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Teachers' Beliefs Regarding the Generalization of Students' Learning and
How to Support the Generalization of Students' Learning

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

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2013

The Dissertation of Jaime Marie Diamond is approved and it is acceptable in quality and

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Chair

University of California, San Diego

San Diego State University

2013

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DEDICATION

This dissertation is dedicated to all of those who strive to be better teachers, learners, and human beings.

To Dr. Joanne Lobato whose mentoring has been invaluable: Your hard work in and dedication to the field of mathematics education is truly inspiring.

To my friends and family whose support and encouragement helped make this journey possible: You are the best kind of cheerleaders!

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ABSTRACT OF THE DISSERTATION

Teachers' Beliefs Regarding the Generalization of Students' Learning and
How to Support the Generalization of Students' Learning

by

Jaime Marie Diamond

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2013
San Diego State University, 2013

Joanne Lobato, Chair

The idea that learning generalizes beyond the conditions of initial learning serves as a basis for our educational system (National Research Council, 2000). That is, educators hope students will use the learning that is generated in the classroom to productively reason about situations they have yet to encounter. One body of research that has examined the generalization of learning is the research on transfer. Traditionally, transfer has been characterized as "how knowledge acquired from one task or situation can be applied to a different one" (Nokes, 2009, p. 2). Transfer has had a rich and varied history wherein researchers have detailed numerous conceptualizations of transfer as well as corresponding ideas about how to best support it. Surprisingly, an extensive search of the literature did not yield any studies aimed to uncover teachers' beliefs about transfer.

Consequently, my research addresses the following question: What beliefs do teachers have about the generalization of students' learning and how to support it?

To answer this two-part question, I asked eight practicing teachers to engage in two 2-hour clinical interviews (Ginsburg, 1997). Qualitative analysis of such data led to the identification of 5 categories (fitting into 3 supercategories) of teachers' beliefs regarding the generalization of learning and 13 categories of teachers' beliefs regarding how to support the generalization of learning. Analysis of data subsequently collected in teachers' classrooms resulted in the elaboration of beliefs identified via interview data and to the identification of beliefs not captured by interview data. The beliefs identified in this study highlight the importance of bringing teachers into the ongoing conversation about transfer.

The mathematical topic of slope provided the content domain in which data on teachers' beliefs were collected. Analysis of the relationship between teachers' beliefs and their mathematical knowledge for teaching (MKT; Silverman & Thompson, 2008) revealed that with the exception of one teacher, there was alignment between the nature of the mathematical content teachers believe generalizes (i.e., a specific meaning for slope versus an association, procedure, or formula) and teachers' personal understanding of slope. To help explain these results, two other components of teachers' MKT were examined.

CHAPTER 1:

The Generalization of Learning

Teachers generally hope that what they do in the classroom matters in terms of their students' performance in and understanding of future situations. In this respect, teachers may have both short-term and long-term objectives. A short-term objective may be that teachers want their students to perform well on an upcoming exam. Thus, a teacher might focus her efforts on helping students to develop meaning for a specific mathematical topic. She might do this by presenting her students with a set of problems that she views as related to the that topic and then asking her students to develop and explain strategies that could be used to solve the set of problems. To see if and how students are able to generalize their learning of the topic, the teacher may introduce a new problem and ask students whether and how it could be solved using their strategies. A long-term goal may be that teachers want their students to think of the situations that arise in their future careers in mathematical terms. Thus, a teacher might help nurture particular ways of thinking and reasoning. She might do this by providing students with rich real-world contexts and accompanying questions that support the development of those ways of thinking. In addition, she may adjust her line of questioning, the nature of the mathematical tasks she uses, and the participatory organization of the classroom in accordance with her observations of and inferences about students' emerging thinking and reasoning.

The idea that students generalize the learning that occurs in the classroom to novel situations serves as a basis for our educational system (Bassok & Holyoak, 1989; Lobato, 2008b; McKeough, Lupart, & Marini, 1995; National Research Council, 2000).

In other words, administrators, teachers, and parents alike hope that their students will be able to extend the learning that emerges and develops within various classroom situations to an ever expanding range of situations. Teachers do not want, for example, learning to be bound to the contexts and conditions in which it arises. On the contrary, teachers want their students to be successful in situations they have yet to encounter as a consequence of their classroom experiences. This vision—that learning should generalize beyond the initial conditions of learning—often shapes teaching practices in various ways.

Different Perspectives on the Generalization of Learning and How to Support It

One body of research that has examined the generalization of knowledge beyond initial learning situations is the research on *transfer* (e.g., Bassok & Holyoak, 1989; Beach, 1999; Bereiter, 1995; Bransford & Schwartz, 1999; Campione, Shapiro, & Brown, 1995; Engle, 2006; Gagne, 1968; Gick & Holyoak, 1983; Lobato, Rhodehamel, & Hohensee, in press; Markman & Gentner, 2000; Singley & Anderson, 1989; Thorndike & Woodworth, 1901; Wagner, 2010). Traditionally, transfer has been characterized as "how knowledge acquired from one task or situation can be applied to a different one" (Nokes, 2009, p. 2). However, there have been many theoretical challenges surrounding the construct of transfer. For example, transfer has been notoriously difficult to produce in laboratory studies (Bransford & Schwartz, 1999; Detterman, 1993; Perkins & Salomon, 1989). Moreover, the numerous critiques waged against transfer have contributed to a growing acknowledgment that, "there is little agreement in the scholarly community about the nature of transfer, the extent to which it occurs, and the nature of its underlying mechanisms" (Barnett & Ceci, 2002, p. 612).

Transfer has had a rich and varied history. In the late 1800s, the doctrine of formal discipline maintained that the mind was comprised of a small set of general faculties; strengthening a faculty via training in one discipline would result in transfer to other disciplines that called upon that faculty. This view was strongly opposed by Thorndike's (1906) theory of identical elements wherein the mind was conceived as a multitude of specialized functions and transfer only occurred to the extent to which identical functions were invoked in initial learning and transfer situations. Thorndike's view was upheld and reformulated in subsequent mainstream cognitive approaches to transfer that dominated the latter half of the 20th century. In such cognitive approaches, Thorndike's "identical elements" were re-conceptualized as shared features of learners' mental representations of initial learning and transfer tasks wherein mental representations were thought to be the result of relatively transparent perceptual processes (Singley & Anderson, 1989). However, dissatisfaction with the mainstream cognitive approach to transfer gained momentum in the late 1980s with Jean Lave's (1988) seminal critique of the culture of such transfer research. Critiques argued that the mainstream cognitive perspective on transfer: (a) privileged the observer's point of view (over the learner's) when making determinations regarding the occurrence of transfer, (b) ignored the realistic conditions that people encounter when faced with solving novel problems, such as being able to solicit help from others, and (c) insufficiently accounted for the role of sociocultural practices, discursive exchanges, and material artifacts in the generalization of learning (Beach, 1999; Bransford & Schwartz, 1999; Lobato, 2006, 2008a; Pea, 1989). As a result, some have argued that the ill-conceived construct of transfer be abandoned altogether (e.g., Carraher & Schliemann, 2002). Others, however,

have opted to develop alternative conceptualizations of and approaches to the study of transfer (e.g. Beach, 1999; Bereiter, 1995; Bransford & Schwartz, 1999; Engle, 2006; Lobato, 2003; Nemirovsky, 2011; Wagner, 2010). For example, Lobato's (2008a, 2012) actor-oriented transfer perspective emphasizes the personal and often idiosyncratic connections learners (actors) make between initial learning and transfer situations as well as the ways in which such connections are organized via practices and artifacts of the social and cultural environments. These differences, in the way transfer and its underlying mechanisms are conceived, matter when it comes to teaching practices that might serve to support the generalization of learning. A brief overview of the teaching practices forwarded by the four approaches to transfer introduced above is presented next, followed by a more extensive overview in Chapter 2.

One idea regarding teaching for transfer has been that teachers should support the development of students' general abilities to think and reason critically. Thus, teaching geometry is valuable because it helps to develop students' critical-reasoning skills, skills that will be useful in other domains requiring general reasoning (e.g., playing chess, programming computers). This idea is attributed to Locke who ascribed to the doctrine of formal discipline wherein the general faculties of the mind needed to be exercised like muscles (Lyans, 1914; Singley & Anderson, 1989). Under this view exertion rather than content was the primary goal of instruction. Transfer was thus conceived of as the influence of the development of a particular ability on performance in domains requiring said ability.

A second idea stands in stark contrast to the one above. Whereas Locke and some early educational psychologists believed that transfer was broad and took place at a more

general level, Thorndike and Woodworth (1901) argued that transfer was a phenomenon

that occurred at a more specific level. In particular, Thorndike and Woodworth believed

that it was rare that knowledge generalized beyond the conditions of initial learning.

Rather, they believed that if a teacher wanted her students to be able to do something in

particular, she should train them how to do it directly. Thus, showing students how to

solve particular kinds of problem, allowing them time to practice the demonstrated

strategy, and providing them with material to check their answers were valuable teaching

practices because they supported students in constructing associations between particular

items of information and reactions made in response to such information. These ideas

derive from an associationist perspective wherein knowledge is thought to manifest in

sets of associations, or connections, between external stimuli and behavioral reactions

articulated in response to those stimuli (Resnick & Ford, 1981). Transfer is then

conceived of as the influence of training in one association on another and is thought to

occur to the extent that initial learning and transfer tasks share identical elements–

generally interpreted as features of the physical environment.

A third idea regarding teaching for transfer has been that a teacher should support

her students in noticing the general principle or strategy underlying a particular example

problem or problem*s* so that students may be supported in (a) applying the principle or

strategy in novel situations or (b) using the exemplar problems as an analog in novel

situations. Thus, asking students to summarize example problems, compare two example

problems, or describe the goals of the example problems are worthwhile teaching

practices because they support the construction of increasingly abstract (i.e.,

decontextualized) mental representations of mathematical tasks. This idea is most

commonly found in information processing accounts, wherein human minds are thought to extract, manipulate, and store symbols of the external environment (e.g., Bassok & Holyoak, 1989; Genter, Loewenstein, & Thompson, 2003; Gick & Holyoak, 1983; Markman & Gentner, 2000; Reeves & Weisberg, 1994; Singley & Anderson, 1989). Here, transfer is conceived of as the application of knowledge acquired in one setting to a different setting. And transfer is thought to occur to the degree that learners' mental representations of learning and transfer tasks are identical, overlap, or can be connected via a mapping or analogy.

A fourth idea regarding teaching for transfer differs in an important way from the three outlined above. The driving force in this fourth conceptualization is a consideration for how the student or the *learner* conceives of learning situations. Thus, the teacher is supported in eliciting student reasoning and, in response, choosing mathematical tasks that attempt to necessitate a focus on productive features of the mathematical environment from among the variety of sources of information that compete for students' attention. Teachers are further supported in attempting to influence discursive practices and mathematical activity in such a way that students' attention is drawn to and remains focused on those productive features. Such practices are worthwhile because they seek to influence students' mathematical noticing. This idea comes from Lobato (2003, 2008a, 2012) whose *actor-oriented transfer* (AOT) perspective draws upon ideas from Piagetian constructivism and situated cognition. Thus, the learners' construal of meaning, the nature of mathematical tasks and activities, as well as the participatory organizations surrounding initial learning and transfer tasks are objects of investigation. Transfer is then defined as the generalization of learning wherein the primary processes involved in

generalizing are learners' construction of similarities or differences between initial learning and transfer situations. More broadly, transfer is understood to be "any influence of prior experiences on learners' activity in novel situations" (Lobato, 2008a, p. 291).

The four ideas discussed above are a subset of the ideas identified in research from a variety of theoretical perspectives regarding what the transfer of learning is and how it can be supported instructionally. Within the mainstream cognitive perspective, for example, there are many more conjectures regarding how to support students in generalizing their learning, each conjecture being accompanied by slightly different conceptualizations of the nature of transfer and its underlying mechanisms. Similarly, within the alternative approaches to transfer, there are several different ideas about how one should teach for transfer, each idea being associated with a different conceptualization of transfer and its underlying processes. These and other conceptualizations of transfer and how to support students in transferring their learning will be discussed in Chapter 2.

Teachers' Beliefs about the Generalization of Learning and How to Support It

As noted above, there is a large body of research highlighting different conceptualizations of the transfer of learning as well as how to best support the transfer of students' learning. This is not surprising since researchers have been hypothesizing, theorizing, and systematically testing ideas for at least one hundred years. Therefore, one might assume that because teachers are the people most typically associated with student learning, some portion of the research on transfer would have identified teachers' beliefs regarding the transfer of their own students' learning. However, a search of this literature

yielded no such results. In fact, I could not find any studies that aimed to identify teachers' beliefs associated with the generalization of learning. Thus, the following question remains: Do teachers even think about the generalization of learning and, if so, what is the nature of teachers' associated beliefs? In response, I have pilot data which suggest that teachers not only think about the generalization of students' learning, but that they have specific beliefs about how it occurs. Moreover, these data indicate that teachers' have specific beliefs regarding how to support the generalization of students' learning. Interestingly, these beliefs appear to vary widely from teacher to teacher.

The pilot data I present below derive from four teacher interviews. Two teachers were interviewed as part of the screening process for a research study funded by the National Science Foundation that focused on the relationship between teachers' practices and the transfer of student learning. The interviews of Joanne, a transfer researcher, and Bonnie, an experienced teacher, were chosen for presentation here due to the similarity in their personal understanding of specific mathematical content and to highlight differences in the ways in which a transfer researcher and a typical classroom teacher think about the teaching of a particular mathematical unit. The other two teachers were interviewed as part of a smaller study that I conducted to explore teachers' beliefs about the generalization of students' learning. The interviews of Sally and Kelly were chosen (from among five teachers participating in the pilot study) for presentation here because their teaching experience was comparable to that of Joanne and Bonnie. Gender-preserving pseudonyms are used for all participants except Joanne. The data presented below highlight some interesting similarities and differences in these teachers' beliefs about the generalization of students' learning.

The Role of the Real World

The real world played a large role in both Sally's and Bonnie's beliefs about the generalization of students' learning, albeit in different ways. On one hand, Sally appeared to believe that students would only be able to generalize their learning of a particular mathematical topic to real-world situations if they could view the real-world situation in terms of the learned topic. For example, Sally explained that Lucy, a fictitious student whose written explanation on a slope task emphasized slope as the result of carrying out the $\frac{rise}{run}$ formula, would probably have difficulty generalizing her understanding of slope to "real world situation[s]" because "she might not be able to figure out how it connects;" for instance, figuring out "how fast she is swimming, um, is probably harder to figure out because she might not connect it with slope." In other words, Sally seemed to believe that Lucy would have difficulty engaging in a real-world situation involving speed due to the unlikelihood that Lucy would be able to connect the notion of how fast one swims to her understanding of slope as defined by the $\frac{rise}{run}$ formula.

On the other hand, Bonnie appeared to believe that students would be able to generalize their understanding of a particular mathematical topic to a novel situation to the extent that they could relate the novel, mathematical situation to familiar aspects of real-world situations or experiences like *rising* out of bed before *running* to the slopes. Specifically, Bonnie explained that one of her main goals in a unit on slope was "build[ing] on the idea that slope is something that they … can relate to." Thus, she explained that in order to keep students from getting "lost" in the unit, she would prompt

students to access familiar real-world experiences by asking questions like "Where have you seen this [slope]?" wherein the expectation was that students would reference "bunny slopes," "skiing," and/or "a mountain which has a lot of steepness." Bonnie then explained that students would be supported in generalizing their learning if mnemonics were created that linked familiar aspects of their experiences to aspects of the slope formula; for instance, Bonnie explained that saying things like "You gotta get out of bed before you can run to the slopes" and "You can't run out of bed; you have to get up and then you go out of bed" gave students an image that "in the future, [they] could refer back to." In sum, Bonnie seemed to believe that students' successful engagement with "future" situations involving slope occurred to the extent that students could remember the relationship between familiar aspects of real-world situations (e.g., *rising* out of bed before *running* to the slopes) and the associated parts of the slope formula (i.e., rise and run).

Interestingly, Sally's and Bonnie's beliefs about the role of the real world in the generalization of students' learning were consistent with their beliefs about *how to support* the generalization of students' learning. Sally emphasized teaching strategies that would support students in seeing real-world situations in mathematical terms. Specifically, she argued that teachers should take specific actions to support students' "ability to transfer their knowledge to real life" because "you can like, for example, teach them, you know, how to find the missing side of a right triangle and they go out in the real world and they don't realize that they can use that out there." Thus, she argued, teachers should make use of "more examples that come from the real world;" for example, "instead of just saying 'this is a triangle,'" and perhaps presenting mathematical

ideas as separate from the real world, she argued that teachers should say "this is a ladder; this is a fence; this is a building," thus pointing out the shape created by the merging of real-world objects and making it more clear to students that mathematical ideas emerge from the world around us. In other words, Sally emphasized strategies that she believed would help students recognize mathematical ideas as arising out of real world situations. Similarly, Bonnie appeared to believe that the use of real world situations facilitated transfer but unlike Sally, Bonnie emphasized the use of real-world situations as a means of helping students to remember how to carry out specific strategies and procedures. Specifically, Bonnie emphasized drawing upon real-world situations that would be familiar to students in order to create mnemonics, analogies, and images that could help guide students' future mathematical performance.

The Role of Concepts versus Procedures

In their beliefs about the generalization of students' learning, Sally and Joanne emphasized the importance of students having conceptually meaningful understandings of mathematical topics while Kelly and Bonnie emphasized the importance of procedural accuracy. In particular, Sally explained that a "rate of change" understanding of slope would be helpful in making sense of topics that arise in future mathematics classes like when students "eventually take calculus" and are asked to figure out "how the tangent line changes over time" as well as in making sense of various situations that arise in the real world like finding the cost of "different phone plans" or "the cost per ride … [in a] taxi." In other words, it seemed that Sally believed that if students could come to interpret slope conceptually as the "rate of change" of a function, then they may be more likely to realize that there are "things that they can solve by using linear functions."

Similarly, Joanne emphasized conceptually meaningful understandings in her belief about the generalization of students' learning. Specifically, Joanne appeared to believe that students generalize their learning based on the mathematical features they notice during instruction. Thus, she argued that students should be supported in focusing on productive features of the mathematical environment. For her this meant features that would support students in "forming a ratio" as a "multiplicative comparison" of two quantities (i.e., attributes one conceives of as being measureable) rather than features that would support students in "just looking for number patterns." Thus, she explained it was important to begin a unit on slope by presenting students with a "real world situation" and then "problematizing the situation in a way that is constructive for them"—in a way that supports a focus on the quantitative relationships involved in a situation.

In contrast, Kelly's belief about the generalization of students' learning appeared to emphasize the importance of students' procedural accuracy. In particular, Kelly seemed to believe that the transfer of learning involved particularized demonstrations or *replications* of learned skills. For example, she explained that her "personal goal as a math teacher" was to nurture students who could "recreate" formulas "on the spot … any time they see a problem." Consequently, when presented with work by a fictitious student (Molly) on a slope task, Kelly focused on the precise way in which Molly determined the slope of a line. Specifically, Kelly explained that while Molly had "an understanding about how to graph points, and to draw a line, and to create *some sort of* triangle," she was concerned that Molly drew her slope triangle as representing "a run first and then a rise (see Figure 1.1)." Thus, she explained that Molly "might have difficulty if you change to a negative slope […] and an indication that that might be

problematic for her is *this* triangle [pointed to Molly's slope triangle recreated in Figure

1.1.] …because she created it in a right-up manner [traced the edges of Molly's slope

triangle in the direction indicated by the arrows shown in Figure 1.1] (emphasis added)."

In other words, when making judgments about Molly's future performance, Kelly

appeared to focus primarily on the fact that Molly did not accurately replicate a particular

skill—the construction of a slope triangle—and to ignore the contents of Molly's written

explanation, which emphasized a rate of change understanding of slope and highlighted

the relationship between the quantities in the context of the slope task. It is not surprising

then that Kelly believed that she could support the generalization of students' learning by

being "a stickler to students" about the proper ways in which to carry out specific skills.
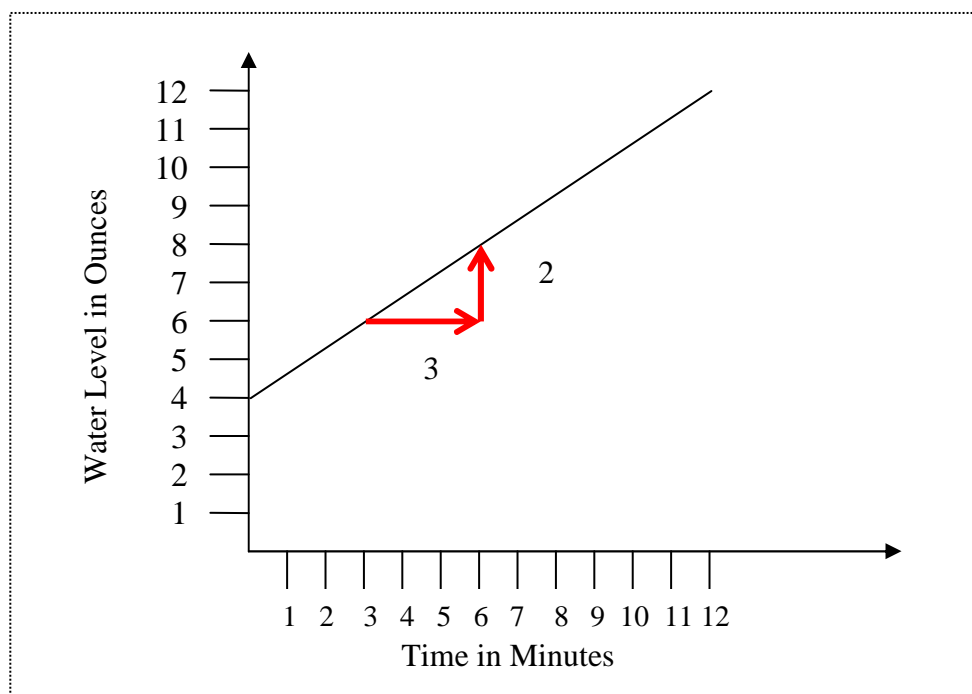
Figure 1.1: Author's re-presentation of Molly's slope triangle.

Bonnie also seemed to emphasize the accuracy with which her students carried

out procedures. As seen in the previous section, Bonnie drew upon students' experiences

surrounding slope to highlight the order in which the slope formula should be

implemented, namely that "rise divided by run" corresponds with the image that one needs to get out of bed (or rise) before one can run to the slopes. Thus, the way in which Bonnie used her students' experiences appeared to indicate that she believed in supporting the transfer of procedures rather than conceptually meaningful understandings.

Summary of Teachers' Beliefs about the Generalization of Students' Learning and How to Support It

The data presented above suggest that these four teachers hold different beliefs about the generalization of students' learning and how to support it. Sally seemed to believe that students generalize their learning when they see a novel situation in terms of something learned and apply associated ways of reasoning; further, she seemed to believe that such generalization could be facilitated by the use of real-world examples. Bonnie appeared to believe that students generalize their learning when they can relate a novel situation to certain aspects of a familiar experience and thus seemed to believe that the transfer of students' learning could be facilitated by the use of mnemonics, analogies, and visuals that relate mathematical procedures to students' experiences. Joanne believed that students (almost) always generalize their learning and do so on the basis of the mathematical features they notice during instruction; moreover, she believed that teachers should try to influence the nature of the mathematical features students notice during instruction. Finally, Kelly seemed to believe that students generalize their learning when they are able to accurately replicate a demonstrated skill in a new situation and believed that such generalization could be supported by teachers' enforcement of the particulars with which students carry out specific skills during instruction.

Despite these differences, there were also some interesting similarities. Both Sally and Bonnie emphasized the role of real-world situations in students' generalization of learning. For Sally, students' ability to see a real-world situation as an instance of something learned was a primary criterion for the occurrence of transfer. For Bonnie, the opposite was true—students' ability to relate mathematical procedures (e.g., slope as $\frac{rise}{run}$) to aspects of their personal experience was of principle concern. In addition, both Sally and Joanne seemed to emphasize the transfer of conceptually meaningful understandings (i.e., for Sally, "rate of change" and for Joanne, "ratio") while Kelly and Bonnie appeared to emphasize skills and procedures (e.g., for Kelly, the particulars of how a slope triangle was constructed and for Bonnie, the order of rise and run in the slope formula).

Limitations of the Pilot Data

While the data presented above suggest that there is variety across teachers with respect to their beliefs about how students generalize their learning experiences and how to support that generalization instructionally, there is only limited amount of data regarding teachers' beliefs about how to *best* support the transfer of student learning. For example, Sally expressed that teachers should use real-world examples, but we don't know whether she believes this is the *only* way or the *best way* transfer can be supported. We only know that the use of real-world examples is *one* way in which Sally believed she could help support her students in generalizing their learning. Furthermore, since Sally was never asked explicitly *how* the use of such examples facilitates transfer, such answers can only be inferred from Sally's other responses. Thus, more work needed to be done to uncover a broader view of the ways in which teachers believe students can be supported in generalizing their learning.

Moreover, teachers were never asked explicitly about their beliefs about *transfer* or *the generalization of students' learning*. Rather, Sally and Kelly were asked questions like "What would you want your students to be able to do with their knowledge of slope?" and "What tasks might be too difficult for Lucy/Molly given her work on the slope task?" to probe their beliefs about transfer and its limitations. While both Sally and Kelly introduced and used the term *transfer* in their interviews, the interviewer only used it one time in response to Sally's statement that students should be able to transfer their learning to the next math class as well as to the real world in order to ask how such transfer could be supported. In contrast, Joanne, Bonnie, nor their interviewer ever introduced the term transfer or the phrase generalization of students' learning into their interviews. Thus, I wanted to find out what kinds of beliefs emerged when such terms and phrases were used explicitly.

Research Question 1

The intriguing finding presented above, namely that there is variety across teachers in their beliefs about the generalization of students' learning and how to support it instructionally, coupled with the limitations of the pilot data previously discussed led to the development of the following two-part research question:

> What are teachers' espoused and inferred beliefs regarding (a) the generalization of students' learning and (b) how to support the generalization of students' learning?

This question seeks to systematically examine teachers' espoused (i.e., stated) and inferred (i.e., attributed) beliefs about what the generalization of learning is and what it takes to support students in generalizing their learning to novel situations. In other words, it seeks to examine teachers' explicit statements about transfer and how to support

transfer. In addition, it seeks to examine the, perhaps less conscious, beliefs teachers hold by posing questions and tasks that provide the means by which teachers' beliefs might be inferred (Philipp, 2007). In the context of mathematics courses, or more specifically a lesson (or set of lessons) on slope and linear functions, this question aims to get at *what* teachers believe generalizes from a lesson on slope and linear functions (e.g., a conceptually meaningful interpretation of slope or a particular formula for slope) and how that generalization occurs (e.g., when a novel situation prompts a learned formula). Moreover, this question aims to get at the ways in which teachers believe they can support the current and ongoing construction of knowledge (e.g., the interpretation of slope as a rate of change) so that it comes to live again in future contexts (e.g., the next math course or a real-world situation). For example, Sally appeared to believe that one would be able to transfer the conceptually meaningful interpretation of slope as a rate of change to situations in the real world if one could see those situations (e.g., the cost of a taxi ride) in terms of their rate-of-change interpretation. Thus, Sally espoused a belief that the generalization of learning could be supported through the use of real-world examples. Further inferences of Sally's beliefs about how to support the generalization of students' learning suggested that she believed teachers should use real-world examples to help students develop conceptually meaningful interpretations of mathematical topics.

A curious reader may wonder: What constitutes a *belief* about the generalization of learning or a *belief* about how to support the generalization of students' learning? Interestingly, there is no consistency in the literature regarding the way in which the construct of beliefs is used or the way in which it is defined (Calderhead, 1996; Philipp, 2007). While some researchers have used the construct interchangeably with notions

such as attitudes, dispositions, judgments, opinions, perceptions, personal theories, and perspectives, other researchers have discussed the construct of beliefs without ever defining it (see reviews by Pajares, 1992 and by Philipp, 2007). When definitions are given, they are often in terms of other constructs (e.g., conceptions, values, or knowledge). For example, Bishop, Seah, and Chin, (2003) defined beliefs in terms of values wherein beliefs are considered distinguishable from values using the following criteria: (a) beliefs are associated with a true/false dichotomy whereas values tend to be associated with a desirable/undesirable dichotomy, and (b) beliefs tend to be context-dependent whereas values tend to be context-independent. For instance, "mathematics is fun" is a belief because it describes a true/false judgment about a particular subject. However, when the first criterion is applied to beliefs regarding the generalization of learning, the ability to apply true/false value is contingent upon numerous variables including the particulars of one's definition and conceptualization of transfer. For example, judging the belief that "supporting the transfer of learning involves engaging students in mentally taxing tasks and activities" as true may require certain epistemological commitments such as the faculty view of mind. Similarly, judging the belief that "supporting the transfer of learning involves pointing out the strategy or principle underlying a pair of exemplar problems" as true may require a commitment to the veridical nature of internal mental representations. Thus, Bishop et al.'s characterization of beliefs does not seem suitable when investigating beliefs associated with the generalization of learning.

In another example, Thompson (1992) defined beliefs in terms of conceptions, perhaps thinking of beliefs as a subset of conceptions. However, she appeared to do so

inconsistently using beliefs to define conceptions and at times using the terms

interchangeably, thus, making the practical use of such a definition difficult. Philipp

(2007) followed Thompson is using conceptions to define beliefs, but he distinguished

beliefs from knowledge. Specifically, he defined a conception (i.e., a general notion or

view) as a belief if one can respect as intelligent and reasonable others' views which are

contradictory to said view and a conception as knowledge if one cannot respect as

intelligent and reasonable others' views which are contradictory to said view. When this

definition is taken from the point of view of the researcher, or the observer, and applied

to ideas about the generalization of learning, its usefulness is revealed. I (as an observer)

can accept the statement that "supporting the transfer of learning involves pointing out

the strategy or principle underlying a pair of exemplar problems" as reasonable for many

teachers (even though it differs from my own views about transfer) given the way in

which traditional mathematics instruction typically unfolds. Thus, I ground the

exploration of the first research question of this dissertation study in Philipp's (2007)

definition of beliefs. I will elaborate on additional aspects of this definition as well as

discuss other ways in which beliefs have been conceived in Chapter 2.

The Relationship between Teachers' Beliefs about the Generalization of Students' Learning and their Mathematical Knowledge for Teaching

As noted above, there appeared to be some interesting similarities across teachers'

beliefs about the generalization of students' learning. While Kelly and Bonnie appeared

to emphasize the transfer of skills and procedures, Sally and Joanne appeared to

emphasize the transfer of conceptually meaningful understandings. A natural question is

whether these differences reflect associated differences in the teachers' understanding of

the mathematics? Is it possible that Kelly and Bonnie understood slope in terms of procedures—slope *is* what results when one carries out a particular set of rules—while Sally and Joanne understood slope to have a particular conceptual meaning—slope *is* a ratio that provides a measure of the rate of change of a function?

In fact, data suggest that Kelly understood slope in terms of procedures. Specifically, when discussing her concerns about Molly's work on a slope task, she said that "the triangle [Molly drew (see Figure 1.1)] represents a run first and then a rise, so my concern is whether they understand that *slope is talking about how things change vertically over horizontally* (emphasis added)." Here, Kelly appeared to interpret slope as the result obtained when one carries out a mathematical procedure (i.e., $\frac{y_2-y_1}{x_2-x_1}$ or $\frac{rise}{run}$) in a particular way, namely by placing the number representing vertical change "over" the number representing horizontal change.

However, data from Bonnie's and Joanne's interviews suggest that they both held a more conceptually meaningful understanding. Thus, I will present data that are suggestive of a relationship between *certain aspects* of teachers' mathematical knowledge and their beliefs about the generalization of students' learning. Specifically, I will show that while Bonnie and Joanne appeared to interpret the meaning of slope similarly (i.e., slope is a ratio measuring the rate of change of a function), they did not seem to have similar understandings of (a) the ways in which students might come to develop this understanding or (b) the actions they might take to support this understanding. Further, these differences seem implicated in Bonnie's and Joanne's beliefs about the generalization of learning.

Teachers' Understandings of Slope

Joanne's and Bonnie's discussions surrounding Angie's (a fictitious student's) work on The Leaky Bucket Task (shown in Figure 1.2) indicated that they both interpreted the meaning of slope as a ratio measuring the rate of change of a function— meaning that they interpreted slope to be an attribute of a function that preserves the relationship between the quantities represented by the dependent and independent variables. (This interpretation of slope as well as others will be discussed in greater detail in Chapter 2.) The Leaky Bucket Task involved a bucket dripping water into a measuring cup that initially contained 4 ounces of water. The water level in the measuring cup was recorded after 0, 3, 6, and 12 minutes and Angie was asked to plot the points represented in the table on a graph and then find the slope of the resulting line. When graphing the data, Angie transposed *elapsed time* and *amount of water in the cup*. Thus, she arrived at an answer of $1\frac{1}{2}$ or $\frac{3}{2}$ rather than $\frac{2}{3}$. Nevertheless, Angie's accompanying explanation exhibited an interpretation of slope as a ratio which provides a measure of rate. Joanne and Bonnie were asked to look over Angie's work, give it score on a scale from 0 – 10, and then explain their rationale for the score.

A leaky bucket drips water into a measuring cup. The water level in the measuring cup is recorded every minute. The table below shows the water level in the cup over time. The cup started out with some water in it before the dripping began. Plot these points on a graph and then find the slope of the line.

Elapsed Time (minutes)	Amount of Water in the Cup (ounces)
0	4
3	6
6	8
12	12

Angie's answer: Slope is 1 ½

Angie's explanation: First you graph the points and then connect them to make a line. Now I pick two points (6,3) and (8,6). Two minutes passed between 6 minutes and 8 minutes. During that time, the water rose in the cup 3 oz. Three oz in 2 minutes is the same as 1 ½ oz in 1 minute. So the slope is 1 ½, which is how much the water rises every minute.

Task to teachers: What score, on a scale from 0 to 10, would you give this student? If the student is on the right track, describe their reasoning and what is good about it. If you deduct points, please explain why and what, in your opinion, was incorrect in the student's reasoning.

Figure 1.2: The Leaky Bucket Task.

Joanne. When responding to Angie's work, Joanne appeared to demonstrate that she interpreted the meaning of slope as a ratio measuring the rate of change of a function. In particular, Joanne explicitly mentioned rate stating, "What I like about her reasoning here is that she is thinking about what rate is." Joanne was then able to elaborate on her interpretation of rate in the context of the leaky bucket explaining that "it's the, in this case, the amount of water to time that has past." Further, Joanne acknowledged and appeared to appreciate Angie's ability to attend to and join the quantities in the leaky-bucket situation to "form this unit of 3 ounces in 2 minutes" as well as Angie's ability to operate on the unit (via "partition[ing]") in a way that preserved the quantitative

relationship to obtain a unit ratio of $1\frac{1}{2}$ ounce in 1 minute. Thus, Joanne appeared to interpret slope as an attribute measuring the constant rate of change of a function.

Bonnie. When asked to respond to Angie's work, Bonnie appeared to demonstrate that she interpreted the meaning of slope in a way that was similar to the way Joanne did. Specifically, Bonnie acknowledged that Angie "graphed the points backwards" but maintained that "conceptually, just based on her explanation, she really understands." This excerpt revealed that first and foremost Bonnie had the ability to look beyond a set of procedures or a numerical answer when thinking about slope. Further, when explaining what in particular Angie understood about slope, Bonnie said, "that rise over run is about change … how much it would change." Here, we see that Bonnie was focused on "how much" something changes and thus appeared to understand slope as something more than just a simple counting technique wherein one finds the rise and the run and then puts one over the other. In addition, Bonnie pointed out that "Angie could relate it to how much of the water [was] rising [over time] ... She can explain to me in terms of the context what slope is in relation to these numbers [pointed to Angie's answer of $\frac{3}{2}$]." Together, these excerpts suggest that Bonnie interpreted slope as a ratio relating two quantities (amount of water and elapsed time) in such a way that a measure of change is provided. Thus it seemed that Joanne and Bonnie interpreted the meaning of slope similarly.

Teachers' Understanding of The Ways in Which Students Might Come to Develop an Understanding of Slope as a Ratio Measuring Rate

In this section, I present data that appear to show Joanne had developed an understanding of the ways in which students might come to interpret the meaning of slope as a ratio measuring rate, while Bonnie had not. Prior to their interviews, Joanne and Bonnie were asked to create a lesson plan on slope for an Algebra 1 class. During their interviews, they were asked several questions in relation to their lesson plans including: (a) What difficulties do you expect students to experience; (b) How will your lesson help students' develop an understanding of linear functions; (c) What is most important when it comes to teaching slope; and (d) What might students say to indicate that they have developed an understanding of slope? Thus, I infer the teachers' understandings about how students come to develop an interpretation of the meaning of slope as a ratio measuring rate from this set of interview questions.

Joanne. Joanne presented evidence that she had an understanding of the ways in which students might come to develop an interpretation of the meaning of slope as a ratio that provides a measure of rate. With respect to developing this interpretation, she explained her thoughts about the ineffectiveness of a particular approach to teaching slope:

> For a lot of people slope is a characteristic of a line… [or people may conceive of] slope as height alone, not even rise, but height… [When] slope is taught as 'rise over run' or as 'y_2 minus y_1 [wrote $y_2 - y_1$] divided or over [wrote $x_2 - x_1$],' this [drew a ring around $y_2 - y_1$] can just be a whole number. This [drew a ring around $x_2 - x_1$] can just be a whole number. Kids can get this right without ever reasoning with ratios at all.

In this excerpt, Joanne explained that using a graphical context and the slope formula as a means by which to support students' in developing an interpretation of slope as a ratio may actually be counterproductive because it may support an interpretation of slope as one whole number over another rather than an interpretation of slope as a ratio (i.e., a multiplicative comparison of two quantities).

In contrast, she explained that the use of real-world contexts may actually serve the development of students' interpretation of slope as an attribute of a function. In particular, she stated that it was important to "think about the slope as being a property of the function" rather than a line. For example, she stated that one could "easily think of slope as measuring speed in a situation in which the graph isn't there." Thus, she described the associated interpretation she hoped such a context would foster; in particular, she described wanting kids to be able to "form a ratio of distance with time but … form the ratio so that ratio is a measure of something—it's a measure of the speed." These excerpts suggest that Joanne had an understanding of the affordances and limitations of different contexts in developing students' interpretation of slope.

Joanne went on to explain how students might come to develop a sophisticated understanding of ratio, moving from an understanding of ratio as a composed unit to an understanding of ratio as a multiplicative comparison. Specifically, she explained how using real-world contexts that involve two different quantities like distance and time might lead to the former understanding but that through the repeated actions of iterating and partitioning, they might develop into the latter understanding. Joanne explained that when two different quantities are used, students may actually "jam them together and form a unit" that they will then iterate (i.e., repeat) or partition (i.e., split into equal parts).

This, she explained, led to an understanding of ratio "as a composed unit instead of a multiplicative comparison." However, she described such an understanding as a "natural" place for students to start off when developing an understanding of ratio. More importantly, Joanne seemed to have an understanding of how students might move from one understanding to the next, namely through the repeated actions of "iterating and partitioning." She explained that through these actions students might discover the "unit ratio," begin to see all iterates and partitions as expressions of some multiple of that unit, and thus see them all as representations of the unit ratio. This, she posited, could lead to the understanding of ratio as a multiplicative comparison. In sum, the data provided here indicate that Joanne had an understanding of how one might develop an interpretation of the slope as a ratio.

Bonnie. Despite the fact that Bonnie seemed to have developed a personal interpretation of slope as a ratio, Bonnie did not discuss such an interpretation when talking about her unit on slope and linear functions. Rather, she spoke about slope in terms of geometric and physical objects. Specifically, when asked to describe what students might say to indicate that they have developed an understanding of slope, Bonnie said: "slope involves a lot of directionality…it's the slant; it's the steepness; it's rise over run." She continually referenced such interpretations of slope throughout the discussion of her lesson plan and its implementation. For example, when asked to anticipate and describe some of the difficulties students might have, Bonnie talked about difficulties associated with graphing and the order of the slope formula (e.g., students remembering "run over rise" rather than "rise over run"). Thus, although it is possible

Bonnie had developed an understanding of the ways in which students might come to interpret slope as ratio, the interview yielded no such evidence.

Teachers' Understandings of Actions That Might Support Students in Developing an Understanding of Slope as a Ratio Measuring Rate

In this section I present data that suggest Joanne had developed an understanding of the actions she might take to support students in developing an interpretation of slope as ratio while Bonnie had not. I will present data from the same set of interview questions used in the previous section.

Joanne. The actions Joanne described as being particularly helpful in fostering students understanding of ratio were (a) using the "same attribute task," (b) asking students to explain *why* (rather than *that*) a particular answer worked, and (c) connecting the mathematical term *slope* to students' emerging understanding of ratio. First, she described the details and associated rationale of the same-attribute task. This task entailed asking her students to find several distance-time pairs that, when entered into speed-simulation software, would allow one character to travel at the same speed as a second character traveling at a given speed (e.g., 10 centimeters in 4 seconds). Her rationale for presenting this task to her students was that the "formation of things [i.e., distance-time pairs] that all have the same attribute [i.e., speed] helps lead to the formation of ratio."

This is not to say that Joanne claimed the task alone would promote the formation of ratio. On the contrary, she claimed that the formation of ratio "comes through [students'] explanation." She thus spoke about asking her students to explain to their peers "*why* something is working, not just *that* it works." Further, she said that she

would "ask all the other kids to keep them [the one explaining] honest." Her rationale for asking students to provide such explanations was that "kids will transition from the number patterns to actually making an explanation that is quantitatively based."

Finally, Joanne explained that she would link the mathematical term slope to students emerging understanding of ratio. In particular, she explained that one should "develop ratio reasoning … and then introduce the term slope … [because] at least they will have gotten started in the right direction where they're doing some co-variational reasoning and know that slope is attached to *that*." Thus, it appeared Joanne had an understanding of the ways in which students might be supported in developing an interpretation of *slope* as ratio.

Bonnie. Despite the fact that Bonnie seemed to personally interpret slope in terms of ratio, the pedagogical actions Bonnie discussed taking in association with her unit on slope and linear functions did not seem to support the development of such an interpretation for students. For example, Bonnie talked about beginning her unit by asking her students to describe what came to mind when they heard the term slope. Such a request is reasonable if one hopes to isolate the everyday meaning of the term slope as incline from the mathematical conceptualization of ratio. However, as mentioned above, Bonnie explained that she would "like to hear them say" things about bunny slopes, mountains, and roadways. Again, such images could be made useful when developing an understanding of ratio if one highlighted the quantitative relationships between the attributes height and length rather than perceived slant; however, there was no evidence to suggest that Bonnie planned to personally emphasize or ask students to attend to such relationships during instruction. Furthermore, Bonnie's mnemonic devices, analogies,

and visuals foregrounded physical and algorithmic views of slope. Specifically, Bonnie talked about using the mnemonic "You rise to the top first and then you run" and the analogy involving having to get up "out of bed before you can run to the slopes" to help students remember the rise over run formula. In addition, Bonnie talked about using visuals such as real-world "mountain slopes" and slanted "hand gestures" to illustrate the meaning of "the positives, negatives, undefined, [and] zero slope." These mnemonics, analogies, and visuals likely serve to background the quantitative relationships necessary to develop an interpretation of slope as ratio. Thus, although it is possible Bonnie had developed an understanding of the actions she could take to support students' ratio reasoning, the interview yielded no such evidence.

In sum, both Joanne and Bonnie appeared to interpret the meaning of slope as a ratio that provides a measure of the rate of change of a function. However, data suggested that Joanne alone had developed an understanding of (a) the ways in which another person might come to interpret slope in this way and (b) the actions she could take to support another person in developing this interpretation of slope.

Relation between Teachers' Beliefs and Their Mathematical Understandings

Interestingly, the above understandings seemed implicated in Joanne's and Bonnie's beliefs about the generalization of students' learning. Specifically, Joanne's belief—that students' generalize their learning on the basis of the mathematical features they notice during instruction—seemed to indicate an awareness of the relationship between features in the mathematical environment (e.g., a graphical representation of a linear function or a complex real-world situation) and the associated ways in which students might come to interpret slope (e.g., as one number over another number, as a

composed unit of two quantities, as a multiplicative comparison of two quantities). Thus, she emphasized the importance of taking actions which support a focus on mathematical features in the environment that have the potential to be productive in developing conceptually meaningful mathematical understandings (e.g., through particular task choices and students' explanations of *why* something works). Similarly, Bonnie's personal understanding of slope coupled with her belief that the generalization of learning occurs when a person relates a novel situation to familiar aspects of real-world experiences (e.g., skiing down a slope) could be productive in supporting students in generalizing an interpretation of slope as ratio. However, the fact that Bonnie did not show evidence of having an understanding of (a) how students come to develop an interpretation of slope as ratio or (b) what to do to support students in developing an interpretation of slope as ratio, suggests that she did not know how to use her personal understanding of slope to support students in generalizing a slope-as-ratio interpretation of slope to novel situations.

Limitations

While the data presented here suggest a relationship between certain aspects of teachers' understanding of the mathematics and their beliefs about the generalization of students' learning, the interviews were not designed with the intention of highlighting teachers' understandings of (a) the ways in which students' might come to develop a particular interpretation of the meaning of a mathematical topic like slope or (b) the actions one might take to support the development of a particular interpretation of a mathematical topic. Thus, it is possible that the interviews did not tap such understandings. In other words, it is possible that Bonnie had developed an

understanding of (a) and (b) but that she was not asked questions that would reveal these understandings. Consequently, a systematic investigation targeting these specific understandings is needed to explore the possible relationship between teachers' beliefs regarding the generalization of students' learning and these aspects of teachers' understandings of the mathematics.

Research Question 2

This dissertation study examines the relationship between teachers' beliefs about the generalization of students' learning and the aspects of teachers' mathematical understandings that were discussed above, by posing the following question:

> What is the relationship between teachers' beliefs regarding the generalization of students' learning and their understanding of (a) a particular mathematical topic, (b) the ways in which students come to understand that topic and (c) the actions they might take to support the development of students' understanding of that topic?

In other words, this question probes the relationship between a teacher's beliefs regarding the generalization of students' learning beyond initial instruction and three aspects of mathematical knowledge for teaching: (a) teachers' personal understandings of the mathematics, (b) teachers' understandings of how someone else might come to hold an understanding that is similar to their own, and (c) teachers' understandings of actions that might support someone else in coming to hold an understanding that is similar to their own. For example, I presented data that suggested Kelly, Joanne, and Bonnie held distinct beliefs about the generalization of students' learning, and that there were similarities and differences in various aspects of their understandings of slope. Moreover, I highlighted some relationships between the teachers' beliefs and various aspects of their mathematical understandings. This question seeks to further understand

the nature of these relationships when each aspect is systematically investigated and examined.

Answering this research question is contingent upon the way in which one conceives of teachers' mathematical understandings. There have been two main approaches to the study of teachers' *mathematical knowledge for teaching* (MKT), one of which will be described here (with details regarding the other influential conceptualization to be presented in Chapter 2). In Silverman and Thompson's (2008) conceptualization, MKT entails at least three different kinds of knowledge: (a) knowledge of the *key developmental understandings* (KDUs) that exist within a particular mathematical content area, (b) knowledge of the ways in which students might come understand those KDUs, and (c) knowledge of the ways in which students might be supported in developing said KDUs. In this conceptualization, MKT is said to be grounded in particularly powerful understandings of specific mathematical concepts, or in KDUs. A KDU is a conceptual advance or a "change in the learner's ability to think about and/or perceive particular mathematical relationships" (Simon, 2006, p. 993). A KDU is particularly powerful because it provides a person with the leverage needed to develop deeper mathematical understandings. In this way, people who possess a KDU tend to see relationships between both directly and indirectly related tasks based on the tasks' underlying conceptual similarities (Silverman & Thompson, 2008). Thus, evidence of such an understanding may be people's ability to solve a variety of different but conceptually related problems (Thompson & Thompson, 1996). For example, consider Kelly, who seemed to understand slope in terms of a particularized set of procedures for finding the rise and run of a line. This understanding may allow Kelly to

view tasks involving graphical representations of linear functions or physical objects showing some sort of slant (e.g., a ramp, roadway, or rooftop) as related to the topic of slope but prevent her from conceiving of tasks involving, for example, speed as being about slope. On the other hand, a person who interprets slope as a ratio may be able to see all of these tasks as being about slope.

According to Silverman and Thompson, understanding a KDU is only the beginning for teachers. For example, a teacher may interpret slope as a ratio but be unaware of the utility of such an interpretation for students or be unable to use such an interpretation as a theme around which to organize classroom activity. Thus, the KDU is said to have pedagogical *potential*. In order for the KDU to have pedagogical *power*, the teacher must construct the associated MKT. The creation of MKT involves a transformation of the KDU from a personal understanding of a particular mathematical concept (e.g., ratio) to an understanding of (a) the ways in which others might come to have this understanding and (b) actions a teacher might take so that her students are supported in developing this understanding. With respect to Joanne and Bonnie, the data presented above suggests that Joanne had transformed her KDU of ratio to MKT while Bonnie had not. (See Chapter 2 for other conceptualizations of MKT and, more generally, teachers' content knowledge.)

