4}\bar{R}\bar{\mathcal{D}}_{\dot\alpha}\omega\bar{\mathcal{D}}^{\dot\alpha}\omega - \frac{1}{12}\mathcal{D}^\alpha X_\alpha\omega\omega \\ &- \frac{1}{4}\omega\mathcal{D}^\alpha\omega W_\alpha + \frac{1}{4}\mathcal{D}^\alpha\omega\omega W_\alpha + \frac{1}{4}\mathcal{D}_{\dot\alpha}\omega W^{\dot\alpha}\omega - \frac{1}{4}\omega W_{\dot\alpha}\mathcal{D}^{\dot\alpha}\omega\Bigg\} + \mathcal{O}(\epsilon) \\ &- \text{conjugate rep} \end{aligned} \tag{5.3.81}$$

Using cyclicity of the trace and the antisymmetry of the 1-forms ω (and the fact that the conjugate rep is the same result after transposition), we find that only a small set of terms survive in the $\epsilon \to 0$ limit, giving

$$\begin{aligned} C &= \frac{1}{32\pi^2}\int E\,\mathrm{Tr}\left(\omega\mathcal{D}^\alpha\omega W_\alpha - \mathcal{D}^\alpha\omega\omega W_\alpha + \omega\mathcal{D}_{\dot\alpha}\omega W^{\dot\alpha} - \mathcal{D}_{\dot\alpha}\omega\omega W^{\dot\alpha}\right) \\ &= \frac{1}{32\pi^2}\int E\,\left(\omega^r\mathcal{D}^\alpha\omega^s W_\alpha^t + \omega^r\mathcal{D}_{\dot\alpha}\omega^s\bar{W}^{\dot\alpha t}\right)\mathcal{A}_{rst} \end{aligned} \tag{5.3.82}$$

where $\mathcal{A}_{rst} \equiv \mathrm{Tr}(\{\mathbf{T}_r,\mathbf{T}_s\}\mathbf{T}_t)$ is the anomaly factor, the symmetrized trace of three generators of the gauge group. This is exactly the same form as the globally supersymmetric result found by McArthur and Osborn [39]. C may also be written

$$C = \frac{1}{16\pi^2}\int E\,\mathrm{Tr}\left(\omega\mathcal{D}^\alpha\omega W_\alpha - \mathcal{D}_{\dot\alpha}\omega\omega W^{\dot\alpha}\right) \tag{5.3.83}$$

[9] The subtraction of the conjugate representation arises because one actually adds the full Hermitian conjugate; in reordering the operators so that U_- appears before U_+ in each term, one finds a sign flip from pushing the one-forms past each other.

by integrating by parts and using $\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha} W^{\dot\alpha}$.

To derive the form of ℓ, we follow exactly the procedure of [39], which is essentially unchanged by the addition of supergravity. We begin by introducing a new function $\mathcal{X}$

$$C = \frac{1}{16\pi^2}\int E\,\mathcal{X}(\omega,\omega,V) \tag{5.3.84}$$

where

$$\mathcal{X}(h_1,h_2,V) \equiv \mathrm{STr}\left(h_1\mathcal{D}^\alpha h_2 W_\alpha - \mathcal{D}_{\dot\alpha} h_1 h_2 W^{\dot\alpha}\right) \tag{5.3.85}$$

is a two form. We define it with a symmetrized and normalized trace of the three generators of the gauge group:

$$\mathrm{STr}\,(ABC) \equiv \frac{1}{2}A^r B^s C^t \mathrm{Tr}\,(\{\mathbf{T}_r,\mathbf{T}_s\}\mathbf{T}_t) \tag{5.3.86}$$

One can show that this two form is both Hermitian and symmetric in its one-form arguments h_1 and h_2. Note $\mathcal{X}$ depends on V implicitly through W_α and the covariant derivative.

Again following McArthur and Osborn, we enlarge the configuration space of V to include a parameter t, with $t=0$ corresponding to $V=0$ and $t=1$ corresponding to the full background V. We denote this parametrized prepotential by V_t. The total variation Ω_t of e^{V_t} is then given by two pieces: $\Omega_t = \omega_t^t + \omega_t$ where, in chiral gauge, $\omega_t = e^{-V_t}\delta e^{V_t}$ and $\omega_t^t = e^{-V_t} d_t e^{V_t}$ for $d_t = dt\,\partial_t$. Since C and therefore $\mathcal{X}$ is exact,

$$(\delta + d_t)\mathcal{X}(\Omega_t,\Omega_t,V_t) = 0 \tag{5.3.87}$$

and one may show (using $dt\wedge dt = 0$)

$$\delta\mathcal{X}(\omega_t,\omega_t^t,V_t) = -\frac{1}{2}d_t\mathcal{X}(\omega_t,\omega_t,V_t) \tag{5.3.88}$$

Then we may construct a local one-form

$$\ell \equiv -\frac{1}{8\pi^2}\int_{I_t}\mathcal{X}(\omega_t,\omega_t^t,V_t) \tag{5.3.89}$$

whose variation is[10]

$$\delta\ell = \frac{1}{8\pi^2}\int_{I_t}\delta\mathcal{X}(\omega_t,\omega_t^t,V_t) = -\frac{1}{16\pi^2}\int_{I_t} d_t\mathcal{X}(\omega_t,\omega_t,V_t) = -\frac{1}{16\pi^2}\mathcal{X}(\omega,\omega,V) \tag{5.3.90}$$

The precise form of ℓ is useful in certain applications – for example, to give a consistent form for the non-Abelian anomaly associated with gauge transformations of V. However, the definition of ℓ is quite path dependent; in particular, ℓ is only defined up to an arbitrary closed form. There are two obvious paths to choose. One is the "gauge coupling" path $V_t = tV$, where t has the immediate interpretation as the strength of the

[10] In these expressions, integration is defined with dt moved to be adjacent to the integration symbol. This generates a sign whenever dt is pushed through another 1-form. Thus $\delta\int_{I_t} = -\int_{I_t}\delta$.

gauge coupling. This is the simplest choice for an Abelian theory. Another reasonable option is the "minimal homotopic" path of $e^{V_t} = (1-t) + te^V$ suggested by Gates, Grisaru, and Penati [46].

Since one is often concerned with Abelian anomalies, we will restrict ourselves briefly to that case and the use of the gauge couping path. This immediately gives

$$\begin{aligned}\ell &= -\frac{1}{8\pi^2}\int_0^1 \mathcal{X}(\omega_t, \omega_t^t, V_t) = \frac{1}{24\pi^2}\left(\delta V \mathcal{D}^\alpha V W_\alpha - \mathcal{D}_{\dot\alpha}\delta V V W^{\dot\alpha}\right) \\ &= \frac{1}{24\pi^2}\left(\delta V \mathcal{D}^\alpha V W_\alpha + \delta V \mathcal{D}_{\dot\alpha} V W^{\dot\alpha} + \delta V V \mathcal{D}_{\dot\alpha} W^{\dot\alpha}\right) \\ &= -\frac{1}{12\pi^2}\left(\delta V \Omega_V\right)\end{aligned} \tag{5.3.91}$$

where we have dropped a total derivative. Here, $W_\alpha = \frac{1}{8}(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha V$ and, it should be recalled, $\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha}$. Ω_V is the Chern-Simons superfield [47] for the Abelian gauge group, obeying $[\Lambda\Omega_V]_D = [\Lambda W^\alpha W_\alpha]_F$ for chiral Λ.

Expression for $\mathrm{Tr}\log D$

We now need to integrate the closed form $L_\epsilon^- + \ell$. We introduce another parameter u which interpolates from $V=0$ to the final value of V. We then take

$$\int_{I_u}\left(L_\epsilon^-(\omega_u^u, V_u) + \ell(\omega_u^u, V_u)\right) = \int_{I_u} L_\epsilon^-(\omega_u^u, V_u) - \frac{1}{8\pi^2}\int_{I_u\times I_t}\mathcal{X}(\omega_{ut}^u, \omega_{ut}^t, V_{ut}) \tag{5.3.92}$$

where V_{ut} denotes the doubly-parametrized V and ω_{ut}^t and ω_{ut}^u are defined in chiral gauge by

$$\omega_{ut}^t \equiv e^{-V_{ut}} d_t e^{V_{ut}}, \quad \omega_{ut}^u \equiv e^{-V_{ut}} d_u e^{V_{ut}} \tag{5.3.93}$$

It is not necessary for the paths parametrized by u and t to be identical. One can show (following McArthur and Osborn) that under an arbitrary variation in the gauge prepotential,

$$\delta\int_{I_u} L_\epsilon^-(\omega_u^u, V_u) = L_\epsilon^-(\omega, V) - \frac{1}{8\pi^2}\int_{I_u}\mathcal{X}(\omega_u, \omega_u^u, V_u) \tag{5.3.94}$$

as well as

$$\delta\int_{I_u\times I_t}\mathcal{X}(\omega_{ut}^u, \omega_{ut}^t, V_{ut}) = \int_{I_t}\mathcal{X}(\omega_t, \omega_t^t, V_t) - \int_{I_u}\mathcal{X}(\omega_u, \omega_u^u, V_u) \tag{5.3.95}$$

The above (5.3.95) is especially simple when the paths parametrized by u and t are identical: then the variation of this term vanishes!

The final expression of the effective action is

$$[\mathrm{Tr}\log D]_\epsilon = \frac{1}{2}[\mathrm{Tr}\log H]_\epsilon + \int_{I_u} L_\epsilon^-(\omega_u^u, V_u) - \frac{1}{8\pi^2}\int_{I_u\times I_t}\mathcal{X}(\omega_{ut}^u, \omega_{ut}^t, V_{ut}) \tag{5.3.96}$$

This shall represent our *definition* for the regulated effective action.

Anomaly for the $U(1)$ ambiguity

Before analyzing the gauge and conformal anomalies, we will consider a different sort of anomaly. Our massless action in the natural path integral variables had the form

$$S = \left[\bar{\eta}\frac{e^V}{X^{1/2}}\eta\right]_D \tag{5.3.97}$$

where η is weight $(3/2, 1)$, X has conformal dimension two, and V is a dimension zero gauge prepotential. Under the replacement

$$e^V \to e^{V+yV_1} \equiv e^{V_y}, \quad X^{1/2} \to X^{1/2}e^{yV_1} \equiv X_y^{1/2} \tag{5.3.98}$$

for a $U(1)$ prepotential V_1, the classical action is invariant for all values of y. Since the gauge and conformal sectors were treated asymmetrically, we expect our definition for the effective action should be anomalous under this transformation; however, if the anomaly is not really physical, then the difference should be a local expression. It turns out this is the case, which we now prove.

We begin with a model where the replacement (5.3.98) has been made for some value of y. The first step is to extract the gauge dependence from $[\mathrm{Tr}\log H]_\epsilon$, writing it as

$$\frac{1}{2}[\mathrm{Tr}\log H]_\epsilon = \frac{1}{2}[\mathrm{Tr}\log H]_{\epsilon,V=0} + \int_0^1 du\, L_\epsilon^+(\omega_{uy}^u, V_{uy}) \tag{5.3.99}$$

The first term on the right can be understood as the effective action in a formally gauge-free background, yet it still depends on the $U(1)$ prepotential V_1 through the compensator X. The second term on the right represents the additional dependence on V_{uy}, the now-doubly parametrized prepotential we have extracted.

The total effective action can be written

$$[\mathrm{Tr}\log D]_\epsilon = \frac{1}{2}[\mathrm{Tr}\log H]_{\epsilon,V=0} + \int_{I_u} L_\epsilon(\omega_{uy}^u, V_{uy}) - \frac{1}{8\pi^2}\int_{I_u\times I_t} \mathcal{X}(\omega_{uty}^u, \omega_{uty}^t, V_{uty}) \tag{5.3.100}$$

where $L_\epsilon = L_\epsilon^+ + L_\epsilon^-$. Recall that in the second and third terms we have introduced auxiliary path variables u and t where $u = 0$ or $t = 0$ correspond to vanishing V and $u = t = 1$ correspond to the full V_y.

Then one can show that by differentiating with respect to y,

$$\partial_y L_\epsilon(\omega_{uy}^u, V_{uy}) = \partial_u \int_\epsilon^\infty d\tau \mathrm{Tr}\left(e^{-\tau H}\tilde{D}\partial_y D\right)_{V_{uy}} + \int_0^\epsilon d\sigma \mathrm{Tr}\left(e^{-\sigma H}\partial_{[u}\tilde{D}e^{-(\epsilon-\sigma)\tilde{H}}\partial_{y]}D\right)_{V_{uy}}$$

where D, $\tilde{D}$, and H are defined in terms of V_{uy}, emphasized by the subscript. (This equation is a special case of (5.3.66).) This immediately implies that

$$\partial_y \int_{I_u} L_\epsilon(\omega_{uy}^u, V_{uy}) = \int_\epsilon^\infty d\tau \mathrm{Tr}\left(e^{-\tau H}\tilde{D}\partial_y D\right)_{V_y} - \int_\epsilon^\infty d\tau \mathrm{Tr}\left(e^{-\tau H}\tilde{D}\partial_y D\right)_{V=0}$$
$$+ \int_0^1 du \int_0^\epsilon d\sigma \mathrm{Tr}\left(e^{-\sigma H}\partial_{[u}\tilde{D}e^{-(\epsilon-\sigma)\tilde{H}}\partial_{y]}D\right)_{V_{uy}}$$

The first term on the right vanishes since $\partial_y(e^{V_y}X_y^{-1/2})$ vanishes. The second term on the right can be simplified by noting that at $V=0$, $\tilde{D}=-D$, and so

$$-\int_\epsilon^\infty d\tau \mathrm{Tr}\left(e^{-\tau H}\tilde{D}\partial_y D\right)_{V=0} = \frac{1}{2}\partial_y \int_\epsilon^\infty \frac{d\tau}{\tau}\mathrm{Tr}\left(e^{-\tau H}\right)_{V=0} = -\frac{1}{2}\partial_y[\mathrm{Tr}\log H]_{\epsilon,V=0}$$

Then the y-derivative of $[\mathrm{Tr}\log D]_\epsilon$ is reduced to

$$\begin{aligned}\partial_y[\mathrm{Tr}\log D]_\epsilon = &+\int_0^1 du \int_0^\epsilon d\sigma\, \mathrm{Tr}\left(e^{-\sigma H}\partial_{[u}\tilde{D}e^{-(\epsilon-\sigma)\tilde{H}}\partial_{y]}D\right)_{V_{uy}} \\ &-\frac{1}{8\pi^2}\int_{I_u\times I_t}\partial_y \mathcal{X}(\omega^u_{uty},\omega^t_{uty},V_{uty})\end{aligned} \tag{5.3.101}$$

which is a local (though divergent) expression. The ambiguity in whether we consider the $U(1)$ as part of the conformal factor or as part of the Yang-Mills factor is therefore a local counterterm allowed by the ambiguities of regularization. We are free to choose whatever parametrization is the most natural.

It is straightforward to evaluate the first term of (5.3.101) using the method of Appendix D. The result is

$$\begin{aligned}-\frac{1}{4\pi^2}\Bigg(&\epsilon\mathrm{Tr}(V_y)V_1 - \frac{1}{4}R\mathrm{Tr}(e^{-V_y}\mathcal{D}^\alpha e^{V_y})\mathcal{D}_\alpha V_1 - \frac{1}{4}\bar{R}\mathrm{Tr}(e^{-V_y}\bar{\mathcal{D}}_{\dot\alpha}e^{V_y})\bar{\mathcal{D}}^{\dot\alpha}V_1 \\ &-\frac{1}{12}\mathcal{D}^\alpha X_\alpha \mathrm{Tr}(V_y)V_1 + \frac{i}{24}\mathrm{Tr}\left(\mathcal{D}_{\{\dot\alpha}(e^{-V_y}\mathcal{D}_{\alpha\}}e^{V_y})\right)\mathcal{D}^{\dot\alpha\alpha}V_1\Bigg)\end{aligned}$$

where we should recall $V_y = V + yV_1$. This is a somewhat deceptive labelling though since the y-dependent compensator X_y is used to define the supergravity superfields R and X_α as well as in the covariant derivatives $\mathcal{D}$. In principle, all of the y (and V_1) dependence may be explicitly expanded.

The second term may be evaluated by noting that $\mathcal{X}$ is independent of the compensator X, and so ∂_y amounts to an arbitrary $U(1)$ shift in the prepotential. Then following (5.3.95),

$$\partial_y\int_{I_u\times I_t}\mathcal{X}(\omega^u_{ut},\omega^t_{ut},V_{ut}) = \int_{I_t}\mathcal{X}(\omega^y_{ty},\omega^t_{ty},V_{ty}) - \int_{I_u}\mathcal{X}(\omega^y_{uy},\omega^u_{uy},V_{uy})$$

which vanishes if the paths parametrized by t and u are identical. Then the only contribution is that of the first term, which is manifestly local and can be integrated in the $U(1)$ deformation parameter y.

Conformal anomaly

The conformal anomaly with which we will be concerned involves the transformation

$$\eta \to e^{-\lambda}\eta, \quad \bar\eta \to \bar\eta e^{-\bar\lambda}, \quad X \to Xe^{-2\bar\lambda-2\lambda} \tag{5.3.102}$$

in the action (5.3.97). Begin by recalling the definition of the effective action:

$$[\mathrm{Tr}\log D]_\epsilon = \frac{1}{2}[\mathrm{Tr}\log H]_\epsilon + \int_{I_u} L_\epsilon^-(\omega_u^u, V_u) - \frac{1}{8\pi^2}\int_{I_u\times I_t} \mathcal{X}(\omega_{ut}^u, \omega_{ut}^t, V_{ut}) \qquad (5.3.103)$$

Under a conformal transformation, $\mathrm{Tr}\log H$ generates the covariant conformal anomaly:

$$\begin{aligned}\frac{1}{2}\delta_\lambda \mathrm{Tr}\log H &= \mathrm{Tr}_+\left(\lambda e^{-\epsilon H_+}\right) + \mathrm{Tr}_+\left(\lambda e^{-\epsilon \tilde{H}_+}\right) + \text{h.c.} \\ &= \frac{1}{16\pi^2}\mathrm{Tr}\left(-\frac{2}{\epsilon}[\lambda]_D + [\lambda A_2]_F\right) + \text{h.c.} \end{aligned} \qquad (5.3.104)$$

Since $\mathcal{X}$ is independent of X, the only other contribution to the conformal anomaly comes from the L_ϵ^- term. It is straightforward to show

$$\delta_\lambda L_\epsilon^- = -\epsilon\int_0^1 d\lambda \mathrm{Tr}\left(e^{-\epsilon\lambda H}\delta_\lambda \tilde{D} e^{-\epsilon\tilde{\lambda}\tilde{H}}\delta_V D\right) + \epsilon\int_0^1 d\lambda \mathrm{Tr}\left(e^{-\epsilon\tilde{\lambda}\tilde{H}}\delta_\lambda D e^{-\epsilon\lambda H}\delta_V \tilde{D}\right)$$

which may be rewritten as

$$\delta_\lambda L_\epsilon^- = -\epsilon\int_0^1 d\lambda \mathrm{Tr}\left(e^{-\epsilon\lambda H}\delta_{[\lambda}\tilde{D} e^{-\epsilon\tilde{\lambda}\tilde{H}}\delta_{V]} D\right)$$

This is easy enough to calculate using the general formula found in Appendix D. The result is a contribution

$$\frac{1}{16\pi^2}\mathrm{Tr}\left[\frac{4}{\epsilon}\lambda V - R\mathcal{D}^\alpha\lambda\mathcal{D}_\alpha V - \frac{1}{3}\lambda\mathcal{D}^\alpha X_\alpha V + \frac{2}{3}\lambda\Box V\right]_D + \text{h.c.} \qquad (5.3.105)$$

which is symmetric with respect to λ and V. The third term may be rewritten to give the missing "cross-term" $W^\alpha X_\alpha$ for the covariant anomaly.