The Relationship between Teachers' Beliefs about the Generalization of Students' Learning and Their Classroom Practices

So far, I have presented some pilot data to suggest that teachers have specific beliefs regarding the generalization of students' learning and that these beliefs vary across teachers. Thus, it is natural to wonder whether there is a relationship between

these kinds of beliefs and teachers' classroom practices. With respect to Joanne and Bonnie, the teachers presented thus far, I do not have data for this directly. However, I do have related data that are suggestive of a relationship between these teachers' beliefs about the generalization of students' learning and their classroom practices.

To introduce the possibility of a relationship between teachers' beliefs about the generalization of students' learning and their classroom practices, I draw upon results published from the larger NSF-funded research study mentioned previously (i.e., Lobato et al., in press). In this article, various aspects of Joanne's and Bonnie's classroom instruction were analyzed in relation to the ways in which their students reasoned on a subsequent transfer task. I therefore start by briefly summarizing the findings regarding the ways in which the students in Joanne's and Bonnie's class reasoned on a transfer task after having engaged in 10 hours of instruction on slope and linear functions. I then briefly present and selectively illustrate the findings relating the differences in students' reasoning and the teachers' classroom practices. Lastly, I discuss the ways in which the differences in Joanne's and Bonnie's practices seem implicated in their beliefs about the generalization of students' learning.

The Nature of Students' Reasoning

After participating in either Joanne's or Bonnie's unit on slope and linear functions, students were presented with a transfer task called The Pool Task (as shown in Figure 1.3). The task was set in a novel water-pumping context, not explored in either class, and also introduced the novelty of uneven spacing between the labeled points on the line. Qualitative analysis of the interview data revealed distinct differences in students' reasoning across the two classes. In particular, 7 of the 8 students in Joanne's

class attended to the quantities represented by the coordinate pairs and/or axes, and 5 of

these 8 students also reasoned with a quantitative relationship to correctly find and

interpret slope in the water-pumping context. In contrast, 8 of the 9 students in Bonnie's

class ignored the quantities represented by the axes, treated the graph as if it were

composed of unit squares or "boxes," and were unable to find and interpret slope in the

water-pumping context.

Water is being pumped through a hose into a large swimming pool. The graph shows the amount of water in the pool over time.

What is the slope of the line? What does the slope tell you about the swimming pool situation?
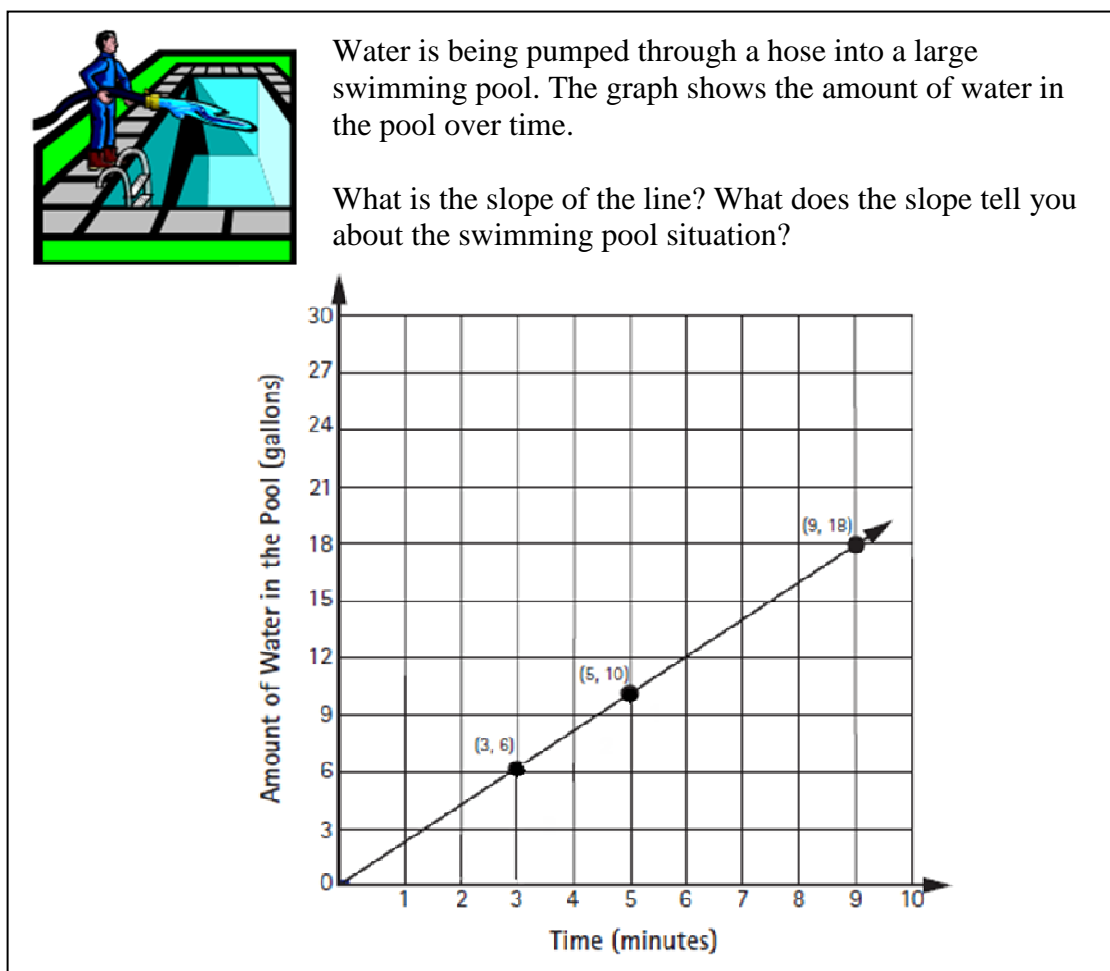
Figure 1.3: The Pool Task.

Teachers' Classroom Practices

In order to explain the differences in students' reasoning on the transfer task,

Lobato et al. (in press) examined several features of the instructional environments

including the teachers' discourse practices and their use of mathematical tasks. In Joanne's class, such features were found to support students in reasoning about quantities and quantitative relationships while in Bonnie's class, such features were found to support students in reasoning about physical objects. To illustrate, I will briefly discuss the way in which graphing was introduced in each class.

Joanne's unit was set within a speed context wherein students spent several lessons figuring out how to find and explain why certain collections of distance-time pairs produced the same speed (e.g., why is traveling 30 cm in 12 s the same speed as 10 cm in 4 s?). To introduce graphing, Joanne asked her students to plot one of these distance-time pairs, namely 30 cm in 12 sec. A student then plotted the pair by locating "12" on the time axis and tracing vertically upwards with her pen until she reached the appropriate height for the point (12,30) as indicated by the distance axis (see Figure 1.4). Joanne then used the mathematical symbols "(12,30)" to rename the distance-time pair as "a point," explaining that one should "write the time first." In a similar fashion, Joanne renamed a collection of distance-time pairs all representing the same speed as "a line" and a speed as "slope." These acts of renaming likely served to keep students' attention focused on quantities while graphing.
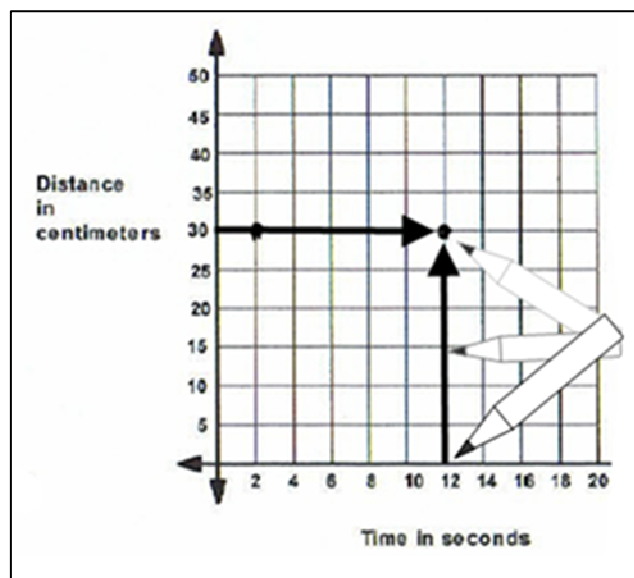
Figure 1.4: Student's gestures while graphing the point representing 30 cm in 12 s.

In Bonnie's class, graphing started out in a similar way with students attending to the quantities represented by the axes. However, the focus changed when a student graphed the point (6,19) without attending to the axes. Instead, she navigated from the previously plotted point (5,16) to (6,19) using the coordinate grid system explaining that she could just "mov[e] one more [to the right] and up three." Bonnie subsequently sanctioned this method of graphing exclaiming, "Wow, you guys are good … I just move over one, and I know to go up to 19 because of the plus three." Furthermore, Bonnie went on to link the "over 1 and up 3" pattern with moving "right 1 street and then go[ing] up 3 streets" on a city map:

> So we have a point. Obviously the points all look the same, right? So they have to kind of give it a name; it's almost like an address. Okay, so it's like how to get to your house. So to get to your house…I can go right 1 street and then go up 3 streets. So it's a way of helping people with directions.

In this excerpt, Bonnie renamed the point (6,19) a "location" and related graphing to navigating using a grid map of city streets. In a similar way, Bonnie renamed several

attributes of physical objects as slope. For instance, a mountain slope was renamed as "mathematical slope" and the risers and runners of a staircase were renamed as the "rise" and "run" values found in the slope formula (see Figure 1.5). These acts of renaming likely supported students in linking the act of graphing as well as the mathematical topic of slope to past experiences with physical objects.
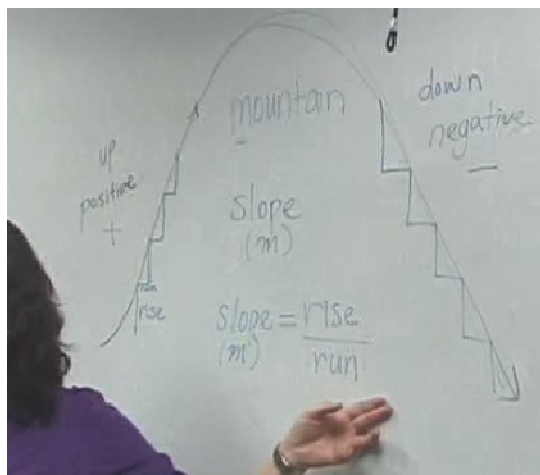
Figure 1.5: Bonnie's diagram linking mountains, stairs, and slope.

In sum, Joanne's acts of renaming and usage of mathematical tasks served to support students' ongoing quantitative reasoning while Bonnie's served to emphasize features of physical objects. Each teacher's pedagogical actions appeared significant in students' subsequent reasoning on the transfer task shown in Figure 1.3 as the majority of the students in each class utilized the associated forms of reasoning.

Relation between Teachers' Beliefs and Their Classroom Practices

Interestingly, the teachers' practices seem to provide elaboration for the beliefs I inferred regarding the generalization of students' learning. For example, Joanne's practice helped give additional meaning to her belief that students generalize their learning on the basis of the mathematical features they notice. In particular, it seemed

that Joanne wanted her students to attend to mathematical objects (as opposed to physical objects), meaning that students would notice quantities (measurable attributes of situations), and then form and attend to quantitative relationships, specifically multiplicative relationships related to slope. Similarly, her practice elucidated her belief that teachers should try to influence the nature of students' mathematical noticing. Specifically, it seemed that she thought teachers should identify the big ideas they want students to develop (e.g., slope as a multiplicative comparison between quantities) and then focus on developing those understandings (e.g., quantitative relationships that are multiplicative in nature) before linking them with mathematical terms like *slope* which could easily be understood by students as slant or inclination. Thus, Joanne seemed to believe that she should enact forms of practice that would help students conceive of future situations involving slope in terms of mathematical objects rather than physical objects.

Bonnie's practice also served to elaborate her beliefs about transfer. For instance, I inferred that Bonnie believed students would be able to generalize their understanding of a particular mathematical topic to a novel situation to the extent that they could relate the novel, mathematical situation to familiar aspects of real-world situations or experiences. This belief was inferred from Bonnie's comments like the one noted above wherein she stated that she wanted "to build on the idea that slope is something that they … can relate to" and that she therefore wanted to give students an image that "in the future, [they] could refer back to." The meaning of this statement became further elucidated during observations of her practice when she linked points on a graph and the act of graphing with locations and navigation. It appeared that the images she referred to

were really experiences that all students could relate to even if they had not experienced them personally. Walking around city blocks arranged in a grid is an experience that is easily accessible to students. Moreover, the role these images might play in guiding future mathematical activity seemed closely aligned with the invoked experience. In particular, navigation is used to help people get from location to location via a sequence of northern, southern, eastern, and western movements. Thus, when asked to graph in a future situation, one merely needs to remember coordinate pairs as locations or addresses and graphing as a way to get from one address to another address via a sequence of up, down, right, and left movements.

Limitations

While the pilot data presented here are suggestive of a relationship between teachers' beliefs about the generalization of students' learning, they are just that—suggestive. As the data were gathered from a study examining the relationship between teachers' practices and their students' subsequent reasoning on a transfer task, there were no questions posed to the teachers explicitly probing their beliefs about the transfer of students' learning or the reasons why particular forms of practice (e.g., renaming) were carried out during instruction. Thus, despite the fact we have data documenting these teachers' classroom practices, its relationship to these teachers' beliefs about transfer specifically is highly inferential. Consequently, a systematic investigation of the relations between various forms of teachers' practices and their beliefs was needed.

Research Question 3

This dissertation study examines the relationship between teachers' beliefs about the generalization of their students' learning and their classroom practices and therefore poses the following question:

What is the relationship between teachers' beliefs regarding the generalization of students' learning and teachers' classroom practices?

In other words, this question asks how teachers' beliefs about the generalization of learning manifest in the classroom and how teachers' beliefs are elaborated and constructed via the process of teaching a lesson. In the context of a lesson or set of lessons on slope and linear functions, I seek to understand the specific beliefs that result as teachers are given the chance to enact their beliefs about the generalization of students' learning of slope and how to support that generalization instructionally. For example, when coupled with her inferred beliefs, Joanne's actions in the classroom seemed to create a more complete picture of how she thought about the generalization of students' understandings associated with slope. In particular, the kinds of mathematical features she hoped she could support students in noticing and subsequently generalizing became clearer.

One may wonder how the term *practice* is being used here. As with *beliefs*, the term *practice* has been used in a variety of different ways by researchers in the field of education (Lampert, 2010). One way in which it has been used is to refer to a profession or occupation as a whole. Thus, the *practice* of teaching refers to a set of activities, beliefs, identities, and so on that are common to all or most teachers. Used in this way, practice refers to the general culture of teaching. For example, the practice of teaching

(in public schools in the U.S.) typically involves interacting with a particular group of students for a period of approximately 9 months about a specific content area or areas. As my interest here lies in identifying the beliefs and actions particular teachers associate with the generalization of students' learning, I will not be using this meaning of the term practice.

A second way in which the term practice has been used in the education literature is to refer to the repeated or continuous exercise of a skill or profession with the result or purpose of attaining proficiency. Thus, the *practice* of teaching could be associated with Kane, Rockoff, and Staiger's (2008) claim that teachers become more effective after experiencing 2 years of teaching, perhaps from practicing a set of skills or repeatedly exerting efforts to achieve a particular outcome. For example, teachers may become more effective at eliciting students' reasoning when they become more familiar with ways to phrase questions and design tasks so that students are provided with an impetus to explain and justify their reasoning. I will not be referencing this meaning of the term in my study because my interest is not in the process of attaining proficiency in supporting students' transfer of learning (although this is an interesting area for future research).

A third way the term practice has been used is to refer to the things that teachers do constantly or habitually. Here, the term can be used in the plural (i.e., practices) wherein the *practices* of teaching refer to teachers' regular rather than idiosyncratic behavior. For example, a teacher may usually ask students to explain and justify their solution strategies; thus, this action may be one of this teacher's many core practices. Since I only plan to observe teachers for at most 2 lessons, I will not assume to have a

clear picture of their habitual ways of teaching and will therefore not invoke this meaning of the term practice here.

A fourth way practice has been used in the literature on education is to simply refer to what a teacher *does* as distinct from what she thinks or knows. Here, the *practice* of teaching is a singular noun that refers to the enactment of teaching. For example, on Monday a teacher might distribute a task, ask students to complete it silently and independently at their desks, and then ask that students turn in their completed work as they leave for the day; on Tuesday the same teacher might distribute a task, ask one student to read the task aloud, ask another student to explain the task in her own terms, ask the class as a whole if there are any questions about the task, ask the class to work in small groups, and then, when everyone is ready, ask each group to present its work one at a time while the rest of the class is told to ask clarification questions as they emerge. Here, practice refers to the distinct ways in which this teacher performed the work of teaching despite the fact that there were inconsistencies across the two days of teaching. This is how I will be using the term practice in my study.

Significance

This section is organized into three parts. In the first part, I discuss the contribution this dissertation study makes to the ongoing research on the transfer of learning and teaching for transfer. In the second part, I discuss the potential of this study for providing insight into linkages that exist among teachers' beliefs, teachers' practices, and the transfer of students' learning. Finally, I discuss the potential contributions of this study to teachers' education and professional development.

The Ongoing Study of Transfer

Transfer, or the generalization of learning beyond the conditions of initial learning, is one of the longest standing research topics in psychology and education (Engle, Nguyen, & Mendelson, 2010). As such, there are many conceptualizations in the transfer literature of what the generalization of learning is and how it can be supported instructionally. Earlier in this chapter, I discussed a few of these ideas. For example, Locke believed transfer occurred at a general level and was the influence of the development of a particular cognitive ability on performance in domains that draw upon that ability; he thus believed teachers should help students to improve their general cognitive abilities by engaging students in mentally challenging activities. In contrast, Thorndike believed transfer occurred at a much more fine-grained level. Specifically, he believed transfer was the influence of training in one association on another and occurred to the extent that learning and transfer tasks share identical elements; he thus believed teachers should support students' in constructing associations between particular external stimuli and behavioral reactions by thoroughly training students how to perform individual skills. In Chapter 2, I present additional ideas from the transfer literature regarding the generalization of learning and associated ideas about how to teach for the generalization of learning.

While there are many ideas in the transfer literature about the generalization of learning, I could not find anything in this literature base regarding how teachers think about the transfer of their students' learning or how to support it. Investigating how teachers think about transfer is important for many reasons, two of which are discussed here. First, teachers are among the people vested with the responsibility of preparing

children for successful engagement in future situations (e.g., future courses and future careers). Teachers spend many hours a day engaging with students hoping that their interactions are in some way valuable to students' future performance in the world. Thus, it is important to know how teachers' beliefs about the generalization of learning as well as their beliefs about how to support it fit within the landscape of the transfer literature. This dissertation study provides the field with information about whether teachers' beliefs are consistent with those already present in the transfer literature or whether they are new.

Second, even when teachers' beliefs turned out to be consistent with beliefs that have been previously articulated by transfer researchers, this dissertation found that teachers' beliefs were infused with a level of detail that only those who are steeped in the context of the mathematics classroom environment can provide. For example, Kelly's belief that transfer is supported by teachers providing instruction for each skill echoes Thorndike's recommendations regarding teaching for transfer. However, when speaking about the particular topic of slope Kelly provided a level of detail not present in Thorndike's theory, namely that a teacher should enforce the particulars with which the skills are to be carried out. Specifically, Kelly appeared to believe that finding the slope of a line is a skill involving several sub-skills including the creation of a slope triangle. For Kelly, it was not sufficient for students to draw a slope triangle; rather, students need to be able to create a slope triangle in a certain manner thereby obtaining a slope triangle that sits *above* rather than *below* a line (see Figure 1.1).

Furthermore, Kelly's belief about the order in which students create their slope triangles seemed to be related to her beliefs regarding the specific kinds of activities

students would and would not be able to successfully engage in. In particular, the hypothetical student's (Molly's) failure to appropriately construct a slope triangle seemed to provide Kelly with evidence that Molly would have difficulty when confronted with future situations involving negative slope. In this way, teachers bring extant beliefs alive in nuanced ways.

Linkages among Teachers' Beliefs, Teachers' Practices, and Students' Transfer of Learning

Emerging research is establishing links between teachers' practices and their students' transfer of their learning experiences (Engle, 2006; Engle, Nguyen, & Mendelson, 2010; Lobato et al., in press). This dissertation study connects teachers' beliefs about the transfer of learning with their teaching practices; thus, contributing to a chain of linkages among teachers' beliefs, teachers' practices, and students' transfer of their learning experiences.

To introduce Research Question 3, I drew upon Lobato et al.'s (in press) study that highlighted relationships between two teachers' classroom practices and the transfer of their students' learning. For example, one teacher's acts of renaming served to keep students' attention focused on mathematical objects like quantities while another teacher's acts of renaming served to keep students' attention on physical objects. When confronted with a transfer task involving slope, the group of students in the first teacher's class reasoned with a quantitative relationship to correctly find and interpret the meaning of slope whereas the group of students in the second teachers' class ignored quantities, treated the graphical representation contained in the transfer task as if it were composed of "boxes," and were unable to find and interpret the meaning of slope. Thus, there

appeared to be conceptual connections between teachers' practices and the ways in which students' reasoned on a subsequent transfer task. This dissertation examines another relationship in a chain of linkages, namely the relationship between teachers' beliefs about the generalization of students' learning and teachers' practices. Future research may then focus on connecting the chain, thereby providing the field with specific insights about relationships that exist between teachers' beliefs, teachers' practices, and students' transfer of learning.

Teachers' Education and Professional Development

One could imagine organizing teacher education and professional development around teachers' beliefs regarding the generalization of students' learning, teachers' beliefs about how to support the generalization of students' learning, and teachers' classroom practices. Several researchers have noted the power of reflection in teacher education and professional development (e.g., Cooney, Shealy, & Arvold, 1998; Mewborn, 1999). Thus, teachers and prospective teachers may be encouraged to articulate and reflect on their beliefs associated with the generalization of students' learning and to attend to the ways in which various beliefs play out in the classroom and/or to test out various ideas regarding teaching actions in classroom situations. The instructor may then work with the teachers and prospective teachers in an effort to shape their beliefs. However, as the previous section pointed out, the field does not yet have the data needed to identify the beliefs that are the most effective in helping students to transfer their learning. This dissertation study thus attempts to obtain a baseline of teachers' currently-held beliefs so that we may one day be in a better position to make

conjectures regarding beliefs that appear the most effective in supporting the

generalization of students' learning

CHAPTER 2:

Literature Review

This chapter is organized into five main sections. In the first section, I review the literature pertaining to the transfer of learning, focusing on approaches to transfer that have articulated suggestions regarding the steps one might take to support the transfer of learning. In the second section, I draw upon the literature to elaborate the conceptualization of beliefs that will be used in this dissertation as well as to provide a rationale for the approach I will use to measure beliefs. In the third section, I review the literature base on the relationships between teachers' beliefs and teachers' practices, focusing on researchers' orientations to the issue of inconsistency/consistency. In the fourth section, I review the literature pertaining to teachers' mathematical knowledge, emphasizing the various ways in which researchers have conceptualized such knowledge. In the fifth section, I review literature on the mathematics relevant to this study. At the end of the chapter, I revisit my research questions in light of this literature review.

The Transfer of Learning

One of the primary goals of this dissertation was to probe teachers' beliefs about the generalization of students' learning (and their corresponding beliefs about how to support the generalization of students' learning). As stated in Chapter 1, the research on *transfer* constitutes one body of research that has studied the generalization of learning beyond the conditions of initial learning. Traditionally, transfer has been characterized as the application of knowledge acquired in one task or situation to another (Nokes, 2009). Over the past century, researchers have articulated numerous conceptualizations of transfer as well as its underlying mechanisms (e.g., Bassok & Holyoak, 1989; Beach,

1999; Bereiter, 1995; Bransford & Schwartz, 1999; Campione, Shapiro, & Brown, 1995; Engle, 2006; Gagne, 1968; Gick & Holyoak, 1983; Lobato et al., in press; Markman & Gentner, 2000; Nemirovsky, 2011; Singley & Anderson, 1989; Thorndike & Woodworth, 1901; Tuomi-Gröhn & Engeström, 2003; Wagner, 2010). A complete review of this literature is beyond the scope of this study. Instead, I focus on approaches to transfer that have emphasized ideas about the ways in which transfer might be supported. These approaches are organized into three main sections: historical approaches to transfer, mainstream cognitive approaches to transfer, and alternative approaches to transfer. In addition, I have included a brief section highlighting some of the critiques of the historical and mainstream cognitive approaches that ultimately inspired the conceptualization and development of some alternative approaches to transfer.

Historical Approaches to Transfer

E. L. Thorndike is often credited as the first to conduct systematic investigations of transfer (Singley & Anderson, 1989). Developed in response to a deep-felt dissatisfaction with the prevailing view of education at the time, namely the doctrine of formal discipline, his ideas came to dominate transfer research for the first two-thirds of the 20th century (Campione et al., 1995). During this time, voices of dissension could be heard offering views of transfer that differed in significant ways from those of Thorndike (e.g., Judd, 1908). However, no voice was as loud or as far reaching as Thorndike's. In this section, I present an overview of Locke's doctrine of formal discipline, Thorndike's (1906) theory of identical elements, and Judd's (1908) theory of general principles. Within each of these subsections, I include implications for instruction as well as some theoretical underpinnings.

Doctrine of formal discipline. Early psychological ideas about transfer were based on the doctrine of formal discipline, a notion that has been credited to Locke (Singley & Anderson, 1989). According to this doctrine, transfer was a general phenomenon that occurred via the strengthening of one's mental musculature. In particular, training in one muscle was thought to lead to improvement in one's ability to perform all tasks requiring the strengthened muscle (Cox, 1997; Singley & Anderson, 1989; Tuomi-Gröhn & Engeström, 2003). Thus, learners would be able to transfer learned skills from one situation to another to the degree that the two situations drew on the same set of mental muscles.

As explained in Chapter 1, educators ascribing to this view believed that difficult and taxing subjects like Latin were of significant value because learners' general abilities to memorize, concentrate, and reason would be strengthened as a consequence of studying the language (Bransford & Schwartz, 1999; Cox, 1997; Lyans, 1914; Singley & Anderson, 1989; Tuomi-Gröhn & Engeström, 2003). In this way, teachers were encouraged to choose intellectually challenging tasks and activities so that students were afforded opportunities to develop general mental abilities. For instance, when teaching Geometry, teachers would emphasize the development of students' proof-writing skills as a means to improve learners' abilities to think critically, reason, and argue. In turn, improvement in these abilities was thought to have effects reaching into other content domains, such as chess playing and computer programming.

The doctrine of formal discipline emerged out of an epistemological commitment to the faculty view of mind wherein the mind was thought to be composed of a set of general faculties, or abilities, such as the ability to observe, reason, discriminate,

concentrate, memorize, and make judgments (Bransford & Schwartz, 1999; Singley & Anderson, 1989; Tuomi-Gröhn & Engeström, 2003). The faculty view of mind thus assumed a fixed and unchanging set of general abilities wherein mental exercise resulted in stronger, not new or different, abilities.

Identical elements. In contrast to the emphasis on the transfer of general mental abilities posited by the doctrine of formal discipline, Thorndike argued that transfer was a phenomenon that occurred at a specific level (Bransford & Schwartz, 1999; Cox, 1997, Singley & Anderson, 1989). In particular, Thorndike argued that learners would be able to transfer their learning of a skill from an initial, learning situation to a novel, transfer situation to the extent that the two situations shared *identical elements* (Thorndike 1903, 1906, 1922, 1924; Thorndike & Woodworth, 1901). While Thorndike never clearly defined the meaning of identical elements, they have come to be interpreted as identical at the level of observable stimuli in the external, physical environment (Brown, 1990; Cox, 1997; Singley & Anderson, 1989; Lobato, 2012). For example, Thorndike and Woodworth (1901) examined the effects of training in estimating areas of a certain shape and size (e.g., rectangles ranging in size from $10 - 100 \text{ cm}^2$) on learners' performance while estimating areas similar in shape but different in size, areas similar in size but different in shape, and areas different in both shape and size. They found that the accuracy learners gained during training decreased with changes made to the data used during training (i.e., with changes in size and/or shape). Thus, they concluded that "no change in the data, however slight, is without effect" on learners' performance and that as the data become more and more unlike the data used during training "there is always a point where the loss [of the improvements gained during training] is complete;" in other

words, Thorndike believed that the "spread of practice occurs only where identical elements are concerned" (Thorndike & Woodworth, 1901, p. 92).

According to the theory of identical elements, training students and giving them sufficient time to practice were two important teacher responsibilities. For example, in one experiment Thorndike and Woodworth (1901) trained learners with a set of 60 rectangles of varying shapes within a particular size range and gave feedback after each of the 60 estimates. This process was repeated 20 times before any tests of transfer were conducted. In another experiment, learners were trained to estimate areas with a set of 102 rectangles and triangles ranging in area from 0.5 to 12.0 in^2. Again, learners were given feedback after every estimate and the process was repeated 32 – 41 times, or until a "certain amount of improvement had been made" (Thorndike & Woodworth, 1901, p. 252). In this way, training supplemented by extensive practice were crucial elements in Thorndike's theory about how to support transfer. If *any* transfer was going to occur, it was imperative that learners first acquire the requisite skills. In other words, the idea was that learners must first become reasonably skilled at estimating, for example, the area of rectangles ranging in size from 10 – 100 cm^2 before they could be expected to demonstrate any transfer to the skill of estimating the area of rectangles ranging in size from 140 – 200 cm^2. Additional teacher responsibilities entailed analyzing the curriculum into specific behaviors and then sequencing students' training and practice so that more basic skills came before the more advanced skills that utilize them (Tuomi-Gröhn & Engeström, 2003).

Thorndike's (1906) view of transfer was inspired by associationist views of learning. Under such views, knowledge is conceived of in terms of stimulus-response

relations. Specifically, knowledge is thought to manifest in the form of associations, or connections, between observable stimuli or events in the external environment and embodied behaviors articulated in reaction to those stimuli (Resnick & Ford, 1981). Associations between particular sets of external stimuli and behavioral responses are thought to form and subsequently grow stronger when given positive reinforcement (Resnick & Ford, 1981; Gagne, 1965). Thus, a learner will learn how to estimate the area of a particular sized rectangle if she is praised or otherwise rewarded every time her estimates show increased accuracy. In this way, the primary method of instruction for those ascribing to such views (e.g., Thorndike) became that of *drill and practice* wherein teachers' responsibilities included choosing sets of stimulus-response pairs and then providing students with enough practice and feedback so that when shown particular stimuli, students were enabled to give the desired responses (Resnick & Ford, 1981).

General principles. Judd (1908, 1936, 1939) disagreed with Thorndike's assertion that transfer was mediated by external, physical similarities and argued instead that transfer was mediated by meaningful principles. Specifically, Judd argued that transfer occurred to the extent that a learner was able to grasp and discern the common idea or "causal structure" underlying learning and transfer tasks (Campione et al., 1995, p. 37). Thus, in Judd's view transfer required mindfulness and did not occur as an effortless reflex or behavioral reaction to external stimuli (Tuomi-Gröhn & Engeström, 2003). In a well-known experiment conducted by Judd and Scholckow, Judd provided evidence for this idea (Judd, 1908). Fifth- and sixth-grade boys were asked to throw darts at a target located 12 in. under water, a skill that requires the thrower to consider how far the target's true location lies from where it appears to lie given a form of visual

deception caused by the refraction, or bending, of light rays as they enter the surface of the water. One group of boys received instruction surrounding the principle of refraction and the other group did not. After an initial adjustment period, both groups performed equally well while the target remained 12 in. under water. When the situation was changed so that the target was only under 4 in. of water, the group of boys that did not receive instruction became confused and made errors that were large and persistent. In contrast, the group who had received instruction adapted quickly to the new conditions. This continued when the situation was changed again so that the target was under 8 in. of water. Judd concluded that the more successful group of boys was able to generalize the newly acquired under-water-dart-throwing skills to various situations because all situations required an understanding of the same principle, namely the principle of refraction.

According to Judd's general-principles approach, the transfer of learning is supported by instruction that targets the principles underlying various learning situations. To illustrate this idea, consider the experiment discussed above wherein Thorndike studied the effects of training learners in estimating the area of rectangles on their performance while estimating the area of triangles. Rather than promoting a great deal of practice with a set of index cards with, for example, a figure on one side and the true area on the other until a certain amount of improvement is gained, Judd might look at the learning tasks and the transfer tasks as a whole and target an underlying principle or idea for instruction. For example, instruction might target the idea that decomposing and reorganizing a particular shape does not change its area. In this way, a learner may be supported in mentally reconfiguring triangles so that they resemble rectangles (see Figure

2.1a). In addition, learners may be supported in seeing the relationship between rectangles and triangles, namely that the area of a triangle is exactly one-half that of a rectangle (see Figure 2.1b). As a consequence of having learned about the conservation of area via decomposition and reorganization, learners may be able to reason about how to find the area of new shapes (e.g., parallelograms). In other words, Judd (1936) argued that the "goal of all education is the development of systems of *ideas* [emphasis added] which can be carried over from the situations in which they were acquired to other situations" (p. 201).
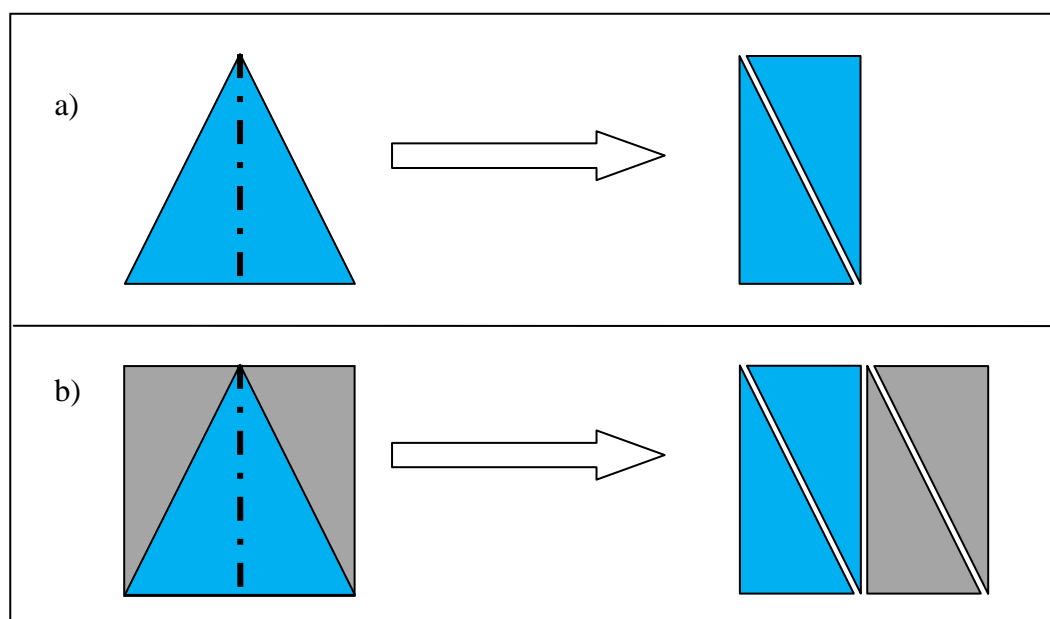
Figure 2.1: (a) Conservation of area achieved through decomposition and reorganization; (b) The relationship between the area of a triangle and the area of a rectangle.

Judd's views are consistent with later Gestaltist theories that stressed knowledge as more than the sum of the stimulus-response associations corresponding to a particular skill or strategy. Specifically, the Gestaltists argued that learners interpret information in the environment as organized or structured wholes in accordance with certain principles (as summarized in Resnick & Ford, 1981). To illustrate this idea, consider Wertheimer's

(1945/1959) famous parallelogram problem. Wertheimer reported that he visited a

classroom in which a teacher had given his students instruction regarding the canonical

method for finding the area of a parallelogram. Specifically, the teacher had told his

students that when presented with a parallelogram and asked to find its area, they should

drop perpendiculars and find the area of the associated rectangle (see Figure 2.2a). The

students were able to use this method to successfully complete a set of practice problems

by mimicking their teacher's demonstrated strategy. The next day, Wertheimer presented

the students with a transfer task, namely the novel parallelogram shown in green in

Figure 2.2b. Many of the students explained that they could not find the area of

Wertheimer's new parallelogram because they had not been taught to do so. Others,

however, incorrectly applied the strategy of dropping perpendiculars (see Figure 2.2b).
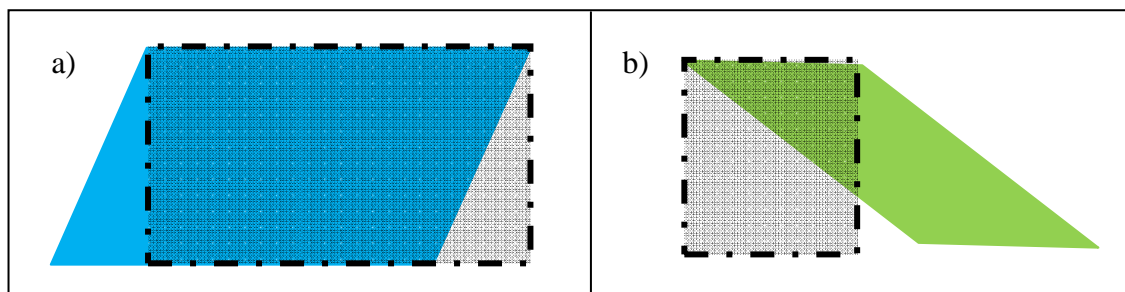
Figure 2.2: (a) How to find the area a parallelogram: Drop perpendiculars and find the
area of associated rectangle; (b) Wertheimer's parallelogram.

In this example, it may be argued that the students had not learned a meaningful

principle that could be used to sensibly organize the task environment with respect to its

goal (i.e., find the area of the parallelogram), as was illustrated by students' behaviors

when they were presented with a transfer task (Wertheimer's parallelogram). Rather, the

students had been taught a standard algorithm in a rote fashion which may have caused at

least some of the students to interpret the bottom of the *page* (as opposed to the bottom of

the parallelogram) as part of the task environment when engaging with the transfer task.

Wertheimer argued that students would not have made such mistakes had they

understood the relationship between the area of a triangle and the area of a parallelogram

(see Figure 2.3), an understanding that would have provided meaning to the algorithm as

well as the meaningful structure students needed to sensibly organize the novel-task

environment with respect to the mathematical features and properties of a parallelogram

(Resnick & Ford, 1981).
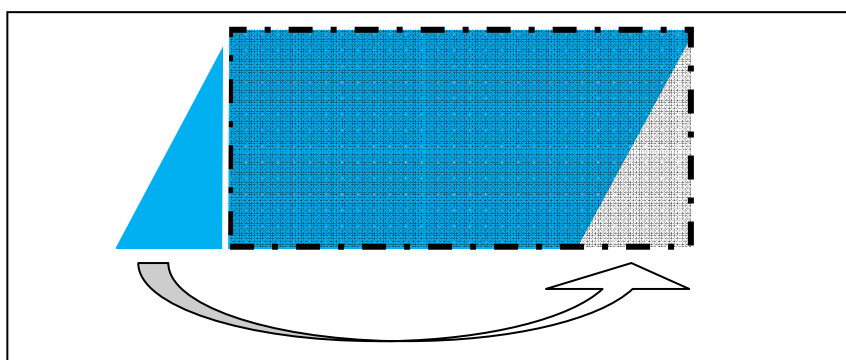
Figure 2.3: Relationship between the area of a parallelogram and the area of a triangle.

Mainstream Cognitive Approaches to Transfer

During the cognitive revolution, researchers began to view human cognition as

more complex than previous theories had suggested. They became increasingly

interested in the non-observable processes involved in learning and knowing (e.g., the

ways in which humans perceive, remember, plan, and reason). Consequently, a view

began to emerge wherein humans were thought to be more like active processors of

symbols than passive responders to external stimuli. During the latter half of the 20th

century, a new wave of *cognitive* approaches came to dominate transfer research. In this

section, I will discuss some of the features that these distinct approaches have in

common, associated implications for instruction, and shared theoretical commitments. I

will also provide specific details for two of these cognitive approaches. Because these

approaches continue to dominate the research on transfer, I have termed them

mainstream.

Mainstream cognitive approaches to transfer posit that transfer occurs to the

extent that the internal, mental representations learners create of initial learning and

transfer situations match, overlap, or can be related via a mapping (Anderson, Corbett,

Koedinger, & Pelletier, 1995; Fuchs et al., 2003; Gentner, Loewenstein, & Thompson,

2003; Gick & Holyoak, 1980, 1983; Novick, 1988; Reeves & Weisberg, 1994; Singley &

Anderson, 1989). Additionally, transfer is thought to be supported to the degree that

learners form sufficiently *abstract* symbolic representations, wherein abstraction is

thought of as a process of decontextualization (Fuchs et al, 2003; Reeves & Weisberg,

1994; Singley & Anderson, 1989). In other words, abstraction is achieved via the mental

deletion of details related to "irrelevant" and "superficial" contextual features of task

situations (i.e., physical aspects like the relative quickness with which one's feet move

when forming the abstraction of "motion through space") (Bassok & Holyoak, 1989).

These abstract representations are said to "avoid contextual specificity so they can be

applied to other instances or across situations" (Fuchs et al., 2003, p. 294).

As a result, researchers operating within the mainstream cognitive paradigm have

forwarded a host of teaching recommendations, three of which will be discussed briefly

here. First, some of these researchers recommend that during initial learning activities,

teachers use several problems all sharing the same underlying *structural* feature (i.e., a

mathematical strategy or principle) and provide students with sufficient time to practice

solving those problems (e.g., Anderson, Kline, & Beasley 1979; Brown, Kane, & Echols,

1986; Carbonell, 1983; Michalski, 1983; Singley & Anderson, 1989). This idea emerges from the assumption that "when several problems similar in solution principle are presented, the cognitive system *automatically* [emphasis added] tabulates the degree of overlap among them and stores the composite of overlapping features as a separate problem representation" (Reeves & Weisberg, 1994, p. 383). These automatically created and stored representations are thought to be more abstract as a function of the number of problems teachers use and students work through. Second, researchers recommend that learners explicitly provide a description or summary of the mathematical strategy or principle underlying an example problem (e.g., Gick & Holyoak, 1980, 1983). This teacher move is thought to support learners in focusing on and subsequently extracting the underlying structural features of the exemplar (rather than the superficial contextual features); thus, this move is thought to support learners in forming increasingly abstract mental representations. Third, researchers recommend using two or more example problems in conjunction with explicit comparisons across those exemplars (e.g., Cantrambone & Holyoak, 1989; Gentner et al., 2003; Markman & Gentner, 2000; Reeves & Weisberg, 1990). In particular, learners are supposed to talk about the *similarities* between exemplars so that learners are assisted in developing representations of the relationships that exist between various features of the exemplars. These relationships are thought to be useful in the construction of mappings between mental representations of learning and transfer situations.

There are several different strands of research within the mainstream cognitive approach to transfer. One prominent perspective is that of identical cognitive elements, perhaps best represented in Singley and Anderson's (1989) work. They explain their

formulation of transfer "is in fact a modern version of Thorndike's theory of identical elements" wherein the modern spin is provided by a cognitive reinterpretation of Thorndike's "elements" (p. 32). Whereas the "elements" of primary concern in Thorndike's theory were generally taken to be features of the external, physical task environment, in Singley and Anderson's reinterpretation they are defined in terms of internal, symbolic-mental structures. Singley and Anderson (1989) assert that these symbolic-mental structures take the form of extensive and integrated networks of *production rules* that associate mental representations of, for instance, task situations in working memory to symbolic operations via condition-action statements (Anderson, 1996). For example, they posit that a piece of declarative knowledge (i.e., factual information) such as "the formula for slope is given by the difference in *y*-values divided by the corresponding difference in *x*-values" can be converted into the following production rule for calculating slope:

IF   the goal is to calculate slope

       and the difference in *y*-values is Δy

       and the difference in *x*-values is Δx

THEN   calculate $\Delta y \div \Delta x$.

According to Singley and Anderson, transfer is then said to occur to the degree that initial learning tasks and novel transfer tasks trigger the same sets of production rules. Thus, Singley and Anderson (1989) posit that with sufficient practice across a set of structurally similar problems, transfer from learning tasks requiring a particular set of production rules to novel situations requiring the same set of production rules is more or less "automatic" (p. 33).

A second influential theory is Gentner and colleagues' structure-mapping approach wherein they argue that transfer involves a *mapping* between mental representations of *relations* rather than, for example, a triggering and subsequent *application* of a learned *production rule* (Gentner et al., 2003; Gentner & Medina, 1998; Gentner & Namy, 1999; Markman & Gentner, 2000). In their structure-mapping theory of transfer, Gentner and colleagues assert that learners' internal mental representations of situations are highly organized propositional networks consisting of entities (e.g., a person or a building), attributes (i.e., descriptive information—tall[person]), and *relations* among them (i.e., associations between two or more entities or attributes—enter[person, building] or cause[fall(water, person), enter(person, building)]). Such relations are thought important because of their potential to carry information about the valuable associations of a domain such as causal relationships and implications.

Markman and Gentner (2000) conducted an experiment to show that when making comparisons across situations, learners actually draw upon representations that preserve relationships across elements and attributes rather than representations consisting of these isolated features. To illustrate using the representational language of the structure-mapping approach, consider judging the relative steepness of two ramps. These researchers posit that learners are likely to draw upon representations involving relations between the height and the length of the ramps such as GREATER[$ratio_2(height_2, length_2)$, $ratio_1(height_1, length_1)$] or GREATER[$m_2(rise_2, run_2)$, $m_1(rise_1, run_1)$] rather than representations involving isolated features of the task situation such as goal, steepness of ramps, first ramp, wooden, inclined plane, length, 3 feet, height, 2 feet, second ramp, steel, inclined plane, length, 4 feet, height, 3 feet. According

to Gentner and her colleagues, transfer from learning to transfer situations is thus supported to the degree that the system of relations a learner acquires during an initial learning situation correspond to the relations the learner perceives in a novel task situation.

In sum, researchers operating within the mainstream cognitive approach to transfer draw upon cognitive structures, namely the relationships between learners' mental representations of initial learning and transfer situations, to provide explanations of transfer. These perspectives on transfer share a commitment to a cognitive architecture that consists primarily of long-term and working memory stores wherein representations (i.e., mental-symbolic structures) interact to encode, process, and store information about the external, physical world (Anderson, 1996; Bruer, 2001; Rittle-Johnson, Siegler, & Alibali, 2001). The encoding of information about the physical world is thought to occur via relatively transparent perceptual processes; thus, under this view, learners' "cognitive processes entail operations on mental representations, which are internal mental structures that correspond to the structure of a segment of the world" (English & Halford, 1995, p. 21).

Critiques

As noted in Chapter 1, Jean Lave's (1988) critique of the culture of mainstream cognitive approaches to transfer highlighted a period of dissatisfaction with transfer research. As a result, some researchers decided to abandon investigations of the phenomenon (e.g., Carraher & Schliemann, 2002; Hammer, Elby, Scherr, & Redish, 2005) while others chose instead to reconceive of the phenomenon, its underlying mechanisms, and ways in which it could be supported (e.g. Beach, 1999; Bereiter, 1995;

Bransford & Schwartz, 1999; Engle, 2006; Lobato, 2006; Nemirovsky, 2011; Tuomi-Gröhn & Engeström, 2003; Wagner, 2010). While there have been many critiques of mainstream cognitive approaches to transfer, only three will be discussed here.

First, mainstream cognitive approaches to transfer have been criticized for privileging the observer's point of view over the learner's when determining *what* transfers (Bransford & Schwartz, 1999; Lave, 1988; Lobato, 2008a; Lobato et al., in press; Lobato & Siebert, 2002). This can be seen when experimenters use expert models of task performance to define, a priori, what counts as evidence of transfer (Lobato, 2012). For example, in one experiment Gentner et al. (2003) showed participants two pieces of written material each exemplifying the concept of negotiation. They then presented participants with a novel task "structured such that three solutions could be formed" (Gentner et al., 2003, p. 396). Participants' work on the transfer task was then scored as to *whether* it showed evidence of one or more of the three pre-determined solutions. In fact, the participants may have made connections between the initial learning and transfer situations that were not captured by any of the available solutions. According to Lave (1988), such methods do not capture the processes that actually occur as learners "naturally bring their knowledge to bear on novel problems" (Lave, 1988). Lobato and colleagues echo this sentiment by arguing that it does not make sense to privilege expert performance since it is well known that novices do not make the same connections that experts do (Lobato, 2008a; Lobato et al., in press; Lobato & Siebert, 2002).

Second, mainstream cognitive approaches to transfer have been criticized for the way in which they view the contexts of initial learning and novel transfer tasks (Lobato,

2008a). As described above, mainstream approaches tend to view learners' mental

representations of task situations as occurring through relatively transparent perceptual

processes, and thus they assume a close correspondence between learners' mental

representations and the task as it appears in the real world. In this way, problem

representations as well as learners' mental representations are treated as unproblematic,

as if production rules or relational networks can be perceived directly (Wagner, 2010). In

the example above, Gentner and colleagues claimed to have "structured" the novel task

"*such that* three solutions could be formed," thus, unifying external task representations

with internal, mental representations. However, transfer researchers who, for example,

draw upon constructivist notions of learning criticize such views for not accounting for

the *interpretive* nature of knowing (Lobato, 2012; Wagner, 2010). For instance,

constructivist approaches to transfer that draw ideas from Piaget (1937/1954) and/or von

Glasersfeld (1990, 1995a, 1995b) hold that what a learner "sees" in any situation is

actively constructed by the learner rather than *passively received* and that learners

develop increasingly *viable* cognitive structures rather than increasingly *veridical* mental

representations of an outside world. Consequently, learners may bring many different

interpretational structures to bear when asked to make sense of the same situation.

Alternately, learners may construct a myriad of interpretational structures when engaging

with the same situation. For example, Lobato et al. (in press) found that the ways in

which students interpreted and subsequently solved a novel slope task were influenced by

the specific kinds of instructional experiences that preceded their engagement with the

slope task.

Third, mainstream approaches have been criticized for insufficiently accounting for the contribution of sociocultural practices, discursive interactions with other people, and material artifacts in the generalization of learning (Bereiter, 1995; Beach, 1999; Engle, 2006; Lave, 1988; Lobato, 2008a; Lobato et al., in press; Packer, 2001; Pea, 1989). In particular, Lave (1988) noted that the "applying knowledge" metaphor mainstream approaches tend to employ suggest a view of knowledge as separable from the environments in which it emerges, develops, and/or is used. Thus, Lave highlighted the tendency of mainstream cognitive transfer researchers to operationalize the notion of setting or situation in terms of problem content and to remain relatively silent about the encompassing socio-cultural settings in which learners' problem-solving activities takes place. However, researchers who draw upon ideas from situated cognition, for example, argue that particular ways of knowing emerge, develop, and are used as a function of activity, history, culture, and social interactions (e.g., Bereiter, 1995; Beach, 1999; Engle, 2006; Lobato, 2006; Tuomi-Gröhn & Engeström, 2003).

Alternative Approaches to Transfer

Several alternative conceptualizations of transfer have been advanced by transfer researchers. In this section, I present alternative perspectives that (a) speak specifically to the critiques discussed above and (b) emphasize ways in which transfer may be supported instructionally. Thus, I present Bransford and Schwartz's (1999) *preparation for future learning* approach, Lobato's (2006) *actor-oriented transfer* approach, Engle's (2006) *framing* approach, and Bereiter's (1995) *dispositional* approach.

Preparation for future learning. Bransford and Schwartz (1999) point out that historical and mainstream cognitive approaches to transfer have utilized what they call

sequestered problem solving, wherein learners are kept isolated from additional resources such as textbooks or colleagues during investigations of transfer and, as a result, prevented from demonstrating their abilities to learn in novel situations by seeking out assistance from such resources. They additionally point to the tendency of these approaches to emphasize transfer as the ability to *directly apply* something that has been previously learned to a new situation. In contrast, Bransford and Schwartz (1999) place an emphasis on the differential preparation learners have to learn in new situations given the nature of their previous learning environments (see also Schwartz, Bransford, & Sears, 2005; Schwartz & Martin, 2004; Schwartz, Sears, & Chang, 2007; Schwartz, Varma, & Martin, 2008). Thus, in their *preparation for future learning* (PFL) approach to transfer, Bransford and Schwartz measure the effectiveness of initial learning situations by the degree to which the learning situations have primed or readied learners to intellectually benefit from other informationally-rich situations.

For example, Bransford and Schwartz describe the results of data collected by Singley and Anderson (1989) using a PFL perspective. The driving question was "How does experience learning one text editor affect learners' abilities to learn a second text editor?" Thus, the focus was on learners' abilities to learn a second set of skills as a function of their previous learning experiences. Data indicated that the effects of learners' experiences with the first text editor were not immediately apparent. In other words, learners' did not directly apply the products of previous learning when confronted with the new text editor. Rather, data collected during the second day of learners' engagement with the second text editor revealed much more evidence of transfer than

data collected during the first, suggesting that benefits of learning experiences may not be noticeable until learners are provided opportunities to learn something new.

Consequently, Schwartz and colleagues argue that teachers should provide students with opportunities that will prepare them to learn from future instruction (Bransford & Schwartz, 1999; Schwartz et al., 2005; Schwartz, Chase, Oppezzo, & Chin, in press; Schwartz & Martin, 2004; Schwartz et al., 2007; Schwartz et al., 2008). One way in which teachers can do this is to begin an instructional unit with an activity wherein students are asked to compare or analyze contrasting cases and provide explanations for their comparisons. For example, before teaching students about slope, teachers may ask their students analyze two tables of data (see Figure 2.4) each representing the amount of soda being bottled over time at two competing companies to determine whether the companies are bottling soda at a constant fastness over time or whether they are speeding up production. When considering Company #1 alone, students may reason univariately with the quantity "total amount of soda," notice that the differences between the accumulated quantities are equal (i.e., 2.5 truckloads), and conclude that the company is bottling soda equally fast over time. However, when considering both companies together, students may attend to the fact that data are collected every day for Company #1 but at irregular time intervals for Company #2. This comparison may make time more prominent for students and prepare them for instruction on covariation, ratio, and slope. Alternately, teachers may ask students to generate their own ideas about phenomena. For instance, rather than presenting them with the tables of data below, teachers may ask their students to come up with ideas regarding how to measure the "fastness" with which a company bottles soda. In this case, students may

attend primarily to the quantity of time suggesting that one find out how long it takes for

a bottle to fill with soda or a truck to be loaded with bottles. In response, teachers may

ask students to test their ideas, again providing students with an opportunity to appreciate

critical features of upcoming instruction.

Company #1		Company #2	
Time (in days)	Total Amount of Soda (in truckloads)	Time (in days)	Total Amount of Soda (in truckloads)
1	2.5	1	5
2	5	3	15
3	7.5	4	20
4	10	7	35

Figure 2.4: An example of contrasting cases.

Subsequent to these preparation activities, teachers implementing a PFL

perspective should either present students with a lecture describing the content students

are supposed to learn or present them with a worked example illustrating a particular

strategy or principle (Schwartz & Martin, 2004). For instance, teachers may follow up on

the activities described above involving the fastness with which two companies are

bottling their respective brands of soda by telling students the slope formula and

explaining the meaning of its various parts. Alternately, teachers may demonstrate how

to solve the question about whether the companies were bottling soda equally fast over

time or whether they were speeding up production by using the slope formula. The

degree to which the preparation activities succeeded in readying students to learn the

slope formula may then be assessed with respect to students' performance on a novel, transfer task involving, for example, slope set in a salary context.

The PFL approach to transfer draws upon Broudy's (1977) construct of *knowing with* (Bransford & Schwartz, 1999; Schwartz et al, 2005). Knowing with is the idea that learners' perceptions, conceptualizations, and judgments in and of novel situations are shaped by their prior experiences. Thus, learners are thought to associate and interpret aspects of new situations in terms of their past experiences. For instance, the students in the example above may have associated time with measures of "fastness" from prior experiences involving races wherein the winner is the person who finishes the race in the shortest amount of time; therefore, when the students saw or heard the word "fast," they may have immediately thought about time. Therefore, the PFL approach to transfer focuses on the kinds of experiences that are transferred "in" to various learning situations and ways in which teachers can influence (e.g., via the use of contrasting cases) what students transfer in.

Moreover, PFL draws upon ideas from perceptual learning theories (Bransford & Schwartz, 1999). In particular, it draws upon Garner's (1974) idea that a single stimulus is perceived, interpreted, and judged in relation to a range of alternatives and that as this occurs particular features of the stimulus become more salient. For example, when asked to describe the image on the left in Figure 2.4, one may describe it as a two-columned table of data. However, when asked to describe the same image in relation to the image on the right of Figure 2.4, specific features may become foregrounded (e.g., the scaling of the first column and the differences between values in the second column). Thus, PFL

offers the use of contrasting cases as a pedagogical tool to support students' noticing of specific features of upcoming instruction.

Actor-oriented transfer. Lobato (2003, 2006, 2008a, 2008b, 2012) points out that the majority of both historical and mainstream approaches to transfer privilege the observer's point of view over the actor's (i.e., learner's) during investigations of transfer. Specifically, she points to the fact that studies involving such approaches generally *pre*-determine the fact, strategy, principle, or heuristic that needs to be demonstrated by the learner in order for a learner's work on a novel task to count as transfer. In contrast, in her *actor-oriented transfer* (AOT) perspective, Lobato characterizes transfer as *any* generalization of learning experiences which may also be understood in terms of the influence of learners' prior activities on their activity in novel situations. What this means is that educators operating within this perspective seek to understand the ways in which learners have generalized their learning even when this generalization leads to incorrect or non-normative mathematical behavior from the observer's point of view (Cui, Rebello, & Bennet, 2006; Hannula & Lehtinen, 2004, 2005; Hohensee, 2011; Lehtinen & Hannula, 2006; Lobato et al., in press; Rebello, Cui, Bennett, Zollman, & Ozimek, 2007).

For example, in a study discussed by Lobato (2008a), students appeared to recognize a novel slope task involving the steepness of a playground slide as necessitating the "rise over run" formula learned in a past classroom situation wherein instruction was dominated by students' measurements of the "risers" and "treads" of physical staircases. The students appeared to make decisions regarding candidates for the "rise" and the "run" of the slope formula based on the fact that they noticed physical

objects like "stair steps" during instruction rather than mathematical objects like the ratio of the amount of the change in the riser relative to the amount of change in the tread. Therefore, some of the students appeared to generalize their learning from the classroom situation to the novel situation via the creation of a "stair step" connection. As a result, these students measured the steps contained in the ladder portion of the slide rather than the steps one might imagine transposing over the steep or "slanty" portion of the slide.

Consequently, the AOT perspective suggests that teachers attend to the nature of students' mathematical noticing during instruction. If, for example, the teacher of the students above were to realize that the students interpreted the meaning of the mathematical topic of slope in terms of attributes of physical objects (such as the slantiness of ramps or the constraint that the size of the risers and treads on any given staircase do not change), then she may be motivated to attempt to shift the locus of the students' attention to a mathematical relationship between the heights and lengths of such objects or to ratio relationships in other situations. Specifically, the AOT perspective posits that discursive practices, mathematical tasks, as well as the nature of mathematical activities contribute to the nature of students' noticing (Lobato et al., in press). Thus, to help her students attend to mathematical rather than physical objects, the teacher above may choose to foster particular forms of discursive practices over others (e.g., verbal interactions and diagrams that emphasize the *measureable* attributes of situations rather than the *physical* attributes of situations). In addition, she may choose to use slope tasks that do not involve physical objects such as tasks involving speed or taste. Lastly, she may make decisions about the nature of the mathematical activity (i.e., the participatory organization that gives rise to particular roles and expectations that informs students' and

teachers' actions) based on how she wants to influence the number and nature of the properties, features, and mathematical objects students notice during instruction.

As noted in Chapter 1, the AOT perspective draws upon ideas from Piagetian constructivism and situated cognition (Lobato, 2003, 2012). Constructivist views of learning hold that learners actively produce knowledge via interpretive engagement with the experiential world and that such engagement serves the development of increasingly viable cognitive structures, or ways of interpreting their interactions with the world that lead to increasingly successful completions of goal-oriented activity as well as to evermore coherent conceptualizations (Piaget 1947/2002; see also von Glasersfeld, 1990, 1995a, 1995b). Due to this emphasis on the interpretive nature of knowing, Lobato argues that rather than using expert models of performance to predetermine what counts as transfer, one should honor the myriad of potentially idiosyncratic ways in which students conceive of similarities across situations. Therefore, she argues that educators attempt to make sense of the connections *learners* make between learning and transfer situations.

Furthermore, the AOT perspective embraces situated perspectives on learning and knowing. Such perspectives tend to focus on the individual as a participant within an interactive system (i.e., situation) and thus view knowledge as inextricable from the situations in which it arises. More specifically, *learning* is seen as a continuous process resulting from one's socially and contextually embedded activity wherein the ultimate goal is appropriating particular forms of participation (Brown, Collins, & Duguid, 1989; Greeno, 1997). In this way, knowledge is said to include, or index, aspects of the situations in which it is learned (Boaler, 1998; Borko, Peressini, Romagnano, & Knuth,

2000; Brown et al., 1989; Carraher, Carraher, & Schliemann, 1985; Greeno, 1997).
Thus, Lobato (2012) argues that when making sense of the nature of the understandings
students transfer from learning to transfer situations, educators scrutinize various aspects
of the learning situation including the nature of the discursive practices, mathematical
tasks, and mathematical activity students are asked to engage in.

Framing. Engle's (2006) ideas about how to support students in generalizing
their learning grew out of a pragmatic concern that mainstream cognitive approaches to
investigations of transfer could not be used to explain why and how learners *choose* to
make use of their learning in novel situations. For example, one may know the slope
formula and know that it is applicable in a speed context, but choose not to make use of
such knowledge (e.g., for fear that another person may not think it appropriate or be
looking for something else). Thus, when conceptualizing supports for transfer, Engle and
her colleagues emphasize the ways in which teachers might influence students' decisions
to make use of relevant understandings; in particular, they posit that teachers can support
the transfer of their students' learning by *framing* initial learning situations as relevant in
and related to other situations occurring across spans of time, groups of people, physical
locations, and topic areas (Engle, 2006; Engle, Nguyen, & Mendelson, 2010).

Engle and her colleagues thus draw upon Bateson's (1972) notion of a *frame*[1]—an
explanatory principle used to interpret the meaning of the activities being engaged in—to
define *framing* as the communicative processes that give rise to a particular frame. For
example, consider a professional "meeting" taking place in a conference room. Inside of

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[1] See also Goffman, 1974; 1981; Tannen, 1993; and Tannen & Wallet, 1987 for elaborations and
expansions on the psychological concept of a *frame*.

this particular location and amongst this set of people, several frames and acts of framing are likely to be present. For instance, prior to the "meeting," the group may engage in an activity commonly referred to as "small talk." During this activity, it may be appropriate for any participant (e.g., an assistant professor) to speak to any other participant (e.g., a dean) about a range of topics. Eventually, one participant is likely to engage in an act of framing wherein there may be movement towards the front of the room and loud verbal articulations calling for attention. In response to this framing move, the other participants may stop talking, find seats, and begin to engage in a "listening" activity. During the "listening" activity, different rules of engagement are likely to be in effect—only one speaker talks at a time, talk seems to be centralized around a singular topic, and the nature of the talk seems more formal, for example, lexical items specific to the profession begin to be used with higher frequencies. Here, we see two frames—"small talk" and "listening." Forms of interaction appropriate in the former may not be appropriate in the latter and vice versa. In addition, we see an act of framing wherein a singular participant conveyed communicative messages (both verbally and non-verbally) to the others that facilitated the change of frames.

Engle and colleagues draw on these notions of frame and framing when suggesting that students are more likely to transfer their learning beyond the conditions of initial learning if the boundaries of learning situations are framed as *expansive* (as opposed to bounded: Engle, 2006; Engle et al., 2010). Consequently, they propose that learning activities be framed (a) temporally as part of an ongoing activity rather than as an isolated event, (b) locally as relevant to other places (e.g., the encompassing community) rather than as relevant to only the immediate learning environment (e.g.,

classroom), (c) participatorally as involving a larger community as opposed to just the teacher and student, (d) topically as related to other classes or curriculum units and as part of a larger content area rather than as an isolated idea, and (e) biographically as emerging out of learners' reasoning rather than out of textbook writers' ideas. Specific framing moves can be used to accomplish such framing. For example, referring to past and future activities and using present progressive verbs (e.g., "you're continuing to think about the speed of the rabbit") can help establish a temporal frame wherein students interpret their emerging understandings as connected to and relevant in past and future events. Similarly, asking students to explain their ideas to parents and classroom visitors can help establish a participatory frame wherein students interpret their developing understandings as appropriate content to be shared and discussed with people outside of the classroom. Asking students to share their own ideas (rather than the textbook's) and subsequently crediting students with the authorship of those ideas can also help establish a biographic frame wherein students interpret their role in situations to be that of a sense-maker and problem-solver. These and other framing moves that serve to expand the boundaries of the initial learning situation are posited to affect the types of future situations students orient to as being appropriate and relevant sites for making use of what they have learned.

Engle (2006) draws upon ideas from situated cognition to offer an explanatory account of how and why learners choose to make use of their learning in novel situations. As noted above, situated accounts of learning tend to focus on the individual as a part of a larger system or situation and on learning a continuous process emerging out of the individual's socially embedded activities; thus, as noted, the goal is appropriating

particularized forms of participation (Brown et al., 1989; Greeno, 1997). Therefore, Engle argues that if the activities learners engage in are socially construed as bounded in terms of time, place, participants, topic, and biography, the appropriated forms of participation may be as well. In contrast, she argues that if learners' activities are socially construed as related to, linked with, and/or part of other times, places, etc., then learners may be more likely to enact appropriated forms of participation outside initial learning environments.

Disposition. According to Bereiter (1995), previous approaches to transfer have emphasized the transfer of facts, strategies, principles, and/or heuristics, and have ignored or backgrounded the transfer of dispositions. In contrast, Bereiter foregrounds the issue of disposition and argues that when *teaching for transfer*, disposition is of primary concern for the success or failure of a lesson depends on whether or not students are supported in transferring their learning of a particular disposition. *Disposition* is used to refer to "some way of approaching things" and the *transfer of disposition* is used to refer to the idea that a particular inclination or way of approaching things carries over into novel situations (Bereiter, 1995, p. 23).

As an example of the transfer of dispositions, consider Boaler's (1998) study of the ways in which the particular inclinations students acquired during instruction were related to the ways in which the students approached novel tasks. She found that students from an instructional environment in which activity consisted of introducing a mathematical procedure or technique and then presenting a range of problems for which students could practice the technique tended to develop a cue-based approach to mathematical problems. For instance, these students explained that they could "get a

trigger, when [the teacher] says like *simultaneous equations* and *graphs*, *graphically* … it pushes that trigger, tells you what to do" (Boaler, 1998, p. 56). Thus, a large portion of these students attempted inappropriately to use equations from trigonometry when they were presented with a novel task that included a picture of a house and asked whether the "angle" of the roof exceeded 70°. In contrast, students from a project-based instructional environment tended to develop a sense-making approach to mathematical problems. For instance, these students explained that they "had to think for [themselves] there and work things out" and that when "presented with "stuff [they] haven't actually done before, [they] try to make as much sense of it as [they] can and try to understand it as best as [they] can" (Boaler, 1998, p. 57). Thus, when presented with the angle task, a large portion of the students drew upon their prior experiences with 90° angles to reason that the presented image represented a 45° angle.

In this way, Bereiter (1995) argues that teachers should focus on the development of students' dispositions and think about the kinds of inclinations, or ways of approaching things, that need to be fostered to support students' successful engagement with novel tasks. This means that teachers should create situations in which desirable forms of thinking and approaching problems take place. For example, when aiming to develop a disposition toward sense-making while teaching arithmetic, teachers may decide to set problems in contexts that allow for students to make sense of the meaning of their actions. Furthermore, teachers may ask students to solve a problem one day that differs in structure from the problem given the previous day so that students' sense-making efforts are supported over time (e.g., one day a problem asks for the total number of

students on the playground and the previous day the given problem asked students to find how many students were sent to the playground).

Additionally, Bereiter (1995) argues that teachers should work towards supporting students in seeing various instances as necessitating the dispositions that have been developed. For example, students should be supported in seeing novel situations as instances requiring sense-making. If, for instance, another teacher merely asks her students to replicate strategies, the student who has developed a disposition towards sense-making must still attend to and act upon his inclination to understand the meaning of his actions despite the absence of the external impetus. Without the kind of external support to continue to nurture this disposition, Bereiter suggests that the learned disposition may not survive. Therefore, the student should go about creating the supportive situation himself. This may mean that the student assembles a study group in which he and the other students engage in sense-making or that the student asks to be switched to another classroom in which he knows sense-making is valued. Therefore, teachers must progressively work towards developing students' abilities to nurture their own productive dispositions.

Like Engle's (2006) and Lobato's (2006) perspectives, Bereiter's view of transfer is inspired by ideas from situated cognition wherein learning is viewed as an ongoing process resulting from socially and contextually embedded activities and learners are said to appropriate ways of thinking and acting that are particularized to specific communities of practice (Brown et al., 1989; Greeno, 1997). Thus, he argues that teachers should strive to develop dispositions that are necessary and/or useful outside of the classroom community. However, unlike Engle who focused on supporting students to enact

appropriated forms of participation *across* classroom and real-world situations, Bereiter focuses on supporting students in seeking out, selecting, and creating situations that complement their appropriated dispositions; therefore, suggesting a transfer of the situation itself wherein productive dispositions are viewed as reappearing as students work to recreate desired situations.

In sum, there have been many distinct approaches to transfer, each suggesting different ideas about how to support the generalization of students' learning. Many of these approaches have been influential in our educational system. For example, Thorndike's emphasis on practice can be seen in the wide-spread appropriation of drill-and-practice methods of instruction by educators across the country. Similarly, Anderson's cognitive tutors have been implemented in mathematics classrooms. Thus, it was important to be aware of these views about the transfer of student learning as it was possible that one or more of them had been adopted by the teachers in the present study. Also, being aware of the range of these ideas helped to illuminate various aspects of the teacher participants' beliefs in this dissertation study, either because they were consistent with the ideas described above or because they brought additional insights and nuances to the discussion of transfer.

Teachers' Beliefs

This dissertation examines teachers' beliefs about the generalization of students' learning and how to support it. Such an examination depends on many factors including the way in which beliefs are conceptualized and the way in which beliefs are measured. In this section, I elaborate on the definition of beliefs that was put forth in Chapter 1. I also discuss the main approaches researchers have taken to the study of teachers' beliefs

with the goal of providing a rationale for the approach taken here. Thus, this section is separated into two sections: conceptualizing beliefs and measuring beliefs.

Conceptualizing Beliefs

As noted in Chapter 1, beliefs have been defined and used in many different ways by researchers. While some researchers view beliefs as indistinguishable from knowledge (e.g., Beswick, 2011, 2012), I have found it helpful to distinguish between the two for the purposes of this dissertation. Thus, as previously noted, I draw upon Philipp's (2007) definitions for beliefs and knowledge. In particular, a conception (i.e., a general notion or view) is said to be a *belief* for someone if he or she can respect as intelligent and reasonable a view which differs and/or contradicts his or her own. In contrast, a conception is said to be *knowledge* for someone if he or she cannot respect as intelligent and reasonable a view which differs and/or contradicts his or her own. However, I use these definitions from the point of view of the researcher. Thus, a conception is a belief for someone if *I* (the researcher) can respect the view as intelligent and reasonable even when it differs from and/or contradicts my own. Similarly, a conception is knowledge for someone if *I* cannot. Before I describe my pragmatic and theoretical purposes for making this change to Philipp's definition, I would like to use these definitions to illustrate the above distinction between beliefs and knowledge. To do this, I will draw upon the espoused and inferred beliefs and knowledge of one of the teachers described in Chapter 1.

Many ideas were espoused in and inferred from Bonnie's interview. Some of these ideas were categorized as beliefs and others as mathematical understandings or knowledge. In what follows, I will summarize three of Bonnie's ideas that were

identified in her interview. I will then use the criteria of "intelligent and reasonable" provided by Philipp's (2007) definition to explain how ideas having to do with the meaning of mathematical topics were categorized as knowledge while ideas having to do with transfer were categorized as beliefs.

First, I provided evidence in Chapter 1 suggesting Bonnie thought students would be able to generalize their learning to novel situations to the extent that a novel situation reminded them of familiar aspects of their real world experiences. Second, I provided evidence that Bonnie espoused a view about how to support the generalization of students' learning, namely, by drawing upon students' real-world experiences to create mnemonics, analogies, or images that could be used to guide their future mathematical performance. Third, data from Bonnie's interview provided evidence that she thought about the mathematical topic of slope as something more than a procedure for placing one number over the other. Specifically, Bonnie looked past an incorrect answer provided by Angie, a hypothetical student, to make sense of the nature of the student's accompanying explanation. Data from Bonnie's comments about the student's explanation were taken as evidence that Bonnie viewed slope in a conceptually meaningful way—as a ratio relating two quantities in such a way that a measure of change is provided.

When making judgments about which of Bonnie's three ideas could be categorized as beliefs and which could be categorized as knowledge, the content of the idea was useful. For example, the third idea summarized above is about mathematics. In particular, it is about the topic of slope. When categorizing ideas about the meaning of a mathematical topic like slope, the standard of mathematical correctness can be used to

assess the reasonableness and/or intelligence of an espoused or inferred idea. In this way, Bonnie's interpretation of Angie's explanation may be considered mathematically correct against the standards of the mathematical community. In contrast, recall that Kelly's description of slope—"slope is talking about how things change vertically over horizontally"—expressed a graphically bounded interpretation of slope. Thus, such an interpretation may be considered mathematically limited (Zaslavsky, Sela, & Leron, 2002). On the other hand, there is no standard for judging the reasonableness or intelligence of ideas related to transfer or how to best support transfer (i.e., the first and second ideas summarized above). This dissertation may be viewed as the first step in a long journey towards achieving such standards. Before such standards can be considered though, one must first gain access to the ideas related to transfer that exist. Thus, espoused and inferred views related to transfer are considered beliefs. In the discussion that follows, I provide additional support for my decision to consider ideas about transfer as *beliefs* about transfer.

I now return to my decision to distinguish between beliefs and knowledge using the researcher's point of view rather than the teacher's point of view. This decision is the result of both theoretical and practical concerns. First, I take an individual's ideas about transfer to be beliefs about transfer to indicate my own orientation towards views of transfer in general. The fact that transfer is one of the most researched topics in education and psychology (Engle et al., 2010), coupled with the fact that there have been many different conceptualizations of transfer over the past 110 years, leads me to believe that transfer is a complex phenomenon and that it may best be studied as a belief wherein

I am supported in making sense of differing ideas about transfer and how to support it rather than casting them aside as unintelligent and/or unreasonable.

Second, when using Philipp's (2007) definition as it was intended (i.e., from the actor's or, in this case, teacher's point of view), many factors may need to be considered. For example, the perceived *authority* of audience members may become important when considering *how* ideas are held (Cooney et al., 1998). In other words, a teacher may hold an idea as knowledge when engaging in tasks involving students' reasoning but hold an idea as a belief when engaging in discussions with me (the researcher). Thus, the teacher may be less likely to accept ideas as intelligent and reasonable if they come from students and more likely when they come from a doctoral student. Similarly, many other factors may become important when I seek to understand the ways in which teachers hold ideas about transfer. This may create a methodological black hole that may be avoided when I (the researcher) view all teachers' ideas about transfer as beliefs.

Lastly, my primary concern is not whether a teacher personally orients to his or her own idea about transfer as a belief or as knowledge. Rather, I am primarily concerned with gaining access to teachers' ideas. In future research, it may become important to use the teacher's point of view when distinguishing between his or her belief about how to support transfer from his or her knowledge about how to support transfer. Before doing that, I first want to be able to systematically investigate the ideas teachers have about supporting transfer and distinguish them from teachers' mathematical *knowledge* for teaching.

Despite my decision to adapt Philipp's (2007) definition with respect to point of view, I follow him in assuming that there exists a complex relationship among teachers'

beliefs, perceptions, and actions. Specifically, Pajares (1992) noted that beliefs shape one's perception of a situation. In other words, beliefs serve as filters which bring forth certain objects, relations, and phenomena while allowing others to fade into the background; in this way beliefs influence what one notices and thus shape one's interpretations (Grant, Hiebert, & Wearne, 1998; Mason, 2002). For instance, a teacher may believe that learning mathematics involves developing conceptually meaningful understandings of various topics and procedures and thus attend to the nature of a student's explanation over the correctness of her final answer. Moreover, beliefs may serve to position one to act in a certain way (Allport cited in McGuire, 1969; Rokeach, 1968). That is, beliefs may draw one towards one action over another. For example, the teacher above may be moved to select tasks and activities that support students in developing understandings rather than activities that help students memorize.

Finally, I follow Philipp and others who have conceived of beliefs as ideas that can be held with varying degrees of conviction (e.g., Rokeach, 1968; Thompson, 1992). For example, some teachers may believe more strongly that a teacher's primary role is to support students in transferring their learning beyond the classroom. This may be indicated when a teacher responds to a general question about teachers' goals with answers about the transfer of student learning. Other teachers may believe this is so but with less conviction as may be indicated by the absence of explicit comments about supporting student transfer and affirmations/elaborations of such statements provided to them.

Measuring Beliefs

Generally speaking, there have been two main methodological approaches to the study of teachers' beliefs: using a beliefs-assessment instrument or using a case-study methodology (Philipp, 2007). In this section, I will draw upon the literature on teachers' beliefs to discuss the ways in which these approaches have been used. In addition, I present some of the trade offs and benefits of each approach. I will conclude this section by discussing a third, less frequently utilized, approach to the study of teachers' beliefs wherein I will highlight some of the features it shares with case-study and beliefs-assessment approaches.

Beliefs-assessment instruments. Beliefs-assessment instruments come in different forms. The most common way in which teachers' beliefs have been measured, however, has been using a Likert-scale survey (Philipp, 2007). For example, Wilkins and Brand (2004) sought to understand the relationship between teacher preparation programs and preservice teachers' beliefs. To study this relationship, they utilized Hart's (2002) 30-item Mathematics Beliefs Instrument (MBI) wherein all of the items were presented to preservice teachers on a 4-point Likert scale (4 = *strongly agree*, 3 = *agree*, 2 = *disagree*, 1 = *strongly disagree*). Sixteen of Hart's 30 items came from Zollman and Mason's (1992) beliefs instrument, which was designed to measure the consistency between a person's beliefs related to the teaching and learning of mathematics and the NCTM Standards (1989). Twelve items were adapted from Schoenfeld's (1989) survey to measure people's beliefs about mathematics and mathematics education in general. The last two items were designed to measure a person's beliefs about his or her effectiveness as a teacher and learner of mathematics. Thus, items on the test included statements like

"Good mathematics teachers show you the exact way to answer the math question you will be tested on" and *"I think I will be very good at teaching mathematics."*

Wilkins and Brand administered the MBI to 89 preservice teachers on the first and last days of their mathematics methods course and found a positive relationship between the methods course and changes in the preservice teachers' beliefs. In particular, they found that teachers' beliefs became more consistent with the philosophies underlying reform efforts. In this way, many researchers have used or adapted previously created Likert-scale assessments, developed their own Likert-scale assessment, or created hybrids (e.g., Fennema, Carpenter, & Loef, 1990; Marbach-Ad & McGinnis, 2008, 2010; McGinnis, 2002, 2003; Peterson, Fennema, Carpenter, & Loef, 1989; Vacc & Bright, 1999).

One of the benefits to using Likert-scale surveys to measure teachers' beliefs is that the surveys can be administered to large groups of teachers at a single moment in time or at several moments over a period of time. In addition, teachers can fill out the surveys without the assistance of a survey administrator; thus, surveys can be mailed to teachers residing at various locations all over the country. Furthermore, Likert-scale surveys are particularly useful when researchers want to test theory (Philipp, 2007). For example, Wilkins and Brand (2004) hypothesized that after taking a particular mathematics methods course, preservice teachers' beliefs about teaching and learning would become more consistent with the philosophies of reform efforts and were able to confirm their hypothesis using the MBI described above.

One tradeoff is that researchers do not have access to the ways in which teachers interpret survey items (Philipp, 2007). For example, in the item "I think I will be very

good at teaching mathematics" (Hart, 2002; Wilkins & Brand, 2004), there is no way for the researcher to know how teachers have interpreted *teaching mathematics*. A teacher may respond differently to this statement depending upon whether she is thinking about presenting a mathematical procedure in a coherent manner, helping students to develop sophisticated mathematical understandings, or orchestrating productive whole-class mathematical discussions. Individual teachers are not typically provided space in which to explain their responses to Likert items. Consequently, researchers can only infer teachers' interpretations.

A common feature of Likert surveys that contributes to the difficulties researchers have when attempting to infer respondents' interpretations of Likert items is that Likert items tend to contain little, if any, context (Philipp, 2007). With respect to the item "I think I will be very good at teaching mathematics," there is no way to interpret the topic or level of mathematics teachers read in the item statement. Teachers might respond differently when thinking about their ability to teach various topics, say the solving of a linear equation versus the meaning of an algebraic expression, or the teaching of Algebra I versus Calculus. Again, teachers' responses to test items provide no indication of the contexts they draw upon in order to answer such questions.

Finally, when using Likert-scale assessments researchers cannot access the relative conviction with which teachers hold the beliefs reflected in test items (Philipp, 2007). For example, when asked to respond to the question "What defines a good mathematics teacher?" teachers may list a variety of features including teachers' inclinations to make sense of student responses before making judgments about correctness, abilities to build upon students' thinking, conceptually meaningful

mathematical understandings, and so on. In other words, they may not say anything to indicate a belief that good mathematics teachers show students "the exact way to answer" math questions. However, presented with the Likert item "Good mathematics teachers show you the exact way to answer the math question you will be tested on" (Hart, 2002; Wilkins & Brand, 2004), a respondent may agree that good mathematics teachers have the ability to provide students with solution strategies even though the respondent may never have thought about this characteristic before.

To respond to these limitations of Likert-scale surveys, Philipp and his colleagues developed a beliefs-assessment instrument as an alternative (Integrating Mathematics and Pedagogy (IMAP), 2003; see also, Ambrose, Clement, Philipp, & Chauvot, 2004). In particular, they created a web-based survey that required *open-ended* responses so that they would have more data with which to infer teachers' interpretations to assessment items. In addition, assessment items provided respondents with contextually-rich situations to interpret. Moreover, assessment items were ordered and respondents were prevented from moving backwards once an item had been answered. This feature was thought to address the issue of the relative importance or intensity with which one holds a particular belief. For instance, a teacher may be asked about the characteristics she thinks make a good mathematics teacher before she is asked about the particular characteristic of showing "the exact way to answer" math questions.

Case studies. Case-study methods provide another means by which to deal with the tradeoffs of Likert-scale surveys. This is because case-study methods rely on rich data sets including (but not limited to) combinations of surveys, clinical interviews, stimulated-recall interviews, classroom observations, and responses to either vignettes or

videotapes of student work. Thus, the data that are collected can be used to provide a detailed description of teachers' beliefs. For example, Borko and her colleagues, like Wilkins and Brand above, were interested in the relationship between teacher preparation courses and preservice teachers' beliefs (Borko et al., 2000). To investigate the relationship, they employed a case-study methodology in which data were collected over a 1½-year period. Data were collected from the classes a particular group of preservice teachers engaged in during their preparatory studies, 10 interviews designed to target issues such as teachers' mathematically specific content knowledge, beliefs, and pedagogy, and classroom observations during the preservice teachers' student-teaching experiences.

As a result of analyzing the data that were collected, Borko and her colleagues were able to provide a detailed description of the relationships between one preservice teacher's (Ms. Savant's) preparatory classes, her beliefs about proof, and the practice that was observed while she was student teaching. For example, Ms. Savant believed that when making judgments about the convincing nature of a mathematical argument, the audience and the speaker were of equal importance—a belief thought to emerge out of the fact that teachers engaged in both formal and informal mathematical experiences in their mathematics methods course. As a result, Ms. Savant appeared to have a more stringent view of "proof" when considering an audience of mathematicians than when considering an audience of students. Her belief was reflected in her teaching practice when she relaxed the requirements for students during derivations of the area formula for a circle.

As a result of using a case-study approach to investigate teachers' beliefs, Borko et al. (2000) were able to identify specific beliefs related to the teaching of proof. In particular, they noted that for one teacher *audience* was an important factor in determining what constituted mathematically acceptable proofs; thus the acceptability of a proof depended on who was viewing and/or writing the proof. Furthermore, the researchers were able to relate such beliefs to specific preparatory experiences, namely, the fact that preservice teachers engaged in both formal and informal discussions and activities surrounding the topic of proof. For these reasons, many studies of teacher beliefs have employed case-study methods (e.g., Cross, 2009; Glasson & Lalik, 1993; Gregg, 1995; Haney & McArthur, 2002; McGinnis, Parker, & Graeber, 2004; Mewborn, 2000; Raymond, 1997; Richardson, Anders, Tidwell, & Lloyd, 1991; Roth-McDuffie, McGinnis, & Graeber, 2000; Skott, 2001; Sztajn, 2003; Thompson 1984).

The tradeoffs to using case-study methods include the fact that they tend to be more expensive than Likert-scale assessments. For example, the large amounts of data that are collected often require various kinds of technical equipment and/or extensive field notes which require many hours of researchers' time. Furthermore, analyses of the data collected using a case-study method take considerably more time than the analyses required by Likert-scale surveys. Lastly, the detailed descriptions that derive from researchers' analyses of collected data chronicle the beliefs of only a small number of teachers.

Interview studies. Some studies of teacher beliefs have utilized a third method, namely that of an interview study (e.g., Bussis, Chittenden, Aramel, 1976; Pressley et al., 1991). This approach takes into account both the benefits and the drawbacks of the

approaches discussed above. For example, Pressley et al. (1991) asked participants to rate their commitments to a particular type of instruction (i.e., strategy instruction) much in the same way one would use a Likert-scale survey to rate the degree to which participants agree or disagree with a particular statement. Pressley and his colleagues were then able to follow up on participants' rankings, asking them to explain their associated reasoning. They were also able to probe participants' explanations for further elaboration. The interview format thus allowed these researchers to better understand the ways in which participants interpreted interview items as well as the meanings participants associated with their responses.

In this dissertation, I approached the investigation of teachers' beliefs about the transfer of students' learning and how to support it using *clinical interviews*. A clinical interview is a *semi*-structured technique for collecting data wherein a participant, here a teacher, is asked, in an interview setting, to engage in and articulate her thinking and reasoning surrounding carefully designed questions and tasks (Ginsburg, 1997). Semi-structured refers to the responsive quality of this kind of interview (Bernard, 1988). While some of the questions are indeed decided upon before the interview begins, others emerge out of the interaction that takes place between the interviewer and the interviewee (i.e., the teacher). Much in the same way Pressley and his colleagues posed follow-up questions to better understand the rankings and explanations participants provided, much care is taken by the interviewer during a clinical interview to understand the ways in which interviewees interpret the interview questions and tasks. Once the interviewer is reasonably certain agreement has been reached regarding the intention and interpretation of a particular question or task, the interviewee is given a chance to respond. Finally, the

interviewer works to elicit the meaning underlying interviewees' responses. Thus, the responsive and probing nature of clinical interviews addresses one of the major drawbacks to using Likert-scale surveys, namely, the researcher's inability to infer respondents' interpretations of survey items. Furthermore, the questions and tasks posed to interviewees can be ordered in such a way that researchers are able to gain better understanding of the relative importance with which respondents view specific ideas (cf., IMAP, 2003).

In addition, clinical interviews provided a means by which I triangulated data about teachers' beliefs associated with the generalization of students' learning. In particular, questions and tasks were posed that allowed teachers to espouse or state their beliefs. Again, the responsive and thus tailored nature of the clinical interviews allowed me to probe teachers about their interpretations of such questions and tasks as well as their responses to such questions and tasks. Data regarding teachers' espoused beliefs was then triangulated with data regarding my inferences of teachers' beliefs. In other words, questions and tasks were designed that allowed me, the researcher, to infer (via teachers' interpretations of and/or responses to the interview tasks) teachers' beliefs about the generalization of students' learning and how it can be supported instructionally (Philipp, 2007).

For example, imagine a teacher espouses a belief about supporting the transfer of students' learning that involves helping students to develop meaning for the procedures that emerge in the classroom. However, the teacher's interpretation of *meaning* may be very different from my interpretation. Thus, teachers' responses to tasks that, for example, ask about the kinds of situations a hypothetical student would be able to

successfully engage in given the student's reasoning on a slope task helped me understand the ways in which teachers viewed meaning. For instance, *meaning* for a teacher who explains that the student would not be able to answer questions involving negative slope because the student did not provide a complete description of the steps she took to solve the problem may involve a student's ability to provide a complete *explanation* of the *steps taken* to solve a problem. In this way, data that allowed me to infer teachers' beliefs were used to gain a clearer image of the beliefs teachers espoused. Moreover, the semi-structured nature of the interview allowed me to create tasks and follow-up questions *in the moment* which provided me with additional opportunities to make inferences regarding the meaning of teachers' beliefs (Ginsberg, 1997).

Despite the fact that this interview approach cannot be used at the scale of a Likert-scale survey, it can be used at a larger scale than case-study approaches. As noted above, case-study methods generally rely on rich data sets that are collected over an extended period of time. Consequently, researchers are only able to provide detailed descriptions of, in most cases, no more than 4 teachers. In contrast, data from clinical interviews can be collected in a matter of minutes or hours thus allowing more time to collect data on a greater number of teachers. (The methods I used to analyze data, however, require extensive time and effort and are described in Chapter 3; thus, I only collected data on 8 teachers.)

In sum, the way in which I chose to study teachers' beliefs about the generalization of students' learning and how it to support it complements my conceptualization of beliefs and addresses some of the drawbacks one faces when using beliefs-assessment instruments or case-study methods. In particular, clinical interviews

provide the freedom and opportunity to make sense of teachers' beliefs about transfer as reasonable and intelligent. In addition, clinical interviews provide a means by which to investigate the complex relation among beliefs, perceptions, and actions. In other words, I was supported in making sense of the ways in which teachers interpreted questions and tasks that were posed to them as well as the responses teachers provided. Moreover, the tasks and questions that were posed were ordered such that I was able to gain some sense of how important various beliefs were to teachers. Finally, as the time required of interview approaches to the study of teachers' beliefs is considerably less than the time required of case-study approaches, I was able to collect data from a greater number of teachers while maintaining many of the same benefits of case-study approaches (e.g., a richer data set from which to infer teachers' beliefs).

The Relationship between Teachers' Beliefs and Teachers' Practices

This dissertation examines the relationship between teachers' beliefs about the generalization of students' learning and teachers' practices. In Chapter 1, I described several ways in which the term practice has been used in educational research and the meanings associated with various usages. In the present study, *practice* refers to the actions teachers take to perform the work of teaching (Lampert, 2010).

Over the years, researchers have become increasingly aware of the influence that teachers' beliefs have on their practices (Nespor, 1987; Pajares 1992; Philipp, 2007; Raymond, 1997; Richardson, 1996; Thompson, 1984, 1992; Wilson & Cooney, 2002). For example, Beswick (2012) described one teacher's beliefs about the nature of mathematics as being consistent with her classroom practice. The teacher, Jennifer, held the belief that mathematics exists in the external world, independently of humans, and

that, over time, humans have come to develop increasingly better understandings of mathematics. Observation of Jennifer's classroom revealed slow-paced, whole-class teacher-directed lessons wherein Jennifer held mathematical authority presenting strategies, asking students to implement the strategies, and assessing students' answers. Thus, it appeared that Jennifer's belief manifested in her practice as she slowly revealed to her students small portions of an already accumulated body of mathematical knowledge (for more on consistencies see Kaplan, 1991; Mewborn, 2000; Peterson et al., 1989).

Others, however, have reported inconsistencies between teachers' beliefs and their practices (e.g., Brown, 1986; Cohen, 1990; Cooney, 1985; Raymond, 1997; Shaw, 1990; Shirk, 1973; Sztajn, 2003; A. G. Thompson, 1984). For instance, Cohen (1990) described Mrs. Oublier's belief that teaching mathematics involves building on student understandings; however, observations of her practice showed that students were never asked to explain their thinking or to demonstrate their strategies. Instead, Mrs. Oublier directed student activity even when it was clear (to the researcher-observer) that students were confused, privileged correct answers by placing them on the board, and ignored incorrect answers. Thus, it appeared that Mrs. Oublier's belief about teaching mathematics did not manifest in her practice.

In this section, I draw upon the literature on the relationship between teachers' beliefs and their practices for the purposes of elucidating various approaches to the consistency/inconsistency issue raised above. Because this issue is more apparent when inconsistencies are found than when consistencies are found, I focus on studies like the second one described above wherein mismatches between beliefs and practice are

explicitly discussed. My goal in this section is to show that when researchers assume consistency between teachers' beliefs and their practices, they may see that beliefs and practices function together and are mutually elaborated and co-constructed.

Explaining Inconsistencies

Researchers have dealt with apparent inconsistencies like the one found in the case of Mrs. Oublier in several different ways. One approach has been to offer explanations for their existence by scrutinizing various aspects of the learning environment. Returning to the case of Mrs. Oublier, one can see an example of how this is done. The story of Mrs. Oublier is the story of a teacher who was inspired by a professional development workshop (Cohen, 1990). Before the workshop, Mrs. Oublier reported that her beliefs about teaching were thoroughly traditional. For instance, teaching mathematics meant providing students with worksheets and supporting them as they memorized facts and procedures; therefore, those were the actions she took while teaching. Then, Mrs. Oublier went to a workshop that focused on building instruction around the students' mathematical understandings and the ways in which students reasoned about mathematical ideas. After the workshop, Mrs. Oublier's espoused belief had changed. In particular, she stated that teaching mathematics meant supporting the development of students' understanding; thus, she sought activities that would engage her students and put forth effort to relate mathematical topics to students' experiences. Mrs. Oublier reported being delighted with her new ways of teaching and by her students' performance.

Moreover, Mrs. Oublier explained that she viewed her classroom as a success in light of the new mathematics framework that was being imposed in her school district.

Consequently, Cohen (1990) told the story of the relationship between Mrs. Oublier's

beliefs and her practice against a backdrop of failed efforts to reform the educational

system. He acknowledged the fact that Mrs. Oublier's practice had indeed changed and

that to some extent it had become aligned with the new reform framework. For example,

Mrs. Oublier worked hard to incorporate concrete objects and manipulatives into her

lessons. She also worked to create open-ended tasks that she thought would help students

to develop mathematical understandings (e.g., *Come up with as many addition sentences*

as you can such that the sum is 14). However, Cohen noted that the framework made no

distinction regarding *how* concrete objects and open-ended activities should be used.

Should open-ended activities be accompanied by discussion? If so, what kind of

discussion and who should contribute to the discussion? Without such guidance, Cohen

argued, Mrs. Oublier was free to tell students how to manipulate objects during

classroom activities and ask that students report answers only. Furthermore, Cohen

explained that the framework did not explicitly define *understanding*. What were the

substantive goals teachers were working towards? Helping students to provide a variety

of *correct answers*? Or were there *ideas* that teachers were supposed to be developing?

Therefore, Cohen argued that Mrs. Oublier was free to act as though understanding

emerged out of students' physical engagement with mathematical tasks and activities.

Cohen thus explained the inconsistency he observed between Mrs. Oublier's belief and

her practice, by pointing to the missed opportunities of reform documents, for example,

the absence of definitions, elaborations, or model examples.

     Another way that researchers have dealt with the inconsistencies they perceive

between teachers' beliefs and practices has been to ask teachers about the goals and

purposes that underlie inconsistent pedagogical moves (Philipp, 2007). For example,

Skott (2001) tells the story of Christopher, a new teacher who believed that teaching

mathematics involved instantiating and facilitating investigative activities and helping

students to become autonomous learners. However, observations of Christopher's

classroom revealed enactments of practice that were contradictory to Christopher's

beliefs. In particular, two students asked for help on an activity in which they were

supposed to find the area of two rectangles that were drawn according to different scales.

In response, Christopher led the students via directive questioning through a series of

computations wherein Christopher highlighted the scaling of each rectangle, explained

the relationship between representations on paper and their real world counterparts, and

asked pointed questions. For instance, after explaining that 1 cm on paper is 2 m in the

real world, Christopher pointed to various lengths in the representation of the rectangle

and asked "How many meters is this? … And this? … Here? … Here?" He then asked

his students what they were going to do with the numbers. When the students said they

would multiply them, Christopher affirmed their resulting answer and then told them to

write it down.

　　　When Christopher was shown video of this teaching episode, he seemed to

provide explanations about larger non-domain specific goals. The first goal seemed

related to sustaining students' confidence. Specifically, Christopher explained that one of

the two students he was helping often gets stuck. He thus went on to describe his need to

think of students in broader terms and to show sensitivity to the development of students'

overall self-confidence. His second goal seemed related to classroom management. In

particular, Christopher stated that, based on his prior experience with the student above,

he knew that if he did not successfully guide the student to the answer the student would keep "coming and asking questions … even [if] I try to get him going on his own" (Skott, 2001, p. 15). Thus, Skott (2001) concluded that the particulars of the situations Christopher engaged in (e.g., the specific students and their associated characteristics) brought forth different goals at different moments and that sometimes those specific goals cast Christopher's belief about his general role as a facilitator into the background.

A third way that researchers have approached apparent inconsistencies between teachers' beliefs and their practices has been to examine a cluster of beliefs (Philipp, 2007). For example, Raymond (1997) provided analyses of one teacher's beliefs about teaching mathematics, learning mathematics, and about the nature of mathematics. The teacher, Joanna, seemed to believe that teaching mathematics involved the facilitation of students' discovery of mathematics and that students should not be shown the mathematics. In addition, Joanna believed that instruction could be successful without explicitly following a textbook and that teachers should draw upon multiple sources during instruction. Moreover, Joanna appeared to believe that learning was best supported through problem solving (which she viewed as a "topic" in mathematics). Finally, Joanna believed that mathematics was mostly facts and procedures and that these facts and procedures should be memorized. She did not believe that mathematics was a creative endeavor. Thus, she saw mathematics as a fixed body of knowledge that was certain and absolute.