Putting everything together, we find a conformal anomaly which may be written (restoring the compensator X)

$$\begin{aligned}\delta_\lambda[\mathrm{Tr}\log D]_\epsilon = &-\frac{1}{8\pi^2\epsilon}\mathrm{Tr}[\lambda X(1-2V)]_D \\ &+ \frac{1}{8\pi^2}\mathrm{Tr}\left[\lambda\left(W_\alpha - \frac{1}{6}X_\alpha\right)^2\right]_F + \frac{1}{12\pi^2}\mathrm{Tr}\left[\lambda W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}\right]_F - \frac{1}{24\pi^2}\mathrm{Tr}\left[\lambda\Omega_\chi\right]_D \\ &+ \frac{1}{16\pi^2}\mathrm{Tr}\left[-R\mathcal{D}^\alpha\lambda\mathcal{D}_\alpha V + \frac{1}{3}\mathcal{D}^\alpha\lambda X_\alpha V - \frac{2}{3}\mathcal{D}^a\lambda\mathcal{D}_a V\right]_D + \text{h.c.} \end{aligned} \qquad (5.3.106)$$

where we have defined

$$\Omega_\chi \equiv G^2 + \bar{\mathcal{P}}R + \mathcal{P}\bar{R} - 2R\bar{R} + \frac{1}{6}\Omega_X + \Omega_L \qquad (5.3.107)$$

with

$$\begin{aligned}[\Omega_X]_D &= [X^\alpha X_\alpha]_F = [X_{\dot\alpha}X^{\dot\alpha}]_{\bar{F}} \\ [\Omega_L]_D &= [W^{\alpha\beta\gamma}W_{\gamma\beta\alpha}]_F = [\bar{W}_{\dot\alpha\dot\beta\dot\gamma}\bar{W}^{\dot\gamma\dot\beta\dot\alpha}]_{\bar{F}}.\end{aligned}$$

The Chern-Simons superfields Ω_X and Ω_L should exist so long as our background gauge sector is topologically trivial [47]. They are not themselves gauge invariant; but since they transform under a gauge transformation into a linear superfield, integrals of expressions like $\phi\Omega_X$ for chiral ϕ *are* gauge invariant.

This expression for the conformal anomaly is fairly simple to understand: the first line which is quadratically divergent is cancelled if we add counterterms to the effective action to remove the original ϵ divergences; the second line is a sensible anomaly with a topological Gauss-Bonnet term; and the third line is an extra contribution to the conformal anomaly in the presence of a gauge sector which is not trace-free and a conformal parameter λ which is not constant.

Gauge anomaly

The gauge anomaly arises from the transformation

$$\eta \to e^{-\Lambda}\eta, \quad \bar{\eta} \to \bar{\eta}e^{-\bar{\Lambda}}, \quad e^V \to e^{\bar{\Lambda}}e^V e^{\Lambda} \tag{5.3.108}$$

in the action (5.3.97). Again we begin by recalling the definition of the effective action,

$$[\mathrm{Tr}\log D]_\epsilon = \frac{1}{2}[\mathrm{Tr}\log H]_\epsilon + \int_{I_u} L_\epsilon^-(\omega_u, V_u) - \frac{1}{8\pi^2}\int_{I_u\times I_t} \mathcal{X}(\omega_{ut}^u, \omega_{ut}^t, V_{ut}) \tag{5.3.109}$$

Under a gauge transformation, $\mathrm{Tr}\log H$ is invariant as it corresponds to the even gauge sector, where the superfields can be combined in a Dirac-like and anomaly-free fashion. The variation of the other two terms can be found from (5.3.94) and (5.3.95) to give

$$\delta_\Lambda[\mathrm{Tr}\log D]_\epsilon = L_\epsilon^-(\omega^\Lambda, V) - \frac{1}{8\pi^2}\int_{I_t} \mathcal{X}(\omega_t^\Lambda, \omega_t^t, V_t) \tag{5.3.110}$$

where $\omega^\Lambda = e^{-V}\bar{\Lambda}e^V + \Lambda$ in the chiral representation and where Λ is conventionally chiral and $\bar{\Lambda}$ is conventionally antichiral. (The precise form of ω_t^Λ is path-dependent but is straightforward to work out.) The first term can be evaluated straightforwardly to give the covariant gauge anomaly

$$\begin{aligned}
L_\epsilon^-(\omega^\Lambda, V) &= \mathrm{Tr}_+\left(\Lambda e^{-\epsilon H_+}\right) + \mathrm{Tr}_+\left(\Lambda^T e^{-\epsilon\tilde{H}_+}\right) + \text{h.c.} \\
&= \frac{1}{16\pi^2}\mathrm{Tr}\left(-\frac{2}{\epsilon}[\Lambda]_D + [\Lambda A_2]_F\right) + \text{h.c.} \\
&= -\frac{1}{8\pi^2\epsilon}[X\mathrm{Tr}\Lambda]_D + \frac{1}{8\pi^2}\left[\mathrm{Tr}\Lambda W^\alpha W_\alpha + \frac{1}{36}\mathrm{Tr}\Lambda X^\alpha X_\alpha\right]_F \\
&\quad + \frac{1}{12\pi^2}\left[\mathrm{Tr}\Lambda W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}\right]_F - \frac{1}{24\pi^2}[\mathrm{Tr}\Lambda\Omega_\chi]_D + \text{h.c.}
\end{aligned} \tag{5.3.111}$$

where we have used $\mathrm{Tr}\Lambda^T = \mathrm{Tr}\Lambda$ as well as $\mathrm{Tr}(\Lambda^T\tilde{A}_2) = \mathrm{Tr}(\Lambda A_2)$. (We have also restored the compensator X in the final equality.) The divergent anomalous term is exactly the gauge variation of the Fayet-Iliopoulos term, which appeared as a divergent contribution to the odd part of the effective action.

This alone is not a consistent anomaly and requires the addition of the term involving $\mathcal{X}$, which is path-dependent and for a non-abelian gauge sector will in general involve an infinite series of terms. We will subsequently neglect this term.

Conspicuous in its absence is anything resembling the cross term $W^\alpha X_\alpha$. This is not found in the covariant part of the gauge anomaly, nor is it found in the term $\mathcal{X}$. Since the $U(1)$ ambiguity implies that a conformal anomaly must be equivalent to a $U(1)$ gauge anomaly up to a local counterterm, it is clear that the missing cross term for the gauge anomaly must be found as the variation of a local counterterm. Indeed, such a term does exist:

$$\frac{1}{2}\delta_\Lambda \left[\mathrm{Tr}(V^2)\mathcal{D}^\alpha X_\alpha\right]_D = \left[\mathrm{Tr}(\Lambda V)\mathcal{D}^\alpha X_\alpha\right]_D + \text{h.c.} \tag{5.3.112}$$

which gives the missing cross term as well as a non-covariant term which depends on the derivative of Λ. This is simply one of the terms of (5.3.105) with the covariant parameter λ replaced by V.

Inclusion of a covariant mass term

The preceding analysis dealt with massless fields, which was sensible since we have been concerned with arbitrary complex representations where a constant mass term would be manifestly forbidden. The models with which we will be concerned, however, do contain covariant mass terms generated both from the superpotential and Kähler potential, so we will need a method to deal with them.

For the case of chiral fermions, the inclusion of a mass term is not terribly difficult. If the operator D has entries μ and $\bar{\mu}$ on the diagonal, one simply constructs $\tilde{D}$ to have entries $\bar{\mu}$ and μ. For chiral superfields, this avenue is not open to us because of the holomorphicity requirement. A generic covariant chiral mass term μ, depending perhaps on the background chiral superfields, simply cannot be used in the antichiral sector. We will therefore restrict ourselves to dealing with mass terms via a perturbative approach.

Given an operator $\det(D+\hat{\mu})$ and the additional operator $\det\tilde{D}$ associated with the massless conjugate, we may formally identify

$$\mathrm{Tr}\log\tilde{D} + \mathrm{Tr}\log(D+\hat{\mu}) = \mathrm{Tr}\log(\tilde{D}D + \tilde{D}\hat{\mu}) \tag{5.3.113}$$

Identifying $H = \tilde{D}D$ and $\tilde{D}\hat{\mu} \equiv V$, this operator at least formally has the structure of $H+V$. Evaluating this perturbatively using a proper time cutoff regulator gives

$$\begin{aligned}\left[\mathrm{Tr}\log(H+V)\right]_\epsilon &= \left[\mathrm{Tr}\log H\right]_\epsilon + \int_\epsilon^\infty d\tau\,\mathrm{Tr}\left(e^{-\tau H}V\right)\\ &\quad - \frac{1}{2}\int_\epsilon^\infty d\tau \int_0^\tau d\sigma\,\mathrm{Tr}\left(e^{-\sigma H}Ve^{-(\tau-\sigma)H}V\right) + \mathcal{O}(V^3)\end{aligned}$$

For our case, $\tilde{D}\hat{\mu}$ has vanishing elements on the diagonal and so only terms even in $\tilde{D}\hat{\mu}$ appear. This leads to the identification

$$\left[\mathrm{Tr}\log(D+\hat{\mu})\right]_\epsilon - \left[\mathrm{Tr}\log D\right]_\epsilon \equiv -\frac{1}{2}\int_\epsilon^\infty d\tau\int_0^\tau d\sigma\,\mathrm{Tr}\left(e^{-\sigma H}\tilde{D}\hat{\mu}e^{-(\tau-\sigma)H}\tilde{D}\hat{\mu}\right) + \mathcal{O}(\hat{\mu}^4) \tag{5.3.114}$$

where $[\mathrm{Tr}\log D]_\epsilon$ is the previous definition we have made. The advantage of (5.3.114) is that the final answer is quite independent of the particular way we have chosen to write (5.3.113); other arrangements of the formal operators lead to an identical regulated result. We may rewrite (5.3.114) as

$$[\mathrm{Tr}\log(D+\hat{\mu})]_\epsilon - [\mathrm{Tr}\log D]_\epsilon = -\int_\epsilon^\infty d\tau \int_0^\tau d\sigma\, Z(\bar{\mu},\mu;\sigma,\tau-\sigma) + \mathcal{O}(\hat{\mu}^4)$$

where Z is as defined in Appendix D. At leading order,

$$Z(\bar{\mu},\mu;\sigma,\tau-\sigma) = \frac{1}{16\pi^2}\frac{1}{\tau^2}\,[\bar{\mu}\mu]_D + \ldots$$

which gives

$$[\mathrm{Tr}\log(D+\hat{\mu})]_\epsilon - [\mathrm{Tr}\log D]_\epsilon = +\frac{\log\epsilon}{16\pi^2}\,[\bar{\mu}\mu]_D + \text{finite} \tag{5.3.115}$$

To calculate anomalies associated with the mass term, observe first that a gauge anomaly acts on the objects D, $\tilde{D}$, and $\hat{\mu}$ via

$$\delta_g D = D\Lambda + \Lambda^T D, \quad \delta_g\tilde{D} = -\tilde{D}\Lambda^T - \Lambda\tilde{D}, \quad \delta_g\hat{\mu} = \hat{\mu}\Lambda + \Lambda^T\hat{\mu}$$
$$\delta_g H = [H,\Lambda], \quad \delta_g(\tilde{D}\hat{\mu}) = [\tilde{D}\hat{\mu},\Lambda]$$

provided that $\hat{\mu}$ transform in a way that leaves the classical action gauge invariant. Given the transformation rules of H and $\tilde{D}\hat{\mu}$, the perturbative expansion of the effective action in terms of $\hat{\mu}$ *must* be free of gauge anomalies. (This is obvious in retrospect since we based our construction on the operator $\tilde{D}D+\tilde{D}\hat{\mu}$, which is manifestly gauge covariant.) Thus

$$\delta_g\Big([\mathrm{Tr}\log(D+\hat{\mu})]_\epsilon - [\mathrm{Tr}\log D]_\epsilon\Big) = 0 \tag{5.3.116}$$

For conformal anomalies, observe that

$$\delta_c D = \{D,\lambda\}, \quad \delta_c\tilde{D} = \{\tilde{D},\lambda\}, \quad \delta_c\hat{\mu} = \{\hat{\mu},\lambda\}$$
$$\delta_c H = \{H,\lambda\} + 2\tilde{D}\lambda D, \quad \delta_c\tilde{D}\hat{\mu} = \{\tilde{D}\hat{\mu},\lambda\} + 2\tilde{D}\lambda\hat{\mu}$$

It follows (after some algebra) that

$$\delta_c\Big([\mathrm{Tr}\log(D+\hat{\mu})]_\epsilon - [\mathrm{Tr}\log D]_\epsilon\Big) = 2\int_0^\epsilon d\sigma\int_0^\sigma d\sigma'$$
$$\mathrm{Tr}\left(e^{-\sigma' H}\lambda e^{-(\sigma-\sigma')H}\tilde{D}\hat{\mu}e^{-(\epsilon-\sigma)H}\tilde{D}\hat{\mu} + \tilde{D}e^{-\sigma'\tilde{H}}\lambda e^{-(\sigma-\sigma')\tilde{H}}\hat{\mu}\tilde{D}e^{-(\epsilon-\sigma)\tilde{H}}\hat{\mu}\right) + \mathcal{O}(\hat{\mu}^4)$$

For our chiral model, the traces under the integrals may be written as

$$\mathrm{Tr}_+\left(e^{-\sigma' H_+}\lambda e^{-(\sigma-\sigma')H_+}\mathcal{P}\bar{\mu}e^{-(\epsilon-\sigma)H_-}\bar{\mathcal{P}}\mu\right) + \text{conjugate rep} + \text{h.c.}$$

where we are using covariant notation for the chiral projectors and the chiral and antichiral mass terms. This is in principle a three point operator, but we don't actually need to evaluate it fully. Simply observing that dimensional counting forbids anything worse than $\lambda\mu\bar{\mu}$ as a D-term, we can first neglect all derivatives on λ to contract the first set of heat kernels and then perform the σ' integration to give

$$\delta_c\Big([\mathrm{Tr}\log(D+\hat{\mu})]_\epsilon - [\mathrm{Tr}\log D]_\epsilon\Big) = 2\int_0^\epsilon d\sigma\,\sigma\,\mathrm{Tr}_+\left(\lambda\mathcal{P}\bar{\mu}e^{-(\epsilon-\sigma)H_-}\bar{\mathcal{P}}\mu e^{-\sigma H_+}\right)$$
$$+\text{ conjugate rep} + \text{h.c.}$$

The operator within the trace is equivalent to Z except for the addition of the factor λ. This immediately yields

$$\delta_c\Big([\mathrm{Tr}\log(D+\hat{\mu})]_\epsilon - [\mathrm{Tr}\log D]_\epsilon\Big) = \frac{1}{8\pi^2}\left[\lambda\bar{\mu}\mu\right]_D + \text{h.c.}$$

Restoring the explicit factors of the gauge and conformal fields gives

$$\delta_c\Big([\mathrm{Tr}\log(D+\hat{\mu})]_\epsilon - [\mathrm{Tr}\log D]_\epsilon\Big) = \frac{1}{8\pi^2}\left[\lambda X\mathrm{Tr}(e^{-V}\bar{\mu}e^{-V^T}\mu)\right]_D + \text{h.c.} \tag{5.3.117}$$

That there is a conformal anomaly involving μ but not a gauge anomaly implies again an asymmetry between whether we include a $U(1)$ factor in the conformal or in the gauge sector. There is an obvious finite counterterm to include whose $U(1)$ gauge variation gives the corresponding $U(1)$ gauge anomaly: one simply puts the $U(1)$ part of the prepotential in place of λ in the above expression.

Summary

We have covered a lot of ground so we briefly review our results. The model we are considering is of the form

$$S = \left[\bar{\eta}\frac{e^V}{X^{1/2}}\eta\right]_D + \frac{1}{2}\left[\eta^T\mu\eta\right]_F + \frac{1}{2}\left[\bar{\eta}\bar{\mu}\bar{\eta}^T\right]_F \tag{5.3.118}$$

The one-loop effective action Γ (with a proper time cutoff) is found by calculating

$$[\Gamma]_\epsilon \equiv -\frac{1}{2}[\mathrm{Tr}\log(D+\hat{\mu})]_\epsilon \tag{5.3.119}$$

The divergences of this effective action are

$$\begin{aligned}[\Gamma]_\epsilon \ni &-\frac{1}{32\pi^2\epsilon}\left[\mathrm{Tr}\,(1-2V)\,X\right]_D \\ &+\frac{\log\epsilon}{96\pi^2}S_\chi - \frac{\log\epsilon}{32\pi^2}\left[X\mathrm{Tr}(e^{-V}\bar{\mu}e^{-V^T}\mu)\right]_D \\ &-\frac{\log\epsilon}{64\pi^2}\left(\left[\left(W^\alpha - \frac{1}{6}X^\alpha\right)^2 + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F + \text{h.c.}\right)\end{aligned} \tag{5.3.120}$$

where

$$S_\chi = \left[G^2 + 2R\bar{R}\right]_D + \left(\frac{1}{12}\left[X^\alpha X_\alpha\right]_F + \frac{1}{2}\left[W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F + \text{h.c.}\right) \tag{5.3.121}$$

We emphasize that the logarithmic divergences are independent of the choice of where to place the $U(1)$ factor.

Of a nearly identical form is the conformal anomaly:

$$\begin{aligned}\delta_c[\Gamma]_\epsilon = &+\frac{1}{16\pi^2\epsilon}\mathrm{Tr}\,[\lambda X(1-2V)]_D \\ &-\frac{1}{16\pi^2}\left[\lambda X\mathrm{Tr}(e^{-V}\bar{\mu}e^{-V^T}\mu)\right]_D + \frac{1}{48\pi^2}\left[\lambda\Omega_\chi\right]_D \\ &-\frac{1}{16\pi^2}\mathrm{Tr}\left[\lambda\left(W^\alpha-\frac{1}{6}X^\alpha\right)^2+\frac{2}{3}\lambda W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F \\ &-\frac{1}{32\pi^2}\mathrm{Tr}\left[-R\mathcal{D}^\alpha\lambda\mathcal{D}_\alpha V+\frac{1}{3}\mathcal{D}^\alpha\lambda\, X_\alpha V-\frac{2}{3}\mathcal{D}^a\lambda\mathcal{D}_a V\right]_D + \text{h.c.}\end{aligned} \tag{5.3.122}$$

where

$$\Omega_\chi \equiv G^2+\bar{\mathcal{P}}R+\mathcal{P}\bar{R}-2R\bar{R}+\frac{1}{6}\Omega_X+\Omega_L \tag{5.3.123}$$

(Recall that $S_\chi = [\Omega_\chi]_D$.) It is worth noting that the finite part of the conformal anomaly is independent of the $U(1)$ ambiguity when λ is a constant.

The part of the gauge anomaly which is covariant and independent of the path comes from

$$\begin{aligned}\delta_g[\Gamma]_\epsilon = &+\frac{1}{16\pi^2\epsilon}\left[\mathrm{Tr}\Lambda\, X\right]_D+\frac{1}{48\pi^2}\left[\mathrm{Tr}\Lambda\,\Omega_\chi\right]_D \\ &-\frac{1}{16\pi^2}\left[\mathrm{Tr}(\Lambda W^\alpha W_\alpha)+\frac{1}{36}\mathrm{Tr}\Lambda\, X^\alpha X_\alpha+\frac{2}{3}\mathrm{Tr}\Lambda\, W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F + \text{h.c.} \\ &+\text{non-covariant piece}\end{aligned} \tag{5.3.124}$$

This differs in three places from the form of the conformal anomaly. Two of them can easily be restored by local counterterms. Both the missing cross term $[W^\alpha X_\alpha]_F$ and the missing divergent term $[\mathrm{Tr}\Lambda V]_D$ can be introduced by using $\delta_g\mathrm{Tr}(V^2)/2 = \mathrm{Tr}(\Lambda V)+\mathrm{Tr}(\bar{\Lambda}V)$. The divergent term is proportional to this directly while the cross term can be generated from $[\mathrm{Tr}(V^2)\mathcal{D}^\alpha X_\alpha]_D$. Note that since these terms are quadratic in the gauge charge, they *cannot* come from the non-covariant piece, which is proportional to the symmetrized trace of three gauge generators. It is interesting that if we restricted to an anomaly free representation (or even just a traceless representation), both of these terms in the conformal anomaly would vanish, since they are proportional to the trace of a single generator, and so there would be no motivation to reintroduce them for the gauge sector.

The mass term, if we assume it should have the form $[X\mathrm{Tr}(\Lambda e^{-V}\bar{\mu}e^{-V^T}\mu)]_D$ is more difficult to generate for an arbitrary gauge transformation Λ. However, one *can* generate this term for the $U(1)$ part of Λ by using $[X(\mathrm{Tr}V)(\mathrm{Tr}e^{-V}\bar{\mu}e^{-V^T}\mu)]_D$, which is enough to verify that the $U(1)$ ambiguity is indeed restricted to local counterterms.

5.4 Old minimal supergravity coupled to chiral matter

In the conformal compensator formalism,

$$S = -3 \times \left[\Phi_0^\dagger e^{-K/3} \Phi_0\right]_D + \left[\Phi_0^3 W\right]_F + \left[\bar{\Phi}_0^3 \bar{W}\right]_{\bar{F}} \tag{5.4.1}$$

Φ_0 is a weight $(1, 3/2)$ conformally chiral superfield, K is weight $(0, 0)$ and Hermitian, and W is weight $(0, 0)$ conformally chiral chiral. There are N chiral matter superfields Φ^i on which K and W depend.

Different gauge choices for Φ_0 correspond to different conformally related flavors of minimal supergravity; in these versions, the quanta of Φ_0 are interpreted as quanta of the gravitational sector. Here we will leave Φ_0 ungauged and its quanta we will interpret at the same level as the other chiral matter. There is some question as to the physicality of this approach; after all, these quanta appear with the wrong sign kinetic term and so their Euclidean path integral is poorly defined.[11] Since the quanta can be removed by a certain gauge choice for diffeomorphisms, any poor behavior of this sector should be accounted for when the entire graviton and Fadeev-Popov sectors are taken into account.

In a previous chapter, we have expanded out the action to second order in the quanta of the chiral, gauge, and supergravity superfields. This action possesses kinetic mixing between the chiral and gravity sectors; in terms of Feynman graphs, the chiral and supergravity quanta mix with a coupling that goes as p^2. The proper procedure then is to find a clever gauge fixing procedure to remove the kinetic mixing (this was the approach taken in [27, 28, 25]) or to find a way to deal with an arbitrary operator on the space of vector *and* chiral superfields.

Either approach is beyond the scope of the tools developed here so we will restrain to a more limited case: we will attempt to calculate divergences and anomalies due *purely to chiral loops*. The analogous procedure in a non-supersymmetric theory would be to calculate loops involving both matter and the conformal mode of the graviton only. There may be some divergences and anomalies found in mixed loops, but we will not attempt to discover those here.

To calculate the effective action due to chiral loops, we must expand Φ^i and Φ_0 as a background plus a quantum superfield. How precisely we do this is a matter of defining quantization and should not affect the final result provided the background fields are taken to satisfy the equations of motion. We will choose

$$\delta\Phi^i = \eta^i, \quad \delta\Phi_0 = \eta_0 \tag{5.4.2}$$

where η^i is weight $(0, 0)$ chiral and η_0 is weight $(1, 3/2)$ chiral.

Denote $Z = -3\bar{\Phi}_0 \Phi_0 e^{-K/3}$ and $P = \Phi_0^3 W$ for generality.[12] Introducing the notation $\Phi^I = (\Phi_0, \Phi^i)$, the action may be written

$$S = [Z]_D + [P]_F + \left[\bar{P}\right]_{\bar{F}} \tag{5.4.3}$$

[11] This is an old problem in the non-supersymmetric gravity literature. The famous paper of Gibbons, Hawking and Perry [48] suggested to Euclideanize the conformal mode of the graviton with an additional factor of i.

[12] The definition of Z differs by a factor of -3 from that used in the previous chapters.

with a first order variation

$$S^{(1)} = \left[\eta^I \mathcal{P} Z_I + \eta^I P_I\right]_F \tag{5.4.4}$$

where $\mathcal{P} = -\nabla^2/4$ is the conformal chiral projector. The equations of motion $\mathcal{P}Z_I = -P_I$ amount to

$$\mathcal{P}\left(\Phi_0\bar{\Phi}_0 e^{-K/3}\right) = \Phi_0^3 W, \quad \mathcal{P}\left(\Phi_0\bar{\Phi}_0 e^{-K/3} K_i\right) = -\Phi_0^3 W_i, \tag{5.4.5}$$

If the gauge choice $\Phi_0 = e^{K/6}$ were adopted these would become

$$2R = e^{K/2} W, \quad -\frac{1}{4}(\bar{\mathcal{D}}^2 - 8R)K_i = -e^{K/2} W_i \tag{5.4.6}$$

The second of these may be rewritten using the first as

$$\frac{1}{4}\bar{\mathcal{D}}^2 K_i = e^{K/2}(W_i + K_i W) \equiv e^{K/2} W_{;i} \tag{5.4.7}$$

In this form, both sides of the equations transform covariantly under Kähler transformations.

The second variation is

$$\frac{1}{2}S^{(2)} = \left[\bar{\eta}^{\bar{I}} Z_{\bar{I}J}\eta^J\right]_D + \frac{1}{2}\left[\eta^I X_{IJ}\eta^J\right]_F + \frac{1}{2}\left[\bar{\eta}^{\bar{I}} \bar{X}_{\bar{I}\bar{J}}\bar{\eta}^{\bar{J}}\right]_F \tag{5.4.8}$$

where

$$X_{IJ} = P_{IJ} + \mathcal{P} Z_{IJ} \tag{5.4.9}$$

Manifest reparametrization invariance has been lost at the second variation. If we wanted to maintain it, we would need to introduce an affine connection on the space of chiral superfields. There is no object in the theory which can serve this purpose (the Kähler affine connection being non-chiral), so we would have to insert one by hand. This seems artificial so we accept the loss of manifest reparametrization invariance and expect it to be restored on shell.

The kinetic matrix $Z_{\bar{I}J}$ is clearly an object which we can treat analogously as e^V, except for the difficulty that its indices carry conformal as well as Yang-Mills charge. This can be remedied by introducing a particular measure for the path integration variables η^I so that each of the η^I are dimension $(3/2, 1)$. Then we could write $Z_{\bar{I}J}$ as $(e^V)_{\bar{I}J}/X^{1/2}$ where X has dimension two and V is dimensionless. In calculating the effective action, V and X would appear differently (as we have previously discussed), but for certain questions we would find answers that were independent of the particular details of this separation. In particular, the *logarithmic* divergences for the theory take the form (including the mass term)

$$\Gamma = -\frac{1}{2}\mathrm{Tr}\log(D + \hat{\mu}) \ni -\frac{\log\epsilon}{64\pi^2}\mathrm{Tr}\left(\left[\Phi_Z + \frac{2}{3}\Phi_W\right]_F + \text{h.c.}\right) + \frac{\log\epsilon}{96\pi^2}S_\chi - \frac{\log\epsilon}{32\pi^2}\mathrm{Tr}\left[\Omega_P\right]_D \tag{5.4.10}$$

where

$$\Omega_P = X_{IJ} Z^{J\bar{J}} \bar{X}_{\bar{J}\bar{I}} Z^{\bar{I}I} \tag{5.4.11}$$

$$\Phi_Z = \left(W^\alpha - \frac{1}{6} X^\alpha \right)^2, \quad \Phi_W = W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \tag{5.4.12}$$

and

$$\begin{aligned} S_\chi \equiv & \left[G^2 + \mathcal{P}\bar{R} + \bar{\mathcal{P}} R - 2R\bar{R} \right]_D + \frac{1}{12} \left[X^\alpha X_\alpha \right]_F + \frac{1}{12} \left[\bar{X}_{\dot{\alpha}} \bar{X}^{\dot{\alpha}} \right]_{\bar{F}} \\ & + \frac{1}{2} \left[W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} \right]_F + \frac{1}{2} \left[\bar{W}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} \bar{W}^{\dot{\gamma}\dot{\beta}\dot{\alpha}} \right]_{\bar{F}} \end{aligned} \tag{5.4.13}$$

There is a distinction between V and X in S_χ and Φ_Z, but the former is a topological invariant independent of small variations in X and the latter is manifestly independent of the distinction, since we may rewrite

$$\Phi_Z = Z^\alpha Z_\alpha, \quad Z_\alpha \equiv \frac{1}{8} \bar{\nabla}^2 \left(Z^{I\bar{K}} \nabla_\alpha Z_{\bar{K}J} \right) \tag{5.4.14}$$

where $Z^{I\bar{J}}$ is the inverse of the kinetic matrix $Z_{\bar{J}I}$. (The Weyl curvature $W_{\alpha\beta\gamma}$ is, of course, independent of X since it is defined in conformal supergravity.)

Only the mass term Ω_P and the field strength Φ_Z are the interesting objects to investigate. We will begin by evaluating Ω_P.

5.4.1 Simplifying Ω_P

To simplify this term, it helps to introduce reparametrization connections and curvatures for the kinetic matrix Z. Observe first that

$$\begin{aligned} \bar{\nabla}^2 Z_{IJ} &= \nabla_{\dot{\alpha}} (Z_{IJ\bar{J}} \nabla^{\dot{\alpha}} \bar{\Phi}^{\bar{J}}) = \nabla_{\dot{\alpha}} (\Gamma(Z)_{IJ}{}^K Z_{K\bar{J}} \nabla^{\dot{\alpha}} \bar{\Phi}^{\bar{J}}) \\ &= R(Z)_{I\bar{J}J\bar{K}} \nabla_{\dot{\alpha}} \bar{\Phi}^{\bar{K}} \nabla^{\dot{\alpha}} \bar{\Phi}^{\bar{J}} + \Gamma(Z)_{IJ}{}^K \bar{\nabla}^2 Z_K \end{aligned}$$

where $\Gamma(Z)$ and $R(Z)$ are analogous to the Kähler connection and curvature but defined with the kinetic matrix Z instead of the Kähler potential. The connections are

$$\begin{aligned} \Gamma(Z)_{ij}{}^k &= \Gamma_{ij}{}^k - \frac{1}{3} \delta_i{}^k K_j - \frac{1}{3} \delta_j{}^k K_i \\ \Gamma(Z)_{ij}{}^0 &= \frac{\Phi_0}{3} \left(\Gamma_{ij}{}^k K_k - K_{ij} - \frac{1}{3} K_i K_j \right) \\ \Gamma(Z)_{0j}{}^k &= \Phi_0^{-1} \delta_j{}^k \\ \Gamma(Z)_{0j}{}^0 &= \Gamma(Z)_{00}{}^k = \Gamma(Z)_{00}{}^0 = 0 \end{aligned}$$

and the curvatures are

$$R(Z)_{ij}{}^{k}{}_{\bar{k}} = R_{ij}{}^{k}{}_{\bar{k}} - \frac{1}{3}\delta_i{}^k K_{j\bar{k}} - \frac{1}{3}\delta_j{}^k K_{i\bar{k}}$$
$$R(Z)_{ij}{}^{k}{}_{\bar{0}} = 0$$
$$R(Z)_{ij}{}^{0}{}_{\bar{k}} = \frac{\Phi_0}{3}\left(R_{ij}{}^{k}{}_{\bar{k}} K_k - \frac{1}{3}K_j K_{i\bar{k}} - \frac{1}{3}K_i K_{j\bar{k}}\right)$$
$$R(Z)_{ij}{}^{0}{}_{\bar{0}} = 0$$
$$R(Z)_{0J}{}^{K}{}_{\bar{L}} = 0$$

In these equations, the quanties on the left have an index structure associated with $Z_{I\bar{J}}$ (i.e. indices are raised and lowered with the kinetic matrix) while the quantities on the right have an index structure associated with the Kähler metric $K_{i\bar{j}}$.

Lowering the indices on the left using the kinetic matrix, we find that the only non-vanishing $R(Z)_{IJ\bar{J}\bar{K}}$ is

$$R(Z)_{ij\bar{j}\bar{k}} = \Phi_0\bar{\Phi}_0 e^{-K/3}\left(R_{ij\bar{j}\bar{k}} - \frac{1}{3}K_{i\bar{j}}K_{j\bar{k}} - \frac{1}{3}K_{i\bar{k}}K_{j\bar{j}}\right) \tag{5.4.15}$$

which is both reparametrization covariant and Kähler invariant. This observation dramatically simplifies calculations involving $R(Z)$.

Using the equation of motion $\mathcal{P}Z_I = -P_I$, we may rewrite

$$\mathcal{P}Z_{IJ} = -\frac{1}{4}R(Z)_{I\bar{J}J\bar{K}}\nabla_{\dot{\alpha}}\bar{\Phi}^{\bar{K}}\nabla^{\dot{\alpha}}\bar{\Phi}^{\bar{J}} - \Gamma(Z)_{IJ}{}^{K}P_K \tag{5.4.16}$$

and then rewrite the “mass term”

$$X_{IJ} = P_{IJ} + \mathcal{P}Z_{IJ} = P_{;IJ} - \frac{1}{4}R(Z)_{I\bar{J}J\bar{K}}\nabla_{\dot{\alpha}}\bar{\Phi}^{\bar{K}}\nabla^{\dot{\alpha}}\bar{\Phi}^{\bar{J}} \tag{5.4.17}$$

in a reparametrization covariant way. The notation $;I$ denotes the covariant field derivative, using the connection $\Gamma(Z)$.

We may easily calculate

$$P_{;00} = 6\Phi_0 W$$
$$P_{;0j} = 2\Phi_0^2 W_j = 2\Phi_0^2\left(W_{;j} - K_j W\right)$$
$$P_{;ij} = \Phi_0^3\left(W_{;ij} - \frac{2}{3}K_i W_{;j} - \frac{2}{3}K_j W_{;i} + \frac{2}{3}K_i K_j W\right)$$

and, raising the left index,

$$P^{\bar{0}}{}_0 = e^{K/3}\Phi_0\left(-2W + \frac{2}{3}K_{\bar{k}}W^{;\bar{k}}\right)$$
$$P^{\bar{i}}{}_0 = 2e^{K/3}\frac{\Phi_0}{\bar{\Phi}_0}W^{;\bar{i}}$$
$$P^{\bar{0}}{}_j = e^{K/3}\Phi_0^2\left(-\frac{2}{3}W_{;j} + \frac{2}{3}K_j W + \frac{1}{3}K_{\bar{k}}W^{;\bar{k}}{}_j - \frac{2}{9}K_{\bar{k}}W^{;\bar{k}}K_j\right)$$
$$P^{\bar{i}}{}_j = e^{K/3}\frac{\Phi_0^2}{\bar{\Phi}_0}\left(W^{;\bar{i}}{}_j - \frac{2}{3}K_j W^{;\bar{i}}\right)$$

The notation $;i$ on the right side of these equations denotes field differentiation covariant with respect to both Kähler transformations *and* reparametrizations. Thus,

$$W_{;i} = D_i W = W_i + K_i W \tag{5.4.18}$$

and

$$W_{;ij} = D_j W_{;i} = \partial_j W_{;i} - \Gamma^k{}_{ji} W_{;k} + K_j W_{;i} \tag{5.4.19}$$

The mass term can then be expanded as

$$\Omega_P = P_{;IJ}\bar{P}^{;JI} - \frac{1}{2}P_{;IJ}R^{IJ\alpha}{}_{\alpha} - \frac{1}{2}\bar{P}_{;\bar{I}\bar{J}}R^{\bar{I}\bar{J}}{}_{\dot{\beta}}{}^{\dot{\beta}} + \frac{1}{16}R^{IJ\alpha}{}_{\alpha}R_{IJ\dot{\beta}}{}^{\dot{\beta}}$$

The relevant quantities we will need are

$$\begin{aligned} P_{;IJ}\bar{P}^{;JI} &= e^{2K/3}\Phi_0\bar{\Phi}_0\left(4W\bar{W} - \frac{8}{3}W_{;j}\bar{W}^{;j} + W_{;ij}\bar{W}^{;ij}\right) \\ P_{;KL}R^{KL}{}_{IJ} &= e^{K/3}\frac{\Phi_0^2}{\bar{\Phi}_0}\left(W_{;k\ell}R^{k\ell}{}_{ij} - \frac{2}{3}W_{;ij}\right) \\ R^{IJ}{}_{KL}R_{IJ\bar{K}\bar{L}} &= R^{ij}{}_{k\ell}R_{ij\bar{k}\bar{\ell}} - \frac{4}{3}R_{k\ell\bar{k}\bar{\ell}} + \frac{2}{9}\left(K_{k\bar{k}}K_{\ell\bar{\ell}} + K_{k\bar{\ell}}K_{\ell\bar{k}}\right) \end{aligned}$$

In the second two formulae, the free indices with 0 or $\bar{0}$ in the slots are understood to vanish. This is due to the particular simplicity of their kinetic matrix.

The mass term can then be written

$$\begin{aligned} \Omega_P = e^{2K/3}\Phi_0\bar{\Phi}_0 &\left(4W\bar{W} - \frac{8}{3}W_{;j}\bar{W}^{;j} + W_{;ij}\bar{W}^{;ij}\right) \\ &- \frac{1}{2}e^{K/3}\frac{\Phi_0^2}{\bar{\Phi}_0}\left(W_{;k\ell}R^{k\ell}{}_{ij} - \frac{2}{3}W_{;ij}\right)\nabla^\alpha\phi^i\nabla_\alpha\phi^j + \text{h.c.} \\ &+ \frac{1}{16}R^{ij\alpha}{}_{\alpha}R_{ij\dot{\alpha}}{}^{\dot{\alpha}} - \frac{1}{12}R^{\alpha}{}_{\alpha\dot{\alpha}}{}^{\dot{\alpha}} + \frac{1}{36}K^{\dot{\alpha}\alpha}K_{\alpha\dot{\alpha}} \end{aligned} \tag{5.4.20}$$

We use here a compact notation where an α in place of an index i denotes saturation with $\nabla_\alpha\phi^i$; thus

$$K_{\alpha\dot{\alpha}} = K_{i\bar{j}}\nabla_\alpha\phi^i\nabla_{\dot{\alpha}}\bar{\phi}^{\bar{j}}, \quad R_{ij\dot{\alpha}}{}^{\dot{\alpha}} = R_{ij\bar{j}\bar{k}}\nabla_{\dot{\alpha}}\bar{\phi}^{\bar{j}}\nabla^{\dot{\alpha}}\bar{\phi}^{\bar{k}}, \quad \text{etc.} \tag{5.4.21}$$

5.4.2 Simplifying Φ_Z

Next we turn to evaluating $\Phi_Z = Z^\alpha Z_\alpha$, where

$$Z_\alpha{}^I{}_J = \mathcal{W}^I_{\alpha J} + \frac{1}{8}\bar{\nabla}^2\left(Z^{I\bar{K}}\nabla_\alpha Z_{\bar{K}J}\right) = \mathcal{W}^I_{\alpha J} + \frac{1}{8}\bar{\nabla}^2\left(\Gamma(Z)^I{}_{JK}\nabla_\alpha\Phi^K\right)$$

We evaluate each term in turn, keeping in mind that Φ_0 is assumed to be a gauge singlet:

$$Z_\alpha{}^0{}_0 = 0 \tag{5.4.22}$$

$$Z_\alpha{}^i{}_0 = \frac{\Phi_0^{-1}}{8}\bar\nabla^2\nabla_\alpha\Phi^i = \Phi_0^{-1}(\mathcal{W}_\alpha\Phi^i) \tag{5.4.23}$$

$$Z_\alpha{}^0{}_j = -\frac{\Phi_0}{24}\bar\nabla^2\left(K_{j\bar k}\nabla_\alpha(K^{\bar k k}K_k) + \frac{1}{3}K_j\nabla_\alpha K\right) \tag{5.4.24}$$

$$Z_\alpha{}^i{}_j = \mathcal{W}_\alpha{}^i{}_j - \Gamma_\alpha{}^i{}_j + \frac{1}{3}X_\alpha\delta^i{}_j - \frac{1}{24}\bar\nabla^2(K_j\nabla_\alpha\phi^i) \tag{5.4.25}$$

where we have defined the effective reparametrization gaugino field strength

$$\Gamma_\alpha{}^i{}_j \equiv -\frac{1}{8}\bar\nabla^2\left(\Gamma^i{}_{jk}\nabla_\alpha\phi^k\right) \tag{5.4.26}$$

The trace of $Z^\alpha Z_\alpha$ can be simplified by extracting $\mathcal{W}_\alpha{}^i{}_j$, $\Gamma_\alpha{}^i{}_j$, and X_α which are invariant under Kähler transformations and treating the non-invariant terms separately. One finds

$$\begin{aligned}[\mathrm{Tr}(Z^\alpha Z_\alpha)]_F = &\left[\mathrm{Tr}\left(\mathcal{W}_\alpha{}^i{}_j - \Gamma_\alpha{}^i{}_j + \frac{1}{3}X_\alpha\delta^i{}_j\right)^2 + \frac{1}{9}X^\alpha X_\alpha\right]_F \\ &+ \left[\frac{1}{72}K^{\dot\alpha\alpha}K_{\alpha\dot\alpha} - \frac{1}{24}R^\alpha{}_{\alpha\dot\alpha}{}^{\dot\alpha} - \frac{1}{6}\nabla^\alpha\mathcal{W}^r_\alpha\left(K_kX_r\phi^k - K_{\bar k}X_r\bar\phi^{\bar k}\right)\right]_D\end{aligned} \tag{5.4.27}$$

where the trace in the first line is to be understood as over the "matter" fields ϕ^i only.