Observations of Joanna's class revealed that while Joanna's practice was inconsistent with her beliefs about the teaching and learning of mathematics, it was consistent with her belief about the nature of mathematics. Students in Joanna's class sat

in rows all facing the chalkboard. Joanna presented material in lecture form and

answered any and all questions students posed. Students were not observed talking or

engaging in discussions with other students. During Joanna's lessons, she rigidly

followed her textbook and then assigned problems from the textbook for students to work

quietly on at their desks. These forms of practice appeared consistent with Joanna's

belief that mathematics was not a creative enterprise but an enterprise consisting of facts

and procedures to be memorized. Raymond was thus able to find consistency by opening

up her investigation to include a range of beliefs.

Assuming Consistencies

In contrast, other researchers have suggested that in order to better understand the

relationship between teachers' beliefs and their practices, one should assume consistency

(Philipp, 2007). When Hoyles (1992) viewed teachers' beliefs and practices as co-

constructed, she found consistency where she once found inconsistency. If we re-

examine Christopher's belief about the teaching of mathematics, namely that teaching

mathematics involved creating and supporting investigative activities and autonomous

learners, in the moment of his teaching (as described above), we might find that the

meanings of "investigative" and "autonomous" were developed further during the

enactment of Christopher's practice. It is possible that before teaching the lesson

Christopher viewed investigative activities as activities involving *no* teacher direction but

that after teaching the lesson he viewed them as activities involving *minimal* teacher

direction. After all, the two students worked independently of the teacher both before

and after the "directive" episode described above. Similarly, it is possible that

Christopher's definition of autonomous learner became elaborated through his teaching

actions. Before the lesson, Christopher may have believed that teaching mathematics meant nurturing students who can solve problems on their own, but after the lesson Christopher's notion of autonomous learner may have broadened to include developing students who are confident about their ability to solve problems on their own.

I revisit one more of the teachers described above to emphasize the ways in which beliefs and practice might co-construct and mutually elaborate each other. Mrs. Oublier espoused a belief that teaching mathematics involved helping students to develop understandings of mathematical ideas. Mrs. Oublier's practice was teacher-directed, involved heavy use of manipulatives and other concrete objects, and was void of student discussion and explanation. Cohen (1990) thus reported inconsistencies between her belief about teaching mathematics and her practice. However, the responses Mrs. Oublier gave to questions Cohen asked her after having observed her lesson indicated that her espoused beliefs were in harmony with her practice. In particular, Cohen reported that Mrs. Oublier believed that "manipulating the materials really helps kids to understand" and that she "seemed quite convinced that these physical experiences *caused* [emphasis added] learning, that mathematical knowledge arose from the activities" (p. 316). Thus, it seemed Mrs. Oublier's practice served to further elaborate her espoused belief. For her, *understanding* was something that emerged as a consequence of simply *using manipulatives* not as something that emerged via problem solving, critical thinking, reasoning, discussion, explanation, or justification.

The reinterpretation of Cohen's finding leads me to wonder whether there were unseen consistencies between Joanna's beliefs about teaching and learning mathematics and her classroom practice. For example, I wonder how Joanna thought about

"facilitation," "discovering" mathematics, or the idea that students should not be "shown" mathematics during or after the lesson Raymond (1997) observed. Did her beliefs become elaborated? Were there consistencies between Joanna's beliefs and practice that were not immediately obvious to the researcher? These concerns have led me to consider practice as a means by which teachers' beliefs become clearer and more precise both to me, the researcher, and to the teachers who hold the beliefs.

<div align="center">**Teachers' Mathematical Knowledge**</div>

Research Question 2 examines the relationship between teachers' beliefs regarding the generalization of their students' learning and teachers' knowledge of the mathematics. Such examination is contingent upon many factors including the way in which teachers' mathematical knowledge is conceptualized. Over the years, educational researchers have conceived of this specific kind of knowledge in a variety of ways (e.g., Baumert et al., 2010; Hill, Ball, & Schilling, 2008; Leinhart & Smith, 1985; Shulman, 1986, 1987; Silverman & Thompson, 2008). In this section, I review some of the main conceptualizations and further develop conceptualizations that best fit the goals of my research question.

Teachers' Knowledge of the Mathematical Content

There is agreement within the literature on teacher education that a core component of teacher competence is a strong understanding of the subject being taught (Baumert et al., 2010). Thus, one way in which teachers' knowledge has been conceived is in terms of their knowledge of the mathematical domain—either teachers' knowledge of the mathematical content to be taught or teachers' knowledge of the mathematical content learned in mathematics courses. This idea is reflected in tests of teacher licensure

that data back to the 1800s (Angus, 2001; Hill, Sleep, Lewis, & Ball, 2007; Shulman, 1986). Items on these tests assessed prospective teachers' proficiency with the subject matter to be taught. With respect to mathematics, test items assessed teachers' computational and problem-solving skills. For example, teachers taking the exam in 1875 in California were asked questions like the following: "divide 88 into two such parts that shall be to each other as 2/3 is to 4/5" (Shulman, 1986, p. 4).

Furthermore, researchers have studied the link between teachers' performance on questions like the one presented above and their students' mathematical achievement. For example, Harbison and Hanushek (1992) administered the same mathematics tests to fourth-grade teachers and their students. They then used the teachers' scores to predict the amount of change they would see in students' scores after having received Grade 4 instruction. In this case, teachers' performance indeed predicted gains in student learning.

However, other studies examining the relationship between teachers' knowledge of the mathematical content and their students' achievement have shown that the relationship is anything but straightforward. These studies have failed to find any statistically significant correlation between various teacher attributes (e.g., level of education, number of post-secondary mathematics courses, major in mathematics, grade point average) and student achievement (Begle, 1972, 1979; Eisenberg, 1977; Gess-Newsome, 1999; Hill et al., 2007; Mewborn, 2001). For example, Begle (1972) showed students' achievement was not affected by their teachers' level of mathematical understanding; in particular, he showed teachers' understanding of abstract algebra (including their knowledge of groups, rings, and fields) had no statistically significant

correlation with student performance on algebraic computations or with students'

understanding of ninth-grade algebra, in general.

Similarly, qualitative studies have shown teachers' computational and problem-

solving skills are not enough to ensure effective instruction. For example, Borko and

colleagues (1992) showed that despite the fact Ms. Daniels, a student teacher, could

successfully demonstrate and implement the rule for dividing fractions, she could not

adequately answer a student's question regarding *why* one must first invert the divisor

and then multiply. In particular, her explanation and accompanying representation

illustrating the meaning of $\frac{3}{4} \div \frac{1}{2}$ showed the multiplication of fractions rather than the

division of fractions—a fact Ms. Daniels could recognize but not remedy (for similar

findings with respect to rates see Thompson & Thompson, 1994, 1996). Thus, it appears

clear that although necessary, knowledge of the mathematical content alone is not

sufficient to ensure effective instruction.

Teachers' Knowledge of Mathematically-Specific Pedagogy

In his influential and transformative work on teacher knowledge, Shulman (1986,

1987) introduced the term *pedagogical content knowledge* (PCK) to address the notion

that content knowledge is not sufficient in supporting effective teaching. In doing so, he

distinguished PCK from teachers' *subject matter knowledge* (SMK). SMK was used to

refer to knowledge of the facts, concepts, and structures that characterize a particular

domain as well as to the ability to not only define a concept but to state and understand

why it is important and how it relates to other concepts in the domain (see also Bruner,

1986 and Schwab, 1961/1978 for similar definitions of subject-matter knowledge).

Shulman then used PCK to refer to knowledge of the subject matter that is needed in the

teaching of that subject matter, for example, knowledge of the most powerful forms of representations, explanations, examples, and counterexamples as well as knowledge of what makes learning a particular concept easy or difficult.

This construct of PCK may now prove useful in explaining Ms. Daniels' instruction surrounding the division of fractions algorithm. Specifically, Ms. Daniels did seem to have the more general pedagogical understanding that a real-world scenario and accompanying diagram might help her students make sense of the mathematical algorithm; thus, Ms. Daniels attempted to create an appropriate story problem and diagram. However, Ms. Daniels did not appear to have an understanding of the relationship between the multiplication and division of fractions nor did she appear to have an understanding of specific ways in which those general pedagogical tools could be used to help make the specific mathematical content comprehensible to her students. In other words, Ms. Daniels seemed to lack some forms of SMK and PCK. Consequently, she was unable to support her students in developing meaning for the algorithm.

Since Shulman's introduction of the term, many researchers within the field of mathematics education have worked to extend and elaborate the idea of PCK. For example, Grossman (1990) emphasizes the ways in which teachers' knowledge of the instructional setting (e.g., grade level, community values, institutional organization) as well as teachers' intellectual and personal dispositions toward the subject matter (including their beliefs about how one learns the subject) contribute to the choices teachers make while teaching (see also Grossman & Richert, 1988; Grossman & Shulman, 1994; Grossman, Wilson, & Shulman, 1989). Others emphasize the need for teachers to have special *mathematical knowledge for teaching* (MKT)—or mathematical

knowledge that is particularly useful in the teaching of mathematics (Ball, 1990; Ball & Bass, 2000; Hill, Ball, & Schilling, 2008; Hill, Sleep, Lewis, & Ball; 2007; Silverman & Thompson, 2008; Thompson & Thompson, 1996). As will be shown below, these researchers have been particularly helpful in identifying specific kinds of mathematical knowledge that raise the quality of teachers' mathematical instruction.

Teachers' Mathematical Knowledge for Teaching

Ball and her colleagues built on Shulman's conception of PCK to define the specialized kinds of knowledge that contribute to effective mathematics instruction as measured by student achievement (Ball & Bass, 2000, 2003; Ball, Hill, & Bass, 2005; Hill et al., 2005; Hill et al., 2007). In particular, Ball and colleagues took as their point of departure the mathematical behaviors teachers enact during "high-quality" mathematical instruction. They then asked themselves the following question: "What do teachers do in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding and skill?" (Ball et al., 2005, p. 17). As a result, Ball and colleagues were able to categorize productive teacher actions and develop an associated 6-part categorization system of teachers' MKT (Hill et al., 2008).

The categorization system includes *common content knowledge* (CCK), *specialized content knowledge* (SCK), *knowledge at the mathematical horizon* (KMH), *knowledge of content and students* (KCS), *knowledge of content and teaching* (KCT), *and knowledge of curriculum* (KC). Each type of knowledge is thought to comprise its own category of teacher actions. For example, SCK is knowledge that is used while teaching and encompasses teachers' abilities to create and make use of representations, provide mathematical explanations and justifications for the algorithms they teach, and make

sense of students' strategies. KCS enables teachers to build on students' mathematical thinking, correct students' errors, and anticipate students' interpretations and misinterpretations of mathematical concepts. KCT allows teachers to purposefully select examples, representations, and activities and to thoughtfully plan and sequence instruction.

Using this categorization system, the *Learning Mathematics for Teaching* (LMT) project (e.g., Hill, 2007; Hill, Ball, & Shilling, 2008; Hill, Schilling, & Ball, 2004) developed multiple-choice instruments that could be used at scale to assess teachers' MKT. Their measures have been the most widely administered due to the fact that the LMT assessment has been shown to (a) locate teachers along a scale corresponding to their knowledge of particular mathematical content and (b) have a positive correlation with students' mathematical achievement (Ball et al., 2005). This conceptualization of MKT is thus particularly helpful for policy makers.

In contrast, Silverman & Thompson (2008) took another approach to the investigation of MKT. Rather than beginning their investigation by categorizing teacher actions, they started by examining the mathematical understandings that give rise to powerful and effective teacher actions (Silverman & Thompson, 2008; Thompson & Thompson, 1996). In other words, they aimed to answer a different question: What mathematical understandings allow a teacher to spontaneously act in a way that is productive in supporting the development of students' conceptual understandings and how do these understandings develop? Thus, this conceptualization of MKT is viewed as being grounded in particularly powerful understandings of specific mathematical concepts.

In Chapter 1, I discussed 3 types of knowledge involved in this conceptualization

of MKT: (a) knowledge of the *key developmental understandings* (KDUs) that exist

within a particular content area, (b) knowledge of the ways in which students come to

develop those KDUs, and (c) knowledge of the actions that could be taken in support of

the development of those KDUs). Additionally, MKT entails two other kinds of

knowledge: (d) knowledge of the various ways in which students *might* understand that

content area (including student understandings that are mathematically incorrect) and (e)

knowledge of the ways in which the KDUs might empower students to learn other,

related mathematical ideas. For the purposes of this dissertation, I have chosen to focus

on the three types of mathematical understandings discussed in Chapter 1. The first type

of understanding—understanding of the KDUs that underlie specific mathematical

content—is said to develop via *reflective abstraction* (Silverman & Thompson, 2008).

Reflective abstraction refers to the process by which one's actions become organized and

coordinated; it is said to occur via the creation and re-creation of one's mental and

physical activities such that properties are drawn from said actions and assimilated to

higher levels of thought (Piaget, 2001; Thompson, 1985, 1994; von Glasersfeld, 1995).

Thus, when a teacher develops a particular KDU, her content knowledge is said to

become "related" to other content knowledge thereby extending the teacher's web of

connections (Silverman & Thomson, 2008).

A teacher's KDU is then transformed into its associated MKT via a second-order

reflective abstraction. In this case, the teacher must put herself in the place of a student

and conceive of the mental and physical activities as well as the constraints that would

have to be in place in order for her students to act in a way that is consistent with the

KDU; thus, conceiving of situations that would afford students opportunities to develop the KDU (Silverman & Thompson, 2008). Through this activity the teacher is said to develop an understanding of how a student might come to develop the KDU as well as an understanding of the teacher actions that could serve to facilitate the development of the KDU.

This conceptualization allows one to investigate the meanings and interpretations that teachers have of specific mathematics and the ways in which those meanings and interpretations become useful (or not) in the classroom. This conceptualization appeared more useful in the current study wherein I attempt to link such understandings with teachers' beliefs about the generalization of students' learning. For example, recall Bonnie whose understanding of slope as ratio allowed her to make sense of Angie's explanation (see Figure 1.5) in terms of ratio. However, the fact that Bonnie's belief regarding the generalization of students' learning (i.e., students will productively generalize their learning of a procedure to a novel situation if the novel situations prompts them to enact the learned procedure via, for example, a mnemonic based on a familiar real-world experience) was inconsistent with the development of such an understanding suggests that she had not engaged in a second-order reflective abstraction. Without an understanding of how students come to conceive of slope as a ratio relating two quantities, Bonnie may have been left thinking about the generalization of students' learning in terms of students mathematically "correct" behavior; thus, her inclination was to support students in linking mathematical topics like *slope* and the "*rise over run*" formula to familiar elements of students' experiences like "*rising* out of bed *before running* to the *slopes*."

Slope

The mathematical topic of slope provides the content domain in which teachers were asked questions about their beliefs regarding the generalization of students' learning as well as their beliefs regarding teaching for the generalization of students' learning. In this dissertation study, teachers' beliefs were inferred from a set of questions involving the teaching artifacts teachers selected from their teaching lives as representing instances in which they had thought about helping students to generalize their learning of slope. Teachers discussed the artifact (most often an activity involving slope) in relation to other novel, activities and answered questions like the following: As a consequence of students' engagement with your activity, what types of novel tasks or future activities do you think students will be enabled to engage with productively? Why? What types of novel tasks or future activities do you think students will experience difficulty engaging with? Why? What could you do as a teacher to support students in generalizing their learning to these activities?

Furthermore, slope is the content domain in which teachers' MKT was assessed. Teachers' MKT was inferred from three sets of questions. The first set of questions was designed to get at teachers' personal understandings of slope and involved teachers' engagement in problem solving situations, responses to hypothetical situations, descriptions of the meaning of various components of the slope formula, and responses to questions like: What does a slope of ½ mean, and what is one measuring when measuring slope? The second set of questions was designed to get at teachers' understandings of the ways in which students might come to develop understandings of slope that are similar to their own and included questions like: What meanings would you like students to have

for slope, how might students develop those meanings, and what sources of difficulties might arise for students? The third set of questions was designed to get at teachers' understandings of the teacher actions that could support development of understandings of slope that are similar to their own and included prompts like: Explain what you would do to help a student understand what slope represents and explain what you would do to help a student understand the meaning of the slope formula.

In sum, my research questions about teachers' beliefs regarding transfer and teachers' MKT are set in the context of the mathematical topic of slope. Thus, in this section, I provide a brief review of the literature on slope. The review is organized into two major sections—research on what slope measures and research on various conceptions of slope. Because the literature base regarding students' understanding of slope is more exhaustive than that for teachers, I draw upon both in this section.

What Does Slope Measure?

Slope is not an isolated concept but is part of the multiplicative conceptual field (Vergnaud, 1983). As such, slope lives at the intersection of ideas related to functions and graphs, proportional thinking (including ratio and rate), and algebraic thinking. Unfortunately the way in which slope is typically described in textbooks does not make this explicit. In particular, slope is typically described in algebra textbooks as a measure of the steepness or "tilt" of a line and is generally presented as a procedure for finding the steepness (Lobato, Ellis, & Muñoz, 2003; Stump, 1999; Walter & Gerson, 2007). For example, students are often taught phrases like "rise over run" as mnemonics for calculating the slope of a line using the slope formula, $\frac{y_2 - y_1}{x_2 - x_1}$ (Stump, 1999, 2001a, 2001b; Walter & Gerson, 2007). This treatment of slope can render slope (a) an attribute of a

physical object rather than a function, and/or (b) a simple counting technique wherein

students use the squares on a coordinate grid system to find two separate numbers, one of

which gets subsequently placed over the other (see Figure 2.5), rather than an expression

of a multiplicative relationship between two quantities (Lobato et al., in press).
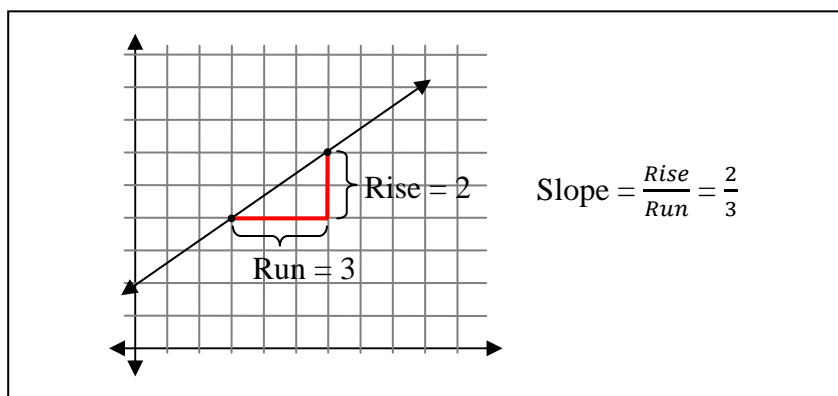
Figure 2.5: A counting technique for finding the slope of a line.

Furthermore, this treatment of slope leads one to ask whether the primary function

of slope is to provide a measure of the steepness of a line (Lobato & Thanheiser, 2002).

If slope does indeed provide a measure of the steepness of a line, then any two lines that

have the same slope should have the same steepness. However, as Figure 2.6 shows, this

is not always the case. Consequently, researchers have noted that students and teachers

alike experience "cognitive conflict" when the scaling of one of the coordinate axes is

changed (Zaslavsky et al., 2002). There are, of course, "fixes" for this problem. For

example, one may set parameters for the scaling of the coordinate system in which one

measures steepness. However, such fixes do not address the question of slope's primary

purpose.
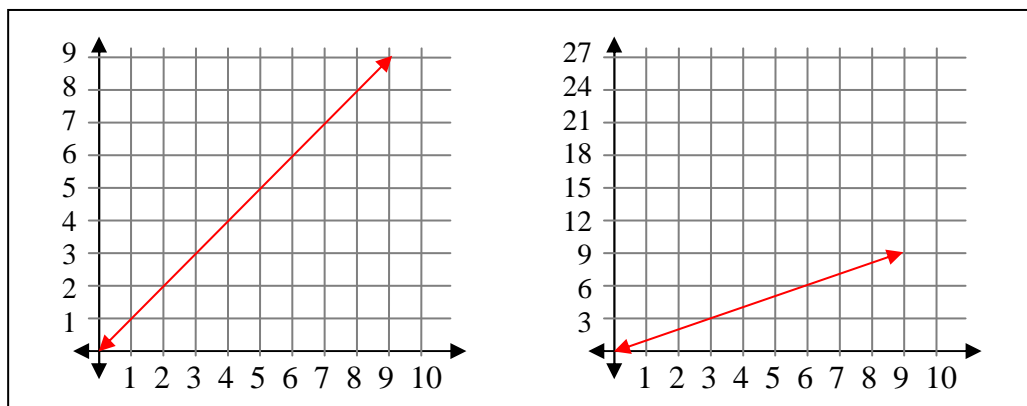
Figure 2.6: Two linear functions with the same slope but different steepnesses.

Inspired by Simon and Blume (1994), Lobato and Thanheiser (2002) proposed an

alternate conception of slope as a measure of the rate of change of two covarying

quantities wherein the amount of change in one quantity is measured with respect to the

amount of change in the other quantity. Quantity is used here to refer to attributes that

one conceives of as being measurable such as height, length, distance, and time (Smith &

Thompson, 2008). Depending on the quantities involved in a situation, slope becomes a

measure of a new attribute and makes sense (a) regardless of the scaling of a coordinate

grid system and (b) in situations where lines are not present. For example, consider

measuring one's motion through space. To obtain an accurate measurement of how fast

one moves through space, one can conceive of the rate of change in distance relative to

the change in time. Thus, slope in this situation provides a measure of the *speed* at which

one moves through space. If one then considers a graphical representation of this

situation, slope remains a measure of the rate of change of the quantity represented by the

vertical axis (i.e., distance) relative to the change in the quantity represented by the

horizontal axis (i.e., time). Thus, meaningful understandings of slope depend on ones'

ability to make sense of the appropriateness of using a ratio to measure something and to

interpret slope as an intensive quantity or ratio rather than as an extensive quantity or direct measure of the steepness of a line (Lobato, 2008a; Lobato, Clarke, & Ellis, 2005; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Schwartz, 1988; Simon & Blume, 1994; Stump, 2001a).

Conceptions of Slope

Slope as a formula. Stump (2001a) found that preservice and inservice teachers' conceptions of slope were dominated by the slope formula. Specifically, when asked to describe what slope is and what it represents, 15 out of 18 preservice teachers and 18 out of 21 inservice teachers referenced the slope formula either as "rise over run," "$\frac{y_2 - y_1}{x_2 - x_1}$," or both. In contrast, only 3 teachers from each group focused on the quantitative relationship between the dependent and independent variables when discussing slope. The dominant conception of slope as a algebraic form constitutes an *instrumental understanding* of slope (i.e., an understanding of a rule and its implementation) as a fraction or simply one whole number positioned above another, wherein the difference between *y*-values is the numerator or the number on top and the difference between the *x*-values is the denominator or the number on bottom (Skemp, 1976; Walter & Gerson, 2007). For example, the majority of the teachers in Stump's (2001a) study could use the slope formula to correctly solve slope tasks when given y_1, y_2, x_1, and x_2. However, two-thirds of the preservice teachers and one-third of the inservice teachers could not find slope when given x_1, y_1, and the angle of inclination between the line passing through (x_1, y_1) and the *x*-axis. The most common response was that the teachers needed "another point in order to determine the change in *y* over the change in *x*" (Stump, 2001a, p. 132). However, this is not the case as information about the first point is already extraneous

information. For example, one could take the given angle of inclination as the argument

for the tangent function and thus find the slope of the line. Therefore, this understanding

of slope may background the *meaning* of slope while foregrounding the computational

aspects of slope.

   Slope as the angle of inclination. Researchers have noted students' propensity

to interpret slope in terms of angles (Lobato et al., 2005; Lobato & Siebert, 2002; Lobato

& Thanheiser, 2002; Moschkovich, 1996; Stump, 2001b; Zaslavsky et al., 2002). For

example, Lobato et al. (2005) and Stump (2001b) found students using the angles at the

tops of various ramps in order to determine their relative steepnesses. However, using

this angle as a measure of steepness may lead students to conclude that shorter, flatter

ramps are steeper as they may have larger angles at the top (see Figure 2.7a). Similarly,

Lobato and Thanheiser (2002) found students using the angle of inclination to determine

the steepness of various ramps (see Figure 2.7b). The fact that this measure can indeed

be used as a valid measure of steepness may account for Stump's (2001b) finding that 15

out of 22 pre-calculus students used the term "angle" to describe slope. The problem

with this conception, mathematically, is that *slope* is not the angle of inclination. Rather,

slope is the tangent of the angle and not the angle itself. In other words, slope is a ratio of

the lengths of the vertical and horizontal sides of a ramp that is created as a result of the

ramp having a particular angle of inclination. Furthermore, this conception of slope may

entail slope as an attribute of a line rather than a function which can lead to problems

when the same function is graphed using two different coordinate systems (see Figure

2.6: Zaslavsky et al., 2002).
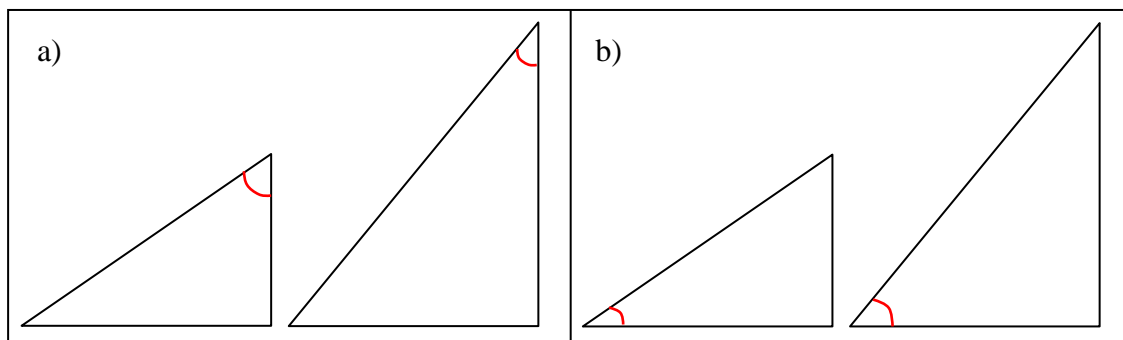
Figure 2.7: (a) Using the angle at the top to determine steepness; (b) Using the angle of inclination to determine steepness.

Slope as a ratio. As noted above, an understanding of slope as ratio is powerful in that it supports one in interpreting the meaning of slope in a variety of situations. Thus, developing and extending students' understanding of ratio to include the mathematical topic of slope is among the Standards for Mathematical Practice (Common Core State Standards for Mathematics, 2010). However, several researchers have noted the fact that students and teachers alike exhibit difficulty conceiving of tasks involving slope in terms of ratio (e.g., Lobato et al., 2005; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Stump, 1999, 2001a, 2001b; Walter & Gerson, 2007; Zaslavsky et al., 2002). This may be due to their tendency to use additive rather than multiplicative strategies when making sense of situations as well as their ability to conceive of ratio as the appropriate form of measure in a given situation (Lobato & Thanheiser, 2002; Simon & Blume, 1994; Stump, 1999, 2001a, 2001b; Thompson, 1994; Walter & Gerson, 2007; Zaslavsky et al., 2002).

Developing an understanding of slope as a ratio is not a trivial matter. It involves a shift in focus from the additive structures involved in a situation to the multiplicative structures involved in a situation (Lobato & Thanheiser, 2002; Simon & Blume, 1994; Thompson, 1994; Walter & Gerson, 2007). To illustrate, consider a classroom containing

12 boys and 8 girls. There are many ways in which one can conceive of the comparison of boys to girls. If one conceives of the situation additively, one might say that there are "4 more boys than girls" or that there are "4 less girls than boys." In contrast, if one conceives of the situation multiplicatively, one might say that there are "one-and-a-half times as many boys as girls" or "two-thirds as many girls as boys." Comparing these quantities (e.g., *number of boys in the class* and *number of girls in the class*) multiplicatively produces a new quantity, either *ratio of the number of boys to the number of girls* or *ratio of the number of girls to the number of boys* (depending on the quantity being used to measure the other) (Lobato & Ellis, 2010; Thompson, 1994). Thus, conceiving of the slope of a function in terms of a ratio involves asking "How many times greater is the quantity represented by the dependent variable than the quantity represented by the independent variable?" (Lobato & Ellis, 2010).

Consequently, an understanding of slope as a ratio depends on one's ability to conceive of ratios, or multiplicative comparisons, as the appropriate form of measure in a given situation. Simon and Blume (1994) noted that developing this ability is a complex process. They further identified some of the difficulties preservice teachers experience in doing so. For example, preservice teachers were observed having difficulty interpreting 3:2 as an expression indicating that the height is 1.5 times the length of the base; consequently, they tended to see ratios as useful descriptions of slope only when being compared to other ratios.

In summary, slope is a mathematical topic that is pervasive to secondary mathematics curricula. For example, it appears in algebra as a measure of the "rate of change" of a linear function, in trigonometry as a measure of a relationship between the

lengths of various sides of a right triangle embedded in a circle (e.g., the tangent function), and in calculus as a measure of how a function changes over time (e.g., the derivative function). However, the way in which slope is typically presented in textbooks—as the steepness of a line (as opposed to a ratio describing the multiplicative relationship between two quantities)—can obscure this fact and lead to the development instrumental understandings of this foundational mathematical topic. These two factors—the pervasiveness of slope to the mathematical curriculum and its ubiquitous treatment in textbooks—appear to have contributed to the emergence of the variety of conceptions of slope described above.

Revisiting the Research Questions

In Chapter 1, the following research questions were first presented:

1. What are teachers' espoused and inferred beliefs regarding (a) the generalization of students' learning, and (b) how to support the generalization of students' learning?

2. What is the relationship between teachers' beliefs regarding the generalization of students' learning and their MKT, meaning their understanding of (a) a particular mathematical topic, (b) the ways in which students might come to develop a particular understanding and (c) the actions they might take to support students in developing a particular understanding?

3. What is the relationship between teachers' beliefs regarding the generalization of students' learning and teachers' classroom practices?

Each research question is now elaborated using ideas from the literature presented above.

Research Question 1

Research Question 1 involves an examination of the explicit comments teachers make about (a) the generalization of students' mathematical learning, and (b) how they believe the generalization of students' mathematical learning can be supported. For

example, Sally said that she believed teachers should use "more examples that come from the real world" because "you can like, for example, teach them, you know, how to find the missing side of a right triangle and they go out in the real world and they don't realize that they can use that out there." Examination of comments like these seemed to indicate Sally's beliefs that (a) students generalize their learning of a learned topic to novel, real-world situations when they view those situations as manifestations of the learned topic, and, therefore, (b) teachers should strive to link the mathematical world with the real world during students' initial learning. However, it was not clear from these comments alone how Sally believed the real-world examples should be used or what particular aspects of the real-world examples should be emphasized.

Thus, Research Question 1 also involves inferring teachers' beliefs about the generalization of students' learning and how to support it. Inferring these teacher beliefs consisted of analyzing the ways in which teachers interpret contextually rich situations (Grant et al., 1998; Mason, 2002; Pajares, 1992). For example, what relationships do teachers conceive of between particular learning and transfer tasks? Moreover, inferences relied on analyses of the actions (Rokeach, 1968) teachers said they would take to support students in a hypothetical situation. For example, recall that Sally predicted Lucy would have trouble when confronted with real-world situations involving rate because Lucy would be unable to connect them to her procedural understanding of slope as rise over run. This prediction seemed to shed light on Sally's belief about the generalization of students' learning—conceptually meaningful understandings support the productive generalization of students' learning—and her belief about the ways in which real-world problems should be used in the classroom—to emphasize conceptually

meaningful interpretations of mathematical formulae. In this way, Sally's beliefs about the generalization of learning seemed to echo Judd's general-principles approach (1908) wherein transfer is thought to be supported by instruction targeting the mathematical ideas that underlie mathematical tasks. Interestingly, I was able to find new ideas as well as nuanced variations of the ideas that have already been highlighted in the transfer literature by (a) explicitly asking teacher participants questions regarding the generalization of their students' learning and the actions they would take to support the productive generalization of students' learning, and (b) inferring the beliefs of the teacher participants from their interpretations of and responses to hypothetical situations.

Research Question 2

Research Question 2 involves examining the understandings teachers have of a specific mathematical topic, namely slope. The first step in this examination was to uncover teachers' personal understandings of slope. This was accomplished during interviews with the teacher participants wherein they were asked to respond to a range of tasks involving slope. For example, some of these tasks involved answering straight-forward questions like "What does slope mean? Provide an example that illustrates its meaning" while others involved teachers' engagement in problem-solving situations (More details about these tasks will be presented in Chapter 3). The second step involved investigating whether and how teachers have transformed these understandings into MKT. In other words, had teachers thought about their own understanding of slope in relation to *how* students might come to also have that understanding? If so, how did they think students come to develop understandings that resemble their own? Furthermore, had teachers thought about their own understanding in relation to the actions they could

take in support of the development of that understanding? If so, what actions did they think support the development of understandings that resemble their own? Analyses carried out in service of these two steps revealed teachers' MKT as defined by Silverman and Thompson (2008). Finally, this question involves relating analyses of teachers' MKT to analyses carried out in service of Research Question 1 for the purposes of answering questions like, what can be said about the ways in which teachers understand a particular mathematical topic with respect to how they think about the generalization of students' learning of that same mathematical topic?

Research Question 3

In assuming consistencies between teachers' beliefs and teachers' practices, Research Question 3 involves a concerted effort to view teachers' actions as manifestations of their beliefs despite how they initially appear. This does not mean forcing consistencies. Rather, it means considering the possibility that teachers' beliefs can be further developed and elaborated through the situations teachers encounter and the actions they take in the classroom. Thus, actions that seemed inconsistent to previously espoused and inferred believes could be viewed as providing clarification to an identified belief. Alternately, the act of teaching was viewed as a means of bringing forth certain features of beliefs that may had been backgrounded during discussions that preceded instruction. Consequently, I attempted to view the ways in which teachers' practices can be seen as: (a) providing the researcher with additional information regarding the meanings of teachers' espoused and inferred beliefs, (b) a means by which the meaning of various aspects of teacher' beliefs may become clearer to the teacher participant, and

(c) bringing forth aspects of teachers' beliefs that were backgrounded during interviews

designed to get at teachers' beliefs.

CHAPTER 3:

Research Methods

In the previous two chapters, I presented and elaborated three research questions, motivating their emergence from pilot data and from the relevant research literature. Specifically, I showed that although there is a large research base on transfer, there appears to be nothing in the literature regarding teachers' beliefs about the transfer of learning or how it can be supported instructionally. Thus, Research Question 1 examines (a) teachers' beliefs about the generalization of students' learning and (b) how to support the generalization of students' learning. Moreover, I made a case for a reflexive relationship between teachers' beliefs regarding the generalization of learning and teachers' practices, wherein teachers' practices may serve to bring forth new facets of teachers' beliefs or to elaborate previously identified facets. Thus, Research Question 3 systematically examines the relationship between these specific teacher beliefs and teachers' practices.

Lastly, I demonstrated a plausible relationship between teachers' beliefs about the generalization of learning and teachers' mathematical knowledge for teaching (MKT), as defined by Silverman and Thompson (2008). In particular, I inferred two teachers' MKT and showed that while both teachers appeared to have similar personal understandings of slope, only one teacher provided evidence that she had developed the additional understandings associated with MKT. The teacher who had transformed her personal understanding of slope into MKT seemed to hold a belief regarding the generalization of students' learning that was consistent with her personal understanding of slope. On the other hand, the teacher who did not show evidence of having transformed her personal

understanding of slope into MKT appeared to hold a belief that was inconsistent with her personal understanding of slope. Thus, Research Question 2 seeks to explore the relationship between teachers' beliefs about the generalization of students' learning and MKT.

In what follows, I present a research design constructed for the purpose of answering these three questions. First, I provide a brief overview of the design. Then, I zoom in to discuss the particular methods used to examine each research question in turn. At the end of the chapter, I discuss the issues of reliability, validity, and disconfirming evidence.

Overview of Research Design

There research design used in this dissertation study has two phases, an interview phase and a classroom phase. The interview phase, or Phase I, was used to respond to all three research questions and the classroom phase, or Phase II, was used to respond to Research Question 3 (see Figure 3.1). In what follows, I describe Phase I and Phase II of the research design in more detail.

Phase I is shown in greater detail in Figure 3.2. In particular, Phase I includes two 2-hour clinical interviews (Ginsburg, 1997). Prior to the first clinical interview, the teacher participants were asked to reflect on the last time they taught a unit on slope and linear functions, go through their teaching materials, and select an item (e.g., a lesson plan, class activity, a test) they believed showed an instance in which they were thinking about supporting the generalization of their students' learning. The particulars of this Teaching Item Activity will be discussed in further detail later. Teacher participants then

emailed me either a description of the item or a photocopy of the item and it was

discussed during the first clinical interview.
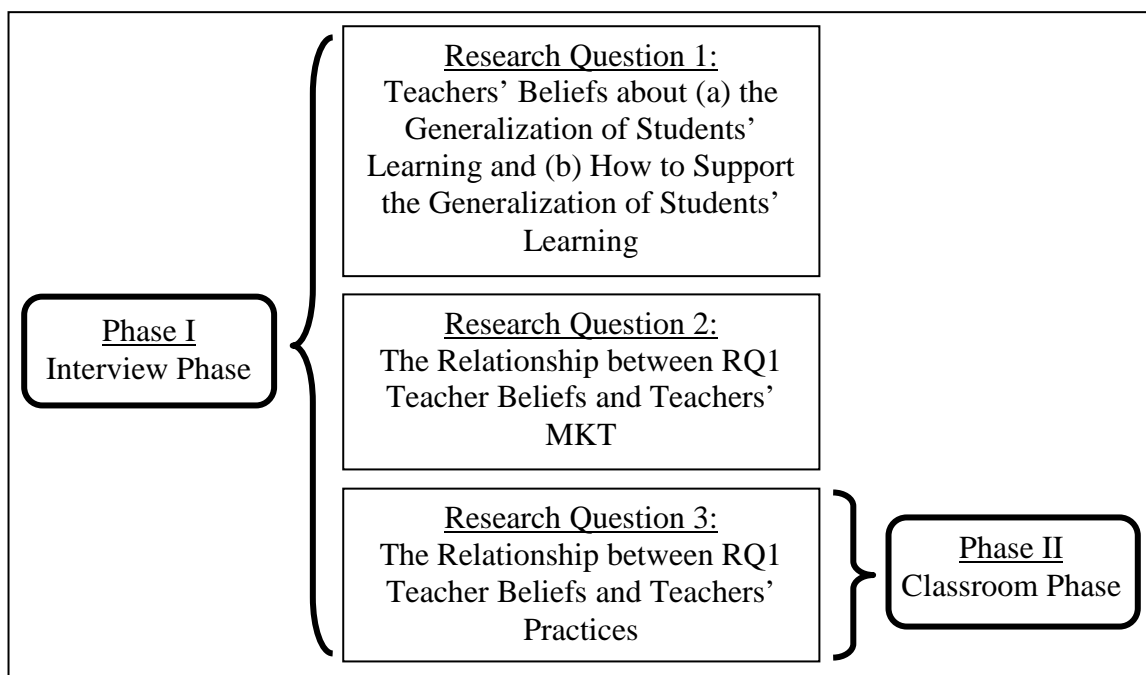
Figure 3.1: Relationship between Phase I and Phase II of the research design and the three research questions.

At the end of the first clinical interview, teacher participants were asked to draw

upon some of their ideas regarding the generalization of students' learning that had been

discussed during the interview to create a lesson plan on slope that prepares students to

generalize their learning beyond the specifics of the lesson. The particulars of this

Lesson Plan Activity will be discussed in further detail later. Teachers' lesson plans were

then discussed at the beginning of the second clinical interview.

In addition to being asked questions about their teaching items and lesson plans,

teachers were asked to elaborate various teaching goals and respond to questions about

slope and linear functions during both of the Phase I interviews (further details about

these questions will be discussed later). The first 2-hour clinical interview as well as the

first portion of the second 2-hour clinical interview was designed to get at teachers' beliefs about the generalization of students' learning and how to support the generalization of students' learning (see Interview #1 and Interview #2 in Figure 3.2). The remaining portion of the second interview was designed to get at different aspects of teachers' mathematical knowledge for teaching: (a) teachers' personal understanding of slope, (b) teachers' understandings of how students might come to develop an understanding of slope that is similar to their own, and (c) teachers' understanding of the actions they could take to help students develop an understanding of slope that is similar to their own (see Interview #2 in Figure 3.2).

Phase II is shown in greater detail in Figure 3.3. In Phase II, teacher participants were asked to teach the lesson on slope that they created and discussed in Phase I. Due to constraints involving being able to handle all the data collection and analysis associated with using all 8 participants for Phase II, half of the participants were asked to engage in the second phase of the study. The particular subset of teachers selected for participation in this phase were selected so that as a set there was variety in teachers' beliefs about the generalization of students' learning (as inferred via preliminary analysis of the Phase I interview data). When it came time for Phase II to begin, one of the teachers dropped out of the study. Thus, 3 teachers actually participated in Phase II which included (a) a classroom observation in which the participants were observed teaching their lessons and (b) a subsequent debriefing in which each participant was asked to discuss events that unfolded during the teaching of the lesson in relation to his or her beliefs as espoused and inferred during Phase I.

Data collected during Phase II were designed to work in conjunction with data collected during Phase I to answer Research Question 3: What is the relationship between teachers' beliefs about the generalization of students' learning and teachers' classroom practices? For example, teachers were reminded during the debriefing about statements made during the Phase I interviews regarding how to support the generalization of students' learning and asked to discuss whether and how they enacted the actions contained in those statements during the teaching of their lessons. Additionally, specific pedagogical actions of interest to the researcher were highlighted for the teachers and the teachers were asked to describe the rationale underlying those actions and whether and how they saw the actions as related to their beliefs (e.g., I noticed that you illustrated several real-world contexts for students. What was your thinking for doing that? What was your goal? Was this action related to your belief about the generalization of students' learning? If so, how?).

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┌─────────────────────────────────────────────────────────────────────────┐
│  ┌──────────────────────────┐          ┌──────────────────────────┐      │
│  │ Teachers receive         │ ══▷      │ Teachers send in a       │      │
│  │ instructions for         │          │ description or copy of   │      │
│  │ The Teaching Item Activity│         │ the selected teaching item│     │
│  └──────────────────────────┘          └──────────────────────────┘      │
│                                                  ║                        │
│                                                  ▽                        │
│  ┌──────────────────────────────────────────────────────────────────┐    │
│  │              Interview #1: Teachers' Beliefs                      │    │
│  │  ┌──────────────────────────────────────────────────────────┐    │    │
│  │  │ Question Set 1: Teaching Item Activity                    │    │    │
│  │  └──────────────────────────────────────────────────────────┘    │    │
│  │  ┌──────────────────────────────────────────────────────────┐    │    │
│  │  │              Question Set 2:                              │    │    │
│  │  │  Teachers' espoused and inferred beliefs about the       │    │    │
│  │  │  generalization of students' learning and how to support it│   │    │
│  │  └──────────────────────────────────────────────────────────┘    │    │
│  │  ┌──────────────────────────────────────────────────────────┐    │    │
│  │  │ Teachers receive instructions for The Lesson Plan Activity│    │    │
│  │  └──────────────────────────────────────────────────────────┘    │    │
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│                                                  ▽                        │
│  ┌──────────────────────────────────────────────────────────────────┐    │
│  │        Interview #2: Teachers' Beliefs and Teachers' MKT         │    │
│  │       ┌──────────────────────────────────────────────┐          │    │
│  │       │ Question Set 3: Lesson Plan Activity          │          │    │
│  │       └──────────────────────────────────────────────┘          │    │
│  │  ┌──────────┐ ┌──────────────────┐ ┌──────────────────┐         │    │
│  │  │Question  │ │Question Set 4**: │ │Question Set 4***:│         │    │
│  │  │Set 4*:   │ │Teachers'         │ │Teachers'         │         │    │
│  │  │Teachers' │ │understanding of  │ │understanding of  │         │    │
│  │  │KDUs      │ │how students      │ │actions that      │         │    │
│  │  │          │ │develop KDUs      │ │support the       │         │    │
│  │  │          │ │                  │ │development of KDUs│        │    │
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Figure 3.2: Phase I of research design.

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│  ┌──────────────────────────────────────────────────────────────────┐    │
│  │                   Inside the Classroom                            │    │
│  │  ┌──────────────────────────┐  ┌──────────────────────────────┐  │    │
│  │  │ Observation:             │  │ Debriefing:                  │  │    │
│  │  │ Watch teachers teach the │  │ Ask teachers questions about │  │    │
│  │  │ Lesson Plan*             │  │ their enactment of the       │  │    │
│  │  │                          │  │ Lesson Plan*                 │  │    │
│  │  └──────────────────────────┘  └──────────────────────────────┘  │    │
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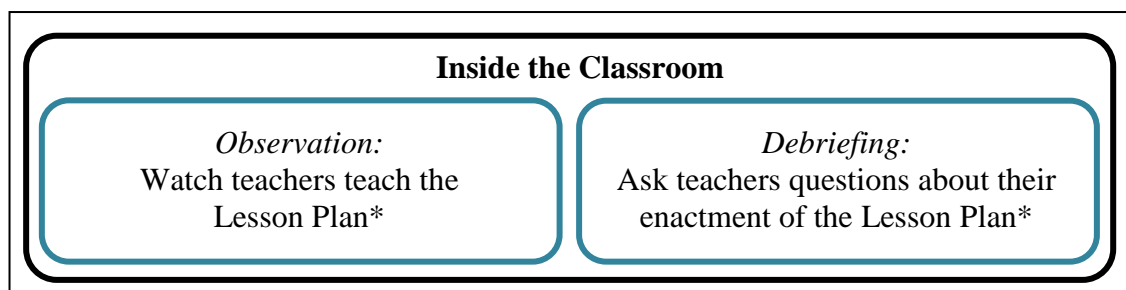
Figure 3.3: Phase II of research design.

Research Question 1

Research Question 1 asks the following question: What are teachers' espoused and inferred beliefs regarding (a) the generalization of students' learning and (b) supporting the generalization of students' learning? In Chapter 2, I made a case for using clinical interviews to study teachers' beliefs regarding the generalization of students' learning and how to support it. This section is organized into four main parts: participants, data collection, instruments, and data analysis.

Participants

Since this dissertation examines *teachers'* beliefs about the generalization of students' learning and how to support the generalization of students' learning, the participants in this study were practicing teachers.   Eight practicing-teacher participants were recruited from local middle and high schools. The rationale for having eight teachers is that it provided me with a manageable data set that was sufficiently large to allow for variation across teachers.

Selection criteria. Teacher participants were selected based on several criteria. First, participants were recruited based on their *current* employment status as a teacher at an accredited middle or high school. *Practicing* teachers were sought out so that when a subset of the participants were asked in Phase II to teach a lesson to a group of students (the details of which will be described later), a setting as well as a group of students was readily available.

Second, participants were recruited based on the nature of the mathematics courses they taught. As discussed in Chapter 2, slope provided the context in which teachers' beliefs were examined. Thus, teachers were chosen based on whether or not

they had the opportunity to develop students' understandings of slope during the 2011-2012 or 2012-2013 school year. Because slope is explicitly addressed as a distinct topic in Algebra I, I primarily targeted Algebra I teachers for recruitment. However, as noted in Chapter 2, students' understanding of slope is also developed in other content areas like trigonometry and calculus when dealing with, for example, tangent functions and derivative functions, respectively. Thus, I also recruited one Geometry/Calculus teacher, one Geometry/Algebra 2 teacher, and one Pre-Algebra teacher based on the fact that *they* saw slope as part of their curriculum.

Finally, I recruited teachers such that as a set, there was variation in the forms of practice that were enacted in their classrooms, the number of years teaching, the amounts of training and professional development received, and the type of school where employed (charter school vs. non-charter school). The rationale for seeking out such variation was that I hoped it would give me a better chance of selecting a group of teachers who held different beliefs regarding the generalization of students' learning and how it could be supported instructionally.

Recruitment source. The sources for locating potential participating teachers were my personal contacts through courses I took at SDSU, contacts with teachers who had previously established a research relationship with faculty working at the Center for Research in Mathematics and Science Education (CRMSE), and contacts with former students (who are also practicing teachers) of mathematics education professors at SDSU. Additionally, one teacher who was contacted in one of these three ways suggested names of other teachers who met the criteria for participation. Specifically, the first teacher I discussed possible participation with was a student with me in the MathEd 604 class at

SDSU. She was a full-time teacher at a charter school in San Diego, met the criteria for participation, and was very interested in being part of my study. She recommended two other teachers from her school. The remaining teachers were identified by SDSU mathematics education professors who recommended practicing teachers from their graduate seminars and CRMSE researchers (or their colleagues) who suggested teachers they had either worked with on research projects or believed would be willing to participate in my dissertation study.

Recruitment method. Once I had a list of potential teacher participants, identified through the process described in the previous section, I contacted the teachers by email or by phone, described the study to them and communicated the selection criteria, double-checked that they were practicing math teachers, and made sure they had opportunities to teach the topic of slope in one of their classes. If they met these two criteria (practicing teachers and had opportunities to teach slope) and were interested in participating, I arranged a site visit to their school to watch them teach a lesson so that I could get an idea of the pedagogical actions they enact during instruction. I also arranged a brief time to talk either before or after the lesson in which I inquired about their years of teaching experience as well as the types of training and professional development they had received. After observing all of the potential teacher participants, taking field notes on the actions they displayed while teaching, and obtaining the background information noted above, I selected a group of 8 teachers who, as a set, demonstrated the most variation.

The teacher participants. The teachers selected for participation in this study are listed in Table 3.1 along with some specifics regarding selection criteria. In

particular, the table shows the content areas (relevant to the present study) each

participant was in charge of teaching during the 2011-2012 and/or 2012-2013 school

year, the number of years of teaching experience each participant had, the topic of some

of each participant's professional development experiences, and a list of some of the

teaching actions that were observed during recruitment. Gender-preserving pseudonyms

are used for all teacher participants.

Table 3.1: The teacher participants.

Teacher Participants	Relevant Classes	Number of Years Teaching	Topic of Professional Development	Teaching Actions
Anne	Geometry Algebra 2	7	• Technology • Gradual Release of Information • Purpose Setting	• Lecturing • Demonstrating Solution Strategies • Accepting Calculational Explanations
Donna	Algebra 1	18	• Mentoring Student Teachers • Coaching New Teachers	• Lecturing • Demonstrating Solution Strategies • Facilitating Group Work
Blake	Algebra 1 Geometry	11	• Teacher Leadership • Pacing • Supplementary Materials	• Demonstrating Solution Strategies • Facilitating Group Work • Accepting Answers without Explanation
Richard	Algebra 1	33	• Implementing the Core-Plus Curriculum	• Facilitating Group Work • Summarizing What Was Learned
Patrick	Algebra 1	26	• Implementing the Interactive Mathematics Program	• Demonstrating Strategies • Facilitating Group Work • Asking Students to Present Their Solutions
Kay	Algebra 1	3	• Curriculum Development and Implementation	• Facilitating Group Work & Student-Student Dialogue • Pressing for Student Explanations
Emma[2]	Pre-Algebra	1	N/A	N/A
Sam	Geometry Calculus	15	• English Language Learners • Assessment • Supplementary Resources	• Facilitating Group Work • Asking Students to Present Their Solutions • Restating Student Solutions

Data Collection

The eight recruited teachers were asked to engage in Phase I of the study which,

as noted above, involved two 2-hour clinical interviews (Clement, 2000; Ginsburg,

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[2] As Emma was a first-year teacher during the 2012-2013 school year, she had yet to engage in professional development and had yet begin teaching during recruitment.

1997). During the first interview and a small portion of the second interview, which informed all three research questions, teachers were asked questions and posed tasks designed to elicit their beliefs about the generalization of students' learning and how to support the generalization of students' learning (see Interview #1 and Interview #2 in Figure 3.2). The same major tasks and questions were posed to each teacher, but the follow-up probes were tailored to individuals (Ginsburg, 1997). During the remaining portion of the second interview, which informed the third research question, teachers were given tasks, with individual follow-ups, to probe their MKT (see Interview #2 in Figure 3.2).

As discussed in Chapter 2, individual clinical interviews can provide researchers access to teachers' interpretations of interview tasks and the meanings they associate with various aspects of their responses (Ginsburg, 1997), which is critical in the process of inferring teachers' beliefs regarding the generalization of students' learning and how it can be supported. Also noted in Chapter 2 was another important feature of clinical interviews, namely *hypothesis testing* (Clement 2000; Ginsburg, 1997). In other words, clinical interviews provide a means by which inferences of teachers' beliefs can be tested *in the moment*. This occurred, for example, in interviews when a teacher provided statements I found ambiguous. During the relevant portions of both interviews, I formulated hypotheses about a teacher's beliefs and attempted to generate data that would either support or refute my hypotheses by posing questions or tasks that further probed the teacher's thinking. Thus, I tried to anticipate some of the follow-up probes but others were generated in the moment to test emergent hypotheses.

The first interview was conducted as soon as the teaching item a teacher had selected had been checked for fit in relation to the instructions and goals stated in The Teaching Item Activity as described in the next section, and as soon as the teacher's schedule permitted. The second interview was conducted according to the availability of each teacher. My rationale for separating these interviews and conducting them on different days was that doing so would help guard against interview fatigue. Further rationale will be presented later. Each clinical interview was recorded with a video camera and a table microphone to obtain a rich record of teachers' responses that could be viewed over and over again. The video camera was aimed to capture teachers' verbal reports, gestures, and written inscriptions. All written work and materials were collected.

Instruments

The major questions and tasks designed to get at teachers' beliefs about the generalization of students' learning and how it can be supported are separated into three main sets, Question Set 1 (questions that are related directly to The Teaching Item Activity), Question Set 2 (major tasks and questions related to teachers' *espoused* beliefs about the generalization of students' learning and how to support it and major tasks and questions designed to generate data that will allow me to *infer* teachers' beliefs), and Question Set 3 (questions that are related directly to The Lesson Plan Activity). I discuss each set of questions below.

Question Set 1: The Teaching Item Activity and associated questions. Prior to engaging in the first of two 2-hour clinical interviews (see Figure 3.2), teacher participants were asked to select an item from a unit they had used to teach the topic of slope and linear functions that they believed showed an instance in which they thought

about supporting their students in being able to generalize or apply their understanding of slope to a new task, activity, or situation (see *Appendix A* for a copy of the instructions that will be given to teachers regarding The Teaching Item Activity). Once an item had been chosen, teacher participants were asked to send me an email describing (a) how they had interpreted the prompt; (b) the item they selected; and (c) why they chose the item. These emails served one primary purpose, namely to help ensure teachers' interpretation of The Teaching Item Activity were consistent with the intentions of the activity. In particular, I checked for teachers' interpretation of "the generalization of students' learning" so that if, for example, a teacher interpreted *generalization* in terms of students' creation of a formula for the nth case, I could provide further clarification regarding the phenomenon of interest prior to the first interview.

During the first 2-hour clinical interview, teacher participants were asked to engage in discussion surrounding their teaching items (see Figure 3.2). The Teaching Item Activity and the associated set of questions teachers answered and discussed (see *Appendix A* for the associated set of questions) were designed to get at teachers' espoused and inferred beliefs regarding the generalization of students' learning and how the generalization of student' learning could be supported instructionally. Specifically, questions 1-4 of Question Set 1 were designed to get at teachers' *espoused* beliefs. For example, question 1 asks teachers to talk about their lesson and to describe how they believe "it shows [they] were thinking about helping students to make future use of their learning."

Questions 5-7 were designed to generate data from which to *infer* teachers' beliefs. In particular, teachers were asked whether they believed that as a consequence of

students' engagement with their teaching item, students would be able to successfully

engage with: The Water Pump Task (adapted from Lobato et al., in press), The Burning

Candle Task (adapted from a transfer task used by Lobato et al., in press), and The Ice

Cream Task (adapted from Lobato, personal communication). These tasks were chosen

for various reasons. First, they all involve real-world contexts. Such contexts were

included to gain insight regarding teachers' beliefs regarding the role of the real world in

the generalization of students' learning. Second, The Water Pump Task asks about "rate"

rather than "slope" and was thus intended to provide insight regarding *what* teachers'

believe generalizes from an initial, learning to a novel, transfer situation (e.g., a formula

for slope, one of the conceptualizations of slope as discussed in Chapter 2). In addition,

The Water Pump Task involves a tabular representation which was included to provide

insight regarding whether and how teachers focus on strategies and procedures affiliated

with such representations (e.g., find the difference between successive rows in the

"Amount of Water" column and divide the resulting number by the difference between

corresponding rows in the "Time" column).

Third, The Burning Candle Task asks explicitly about slope and involves another

tabular representation. Thus, The Burning Candle Task like The Water Pump Task was

selected to support teachers in thinking about what students generalize from an initial,

learning situation to a novel, transfer situation. The fact that The Water Pump Task

involves positive slope and The Burning Candle Task involves negative slope was also

important in that it was meant to highlight teachers who focused on the generalization of

procedures or formulas since a teacher could see the tasks as involving distinct

procedures and skill sets.

Finally, I included The Ice Cream Task to further probe teachers' beliefs about transfer. The Ice Cream Task is unique among the set in that it does not involve conventional representations of tables, graphs, or equations. Rather, it involves a real-world situation and the attributes of slant and taste. Further, "slope" is nowhere in the problem statement. Thus, like The Water Pump Task, The Ice Cream Task provides insight regarding what teachers believe generalizes from a learning situation to a transfer situation and how the generalization of that learning could be supported.

Question Set 2: Teachers' beliefs about the generalization of students' learning and how to support the generalization of students' learning. Question Set 2 (see Figure 3.2) was designed to get at teachers' beliefs about the generalization of students' learning as well as their beliefs regarding how the generalization of students' learning can be supported instructionally. This set of questions is not associated with The Teaching Item Activity or The Lesson Plan Activity. Question Set 2 involves additional questions designed to reveal teachers' *espoused* beliefs about the generalization of their students' learning and how to support it. Question Set 2 also involves additional questions that are intended to generate data from which to *infer* teachers' beliefs. The questions in Question Set 2 were posed during the first interview after Question Set 1.

The rationale for posing most of the teacher beliefs questions during the first interview was that all three of my research questions rely on analyses of data regarding teachers' beliefs about the generalization of students' learning. Thus, posing the questions in Question Sets 1 and 2 during the first interview provided me with a small window of time in which I could generate hypotheses regarding teachers' beliefs as well as clarification questions regarding their beliefs. These hypotheses and clarification

questions were noted and attended to during the discussion of Question Set 3 (i.e., The Lesson Plan Activity) in Interview #2.

Question Set 2 includes questions involving hypothetical students' responses to a slope task (adapted from a task used during the screening interview mentioned in Lobato et al., in press; see *Appendix B* for a copy of the hypothetical students' work as well as the initial prompt that was presented to teachers). In particular, teachers were presented with Molly's and Lucy's work on The Leaky Bucket Task discussed in Chapter 1. In this version, both Molly and Lucy provide correct answers. However, Molly's explanation demonstrates an understanding of *slope as ratio* (see Chapter 2 for details) and Lucy's explanation demonstrates an understanding of *slope as a formula* (see Chapter 2 for details). Teachers were shown one student's work, asked to make predictions about the kinds of tasks that the student would be able to successfully engage with as well as the kinds of tasks that may prove too difficult for the student, and asked about teacher actions that could be used to support the student so that she will be able to successfully engage with both kinds of tasks. This process was then repeated with the other student's work. Teachers' answers to questions like "What would you do with Molly during The Leaky Bucket Task to support her engagement with the more difficult tasks?" were taken as evidence of teachers' espoused beliefs.

After the teachers had discussed Molly's and Lucy's work, made predictions, and offered supportive teacher actions, they were presented a set of three tasks and asked a similar series of questions as the questions mentioned above. (See *Appendix B* for a copy of the interviewer questions for this task.) The rationale for presenting teachers with a set of three specific tasks is that it gave the teachers something tangible to base their

predictions regarding the generalization of students' learning and associated teacher actions on. Moreover, because the tasks were chosen in a manner similar to the way in which the tasks were chosen for The Teaching Item Activity, they provided contexts and situations from which teachers' beliefs were inferred.

In addition, Question Set 2 contains a set of three hypothetical teachers' activities. Teacher A's activity focuses on angles and steepness. Teacher B's activity focuses on making connections between the conventional representations tables, equations, and graphs. Teacher C's activity focuses on the idea of rate by asking students to reason about a runner's speed during a marathon. Teachers were presented with these activities and asked whether and how they would use each of the activities if their primary goal were to support students in generalizing their understanding of slope (see *Appendix B* for the activities and associated questions).

Question Set 3: The Lesson Plan Activity and associated questions. During the first 2-hour interview, teacher participants were asked if they thought they could create a lesson (or adapt an existing lesson) such that the lesson implements some of their ideas regarding the generalization of students' learning and how to support it (see question 8 in Question Set 1 in *Appendix A*). All teachers responded in the affirmative and were therefore asked to do so and to bring the lesson to the second 2-hour interview (see *Appendix C* for a copy of the instructions that were given to teachers regarding The Lesson Plan Activity). The Lesson Plan Activity and the associated set of questions teachers answered and discussed (see *Appendix C* for the associated set of interviewer questions) were designed to get at teachers' beliefs regarding the generalization of students' learning and how it could be supported instructionally. Moreover, as this

activity was called The *Lesson Plan* Activity and the teachers were told that they may be asked to participate in Phase II of the dissertation study (in addition to participating in the Phase I interviews) wherein they would be observed carrying out this lesson, the activity was designed to generate data regarding the actions teachers might carry out in their current classrooms with their current students and teachers' beliefs surrounding those actions. It was also intended to highlight candidates for participation in Phase II.

Data Analysis

One goal in analyzing the data collected during the interviews is to identify teachers' beliefs about the generalization of students' learning and how to support it. Interview #1 data were thus analyzed using a mixed methods approach, or what Miles and Huberman (1994) describe as "partway between a priori and inductive coding" (p. 61). Categorizing teachers' beliefs involved drawing upon the transfer literature discussed in Chapter 2. For example, some of the teacher participants expressed the belief that teachers should make use of multiple exemplars as a way of supporting the generalization of learning—a belief found in the research literature from the mainstream cognitive perspective on transfer (e.g., Gentner et al., 2003; Markman & Gentner, 2000). Other categories of teachers' beliefs were induced using *open coding* from grounded theory (Strauss, 1987). Strauss and Corbin (1990) acknowledge the use of literature as legitimate within a grounded theory approach, as long as one is not overly constrained by existing categories.

Open coding involves fracturing the data or breaking them apart into discrete segments. Segments that share similarities are grouped together and each group is labeled or given a category name so that it may be conceived of and discussed as a belief.

One reason for labeling or naming groups of similar segments is to reduce the data from a fractured set containing many individual segments to a set containing related segments and thus much fewer categories.

A key feature of grounded theory is *the constant comparative method* (Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1990, 1994). The constant comparative method refers to a process of revisiting segments and groups of data in an effort to refine and revise their possible meanings as well as the ways in which they might be similar and/or different (Corbin & Strauss, 1990; Merriam, 2009). It is not the case, for example, that once a segment of data has been analyzed for meaning and categorized that the results of those analyses remain unchanged. Rather, as more and more data get analyzed and categorized, previous analyses are reexamined and revised as necessary in light of new data and emerging categorization schemes.

Research Question 2

Research Question 2 asks the following question: What is the relationship between teachers' beliefs regarding the generalization of students' learning and teachers' MKT, as defined by Silverman & Thompson (2008). For the purposes of this dissertation, examining teachers' MKT involved investigating three different types of understanding: (a) teachers' personal understanding of slope, (b) teachers' understanding of how students' might come to develop an understanding of slope that is similar to their own, and (c) teachers' understandings of the actions they can take to help support students in developing an understanding of slope that is similar to their own. In this section, I discuss data collection, instruments, and data analysis.

Data Collection

The second interview was conducted after the first, according to teachers' availability. As noted above, the time between interviews provided me with a period of time in which to generate clarification questions with respect to teachers' beliefs about the generalization of students' learning. These questions were posed at the beginning of the second interview and prior to the tasks from Question Set 4. To obtain a rich record of the data related to teachers' MKT, the second interview was recorded with a video camera and a table microphone. As was the case with the first interview, the video camera was aimed to capture teachers' verbal reports, gestures, and written inscriptions. All written work was collected.

Instruments

Teachers' MKT was examined during the second interview of Phase I of the research design (see Figure 3.2). The second interview followed the clinical interview format discussed earlier. In particular, teachers were posed the same major set of tasks and questions, but the follow-up probes were tailored to individual teachers (Ginsburg, 1997). Follow-up probes were constructed so that I was afforded further opportunities to investigate teachers' interpretations and meanings, and so that I could test emergent hypotheses regarding various aspects of teachers' MKT (Clement 2000; Ginsburg, 1997).

In what follows, I describe the major tasks and questions that were posed to teachers in order to gather data on teachers' MKT. Because I investigated three types of knowledge associated with MKT, this section is separated into three main parts such that each part is associated with a particular type of knowledge. The organization here is not

intended to be reflective of the order in which the questions were posed during the interview.

   Teachers' personal understanding of slope. Teachers' personal understanding of slope was investigated during the second interview using Question Set 4 (see Figure 3.2). The questions in Question Set 4 that were designed to get at teachers' personal of slope are indicated with a single asterisk (i.e., *) and involve questions about the meaning of slope and various values for slope (see *Appendix D* for this set of questions). These questions tended to avoid instructional situations and situations involving hypothetical students. Rather, they were designed to be direct questions that required teachers to provide responses indicating their own understanding of the mathematical topic of slope. This was so that this set of questions could be conceived of as being about a particular kind of understanding—teachers' personal understandings of slope. For example, Question Set 4 contains questions like: What does slope mean; what does slope represent; what is one measuring when measuring slope; what does it mean to have a slope of ½; and what does it mean to have a slope of -1?

   Question Set 4 also includes a task wherein teachers were asked to graphically represent a linear function and to create an associated story. Teachers were then asked to respond to several questions about the meaning of various quantities, operations, and terms in relation to the context provided by their story. For example, what does "4-2" mean in the context of your story? What does the division in the slope formula accomplish in relation to your story? What does slope mean in the context of your story?

Question Set 4 additionally asks teachers to consider a situation involving a hypothetical teacher. In The Hypothetical Teacher Situation[3], a hypothetical teacher expresses concern about the definition of slope after noticing that a graphically represented function looks steeper on one set of axes than it does on another. The teacher participants were asked to provide a response to the hypothetical teacher. This activity was included to generate data regarding the ways in which teachers interpret the meaning of slope.

Lastly, Question Set 4 asks teachers to consider an additional measure of the steepness of a ramp (in addition to the ratio of height to length). In particular, it asks teachers to consider whether the ratio of "slant height" (i.e., the length of the hypotenuse of a right triangle) to length could be used as a measure of steepness of a ramp. This task[4] was included to probe teachers' understanding of ratio.

Teachers' understanding of how students' come to develop an understanding of slope that is similar to their own. Teachers' understanding of how students come to develop an understanding of slope that is similar to their own was investigated during the second interview using Question Set 4 (see Figure 3.2). The questions in Question Set 4 that were designed to get at this type of understanding are indicated with a double asterisk (i.e., **) and involve a focus on students' understanding of slope and how those understandings develop (see *Appendix D* for this set of questions). For instance, after providing descriptions for the ways in which teachers personally interpret the meanings

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[3] The Hypothetical Teacher Situation was adapted from a task developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.
[4] The Slant Height Task was developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.

of various aspects of the slope formula (e.g., differences of the y_2-y_1 and the $\div$ in $\frac{y_2-y_1}{x_2-x_1}$)

by drawing upon the contexts provided by their stories (as described in the previous

section), they were asked questions like: How might *students* come to develop that

meaning for division? In addition, teachers were provided with tasks involving linear

functions and asked questions like: What must a student understand to answer the

question in the task; What difficulties or sources of confusion might students have in

responding to the task? Moreover, teachers were provided with hypothetical student

responses to tasks and asked questions like: What understandings underlie such a

response; What might the student have been thinking?

These questions were meant to differentiate teachers like Bonnie in Chapter 1

who appeared to have developed mathematically sophisticated understandings of slope

but not to have developed associated understandings of how students come to hold those

understandings from teachers like Joanne who seemed to have developed both kinds of

understandings. Recall that in Chapter 1 I presented evidence suggesting Bonnie

understood slope as a ratio measuring the rate of change of a function but that she did not

talk about this understanding when discussing the difficulties students might face or when

giving examples of what students might say to indicate they had developed an

understanding of slope. Rather Bonnie spoke of difficulties associated with remembering

the order of the "rise over run" formula and student statements involving the terms

"slant" and "steepness." In contrast, Joanne talked about difficulties associated with

forming a ratio and with conceiving of the slope formula in terms of a ratio.

Lastly, Question Set 4 includes instructional situations. For example, teachers

were asked whether a given task involving a truck's motion through space could be used

to develop students' understanding of slope[5]. It was included as a means of highlighting

the ideas teachers view as important when developing students' understanding of slope.

For example, did teachers think the task is useful inasmuch as it can be represented

graphically and discussed in terms of steepness or did teachers find this task useful

because students can use unit ratios to solve the problem? Again this question was meant

to differentiate teachers who have yet to develop an understanding of how students come

to have conceptually meaningful understandings of slope from those who have.

Teachers' understandings of the actions they can take to help support

students in developing an understanding of slope that is similar to their own.

Teachers' understandings of actions that support students in developing an understanding

of slope that is similar to their own was investigated during the second interview using

Question Set 4 (see Figure 3.2). The questions in Question Set 4 that were designed to

get at this type of understanding are indicated with a triple asterisk (i.e., ***) and involve

a focus on teacher actions (see *Appendix D* for this set of questions). For example,

Question Set 4 contains questions like: What can you do to help students develop an

understanding of what the division used in the slope formula accomplishes; What can you

do to help students develop an understanding of slope? As was the case in the previous

section, these questions involve hypothetical students and hypothetical student responses,

but this time they ask about teaching actions, for example, What would you do or say

with this student to support the development of her understanding? As before, these

questions were posed to differentiate teachers like Bonnie and Joanne. Remember that

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[5] This task was adapted from a task developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.

Bonnie suggested teacher moves (e.g., creating mnemonics based on experiences that would be realizable and familiar for students) that would help students remember the order in which to carry out the slope formula while Joanne suggested teacher moves (e.g., using the same attribute task) that would help students conceive of and focus on quantitative relationships.

Data Analysis

Another goal in analyzing the data collected during the interviews is to identify the relationship between teachers' beliefs about the generalization of students' learning and teachers' MKT. As noted above, teachers' beliefs were analyzed using a mixed methods approach (Miles & Huberman, 1994) wherein some categories were derived from the transfer literature and some were induced using open coding from grounded theory (Strauss, 1987). Similarly, teachers' MKT was analyzed using a mixed methods approach. Specifically, categorizing teachers' personal understandings of slope involved drawing upon understandings highlighted in the mathematics education literature like the understandings outlined in Chapter 2 (e.g., *slope as the angle of inclination* and *slope as a ratio*) and inducing codes from data obtained during the second interview (see Figure 3.2). Categorizing the other two kinds of understandings—teachers' understanding of how students come to develop an understanding of slope that resembles their own and teachers' understanding of the actions they can take in order to support students in developing understandings of slope that resemble their own—relied more heavily on open coding although relevant literature was drawn upon when appropriate.

Establishing relationships between teachers' beliefs and teachers' MKT involved *axial coding* (Strauss & Corbin, 1990). Axial coding follows open coding wherein data

are fractured or split apart into smaller segments and categorized according to the phenomenon of interest (here, some data were categorized according to teachers' beliefs and other data were categorized according to teachers' MKT). During axial coding, the fractured data are put back together again by establishing relationships or connections between categories. Here, I was seeking to establish relationships between categories of a teacher's beliefs and categories of a teacher's MKT. In seeking to establish such relationships, axial coding involved scrutinizing the data to determine the following features: the conditions that gave rise to the teachers' beliefs and MKT, the multiple contexts in which such data were collected, the action/interactions through which teachers' beliefs and MKT were discussed and assessed, and its consequences. These features are typically identified during open coding but now become associated with the phenomena of interest—relationships between teachers' beliefs about the generalization of students' learning and teachers' MKT. Much in the same way open coding involves labeling or naming segments of data, axial coding involves making statements about the relationships between categories and then verifying those statements within the data (Corbin & Strauss, 1990). Thus, the constant comparative method described earlier continues to play a large role in axial coding.

Research Question 3

Research Question 3 asks the following question: What is the relationship between teachers' beliefs regarding the generalization of students' learning and teachers' classroom practices? More specifically, this question examines the ways in which teachers' classroom practices illustrate, elaborate and/or extend teachers' beliefs about

the generalization of students' learning. In this section, I present the data collection and analysis methods that were used to carry out this examination.

Data Collection

Because the purpose here was to examine teachers' practices in order to further understand their beliefs about the generalization of learning, examination of teachers' practices were tailored to individual teachers. Thus, there were three steps to the data collection process: (a) specifying the dimensions of teachers' practices that were initially attended to during classroom observations; (b) conducting classroom observations; and (c) conducting post-observation debriefings. Each step of the data collection is discussed in turn. Moreover, I illustrate each step by drawing upon data collected during Bonnie's interview wherein her belief about the generalization of learning was inferred (see Chapter 1) as well as data yielding from videotaped observations of Bonnie's classroom instruction. As there is no data from post-instruction interviews with Bonnie, I illustrate the third step of the data collection process by describing the kinds of questions I would have asked and the kinds of data that might have resulted.

Specifying dimensions of teachers' practices. Prior to conducting Phase II of the research design wherein I observed teachers' classroom practices and engaged teachers in a subsequent debriefing (see Figure 3.3), I watched the videotaped records of teachers' clinical interviews (see Figure 3.2). While watching a recording of a particular teacher's interviews, I identified the teacher's beliefs regarding the generalization of learning by making preliminary inferences from the discussion of Question Sets 1, 2, and 3. Then I used these preliminary inferences to orient me to particular aspects of the teacher's practice during the classroom observation portion of Phase II. In other words, I

watched the interviews for the explicit purpose of identifying themes that seemed to be of central importance to a teacher's beliefs about the generalization of students' learning (à la Bowen, 2006). The identification of themes had the effect of reducing the set of possible teacher actions that could actively be attended to during Phase II.

For each theme that was identified, I watched teachers' practices for four specific purposes. First, I watched for the *form* in which the theme appeared. In other words, how did the theme get instantiated during the lesson (e.g., a verbal statement, a mathematical activity, a drawing)? Second, I watched for *meaning*. In other words, based on the form in which the theme appeared, what could be said about how the teacher seemed to be interpreting the theme? Third, I watched for the *function*[6] that the theme seemed to serve in supporting the generalization of students' learning. In particular, what role did the theme seem to play in helping students to make future use of their learning (e.g., to tell students that a mathematical idea is relevant outside of the classroom; to necessitate sense making; to make a mathematical idea seem familiar; to help students memorize)? Finally, I attempted to infer via observations of a teacher's pedagogical actions, *what* the teacher believed she was helping her students to generalize (e.g., a procedure, an idea, an experience, a heuristic)?

To illustrate this step of the data collection process, consider Bonnie's beliefs regarding the generalization of learning that were presented in Chapter 1. Specifically, an individual interview with Bonnie revealed the importance of the real world in Bonnie's beliefs regarding the generalization of learning. In Chapter 1, I presented data

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[6] *Form*, *meaning*, and *function* are terms Lakoff (1990) used to describe properties of language. I am using them here to describe properties of themes that seem to be of central importance to teachers' beliefs about how to support the generalization of students' learning.

that showed Bonnie's inclination to connect the mathematical topic of slope with familiar

aspects of students' real world experiences and her belief that mnemonics, analogies, and

images emerging out of these real world experiences would help guide students' future

mathematical performance. Thus, the *real world* seemed to be a theme of central

importance in Bonnie's belief about the generalization of students' learning.

If I were to observe Bonnie's teaching practice, I would begin by looking for the

forms in which the identified theme appears. Specifically, how does the real world

appear in Bonnie's lesson? Secondly, I would observe Bonnie's lesson for indications of

how she seems to be interpreting the meaning of the identified theme. For instance, does

Bonnie seem to be interpreting the real world as a world that is separate from the

mathematical or classroom world? Third, I would examine the functions the theme

seems to serve with respect to helping students generalize their learning. Is the real world

used merely as a source for the creation of mnemonics or is it used in some other way

that helps to facilitate the generalization of learning? Lastly, *what* did Bonnie's

pedagogical actions seem to support her students in transferring (e.g., procedures,

meaningful understandings, heuristics)? As I do not have access to Bonnie's current

teaching practice, I will draw upon recordings of lessons she has previously taught in

continuing to illustrate the methods of data collection that were used to answer Research

Question 3.

Conducting classroom observations. Teachers' practices were observed during

the execution of the lesson that teachers planned during Phase I of the study. By asking

teachers to demonstrate a lesson explicitly created for the purposes of supporting transfer,

I aimed to foreground the issue of teaching for the generalization of students' learning

and background other constraints that may heavily influence teachers' everyday practices like pacing guides and state tests. Teachers were observed during the 2012-2013 school year during normal school hours. Moreover, with the exception of one teacher, teachers were observed teaching their lessons on slope in sequence, that is, during the slope and linear functions unit of their normal curricula.

During classroom observations, I set up a video camera so that it recorded the majority of teachers' actions during instruction. I solicited help from the teachers regarding where to place the camera. For example, if a teacher planned to lecture at the board, I set the camera up so that it captured the front of the classroom; if a teacher planned to have students work in small groups, I held the camera and followed the teacher from group to group while keeping the camera focused on the teacher. Furthermore, I equipped teachers with a wireless microphone to ensure that all of the comments teachers made to their students were recorded.

I also took fieldnotes organized by the list of themes identified from the Phase I interviews and by the four criteria of form, meaning, function, and what the theme seemed to support the transfer of. To illustrate the nature of the fieldnotes using Bonnie from Chapter 1, I watched one of her 10 one-hour lessons (recorded for the purposes of the larger NSF funded study mentioned in Chapter 1) and took notes on the theme identified above (i.e., the real world). It should be noted that Bonnie was not explicitly told to teach for transfer. Thus, it is impossible to know what would have happened if her explicit purpose had been to prepare students for successful engagement in novel situations. Nonetheless, I chose to revisit the lesson referenced in Chapter 1 wherein Bonnie talked about points, or coordinate pairs, in terms of house addresses. I

specifically watched this lesson for the theme of the real world and took notes on the

form in which the theme appeared, the *meaning* the theme seemed to have for Bonnie, the

function the theme seemed to serve with respect to supporting the generalization of her

students' learning, and the *what* that Bonnie's actions seemed to be helping her students

transfer.

In the events leading up to the first time the real world appeared in Bonnie's

lesson, students explored patterns in the context of a visually growing pattern (see Figure

3.4a) and created associated tables of data (see Figure 3.4b). Bonnie then asked students

to go up to the overhead and graph the data recorded in their tables. After several points

had been plotted, a line had been drawn connecting the points, and the class had

discussed their observations of the line (e.g., "the line is slanted"), Bonnie highlighted the

point (0, 1) and labeled it "the intercept."
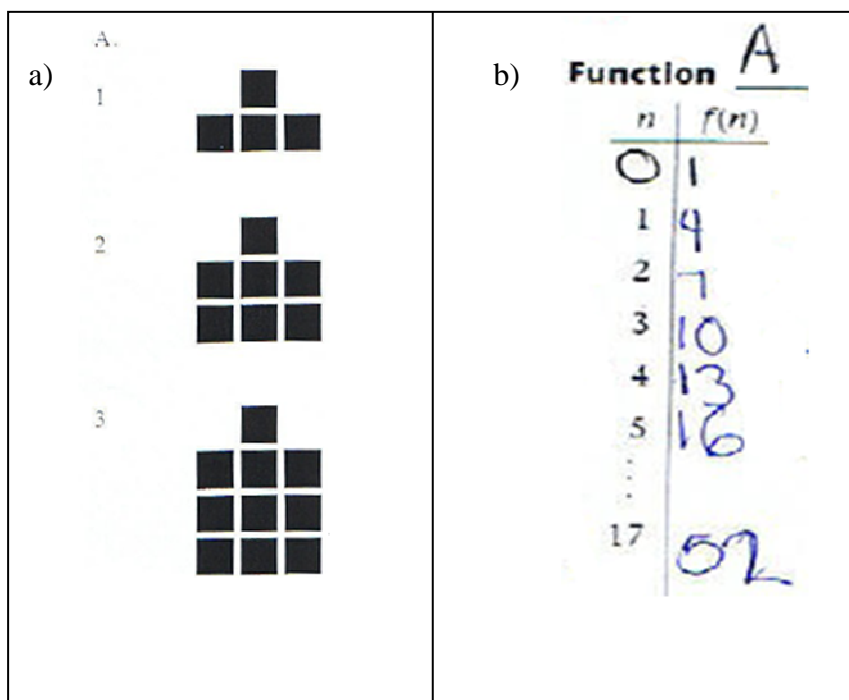
Figure 3.4: (a) A visually growing pattern used during Bonnie's lesson; (b) The table
associated with the visually growing pattern shown in (a).

The real world first appeared in response to a student's question regarding the assigned label of "intercept." Specifically, a student asked, "Isn't it called a point?" The student went on to explain that his science teacher often says "go to the point" and that when his science teacher says this, he could be referring to any of the points not just the intercept. Bonnie responded by exclaiming, "Oh, the coordinate point!" She then asked the students if they had seen coordinate points before and wrote "(0,1)" and "(1,4)" on the overhead. She said "this is how they write a point instead of using a table; sometimes they do write it as a coordinate like this [pointed to "(0,1)" and "(1,4)" written on the overhead] to represent your point." She then highlighted the fact that "the points all look the same" and explained that in order to distinguish the points from one another mathematicians "had to give [the points] some names; it's almost like an *address*, like *how to get to your house*."

Here, the theme of the real world, identified while watching Bonnie's interview, took the *form* of a verbal reference about home "addresses" and "get[ting] to your house." Bonnie explained the coordinate pair representation—(x,y)—in terms of an address rather than in terms of its relationship to some reference point called the origin. In this way, it seemed that Bonnie drew on real-world situations that students would be familiar with; thus, Bonnie may view the *meaning* of real-world situations as situations that are easily relatable for students. Further, the form in which the real world appeared seemed to serve a particular *function*. Specifically, Bonnie acknowledged that "coordinate point[s]" could be used to label dots on the graph and seemed to liken coordinate points to addresses *so that* students would be more likely to remember why and how they were useful. This could also be seen later in the lesson when Bonnie

explained that coordinates are "a way of helping people with directions" and that they are actually called "ordered pairs" because the order in which the pair is listed matters. She asked her students, "If I switch this and go 4 and then 1, am I in the same place? … No! I'm nowhere near [the point (1,4)]!" Thus, Bonnie seemed to link or make connections between ordered pairs and addresses so that in the future students would remember that an ordered pair provides detailed information about a location on a graph.

In a similar turn of events, the real world appeared a second time when Bonnie used "the variable *m*" to rename the growth between successive figures in the visually growing pattern shown in Figure 3.4a. In particular, Bonnie pointed to the number "3" which had been used to represent the successive growth between figures, explained that it was called the *slope*, and said that mathematicians use *m* to represent slope. One student asked why they used the letter *m* and another student asked if it was their way of "trying to confuse us." Bonnie responded by asking the class, "where have you heard of the word slope?" The students said "mountains." Bonnie repeated, "mountains, like skiing down the slopes, so they [mathematicians] thought *m.*"

In this situation, the real world took the *form* of a verbal reference made by the students in response to a question posed by the teacher, namely "where have you heard of the word slope." In this situation Bonnie appeared to intentionally pose a question that would prompt her students to access their own real-world experiences. Thus, it seemed again that Bonnie viewed the *meaning* of real-world situations as situations that would be familiar to her students. She further elaborated on the form introduced by the students saying, "mountains, like skiing down the slopes." This elaboration seemed to serve a

particular *function,* namely to connect a specific real-world experience (i.e., skiing down mountain slopes) with the mathematical term "slope" and the attribute of steepness.

In both instances discussed above, the form, meaning, and function of the real world seemed to facilitate the transfer of a particular *what,* namely meanings for standardized terms and representations. In the first situation, Bonnie linked ordered pairs of the form (x,y) to addresses and finding houses for the purpose of helping students to remember what information was being delivered through this mathematical representation—information about location. In the second situation, Bonnie linked the mathematical term *slope* and the letter m used to represent slope to skiing down real-world mountain slopes for the purpose of helping students remember that both the term slope and the letter used to represent it provide information about steepness. Both of the real-world situations arose in response to students' questions—"Isn't it called a [coordinate] point?" and is m mathematicians' way of "trying to confuse us?" Bonnie's subsequent actions, thus, seemed to imply that she wanted her students to generalize their knowledge of the meanings or definitions of various mathematical terms and representations. The forms in which identified themes seemed to manifest during teachers' lessons as well as associated inferences regarding meaning, function, and what would therefore be a focus of the subsequent debriefing.

As a consequence of watching Bonnie's lesson, it is possible that certain features of her belief about transfer became elaborated. For example, Bonnie may believe that the *forms* of the real world mentioned above *functioned* to link mathematical terms and representations to aspects of specific real-world experiences. Thus, Bonnie may believe that her students now are primed to, for example, remember mountain slopes when they

see *m* or read "slope" in a problem statement. Bonnie may believe that such images of mountain slopes facilitate the generalization of students' learning by helping them to remember *what* a particular question is asking about—it is asking about steepness. Thus, the real-world experience may serve as a bridge for students' future mathematical performance. Such inferences would therefore be another focus of the debriefing that would follow the observation of Bonnie's lessons.

Conducting post-observation debriefings. Debriefings lasted about 30 minutes and closely followed observations of teachers' classroom lessons. They were either conducted during one of the teacher's "free" periods or after school. Debriefings were videotaped and took place in teachers' classrooms. The video camera was placed on a tripod and aimed to capture teachers' verbal reports, gestures, and written inscriptions.

During the debriefing, I asked teachers to reflect on their lessons specifically with respect to the goal of supporting students to generalize their learning of slope and asked them how they saw their lessons as serving that goal. I also asked teachers if anything new or different came up and whether their beliefs changed in any way. I asked them again how they thought about the generalization of students' learning and supporting the generalization of students' learning. Then, I referenced some of the explicit statements teachers made during Phase I interviews regarding the generalization of students' learning and asked how teachers saw such statements in relation to their lessons. Similarly, I described the themes I identified by watching the Phase I interviews and asked how teachers viewed those themes in relation to their lesson. Lastly, I identified particular teacher actions from the lesson and asked how teachers saw those actions in

relation to how they think about the generalization of students' learning and how to support it.

If my observations of Bonnie's classroom lesson were to be followed by a debriefing in which I could further probe Bonnie's beliefs regarding the generalization of her students' learning, I would ask her the first three questions in the paragraph above. Then, I would explain to her that as a result of watching the initial interview with her, it seemed that the real-world played a big role in her beliefs about the generalization of students' learning. I would then ask her whether and how she saw the real world in relation to her lesson, her reasons for bringing specific experiences into the lesson, and the role she saw the real world playing in helping students to generalize their learning. If she did not mention the instance in which she likened ordered pairs to addresses or the instance in which she linked the mathematical term slope (and m) to skiing down real-world slopes, I would reference these instances and ask her about her motivations and whether and how she saw those moves in relation to the generalization of students' learning.

In sum, there were three steps to the data collection process. In the first step, I specified the themes that seemed to be of central importance in a teacher's beliefs about the generalization of students' learning and how it could be supported instructionally. In the second step, I watched a teacher's lesson for the form, meaning, and function of the themes identified in the first step, as well as for what teachers seemed to be supporting the transfer of. In the last step, I engaged the teacher in a 30-minute debriefing wherein she contemplated what occurred during her classroom lesson in relation to how she thought about the generalization of students' learning and how to support it.

Data Analysis

My goal in analyzing data yielding from Phase II of the research design (see Figure 3.3) was to further flesh out each teacher's beliefs about the generalization of students' learning. Thus, data collected during Phase II were analyzed using the constant comparative method from grounded theory (Strauss, 1987; Strauss and Corbin, 1990). As data were analyzed and categorized, they were compared to the analyses of the Research Question 1 data. Such comparisons resulted in the elaboration of meanings given to a previously established category and the identification of additional beliefs for particular teachers.

<div align="center">

Reliability, Validity, and Disconfirming Evidence

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To make a case for the quality of the findings that derive from this study, I address three issues here. The first issue is the reliability of the categories of teachers' beliefs about the generalization of learning (and how to support it) that emerge as a result of inducing codes from the data using open coding from grounded theory. The second issue is the validity of inferences that are made while analyzing data. The third issue is the way in which disconfirming evidence is dealt with.

One way to ensure reliability of the categories that emerge during coding and analysis of data is peer review (Confrey & Lachance, 2000). Graduate students enrolled in *MSE 830: Research Seminar* at SDSU during the Fall semester of the 2012-2013 school year served as peers in the reliability check of some of the categories that emerged from the data that were collected. In particular, I prepared and presented PowerPoint slides for Research Question 1 describing emergent categories of beliefs. I drew upon the data that were collected to present evidence of each category. I asked for feedback

regarding whether my peers saw the same regularities in the data that I did. In other words, I asked whether they agreed that grouped segments of data belonged together. Moreover, I asked for feedback regarding whether the labels given to each category captured the regularities. I repeated the process described here for all research questions and all categories for the chair of my dissertation committee.

The second issue of validity corresponds to the issue of *fit* in grounded theory. Fit refers to how faithful categories and relationships among categories are to the phenomena they represent (Glaser & Strauss, 1967; Strauss & Corbin, 1994). Assessing "fitness" requires expertise in the area of study (Confrey & Lachance, 2000). Thus, one way to ensure validity or fitness of induced categories as well as inferences regarding relationships among categories would be to make presentations (similar to the presentations described in the previous paragraph) to an expert at different times during the analytic process. For this dissertation study, such presentations were made periodically to Dr. Joanne Lobato during the open and axial coding processes. The focus of these presentations was how closely inferential claims represented the associated data. Therefore, Dr. Lobato was asked to provide critical feedback regarding the strength of each claim.

As noted earlier, grounded theory approaches to data analysis make use of the constant comparison method wherein segments of data are constantly compared against each other for similarities and differences and similar segments of data grouped to form categories (Corbin & Strauss, 1990; Strauss & Corbin, 1994). Furthermore, grounded theory involves constantly comparing segments of data to inferences regarding relations between categories. This method allows the researcher to uncover disconfirming

evidence. Disconfirming evidence is used to help the researcher further refine, modify, and develop categories, subcategories, and relationships between (sub)categories (Corbin & Strauss, 1990). This process results in categories and inferences regarding the relationships that exist among categories that perfectly fit the data set from which they emerge (Borgatti, 1996). Any disconfirming evidence that could not be reconciled by categories (of beliefs or MKT) are reported directly in the results.

CHAPTER 4:

Results on Teachers' Beliefs Regarding the Generalization of Students' Learning

and How to Support It

In this chapter, I present the findings from my analysis of the interview data,

which address the first research question:

> What are teachers' beliefs about (a) the generalization of students'
> learning and (b) how to support the generalization of students'
> learning?

Specifically, I offer evidence to support the claim that there are five categories of

participating teachers' beliefs about the generalization of students' learning. These

categories fit within the three super-categories: content, disposition, and students' affect

(as shown in Figure 4.1). Content refers to the mathematical knowledge that students

generalize; disposition refers to the general orientation towards or outlook on problem

solving that students generalize; and students' affect refers to student-held beliefs that

function to facilitate the generalization of their learning (details to be provided in this

chapter). The number of teachers holding each belief is also shown in Figure 4.1.

Because teachers seemed to hold more than one belief about the generalization of

students' learning, the total number of teachers across categories exceeds 8 (i.e., the

number of teachers in the study). For each category of belief about the generalization of

students' learning, I also present evidence of 2-3 beliefs regarding how to support the

generalization of students' learning.

Content	Disposition	Students' Affect
Category 1: Associations, Procedures, Formulas (3)		**Category 4:** Students' View of Self (6)
Category 2: Meaning (4)	**Category 3:** Orientation towards Problem Solving (3)	**Category 5:** Students' View of Mathematics (3)
(7)	(3)	(7)
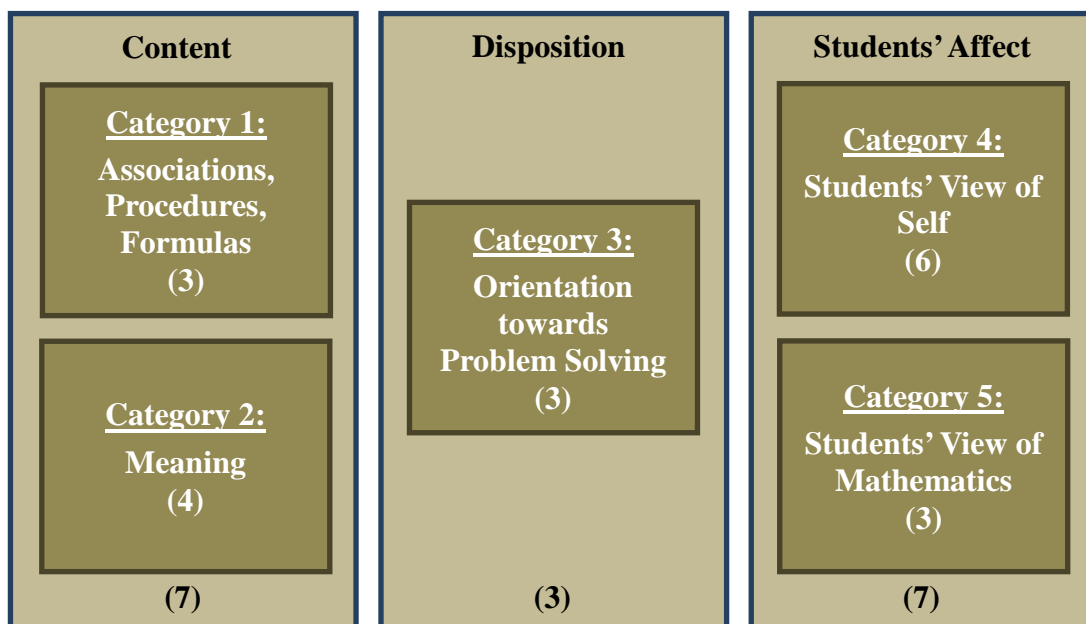
Figure 4.1: Three super-categories of teachers' beliefs about the generalization of students' learning with the number of participants holding each belief in parentheses.

This evidence, along with a reflection on the existing transfer literature, will

support the following conclusions:

- Half of the teachers in this study emphasized the meaning of mathematical topics

  like slope and just under half emphasized associations, procedures, or formulas in

  their beliefs regarding how specific mathematical content generalizes.

- Equal numbers of teachers (7 of 8) had specific beliefs about how mathematical

  content facilitates the generalization of learning and less specified, global ideas

  about how students' affect supports the generalization of learning.

- The role affect plays has been absent from the transfer literature; yet it was

  present in the espoused beliefs of all but one participating teacher.

- While less than half of the teachers expressed a belief regarding the role that

  disposition plays in the generalization of learning, their ideas contribute to the

transfer literature by extending the writings on dispositional approaches to transfer into the domain of mathematics education.

- The teachers identified a number of instructional actions aimed at supporting the generalization of learning, which had not been previously reported in the transfer literature. Furthermore, those pedagogical actions that did share some elements with what is reported in the transfer literature had significant practice-based differences from what is recommended by transfer researchers.

Taken together, these conclusions will help demonstrate the importance of asking *teachers* about their beliefs regarding the generalization of students' learning and how to support it instructionally.

Finally, the chapter is organized into three major sections as defined by the super-categories illustrated in Figure 4.1 (content, disposition, and students' affect). Discussions will follow at the end of each of these three sections. Moreover, the three sections of this chapter are separated into subsections according to the five categories of teacher beliefs shown in Figure 4.1. In other words, the first section (content) includes the two major subsections corresponding to the first and second categories of teacher beliefs about the generalization of students' learning (*associations, procedures, formulas* and *meaning*); the second section (disposition) includes one major subsection corresponding to the third category of teacher belief (o*rientation towards problem solving*); the third section (students' affect) includes two major subsections corresponding to the fourth and fifth categories of teacher beliefs (*students' view of self* and *students' view of mathematics*) (see Figure 4.1). Within each of these five subsections, I (a) provide an elaborated statement of the particular belief regarding the generalization of

students' learning, (b) illustrate the belief with evidence from one of the participating teachers in the category, (c) present evidence from the other participating teachers in the category to show that they also seemed to hold the belief, and (d) present the participating teachers' corresponding beliefs about how to support the generalization of students' learning.

Content

Categories 1 and 2 involve beliefs regarding the nature of the mathematical content that will generalize for students. Teachers in Category 1 appeared to believe that students will be able to generalize their learning to a novel situation if the novel situation prompts students to make use of a learned association, procedure, or formula. Teachers in Category 2 seemed to believe that students will be able to productively generalize their learning to a novel situation if they develop mathematically valid interpretations of the meaning of mathematical topics (e.g., slope is a ratio describing the multiplicative relationship between two quantities (i.e., measurable attributes of an object or event (Smith & Thompson, 2008))). In other words, the beliefs about the generalization of learning presented in this section, as well as the associated teaching actions, focus on the generalization of particular types of student knowledge—procedural knowledge in Category 1 and conceptual knowledge in Category 2.

Category 1: Associations, Procedures, or Formulas Transfer

Statement of the belief. Teachers in this category (Anne, Blake, and Donna) appeared to believe that students will be able to productively generalize their learning to a novel situation if the novel situation prompts students to make use of a learned association, procedure, or formula. *Association* is used to refer to the idea that a student

links a specific word, phrase, or image with a particular mathematical response. For

example, when confronted with a set of three graphs and asked which one represents

"positive slope," the student might circle the graph showing a line slanted "up" (as one

looks from left to right), because of class activities in which he learned to associate the

mathematical term "positive slope" with the image of an "uphill" journey. *Procedure*

refers to the use of a pre-determined set of steps to solve a given problem. For example,

upon being presented with a linear function of the form *y*=m*x*+b and asked to create a

graph representing the function, a student might locate "b" on the *y*-axis (i.e., (0, b)),

place a point, then go "up" m units on the *y*-axis and to the right one unit, place a second

point, and then draw a line through the two plotted points. *Formula* refers to the

employment of a conventional rule (e.g., $slope = \frac{y_2 - y_1}{x_2 - x_1}$ or $slope = \frac{rise}{run}$) to solve a

problem. Additionally, teachers in this category did not appear to prioritize the

conceptual underpinnings of such associations, procedures, or formulas. In other words,

teachers did not explicitly focus on the meanings, concept images, or interpretations

students develop while engaging in mathematical activities involving topics like slope.