For reference, we have defined

$$K_{\alpha\dot\alpha} = K_{k\bar k}\nabla_\alpha\phi^k\bar\nabla_{\dot\alpha}\bar\phi^{\bar k} \tag{5.4.28}$$

$$R^\alpha{}_{\alpha\dot\alpha}{}^{\dot\alpha} = R_{jk\bar j\bar k}\nabla^\alpha\phi^j\nabla_\alpha\phi^k\bar\nabla_{\dot\alpha}\bar\phi^{\bar j}\bar\nabla^{\dot\alpha}\bar\phi^{\bar k} \tag{5.4.29}$$

$$\Gamma_\alpha{}^i{}_j = -\frac{1}{8}\bar\nabla^2\left(K^{i\bar k}\nabla_\alpha K_{\bar k j}\right) \tag{5.4.30}$$

The appearance of the combination $\mathcal{W}_\alpha{}^i{}_j - \Gamma_\alpha{}^i{}_j$ as a field strength is gratifying. In a component calculation, we have (after applying the equations of motion for the auxiliary fields) a reparametrization connection for the component fields, and so we would expect $\Gamma_\alpha{}^i{}_j$ to appear in the final answer with the Yang-Mills connection, which it here does. Moreover, this specific combination is necessary in order to have covariance under a full gauged isometry [6].

5.4.3 Summary: Chiral loop logarithmic divergences

The logarithmic divergences of the theory can be written in the following way:

$$\Gamma \ni -\frac{\log\epsilon}{64\pi^2}\left(\left[\Phi_1 + \frac{2}{3}(N+1)\Phi_W\right]_F + \text{h.c.}\right) + \frac{\log\epsilon}{96\pi^2}(N+1)S_\chi - \frac{\log\epsilon}{32\pi^2}\left[\Omega_1+\Omega_2+\Omega_3\right]_D \tag{5.4.31}$$

where

$$\Phi_1 = \mathrm{Tr}\left(\mathcal{W}_\alpha{}^i{}_j - \Gamma_\alpha{}^i{}_j + \frac{1}{3}X_\alpha\delta^i{}_j\right)^2 + \frac{1}{9}X^\alpha X_\alpha$$

$$\Phi_W = W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} \tag{5.4.32}$$

The curvatures appearing in the trace in Φ_1 can be understood as the effective curvatures (after equations of motion are applied) for the underlying component theory. For example, $\Gamma_\alpha{}^i{}_j$ has the interpretation as the Kähler reparametrization curvature and $X_\alpha\delta^i{}_j$ is the effective $U(1)_R$ curvature.

There are additional D-terms which are more difficult to interpret:

$$\Omega_1 = e^{2K/3}\Phi_0\bar{\Phi}_0\left(4W\bar{W} - \frac{8}{3}W_{;j}\bar{W}^{;j} + W_{;ij}\bar{W}^{;ij}\right) \tag{5.4.33}$$

$$\Omega_2 = -\frac{1}{2}e^{K/3}\frac{\Phi_0^2}{\bar{\Phi}_0}\left(W_{;k\ell}R^{k\ell}{}_{ij} - \frac{2}{3}W_{;ij}\right)\nabla^\alpha\phi^i\nabla_\alpha\phi^j + \text{h.c.}$$

$$+\frac{1}{16}R^{ij\alpha}{}_\alpha R_{ij\dot{\alpha}}{}^{\dot{\alpha}} + \frac{1}{24}K^{\dot{\alpha}\alpha}K_{\alpha\dot{\alpha}} - \frac{1}{8}R^\alpha{}_{\alpha\dot{\alpha}}{}^{\dot{\alpha}} \tag{5.4.34}$$

$$\Omega_3 = -\frac{1}{6}\nabla^\alpha\mathcal{W}_\alpha^r\left(K_kX_r\phi^k - K_{\bar{k}}X_r\bar{\phi}^{\bar{k}}\right) \tag{5.4.35}$$

Although Ω_1 can be thought of as a renormalization of the Kähler potential, the others cannot since they involve derivatives of the background fields and we usually consider the Kähler potential to be derivative-free.

Finally there is a topological term

$$S_\chi = \left[G^2 + 2R\bar{R}\right]_D + \mathrm{Re}\left[W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} + \frac{1}{6}X^\alpha X_\alpha\right]_F \tag{5.4.36}$$

which is the superspace version of the Gauss-Bonnet term.

5.4.4 Chiral loop quadratic divergences

The logarithmic divergences considered previously are the physical divergences of the theory, in the sense that they are independent of the particular form of our regularization prescription. This is not true of the quadratic divergences, which for our generic model take the form

$$\Gamma = -\frac{1}{32\pi^2\epsilon}\left[\Omega_X + \Omega_V\right]_D \tag{5.4.37}$$

where

$$\Omega_X = (N+1)X, \quad \Omega_V = -2X\mathrm{Tr}V \tag{5.4.38}$$

These clearly depend on the precise choice of X, which is itself partly determined by the choice of path integration measure.

Focusing on the D-term, we note that the kinetic matrix is

$$Z_{\bar{I}J} = e^{-K/3} \begin{pmatrix} -3 & \Phi_0 K_j \\ K_{\bar{i}} \bar{\Phi}_0 & \Phi_0^\dagger \Phi_0 \left(K_{\bar{i}j} - \frac{1}{3} K_{\bar{i}} K_j\right) \end{pmatrix}$$

We haven't as yet specified the precise measure. If we take the point of view that the field Φ_0 is to be truly used as a compensator, then the simplest approach is to define the measure to include various factors of Φ_0 so that the effective path integral variables are of dimension $(3/2, 1)$. Performing such a rescaling involves taking $\eta^i \to \frac{1}{\Phi_0^{3/2}} \eta^i$ and $\eta^0 \to \frac{1}{\sqrt{\Phi_0}} \eta^0 \times \frac{1}{\sqrt{3}}$ (the additional $\sqrt{3}$ factor to normalize the kinetic term of η^0):

$$Z'_{\bar{I}J} = \frac{e^{-K/3}}{(\Phi_0 \bar{\Phi}_0)^{1/2}} \begin{pmatrix} -1 & \frac{1}{\sqrt{3}} \bar{K}_j \\ \frac{1}{\sqrt{3}} K_{\bar{i}} & \left(K_{\bar{i}j} - \frac{1}{3} K_{\bar{i}} K_j\right) \end{pmatrix}$$

where now the fields η'^i and η'^0 have the same dimension.

Unfortunately, η^0 still conspicuously has the wrong sign kinetic term. The approach advocated in [48] would involve taking $\eta^0 \to \beta \eta^0$, $\bar{\eta}^0 \to \bar{\beta} \bar{\eta}^0$ with $\beta \bar{\beta} = -1$, requiring that the naive understanding of conjugation be modified after Euclideanizing this mode. We will take this approach here, leaving β and $\bar{\beta}$ arbitrary except for the requirement that $\bar{\beta}\beta = -1$.

This leads to

$$Z'_{\bar{I}J} = \frac{e^{-K/3}}{(\Phi_0^\dagger \Phi_0)^{1/2}} \begin{pmatrix} 1 & \frac{\bar{\beta}}{\sqrt{3}} K_j \\ \frac{\beta}{\sqrt{3}} K_{\bar{i}} & \left(K_{\bar{i}j} - \frac{1}{3} K_{\bar{i}} K_j\right) \end{pmatrix} \tag{5.4.39}$$

The precise choice of β and $\bar{\beta}$ should not have an effect on the final answer.

We still must separate this kinetic matrix into conformal and gauge terms. The most physically sensible choice is to identify X as the quantity in the classical theory which is gauged to unity, that choice here being

$$X = \Phi_0 \bar{\Phi}_0 e^{-K/3} \tag{5.4.40}$$

Given that choice, the non-Yang-Mills part of V is defined by

$$e^V = e^{-K/2} \begin{pmatrix} 1 & \frac{\bar{\beta}}{\sqrt{3}} K_j \\ \frac{\beta}{\sqrt{3}} K_{\bar{i}} & \left(K_{\bar{i}j} - \frac{1}{3} K_{\bar{i}} K_j\right) \end{pmatrix} \tag{5.4.41}$$

which yields

$$\mathrm{Tr} V = \mathrm{Tr} \mathbf{V} - \frac{N+1}{2} K + \mathrm{Tr} \log K_{k\bar{k}} \tag{5.4.42}$$

where $\mathbf{V}$ is the true Yang-Mills prepotential. We have then the quadratic divergences

$$\Gamma = -\frac{1}{32\pi^2 \epsilon} \left[\Omega_X + \Omega_V\right]_D \tag{5.4.43}$$

where

$$\Omega_X = \Phi_0\bar{\Phi}_0 e^{-K/3}(N+1), \quad \Omega_V = \Phi_0\bar{\Phi}_0 e^{-K/3}\Big(-2\mathrm{Tr}\mathbf{V} + (N+1)K - 2\mathrm{Tr}\log K_{k\bar{k}}\Big) \tag{5.4.44}$$

In the gauge where $\Phi_0 = e^{K/6}$, one can easily check that in the absence of fermions for a generic (0,0) superfield V

$$[\Phi_0\bar{\Phi}_0 e^{-K/3}V]_D = -\frac{1}{3}V\mathcal{L}_{sg+m} + \frac{1}{16}\mathcal{D}^\alpha(\bar{\mathcal{D}}^2 - 8R)\mathcal{D}_\alpha V - 8\bar{R}\bar{\mathcal{D}}^2V - 8R\mathcal{D}^2V$$

where $\mathcal{L}_{sg+m}$ is the normal Lagrangian of supergravity coupled to a Kähler potential. Assuming Wess-Zumino gauge for reparametrizations, Yang-Mills, and Kähler transformations, we conclude

$$[\Omega_V]_D = -2\mathrm{Tr}\mathbf{D} - \frac{1}{2}(N+1)\mathcal{D}^\alpha X_\alpha + \mathcal{D}^\alpha\Gamma_\alpha{}^j{}_j$$

This coincides with component field calculations [1], which isn't too surprising, since our choice of X corresponds to the natural choice of a Weyl-rescaled metric at the component level.

In addition, using the superfield equations of motion and neglecting all fermions

$$-\frac{1}{3}\mathcal{L}_{sg+m} = [1]_D = [2R]_F = \Big[e^{K/2}W\Big]_F = -e^K W_{;k}\bar{W}^{;k} + 3e^K W\bar{W}$$

and so

$$[\Omega_X]_D = (N+1)\times\Big(-e^K W_{;k}\bar{W}^{;k} + 3e^K W\bar{W}\Big) = -(N+1)\hat{V}$$

This result differs from a corresponding result in [1], where Gaillard, Jain, and collaborators found $\hat{V} + M^2$, where M^2 is the gravitino mass squared, using a momentum cutoff calculation. The deviation seems likely due to a breakdown in supersymmetry due to the cutoff.[13]

5.4.5 Anomalies

There are a number of classical symmetries respected by the action (5.4.1) which are not manifestly respected by the measure.[14] These are

1. Kähler transformations

$$\Phi_0 \to e^{F/3}\Phi_0, \quad K \to K + F + \bar{F}, \quad W \to e^{-F}W \tag{5.4.45}$$

[13] A subsequent analysis with Pauli-Villars regulators[2] found a supersymmetric divergence, but the original analysis with a momentum cutoff is closer in spirit to the analysis performed here.

[14] Of these, only Yang-Mills gauge transformations are physical and thus the only one which must be anomaly-free to yield a consistent theory. However, in string-inspired supergravity theories, modular transformations in the underlying string theory manifest themselves in the effective supergravity theory as a certain combination of reparametrization and Kähler transformations. Thus it seems useful to consider the general class of symmetries described here.

2. Reparametrizations of the chiral matter

$$\Phi^i \to \Lambda^i(\Phi) \tag{5.4.46}$$

3. Yang-Mills gauge transformations

$$\Phi^i \to \exp(\Lambda^r \mathbf{T}_r)^i{}_j \Phi^j, \quad e^V \to e^{\bar{\Lambda}} e^V e^{\Lambda} \tag{5.4.47}$$

Our choice of $X = \bar{\Phi}_0 \Phi_0 e^{-K/3}$ is conspicuous in being the choice which is Kähler invariant in addition to being Yang-Mills and reparametrization invariant. This means that each of these transformations manifests itself as a *gauge* anomaly in the way we defined the effective action.

This is not the only reasonable choice. We could have chosen, for example, $X = \bar{\Phi}_0 \Phi_0$, which would correspond to a calculation in conventional (i.e. non-Kähler) Poincaré supergravity. The Kähler anomaly in such a calculation would be a purely conformal anomaly. Another choice would be to place *all* of the e^K factors into X; this would yield a combination of conformal and gauge anomalies which together give the Kähler anomaly. However, as we have shown, the difference between any of these approaches is a local (though infinite) counterterm and so there is no particular need to choose one over any other.

Since the above set of transformations may all be interpreted as gauge transformations, we can treat them in one step. Taking into account the rescalings we have made, we find the transformations

$$\delta\eta'_0 = \frac{F}{2}\eta_0 + \frac{1}{\beta\sqrt{3}} F_i \eta'^i + \mathcal{O}(\eta^2), \quad \delta\eta'^i = \frac{F}{2}\eta'^i + \Lambda^i{}_j \eta'^j + \Lambda^r \mathbf{T}_r{}^i{}_j \eta'^j + \mathcal{O}(\eta^2)$$

The kinetic matrix associated with our variable choice is

$$\frac{1}{X^{1/2}} e^V = \frac{e^{-K/2}}{X^{1/2}} \begin{pmatrix} 1 & \frac{\bar{\beta}}{\sqrt{3}} K_j \\ \frac{\beta}{\sqrt{3}} K_{\bar{i}} & \left(K_{\bar{i}j} - \frac{1}{3} K_{\bar{i}} K_j\right) \end{pmatrix} \tag{5.4.48}$$

where $X = \Phi_0 \bar{\Phi}_0 e^{-K/3}$. This choice of X is particular in being totally invariant under the combined Kähler and reparametrization symmetries. The anomaly associated with these is then simply a gauge anomaly. Taking the regulated effective action (i.e. the ϵ-divergent effective action with a simple subtraction to remove the ϵ divergences), the covariant part of the one-loop anomaly is

$$\begin{aligned} \delta_g[\Gamma]_{reg} = & -\frac{1}{16\pi^2} \left[\mathrm{Tr}(\Lambda \hat{Z}^\alpha \hat{Z}_\alpha) + \frac{2}{3} \mathrm{Tr} \Lambda W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} \right]_F + \mathrm{h.c.} \\ & + \frac{1}{48\pi^2} [\mathrm{Tr}\Lambda \Omega_\chi]_D + \text{non-covariant piece} \end{aligned} \tag{5.4.49}$$

with infinitesimal gauge parameter

$$\Lambda^I{}_J = \begin{pmatrix} -\frac{1}{2} F & \frac{\bar{\beta}}{\sqrt{3}} F_j \\ 0 & -\frac{1}{2} F \delta^i{}_j - \Lambda^i{}_j - \Lambda^r \mathbf{T}_r{}^i{}_j \end{pmatrix} \tag{5.4.50}$$

In the expression for the anomaly, we have "completed the square" for the curvature piece by introducing the local counterterm whose gauge variation includes $W^\alpha X_\alpha$. In the variables we are using, $\hat{Z}_\alpha$ has the components

$$\hat{Z}_\alpha{}^0{}_0 = 0 \tag{5.4.51}$$

$$\hat{Z}_\alpha{}^i{}_0 = \frac{\beta}{\sqrt{3}}(\mathcal{W}_\alpha \Phi^i) \tag{5.4.52}$$

$$\hat{Z}_\alpha{}^0{}_j = -\frac{\sqrt{3}}{24\beta}\bar{\nabla}^2 \left(K_{j\bar{k}} \nabla_\alpha (K^{\bar{k}k} K_k) + \frac{1}{3} K_j \nabla_\alpha K \right) \tag{5.4.53}$$

$$\hat{Z}_\alpha{}^i{}_j = \mathcal{W}_\alpha{}^i{}_j - \Gamma_\alpha{}^i{}_j + \frac{1}{3} X_\alpha \delta^i{}_j - \frac{1}{24}\bar{\nabla}^2 (K_j \nabla_\alpha \phi^i) \tag{5.4.54}$$

where $\beta\bar{\beta} = -1$. We have neglected the part of the anomaly arising from the path-dependent piece.

The covariant part of the Kähler anomaly is

$$\begin{aligned}
\delta_g[\Gamma]_{reg} \ni &+ \frac{1}{32\pi^2}\left[F\Phi_1 + \frac{2}{3}F(N+1)W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} \right]_F - \frac{1}{96\pi^2}\left[F(N+1)\Omega_\chi \right]_D \\
&+ \frac{1}{32\pi^2}\left[-\frac{1}{3}FK_{i\bar{j}}\nabla^\alpha\phi^i W_\alpha \bar{\phi}^{\bar{j}} - \frac{1}{24}FR^\alpha{}_{\alpha\dot{\alpha}}{}^{\dot{\alpha}} + \frac{1}{72}FK^{\alpha\alpha}K_{\alpha\dot{\alpha}} \right]_D \\
&+ \frac{1}{32\pi^2}\left[-\frac{1}{3}F_j(\mathcal{W}^\alpha - \Gamma^\alpha)^j{}_k \nabla_\alpha \phi^k + \frac{1}{9}\nabla^\alpha F K_k \mathcal{W}_\alpha \phi^k \right]_D + \text{h.c.}
\end{aligned} \tag{5.4.55}$$

where

$$\Phi_1 = \text{Tr}\left(\mathcal{W}_\alpha{}^i{}_j - \Gamma_\alpha{}^i{}_j + \frac{1}{3}X_\alpha \delta^i{}_j \right)^2 + \frac{1}{9}X^\alpha X_\alpha \tag{5.4.56}$$

The first two lines of this expression are quite similar to the expression for the logarithmic divergences given in (5.4.31). Φ_1 is as defined there, for example, and $FK_{i\bar{j}}\nabla^\alpha\phi^i\mathcal{W}_\alpha\bar{\phi}^{\bar{j}}$ is equivalent to that equation's Ω_3 after integrating the latter by parts. As before, the Yang-Mills curvature appears only in the reparametrization-covariant combination $\mathcal{W}_\alpha - \Gamma_\alpha$.

One expects the Kähler anomaly to encode the same information as the log divergences, up to the addition of local counterterms. We can check here that this is indeed the case. The major difference between (5.4.55) and (5.4.31) (aside from the path-dependent terms that we neglect) is the lack of a mass term Ω_P as well as the addition of the third line in (5.4.55). It turns out, however, that these amount to variations of finite counterterms. For example, the "missing" term involving Ω_P can be introduced simply by adding the finite counterterm $[K\Omega_P]_D$ with the appropriate normalization. Similarly, the third line of (5.4.55) (as well as the second!) may be removed via the addition of local counterterms involving K. The only honest Kähler anomalies (i.e. ones that cannot be cancelled by local counterterms) are the field strength terms involving Φ_1 and $W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$. The reason for this is that while these terms *can* be written as D-terms, say $F\Omega$ where Ω is an appropriate Chern-Simons superfield, the candidate counterterm $K\Omega$ is *not* gauge invariant under gauge transformations associated with Ω. For example, the Lorentz Chern-Simons term Ω_L, whose chiral projection is $W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$, transforms under a Lorentz transformation by a term which is a linear superfield, $\delta_{Lor}\Omega_L = L$, and while the integral of FL vanishes, the integral of KL does not. It seems hardly productive to trade one anomaly for another, so we will leave these terms be.

Note that we have kept the combination

$$\Omega_\chi \equiv G^2 + \bar{\mathcal{P}}R + \mathcal{P}\bar{R} - 2R\bar{R} + \frac{1}{6}\Omega_X + \Omega_L \tag{5.4.57}$$

together as a single object since its D-term integral (without an overall F factor) is topological. However, in simplifying the Kähler anomaly as much as possible, one should probably eliminate the G^2 and $\bar{\mathcal{P}}R + \mathcal{P}\bar{R}$ terms with the local counterterms KG^2 and $K\bar{\mathcal{P}}R + K\mathcal{P}\bar{R}$. In doing so, the Kähler anomaly for pure chiral loops is reduced to one purely described by F-term field strength expressions. This overlaps nicely with the calculations of Ovrut and Cardoso [40] and one may check that the coefficients of $\mathcal{W}^\alpha\mathcal{W}_\alpha$ and $W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$ agree with those results. (One must be sure to count the contributions of $W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$ from Ω_χ.) However, while those authors worked essentially to first order in K, the conformal terms we have found are inherently non-perturbative in K. Of course, the rest of the anomaly involving path-dependent non-covariant terms we have not said much about, since these in our approach are dependent strongly on the precise prescription one uses to integrate the effective action. Thus we have not checked the level of agreement between our path-dependent non-conformal terms and the corresponding non-conformal terms found in [40] since there is no particular reason for these to match.

This approach also gives the covariant form of the reparametrization and Yang-Mills anomalies, which may be collectively written

$$\begin{aligned}
\delta_g[\Gamma]_{reg} \ni &+ \frac{1}{16\pi^2}\left[\Lambda^i{}_j \Phi_1{}^j{}_i + \frac{2}{3}\Lambda^i{}_i W^{\gamma\beta\alpha}W_{\gamma\beta\alpha}\right]_F - \frac{1}{48\pi^2}\left[\Lambda^i{}_i\Omega_\chi\right]_D \\
&+ \frac{1}{16\pi^2}\left[\Lambda^i{}_j \nabla^\alpha\phi^j\left(-\frac{1}{6}K_{i\bar{j}}\mathcal{W}_\alpha\bar{\phi}^{\bar{j}} - \frac{1}{48}R_{i\alpha\dot{\alpha}}{}^{\dot{\alpha}} + \frac{1}{72}\nabla^{\dot{\alpha}}K_iK_{\alpha\dot{\alpha}}\right)\right]_D \\
&+ \frac{1}{16\pi^2}\left[-\frac{1}{18}\Lambda^i{}_j\nabla^\alpha\phi^j K_i\, K_k\mathcal{W}_\alpha\phi^k + \frac{1}{6}\Lambda^i{}_j(\mathcal{W}^\alpha - \Gamma^\alpha + \frac{1}{3}X^\alpha)^j{}_k\nabla_\alpha\phi^k K_i\right]_D \\
&+ \text{h.c.}
\end{aligned} \tag{5.4.58}$$

where $\Lambda^i{}_j$ consists of both the chiral reparametrization parameter $\Lambda^i{}_j = \partial_j\Lambda^i$ and the chiral Yang-Mills parameter $\Lambda^r\mathbf{T}_r{}^i{}_j$.

The terms involving the trace $\Lambda^i{}_i$ correspond to the chiral part of the variation of $\log\det K_{i\bar{j}} = \mathrm{Tr}\log K_{i\bar{j}}$ and were previously reported in [40] and elsewhere. The additional terms involving the general matrix $\Lambda^i{}_j$ are not dissimilar in form to those found in the Kähler anomaly, and one expects that certain of these should be local counterterms as well, but there seems no generic requirement that this should be so.

Chapter 6

Ongoing work and conclusion

We have shown how the effective action due to chiral loops may be defined in a manifestly supersymmetric way, thus enabling a calculation of the covariant part of the various anomalies in the classical theory. In principle, we have also a prescription for the calculation of the non-covariant part of the anomalies, but this is a path-dependent prescription as in the globally supersymmetric case. One critical feature that we have uncovered is the the overlap between the $U(1)$ part of supergravity and a corresponding $U(1)$ in the gauge sector. While the difference between these two is only a local counterterm in the calculation we have performed here, it undoubtedly affects details of the non-covariant part of the calculation, which we have not attempted to define precisely. A UV complete theory would undoubtedly shed light on these issues.

One possible method for UV completion is to include massive Pauli-Villars chiral superfields to regulate the divergences in a manifestly supersymmetric way. This was the point of view taken in [2], where it was shown at the component level that the divergences in general supergravity models may be regulated via the introduction of PV supermultiplets. Recently it has been shown [49] that the form of the anomalies in such theories has a structure similar to that of (5.3.122), with the anomalous Pauli-Villars masses contributing to the compensator field X defining the Gauss-Bonnet term and the $U(1)$ field strength X_α. It seems plausible that a generalization of the Green-Schwarz anomaly cancellation mechanism should be applicable here.

Having constructed the one-loop chiral contributions, one naturally turns next to the gauge and supergravity loops. The former are quite straightforward to deal with, while the latter are more troublesome. The difficulty in the gauge-fixing procedure, which leads to ghosts with additional gauge invariances and non-minimal Lagrangians, has long ago been overcome in the context of supergravity coupled to a single chiral compensator on-shell [25], where the on-shell conditions eliminate the superfields R and G_c and drastically simplify the commutators of the various derivatives. Our task is a more difficult one since for arbitrary couplings to matter and gauge fields, there is no similar simplification on-shell. However, the basic program of [25] may still be applied with some modifications, which we are currently in the process of exploring and hope to complete in the near future.

Appendix A

Solution to the Bianchi identities

A.1 General solution to gauge constraints

The constraints chosen for conformal supergravity include a set of constraints we shall call the "gauge" constraints for their similarity to the constraints imposed on internal gauge theories in superspace:

$$\{\nabla_\alpha, \nabla_\beta\} = \{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\} = 0$$
$$\{\nabla_\alpha, \nabla_{\dot\alpha}\} = -2i\nabla_{\alpha\dot\alpha}$$

where $\nabla_A \equiv E_A{}^M \left(\partial_M - h_M{}^{\underline{b}} X_{\underline{b}}\right)$ is the covariant derivative. Here $X_{\underline{b}}$ is any non-translation symmetry generator; for the conformal group it consists of scalings D, chiral rotations A, Lorentz rotations M_{ab}, and the special conformal transformations K_C. In principle, it may also include any internal symmetries (eg. Yang-Mills), but we will not be explicitly concerned with those here. Since they commute with the conformal group, it is quite easier to add these symmetries later when needed.

The gauge constraints enforce relationships between the various fermionic connections. One could attempt to solve these constraints in terms of prepotentials and then give all the connections and curvatures in terms of these prepotentials. In the case of internal symmetries, this is quite straightforward to do; one finds the prepotentials take the form of adjoint Hermitian superfields $V = V^r X_r$ where X_r is the internal symmetry generator. These in turn possess a gauge invariance of the form $V \to V + \Lambda + \bar\Lambda$ for chiral superfields Λ. When the symmetry group fails to commute with translations, this approach is more difficult (though not impossible). Moreover, in practice one is only concerned with calculating the curvatures themselves. It turns out the simpler procedure is usually to derive the constraints the curvatures obey and to solve the curvatures in terms of some unconstrained superfields. In this latter procedure, one finds chiral gaugino superfields $\mathcal{W} = \mathcal{W}^r X_r$ whose lowest components are the gauginos and which transform homogeneously under the gauge transformation. (These, of course, can be written in terms of the gauge prepotentials, but this is usually not necessary to do.) It is this latter procedure which we will follow here.

The starting point to deriving constraints on the curvatures is the Bianchi identity

$$0 = \sum_{[ABC]} [\nabla_A, [\nabla_B, \nabla_C]]$$

where the sum is over (graded) cyclic permutations of the indices. Both the permutation and the commutator carry an implicit grading which gives an extra sign whenever two fermionic indices are pushed past each other. We shall examine each case in turn, in a treatment roughly analogous to that of [7].

The case of $\alpha\beta\gamma$ is trivial. All terms in the sum vanish.

The second case is $\alpha\beta\dot{\gamma}$. The Bianchi identity reads

$$\begin{aligned} 0 &= [\nabla_\alpha, \{\nabla_\beta, \nabla_{\dot{\gamma}}\}] + [\nabla_{\dot{\gamma}}, \{\nabla_\alpha, \nabla_\beta\}] + [\nabla_\beta, \{\nabla_{\dot{\gamma}}, \nabla_\alpha\}] \\ &= -2i[\nabla_\alpha, \nabla_{\beta\dot{\gamma}}] + 0 - 2i[\nabla_\beta, \nabla_{\alpha\dot{\gamma}}] \\ &= +2iR_{\alpha(\beta\dot{\gamma})} + 2iR_{\beta(\alpha\dot{\gamma})} \end{aligned}$$

This implies the curvature is antisymmetric in the undotted indices. We therefore may define the "gaugino" superfield $\mathcal{W}$ by

$$R_{\alpha(\beta\dot{\beta})} = 2i\epsilon_{\alpha\beta}\mathcal{W}_{\dot{\beta}}, \quad R_{\dot{\alpha}(\beta\dot{\beta})} = 2i\epsilon_{\dot{\alpha}\dot{\beta}}\mathcal{W}_\beta \tag{A.1.1}$$

We have included the analogous formulae for the complex conjugate. Note that $\mathcal{W}_\beta^\dagger = -\mathcal{W}^{\dot{\beta}}$ under this definition.

The third case of interest is $\alpha\beta c$. One finds

$$\begin{aligned} 0 &= \{\nabla_\alpha, [\nabla_\beta, \nabla_c]\} + [\nabla_c, \{\nabla_\alpha, \nabla_\beta\}] - \{\nabla_\beta, [\nabla_c, \nabla_\alpha]\} \\ &= -\{\nabla_\alpha, R_{\beta c}\} + 0 - \{\nabla_\beta, R_{\alpha c}\} \end{aligned}$$

Writing R in terms of $\mathcal{W}$ and contracting with $\sigma^c_{\gamma\dot{\gamma}}$ gives

$$0 = -2i\epsilon_{\beta\gamma}\{\nabla_\alpha, \mathcal{W}_{\dot{\gamma}}\} - 2i\epsilon_{\alpha\gamma}\{\nabla_\beta, \mathcal{W}_{\dot{\gamma}}\}$$

A further contraction with $\epsilon^{\gamma\beta}$ gives

$$0 = \{\nabla_\alpha, \mathcal{W}_{\dot{\alpha}}\} = \{\nabla_{\dot{\alpha}}, \mathcal{W}_\alpha\} \tag{A.1.2}$$

where we have included the conjugate result as well. This generalizes the chirality condition of the normal Yang-Mills case, but this is not quite the conventional chirality. To wit,

$$0 = \{\nabla_\alpha, \mathcal{W}_{\dot{\alpha}}{}^B X_B\} = (\nabla_\alpha \mathcal{W}^B_{\dot{\alpha}})X_B - \mathcal{W}_{\dot{\alpha}}{}^C f_{C\alpha}{}^B X_B$$

$\mathcal{W}_{\dot{\alpha}}$ is antichiral in the conventional sense only when the second term vanishes, which is the case when the symmetry group under consideration is internal (i.e. one that commutes with translations). Nevertheless, it is useful to retain the term "chiral" to describe $\mathcal{W}_\alpha$ and "antichiral" for $\mathcal{W}_{\dot{\alpha}}$.

The fourth case of interest is $\alpha\dot{\beta}c$. We find

$$\begin{aligned} 0 &= \{\nabla_\alpha, [\nabla_{\dot{\beta}}, \nabla_c]\} + [\nabla_c, \{\nabla_\alpha, \nabla_{\dot{\beta}}\}] - \{\nabla_{\dot{\beta}}, [\nabla_c, \nabla_\alpha]\} \\ &= -\{\nabla_\alpha, R_{\dot{\beta}c}\} - 2i[\nabla_c, \nabla_{\alpha\dot{\beta}}] - \{\nabla_{\dot{\beta}}, R_{\alpha c}\} = -\{\nabla_\alpha, R_{\dot{\beta}c}\} + 2iR_{c(\alpha\dot{\beta})} - \{\nabla_{\dot{\beta}}, R_{\alpha c}\} \end{aligned}$$

which serves to define the bosonic curvature:

$$2iR_{b(\alpha\dot{\alpha})} = \{\nabla_\alpha, R_{\dot{\alpha}b}\} + \{\nabla_{\dot{\alpha}}, R_{\alpha b}\}$$

Rewriting the right-hand side in terms of $\mathcal{W}$ gives

$$R_{(\beta\dot{\beta})(\alpha\dot{\alpha})} = +\epsilon_{\dot{\alpha}\dot{\beta}}\{\nabla_\alpha, \mathcal{W}_\beta\} + \epsilon_{\alpha\beta}\{\nabla_{\dot{\alpha}}, \mathcal{W}_{\dot{\beta}}\}$$

The left-hand side is antisymmetric under interchange of the pairs $(\beta\dot{\beta})$ and $(\alpha\dot{\alpha})$ and so the right-hand side must be as well. It is easy to check that this requires the additional condition

$$\{\nabla^\alpha, \mathcal{W}_\alpha\} = \{\nabla_{\dot{\alpha}}, \mathcal{W}^{\dot{\alpha}}\} \tag{A.1.3}$$

This generalizes the analogous property for the Yang-Mills case much as the chirality condition has been generalized. Using this constraint one may rewrite the curvature in the manifestly antisymmetric form

$$R_{(\beta\dot{\beta})(\alpha\dot{\alpha})} = -\frac{1}{2}\epsilon_{\dot{\beta}\dot{\alpha}}\{\nabla_{\{\beta}, \mathcal{W}_{\alpha\}}\} - \frac{1}{2}\epsilon_{\beta\alpha}\{\nabla_{\{\dot{\beta}}, \mathcal{W}_{\dot{\alpha}\}}\} \tag{A.1.4}$$

The remaining cases to check are αbc and abc. These turn out to follow from the previous conditions on $\mathcal{W}$ (just as in the Yang-Mills case) and so we do not include them here. All other cases are conjugates of those given above, and so the constraints have been solved.

It is useful to derive how the symmetry generator $X_{\underline{d}}$ acts on $\mathcal{W}_\beta$. In order to do this, it is helpful to have a set of constraints on the structure constants consistent with the Jacobi identities. The easiest way to proceed is from the general formula (2.1.44), specializing to the cases of CB equal to $\gamma\beta$ and $\gamma\dot{\beta}$. For the first case, one finds

$$0 = \sum_{(\gamma\beta)} \left(-f_{\underline{d}\gamma}{}^F R_{F\beta} - f_{\underline{d}\gamma}{}^{\underline{f}} f_{\underline{f}\beta}{}^A X_A\right) \tag{A.1.5}$$

where $R_{F\beta} = R_{F\beta}{}^A X_A$ where X_A in this and the above formula consists of both the translations P_A and the non-translation symmetries $X_{\underline{a}}$. For the second case, one finds

$$0 = 2if_{\underline{c}(\beta\dot{\beta})}{}^A X_A - f_{\underline{c}\beta}{}^D R_{D\dot{\beta}} - f_{\underline{c}\dot{\beta}}{}^D R_{D\beta} - f_{\underline{c}\beta}{}^{\underline{d}} f_{\underline{d}\dot{\beta}}{}^A X_A - f_{\underline{c}\dot{\beta}}{}^{\underline{d}} f_{\underline{d}\beta}{}^A X_A \tag{A.1.6}$$

(We have relabelled $\underline{d}$ to $\underline{c}$ and γ to β since β and $\dot{\beta}$ naturally go together to form a vector index.)

One set of additional constraints is also useful. For any theory in superspace, we would like to be able to write down chiral integrals; the existence of these implies the structure constant constraints (2.1.84)

$$f_{\underline{a}\beta}{}^c = f_{\underline{a}\beta\dot{\gamma}} = 0, \quad f_{\underline{a}\beta}{}^{\underline{c}}\left(f_{\underline{c}d}{}^d + f_{\underline{c}\dot{\delta}}{}^{\dot{\delta}}\right) = 0$$

as well as their complex conjugates

$$f_{\underline{a}\dot{\beta}}{}^c = f_{\underline{a}\dot{\beta}}{}^\gamma = 0, \quad f_{\underline{a}\dot{\beta}}{}^{\underline{c}}\left(f_{\underline{c}d}{}^d - f_{\underline{c}\delta}{}^\delta\right) = 0$$

Applying these constraints to (A.1.5) gives

$$0 = \sum_{(\gamma\beta)} \left(-f_{\underline{d}\gamma}{}^{\nu} R_{\nu\beta} - f_{\underline{d}\gamma}{}^{\underline{f}} f_{\underline{f}\beta}{}^{A} X_A \right) = -\sum_{(\gamma\beta)} f_{\underline{d}\gamma}{}^{\underline{f}} f_{\underline{f}\beta}{}^{A} X_A \tag{A.1.7}$$

which is a further constraint on the structure constants. Note that this constraint is equivalent to

$$f_{\underline{d}\gamma}{}^{\underline{f}} f_{\underline{f}\beta}{}^{A} X_A = \frac{1}{2}\epsilon_{\gamma\beta} f_{\underline{d}}{}^{\phi\underline{f}} f_{\underline{f}\phi}{}^{A} X_A \tag{A.1.8}$$

Applying the constraints to (A.1.6) gives $f_{\underline{c}b}{}^{A}$ in terms of $f_{\underline{c}\beta}{}^{A}$ and $f_{\underline{c}\dot{\beta}}{}^{A}$:

$$f_{\underline{c}(\beta\dot{\beta})(\alpha\dot{\alpha})} = 2\epsilon_{\dot{\beta}\dot{\alpha}} f_{\underline{c}\beta\alpha} - 2\epsilon_{\beta\alpha} f_{\underline{c}\dot{\beta}\dot{\alpha}} \tag{A.1.9}$$

$$f_{\underline{c}(\beta\dot{\beta})}{}^{\alpha} = -\frac{i}{2} f_{\underline{c}\dot{\beta}}{}^{\underline{d}} f_{\underline{d}\beta}{}^{\alpha} \tag{A.1.10}$$

$$f_{\underline{c}(\beta\dot{\beta})}{}^{\dot{\alpha}} = -\frac{i}{2} f_{\underline{c}\beta}{}^{\underline{d}} f_{\underline{d}\dot{\beta}}{}^{\dot{\alpha}} \tag{A.1.11}$$

$$f_{\underline{c}(\beta\dot{\beta})}{}^{\underline{a}} = -\frac{i}{2} f_{\underline{c}\beta}{}^{\underline{d}} f_{\underline{d}\dot{\beta}}{}^{\underline{a}} - \frac{i}{2} f_{\underline{c}\dot{\beta}}{}^{\underline{d}} f_{\underline{d}\beta}{}^{\underline{a}} \tag{A.1.12}$$