Illustration of the belief. I illustrate this category with evidence from Anne.

The evidence includes Anne's discussions of the three activities[7] that she brought to the

interviews: The Wall Activity (a teaching activity), The Museum Activity (an activity

that allowed Anne to assess what was learned during The Wall Activity), and The

Comparing Values Activity (an activity aimed at the generalization of students' learning).

Anne brought The Wall Activity and The Museum Activity to both interviews—to

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[7] The three activities Anne selected and brought to the interviews (The Wall Activity, The Museum
Activity, and The Comparing Values Activity) were drawn from the reform-oriented book called *A Visual
Approach to Functions* (Van Dyke, 2002).

Interview 1 as her "teaching artifact" to illustrate the way in which she thought about supporting students "so that they would be able to generalize their understanding of slope to a new task, activity, or situation," and to Interview 2 as activities included as part of her "Lesson Plan Activity" in which teachers created a lesson to illustrate how they would "help students generalize their understanding of slope." The Comparing Values Activity was the "new task or activity" Anne chose to include for the Lesson Plan Activity in Interview 2.

In The Wall Activity, Anne explained that her goal was to support students in creating graphs representing a boy's motion through space, labeling each of the graphs with the appropriate sentence (i.e., "Bobby walks towards the wall," "Bobby walks away from the wall," or "Bobby stands in the middle of the room"), and recording the graph-label associations in their notes. See Figure 4.2 for specific graph-label associations.

"Bobby walks *away* from the wall."

"Bobby walks *towards* the wall."

"Bobby *stands* in the middle of the room."
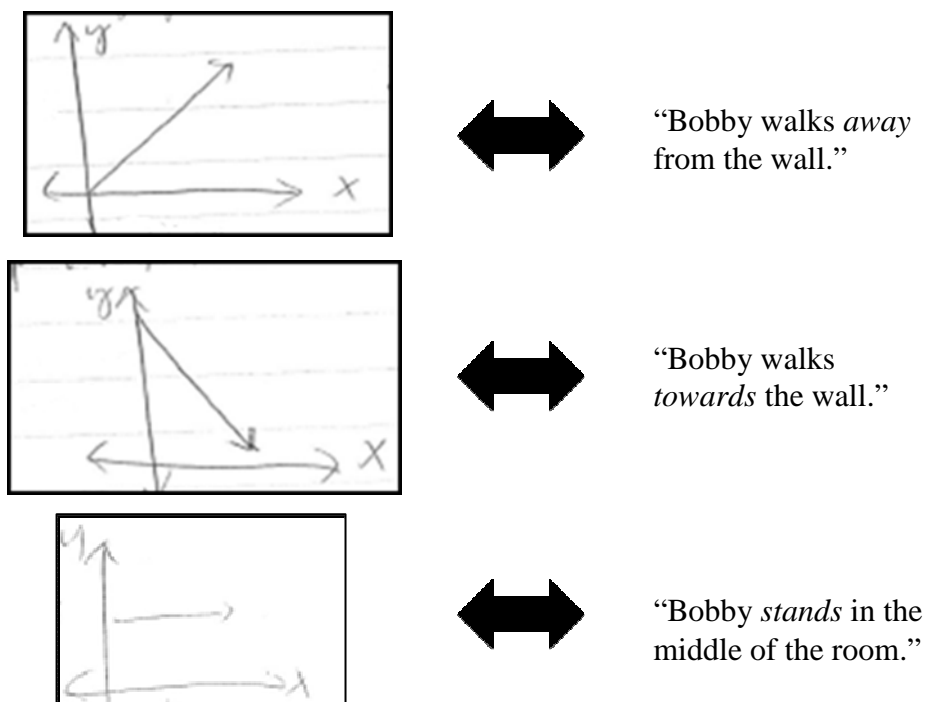
Figure 4.2: Author's reproduction of Anne's graph-label associations.

As a consequence of having engaged in The Wall Activity, Anne explained that she believed her students would be able to productively engage with The Museum Activity (see Figure 4.3 for a portion of the activity) because specific words in The Museum Activity like "towards" and "away" should prompt students' selection of the appropriate graphical image. For example, Anne explained that her students would be successful in answering the first question in The Museum Activity which reads, "We walked at a steady pace toward the museum," because they would think "which [graph] is '*toward* the museum' and which one is not?" She believed her students would "choose graph B because that's [points to Graph B] *towards*; it's the same as this is *towards* [points to the second graph-label association recorded in her notes and reproduced in Figure 4.2]." In other words, Anne stated that "what they do is they have to *match* [emphasis added] the sentence to the picture." Thus, Anne appeared to believe her students would be able to productively engage with The Museum Activity by making use of the graph-label associations learned during The Wall Activity.

Moving Around the Museum

Each sentence tells how far a person is from an object. Decide which graph matches the sentence. In Questions 1, 2, and 4 the "object" is a museum. In Question 3 it is a painting.

1. We walked at a steady pace toward the museum.

4. Afterward, we raced down the hill away from the museum. (Be careful here.)
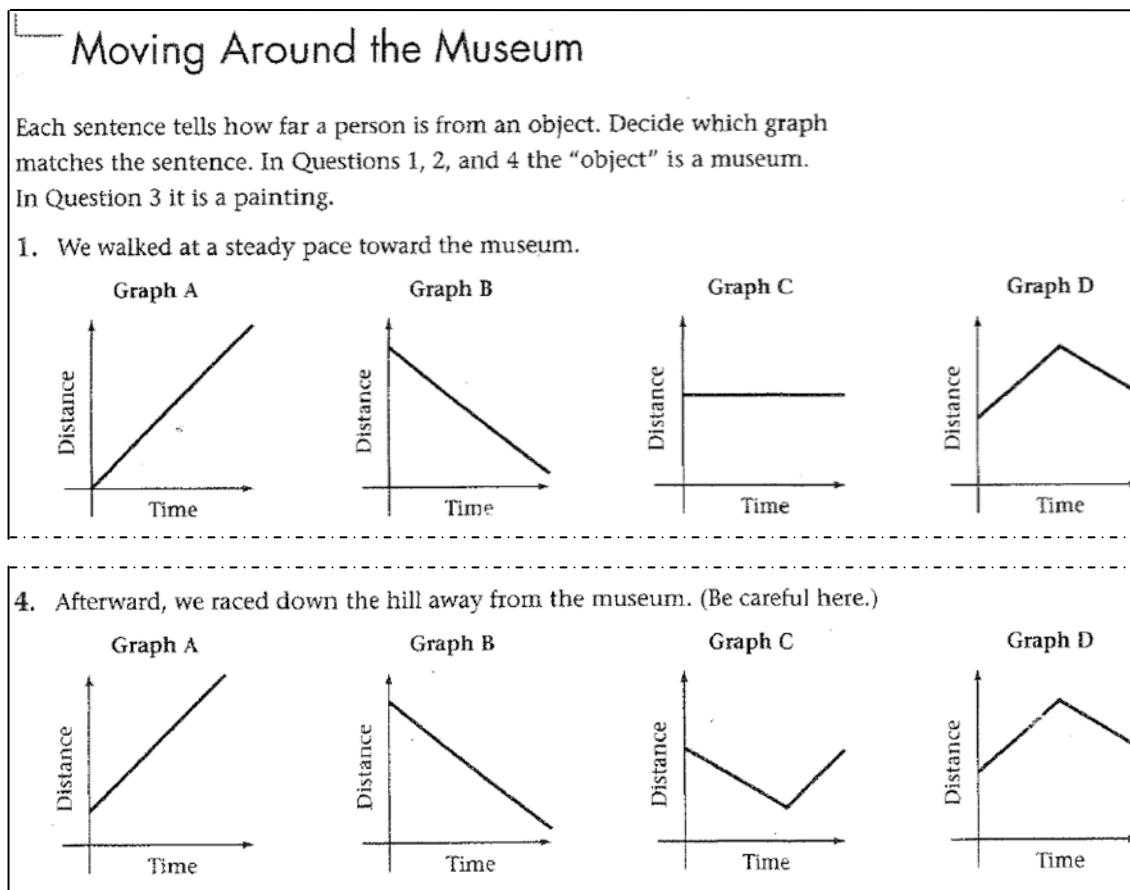
Figure 4.3: A portion of Anne's Museum Activity.

Interestingly, Anne explained that the fourth question in The Museum Activity would serve as a test of whether her students "understand" The Wall Activity. The fourth question reads, "Afterwards, we raced down the hill away from the museum" (see Figure 4.3). Anne seemed to wonder whether her students would focus on the phrase "down the hill" rather than the word "away." Anne stated that "if 'down the hill' means this [points to Graph B], then they don't understand the lesson … Negative slope is *not* down the hill." Thus, it appeared that Anne believed her students would not be able to generalize their learning to the fourth question if they did not make use of the particular association learned during The Wall Activity. In other words, if students focused on "down the hill" rather than "away" when confronted with the fourth question, they would not be

prompted to make use of the learned association and would therefore be unsupported in choosing the correct graph, Graph A.

To further illustrate the way in which Anne appeared to believe the associations shown in Figure 4.2 (rather than, for example, the quantitative meanings individual students develop) served to mediate the generalization of students' learning, I now present and discuss Anne's Comparing Values Activity (see Figure 4.4 for a portion of this activity). Anne brought this activity to Interview 2 as an example of a novel task that she thought her students could solve as a consequence of engaging in The Wall and Museum Activities of her lesson (i.e., a task that researchers could call a "transfer task").

Comparing Values

1. The value of Lynn's house first decreased, then remained the same. Finally, the value began to increase.

   a. Decide which of these graphs best shows the value of Lynn's house.
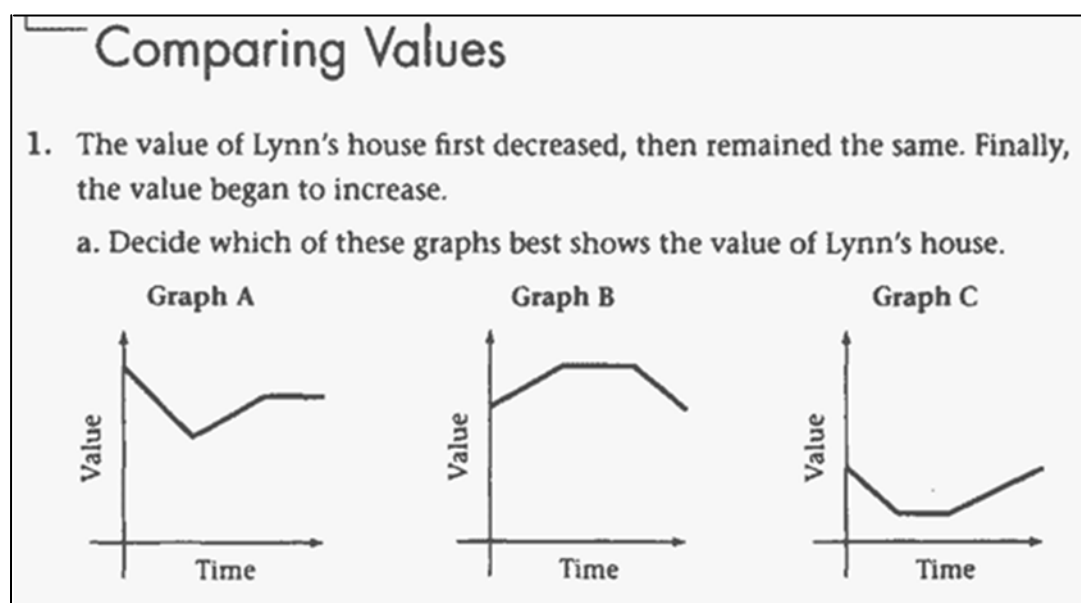
Figure 4.4: A portion of Anne's Comparing Values Activity.

Anne chose The Comparing Values Activity because it involved a new quantity (home value), thus, making it "totally different" from The Wall and Museum Activities (with their focus on distance). As Anne explained, students would have to "apply" their understanding "to a situation where we're talking about positive and negative [slope] but now its value, not distance." When asked how she thought her lesson prepared students

for their subsequent engagement with The Comparing Values Activity, Anne explained

that as part of her Wall Activity she often asks various students to create "journeys" (i.e.,

piecewise-linear graphs) for other students to walk and that the journeys are enacted by a

student who looks at "each section of the graph" and uses his or her notes (reproduced in

Figure 4.2) to answer the question "what does this part mean?" Anne seemed to believe

that students' engagement with the journeys exercise would prepare them for the novel

Comparing Values Activity because they could map the association "walking away" ↔

"positive slope" onto the association "positive slope" ↔ "the value is going up" to derive

an association of "walking away" with "the value is going up." In particular, Anne said:

> I've had students in the past where they actually write on the graph
> like this means 'going away' or this means … 'the value is going up or
> down' or whatever. And so they can do the same thing and interpret
> the graph, each piece of the graph.

The fact that Anne seemed to approve of her students writing "walking away" on a graph

that represented monetary home value over time (rather than distance over time) suggests

that her focus was on the formation of associations between graphical shapes and written

phrases that resulted in "correct" responses but not necessarily on the underlying meaning

of graphs.

Anne's prioritization of associations over conceptual underpinnings is further

indicated by her discussion of the fourth task in The Museum Activity. One could argue

that the generalization of students' learning during The Wall Activity is mediated by an

interpretation of the distance decreasing as the time increases (Figure 4.2), which is

similar to a possible interpretation for the fourth question in The Museum Activity

(Figure 4.3). After all, the vertical axes in The Museum Activity are labeled "Distance."

However, it is unclear whether distance refers to "distance from the bottom of the hill," "distance to the museum," or "total distance traveled." In fact, Graph B *could* be the correct answer to the fourth question if a student has interpreted the vertical axis to be distance from the bottom of the hill, since that distance would decrease as one raced down the hill. Anne's statement that "negative slope is *not* down the hill" suggests that she has prioritized associations like "negative slope" with phrases such as "getting closer" (rather than "down the hill") over the possible quantitative meanings of different graphs.

Furthermore, when Anne was asked how she thought her students would do on The Burning Candle Task (see Figure 4.5), as a consequence of their engagement with her planned lesson, she indicated that many students would be able to find the slope of the graph (Part A of the task) but not interpret its meaning in the context (Part B). Specifically, Anne said, "They can't tell you what that means in this context because the candle burning is very strange." She apparently did not think her students would be able to interpret the slope in terms of a decrease in the candle's height over time, which is another indication that her teaching of the association of "walking toward a wall" with "negative slope" in The Wall Activity (Figure 4.2) was unlikely grounded in a quantitative interpretation that the distance between the walker and the wall decreased over time.

Burning Candle Task

The height of a candle is measured over time as the candle burns. The information is recorded in a table and in a graph. For example, after 3 hours the candle is 32.5 centimeters high. A) Find the slope of the function represented below. B) Explain what the slope means in this situation. C) Predict when the candle will burn out.

Time (in hours)	Height (in cm)
0	40
3	32.5
4	30
5.5	26.25

Figure 4.5: The Burning Candle Task.

The other teachers. The other two teachers in this category—Blake and Donna—provided similar evidence which seemed to suggest that they believed students would be supported in generalizing their learning to a novel situation if the novel situation prompts them to utilize a learned association, procedure, or formula. Blake, for example, explained that his teaching artifact (a collection of six activities) focused on introducing students to the following slope formula for lines represented graphically: $slope = \frac{rise}{run}$. After describing his teaching artifact, Blake was asked how (as a consequence of their engagement with his teaching artifact) his students would do on two tasks that the researcher conceived of as transfer tasks—The Burning Candle Task (see Figure 4.5) and The Water Pump Task (see Figure 4.6). Blake explained that his students

would be better supported in productively generalizing their learning from his lesson to

The Burning Candle Task than to The Water Pump Task because The Burning Candle

Task contains a graphical representation; thus, students would be prompted to employ the

learned formula (i.e., they would choose two points on the line, count the corresponding

rise and run, and place the rise over the run). On the other hand, Blake did not think his

students would be able to productively generalize their learning to The Water Pump Task

because the task does not contain a graphical representation and does not explicitly tell

students to find "slope;" thus, he did not believe his students would know how to

proceed. In this way, Blake appeared to believe that students' generalize their learning to

novel tasks when such novel tasks prompt them to use learned formulas.

Furthermore, Blake did not appear to attend to the meanings and interpretations

that *students* develop for formulas like $slope = \frac{rise}{run}$ but rather to whether they use the

pre-determined formula. As evidence for this claim, consider that Blake wanted his

students to select specific points on the graph (from the graph shown in Figure 4.5) to

plug into the slope formula, as indicated by the following statement: "I would hope that

they would figure out that the easiest way to get the slope is to rise here and run that way

[points to the two coordinate pairs which have whole-number components]." Because he

was not sure they would do this, he described how he would provide guidance:

> If you ask the right question, they will be thinking in the right
> direction. I can't expect them to just automatically hone in on what it
> is that I want. You have to ask the right question. … I might say
> 'which two points do you think would be the best points; there's two
> points that are good points to pick; there's only two of them; there are
> four points on there, but which two do you think would be the best to
> figure out what the slope it?'

Blake's focus on one best way to employ the formula seemed to suggest that he did not explicitly attend to the ways in which students might conceive of the slope formula or its implementation. For example, he did not appear concerned with the points *they* would choose or why they would choose them (which may have indicated deeper conceptual understanding), but rather with guiding students towards the "only two" good points on the graph.

Water Pump Task
Water is being pumped through a hose into a large swimming pool. The table below shows the total amount of water in the pool to begin with as well as the total amount of water in the pool after 3, 5, and 9 minutes. The amount of water is measured in gallons. The time is measured in minutes.

Time (minutes)	Amount of Water (gallons)
0	0
3	4
5	$6\frac{2}{3}$
9	12

Does the data above represent a water pump that is pumping equally fast over time? If so, find the rate at which the pump is pumping. If not, explain what is happening (i.e., the pump is speeding up; the pump is slowing down; the pump is speeding up and then slowing down).

Figure 4.6: The Water Pump Task.

Similarly, Donna explained that she believed her teaching artifact helped students learn how to use a particular procedure as well as the slope formula. Specifically, it supported students in finding the slope of a line by: (a) creating a line from a tabular set of data (and if necessary, using the slope formula, $slope = \frac{y_2 - y_1}{x_2 - x_1}$, to fill in missing entries in the table); (b) identifying the rise between consecutive points on the graph

(students do not have to look for the run because the run will be 1 when the table is completely filled in); and (c) writing a rule of the form $y=mx+b$ (wherein m takes the value obtained during the previous step). Like Blake, Donna was then asked how she thought her students would do with The Burning Candle Task and The Water Pump Task as a consequence of their engagement with her lesson. Donna explained that her students would be able to productively engage with part B of The Burning Candle Task as well as The Water Pump Task because both tasks contain tabular representations which should prompt students to make use of the previously learned procedure and formula. For instance, students could begin their response to part B of The Burning Candle Task by explaining that slope means you start by filling in the table (a step included in part (a) of the procedure demonstrated in class). Like Blake, Donna did not explicitly discuss the ways in which students might come to interpret the procedures and formulas she presented but rather discussed the importance of attending to whether students could correctly implement the pre-determined procedure and formula.

Associated instructional moves. There were two instructional moves that teachers in this category apparently believed would support the generalization of learned associations, procedures, or formulas: (a) tell students the association, procedure, and/or formula you want them to use; and (b) use multiple examples. I present evidence of each pedagogical belief related to the generalization of learning in turn.

Tell students the desired association, procedure, or formula. The teachers in this category appeared to believe that teachers could support students in generalizing their learning by explicitly showing or telling them the associations, procedures, and/or

formulas that could be used to solve particular types of problems. For example, when

discussing his teaching artifact (a collection of six activities), Blake explained:

> I start with two points on a graph and I'll define, I will give them an
> equation, you know, the y_2 minus y_1, but, I really want them to be able
> to find the slope of a line by identifying two points and counting rise
> and run.

In this excerpt, Blake stated that he starts his slope activity by explicitly giving his

students the formula he wants them to use when finding slope. Moreover, as noted

above, Blake made decisions regarding students' productive engagement with The

Burning Candle and Water Pump Tasks based on whether he believed his students would

be prompted to utilize the learned formula.

Similarly, when explaining their teaching artifacts, both Anne and Donna

explicitly stated that they tell their students precisely which association, procedure, or

formula to use when engaging in problems involving slope. As Anne said, students "are

not going to make the connection[s] unless [they're] explicit." Thus, as noted above,

Anne told her students precisely what they should record in their notes during The Wall

Activity (see Figure 4.7 for an example from Anne's notes). Both Anne and Donna

subsequently made decisions regarding students' productive engagement with subsequent

tasks based on whether they believed their students would make use of the recorded

associations (in Anne's case) or the demonstrated procedure and formula (in Donna's

case).
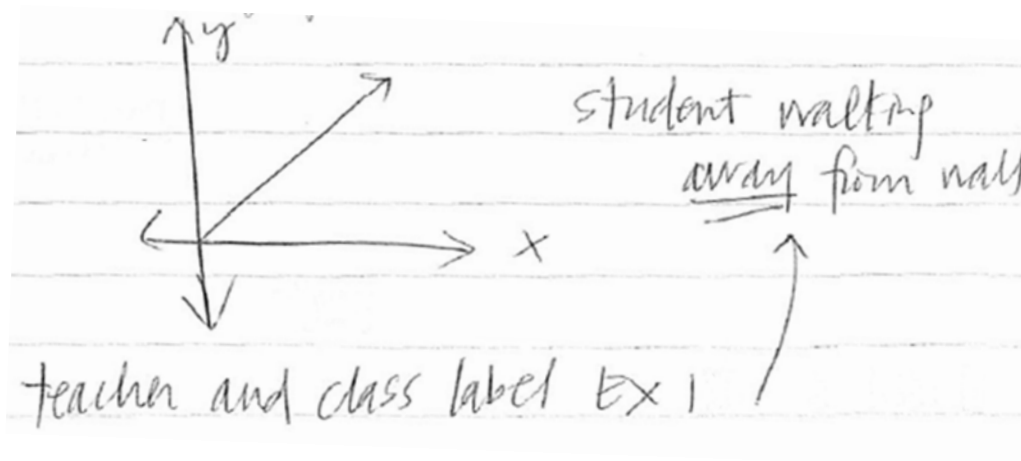
Figure 4.7: An excerpt from Anne's notes for The Wall Activity.

Use multiple examples. Two teachers (Donna and Blake) in this category appeared to believe that students would be supported in generalizing associations, procedures, and/or formulas if multiple examples were used during initial learning. Donna seemed to believe that the use of multiple examples would support her students in knowing *when* to use a learned procedure or formula and Blake seemed to believe that the use of multiple examples would support students in *seeing* the pre-determined formula. For instance, Donna's teaching artifact consisted of a lesson on arithmetic sequences and their connection to slope. The lesson began with multiple examples (see Figure 4.8) of arithmetic sequences and the query to her students, "What do they all have in common?" Specifically, Donna said:

> I just give them "55, 49, 43," [and tell them to] continue the pattern.
> Ok. And most kids will go, "Oh, it's going down by 6." So then we
> do a couple of those (see examples in Figure 4.7). It can be decimals.
> It can be fractions. It can be—and I want it to be added or subtracted.
> It doesn't matter. Ok. And we do a couple of those and I say, 'OK,
> what do they all have in common? They all have in common that this
> difference, whether it be subtracting 6 from each term or adding 5 to
> each term, *that* [the difference] is constant.

Thus, Donna's lesson began with arithmetic sequences that varied in terms of number type (e.g., whole numbers, decimals, or fractions) but that all shared a constant difference in consecutive terms. Donna explained that after students identify the constant difference as the feature shared by all of the sequences, students either spontaneously or via a sequence of guided prompts link arithmetic sequences to slope, at which point Donna demonstrates the slope formula discussed earlier. Thus, it appeared that Donna used multiple examples to illustrate to students *when* it is appropriate to use the slope formula, namely when there is a constant difference between successive terms of a sequence[8].
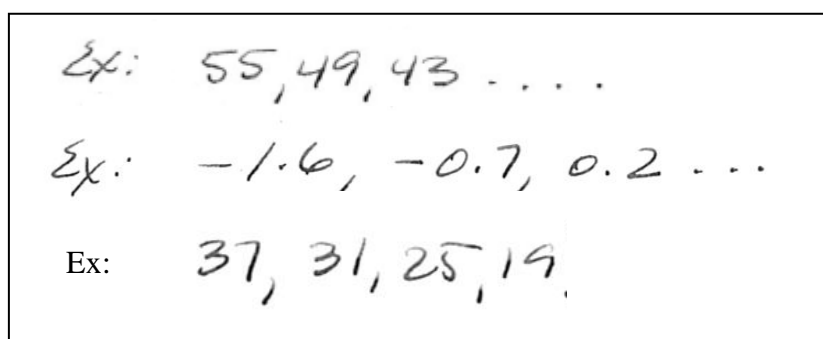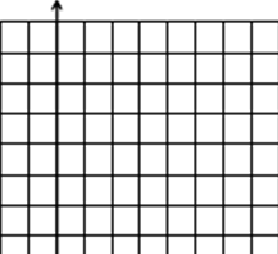
Figure 4.8: A sample of Donna's multiple examples.

Blake seemed to make use of multiple examples for a different purpose. Rather than helping students to determine whether a particular situation necessitated a demonstrated formula, Blake seemed to believe multiple exemplars helped students to *see* the formula itself. Blake's teaching artifact was a collection of six activities wherein the contexts varied but the formatting of the activity and wording of various questions remained the same (see Figure 4.9 for two of these activities). As Blake explained, "this [pointed to his collection of activities] is trying to create, I am trying to show them a

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[8] The independent variable, with respect to Donna's arithmetic sequences, is taken to be the ordinal placement of a term in the sequence; thus, the "run" or the "$x_2 - x_1$" in the slope formula is 1, which means that the slope in this case, calculationally, is the same as the constant difference between consecutive terms of an arithmetic sequence.

pattern" and as noted previously, Blake said "I really want them to be able to find the slope of a line by identifying two points and counting rise and run." In this excerpt, Blake seemed to express his belief that by making use of multiple examples problems he was actually helping his students learn to recognize and then reinforce the particular "pattern" (i.e., formula) for finding slope, namely $slope = \frac{rise}{run}$, he had shown them previously. In particular, students are to take two coordinate pairs from part (d), plot them on the graph in part (e), and recognize that slope is the "rise" between the plotted points over the "run" between the plotted points.

Algebra I – Chapter 5 Application Problems   Name _____
Problem #3: Meal Deal   Date _____ Period _____

Fastfood meal deals are $3.00 each. Kyra wants to purchase one for each of her friends.

a) Identify a variable to represent the number of meal deals Kyra purchases.	b) Identify a variable to represent Kyra's total cost of buying meal deals for her friends.	c) Write an equation to represent Kyra's expenses for purchasing meal deals for her friends.

d) Complete the table with at least 5 entries.

(,)

e) Label and scale your graph. Then, graph your equation.

Algebra I – Chapter 5 Application Problems   Name _____
Problem #4: Car Depreciation   Date _____ Period _____

The value of a new car is $21,000. The value decreases $2,000 per year.

a) Identify a variable to represent the number of years the car is owned.	b) Identify a variable to represent the value of the car.	c) Write an equation to represent the value of the car over time.

d) Complete the table with at least 5 entries.

(,)

e) Label and scale your graph. Then, graph your equation.

Figure 4.9: Two of Blake's six activities.

Category 2: Meanings Transfer

Statement of the belief. Teachers in this category (Richard, Emma, Patrick, and Kay) seemed to believe that the generalization of students' learning is based on how students interpret their mathematical activity and the meanings they develop for mathematical topics like slope. Moreover, these teachers appeared to believe that the

productive generalization of students' learning was supported by mathematically valid interpretations of the meanings of topics like slope, for example, *slope is a ratio which provides a description of the multiplicative relationship between two quantities*. Thus, teachers in this category tended to make predictions regarding the generalization of students' learning based on the meanings they thought a student might develop for a particular topic rather than on whether they thought a particular task would prompt a student to make use of a pre-determined association, procedure, or formula.

Illustration of the belief. I illustrate this category with evidence from Patrick. The teaching artifact Patrick brought to the first interview was The Previous Travelers Activity (see Figure 4.10), an activity from "The Overland Trail" unit of the reform-oriented and National Science Foundation funded Interactive Mathematics Program (IMP) curriculum. In the activity, Patrick explained that students are provided with data regarding the number of pounds of beans, sugar, and gunpowder families of various sizes needed in order to successfully complete the trip from modern day Kansas City to Sutter's Fort in California. Students are then asked to plot the data, construct a line of best fit, create a table using points from their line, write an algebraic rule representing the relationship between number of people and pounds of beans (or sugar or gunpowder), and then figure out how many pounds of each item they would need given they are traveling with a family of a particular size (see Figure 4.11 for Patrick's notes on the "beans compared to people" portion of the activity).

Dear friends,

We've arrived in Fort Hall after many adventures, both good and bad. It would take me forever to describe all that happened. We can talk about all of that when we meet up again in California.

I know that you're anxious for some practical information for your own trip. Several of the families on our wagon train kept track of the quantities of various goods they actually needed on the journey. The families were of different sizes, so this information should help you and your friends decide how the amounts vary from group to group. I know this won't answer all your questions, but it's a start.

Previous Travelers

The first settlers had to make the long journey west without advice from previous travelers. However, later wagon trains used information from the early travelers to decide on the supplies to buy for their journey.

While in Fort Laramie, Wyoming, you get a letter from friends describing the supplies they and others used on the leg of the trip from Fort Laramie to Fort Hall, Idaho. Use the information in the letter to answer these questions.

1. Make a graph displaying all the data relating the number of people and the amount of beans they used. Use appropriate scales for the axes. Then make a similar graph for sugar and another for gunpowder.

2. Do these steps for each graph in Question 1.
   a. Sketch what you consider to be the **line of best fit** for the graph; that is, find the straight line that you think best fits the data.
   b. Make an In-Out table *from your line*. Determine a rule that fits your table or that comes reasonably close.
   c. Use either the In-Out table or your graph to find the quantity of each item needed for each of your group's four family units.

Number of people	Pounds of beans	Pounds of sugar	Pounds of gunpowder
5	61	20	3
8	95	50	4
6	56	30	2.5
7	75	23	4.1
11	125	60	5
10	135	40	5.8
5	80	39	1.8
7	100	44	3.8
10	103	53	4.3
6	75	35	3.6
8	100	35	3.2
7	105	36	3.1
9	125	45	4.7
12	150	55	6.1
10	125	31	5.2

Well, good luck to you all!

The Helmicks

Figure 4.10: Patrick's Previous Travelers Activity.

Patrick explained that he hoped students' engagement with The Previous Travelers Activity would support them in developing an interpretation of slope as a ratio, or a multiplicative comparison, of two quantities. Specifically, he reported that he hoped his students would come to interpret slope as the number of pounds of beans per person, by comparing how many times greater the quantity *pounds of beans* is than the quantity *number of people*:

> The number of people in our family times 12 is a pretty good predictor of how many beans we need because those are the people that survived

and lived. And that's kind of a decent rule for what they did. So, yeah, the slope is 12; and we talk about well, what does that mean? Well, for every person, you need 12 pounds [of beans].

By noting that there are 12 times as many pounds of beans as there are people, students are forming a ratio as a multiplicative comparison between two quantities. This multiplicative relationship between the two quantities can also be expressed as 12 pounds of beans per person, which is the slope of the function. Patrick went on to extend this meaning of slope to a ratio of the *change* in the pounds of beans to the *change* in the number of people. Referring to the graph shown in Figure 4.11, Patrick stated that he wants his students to (a) look at two points on the graph, (b) think "We're at this many pounds of beans and we're going to this many pounds of beans," (c) form a difference in the amount of beans by asking "How many pounds of beans is that an increase of," and (d) form an associated difference in the number of people by asking "How many people is that"? As a result, students will have still formed a "ratio of pounds of beans for so many people."
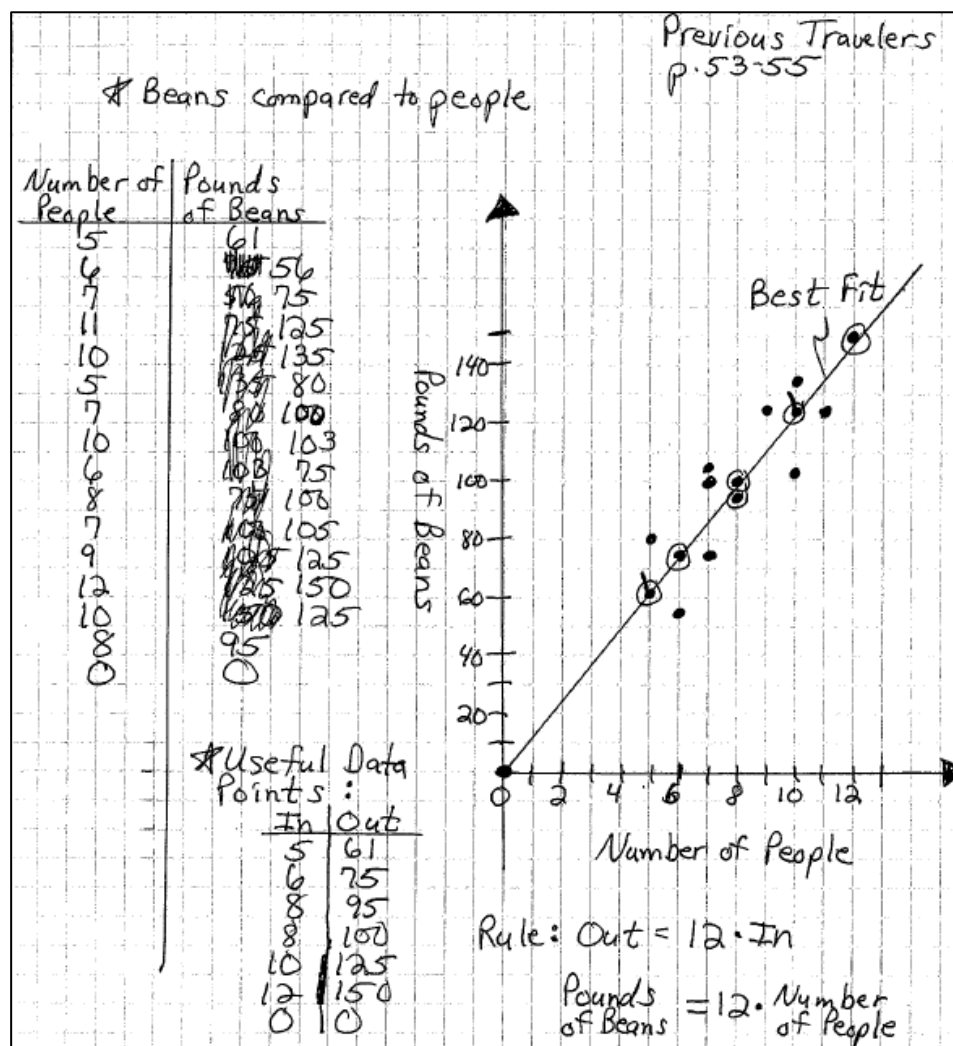
Figure 4.11: Patrick's notes for the "beans compared to people" portion of his teaching artifact.

Patrick appeared to believe that his students would be able to generalize this interpretation of slope as a ratio of two quantities to other situations. When asked if his students would be able to productively engage with The Burning Candle Task (Figure 4.5) as a consequence of participating with The Previous Travelers Activity, Patrick described a similar interpretation for the graphical representation shown in Figure 4.11. Specifically, Patrick wanted his students to look at the two points (0,40) and (3,32.5), and think "Now I'm at 40; now I'm at 32.5; how much [height] did I lose? And that took me

3 hours." This excerpt seemed to again emphasize Patrick's focus on students' interpretation of slope as necessitating the coordination of two quantities in such a way that one forms, in this case, a ratio of elapsed height (e.g., 40cm-32.5cm = 7.5 cm) to elapsed time (e.g., "3 hours").

Despite the fact that Patrick saw this connection (i.e., slope as a ratio of two quantities) between his Previous Travelers Activity and The Burning Candle Task, he wondered whether his students would recognize the same connection. For instance, Patrick stated that "to explain what it [slope] means in this situation would be a challenge" for students. He predicted that his students "would be able to say 'the candle is shrinking' or the 'candle isn't as tall'," but stated "I don't know if they would get at 'oh, let's see, what's our time frame and if it's per something;' that would be a challenge, I think, to know that it's per hour." In this way, Patrick was unsure whether his students would have fully developed an interpretation of slope as a ratio of two quantities and whether they could explain slope in terms of a unit ratio (as a loss in the height of the candle per hour). Consequently, Patrick suggested that he implement particular teaching actions (which will be discussed later) during The Previous Travelers Activity that he believed would better support students in developing the desired interpretation of slope and thus better support students' subsequent engagement with The Burning Candle Task.

In sum, it appeared Patrick believed students' ability to *productively* generalize their learning to The Burning Candle Task depended on whether they had developed a mathematically valid interpretation of slope as a ratio of two quantities. This is different than, for example, Blake from Category 1 who appeared to believe his students would be able to productively engage with The Burning Candle Task because the task contains a

graphical representation which would prompt his students to make use of a learned

formula, namely $slope = \frac{rise}{run}$.

The other teachers. The other three teachers in this category—Emma, Richard,

and Kay—provided similar evidence, which appeared to suggest that they believed

students' ability to *productively* generalize their learning to novel situations was

dependent upon mathematically valid interpretations of the meaning of topics like slope.

For example, Emma explained that she believed her teaching artifact[9] (see Figure 4.12)

would serve to support students in conceiving of slope in terms of the coordination of the

change in two quantities (e.g., the amount of distance traveled and the amount of time

passed). For example, she hoped students would come to interpret zero slope, in the

context of a person's motion through space, as: "With the horizontal line, they are

standing still; they are having the same distance and time is going by." When

subsequently presented with The Burning Candle Task (Figure 4.5), Emma believed her

students would be able to productively engage with the task by similarly interpreting

slope in the new situation as a coordination in the differences of two quantities:

> With the distance versus time thing, for this sort of slope, it's asking
> the same question just like "What does the slope mean when it's
> straight across?" And so I'm hoping they could apply that and say
> "What's happening from here to here [points to two points on the
> graphical representation of zero slope in her teaching artifact]? Well
> the distance stayed the same but the time is going by." So then "What
> happened from here to here [points to two points on the graphical
> representation in The Burning Candle Task]? Well the height of the
> candle is going down and the time went by."

[9] The axes in Emma's teaching artifact are not pre-labeled with distance and time. Emma's hope is that such quantities come up during the class's discussion of the activity. In particular, the following note was attached to her artifact: "Students could produce a distance versus time graph and highlight the fact that if the *x*-axis is time and the *y*-axis is distance, then a vertical line cannot make sense – it would portray the scenario of changing distances within a single moment of time. If this argument did not come up, I would introduce it and discuss why a horizontal line makes sense in this scenario and a vertical line cannot."

In this excerpt, Emma seemed to express recognition of a connection between students'

prior engagement with her teaching artifact and their subsequent engagement with the

novel Burning Candle Task, namely that both necessitate students' coordination of two

quantities—distance and time (with respect to her teaching artifact) and height and time

(with respect to The Burning Candle Task).

However, like Patrick, Emma was uncertain her students would conceive of the

connection that she saw. For instance, when referencing Part B of The Burning Candle

Task, Emma predicted, like Patrick, that her students might focus on a single quantity and

say "the slope means the height of the candle is decreasing." She, therefore, suggested

teacher actions (to be discussed later) that would better support the desired interpretation

of slope as a coordination of two quantities.

Both Richard and Kay also believed that their teaching artifacts would support

students in developing a mathematically valid interpretation of the meaning of slope.

Richard predicted that his students would be able to productively generalize their learning

to The Burning Candle Task if they had developed an interpretation of slope as the rate at

which two quantities change in relation to one another. In particular, Richard noted that

The Burning Candle Task was "the *same* type of situation" as the one used in his teaching

artifact (i.e., The Stretching Things Out Activity[10]) wherein students reasoned about the

length of a rubber band in relation to the number of weights attached to the end of the

rubber band. Richard went on to elaborate what he meant by "the same," stating that

both situations involve "something changing," or "rate of change." Thus, Richard

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[10] The Stretching Things Out Activity was drawn from the reform-oriented and National Science
Foundation funded Core-Plus curriculum.

seemed to recognize a particular connection between the two situations in terms of the interpretation of slope he hoped his students would develop during The Stretching Things Out Activity, and believe that students would be able to productively generalize their learning if they had developed a rate-of-change interpretation of slope.

Similarly, Kay predicted that her students would be able to productively generalize their learning to The Water Pump Task (see Figure 4.6) if they had developed a meaning for slope in a linear situation as the constant rate at which two quantities change in relation to one another. Kay started by explaining her teaching artifact, an activity about a girl removing an unknown, but constant amount, from her piggy bank every day. During this activity, students are provided information regarding the total amount of money in the piggy bank on day 17 and on day 37 and are asked to figure out how much money is being removed every day. Kay explained that she hoped her students would come to interpret slope in a linear situation as "a constant rate of change, that something is changing at a constant rate." In The Piggy Bank Task, Kay explained that the slope is the consistent amount of money being removed from the bank every day and that no matter which two consecutive days you choose, the amount of money removed from the piggy bank will be the same. Kay appeared to believe that her students would be able to generalize this interpretation of slope to other situations, for instance, to The Water Pump Task by reasoning that the rate at which the water is being pumped into the pool is consistent if the amount of water going into the pool every minute remains the same across the set of data provided.

Task: Explain why the line in graph #1 has an undefined slope and why the line in graph #2 has zero slope.

Graph #1

Graph #2

Your explanation can range from the formulas or definition of slope, or you may use a real world example to create your argument. If you think of some other argument to form, great! I want your argument to be creative and convincing!
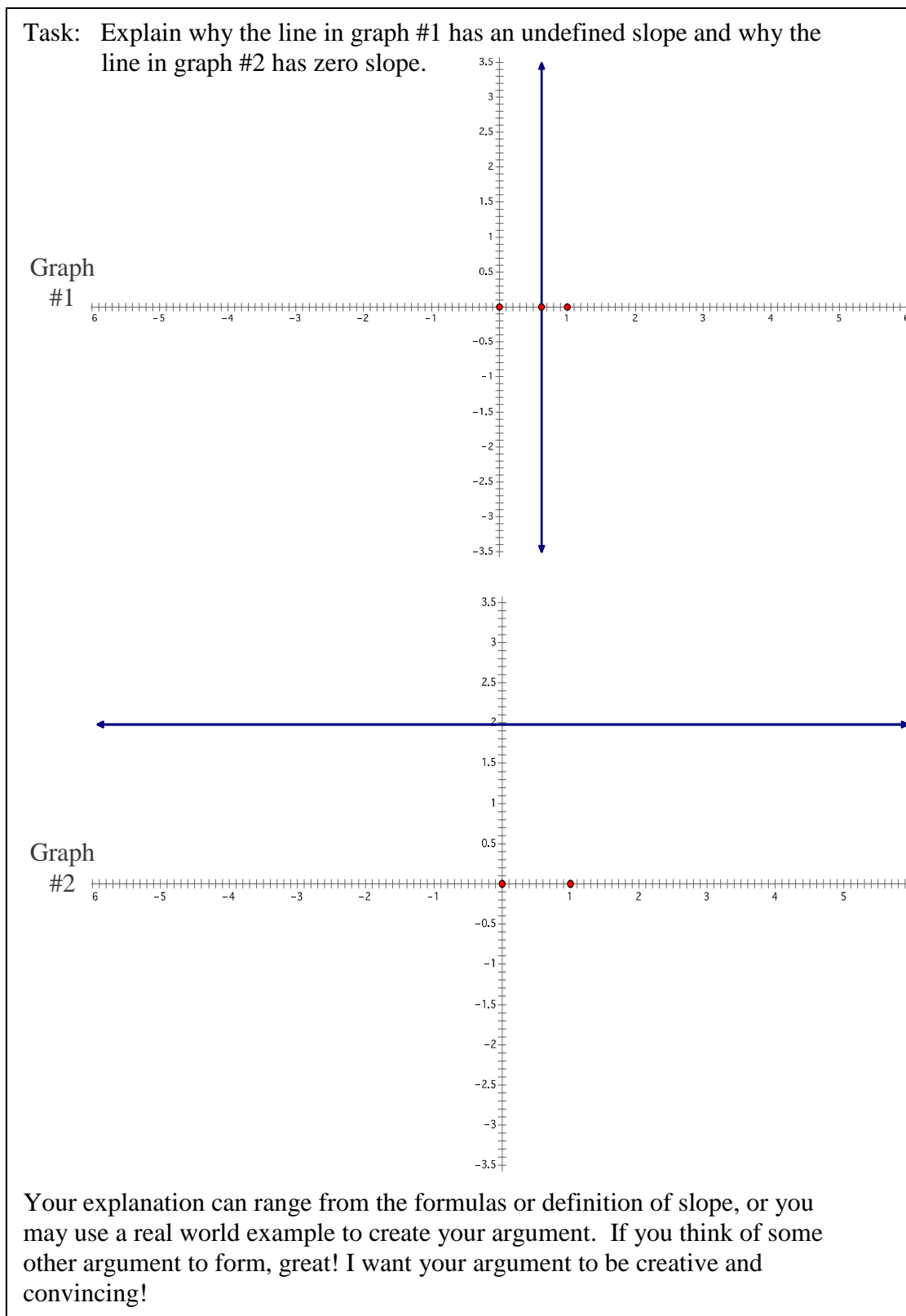
Figure 4.12: Emma's teaching artifact.

Kay provided some evidence during her interviews that one could argue is consistent with the Category 1 belief about the generalization of students' learning. Specifically, when discussing the generalization of students' learning in relation to the topics of slope and linear functions, Kay explained that she would repeatedly remind her students that "slope is the change in y over the change in x." One could interpret Kay's statement as a verbalization of the formula $slope = \frac{\Delta y}{\Delta x}$ or $slope = \frac{y_2 - y_1}{x_2 - x_1}$. Thus, one might argue that Kay is more like Blake from Category 1 who seemed to believe that students would be able to productively generalize their learning to a novel task if the task prompted students to make use of a learned formula. However, further evidence suggested that the statement was used as shorthand and intended to remind students that slope should be interpreted in terms of its meaning as the *rate* at which *two* quantities change *in relation to one another*. Specifically, Kay explained that The Piggy Bank Task would serve to help ground students' interpretation of slope and its associated formula:

> OK, well now, you know, slope is the change in y over the change is x and … I said, "So, if we were to graph, what would be on the y-axis? What would be on the x-axis?" and then we're always going back to what these numbers represented. So, they had, they already felt comfortable with this problem, so they had something contextual that they could tie these really abstract ideas to, you know, "What is y_2 minus y_1? What does that *mean*?" Well, that's not a big deal. That's just, you know, the number of, the amount of money, and then "What's x_2 minus x_1?" Well that's just the difference in days.

Kay, therefore, seemed to want her students to think of the quantitative meanings underlying the phrases "change in y" and "change in x," namely changes in the amount of money in the piggy bank and in the number of days, respectively. Thus, it did not appear Kay hoped her students would generalize a formula, but the meanings associated with the formula.

Associated instructional moves. There were three instructional moves that teachers in this category apparently believed would support the generalization of mathematically valid interpretations of slope: (a) choose tasks and pose questions that emphasize students' quantitative reasoning; (b) use a curriculum that progresses from contextualized problem situations to more decontextualized problems and formulas; and (c) provide students with opportunities to explain their reasoning. Each instructional move will be discussed in turn.

Choose tasks and pose questions that emphasize quantitative reasoning. Patrick, Emma, Richard, and Kay appeared to believe that teachers could support students in generalizing their learning by choosing tasks and posing questions that provide students with opportunities to reason quantitatively (in the spirit of Smith & Thompson, 2008). This pedagogical action is related to these teachers' belief that their students would be able to productively generalize their learning if they developed a mathematically valid interpretation of slope as a ratio of two coordinated quantities. Specifically, these teachers appeared to believe that their teaching artifacts would support students in conceiving of slope in terms of *two relevant quantities* (e.g., *the number of pounds of supply* and *the number of people in a family*) as well as the multiplicative *relationship* that exists between those quantities (e.g., we need 12 times as much supply as there are number of people in our family).

For example, Patrick explained that he tends to avoid using more "traditional," decontextualized problems and instead selects tasks set in real-world contexts and poses accompanying questions to get students thinking about the measurable attributes of objects in those situations. In his words:

Instead of collecting data in a traditional sense where it's *x* and *y* tables and it's we're graphing *x* and *y*, we are graphing *distance* versus *time* or we are graphing *how much coffee they have* as *days pass* and it's going down; they are using up supplies. Or *how much water do they have in storage* and why does the graph go up and then go down?

Patrick seemed to believe that by posing questions about the quantities in real-world situations (e.g., "How much water do they have in storage?"), students would begin to focus on measurable attributes such as *the amount of water*. Once students are able to identify relevant quantities in various contexts, Patrick explained that he would support students in conceiving of the relationships that exist between those quantities by asking questions like "How much coffee are they using every day?" and "How many miles have they traveled each day?" As a result, students may come to interpret slope as "how much change is happening" over time, in a particular situation.