We are now in a position to derive the general gauge transformation property of $\mathcal{W}_{\dot{\beta}}$. To proceed, first note that in principle $R_{\gamma(\beta\dot{\beta})} = f_{\gamma(\beta\dot{\beta})}{}^{A} X_A + \Delta R_{\gamma(\beta\dot{\beta})}$ where the first term on the right is a structure constant in the global theory and the second term is the local correction. (In practice, the first term usually vanishes.) It follows that a similar decomposition of $\mathcal{W}$ takes place, giving $\mathcal{W}_{\dot{\beta}}{}^{A} = f_{\dot{\beta}}{}^{A} + \Delta\mathcal{W}_{\dot{\beta}}{}^{A}$. Since the first term is a structure constant, it necessarily is gauge invariant; we therefore need only calculate the gauge transformation of the local correction. Using equation (2.1.45), for the case of $CB = \gamma b$ gives

$$2i\epsilon_{\gamma\beta} X_{\underline{d}} \mathcal{W}_{\dot{\beta}}{}^{A} = -2i\epsilon_{\gamma\beta} \Delta\mathcal{W}_{\dot{\beta}}{}^{F} f_{F\underline{d}}{}^{A} + 2i f_{\underline{d}\gamma\beta} \Delta\mathcal{W}_{\dot{\beta}}{}^{A} - i f_{\underline{d}(\beta\dot{\beta})(\gamma\dot{\gamma})} \Delta\mathcal{W}^{\dot{\gamma}A} \tag{A.1.13}$$

Using (A.1.9) allows one to show the right-hand size is proportional to $\epsilon_{\gamma\beta}$. The final result is

$$X_{\underline{d}} \mathcal{W}_{\dot{\beta}}{}^{A} = -\Delta\mathcal{W}_{\dot{\beta}}{}^{F} f_{F\underline{d}}{}^{A} - f_{\underline{d}\phi}{}^{\phi} \Delta\mathcal{W}_{\dot{\beta}}{}^{A} - f_{\underline{d}\dot{\beta}\dot{\gamma}} \Delta\mathcal{W}^{\dot{\gamma}A}$$

The first term on the right hand size can be combined with the left-hand side to yield the compact formula

$$[X_{\underline{d}}, \Delta\mathcal{W}_{\dot{\beta}}] = -f_{\underline{d}\phi}{}^{\phi} \Delta\mathcal{W}_{\dot{\beta}} - f_{\underline{d}\dot{\beta}\dot{\gamma}} \Delta\mathcal{W}^{\dot{\gamma}} \tag{A.1.14}$$

The complex conjugate is

$$[X_{\underline{d}}, \Delta\mathcal{W}_{\beta}] = +f_{\underline{d}\dot{\phi}}{}^{\dot{\phi}} \Delta\mathcal{W}_{\beta} - f_{\underline{d}\beta}{}^{\gamma} \Delta\mathcal{W}_{\gamma} \tag{A.1.15}$$

We include the precise definition of the covariant derivative of the local gaugino superfields for completeness:

$$\nabla_C \Delta\mathcal{W}_{\beta}{}^{A} = E_C{}^{M} \partial_M \Delta\mathcal{W}_{\beta}{}^{A} + h_C{}^{\underline{d}} \left(\Delta\mathcal{W}_{\beta}{}^{F} f_{F\underline{d}}{}^{A} - f_{\underline{d}\dot{\phi}}{}^{\dot{\phi}} \Delta\mathcal{W}_{\beta}{}^{A} + f_{\underline{d}\beta}{}^{\gamma} \Delta\mathcal{W}_{\gamma}{}^{A} \right) \tag{A.1.16}$$

$$\nabla_C \Delta\mathcal{W}_{\dot{\beta}}{}^{A} = E_C{}^{M} \partial_M \Delta\mathcal{W}_{\dot{\beta}}{}^{A} + h_C{}^{\underline{d}} \left(\Delta\mathcal{W}_{\dot{\beta}}{}^{F} f_{F\underline{d}}{}^{A} + f_{\underline{d}\phi}{}^{\phi} \Delta\mathcal{W}_{\dot{\beta}}{}^{A} + f_{\underline{d}\dot{\beta}\dot{\gamma}} \Delta\mathcal{W}^{\dot{\gamma}A} \right) \tag{A.1.17}$$

(The covariant derivative of the constant part of $\mathcal{W}$ vanishes.)

A.2 Conformal supergravity solution

From the result of the previous section, we may define *maximal* conformal supergravity as the theory with the Yang-Mills constraints

$$\{\nabla_\alpha, \nabla_\beta\} = \{\nabla_{\dot\alpha}, \nabla_{\dot\beta}\} = 0$$
$$\{\nabla_\alpha, \nabla_{\dot\alpha}\} = -2i\nabla_{\alpha\dot\alpha}.$$

It follows that the remaining curvatures are of the form

$$\begin{aligned}R_{\alpha(\beta\dot\beta)} &= 2i\epsilon_{\alpha\beta}\mathcal{W}_{\dot\beta}\\ R_{\dot\alpha(\beta\dot\beta)} &= 2i\epsilon_{\dot\alpha\dot\beta}\mathcal{W}_{\beta}\\ R_{(\beta\dot\beta)(\alpha\dot\alpha)} &= -\frac{1}{2}\epsilon_{\dot\beta\dot\alpha}\{\nabla_{\{\beta}, \mathcal{W}_{\alpha\}}\} - \frac{1}{2}\epsilon_{\beta\alpha}\{\nabla_{\{\dot\beta}, \mathcal{W}_{\dot\alpha\}}\}\end{aligned}$$

where the superfields $\mathcal{W}$ obey the constraints

$$\{\nabla_{\dot\alpha}, \mathcal{W}_\alpha\} = \{\nabla_\alpha, \mathcal{W}_{\dot\alpha}\} = 0$$
$$\{\nabla^\alpha, \mathcal{W}_\alpha\} = \{\nabla_{\dot\alpha}, \mathcal{W}^{\dot\alpha}\}$$

The $\mathcal{W}$ here is understood as

$$\mathcal{W}_\alpha = \mathcal{W}(P)_\alpha{}^B P_B + \frac{1}{2}\mathcal{W}(M)_\alpha{}^{cb}M_{bc} + \mathcal{W}(D)_\alpha D + \mathcal{W}(A)_\alpha A + \mathcal{W}(K)_\alpha{}^B K_B$$

That is, there is a $\mathcal{W}$ associated with each symmetry in the conformal group. These $\mathcal{W}$ are not conformally primary but are rotated into each other by the action of the conformal group. In this case, the global theory is characterized by $\mathcal{W} = 0$ and so no decomposition of $\mathcal{W}$ into global and local parts is necessary.

The chirality condition $\{\nabla_{\dot\alpha}, \mathcal{W}_\alpha\} = 0$ reads

$$\begin{aligned}0 &= \nabla_{\dot\alpha}\mathcal{W}(P)_\alpha{}^B\nabla_B - \mathcal{W}(P)_\alpha{}^C T_{C\dot\alpha}{}^B\nabla_B + \mathcal{W}(M)_{\alpha\dot\alpha\dot\beta}\nabla^{\dot\beta} + \frac{1}{2}\mathcal{W}(D)_\alpha\nabla_{\dot\alpha} + i\mathcal{W}(A)_\alpha\nabla_{\dot\alpha}\\ 0 &= \frac{1}{2}\nabla_{\dot\alpha}\mathcal{W}(M)_\alpha{}^{cb}M_{bc} - \frac{1}{2}\mathcal{W}(P)_\alpha{}^D R_{D\dot\alpha}{}^{cb}M_{bc} - 2\mathcal{W}(K)_{\alpha\dot\beta}M^{\dot\beta}{}_{\dot\alpha}\\ 0 &= \nabla_{\dot\alpha}\mathcal{W}(K)_\alpha{}^B K_B - \mathcal{W}(P)_\alpha{}^C R(K)_{C\dot\alpha}{}^B K_B + i\mathcal{W}(K)_{\alpha(\dot\alpha}{}^{\beta)}S_\beta\\ 0 &= \nabla_{\dot\alpha}\mathcal{W}(D)_\alpha - \mathcal{W}(P)_\alpha{}^B R(D)_{B\dot\alpha} - 2\mathcal{W}(K)_{\alpha\dot\alpha}\\ 0 &= \nabla_{\dot\alpha}\mathcal{W}(A)_\alpha - \mathcal{W}(P)_\alpha{}^B R(A)_{B\dot\alpha} - 3i\mathcal{W}(K)_{\alpha\dot\alpha}\end{aligned} \tag{A.2.1}$$

For the last two equations we have omitted the generators D and A respectively. The curvatures in these expressions are defined in terms of $\mathcal{W}$; therefore, these formulae possess both linear and quadratic terms in $\mathcal{W}$.

The condition $\{\nabla^\alpha, \mathcal{W}_\alpha\} = \{\nabla_{\dot\alpha}, \mathcal{W}^{\dot\alpha}\}$ reads

$$\begin{aligned}&\nabla^\alpha\mathcal{W}(P)_\alpha{}^B\nabla_B + \mathcal{W}(P)^{\alpha C}T_{C\alpha}{}^B\nabla_B - \mathcal{W}(M)^\alpha{}_\alpha{}^\beta\nabla_\beta - \frac{1}{2}\mathcal{W}(D)^\alpha\nabla_\alpha + i\mathcal{W}(A)^\alpha\nabla_\alpha\\ &= \nabla_{\dot\alpha}\mathcal{W}(P)^{\alpha B}\nabla_B + \mathcal{W}(P)_{\dot\alpha}{}^C T_C{}^{\dot\alpha B}\nabla_B - \mathcal{W}(M)_{\dot\alpha}{}^{\dot\alpha}{}_{\dot\beta}\nabla^{\dot\beta} - \frac{1}{2}\mathcal{W}(D)_{\dot\alpha}\nabla^{\dot\alpha} - i\mathcal{W}(A)_{\dot\alpha}\nabla^{\dot\alpha}\end{aligned} \tag{A.2.2}$$

$$\frac{1}{2}\nabla^{\alpha}\mathcal{W}(M)_{\alpha}{}^{cb}M_{bc} + \frac{1}{2}\mathcal{W}(P)^{\alpha D}R_{D\alpha}{}^{cb}M_{bc} + 2\mathcal{W}(K)^{\alpha\beta}M_{\beta\alpha}$$
$$= \frac{1}{2}\nabla_{\dot\alpha}\mathcal{W}(M)^{\dot\alpha cb}M_{bc} + \frac{1}{2}\mathcal{W}(P)_{\dot\alpha}{}^{D}R_{D}{}^{\dot\alpha cb}M_{bc} + 2\mathcal{W}(K)_{\dot\alpha\dot\beta}M^{\dot\beta\dot\alpha} \tag{A.2.3}$$

$$\nabla^{\alpha}\mathcal{W}(K)_{\alpha}{}^{B}K_{B} + \mathcal{W}(P)^{\alpha C}R(K)_{C\alpha}{}^{B}K_{B} - i\mathcal{W}(K)^{\dot\alpha}{}_{\dot\alpha}{}^{\beta}S_{\beta}$$
$$= \nabla_{\dot\alpha}\mathcal{W}(K)^{\dot\alpha B}K_{B} + \mathcal{W}(P)_{\dot\alpha}{}^{C}R(K)_{C}{}^{\dot\alpha B}K_{B} - i\mathcal{W}(K)_{\dot\alpha}{}^{(\dot\alpha\beta)}S_{\beta} \tag{A.2.4}$$

$$\nabla^{\alpha}\mathcal{W}(D)_{\alpha} + \mathcal{W}(P)^{\alpha B}R(D)_{B\alpha} + 2\mathcal{W}(K)^{\alpha}{}_{\alpha} = \nabla_{\dot\alpha}\mathcal{W}(D)^{\dot\alpha} + \mathcal{W}(P)_{\dot\alpha}{}^{B}R(D)_{B}{}^{\dot\alpha} + 2\mathcal{W}(K)_{\dot\alpha}{}^{\dot\alpha} \tag{A.2.5}$$

$$\nabla^{\alpha}\mathcal{W}(A)_{\alpha} + \mathcal{W}(P)^{\alpha B}R(A)_{B\alpha} - 3i\mathcal{W}(K)^{\alpha}{}_{\alpha} = \nabla_{\dot\alpha}\mathcal{W}(A)^{\dot\alpha} + \mathcal{W}(P)_{\dot\alpha}{}^{B}R(A)_{B}{}^{\dot\alpha} + 3i\mathcal{W}(K)_{\dot\alpha}{}^{\dot\alpha} \tag{A.2.6}$$

This is a very complicated structure that simplifies a great deal when we apply the further constraints of conformal supergravity. These are $F_{\alpha b} = 0$, $H_{\alpha b} = 0$, and $T_{\gamma b}{}^{A} = 0$ along with their complex conjugates. (In addition, we want $T_{cb}{}^{a} = 0$ but this turns out to be a consequence of the other constraints.) These constraints clearly force $\mathcal{W}(A)_{\alpha}$, $\mathcal{W}(D)_{\alpha}$, and $\mathcal{W}(P)_{\alpha}{}^{B}$ to zero. Since this set of constraints is conformally invariant (i.e. $S_{\gamma}\mathcal{W}(D)_{\beta} = +2\mathcal{W}(P)_{\beta\gamma} = 0$), it follows that the covariant derivative of any of these also vanishes.

The only non-vanishing gaugino superfields are then $\mathcal{W}(M)$ and $\mathcal{W}(K)$. In terms of these, the chirality constraints (A.2.1) read

$$\begin{aligned}
0 &= \mathcal{W}(M)_{\alpha\dot\alpha\dot\beta}\nabla^{\dot\beta} \\
0 &= \frac{1}{2}\nabla_{\dot\alpha}\mathcal{W}(M)_{\alpha}{}^{cb}M_{bc} - 2\mathcal{W}(K)_{\alpha\dot\beta}M^{\dot\beta}{}_{\dot\alpha} \\
0 &= \nabla_{\dot\alpha}\mathcal{W}(K)_{\alpha}{}^{B}K_{B} + i\mathcal{W}(K)_{\alpha(\dot\alpha}{}^{\beta)}S_{\beta} \\
0 &= -2\mathcal{W}(K)_{\alpha\dot\alpha} \\
0 &= -3i\mathcal{W}(K)_{\alpha\dot\alpha}
\end{aligned}$$

It follows that $\mathcal{W}(M)_{\alpha\dot\beta\dot\gamma}$ and $\mathcal{W}(K)_{\alpha\dot\alpha}$ vanish. Furthermore, $\mathcal{W}(M)_{\alpha\beta\gamma}$ is chiral and $\nabla_{\dot\alpha}\mathcal{W}(K)_{\alpha}{}^{\beta} = -i\mathcal{W}(K)_{\alpha\dot\alpha}{}^{\beta}$.

Considering the remaining constraints, we have (A.2.2)

$$-\mathcal{W}(M)^{\alpha}{}_{\alpha}{}^{\beta}\nabla_{\beta} = -\mathcal{W}(M)_{\dot\alpha}{}^{\dot\alpha}{}_{\dot\beta}\nabla^{\dot\beta}$$

This implies that $\mathcal{W}(M)^{\alpha}{}_{\alpha\gamma} = 0$. Therefore, $\mathcal{W}(M)_{\alpha\beta\gamma}$ is totally symmetric in its indices. Similarly for the conjugate.

Next is (A.2.3)

$$\frac{1}{2}\nabla^{\alpha}\mathcal{W}(M)_{\alpha}{}^{cb}M_{bc} + 2\mathcal{W}(K)^{\alpha\beta}M_{\beta\alpha} = \frac{1}{2}\nabla_{\dot\alpha}\mathcal{W}(M)^{\dot\alpha cb}M_{bc} + 2\mathcal{W}(K)_{\dot\alpha\dot\beta}M^{\dot\beta\dot\alpha}$$

which implies that $\mathcal{W}(K)_{\beta\gamma} = -\frac{1}{4}\nabla^\alpha \mathcal{W}(M)_{\alpha\beta\gamma}$ (as well as its conjugate). Since we already know that $\mathcal{W}(K)_{\beta(\alpha\dot\alpha)} = i\nabla_{\dot\alpha}\mathcal{W}(K)_{\beta\alpha}$, it follows that $W(K)_{\beta(\alpha\dot\alpha)} = -\frac{1}{2}\nabla^\phi{}_{\dot\alpha}\mathcal{W}(M)_{\phi\beta\alpha}$.

Equation (A.2.4) implies

$$\nabla^\alpha \mathcal{W}(K)_\alpha{}^B K_B - i\mathcal{W}(K)^{\dot\alpha}{}_{\dot\alpha}{}^\beta S_\beta = \nabla_{\dot\alpha}\mathcal{W}(K)^{\dot\alpha B}K_B - i\mathcal{W}(K)_{\dot\alpha}{}^{(\dot\alpha\beta)}S_\beta$$

which, when we insert our existing formulae, gives a new identity

$$\nabla^\beta\nabla^\phi{}_{\dot\alpha}\mathcal{W}(M)_{\phi\beta\alpha} = \nabla^{\dot\beta}\nabla^{\dot\phi}{}_\alpha \mathcal{W}(M)_{\dot\phi\dot\beta\dot\alpha}$$

Finally, we note that the final two constraints (A.2.5) and (A.2.6) give

$$+2\mathcal{W}(K)^\alpha{}_\alpha = 2\mathcal{W}(K)_{\dot\alpha}{}^{\dot\alpha}$$

and

$$-3i\mathcal{W}(K)^\alpha{}_\alpha = +3i\mathcal{W}(K)_{\dot\alpha}{}^{\dot\alpha},$$

which are satisfied trivially. (Both sides vanish.)

All of the curvatures are then specified in terms of a single totally symmetric chiral superfield $\mathcal{W}(M)_{\alpha\beta\gamma}$ as well as its conjugate, which together obey a Bianchi identity. Furthermore, from the transformation rules of the $\mathcal{W}$ found in the previous section, $\mathcal{W}(M)_{\alpha\beta\gamma}$ is conformally primary of scale dimension 3/2 and $U(1)$ weight $+1$. To make contact with the conventional normalizations and reality conditions, we define a new superfield $W_{\alpha\beta\gamma}$ via $\mathcal{W}(M)_{\alpha\beta\gamma} = -2W_{\alpha\beta\gamma}$ and $\mathcal{W}(M)_{\dot\alpha\dot\beta\dot\gamma} = +2W_{\alpha\beta\gamma}$ and summarize our results in terms of it:

$$\begin{gathered}
\mathcal{W}(P)_\alpha{}^B = \mathcal{W}(P)_{\dot\alpha}{}^B = 0 \\
\mathcal{W}(D)_\alpha = \mathcal{W}(D)_{\dot\alpha} = 0 \\
\mathcal{W}(A)_\alpha = \mathcal{W}(A)_{\dot\alpha} = 0 \\
\mathcal{W}(M)_{\alpha\dot\beta\dot\gamma} = \mathcal{W}(M)_{\dot\alpha\beta\gamma} = 0 \\
\mathcal{W}(M)_{\alpha\beta\gamma} = -2W_{\alpha\beta\gamma}, \quad \mathcal{W}(M)_{\dot\alpha\dot\beta\dot\gamma} = +2W_{\dot\alpha\dot\beta\dot\gamma} \\
\mathcal{W}(K)_{\alpha\beta} = \frac{1}{2}\nabla^\phi W_{\phi\alpha\beta}, \quad \mathcal{W}(K)_{\dot\alpha\dot\beta} = \frac{1}{2}\nabla^{\dot\phi}W_{\dot\phi\dot\alpha\dot\beta} \\
\mathcal{W}(K)_{\alpha\dot\beta} = \mathcal{W}(K)_{\dot\alpha\beta} = 0 \\
\mathcal{W}(K)_{\alpha(\beta\dot\beta)} = \nabla^\phi{}_{\dot\beta}W_{\phi\alpha\beta}, \quad \mathcal{W}(K)_{\dot\alpha(\beta\dot\beta)} = \nabla^{\dot\phi}{}_\beta W_{\dot\phi\dot\alpha\dot\beta}
\end{gathered}$$

$W_{\alpha\beta\gamma}$ is a totally symmetric chiral primary superfield obeying a Bianchi identity

$$\nabla^\beta\nabla^\phi{}_{\dot\alpha}W_{\phi\beta\alpha} = -\nabla^{\dot\beta}\nabla^{\dot\phi}{}_\alpha W_{\dot\phi\dot\beta\dot\alpha}$$

From the above definitions of $\mathcal{W}$ and of the curvatures R in terms of $\mathcal{W}$, one can quite easily derive the curvatures in terms of W. These are given fully in Section 2.2.6.

Appendix B

Global superconformal transformations

In the literature on the conformal group, the generators on the fields in the global approach are given at an arbitrary point x. For example, D is defined as $\Delta + x \cdot \partial$. (See for example [19].) For completeness, we present the global superconformal generators in the same global picture.

The action of a generator g on a field Φ may be defined at the origin. One takes the defining relations for a primary superfield Φ as

$$\begin{gathered} P_a\Phi(0) = \partial_a\Phi(0), \quad Q_\alpha\Phi(0) = D_\alpha\Phi(0), \quad \bar{Q}^{\dot\alpha}\Phi(0) = \bar{D}^{\dot\alpha}\Phi(0) \\ M_{ab}\Phi(0) = \mathcal{S}_{ab}\Phi(0), \quad D\Phi(0) = \Delta\Phi(0), \quad A\Phi(0) = iw\Phi(0) \\ K_a\Phi(0) = 0, \quad S_\alpha\Phi(0) = 0, \quad \bar{S}^{\dot\alpha}\Phi(0) = 0 \end{gathered}$$

The action of the supersymmetry translation generators Q_α at the origin are formally defined to be the same as D_α. This is certainly allowed by the discussion in Wess and Bagger since both are equivalent to ∂_α there; however, it will soon be apparent that the *intrinsic* action of Q_α on a field anywhere is to be found by the action of D_α.

In order to find the action of g elsewhere, conjugation by the translation operator is necessary. That is, in order to calculate $g\Phi(z)$, one must commute g past the translation element, $g\Phi(z) = ge^{zP}\Phi(0) = e^{zP}\tilde{g}(z)\Phi(0)$ where $\tilde{g}(z) \equiv e^{-zP}ge^{zP}$, and the elements in the expansion of g' are to be taken to act on Φ at the origin. One may calculate the effect of conjugation by the translation element on each of the generators:

$$
\begin{aligned}
\tilde{P}_a(z) =& P_a \\
\tilde{Q}_\alpha(z) =& Q_\alpha - 2i(\sigma^a\bar{\theta})_\alpha P_a \\
\tilde{\bar{Q}}^{\dot{\alpha}}(z) =& \bar{Q}^{\dot{\alpha}} - 2i(\bar{\sigma}^a\theta)^{\dot{\alpha}} P_a \\
\tilde{D}(z) =& D + x^a P_a + \frac{1}{2}\theta Q + \frac{1}{2}\bar{\theta}\bar{Q} \\
\tilde{A}(z) =& A - i\theta Q + i\bar{\theta}\bar{Q} - 2(\theta\sigma^a\bar{\theta})P_a \\
\tilde{M}_{ab}(z) =& M_{ab} - x_{[a}P_{b]} + (\theta\sigma_{ab}Q) + (\bar{\theta}\bar{\sigma}_{ab}\bar{Q}) + P_c\epsilon_{abcd}(\theta\sigma_d\bar{\theta}) \\
\tilde{K}_a(z) =& K_a + 2x_a D - 2x_b M_{ab} + i(\theta\sigma_a\bar{S}) + i(\bar{\theta}\bar{\sigma}_a S) + 2x_a x_b P_b - x^2 P_a \\
& + x_a(\theta Q) - 2x_b(\theta\sigma_{ab}Q) + x_a(\bar{\theta}\bar{Q}) - 2x_b(\bar{\theta}\bar{\sigma}_{ab}\bar{Q}) + 3\zeta_a A + \epsilon_{abcd}\zeta_b M_{cd} \\
& - 2\epsilon_{abcd}\zeta_b x_c P_d - 2i\zeta_a(\theta Q) + 2i\zeta_a(\bar{\theta}\bar{Q}) - 2\zeta^a P_a \\
\tilde{S}_\alpha(z) =& S_\alpha + ix_a(\sigma_a\bar{Q})_\alpha - 2\theta_\alpha D + 3i\theta_\alpha A + 2(\sigma^{ba}\theta)_\alpha M_{ab} \\
& - 2\theta_\alpha x^a P_a + 4(\sigma^{ab}\theta)_\alpha x_a P_b - 2\theta^2 Q_\alpha - 2\theta_\alpha(\bar{\theta}\bar{Q}) + 2i\theta^2(\sigma^a\bar{\theta})_\alpha P_a \\
\tilde{\bar{S}}^{\dot{\alpha}}(z) =& \bar{S}^{\dot{\alpha}} + ix_a(\bar{\sigma}_a Q)^{\dot{\alpha}} - 2\bar{\theta}^{\dot{\alpha}} D - 3i\bar{\theta}^{\dot{\alpha}} A + 2(\bar{\sigma}^{ba}\bar{\theta})^{\dot{\alpha}} M_{ab} \\
& - 2\bar{\theta}^{\dot{\alpha}} x^a P_a + 4(\bar{\sigma}^{ab}\bar{\theta})^{\dot{\alpha}} x_a P_b - 2\bar{\theta}^2\bar{Q}^{\dot{\alpha}} - 2\bar{\theta}^{\dot{\alpha}}(\theta Q) + 2i\bar{\theta}^2(\bar{\sigma}^a\theta)^{\dot{\alpha}} P_a
\end{aligned}
$$

where $\zeta^a \equiv \theta\sigma^a\bar{\theta}$.

The first set of definitions imply

$$
\begin{gathered}
P_a\Phi(z) = \partial_a\Phi(z) \\
Q_\alpha\Phi(z) = D_\alpha\Phi(z) - 2i(\sigma^a\bar{\theta})_\alpha\partial_a\Phi(z), \quad \bar{Q}^{\dot{\alpha}}\Phi(z) = D^{\dot{\alpha}}\Phi(z) - 2i(\bar{\sigma}^a\theta)^{\dot{\alpha}}\partial_a\Phi(z)
\end{gathered}
$$

which is consistent with the standard definitions in the literature [7].

Appendix C

A brief note on implicit grading

We make use of the convention of [7] with respect to superspace indices and their contractions. Furthermore, we adopt an implicit grading scheme to avoid cumbersome notation. In any formula involving capital Roman superindices (A, B, C,...), an order is set by the uncontracted indices of the first term; all other terms, if not in the order given, must be supplemented with a grading to flip the indices to the appropriate order. In addition, all index contractions are to be done high to low between adjacent indices; any other configuration of indices must be swapped into this configuration.

A few examples help a good deal. First the commutator:

$$[\nabla_B, \nabla_A] = -R_{BA}$$

Explanding this out gives

$$\nabla_B \nabla_A - \nabla_A \nabla_B = -R_{BA}$$

The first term sets the order to be B then A; the second term has the wrong order and so a grading must be inserted. The final result is

$$\nabla_B \nabla_A - (-)^{AB} \nabla_A \nabla_B = -R_{BA}$$

The commutator is really an anticommutator if both A and B are fermionic.

Next, a more involved example:

$$V_C{}^B \nabla_B W_A + V_C{}^B \nabla_A W_B = F_{AB}{}^{BD} G_{CD}$$

The first term sets the order: C then A. The B contraction is properly done, so no grading is necessary for the first term. The second term has C and A in the correct order, but the B contraction is done through the A. One must swap the A with either B to get an adjacent contraction, giving a grading $(-)^{AB}$. The third term on the right side has the B contraction done in the wrong order. This requires we place a grading of $(-)^B$. In addition, the D contraction is done through the index C, giving a grading of $(-)^{CD}$. Finally, the overall order of indices is A then C; swapping them to the correct order gives a grading $(-)^{AC}$. The final result with the gradings restored is

$$V_C{}^B \nabla_B W_A + (-)^{AB} V_C{}^B \nabla_A W_B = (-)^{B+CD+AC} F_{AB}{}^{BD} G_{CD}.$$

Now suppose G were a two-form. Then the form indices CD can be swapped at the cost of a sign if they are not both fermionic; this gives

$$V_C{}^B \nabla_B W_A + (-)^{AB} V_C{}^B \nabla_A W_B = -(-)^{B+AC} F_{AB}{}^{BD} G_{DC}.$$

We would have compactly written this without the gradings as

$$V_C{}^B \nabla_B W_A + V_C{}^B \nabla_A W_B = -F_{AB}{}^{BD} G_{DC}.$$

which is equal to the first equation, provided we take $G_{DC} = -G_{CD}$ which is true modulo the grading.

The advantage of this notational method is that in any calculation involving superindices, one may naively treat them as if they were all regular bosonic indices. Then when one wishes to actually insert the components, the gradings can be added on the fly subject to the rules we have given.

Appendix D

Calculation of the two-point chiral heat kernel

A common expression that we've come across is

$$Z(\omega_2, \omega_1; \tau_+, \tau_-) = \int E' \int E \ \mathrm{Tr}\left(\omega_2(z')U_-(z', z, \tau_-)\omega_1(z)U_+(z, z', \tau_+)\right)$$

which is a functional of two local superfields ω_1 and ω_2 and a function of two heat kernel parameters τ_+ and τ_-. We are interested in a small τ_+ and τ_- local expansion. Without loss of generality, we can define $\tau_+ \equiv \epsilon\lambda$ and $\tau_- \equiv \epsilon\tilde{\lambda}$ with $\lambda + \tilde{\lambda} = 1$. Then ϵ is taken to be our small parameter.

The first step is to use the symmetry of H_- to swap the coordinates of U_- so that z is the leading coordinate in both bi-scalars. Due to (5.3.45), this induces a change in the representation of $W^{\dot{\alpha}}$ within U_-. Then one could choose to work in a normal coordinate system for z about z'. The difficulty in doing the calculation this way is that U_- involves an exponential in $\bar{\Sigma}$ and U_+ in Σ, but $\bar{\Sigma}$ and Σ are only both $y^2/2$ when in their respective antichiral and chiral gauges. However, in performing the z integration we can certainly choose to do it in a conventional way by doing the Grassmann integrations, reducing the expression to one in terms of y with η and $\bar{\eta}$ vanishing. In the case of vanishing η and $\bar{\eta}$ gauge it is not hard to see that both Σ and $\bar{\Sigma}$ reduce to $y^2/2$. We will show this in due course.

We perform the Grassmann integrations in a covariant way, using

$$\int E\,\Omega = -\frac{1}{4}\int \bar{\mathcal{E}}\,(\mathcal{D}^2 - 8R)\Omega = \int d^4y\, e\, \left(\bar{\mathbf{f}} + i\psi_a\sigma^a\bar{\mathbf{s}} - \psi_a\sigma^{ab}\psi_b\bar{\mathbf{r}}\right)$$

where $\bar{\mathbf{f}}$, $\bar{\mathbf{s}}$ and $\bar{\mathbf{r}}$ are defined in terms of Ω as

$$\bar{\mathbf{r}} = -\frac{1}{4}(\mathcal{D}^2 - 8\bar{R})\Omega, \quad \bar{\mathbf{s}}^{\dot{\alpha}} = -\frac{1}{8}\mathcal{D}^{\dot{\alpha}}(\mathcal{D}^2 - 8\bar{R})\Omega, \quad \bar{\mathbf{f}} = +\frac{1}{16}(\bar{\mathcal{D}}^2 - 24R)(\mathcal{D}^2 - 8\bar{R})\Omega$$

We have elected to evaluate the D-term integral via an $\bar{F}$-term. This will give the same result as using an intermediate F-term up to a total derivative.

The quantity Ω has two leading prefactors of the form

$$P_+ = \frac{1}{(4\pi\epsilon\lambda)^2}\Delta^{1/2}\exp\left(-\frac{\Sigma}{2\epsilon\lambda}\right) \quad \text{and} \quad \bar{P}_- = \frac{1}{(4\pi\epsilon\tilde{\lambda})^2}\bar{\Delta}^{1/2}\exp\left(-\frac{\bar{\Sigma}}{2\epsilon\tilde{\lambda}}\right)$$

and it may be written as

$$\Omega = P_+\bar{P}_- \times \omega_2\bar{F}_-\omega_1 F_+$$

$\bar{\mathbf{f}}$, $\bar{\mathbf{s}}$ and $\mathbf{r}$ will also have these prefactors, so we extract the common term P_-P_+, defining the superfield T by

$$P_-P_+T \equiv \left(\bar{\mathbf{f}} + i\psi_a\sigma^a\bar{\mathbf{s}} - \psi_a\sigma^{ab}\psi_b\bar{\mathbf{r}}\right)$$

Having performed the Grassmann integrations, the remaining y integration can be done in any coordinate system of our choosing subject to the constraint that $\eta = \bar{\eta} = 0$. We will take as our coordinate system the normal coordinate system defined by expanding any function of y in a Taylor series, using

$$F(y) = F + y^a\mathcal{D}_aF + \frac{1}{2}y^ay^b\mathcal{D}_a\mathcal{D}_bF + \ldots$$

Recall that in chiral gauge Σ obeys $[\mathcal{D}_a\mathcal{D}_b\Sigma] = \eta_{ab}$ with any number of other purely bosonic (symmetrized) derivatives vanishing, it follows that in this normal coordinate system $\Sigma = y^2/2$ as well. Similarly for $\bar{\Sigma}$. This simplifies the exponential part of the prefactors, leading to the integration

$$\frac{1}{(4\pi)^4}\frac{1}{\epsilon^4\lambda^2\tilde{\lambda}^2}\int d^4y\,\exp\left(-\frac{y^2}{4\epsilon\lambda\tilde{\lambda}}\right)\Gamma(y)\,T(y)$$

where $\Gamma(y) \equiv \Delta^{1/2}(y)\bar{\Delta}^{1/2}(y)e(y)$.

The Gaussian integration is simple, keeping in mind we want only the diverging terms in ϵ:

$$\frac{1}{16\pi^2\epsilon^2}\left([\Gamma T] + \epsilon\lambda\tilde{\lambda}[\mathcal{D}^a\mathcal{D}_a(\Gamma T)]\right) + \mathcal{O}(1)$$

Recall that $\Delta = \det(E_{\mathcal{A}}{}^{\mathcal{M}}) = \det(E_a{}^m)/\det(E_\alpha{}^\mu - E_\alpha{}^m E_m{}^a E_a{}^\mu)$, giving

$$\begin{aligned}\Gamma &= \exp(-\frac{1}{2}\mathrm{Tr}\log\det(E_\alpha{}^\mu - E_\alpha{}^m E_m{}^a E_a{}^\mu) + \text{h.c.})\\ &= \exp(y^2R\bar{R} + \mathcal{O}(y^3))\end{aligned}$$

This simplifies the expression we seek to

$$\frac{1}{16\pi^2\epsilon^2}\left([T] + 8\epsilon\lambda\tilde{\lambda}R\bar{R}[T] + \epsilon\lambda\tilde{\lambda}[\mathcal{D}^a\mathcal{D}_aT]\right) + \mathcal{O}(1)$$

The task remains to determine $[T]$ and $[\mathcal{D}^a\mathcal{D}_aT]$, which will both depend on ϵ, λ, and $\tilde{\lambda}$. We begin with the expansion for $[T]$, which we will need to first order in ϵ. In deriving $[T]$, a number of terms will appear. They will involve U_+ and U_- with at most two derivatives. By cleverly ordering the derivatives, it will be possible to write $[T]$ in terms of $[U_+]$, $[\mathcal{D}_\alpha U_+]$, $[\bar{\mathcal{P}} U_+]$, $[\mathcal{D}_\alpha \mathcal{D}_b U_+]$, $[\mathcal{P}\bar{\mathcal{P}} U_+ = dU_+/d\tau_+]$ and also in terms of $[U_-]$, $[\mathcal{D}^{\dot\alpha} U_-]$, and $[\mathcal{P} U_-]$. But only certain combinations of these terms will contribute. Using $[A_1] = -2R$, $[\mathcal{D}_\alpha A_1] = -\mathcal{D}_\alpha R + 2W_\alpha$, and $[\mathcal{D}^2 A_1] = 2\mathcal{D}^\alpha W_\alpha + \frac{1}{3}\mathcal{D}^\alpha X_\alpha - 8R\bar{R}$ as well as $[\mathcal{D}_\alpha \log\Delta] = 0$ and $[\mathcal{D}^2 \log\Delta] = 8\bar{R}$,

$$\begin{aligned}
[U_+] &= P_+\left([F]\right) = P_+\left(-2\epsilon\lambda R + \mathcal{O}(\epsilon^2)\right) \\
[\mathcal{D}_\alpha U_+] &= P_+\left([\mathcal{D}_\alpha F] + \ldots\right) = P_+\left(-\epsilon\lambda \mathcal{D}_\alpha R + 2\epsilon\lambda W_\alpha + \mathcal{O}(\epsilon^2)\right) \\
[\bar{\mathcal{P}} U_+] &= P_+\left([\bar{\mathcal{P}} F] - \frac{1}{8}[\mathcal{D}^2 \log\Delta\ F] + \ldots\right) = P_+\left(1 - \frac{\epsilon\lambda}{2}\mathcal{D}^\alpha W_\alpha - \frac{\epsilon\lambda}{12}\mathcal{D}^\alpha X_\alpha + \mathcal{O}(\epsilon^2)\right) \\
[\mathcal{D}_b U_+] &= P_+\left([\mathcal{D}_b F] + [\mathcal{D}_b \log\Delta\ F]\right) = P_+\left(0 + \mathcal{O}(\epsilon)\right) \\
[\mathcal{D}_\alpha \mathcal{D}_b U_+] &= P_+\left([\mathcal{D}_\alpha \mathcal{D}_b F] + \frac{1}{2}[\mathcal{D}_\alpha \mathcal{D}_b \log\Delta F] + \frac{1}{2}[\mathcal{D}_b \log\Delta \mathcal{D}_\alpha F] + \ldots\right) = P_+\left(0 + \mathcal{O}(\epsilon)\right) \\
[dU_+/d\tau_+] &= P_+\left(-\frac{2}{\tau_+}[F] + \frac{d[F]}{d\tau_+}\right) = P_+\left(2R + \mathcal{O}(\epsilon)\right)
\end{aligned}$$

The last three terms we have expanded only to first order in ϵ as that is all we will need. We also require

$$\begin{aligned}
[U_-] &= P_-\left(-2\epsilon\tilde{\lambda}\bar{R} + \mathcal{O}(\epsilon^2)\right) \\
[\mathcal{D}^{\dot\alpha} U_-] &= P_-\left(-\epsilon\tilde{\lambda}\bar{\mathcal{D}}^{\dot\alpha}\bar{R} + 2\epsilon\tilde{\lambda} W^{\dot\alpha} + \mathcal{O}(\epsilon^2)\right) \\
[\mathcal{P} U_-] &= P_-\left(1 - \frac{\epsilon\tilde{\lambda}}{2}\bar{\mathcal{D}}_{\dot\alpha} W^{\dot\alpha} - \frac{\epsilon\tilde{\lambda}}{12}\bar{\mathcal{D}}_{\dot\alpha} X^{\dot\alpha} + \mathcal{O}(\epsilon^2)\right)
\end{aligned}$$

Note that the terms involving $W^{\dot\alpha}$ in derivatives of U_- have the same sign as the corresponding terms involving W_α in derivatives of U_+. The reason for this is that U_- naturally is conjugate to U_+ and so the formulae involving the operators W_α would normally be replaced by their conjugates $-W^{\dot\alpha}$ (since the operator W_α is formally anti-Hermitian in our convention). However, in swapping the coordinates of U_- we have conjugated a second time, yielding $+W^{\dot\alpha}$.

In expanding out $[T]$, we note that $[\psi] = 0$ and so we need only calculate

$$P_+ P_- T = \omega_2 \times \frac{1}{16}(\bar{\mathcal{D}}^2 - 24R)(\mathcal{D}^2 - 8R)\left(U_- \omega_1 U_+\right)$$

Using the above rules and working to linear order in ϵ one finds

$$\begin{aligned}
[T] =& \omega_2\omega_1 + \epsilon\lambda\omega_2\left(\frac{1}{2}\mathcal{D}^2\omega_1 R + \frac{1}{2}\mathcal{D}^\alpha\omega_1\mathcal{D}_\alpha R - \mathcal{D}^\alpha\omega_1 W_\alpha - \frac{1}{2}\omega_1\mathcal{D}^\alpha W_\alpha\right) \\
&+ \epsilon\tilde{\lambda}\omega_2\left(+\frac{1}{2}\bar{\mathcal{D}}^2\omega_1\bar{R} + \frac{1}{2}\bar{\mathcal{D}}_{\dot\alpha}\omega_1\bar{\mathcal{D}}^{\dot\alpha}\bar{R} - W_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\omega_1 - \frac{1}{2}\bar{\mathcal{D}}_{\dot\alpha} W^{\dot\alpha}\omega_1\right) \\
&- \frac{\epsilon}{12}\omega_2\omega_1\mathcal{D}^\alpha X_\alpha - 8\epsilon\tilde{\lambda}R\bar{R}\omega_2\omega_1
\end{aligned}$$

Next we must work out $[\mathcal{D}^a\mathcal{D}_a T]$ to zeroth order in ϵ. This is more difficult than it first appears since $\mathcal{D}^2\Sigma/2\epsilon\lambda$ survives under two bosonic derivatives and thus decrements the overall ϵ order of the expression. However, since it multiplies $F = \epsilon A_1 + \ldots$, the inverse ϵ is immediately used up. More pernicious is the term $dU_+/d\tau_+$, which gives $\Sigma/2\epsilon^2\lambda^2$. Thankfully $dU_+/d\tau_+$ multiplies only U_- and so only U_- need be written to linear order in ϵ.