In fact, Patrick explained that to support the productive generalization of students' learning, next time he used the Previous Travelers Activity (Figure 4.10), he planned to be more explicit about supporting students in conceiving of the numerical value of slope in terms of the quantities in a specific situation. Specifically, when students produced 12 as the slope of the function, he planned to ask, "What does the 12 mean?" Furthermore, he relayed that he would give his students "time to think about 'well, 12 what? It's 12 what? And why is it 12 of something?'" Moreover, Patrick reported that he would ask his students *where* they see the slope of 12 in the graph shown in Figure 4.11, as a way to help students conceive of and coordinate both quantities in the ratio (12 pounds of beans for each 1 person).

Similarly, Emma believed that by asking her students to conceive of vertical and horizontal lines in terms of a person's motion through space, her activity would support

students in conceiving of slope in terms of a relationship between the quantities *distance* and *time*. Furthermore, she suggested that to support students in conceiving of both quantities as well as the relationship between those quantities, she would point out particular points on a line and ask "What does it mean when you are at (0,0)? What if you're at (2,0)? What does *that* mean?" and "What does the slope mean when [the line] is straight across?" Richard believed asking students to engage with his activity would support them in conceiving of slope in terms of the relationship between the quantities *the length of a rubber band* and *the amount of weight attached to the rubber band*. He also suggested that asking questions like "What is [the slope] telling us?" and "What does the slope mean?" would help students think about slope quantitatively. Similarly, Kay believed posing the question "What does the slope mean?" to students while they engage with her teaching artifact would support students in conceiving of slope in terms of the relationship between *the amount of money being withdrawn from a piggy bank* and *the number of days that have passed*. In this way, these teachers appeared to believe that choosing tasks and posing questions that emphasized students' quantitative reasoning would support students in productively generalizing their learning because such actions can be used to support students in developing mathematically valid interpretations of slope.

Progress from the contextualized to the decontextualized. According to Patrick and Kay, curricula that progress from contextualized problem situations to decontextualized tasks and symbolic formulas will support the generalization of students' learning. Patrick seemed to believe that such a progression would be particularly helpful in supporting students to make use of the meanings that they developed while engaging in

contextualized problem situations when they are eventually confronted with decontextualized tasks. That is, students would be supported in generalizing their understanding of slope because they could leverage their meaningful interpretations of slope to make sense of decontextualized problem situations (as "transfer" tasks). In particular, Patrick explained how such a progression could be powerful for students:

> The first problems they do are all conceptual. So, [for example] they do one and it says "You need four inches of shoe lace for every child" and they're graphing number of children and amount of shoe lace … So those are the variables and then we go into some traditional examples where it's purely *x* and *y* and it's an algebraic rule; it's just the typical Cartesian coordinate grid. But they still have to identify where you start at and what is the change? So they realize "Oh, it's the same; it's either in a word problem, a context, or it's not." So, that's what I hope they would be able to do later is still be able to, we call that [slope], but "Oh, no, it's the same thing."

Thus, by beginning a unit with a problem which supports students in thinking about slope in terms of quantities like *the amount of shoelace* and *the number of children* and, in particular, as the unit ratio 4 inches of shoelace for every 1 child, students are first supported in developing a mathematically valid interpretation of slope as a ratio of two quantities. Students can then make use of such ways of reasoning when presented with "traditional examples where it's purely *x* and *y*," eventually realizing that *both* kinds of situations involve slope, or a ratio of two quantities. In this way, it appeared that Patrick believed a progression from contextualized to decontextualized problem situations would support students in being able to recognize both types of problem situations as necessitating the interpretation of slope students develop during initial learning activities.

Similarly, Kay discussed starting each unit by asking students to engage with a contextualized problem situation and then leveraging the meaning students develop

during their engagement to make sense of decontextualized tasks as well as formulas. In particular, Kay explained that she starts every unit by asking her students to work in groups to engage with a contextualized problem situation. With respect to supporting students in generalizing their understanding of slope, Kay explained that The Piggy Bank Activity was a great way to start a unit on slope because students could make sense of the relevant quantities in the context as well as the relationships that exist between those quantities and then later refer back to those quantities and relationships when asked to make sense of and use the conventional slope formula, $slope = \frac{y_2 - y_1}{x_2 - x_1}$. As discussed previously, Kay expected that students would map the number of elapsed days in the piggy bank situation (Day 37 – Day 17 = 20 days) to $x_2 - x_1$ in the slope formula and the corresponding change in amount of money in the piggy bank ($150.50 – $325.50 = -$175.00) to $y_2 - y_1$ (which would follow the contextual problem). Using this example, Kay expressed her belief that she could support students in seeing the decontextualized slope formula as a representation of *their* reasoning in a contextually-rich situation.

Provide students with opportunities to explain their reasoning. Richard and Emma spoke explicitly about their belief that supporting students in explaining (verbally or in writing) their interpretation of a particular problem or mathematical topic would be useful when aiming to support the generalization of students' learning. As Richard stated, "If they never can explain it [i.e., slope] in something that they do know, I think they are going to have a really hard time later on trying to take it to something that is new to them."

Similarly, Emma expressed her belief that "having students explain their answers" ("right or wrong") supports students in being able to make future use of their learning.

Apparently, she believed such explanations would help make students' ways of reasoning more explicit to themselves: "I think it's nice to really talk through what went through their mind [be]cause it may make [their thinking] more obvious for them." Once ways of reasoning are made more obvious to students, Emma believed they would be available for future use. For instance, Emma stated that if students are supported in reasoning about and verbally explaining the meaning of a particular slope value, then they should be able to make use of those same ways of reasoning to explain the meaning of additional slope values.

Discussion of Categories 1 and 2

In this section, I discuss points of contact and divergence between teachers' beliefs about the generalization of students' learning and transfer researchers' beliefs about the generalization of students' learning in an effort to highlight the added value in investigating *teachers'* beliefs. The first category of belief about the generalization of learning (as well as the corresponding beliefs about how to support the generalization of learning) has points of contact with both Thorndike's theory of identical elements and mainstream cognitive approaches to transfer.

Similarly to Thorndike's associationist view of transfer (i.e., a desired response spreads from learning to transfer situations to the extent that both situations share observable stimuli), Anne (from Category 1) appeared to believe that the generalization of learning was mediated by learned associations (such as the word-graph association of "towards" and "line with negative slope"). Similarly to a mainstream cognitive approach to transfer Blake and Donna (also from Category 1) seemed to believe that transfer is supported instructionally through the use of multiple examples during initial learning (cf.,

Anderson, Kline, & Beasley 1979; Brown, Kane, & Echols, 1986; Carbonell, 1983;

Michalski, 1983; Singley & Anderson, 1989).

However, there were four significant differences between the views of these

teachers and the transfer researchers. First, Thorndike emphasized extensive practice

when learning associations; Anne did not. For example, to promote the transfer of the

ability to estimate the area of any rectangle, Thorndike and Woodworth (1901) used 1200

practice items with students. In contrast, Anne's "training" process only asked students

to record three different graph-label combinations in their notes (and then to use these

notes to answer novel "transfer" tasks). Second, Thorndike and colleagues utilized the

drill and practice approach that became central to the traditional mathematics

instructional approach inspired by associationism and behaviorism. In contrast, Anne's

teaching artifacts were drawn from a reform-oriented book called A Visual Approach to

Functions (Van Dyke, 2002), which emphasizes the use of global properties of graphical

representations of functions to interpret their meaning in context (Kieran & Yerushalmy,

2004; Leinhardt, Zaslavsky, & Stein, 1990; National Council of Teachers of

Mathematics, 2000, 2006). Thus, Anne's apparent belief in the Thorndikean idea of

transfer being mediated by common associations finds an unusual home in the

constructivist-inspired activities chosen by Anne to supplement her school's mathematics

curriculum. This suggests practice-based decision-making that transcends the use of a

curricular approach aligned with Thorndikean ideas.

Third, researchers from a mainstream cognitive perspective on transfer tend to

create systematic variation in the superficial details of the use of multiple examples in

instruction, to encourage the encoding of the similar structure across the examples (e.g.,

Cantrambone & Holyoak, 1989; Reeves & Weisberg, 1990, 1994). In contrast, Blake

apparently believed that sameness in the surface details across multiple examples (in

terms of formatting and wording) was essential for transfer and instead sometimes varied

the structure (in this case the solution method—see the two examples shown in Figure

4.9, where solving part e of each problem requires a different procedure). Finally, Donna

appeared to believe, like some researchers operating from a mainstream cognitive

approach to transfer (e.g., Gentner et al., 2003; Markman & Gentner, 2000; Reeves &

Weisberg, 1990), that teachers should provide their students with opportunities to identify

and discuss similarities across the multiple example problems. However, for the transfer

researchers, the purpose of talking about such similarities is to help students attend to

important relationships that will be useful to the learner when later constructing mappings

between initial learning and future transfer situations. The purpose for Donna seemed to

be to help students *choose when* to use a learned procedure or formula—something that,

according to Engle (2006), mainstream approaches to transfer do not tend to concern

themselves with.

The second category of belief about the generalization of students' learning has

some points of contact with Lobato's (2006, 2012) actor-oriented perspective on transfer.

In particular, teachers in Category 2 seemed to embrace a basic tenet of the approach by

being sensitive to the different interpretations that students can form of mathematical

topics during learning situations and acknowledging the consequences of such for their

reasoning in novel situations. For example, Patrick conjectured that some of his students

may interpret slope in terms of a single quantity (rather than two quantities) during initial

learning and thus reason about slope in the novel Burning Candle Task in terms of the

quantity height (rather than height *and* time). However, there is insufficient evidence to claim Patrick, or the other teachers in this category, would have considered such mathematically incorrect reasoning as evidence that students *had* transferred or generalized their learning, as is the case in the actor-oriented transfer perspective. In other words, the teachers in Category 2 appeared to embrace an actor-oriented perspective but not necessarily an actor-oriented perspective on *transfer*.

However, the teachers in Category 2 significantly contributed to the pedagogical actions previously reported in the literature as being related to instances of productive generalizing from an actor-oriented perspective (Lobato, Hohensee, & Rhodehamel, in press; Lobato et al., in press). Specifically, they articulated the importance of quantitative reasoning to the generalization of learning and illustrated this idea using a variety of curricular materials that practicing teachers currently use. Furthermore, they elaborated potentially important roles for the sequencing of activities and students' explanations in the transfer of learning. In sum, across both Categories 1 and 2, there appeared to be added value in investigating teachers' beliefs about the generalization of students' learning and how to support it.

Disposition

Category 3: Dispositions Transfer

Statement of the belief. While teachers in the previous two categories of belief emphasized particular mathematical content (e.g., the interpretation of the meaning of slope as the rate at which two quantities change in relation to one another) when conceiving of the generalization of students' learning, teachers in this category (Emma, Kay, and Donna) seemed to emphasize students' dispositions toward problem solving in

general. In the spirit of Gainsburg (2007), I use the term *disposition* to refer to a personal

orientation towards or outlook on problem solving, which includes what problem solving

is about. Here, disposition does not refer to a heuristic, or a general set of steps one uses

to solve mathematical problems (e.g., draw a diagram, identify special cases, and then

solve a simpler problem). Two contrasting dispositions toward problem solving,

identified in previous research, are a sense-making orientation (Schoenfeld, 1991) versus

a "matching game" orientation (in which one matches the text in a problem statement

with mathematical symbols and operations, often via the use of key words; Sowder,

1989). Teachers in this category seemed to believe that students will be able to

productively generalize their learning to a novel situation if they develop and make use of

particular dispositions. Moreover, these teachers seemed to believe that one's disposition

carries over to novel situations and functions to facilitate the generalization of learning.

 Illustration of the belief. I illustrate this category with evidence from Emma.

Remember from the previous category of belief that Emma believed the productive

generalization of students' learning depended on the development of mathematically

valid interpretations of the meanings of mathematical topics like slope. Here, I present

evidence to show Emma appeared hold a second belief about the generalization of

students' learning, namely that students will be supported in generalizing their learning if

they develop and make use of a specific disposition. In particular, Emma seemed to

believe that the productive generalization of students' learning was facilitated by a

visualization and sense-making orientation towards problem solving wherein students

imagine themselves in a problem scenario in order to reason about what is happening in

the problem. Here, it is the disposition (rather than specific mathematical content) that

generalizes to novel situations and functions to support students' productive engagement

with such situations.

Remember that, at the end of the first interview, teachers were asked to create a

lesson plan "to help students generalize their understanding of slope." The lesson plan

was subsequently discussed during the second interview. Emma's lesson plan included a

class activity about Andy's trip to the gas station (see Figure 4.13). When asked how she

thought her activity served to support students in being able to generalize their

understanding of slope, Emma said that her activity allows students to "actually place

themselves into the problem." Emma elaborated saying pretending to perform the actions

of the people in the situation "allows them [her students] to reason a little better."

Specifically, the excerpt below suggests Emma believed her activity, specifically

question 3 (see Figure 4.13), would prompt students to visualize themselves in a car on

the way to the gas station and as a consequence of imagining their own trips to the gas

station students would be supported in "reason[ing]" about what was happing in the

situation:

> That part [points to question 3] was more to have them really visualize
> what's going on and put themselves into the problem … Having them
> place themselves into the problem and visualize what's actually, like
> think through [the problem] as if they were actually taking their car to
> the gas station and going back home … it's like "Just pretend we are
> doing this" and then it allows for them to reason a little better, so then
> they can step into it.

Emma, thus, emphasized a general approach towards solving problems wherein students

pretend they are performing the actions described in the problem so that they can reason

through the problem and consequently figure out what to do. In this way, Emma went on

to express her belief that such visualizing occurs within a sense-making orientation

towards problem solving in which students ask questions like "What is actually going on

here?" as opposed to "What equation do I use to solve this—what is the formula?"

Emma believed that when students make use of a visualization and sense-making

orientation, they are better able to "assess where to go" and "find the solution."

The graph above shows Andy's car ride to go put gas in his brand new car.

1) Using the graph, explain Andy's trip in as much detail as you can. You can be creative and use your imagination to explain any peculiar points in the graph. But remember, the graph is about Andy filling up his gas tank, so you must include filling up his tank in your story.

2) Andy heard on the news that driving less than 55 miles per hour can help conserve gas. Andy loves to drive fast, but also can't afford the high gas prices so he tries to drive less than 55 miles per hour on the way home. Is he successful? Explain.

3) Let's say I take a GPS and attach it to Andy's car and actually plot his distance and time during his trip to a gas station. What would be different on this more realistic graph? What would be the same?
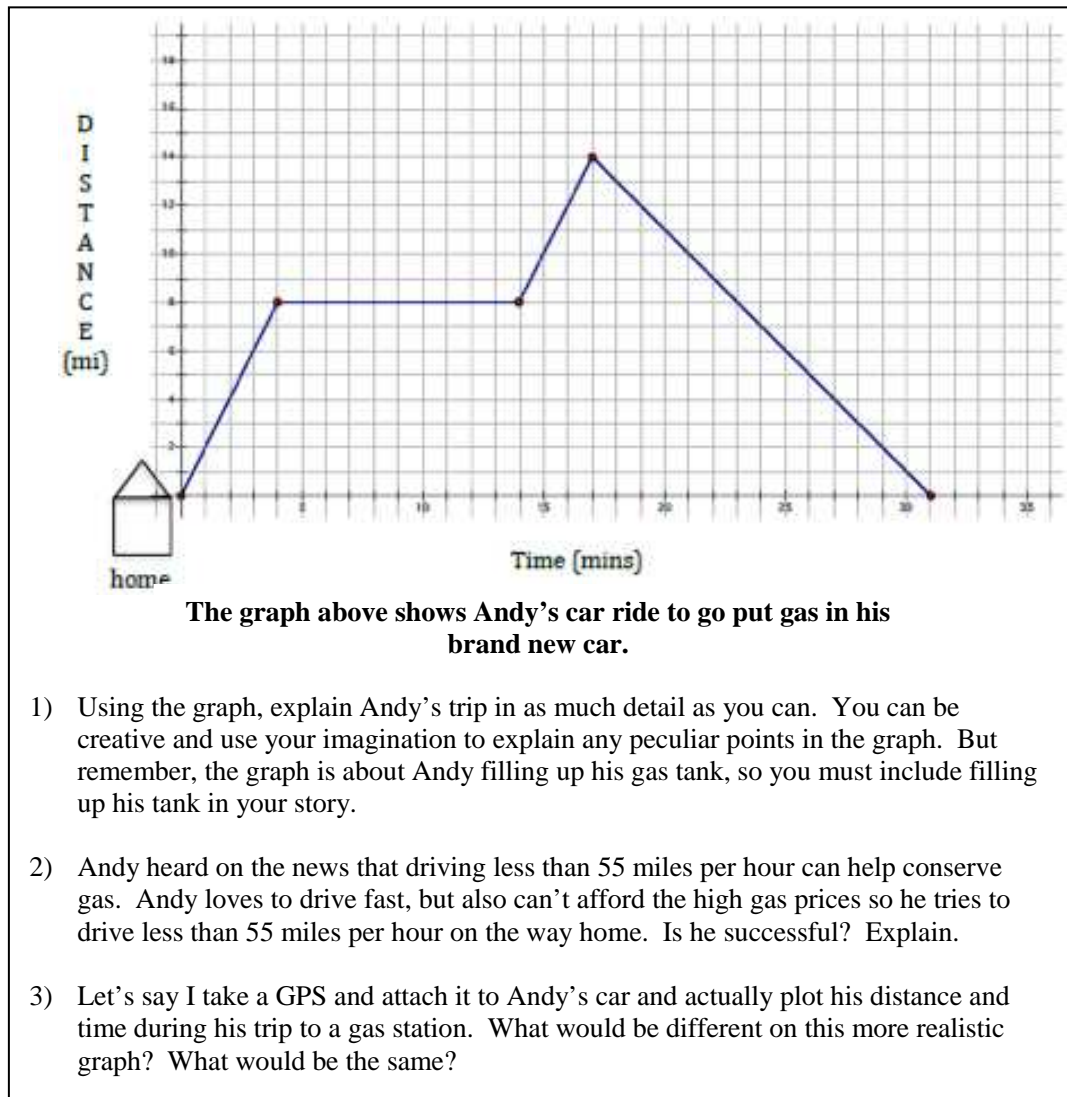
Figure 4.13: Emma's activity about Andy's trip to the gas station.

To see how the visualization and sense-making orientation towards problem

solving functioned to support students' productive engagement with novel tasks, consider

the new task that Emma brought to the second interview, which she thought her students

would be able to successfully engage with as a consequence of her lesson on slope. As shown in Figure 4.14, The Vase Task presents an image of a triangular flower vase and asks students to create a graph showing the relationship between the height of the water in the vase and the volume of the water in the vase. Emma explained that students would be able to successfully engage with the task because it provides them with an opportunity to make use of the disposition developed during students' engagement with the task involving Andy's trip to the gas station:

> What I would want them to take from this [points to the task shown in Figure 4.13] initially is to take that initial first step of "I am going to picture filling up this triangular vase and I am going to imagine the water rising versus how much I am actually, like the volume of the water that I've poured in there." … I'm hoping they will make a curve.

Evidently Emma wanted her students to orient towards the new task in the same way as they oriented towards the task about Andy's trip to the gas station, namely by projecting themselves into the situation and visualizing, or imagining, what is happening to the water level in the vase in order to make sense of the new situation. In other words, it seemed Emma hoped her students would generalize a particular disposition—the visualization and sense-making disposition. It appeared that Emma believed by visualizing the water as it is being poured into the flower vase, students would be supported in making sense of the relationship between the height of the water and the amount of water that had been poured into the vase. As a consequence of using this disposition, Emma hoped her students would "make a curve."

Lacey just bought a brand new vase to hold all the flowers her husband buys her. Her husband recently brought home Blue Siberian Irises, flowers notorious for needing lots of water. To make sure her flowers last, Lacey wants to fill her new vase with water all the way to the top. Assuming Lacey pours water into the vase at a constant rate, create a graph showing the water height versus the volume of water in the vase.

Figure 4.14: Emma's "new" task: The Vase Task.

It is interesting to note that the activity used in the initial lesson involved several linear relationships between distance and time whereas the new task involved a nonlinear relationship between height and volume. Emma noted this fact, stating "the curve is the biggest thing that makes it novel, that it won't be a linear graph." It, therefore, appeared Emma believed the use of a particular disposition, namely a visualization and sense-making approach towards problem solving, transcends the specific mathematical content of tasks thus supporting students in generalizing their learning from, for example, situations involving linearly related quantities to situations involving nonlinearly related quantities.

The other teachers. Like Emma, Kay seemed to believe that students would be able to productively generalize their learning if they developed and made use of a sense-making orientation towards problem solving. In particular, Kay explained that she believed students would be able to generalize their learning "if they are able to reason about the problem … for me it's whether they can apply their reasoning that they learned from going through this problem to other situations, not necessarily what they learned

about slope." Here, Kay seemed to emphasize the development and subsequent use of a particular orientation towards or outlook on problem solving, namely problem solving involves reasoning about a problem as opposed to remembering something specific about slope. Kay went on to elaborate what she meant by "reasoning," saying:

> Reasoning skills in general; just taking a problem and breaking it down … So, again I think it is more about building reasoning skills and if they can build them, I would hope it would help them to solve future problems.

The above excerpt seemed to illustrate Kay's belief that a disposition that involves breaking a problem down to reason about what is happening in it will help enable students to "solve future problems." Kay, thus, predicted that as a consequence of developing a sense-making orientation towards problem solving, students would be supported in productively generalizing their learning of slope and linear functions to a novel, quadratic-motion situation because students would "have to think about" and "reason" through the quadratic-motion situation to solve it (i.e., they would make use of the same sense-making disposition).

Unlike Emma and Kay who emphasized a sense-making disposition, Donna seemed emphasize a group-brainstorming disposition. In other words, Donna appeared to believe that a student will be supported in productively generalizing her learning if her orientation towards problem solving involves working in a collaborative setting wherein people exchange ideas about how to get started solving problems. Specifically, Donna explained that to problem solve, students simply need to get started in the problem solving process by putting forward "any kind of crazy ideas" about how to proceed.

Donna went on to provide an example of how this disposition might look in a problem-solving situation involving linear functions:

> Like I could say "gosh, we should do this with linear functions" and you would say "oh, but why not use the formula" and then somebody else might say "gosh, why don't we graph it and really see."

Thus, it appeared that Donna believed such a disposition would provide students with *multiple* ideas regarding what to do to solve a problem thereby increasing their chances of successfully solving the problem. As Donna stated, the "goal is to come together and get through that problem." Moreover, as a consequence of developing such a disposition, Donna believed her students would be able productively engage with a novel Cell-Phone Task wherein students are asked to work in groups to gather and graphically represent data regarding the cost of different cell-phone plans and to subsequently decide if the plans are linear. Because the task requires students to work in groups, Donna believed students would be supported in making use of their group-brainstorming disposition and thus generalizing their learning.

Associated instructional moves. Each of the three teachers in this category (Emma, Kay, and Donna) seemed to emphasize a different instructional move. These three instructional moves will be discussed in this section: (a) apply a real-world context to a problem if one is not already provided; (b) avoid problem statements that tell students what they need to do to solve the problem; and (c) model the desired orientation towards problem solving.

Apply real-world contexts to problems that do not already have them. Emma suggested that to support the generalization of students' learning, teachers should encourage students to apply real-world situations to problems that are not already set

within such contexts. Emma seemed to believe that by applying real-world situations to decontextualized problems, students would be enabled to visualize themselves in the problems and thus be supported in making use of their visualization and sense-making disposition. To illustrate, recall that Emma's teaching artifact (see Figure 4.12) was originally void of a real-world context. She hoped students would apply such a context to the artifact in order to solve it, but if they did not do so, Emma explained that *she* would introduce the possibility and help them to apply a distance versus time scenario to both graphical representations in order to make sense of them. Emma went on to say that applying a real-world context to a decontextualized situation "could also be something students could use [in the future], like if they are grappling with something, put it in a real-world context to explain it more." Emma therefore reported that students who learn to approach decontextualized problems by asking themselves "What if I just attach some sort of real-world situation to this? How would that help me? Would that reveal more about this slope?" will be supported in making use of their visualization and sense-making disposition and consequently solving novel problems.

   Avoid giving problems that outline what students need to do to solve them. Kay suggested teachers provide students with problems that are open-ended and that avoid step-by-step instructions regarding what to do to solve them (e.g., *graph the points* and then *calculate the slope of the resulting line*). She seemed to believe that open-ended problems would support students' sense-making disposition because they encourage students to figure out what to do on their own. As Kay explained, problems that explicitly tell students to, for example, "calculate slope" may actually "create a roadblock for [students]" since they call for specific steps that students may not know how to carry

out. Moreover, Kay provided evidence suggesting that she believed such problems might

actually prevent students from making use of a sense-making disposition. Specifically,

Kay said:

> If you look at a lot of textbooks […] a lot of the problems are broken
> down into steps. 'Step 1, find slope. Step 2, graph it. Step 3-,' you
> know, OK great, where is the thinking?

Thus, it appeared Kay believed that problems with "steps" to follow deny students

opportunities to utilize a sense-making disposition because they directly tell students

what they need to do rather than allowing students to figure it out for themselves. In this

way, Kay explained the key to the problems she designs:

> They don't tell you, they don't have any steps for you to do. You have
> to figure what steps you need to, whatever it is that you think you need
> to do, you need to figure it out.

This excerpt seemed to highlight Kay's belief that if students are to be supported in

utilizing a sense-making disposition, and thus to be supported in generalizing their

learning to novel situations, teachers should provide students with opportunities to

engage with problem situations that are open ended in nature meaning that they avoid

directive statements like *find the slope* which deny students opportunities to make use of

their sense-making disposition.

> ***Model the disposition.*** Donna suggested that teachers model or enact the

disposition they want their students to develop since, as she noted, productive group

brainstorming "is not going to happen on Day 1." To help students develop a productive

or effective group-brainstorming disposition, Donna explained that she enacts its various

components, for example, demonstrating to students how to share ideas in a group

setting. For instance, when confronted with a problem-solving situation in the classroom,

Donna explained that she tells students what the problem reminds her of, what she

attends to, and how she comes up with ideas regarding how to solve the problem:

> We do a lot of teacher modeling – "This is how I'm thinking; look I saw this; oh, this reminds me of this," you know, so I do a lot of modeling of how I just kind of reach out and grab these things. … So really once you get that going, then they start problem solving.

Donna seemed to believe that if students are exposed to productive aspects of a

disposition, they will be enabled to re-enact them on their own, thus, making such

productive elements part of their own disposition. In this excerpt, Donna also seemed to

express her belief that once students develop such a disposition, they will "*start* problem

solving."

Discussion of Category 3

This category of belief is similar to Bereiter's (1995) *dispositional* approach

towards problem solving wherein, as discussed in Chapter 2, Bereiter argued that when

teaching for the transfer of learning, students' "way of approaching things" is of primary

concern (p. 23). In the spirit of Bereiter, teachers in this category of belief emphasized

the particular ways in which students orient towards problem solving and the ramification

of their dispositions for the generalization of students' learning. However, Bereiter did

not focus on the mathematical domain; instead, he illustrated his ideas with examples

from reading, moral education, and science education. Thus, the particular dispositions

that the teachers in this category articulated—a *visualization and sense-making*

disposition, a *sense-making* disposition, and a *group-brainstorming* disposition—are new

to the transfer literature. In other words, by talking to teachers of mathematics, I was

able to get specific ideas about dispositional approaches to transfer that are related to the field of mathematics education.

Students' Affect

Categories 4 and 5 involve the affective domain, specifically teachers' beliefs about students' beliefs. To avoid confusion arising from use of the word "belief" twice, I use the term "view" instead of belief in reference to the students. Thus, this section addresses teachers' beliefs about students' views. Category 4 involves the role that *students' view of self* plays in the generalization of learning (according to the teacher participants in this study) and Category 5 involves *students' view of mathematics*. Following McLeod (1992), who conceived of both views of self and views of mathematics under the affective domain in mathematics, I group Categories 4 and 5 together as involving students' affect (versus mathematical content or a mathematical disposition). Despite the fact that the role of affect has been absent from the transfer literature, the results presented in this section indicate that 7 of the 8 teachers subscribed to a belief about the role of students' affect in promoting the generalization of learning.

Category 4: Students' View of Self Facilitates Transfer

Statement of the belief. Teachers in Category 4 seemed to believe that students will be supported in generalizing their learning to novel situations if students' develop confidence in their abilities to engage in mathematical activity. Confidence here refers to a person's view of his or her "competence in mathematics" (McLeod, 1992, p. 583) or the "belief that one can learn to do that which is expected of one" (Broekmann, 1998, p. 18). As will be demonstrated below, the teachers' language reflected a conception of confidence as an internalized and individual psychological construct (i.e., confidence is a

view about one's own mathematical capabilities) rather than as a normative practice-oriented construct (i.e., confidence is an interconnected and mutually defining process and product of engagement within a particular community (cf., Graven, 2004)). This is another reason why the Category 4 belief is associated with affect rather than a construct such as identity (which is often conceptualized as a product of one's socially-situated engagement in and negotiation with a community; Bishop, 2012; Nasir, 2002). The choice to characterize confidence at the individual psychological level reflects an attempt to remain consistent with the language used by the teachers in this study.

All but two teachers (Anne, Donna, Blake, Emma, Patrick, and Sam) seemed to believe that the generalization of students' learning is contingent upon how confident students are that they can engage in mathematical activity. In other words, students who view themselves as capable and therefore think "I *can* do this" when confronted with a mathematical problem-solving situation will be supported in generalizing their learning while students who do not will shut down and thus be prevented from generalizing their learning. This belief about the generalization of students' learning seemed vague or imprecise in the sense that as a mechanism for supporting the generalization of students' learning, it was not well specified. In other words, teachers appeared to believe that if students are confident in their ability to engage in mathematical activity, then their learning would simply generalize to new situations. It could be that teachers believed students' confidence acted like a key to unlock the door to students' engagement with novel problems thereby allowing them the opportunity to apply mathematically-specific understandings like those discussed in Categories 1 and 2. Alternately, it could be that students' confidence acted at the more general level allowing students to productively

engage with novel situations that transcended their learning of a particular mathematical topic like slope. In this way, the limits of this belief about the generalization of students' learning are unclear.

Illustration of the belief. I illustrate this category with evidence from Donna. Donna seemed to believe that the generalization of students' learning was dependent on how confident students are that they can engage in mathematical activity. For example, Donna said, "It's hard to get kids to generalize [their learning] … because you have to break down their beliefs of 'I just suck at this; I don't know anything.'" Moreover, Donna went on to say:

> For me, it is making that "Ah-ha" like "Oh, I *can* do this; I *do* know how to do this." So, that's what I'm going for. I want them to realize that yes they can really do this … You have to build self-esteem into those learners like "No, you're not stupid."

Thus, Donna seemed to believe that for students to generalize their learning, they need to cast away or break free of the notion that they are incapable of engaging in mathematical activity and, instead, take on the view that they *are* able to productively engage in mathematical activity.

When asked how she thought her students would engage with parts A and C of The Burning Candle Task (i.e., *find the slope of the function represented below* and *predict when the candle will burn out*), Donna appeared to emphasize students' confidence in their mathematical abilities (rather than mathematical content or disposition). In responding to the question, Donna talked about her students in terms of grade levels (i.e., ninth, tenth, and eleventh graders), compared their relative levels of confidence, and decided that because eleventh graders are generally more confident or

"comfortable" with their own mathematical abilities they would simply *be* more

successful whereas ninth and tenth graders, who are generally less confident, would shut

down at the sight of the "scary" problem. Specifically, Donna said that when confronted

with The Burning Candle Task her eleventh graders who "are more comfortable" would

jump in and "just find the slope." Donna contrasted this with how she thought her

younger students would approach the same task, saying "I don't think ninth and tenth

graders are comfortable here yet; this whole problem is just scary looking … and we talk

about that, I don't want them shutting down because it looks scary." In these excerpts,

Donna seemed to express her belief that her ninth and tenth graders would be unable to

engage with the novel Burning Candle Task because they do not have confidence in their

abilities to engage with such a problem (i.e., they do not view themselves as capable)

whereas her eleventh graders, who do have confidence in their mathematical abilities,

would be supported in productively engaging with the task. In this way, Donna appeared

to believe that lower levels of confidence would inhibit the generalization of students'

learning whereas higher levels of confidence would somehow function to facilitate the

generalization of students' learning.

The other teachers. The other five teachers in this category also appeared to

believe that students' confidence in their mathematical abilities supports the

generalization of their learning. For example, Sam, the only teacher in Category 4 but

none of the other categories, explained that his teaching artifact (an activity involving

slope) supported the generalization of students' learning because it helped them develop

confidence in their ability to find slope. In particular, he said that the activity would help

students realize that "figur[ing] out the slope" is "not a hard thing to do." He went on to

say, "I don't know how much I focused on the meaning of this [slope], but, you know, just getting them to feel confident." Thus, for Sam, it is not particular meanings for slope or even a procedure for finding slope that generalizes to novel situations, but confidence in one's mathematical abilities that provides access to one's engagement with novel situations.

Emma also seemed to believe that confidence in one's mathematical abilities supports the generalization of one's learning. However, unlike Sam, she also provided evidence consistent with beliefs that students' mathematical meaning generalizes, as do their dispositions regarding problem solving. It may be the case that Emma (and the other teachers in this category) believed several different factors functioned together to support the generalization of students' learning. Perhaps students' confidence allows students to engage with novel tasks while the dispositions and mathematically valid meanings students develop for mathematical topics like slope allow students to *productively* engage with novel tasks.

Associated instructional moves. There were several ways in which teachers in this category appeared to believe they could support students in feeling confident about their ability to engage in mathematical activity, and thus to support the generalization of students' learning. I present the three most prevalent instructional moves: (a) use language of "understanding/not understanding" or "participating/not participating" versus "easy/hard" or "good/bad", (b) let students figure out problems on their own and use their own strategies, and (c) and use contextually-meaningful tasks.

Use language of "understanding/not understanding" or "participating/not participating" versus "easy/hard" or "good/bad." When considering ways in which

teachers' could support their students in feeling confident about their ability to engage in mathematical activity, and thus support the generalization of students' learning, Donna and Blake focused on the nature of the language used in the classroom. In particular, they suggested teachers attend to the ways in which people in the classroom verbalize the relative ease or difficulty they experience while engaging in mathematical activity. As Donna explained, she and the students in her classroom are "very very careful of words." Donna provided the following example:

> In my classes, we don't say "It's easy." I don't let them say that. We say "I understand that" or "I don't understand that." And if someone says, "Oh, that's so easy," I'm like "Excuse me?" and then they're all like, "You can't say that; you can't say that."

In this excerpt, Donna seemed to distinguish between *a problem* being an "easy" problem and the relative ease or difficulty *an individual* experiences when confronted with that problem. In the former, the relative ease or difficulty of a problem is an inherent characteristic of the problem itself and in the latter, the relative ease or difficulty of a problem depends on the individual. It seemed Donna aimed to support students in reflecting on and articulating the latter rather than the former. For if one student in the class categorizes *a problem* as "easy," then other students in the class may feel stupid for not understanding the "easy" problem. However, if a student says "*I* understand this," then the other students may be less affected by the statement as it does not imply that another student *should* also be able to understand it. Moreover, Donna explained that she does not perpetuate the view that courses like calculus are "hard" but rather tells her students "*everybody* can take calculus." In other words, Donna wanted to support her

students in viewing themselves as competent like "oh, I *can* do that" regardless of the topic or content area and used language as a vehicle to support such a view.

Similarly, Blake made use of personal anecdotes to let students know that the relative ease or difficulty they experience, for example, on tests does not dictate who they are mathematically but is simply a part of what it means to participate in life:

> I'm just letting them know that this grade that you get on one particular test isn't the end all and be all and this isn't who you are. You are not, in my mind, a person who is "bad at math" or a person who is "good at math." You are just somebody who is participating in life.

Blake seemed to want students to reject the notion that grades define *who* people are (e.g., an F means I am bad at math whereas an A means I am good at math). Rather, he wanted students to view being evaluated as a normal part of participating life and as providing one with an opportunity to choose to participate differently. Thus, Blake shared personal stories of starting out various semesters in his career as a student with Ds or Fs in his math classes, subsequently deciding to change the way he participated, eventually receiving As and Bs in said classes, and finally becoming a math teacher. In this way, Blake used language as a vehicle to convey the message that students' *are* capable of engaging in mathematical activity regardless of the grades they receive.

Let students figure out problems on their own and use their own strategies.

Sam and Emma suggested that teachers reorganize the roles and obligations of the teacher and students in a classroom so that students are the ones responsible for figuring out how to solve problems. They apparently believed that taking on such roles and obligations would develop students' confidence in their mathematical abilities. For example, Sam suggested teachers provide students with opportunities to answer their own

questions. To illustrate, consider the two-part activity Sam brought to the first interview.

Sam explained that the first part, which consists of 6 questions, is to be completed as a

class while the second part, which consists of questions 7-12 is to be completed by

students in small groups. In each of the first 6 questions, students are provided with (a)

two coordinate points (x_1, y_1) and (x_2, y_2), wherein x_1, x_2, y_1, and y_2 are all known values,

(b) space in which to plug those values into the slope formula, and (c) a Cartesian plane

on which to graph the associated line. However, in questions 7-12, students are provided

with a different set of information. For instance, in question 7, (a) y_1 is an unknown, (b)

the slope is given, and (c) a line has already been graphed.

When asked how he thought the first part of the activity supported students in

productively engaging with the second part of the activity, Sam did not focus on

relationships between mathematical details of the two parts of the activity, but rather on

providing students with opportunities to figure out the second part on their own. For

example, he said:

> Kids would be like "What do we do about number 7?" and I'm like "I
> don't know; figure it out, you know this number [pointed to the
> missing *y*-value in question 7] could be anything," and I just walk off,
> you know, and then I come back 5 minutes later and they're like
> "Yeah, we figured it out," like I didn't have to teach anything, you
> know, and just try to like change the problem in such a way where they
> have to like take some process that they thought they understood pretty
> well here [pointed to the first part of the activity] and into some new
> situation and it's not that much different [pointed to the second part of
> the activity].

In this excerpt, Sam seemed to express his belief that allowing students the time and

opportunity to answer their own questions (e.g., by walking away after hearing a

students' question rather than answering the question directly) would support the

generalization of their learning. Sam went on to explain that he hoped such a move

would support students in feeling "confident" about their ability to engage in

mathematical activity and, in particular, novel mathematical tasks.

Similarly Emma suggested that teachers support students in developing their own

strategies for solving problems. In the following, Emma explained how she believed

encouraging students to use their own methods would support students' confidence:

> Allowing them to use their own way of thinking and having that
> reasoning, or whatever it is, if you can, have it sort of come up
> organically from the student … because now they see sort of the steps
> it takes to get to the solution and they did it themselves. So you have
> that added sort of confidence in it like "I can tackle this; I can do this;
> I got it."

In this excerpt, Emma appeared to express the belief that encouraging students to come

up with their own strategies, as opposed to simply providing students with procedures or

formulas to use, would serve to build students' confidence because they would actually

see themselves as capable of successfully solving math problems on their own. As a

consequence of having increased levels of confidence in their mathematical problem-

solving abilities, Emma seemed to believe that students would be more willing to engage

with novel problem situations.

Use contextually-meaningful tasks. Patrick suggested the use of rich real-world

problem situations as a way to support students in feeling confident about engaging in

mathematical activity, and in particular, in answering mathematical questions. Patrick

explained that the contexts used in real-world problem situations can be used as tools to

convince students who are "marginally successful" that they do understand what is going

on and that they are therefore capable of engaging in mathematical activity. Specifically,

he said, "I think the context, the contexts definitely helps me help students understand what we are doing. So, I can give them that confidence because I can convince them they understand what is going on." Patrick further elaborated that he believed such problems were particularly powerful in developing confidence because they make it easier for students to answer questions when they are stuck. Specifically, Patrick said, "If they're really stuck, in a context, they can answer your questions because it is a fairly simple context and they can find success in that." Thus, he explained that real-world problems can be used as a tool "to use to get everybody successful doing something with slope." In other words, Patrick seemed to believe that contextualized problem situations make it easier for students to answer mathematical questions, which in turn provides students with opportunities to feel successful and confident in their overall ability to engage in mathematical activity.

Category 5: Students' View of Mathematics Facilitates Transfer

   Statement of the belief. Teachers in this category (Richard, Patrick, and Blake) seemed to believe that the generalization of students' learning was mediated by students' views about mathematics. In particular, teachers appeared to believe that students would be supported in generalizing their learning to novel situations if they view mathematics as relevant to and useful outside of the mathematics classroom.

   As with the Category 4 belief, the specific details of how this belief functions remain unclear as teachers tended to speak in vague terms regarding the particular ways in which students' views about the relevance and usefulness of mathematics promoted the generalization of learning. It may be the case that teachers in this category believed students' views of mathematics would support the generalization of learning because they

would frame students' current activity inside the classroom as relevant to future activity both in and out of the classroom. For example, if a student views mathematics as relevant and useful to what a person does in some real-world situation outside of the mathematics classroom, then that student will be more likely to engage successfully when confronted with a novel situation involving said situation. In this way, the Category 5 belief may function along with teachers' other beliefs. On the other hand, it may be that viewing mathematics as relevant and useful outside of the classroom functions at a more general level, allowing students to productively engage with novel situations that transcend the learning of particular procedures, concepts, or dispositions. In short, the particular ways in which beliefs involving students' affect (both Categories 4 and 5) functioned to promote the generalization of learning were underspecified by the teachers.

Illustration of the belief. I illustrate this category with evidence from Richard. Richard appeared to believe that students would be supported in generalizing their learning if they develop the view that the mathematics learned in the classroom is relevant and useful outside of the classroom. As Richard stated, students would be supported in generalizing their learning if they are able "to see how math is used in the real world." As he explained, students who view mathematics as useful in the real world are supported in making future use of their learning *because* they are able to make connections between current and future situations:

> If a kid can get a grasp of "How am I going to use this?" "What does it mean in everyday life?" it's sure going to help them … to make a connection later on when you're delving into the higher mathematics.

In this excerpt, Richard seemed to express his belief that students who have recognized everyday uses for the mathematical interpretations they develop in the classroom will be

supported in making connections between that interpretation and the interpretations they will be expected to develop in later, higher-level mathematics classes. However, it is important to note that he didn't specify the particular connection that he expects students to make. Thus, this belief seems to function at a more global level to promote the generalization of learning than the beliefs involving specific mathematical content or dispositions (Categories 1-3).

The teaching artifact[11] Richard brought to the first interview was an activity called "Stretching Things Out" (See Appendix A). The activity asked students to consider how the approximately linear relationship between the length of a rubber band (and subsequently a coil spring) and the amount of weight on the end of a rubber band would manifest in various mathematical representations (e.g., graphical, tabular, symbolic). When asked how he believed The Stretching Things Out Activity served to support the generalization of students' learning, Richard pointed to the fact that the activity is set in a "real life" context. In particular, he noted that rather than it being a traditional, decontextualized activity, it actually makes use of phenomena one could reasonably expect to encounter in the real world: "It's real life; rather than just 'here is a line; let's give it the equation $y=mx+b$,' it was 'what's happening with the weight? What's happening with the length?'" Thus, as Richard explained, "the kid sees 'Oh, somebody is going to have to do this [in real life]; I mean, it's something that is done.'" Moreover, Richard reflected on a time in which he did use the activity in his teaching:

> You never had kids asking "When am I going to use this?" I mean it
> was fun not to have to answer that question every day where in a

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[11] This activity is from the reform-oriented curriculum Core Plus (Coxford et al., 1998).

traditional class, I think, you get asked that every day: "Why do I need this?"

In these excerpts, Richard seemed to express his belief that his teaching artifact supported students in viewing mathematics as relevant and useful in the real world.

Interestingly, Richard said he thought his students would be able to productively engage with a quadratic-motion situation as a consequence of their engagement in The Stretching Things Out Activity. When providing his reasoning for this, Richard appeared to emphasize students' view of the usefulness of mathematics in the real world (rather than mathematical content or a mathematical disposition). In particular, Richard said that after having engaged in The Stretching Things Out Activity wherein two different mediums are examined (i.e., a rubber band and a coil spring) and various real-world problems arise (e.g., a rubber band breaks), students come to realize the world is full of different situations. In his words: "It [The Stretching Things Out Activity] helps them realize that there's not just one type of thing going on … there are other things that happen … look at house prices, look at different things." Here, Richard seemed to express his belief that students would be enabled to productively engage with a novel problem involving quadratic motion because of a view they had developed during The Stretching Things Out Activity, namely that that mathematics is relevant and useful in *many different* real-world contexts.

The other teachers. Patrick also seemed to believe that the generalization of students' learning would be facilitated by a view that mathematics is relevant and useful outside of the mathematics classroom. Patrick reflected on the evolution of his own teaching when discussing the generalization of students' learning. Because of a

conversation with the economics teachers at his school, Patrick realized that the curriculum he was using was organized into disjoint topics with "no purpose." In this conversation, the economics teacher contended that the seniors in his class were unable to recognize linear relationships unless they were expressed in the $y = mx + b$ form, and this was despite the students having "taken quite a bit of math." Patrick was surprised that his students were unable to generalize their learning from their prior mathematics classes to their economics class. Patrick began to realize that teachers are "missing something if [they] are not supporting their [students in] using math elsewhere" and that he, as a teacher, "should be supporting math being used in physics and in chemistry and in economics and all the rest of the disciplines." This seemed to be the impetus for Patrick's belief that to generalize their learning of mathematics, kids need to "feel purposeful in learning math," see "*that's* why we are doing this math," and realize that they can "actually *use it* on something." After that conversation with the economics teacher, Patrick decided that he would "still teach all the math but [it] would have this purpose behind it as opposed to 'we're just doing topics.'"

Blake also seemed to believe that students with the view that mathematics is useful outside of the classroom would be able to productively generalize learned associations, procedures, and formulas to novel situations. For instance, he said that in order for students to "generalize slope to other situations," they need to realize "the fact that it's everywhere" and "learn to see" slope (i.e., "a certain amount horizontally, a certain amount vertically") in the world around them. He went on to say:

> I'm hoping that when they walk away with it that day that they can
> have some understanding of it as a certain amount horizontally, a

certain amount vertically, … [and] be able to look at the world and say "oh, I see different angles everywhere that I could call slope."

In the above excerpts, Blake seemed to express his belief that students will be able to make future use of their learning of slope if they become aware that it can be seen in the world around them and learn how to identify it in terms of a vertical rise and a horizontal run. Thus, Blake seemed to believe that the generalization of students' learning is facilitated by a particular view of mathematics, namely that it is relevant outside of the classroom. Interestingly, Blake acknowledged that there were some students who would be able to productively generalize their learning to novel situations even if they did not develop such a view of mathematics. These are the "kids who just love the puzzle of it…who just love the rules;" for them, "it doesn't have to connect to anything." But for the "other ones who just hate the rules," Blake felt that it was crucial to help see the usefulness of mathematics in the real world.

Associated instructional moves. There were two ways in which the teachers in this category appeared to believe they could support students in developing a view of mathematics as useful and relevant outside of the mathematics classroom: (a) make use of real-world situations, and (b) ask students to come up with real-world situations involving a particular mathematical topic. Each instructional move is discussed in turn.

Make use of real-world situations. Each of the three teachers in this category spoke about making use of real-world situations as a way to support students in viewing mathematics as useful and relevant outside of the classroom. Patrick appeared to believe that students would be supported in generalizing their learning if they are asked to engage with real-world problem situations that spark students' curiosity and that students find

personally "interesting" and "relevant." In particular, when explaining how he developed

his lesson plan so that it helped to enable students to make future use of their learning,

Patrick said that he was not inclined to choose traditional "practice" problems or to use

worksheets, but rather to choose problems that would support students in thinking about

why they would want to learn about slope and how it could be useful in their lives outside

of the classroom. Specifically, Patrick said:

> I wanted it to be interesting. I wanted it to be something they would
> be curious to figure out. And kind of relevant … My inclination
> wasn't to say "Well, let's do a worksheet" or "let's do more practice."
> It was "Alright, so when would you ever come across slope; when
> would you ever want to even have a thought about something that's
> steep or there's changing amounts?" So, that's when I thought about
> cars, working so much, making money, so much per week.

This excerpt appeared to highlight Patrick's belief that selecting particular kinds of

problems would support students in viewing mathematics as relevant, namely problems

that are interesting to students and that help students identify instances in their lives that

involve mathematical topics like slope.

Similarly, Richard and Blake stressed the importance of making students aware of

how mathematics can be seen and used in their individual lives. Richard drew upon

students' real-life interests, such as football or music, to help students see connections

between the mathematics discussed in the classroom and phenomena that exist outside of

the classroom. Blake said that he provides students with several examples of real-world

situations involving the specified mathematical content, for instance using home-buying

and purchasing situations, to illustrate linear functions.