The terms which we will need then are

$$\begin{aligned}
\frac{U_+}{P_+} &\sim 0 + \mathcal{O}(\epsilon) \\
\frac{\mathcal{D}_\alpha U_+}{P_+} &\sim 0 + \mathcal{O}(\epsilon) \\
\frac{\bar{\mathcal{P}} U_+}{P_+} &\sim -\frac{1}{4}\mathcal{D}^2 A_0 + \frac{1}{8}\mathcal{D}^2\Sigma\, A_1 + \mathcal{O}(\epsilon) \\
\frac{\mathcal{D}_b U_+}{P_+} &\sim -\frac{1}{2}\mathcal{D}_b\Sigma\, A_1 + \mathcal{O}(\epsilon) \\
\frac{\mathcal{D}_\alpha\mathcal{D}_b U_+}{P_+} &\sim -\frac{1}{2}\mathcal{D}_b\Sigma\,\mathcal{D}_\alpha A_1 - \frac{1}{2}\mathcal{D}_\alpha\mathcal{D}_b\Sigma\, A_1 - \frac{1}{4}\mathcal{D}_b\Sigma\,\mathcal{D}_\alpha \log\Delta\, A_1 + \mathcal{O}(\epsilon) \\
\frac{1}{P_+}\frac{dU_+}{d\tau_+} &\sim -A_1 + \frac{\Sigma}{2\epsilon\lambda}A_1 + \frac{\Sigma}{4}A_2 + \mathcal{O}(\epsilon)
\end{aligned}$$

as well as

$$\begin{aligned}
\frac{U_-}{P_-} &\sim \epsilon\tilde{\lambda}\bar{A}_1 + \mathcal{O}(\epsilon^2) \\
\frac{\bar{\mathcal{D}}^{\dot{\alpha}} U_-}{P_-} &\sim 0 + \mathcal{O}(\epsilon) \\
\frac{\mathcal{P} U_-}{P_-} &\sim -\frac{1}{4}\bar{\mathcal{D}}^2 \bar{A}_0 + \frac{1}{8}\bar{\mathcal{D}}^2\bar{\Sigma}\, \bar{A}_1 + \mathcal{O}(\epsilon)
\end{aligned}$$

The terms generated by $\mathbf{r}$ are easy to dispense with since the two bosonic derivatives must be expanded on the ψ terms and the remaining terms generated involving U_+ and U_- have insufficient derivatives. Similarly, $\mathbf{s}$ will also fail to contribute anything. As before, the only relevant terms come from $\mathbf{f}$, with

$$\mathcal{D}^a\mathcal{D}_a T \sim \omega_2 \mathcal{D}^a\mathcal{D}_a \left(\frac{1}{P_+P_-}\frac{1}{16}(\bar{\mathcal{D}}^2 - 24R)(\mathcal{D}^2 - 8R)\,(U_-\omega_1 U_+)\right)$$

and only two terms from this expression can contribute:

$$\mathcal{D}^a\mathcal{D}_a T \sim \omega_2 \mathcal{D}^a\mathcal{D}_a \frac{1}{P_+P_-}\left(\mathcal{P}U_-\omega_1\bar{\mathcal{P}}U_+ + U_-\omega_1\frac{dU_+}{d\tau_+}\right)$$

Using

$$\begin{gathered}
[\mathcal{D}^a\mathcal{D}_a\Sigma] = 4, \quad [\mathcal{D}^a\mathcal{D}_a\mathcal{D}^2\Sigma] = -32\bar{R} \\
[\mathcal{D}^2 A_0] = -4, \quad [\mathcal{D}_a\mathcal{D}^2 A_0] = -8iG_a, \quad [\mathcal{D}^a\mathcal{D}_a\mathcal{D}^2 A_0] = -8i\mathcal{D}^a G_a + 16G^2 + 32R\bar{R}
\end{gathered}$$

we find a large number of cancellations yielding

$$\mathcal{D}^a\mathcal{D}_aT \sim \omega_2\left(\mathcal{D}^a\mathcal{D}_a\omega_1 + 8\frac{\tilde{\lambda}}{\lambda}R\bar{R}\omega_1\right)$$

Putting everything together, we find

$$\frac{1}{16\pi^2\epsilon^2}\Bigg\{\omega_2\omega_1 + \epsilon\lambda\omega_2\left(\frac{1}{2}\mathcal{D}^2\omega_1 R + \frac{1}{2}\mathcal{D}^\alpha\omega_1\mathcal{D}_\alpha R - \mathcal{D}^\alpha\omega_1 W_\alpha - \frac{1}{2}\omega_1\mathcal{D}^\alpha W_\alpha\right)$$
$$+ \epsilon\tilde{\lambda}\omega_2\left(+\frac{1}{2}\bar{\mathcal{D}}^2\omega_1\bar{R} + \frac{1}{2}\bar{\mathcal{D}}_{\dot{\alpha}}\omega_1\bar{\mathcal{D}}^{\dot{\alpha}}\bar{R} - W_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\omega_1 - \frac{1}{2}\bar{\mathcal{D}}_{\dot{\alpha}}W^{\dot{\alpha}}\omega_1\right)$$
$$- \frac{\epsilon}{12}\omega_2\omega_1\mathcal{D}^\alpha X_\alpha + \epsilon\lambda\tilde{\lambda}\omega_2\mathcal{D}^a\mathcal{D}_a\omega_1\Bigg\}$$

which after integrating by parts gives our final expression

$$Z = \frac{1}{16\pi^2\epsilon^2}\int E\,\mathrm{Tr}\Bigg\{\omega_2\omega_1 - \frac{\epsilon\lambda}{2}R\mathcal{D}^\alpha\omega_2\mathcal{D}_\alpha\omega_1 - \frac{\epsilon\tilde{\lambda}}{2}\bar{R}\bar{\mathcal{D}}_{\dot{\alpha}}\omega_2\bar{\mathcal{D}}^{\dot{\alpha}}\omega_1$$
$$- \frac{\epsilon}{12}\mathcal{D}^\alpha X_\alpha\omega_2\omega_1 - \epsilon\lambda\tilde{\lambda}\mathcal{D}^a\omega_2\mathcal{D}_a\omega_1$$
$$+ \frac{\epsilon\lambda}{2}(\mathcal{D}^\alpha\omega_2\omega_1 W_\alpha - \omega_2\mathcal{D}^\alpha\omega_1 W_\alpha)$$
$$+ \frac{\epsilon\tilde{\lambda}}{2}(\bar{\mathcal{D}}_{\dot{\alpha}}\omega_2 W^{\dot{\alpha}}\omega_1 - \omega_2 W_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\omega_1) + \mathcal{O}(\epsilon^2)\Bigg\}$$

where we have relabelled z' to z.

We note that the coefficients of these terms can be checked in several ways. The case of constant ω_2 and ω_1 is easy enough to rearrange into a trace over a single chiral or antichiral heat kernel. For $\lambda = 0$ or $\tilde{\lambda} = 0$ one can similarly evaluate the resulting expression immediately. The only cases not covered by either of these is the term $\mathcal{D}^a\omega_2\mathcal{D}_a\omega_1$; but this expression can be checked in the case of global supersymmetry where the calculation is quite easier.

Appendix E

Arbitrary linear and chiral superfield models at first order

We have expanded the actions for arbitrary chiral models to second order in the quantum fields to enable quantization. The structure they possess is fairly interesting and is reflected in the minimal model of a linear compensator coupled to supergravity and a Kähler potential. We will briefly consider the generalization to an *arbitrary* coupling of a linear superfield L to chiral multiplets Φ^i in the context of conformal supergravity. Although we will assume only a single linear superfield L, the generalization to several is straightforward.

The interesting part will be contained in the D-term action

$$S_D = -3 \int E\, \mathcal{F}(L, \Phi^i, \bar{\Phi}^{\bar{j}})$$

The -3 is chosen so that if $\mathcal{F}$ is independent of L, a canonical Einstein-Hilbert term is reproduced for the choice $\mathcal{F} = 1$. Observing that

$$\begin{aligned} D\mathcal{F} = 2\mathcal{F} &= \mathcal{F}_i \Delta_i \Phi^i + \mathcal{F}_{\bar{j}} \Delta_{\bar{j}} \bar{\Phi}^{\bar{j}} + 2\mathcal{F}_L L \\ -\frac{3i}{2} A\mathcal{F} = 0 &= \mathcal{F}_i \Delta_i \Phi^i - \mathcal{F}_{\bar{j}} \Delta_{\bar{j}} \bar{\Phi}^{\bar{j}} \end{aligned}$$

and that the Einstein-Hilbert term is contained within

$$S_D \ni -\frac{3}{2} \mathcal{F}_i \Box \Phi^i - \frac{3}{2} \mathcal{F}_{\bar{j}} \Box \bar{\Phi}^{\bar{j}}$$

where $\Box$ are superconformal (and thus contain $\mathcal{R}/6$ weighted by the scaling dimension of the field on which $\Box$ acts), it is easy to see that the normalization of the Einstein-Hilbert term is

$$X = \frac{1}{2} \mathcal{F}_i \Delta_i \Phi_i + \frac{1}{2} \mathcal{F}_{\bar{j}} \Delta_{\bar{j}} \bar{\Phi}^{\bar{j}} = \mathcal{F} - L\mathcal{F}_L$$

It is clear that the field multiplying the Einstein-Hilbert term is the proper conformal compensator to use for our theory, so we have chosen to label the above combination as X.

Expanding S_D to first order in quantum fields using the tools we have developed is straightforward. One finds

$$\begin{aligned}\delta S_D =&3i\nabla^\alpha(V_\alpha \mathcal{F}) - 3i\nabla_{\dot\alpha}(V^{\dot\alpha}\mathcal{F}) + 3\Delta_b(V^b L)\mathcal{F}_L + (\Delta_b V^b)(\mathcal{F} - L\mathcal{F}_L)\\ &+ 3iV^b\left(\mathcal{F}_i\nabla_b\Phi^i - \mathcal{F}_{\bar j}\nabla_b\bar\Phi^{\bar j}\right) + 3i\Sigma^r\left(\mathcal{F}_i X_r\Phi^i - \mathcal{F}_{\bar j}X_r\bar\Phi^{\bar j}\right) - 3\mathcal{F}_L\mathcal{L}\end{aligned}$$

where Δ_b is conformally covariant, as are all the other derivatives. Integrating by parts (and taking care that the special conformal connections vanish) gives

$$\begin{aligned}\delta S_D =& + 3V^b L\Delta_b\mathcal{F}_L + V^b\Delta_b(\mathcal{F} - L\mathcal{F}_L) + 3iV^b\left(\mathcal{F}_i\nabla_b\Phi^i - \mathcal{F}_{\bar j}\nabla_b\bar\Phi^{\bar j}\right)\\ &+ 3i\Sigma^r\left(\mathcal{F}_i X_r\Phi^i - \mathcal{F}_{\bar j}X_r\bar\Phi^{\bar j}\right) - 3\mathcal{F}_L\mathcal{L}\end{aligned}$$

Using

$$\begin{aligned}V^b\Delta_b\mathcal{F} =&V^b\mathcal{F}_L\Delta_b L - iV^b\mathcal{F}_i\nabla_b\Phi^i + iV^b\mathcal{F}_{\bar j}\nabla_b\bar\Phi^{\bar j}\\ &+ \frac{1}{2}\mathcal{F}_{I\bar J}\nabla_\alpha\Psi^I\nabla_{\dot\alpha}\Psi^{\bar J}\end{aligned}$$

where Ψ^I denotes the set (Φ^i, L) and $\Psi^{\bar J}$ the set $(\bar\Phi^{\bar j}, L)$, we can write the variation as

$$\begin{aligned}\delta S_D =& - 2V^b\Delta_b(\mathcal{F} - L\mathcal{F}_L) + \frac{3}{2}V^{\dot\alpha\alpha}\mathcal{F}_{i\bar j}\nabla_\alpha\Phi^i\nabla_{\dot\alpha}\bar\Phi^{\bar j} - \frac{3}{2}V^{\dot\alpha\alpha}\mathcal{F}_{LL}\nabla_\alpha L\nabla_{\dot\alpha}L\\ &+ 3i\Sigma^r\left(\mathcal{F}_i X_r\Phi^i - \mathcal{F}_{\bar j}X_r\bar\Phi^{\bar j}\right) - 3\mathcal{F}_L\mathcal{L}\end{aligned}$$

This form is immediately reminiscent of that we have discussed before. Since $X \equiv \mathcal{F} - L\mathcal{F}_L$ is to be identified as the compensator, we define $G_a \equiv -X^{1/2}\Delta_a X^{-1/2}$ as before. This immediately yields

$$\begin{aligned}\delta S_D =& - 4XV^b G_b + \frac{3}{2}V^{\dot\alpha\alpha}\mathcal{F}_{i\bar j}\nabla_\alpha\Phi^i\nabla_{\dot\alpha}\bar\Phi^{\bar j} - \frac{3}{2}V^{\dot\alpha\alpha}\mathcal{F}_{LL}\nabla_\alpha L\nabla_{\dot\alpha}L - \frac{3}{2}V^{\dot\alpha\alpha}X^{-1}\nabla_\alpha X\nabla_{\dot\alpha}X\\ &+ 3i\Sigma^r\left(\mathcal{F}_i X_r\Phi^i - \mathcal{F}_{\bar j}X_r\bar\Phi^{\bar j}\right) - 3\mathcal{F}_L\mathcal{L}\end{aligned}$$

To maintain the analogy, we should make the identifications

$$\begin{aligned}K_{\alpha\dot\alpha} &\equiv -3X^{-1}\mathcal{F}_{i\bar j}\nabla_\alpha\Phi^i\nabla_{\dot\alpha}\bar\Phi^{\bar j} + 3X^{-1}\mathcal{F}_{LL}\nabla_\alpha L\nabla_{\dot\alpha}L + 3X^{-2}\nabla_\alpha X\nabla_{\dot\alpha}X\\ K_r &\equiv +3iX^{-1}\left(\mathcal{F}_i X_r\Phi^i - \mathcal{F}_{\bar j}X_r\bar\Phi^{\bar j}\right)\end{aligned}$$

which would give

$$\delta S_D = - 4XV^b G_b + XV^b K_b + X\Sigma^r K_r - 3\mathcal{F}_L\mathcal{L}$$

We would like to think of terms involving V^a to consist of a "supergravity term" G_b and the "matter term" K_b, so it is sensible to expand K_b out entirely in terms of the fields. We find

$$\begin{aligned}K_{\alpha\dot\alpha} =& - 3\nabla_\alpha\Phi^i\nabla_{\dot\alpha}\bar\Phi^{\bar j}\left(X^{-1}\mathcal{F}_{i\bar j} - X^{-2}X_i X_{\bar j}\right)\\ &+ 3\nabla_\alpha\Phi^i\nabla_{\dot\alpha}L\left(X^{-2}X_i X_L\right) + 3\nabla_\alpha L\nabla_{\dot\alpha}\Phi^{\bar j}\left(X^{-2}X_L X_{\bar j}\right)\\ &+ 3\nabla_\alpha L\nabla_{\dot\alpha}L\left(X^{-1}\mathcal{F}_{LL} + X^{-2}X_L X_L\right)\end{aligned}$$

where $X_i = \mathcal{F}_i - L\mathcal{F}_{Li}$, $X_{\bar{j}} = \mathcal{F}_{\bar{j}} - L\mathcal{F}_{L\bar{j}}$ and $X_L = -L\mathcal{F}_{LL}$.

Before moving on, we should make one more generalization. Up until now we have assumed L to be a normal linear multiplet. However, we may instead choose for L to obey the modified linearity condition

$$\mathcal{P}L = -\frac{1}{4}\bar{\nabla}^2 L = -\frac{1}{2}k\mathrm{Tr}(W^\alpha W_\alpha)$$

This amounts to choosing $L = L_0 + k\Omega$, where L_0 is a normal linear superfield and Ω is the Chern-Simons superfield [6]. L is chosen to be gauge invariant, so the gauge transformation of Ω, which is itself a linear superfield, must be cancelled by the transformation of L_0.

The Yang-Mills term then receives contributions from the D-term of $\mathcal{F}$:

$$-3\int E\,\mathcal{F} = \frac{3k}{4}\int \mathcal{E}\left(\mathcal{F}_L\mathrm{Tr}(W^\alpha W_\alpha) + \ldots\right) + \mathrm{h.c.}$$

This contributes to f_{rs} (effectively) a non-holomorphic factor of $3k\delta_{rs}\mathcal{F}_L$ and thus to G_{rs} a factor of $6k\delta_{rs}\mathcal{F}_L$.

The quanta of L which we previously denoted $\mathcal{L}$ should now be understood as

$$\mathcal{L} = \mathcal{L}_0 - ik\nabla^\alpha(W_\alpha\Sigma) - ik\nabla_{\dot{\alpha}}(W^\alpha\Sigma) + ik(\nabla^\alpha W_\alpha)\Sigma - kV^{\dot{\alpha}\alpha}W_\alpha\bar{W}_{\dot{\alpha}}$$

where $\mathcal{L}_0$ is linear. This formula is determined by requiring the chiral quantum variation of both sides of the modified linearity condition to coincide.

One can easily check that

$$\begin{aligned}-3\mathcal{L}\mathcal{F}_L &= -3\mathcal{F}_L\mathcal{L}_0 + 3ik\mathcal{F}_L\nabla^\alpha(W_\alpha\Sigma) + 3ik\mathcal{F}_L\nabla_{\dot{\alpha}}(W^\alpha\Sigma) - 3ik\mathcal{F}_L(\nabla^\alpha W_\alpha)\Sigma \\ &\quad + 3k\mathcal{F}_L V^{\dot{\alpha}\alpha}W_\alpha\bar{W}_{\dot{\alpha}} \\ &= -3\mathcal{F}_L\mathcal{L}_0 + \Sigma^r\mathcal{Y}_r + V^b\mathcal{Y}_b\end{aligned}$$

where

$$\begin{aligned}\mathcal{Y}_r &= -3ik(\nabla^\alpha\mathcal{F}_L)W_{\alpha r} - 3ik(\nabla^{\dot{\alpha}}\mathcal{F}_L)\bar{W}^{\dot{\alpha}}{}_r - 3ik\mathcal{F}_L(\nabla^\alpha W_\alpha)_r \\ \mathcal{Y}_{\alpha\dot{\alpha}} &= -6\mathcal{F}_L k\mathrm{Tr}(W_\alpha\bar{W}_{\dot{\alpha}})\end{aligned}$$

This agrees with the previous definition for these objects provided we rewrite them solely in terms of $G_{rs} = f_{rs} + \bar{f}_{rs}$ Then taking into account the contribution from the linear multiplet gives $G'_{rs} = f_{rs} + \bar{f}_{rs} + 6\mathcal{F}_L k\delta_{rs}$.

The first order structure can then be written

$$\delta S_D = V^b\left(-4XG_b + XK_b + \mathcal{Y}_b\right) + \Sigma^r\left(XK_r + \mathcal{Y}_r\right) - 3\mathcal{F}_L\mathcal{L} - 3\mathcal{F}_i\eta^i - 3\mathcal{F}_{\bar{j}}\bar{\eta}^{\bar{j}}$$

where we have included also the chiral superfield variations.

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