Ask students to come up with real-world situations that involve particular

mathematical topics. Patrick and Richard appeared to believe that to support students in

seeing mathematics as relevant and useful outside of the mathematics classroom, and thus support students in generalizing their learning, teachers should ask *students* to come up with real-world examples that involve mathematical topics. When explaining how the goal of supporting students to make future use of their learning contributed to the way in which he thought about his lesson plan on slope, Patrick said it led him to ask questions like: "Are there any other instances where you hear something that is stated like this rate of pounds of beans per person; what else have you ever heard or seen that is so much of an amount per something?" Patrick went on to provide reasoning for employing such a move, namely that when students share that they have heard "miles per hour" or seen a commercial advertising a car's mileage per gallon, they often realize, "Wow, there is a lot of rates that I hear or I know about, but I never think about them as it's *this* much *per* something." Similarly, Richard said that he believed asking students to identify jobs wherein employees make regular use of mathematics and to write reports responding to the question "How do you use mathematics in your life?" would help students view mathematics as relevant and useful in the real world. Thus, Patrick and Blake appeared to believe that their students would come to view mathematics as relevant and useful outside of the classroom, and therefore be supported in generalizing their learning, as a consequence of being asked to provide examples of real-world instantiations of mathematical topics.

Discussion of Categories 4 and 5

These categories about the generalization of students' learning and the corresponding beliefs how to support it do not appear in the literature base on transfer. The absence of an emphasis on students' confidence regarding their mathematical

abilities as well as students' views of mathematics would indicate that transfer

researchers have not identified such views as being important factors with respect to the

generalization of students' learning. However, the fact that 7 out of the 8 teachers who

participated in the present study provided evidence which appeared to indicate that they

believe students who (a) feel confident in their ability to engage in mathematical activity

and/or (b) view mathematical as relevant and useful outside of the classroom, are more

likely to generalize their learning to novel situations seems to show that for teachers,

these student views are crucial factors in the generalization of students' learning. Thus, it

seems that by investigating the beliefs of those who interact with student learners on a

daily basis in the context of a mathematics classroom, as opposed to, for example, those

who interact with student learners intermittently in the context of a laboratory setting, we

are afforded the opportunity to gain access to novel beliefs about the generalization of

students' learning and how to support it.

Chapter Conclusion

In this chapter, I identified 5 categories (fitting into 3 supercategories) of

teachers' beliefs regarding the generalization of students' learning and 13 associated

beliefs regarding how to support the generalization of students' learning (see Table 4.1).

Table 4.1: Summary of findings regarding teachers' beliefs about (a) the generalization of students' learning and (b) how to support the generalization of students' learning.

Categories of Teachers' Beliefs About the Generalization of Learning				
		Statement of Belief	**Teachers**	**Associated Instructional Moves**
Content	**1**	Students will productively generalize their learning to a novel situation if the novel situation prompts students to make use of a learned association, procedure, or formula.	Anne Blake Donna	• Tell students the desired association, procedure, or formula • Use multiple examples
	2	Students will productively generalize their learning to a novel situation if they develop mathematically valid interpretations of the meaning of topics like slope.	Richard Emma Patrick Kay	• Choose tasks and pose questions that emphasize quantitative reasoning • Use a curriculum that progresses from contextualized to decontextualized situations • Provide students with opportunities to explain their reasoning
Disposition	**3**	Students will productively generalize their learning to a novel situation if they develop and make use of particular dispositions.	Emma Kay Donna	• Apply a real-world context to a problem if one is not already provided • Avoid giving problems that outline what students need to do to solve them • Model the desired disposition
Students' Affect	**4**	Students will productively generalize their learning to a novel situation if they develop confidence in their abilities to engage in mathematical activity.	Anne Donna Blake Emma Patrick Sam	• Use language of "understanding/not understanding" or "participating/not participating" rather than "easy/hard" or "good/bad" • Let students figure out problems on their own and use their own strategies • Use contextually-meaningful tasks
	5	Students will productively generalize their learning to a novel situation if they develop the view that mathematics is relevant and useful outside of the mathematics classroom.	Blake Patrick Richard	• Make use of real world situations • Ask students to come up with real-world situations involving a particular mathematical topic

By looking at the summary provided in Table 4.1, one can see that all but one of

the participating teachers in this study emphasized the role of mathematical content in

their beliefs regarding the generalization of students' learning. Specifically, half of the

teachers emphasized the importance of a mathematically valid *meaning* for mathematical topics and just under half of the teachers emphasized the importance of *associations, procedures, and/or formulas* in their beliefs regarding the generalization of learning. These beliefs regarding the role of content seemed specific in nature in the sense that as a mechanism for supporting the generalization of students' learning, content (e.g., a meaning or an association) appeared to function in a particular way for each teacher; for example, one teacher believed that if a novel problem prompted a student to make use of a learned association (e.g., a link between a graphical image of positive slope and the word "away") then that student would generalize her learning by making use of said association.

Similarly, Table 4.1 shows that all but one of the participating teachers emphasized the role of students' affect in their beliefs regarding the generalization of students' learning. In particular, three-quarters (or 6 out of 8) of the teachers stressed the importance of students' confidence in their ability to engage in mathematical activity and 3 out of 8 of the teachers stressed the importance of students' view that mathematics is relevant and useful outside of the mathematics classroom in their beliefs regarding the generalization of learning. Such beliefs regarding the role of students' affect appeared less specific than the content beliefs in the sense that as a mechanism for the generalization of students' learning, the way in which, for example, students' confidence functioned to support the generalization of students' learning was vague. Moreover, as a mechanism for the generalization of students' learning, students' affect seemed to function at a more global level than content allowing students to generalize their learning to novel situations that transcended the learning of particular mathematical topic.

The fact that the role of students' affect in the generalization of students' learning is absent from the transfer literature makes the above finding significant despite the underspecified nature of these beliefs. Likewise, the finding that 3 out of the 8 teachers emphasized the importance of disposition in their beliefs adds to the transfer literature by extending the writings on dispositional approaches to transfer into the domain of mathematics education. These findings point to the importance of talking to *teachers* about their beliefs regarding the generalization of students' learning.

Similarly, the findings regarding teachers' beliefs about how to support the generalization of students' learning point to the importance of talking to teachers as several of the instructional beliefs forwarded by teachers in this study had not been identified by transfer researchers. For example, the teachers in this study used a variety of curricular materials that many other teachers currently use to suggest (from an actor-oriented perspective) that to support the productive generalization of learning, teachers focus on students' quantitative reasoning as well as the sequencing of activities, using a curriculum that progresses from contextualized to decontextualized problems. Moreover, the pedagogical actions that did share some elements with what is reported in the transfer literature seemed to entail practice-based differences. For instance, Anne, who seemed to believe in the Thorndikean idea of transfer being mediated by common associations, believed that drawing upon multiple reform-oriented and constructivist-inspired activities would support the generalization of students' learning. Her instructional belief therefore suggests practice-based decision-making that transcends a Thorndikean approach to curricula. Thus, the findings presented in this chapter highlight the significance of talking to teachers about their beliefs regarding the generalization of learning.

In the next chapter, I follow up on the finding that about half of the teachers emphasized the meaning of mathematical topics while the other half focused on associations, formulas or procedures when discussing the role of mathematical content in the generalization of learning, by exploring the second research question:

> What is the relationship between teachers' beliefs regarding the generalization of students' learning and their mathematical knowledge for teaching (MKT)?

In the second interview of this study, teachers were asked a set of questions designed to explore their mathematical knowledge for teaching (MKT). Following, Silverman and Thompson (2008), I investigated teachers' personal understandings of slope and explored how those understandings might be transformed into an awareness of (a) the ways in which *others* might come to develop those understandings and (b) the actions they might take to support the development of those understandings. In particular, I now ask, Will there be an alignment between teachers' membership in belief Categories 1 or 2 and the teachers' personal knowledge of slope. That is, will the three teachers who appeared to hold beliefs consistent with Category 1 about the generalization of learning be restricted to procedural knowledge of slope while the teachers in Category 2 have developed deeper meanings for slope? Additionally, has either set of teachers transformed their personal understandings into MKT? Such questions will be answered in the next chapter.

CHAPTER 5:

Results on the Relationship between Teachers' Beliefs Regarding the Generalization

of Students' Learning and Teachers' MKT

In this chapter, I address the second research question:

What is the relationship between teachers' beliefs regarding the generalization of students' learning and their mathematical knowledge for teaching (MKT)?

This study utilizes Silverman and Thompson's (2008) characterization of MKT as having multiple components. Stated in terms of the mathematical focus in this study (i.e., slope), three components of MKT are: (a) teachers' personal understanding of slope, (b) teachers' understanding of how students' come to understand slope, and (c) teachers' understanding of the actions they can take to support the development of students' understanding of slope. Consequently, my response to the second research question addresses each of these components of MKT for the participating teachers, as exhibited in their work on the mathematics-focused tasks from the second interview. Then I relate each component of the teachers' MKT to their beliefs about the generalization of students' learning (as presented in Chapter 4).

Specifically, in the first half of this chapter, I present evidence for the following three categories of the first component of MKT—teachers' personal understanding of slope: (a) no evidence of ratio reasoning, along with the conception of slope as steepness; (b) limited evidence of ratio reasoning that is disconnected from slope, along with the conception of slope as steepness; and (c) strong evidence of ratio reasoning connected to slope, along with steepness emphasized only secondarily or not at all (see Figure 5.1). Additionally, I show that the teachers who provided little or no evidence of understanding

slope as a ratio also appeared to believe that the generalization of learning is mediated by

associations, procedures, or formulas (Category 1 from Chapter 4). Furthermore, the

teachers who showed strong evidence of understanding slope in terms of ratio (and who

provided evidence of a content belief about the generalization of learning) also appeared

to believe that productive generalizing is supported by mathematically valid

interpretations of the meanings of topics like slope (Category 2 from Chapter 4), except

for one teacher (Blake), as shown in Table 5.1. As a result of these findings, I argue that

developing a personal understanding of slope as a ratio, with a de-emphasis on slope as

steepness, is a necessary but not sufficient condition for belonging to the Category 2

belief about the generalization of students' learning.

Table 5.1: Relationships between teachers' beliefs regarding the generalization of students' learning and teachers' personal understanding of slope.

		Teachers' Personal Understanding of Slope	
Category 1 Teachers	Donna	No evidence of ratio reasoning	Slope is steepness
	Anne	Limited evidence of ratio reasoning but ratio is not linked to slope	
	Blake	Strong evidence of ratio reasoning and ratio is linked to slope	Slope is not steepness
Category 2 Teachers	Richard		
	Patrick		Slope may also be considered steepness
	Emma		
	Kay		
Other	Sam		

In the second half of the chapter, I draw upon the second and third components of

MKT to illuminate the findings regarding Blake, namely that he did not fit the alignment.

In particular, I show that while the teachers from Category 2 in Chapter 4 *consistently*

emphasized the development of students' quantitative interpretations of the meaning of

slope and its associated formula, Blake did not. Moreover, I show that the teachers from

Category 2 in Chapter 4 offered a richer and more varied set of actions than Blake to

support the development of students' quantitative interpretations and that in contrast to

the teachers from Category 2 in Chapter 4, Blake suggested teaching actions that

supported the development of a numeric interpretation of slope. These findings taken

together suggest, for example, Blake may be unaware that an emphasis on numbers and

numerical patterns contradict an emphasis on quantitative reasoning.

Teachers' Personal Understanding of Slope

In this section I present the three categories of teachers' personal understanding of

slope articulated above. As noted, there were two parts to teachers' understanding of

slope—ratio and steepness—wherein there appeared to be a continuum for each. With

respect to steepness: two teachers seemed to understand slope as the steepness of a line or

other physical object; four teachers appeared to understand how one *could* interpret slope

as the steepness of a line; and two teachers rejected the notion that one interpret slope as

the steepness of a line. With respect to ratio: one teacher exhibited no evidence of ratio

reasoning; one teacher showed limited evidence of ratio reasoning; and six teachers

showed strong evidence of ratio reasoning. Evidence for ratio reasoning was considered

strong if teachers provided evidence of each of the following:

- coordinating two quantities (or the changes in two quantities) to form a unit

  composed of two quantities (e.g., joining 10 cm with 2.5 s to form a speed);

- iterating or partitioning a unit composed of two quantities (e.g., cutting the

  composed unit of 10 cm in 2.5 s in half to argue that both pairs of values—10

  cm in 2.5 s and 5 cm in 1.25 s—represent the same speed);

- reasoning with an equivalent set of ratios (e.g., 10 cm in 2.5 s is the same

  speed as 4 cm in 1 s and as 6 cm in 1.5 s); and

- interpreting the meaning of ratio in a variety of situations (e.g., in situations with and without an inherent steepness—the relationship between the height and length of a ramp represents the steepness of the ramp and the relationship between the amount of distance traveled and the corresponding amount of time represents speed).

Researchers point to the above criteria as evidence of ratio reasoning[12] (e.g., Lamon, 1995; Lobato & Ellis, 2010; Simon & Blume, 1994). Evidence for ratio reasoning was considered limited if teachers did not provide evidence of all four criteria.

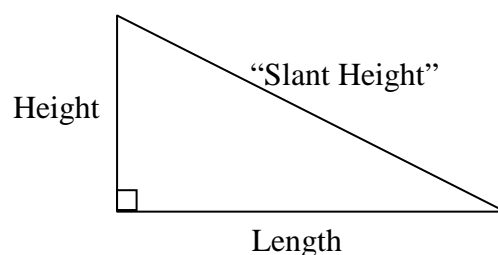**No Evidence of Ratio Reasoning; Slope is Steepness**

Donna, from Category 1 in Chapter 4, was the only teacher who did not exhibit any evidence of ratio reasoning. She, therefore, did not appear to interpret slope as a ratio. Rather, she seemed to interpret slope in terms of the steepness of a line or other physical object. To illustrate, I present data from two tasks: The Slant Height Task (see Figure 5.1) and The Slope of One-Half Task. In the Slant Height Task, teachers were provided with a diagram of a ramp with all three sides labeled: height, length and slant height. Teachers were then asked to consider whether the ratio of the slant height to the length of a ramp (called the *ramp number* in this task), could be used as a good measure of the steepness of the ramp (in addition to *slope*, or the ratio of height to length). The goal of this task was to ask teachers to reason with two quantities not typically associated with steepness, thus putting them in an unfamiliar situation, to examine the ways in

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[12] There are other researchers who would consider at least a subset of the criteria as evidence of *pre-ratio reasoning* (e.g., Lesh, Post, & Behr, 1988).

which they reasoned about the relationship between two quantities. The specific focus was on teachers' ratio reasoning.

In the Slope of One-Half Task (which was stated verbally without a task sheet), teachers were asked to provide an explanation for the meaning of a slope of ½. The goals of this task were to examine the quantities teachers would draw upon to interpret the meaning of a slope of ½ and to examine whether and how teachers conceive of the relationship between two quantities (e.g., a hill rises 1 foot vertically for every 2 feet it runs horizontally or a mixture contains ½ as much salt as sugar). Solely using quantities with inherent steepness qualities to interpret the meaning of slope could indicate a teacher who understands slope as the steepness of a physical object. Interpreting a slope of ½ as a relative description of some quantity (e.g., not very salty) could indicate a teacher with limited or no ratio reasoning whereas joining the 1 and the 2 to form a unit of ½ could indicate a teacher with limited or strong ratio reasoning.

Can the ratio of "slant height" to length also be used as a measure of the steepness of a ramp (in addition to the ratio of height to length which is called slope)?

Let's give the new ratio a name (i.e., *ramp number*) so as to avoid confusing it with *slope* which is the ratio of height to length.

Examine several *ramp numbers* (e.g., 4:3; 5:4; 6:9). Test whether or not a *ramp number* is a good measure of steepness. This will involve thinking about what makes something a good measure. How can you test this measure to see if it's actually a measure of the steepness of the ramp? Think of a way and then test it.

DON'T USE ALGEBRAIC FORMULAS and DON'T USE YOUR KNOWLEDGE OF SLOPE!!!

Use only the kinds of tools (e.g., paper, pencil, straight edge) that students would have easy access to.

Figure 5.1: The Slant Height Task.

When responding to The Slant Height Task, Donna experimented with several different ramp numbers (see Figure 5.2) to figure out whether they would tell her anything about the relative steepness of a ramp. She began by "doing ones," meaning that she explored ramp numbers with a ratio over 1 (but less than 2) (see Figure 5.2a). Donna appeared curious about the relative steepness of a ramp with number of, for example, $\frac{4}{3}$ as compared to a ramp with a ramp number of $\frac{6}{5}$ as evidenced by her utterance "will these [the steepnesses] change if you keep your improper fraction with a 1." However, she did not coordinate the quantities *slant height* and *length* to reason about the steepness of a ramp. For instance, by coordinating the quantities of slant height and

length, one could reason that a ramp number of $\frac{4}{3}$ is steeper than a ramp with a ramp

number of $\frac{6}{5}$, because the former ramp has $1\frac{1}{3}$ units of slant height for every 1 unit of

length while the latter ramp has $1\frac{1}{5}$ units of slant height for every 1 unit of length.

Instead, Donna converted all the improper fractions to mixed numbers (as shown in

Figure 5.2) and reasoned with the "whole number" part of the mixed numbers to

conclude that ramps with ramp numbers $1\frac{1}{3}$, $1\frac{1}{4}$, and $1\frac{1}{5}$ indicate one *class* of steepness

(because they all have a whole number part of 1) while a ramp with a ramp number of $2\frac{2}{3}$

indicate a different class of steepness (because it has a whole number part of 2). In her

own words, Donna explained:

> The whole number is showing me that … as soon as that [points to the
> underlined whole numbers in Figure 5.2] goes up to the next integer, 2,
> it got steeper. So, right here [points at the ramps in Figure 5.2a], I
> know these are all close … these are all so similar; it doesn't look like
> they're changing that much, but when we got to 2 [points at the ramp
> in figure 5.2b], that bumped it up to *steeper*.

Hence, when reasoning about steepness, Donna did not show evidence of coordinating

two quantities but rather of attending to a particular numerical value, specifically the

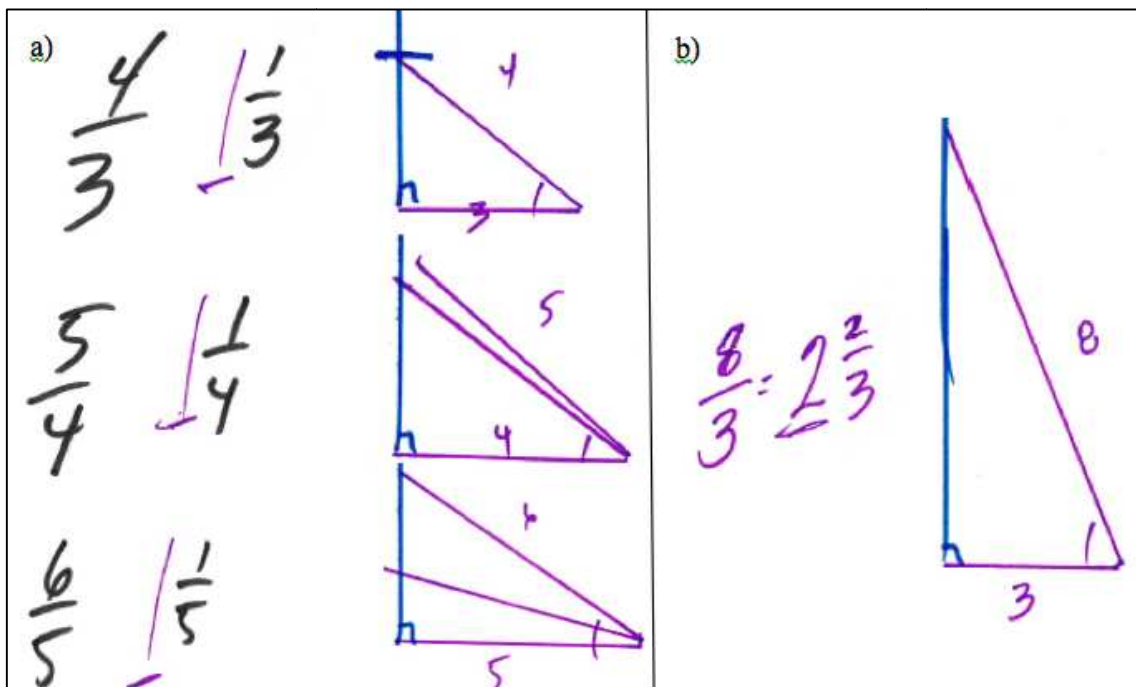whole number portion of a mixed number.

Figure 5.2: (a) Ramp numbers with whole numbers of 1; (b) Ramp numbers with whole numbers of 2.

Furthermore, when asked during The Slope of One-Half Task what a slope of ½ might mean, Donna's response suggested that she interpreted the meaning of slope as steepness and the value associated with slope as an indication of the relative steepness of a line. In particular, Donna said:

> I think it's [a slope of ½] *not* steep … I just think of it as a fraction …
> The smaller the fraction is to me, the less steep the slope is; the
> greater, it would be very steep … I automatically pop it into a
> coordinate graph … So, if we have 1, however you'd like, *y=x*, this is
> like, I call it a reference graph. As soon as you start changing this
> number, that's going to affect where it goes. So, if it's below 1, it's
> going to fall. If it's above 1, it's going to get steeper.

This excerpt highlights the fact that when conceiving of slope, Donna thinks of a single quantity, steepness, rather than a relationship between two quantities. In the context of steepness, a slope of ½ could represent a relationship between the quantities of height and length (e.g., a hill increases in height ½ ft for each one foot of length or the height of hill

is ½ the length of its base). In contrast, Donna seemed to conceive of particular values for slope as an indication of relative steepnesses with respect to a line on a coordinate graph with a slope of 1 (i.e., *y=x*).

Interestingly, Donna did not seem to understand that being given the value of ½ as the slope of a physical object with steepness (such as a ramp) determines the steepness of that object. Specifically, Donna concluded that a room full of carpenters, each instructed to build ramps with slopes of ½, would not build ramps with the same steepness. Her misunderstanding seemed linked to her association of steepness with the angle of inclination of a ramp (i.e., the angle between the slant height and the base of a ramp) rather than to the coordination of two quantities. She explained that the angle of inclination would depend on the units each carpenter chose to use. Specifically, when asked whether the steepness of the carpenters' ramps would be the same, Donna said:

> No because half is, is it half a centimeter? Is it half an inch? Is it half a foot? *What's half represent? …I just can't imagine what they would do with ½* [emphasis added], well because I'm thinking, I don't know, because the angle would be different.

The fact that Donna attached units to the slope of ½ such as ½ cm or ½ ft is consistent with conceiving of slope as an extensive quantity or a direct measure rather than as an intensive quantity or ratio (Lobato & Thanheiser, 2002; Schwartz, 1988). Furthermore, Donna's conclusion that a shorter direct measure of ½ cm versus a longer measure of ½ ft would affect the angle is reminiscent of the misconception that angle measure is related to the length of the line segments that form its sides (Clements, Battistia, Sarama, & Swaminathan, 1996). In sum, the evidence taken together indicates that Donna conceived of slope as a single quantity, namely steepness.

Limited Evidence of Ratio Reasoning that is Disconnected from Slope; Slope is

Steepness

Anne, from Category 1 in Chapter 4, showed some evidence of ratio reasoning.

The strongest evidence for such reasoning emerged during The Slant Height Task shown

in Figure 5.1. Initially, Anne was unsure whether the ratio of slant height to length could

be used as a measure of the steepness of a ramp. However, when asked about the relative

steepness of a ramp with a ramp number of $\frac{7.5}{7.4}$ (i.e., a slant height of 7.5 and a length of

7.4) and a ramp with a ramp number of $\frac{3.5}{3.4}$, Anne concluded that the second ramp would

be steeper. She first noticed that the slant height in each case was 0.1 units longer than

the length of the corresponding base. Then she appeared to consider this difference

relative to the length of both bases:

> 0.1 in each of these [ramps] is a different amount, meaning that the
> ratio between 0.1 and, like if I were to divide them, *that* 0.1 is divided
> by more [points to "7.5 to 7.4" written on a piece of paper in front of
> her]. So, it's, within this context, it's less than 0.1 here [points to "3.5
> to 3.4" written on a piece of paper in front of her]. So, 3.5 over 3.4,
> *that* 0.1 is a different amount.

This transcript excerpt suggests that Anne conceived of the relationship between 0.1 and

7.4 and the relationship between 0.1 and 3.4. She seemed to understand that sharing or

separating 0.1 units across a larger amount would result in a smaller amount, thus

concluding that a ramp with a ramp number of $\frac{7.5}{7.4}$ is less steep than a ramp with a ramp

number of $\frac{3.5}{3.4}$.

One could alternately argue that Anne was not coordinating quantities or

reasoning about the relationship between them, but was simply dividing two numbers,

knowing that $\frac{a}{b} < \frac{a}{c}$ if $b > c$. However, there was an additional piece of evidence that suggested Anne might be reasoning via ratio. When asked to produce additional measurements for a ramp that would have the same steepness as a ramp with a ramp number of $\frac{3.5}{3.4}$, Anne produced $\frac{7}{6.8}$, thus, exhibiting evidence of iterating, or repeating, a unit composed of 3.5 units of slant height and 3.4 units of length to get a ramp with an equivalent steepness (i.e., 7 units of slant height to 6.8 units of length).

Additional evidence that Anne does not belong in the previous category with Donna emerged during The Slope of One-Half Task. When asked to describe how she thought about the meaning of a slope of ½, Anne said "That means that whatever the x-value is, y is ½ of it." Here, Anne appeared to conceive of a slope of ½ in terms of a multiplicative comparison between two quantities (as opposed to a singular quantity as was the case with Donna). However, she was unable to provide a real-world situation in which there might be a slope of ½, when asked.

Despite some evidence of ratio reasoning, Anne was adamant that slope and ratios were unrelated, making comments like "It's a ratios task and not a slope task." For example, Anne was asked about the meaning of slope in the following two situations:

1. A mountain increases 10 feet in height for every horizontal increase of 4 feet.

2. A mixture of lemonade is 7 teaspoons of lemon juice for every 2 cups of water.

She responded, "Number 1 is more about slope; Number 2 is more about a ratio." Anne elaborated the meaning of her statement, saying:

> Part of it is visual where I actually think of a mountain [drew image shown in Figure 5.3] … For every, you know, this is 4 [gestured along the horizontal axis] and then I need to go up 10 [gestured along the vertical dotted line segment] … I think of it like, it looks like a *line* to me … The feet, the horizontal and the vertical are more related [to slope].

Anne's verbal and written responses suggested that, for Anne, the topic of *slope* is related

to the steepness of some physical object like a mountain or a line on a coordinate graph

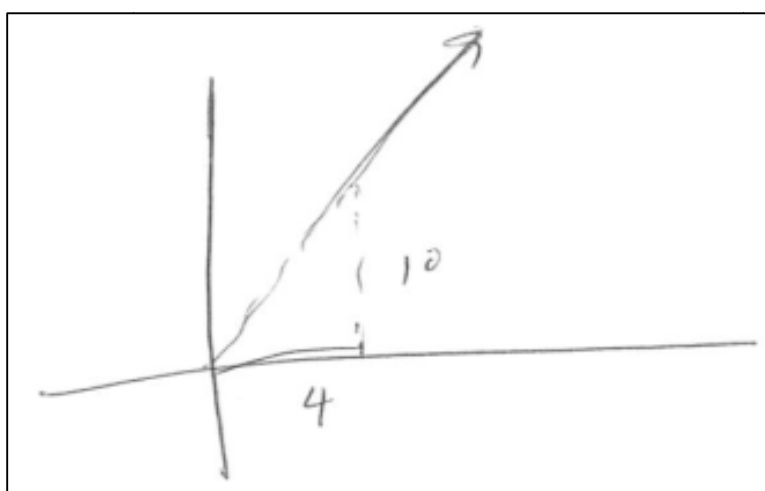wherein steepness is determined by the lengths of vertical and horizontal segments.

Figure 5.3: Anne's picture for Situation 1.

Anne went on to contrast Situation 1 with Situation 2, saying:

> Whereas with the lemonade, it's a *ratio*, so what if I only have 3 [wrote "————   —" (see Figure 5.4)], then I need to think—to me, it's more like equivalent fractions. So, I'm thinking, okay, I need to maintain this ratio 2 to 7, so I am going to multiply this [pointed to the "2" in "2 cups"] times 1.5 [wrote "1.5" above the "~" (see Figure 5.4)] … I'm looking at this one [pointed back and forth between "2 cups H_2O" and "3"] and what's the multiplication factor? I need to multiply by 1.5. So this means I need to multiply by 1.5 [pointed to "7 tsp lemon" and wrote "1.5" below the "~"], so three-halves is 21 over 2 teaspoons [wrote "21/2 tsp" under "3"].

Here, Anne did not draw a line or an image of an object. Rather she set up "a ratio" and

asked herself questions about what she would have multiply 2 by to get 3 and therefore

what she would have to multiply 7 by to obtain a ratio equivalent to $\frac{2}{7}$. The fact that Anne

made distinctions between slope and ratio on several occasions throughout her interviews

and related the slope to objects with physical slants or steepness and ratio to situations in

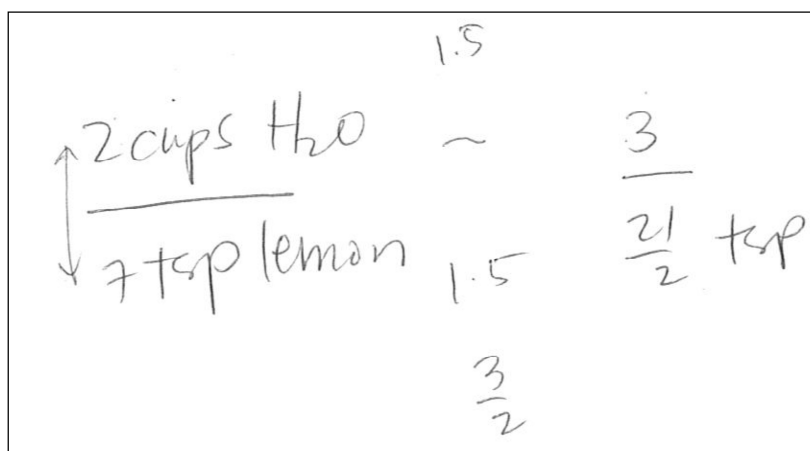which there were no slanty or steep objects suggests that slope *is* steepness for Anne.

Figure 5.4: Anne's picture for Situation 2.

Strong Evidence of Ratio Reasoning Connected to Slope; Steepness Downplayed

The remaining six teachers demonstrated strong evidence of ratio reasoning and

appeared to conceive of slope as a ratio. This includes all of the teachers who appeared

to hold beliefs about the generalization of learning from Category 2 in Chapter 4, one of

the teachers from Category 1 in Chapter 4 (Blake), and Sam (whose belief about the

generalization of learning was not categorized as a content belief). In what follows, I

first illustrate the four components of ratio reasoning identified above using evidence

from Patrick. Then, I present evidence that, in contrast to Anne, slope and ratio were

connected for these teachers. Finally, I provide evidence that these six teachers

downplayed steepness in their conceptions of ratio, either by (a) acknowledging that

slope can be conceived in terms of steepness but secondarily to thinking of slope as a ratio, or (b) rejecting slope as being the same as steepness.

Ratio reasoning. In his response to The Slant Height Task, Patrick started out similarly to Donna, by investigating whether a change in ramp number would indicate a change in the steepness of a ramp. In contrast to Donna (who grouped ramp numbers into classes of steepness), Patrick reasoned that unequal ramp numbers (e.g., $\frac{4}{3}$ and $\frac{6}{5}$) would result in unequal steepnesses saying, "If you get a larger value for your ramp number, then the ramp should be steeper and a smaller value, the ramp should be not as steep." He then set out to investigate whether ramp numbers representing equivalent ratios (e.g., $\frac{3}{1}$, $\frac{6}{2}$, and $\frac{9}{3}$) would represent ramps with equivalent steepnesses. For example, Patrick said:

> Okay, so we have a ramp number of 3; so we can draw 2 ramps [drew two ramps] and 3 can be figured out in many ways; so, you could do 3 over 1 or we could do 6 over 2 [labeled the slant heights and lengths of the ramps 3 and 1 and then 6 and 2, respectively (see Figure 5.5a)].

Here, Patrick interpreted a ramp number of 3 in terms of two quantities—slant height and length. Patrick then went on to visually duplicate the ramp on the left of Figure 5.5a (see Figure 5.5b) to show that the two ramps joined together in Figure 5.5a (and others) have the same steepness. In particular, Patrick stated:

> If we extended [extended the slant height with a dotted line (see Figure 5.5b)] and this was a distance of 2 [gestured along the length of the ramp], so if we went 1 more [gestured along the dotted line extending from the length of the ramp], then we would go up twice [gestured along the entire length of the new slant height] … As long as you have this ramp number of 3, if this [pointed to the slant height] has to be bigger by some value and that means this [pointed to the length] has to be bigger by some value, like this [pointed to the ramp with a slant height of 3 and a length of 1] is doubling … that means you have a

> bunch of baby ramps [drew a third ramp with a slant height of 9 and a
> length of 3 (see Figure 5.5c), then pointed to each of the 3 ramps with
> slant heights of 3 and lengths of 1].

In the above excerpt, Patrick appeared to conceive of the original ramp of slant height 3

and length 1 as a composed unit that could be iterated multiple times, resulting in ramps

of different sizes but with the same steepness. As Patrick concluded, "[If] we all had to

build a ramp with a ramp number of 3, we would all build a ramp with the same

steepness, different size ramps, but the same steepness." Thus, Patrick appeared to

understand, unlike Donna, that it was not the steepness of the ramp relative to some

benchmark that mattered but rather the multiplicative relationship between the slant

height and the length.
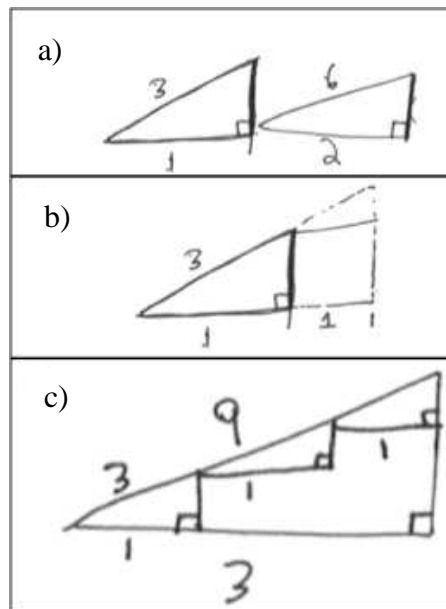
Figure 5.5: (a) Ramps with a ramp number of 3; (b) Patrick iterates a ramp with a slant
height of 3 and a length of 1 once to show that the ramps in (a) are the same steepness;
(c) Patrick iterates a ramp with a slant height of 3 and a length of 1 twice to find
additional ramps with the same steepness as the ramps in (a).

In the above situation, Patrick reasoned with a set of equivalent ratios (i.e., $\frac{3}{1}, \frac{6}{2}$,

and $\frac{9}{3}$) to conclude that if multiple people were told to build ramps with a ramp number of

3, those people would build ramps with equivalent steepnesses. Furthermore, as is characteristic of the teachers in this category, Patrick not only demonstrated the ability to interpret ratio in this situation but in many situations. Whereas Anne only associated ratio with situations without a slanty or steep object, the teachers in the present category were able to interpret the meaning of ratio appropriately across a variety of contexts. For example, Patrick interpreted the meaning of ratio in several situations including the steepness of a line on a coordinate grid (e.g., as "gaining so much vertically at the expense of so much horizontally"), the rate at which supplies are being consumed (e.g., as "using up 1 pound per day"), the speed at which one travels (e.g., as "you've covered a mile in a day"), and the size of a investment package (e.g., as "every person would give $5,000"). In sum, Patrick, like the other 5 teachers in this category, was considered to have demonstrated strong evidence of ratio reasoning because he: (a) coordinated and joined two quantities to form a unit composed of two quantities, (b) iterated a unit composed of two quantities to form an equivalent unit, (c) reasoned with a set of equivalent ratios, and (d) interpreted the meaning of ratio in a variety of situations.

Slope connected to ratio reasoning. In contrast to Anne who exhibited some evidence of ratio reasoning but was explicit about not connecting slope with ratio reasoning, all of the teachers in the present category appeared to interpret the slope of a linear function in terms of ratio. For example, in the investment package example mentioned above, Patrick responded to a question about the meaning of the mathematical topic of *slope* by providing an example in which he coordinated two quantities (e.g., amount of money and number of people), partitioned a unit composed of two quantities to form an equivalent ratio (e.g., if 2 investors leads to a gain of $10,000, that means each

person gave $5,000), and reasoned with a set of equivalent ratios (e.g., $\frac{\$5,000}{1\ person}$, $\frac{\$10,000}{2\ people}$,

$\frac{\$15,000}{3\ people}$, $\frac{\$20,000}{4\ people}$). In other words, when asked explicitly about slope, Patrick

demonstrated the types of reasoning presented in the previous section.

      Slope may be interpreted as the steepness of a line. While all of the teachers in

this category exhibited strong evidence of ratio reasoning, they differed in the way they

thought about the relationship between slope and steepness. Four of the teachers in this

category (Patrick, Emma, Kay, and Sam) appeared to give primacy to an interpretation of

slope as ratio, but also accepted an interpretation of *slope as the steepness of a line*. In

contrast, Blake and Richard rejected the notion that one interpret slope as the steepness of

a line.

      Teachers were presented with The Hypothetical Teacher Situation shown in

Figure 5.6, wherein a hypothetical teacher notices that a graphically represented function

looks steeper on one set of axes than it does on another and thus expresses confusion

about the definition of slope typically presented in textbooks—slope is the steepness of a

line. The four teachers in this subcategory pointed to the fact that the scaling of the axes

on which a function is represented affects the steepness of the line used to represent that

function and that one cannot compare the relative steepnesses of graphically represented

functions unless the scaling is uniform. For example, Patrick explained that slope

manifests as the steepness of a line when a function is graphically represented and said

that he would tell the teacher "If you alter the scale of the grid, you can represent data

differently, so it [slope] will appear different but the value is the same." He elaborated

with an example:

> Let's say you make $10 an hour and let's try to impress your friends
> by how much money you are going to have. How can you change the
> scale to make the amount of money you've made so far look really
> steep *or* [like] "Are you kidding?! That's all the money you have?" If
> you make $10 an hour and you make your vertical axis count $50 each,
> your line of how much money you are earning is like "Are you
> kidding, you are barely making anything" … So, just to have them
> realize that "Oh, it's the same amount of money per hour; it's just
> pictured differently."

By acknowledging that the slope of a function manifests as steepness when graphically

represented and using the word *steep* in relation to the amount of money a person makes

per hour, Patrick seemed to accept *slope as the steepness if a line* as a legitimate

interpretation of slope. He apparently understood that one could use the axes of a

graphical representation to alter the visual appearance of slope. However, he continued

to emphasize the relationship between two quantities (i.e., amount of money and time)

when discussing steepness, thus, simultaneously illustrating his understanding of slope as
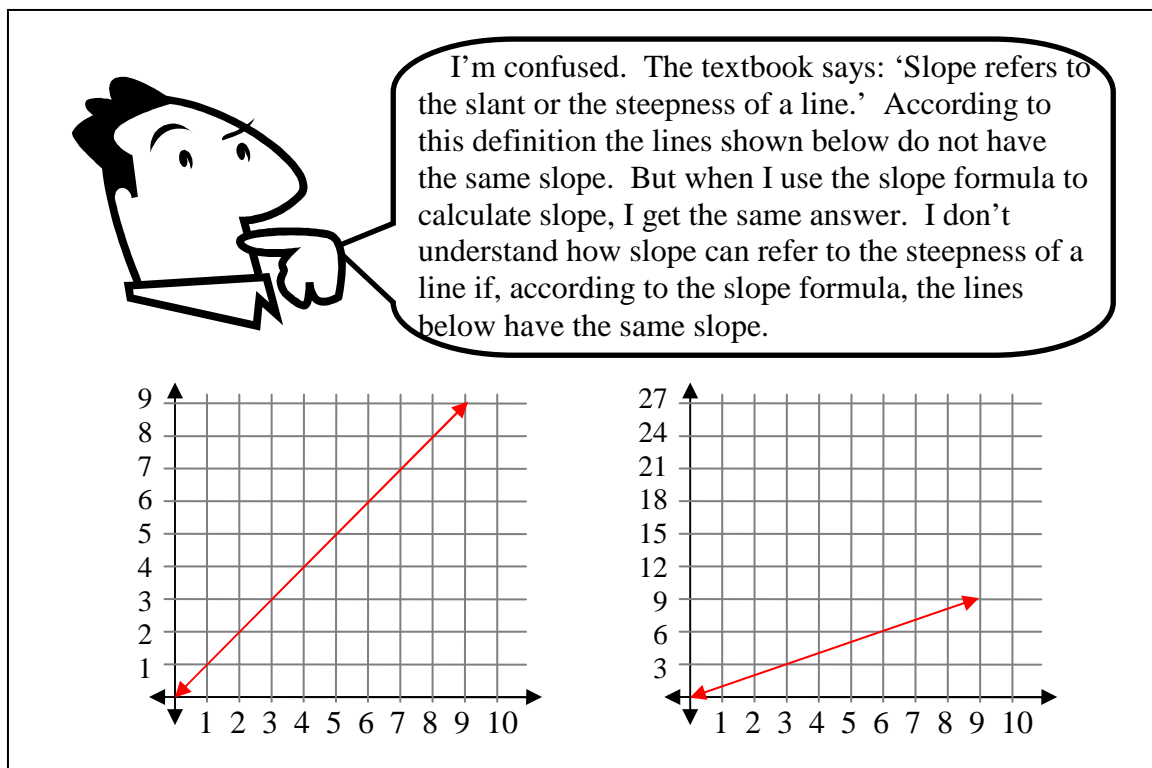
ratio.

Figure 5.6: The Hypothetical Teacher Situation.

Slope is not steepness. In contrast, when presented with The Hypothetical

Teacher Situation, Blake and Richard explicitly discouraged the teacher from thinking

about slope in terms of the steepness of a line. Instead, they encouraged the teacher to

conceive of slope in terms of the rate at which a function changes. As Blake explained,

he would have the teacher focus on the fact that both graphs represent a function with a

slope, or a rate of change, of 1. In his words:

> I would want to emphasize that it's [slope] a description of the rate of
> change … they [the graphically represented functions] are still
> growing at the same rate. They *appear* to be different steepnesses, but
> the rates are *the same*. So, with respect to the rates, they are the same.
> With respect to the *picture*, they're not. But *the number* we use to
> measure that rate is the same.

Thus, Blake rejected the idea that the teacher interpret slope as the steepness of a line since the appearance of a graphically represented function can be altered without affecting the rate of change of the function. Similarly, Richard said:

> We are not changing the slope but we are changing the *look* of it, the steepness of it … I think that's where "rate of change" is a better word than "steepness of a line" … because "rate of change" is comparing both your *y*s and your *x*s where I can change the "steepness" of a line by just changing the scale [of the axes].

Here, Richard was explicit in saying he preferred "rate of change" to the "steepness of a line." In sum, these teachers explicitly discouraged the hypothetical teacher from interpreting slope as the steepness of a line.

Discussion: Relationships between Teachers' Personal Understanding of Slope and Their Beliefs Regarding the Generalization of Students' Learning

With respect to the seven teachers in Categories 1 and 2 in Chapter 4, there was alignment (with the exception of one teacher) between teachers' personal understanding of slope and their beliefs about the generalization of students' learning. Teachers who exhibited little or no ratio reasoning and appeared to understand slope in terms of steepness were the teachers who appeared to believe that productive generalizing is supported by learned associations, procedures, or formulas (Category 1 in Chapter 4). Teachers who exhibited strong ratio reasoning and further seemed to have that reasoning connected to the mathematical topic of slope were the teachers who appeared to believe that productive generalizing is supported by mathematically valid interpretations of the meanings of topics like slope (Category 2 in Chapter 4). The exception was Blake who demonstrated strong evidence of ratio reasoning wherein such reasoning appeared linked to slope but was in Category 1 in Chapter 4. Thus, having a personal understanding of

slope as ratio, with a de-emphasis on slope as steepness seems to be a necessary but not sufficient condition for membership in Category 2 in Chapter 4.

With respect to the mathematical topic of slope, Donna and Anne (from Category 1 in Chapter 4) emphasized the generalization of associations, procedures, or formulas which involved graphical representations. This seems consistent with their personal understanding of slope as the steepness of a line. For example, Donna's focus (in Chapter 4) on students' being able to find a value for slope via a procedure that involved graphing rather than on the development of their understanding of the relationship between, for example, values located on the axes of a graph seems to correspond with her personal understanding of the numerical values used to represent slope, namely they are indicative of the *relative* steepness of a line in relation to some reference graph (e.g., $y=x$). Anne's emphasis on the generalization of associations involving images of graphical representations of positive, negative, and zero slope (and not the meanings underlying those representations) may correspond to her personal understanding of slope as the steepness of a line and her insistence that slope and ratio are unrelated.

All of the teachers from Category 2 in Chapter 4 emphasized the generalization of mathematically valid interpretations of the meanings of topics like slope which may correspond to having a strong understanding of ratio and linking that understanding to the mathematical topic of slope. These teachers highlighted (in Chapter 4) students' interpretation of the meaning of slope as involving a relationship between two quantities which seems consistent with the their showing evidence of interpreting the meaning of ratio via the coordination of two quantities.

What is surprising is that Blake (from Category 1 in Chapter 4) emphasized students' generalization of the formula $slope = \frac{rise}{run}$ in conjunction with a graphical representation, but showed strong evidence of ratio reasoning and, in The Hypothetical Teacher Situation, explicitly rejected an interpretation of slope as the steepness of a graphically represented line in favor of an interpretation of slope as the rate of change of a function. Perhaps, he thought if students learned the formula, they would spontaneously develop and subsequently generalize an interpretation of slope as the rate at which the two quantities change in relation to one another. A possible explanation for why Blake would emphasize a formula when he appeared to personally interpret slope in terms of a mathematically valid meaning comes from looking at his understanding of how students come to develop an understanding of slope and his understanding of the actions he could take to support students in developing an understanding of slope.

Teachers' Understanding of Two Other Components of MKT

As I argued in Part 1 of this chapter, all of the teachers from Category 2 in Chapter 4 and Blake, from Category 1 in Chapter 4, exhibited strong evidence for a personal understanding of slope as a ratio. However, when discussing how students come to develop an understanding of slope, the teachers in Category 2 *consistently* emphasized the development of a quantitative interpretation of the meaning of slope while Blake did not. Moreover, when discussing the actions teachers could take to support the development of a quantitative interpretation of the meaning of slope, the teachers in Category 2 described a richer and more varied set of actions than Blake. In other words, as discussed in the previous section, Blake exhibited a personal understanding of the slope as ratio. However, when discussing what he would expect

from students, Blake began to reveal another interpretation of slope he found important

(i.e., a numeric interpretation) and, unlike the teachers in Category 2, did not appear

consistent with respect to expecting and supporting the development of quantitative

meanings from students.

In this section, I present evidence from The Linear Situation Task wherein

teachers were asked to graphically represent a linear function of their choosing and to

"create a story" to accompany their graph. Teachers were subsequently asked to provide

explanations for ways in which students come to develop meaning for various

components of the slope formula (such as the differences y_2-y_1 and x_2-x_1 in $\frac{y_2-y_1}{x_2-x_1}$ and the

meaning of $\div$ in this formula). They were also asked what they could do to support the

development of such meanings. I present evidence from a Category 2 teacher's responses

to this task to illustrate that teachers in Category 2 consistently emphasized the

development of a quantitative interpretation of slope that involved slope as a description

of the relationship between two quantities. Furthermore, these teachers suggested a rich

and varied set of actions to support the development of said interpretation. I then present

evidence from Blake to show he did not consistently emphasize the development of a

quantitative interpretation of slope but also emphasized a numeric interpretation. I end

this section with a brief discussion of Donna and Anne, the other teachers in Category 1.

It should be noted that teachers seemed to have a difficult time either conceptualizing or

discussing the ways in which students come to develop an understanding of slope (i.e.,

the second component of MKT); thus, the majority the data collected and presented here

involved supportive teacher actions (i.e., the third component of MKT).

Category 2 Teachers

When asked, in The Linear Situation Task, to graphically represent a linear function and to create an accompanying story, Emma drew the function shown in Figure 5.7a and provided the following story:

> Someone went out to get the mail, so they walked to their mailbox, which is … 4 feet away. They got there in 4 seconds. Then they stopped to get the mail, which took them 4 seconds. Then they walked back to their house.

She then expressed several ideas about what students would have to understand in order to develop meaning for the slope formula. For example, she said students would have to interpret numerical differences of the form $y_2 - y_1$ (e.g., 4-2) and $x_2 - x_1$ (e.g., 10-8) in terms of the elapsed quantities distance and time, respectively, and make use of those meanings when interpreting graphical representations. Specifically, when explaining how students might come to understand the meaning of a difference in y-values (i.e. 4-2) Emma highlighted where she saw 4-2 on her graph (see the arrow she drew in Figure 5.7a) and said students should come to interpret 4-2 as:

> That means he's going back home. … I think that's when they would know that that's [4-2] a change in distance … I think just having a general idea of what is going on in this graph will help them have that meaning.

Emma seemed to emphasize students' abilities to interpret graphical representations quantitatively as an important step towards interpreting the meaning of the slope formula (as a ratio). As Emma explained, if a student understands that "the x-axis is time, [then] if you are subtracting two x-values, you are finding a difference of time." Here, Emma again emphasized the importance of students' abilities to reason quantitatively.
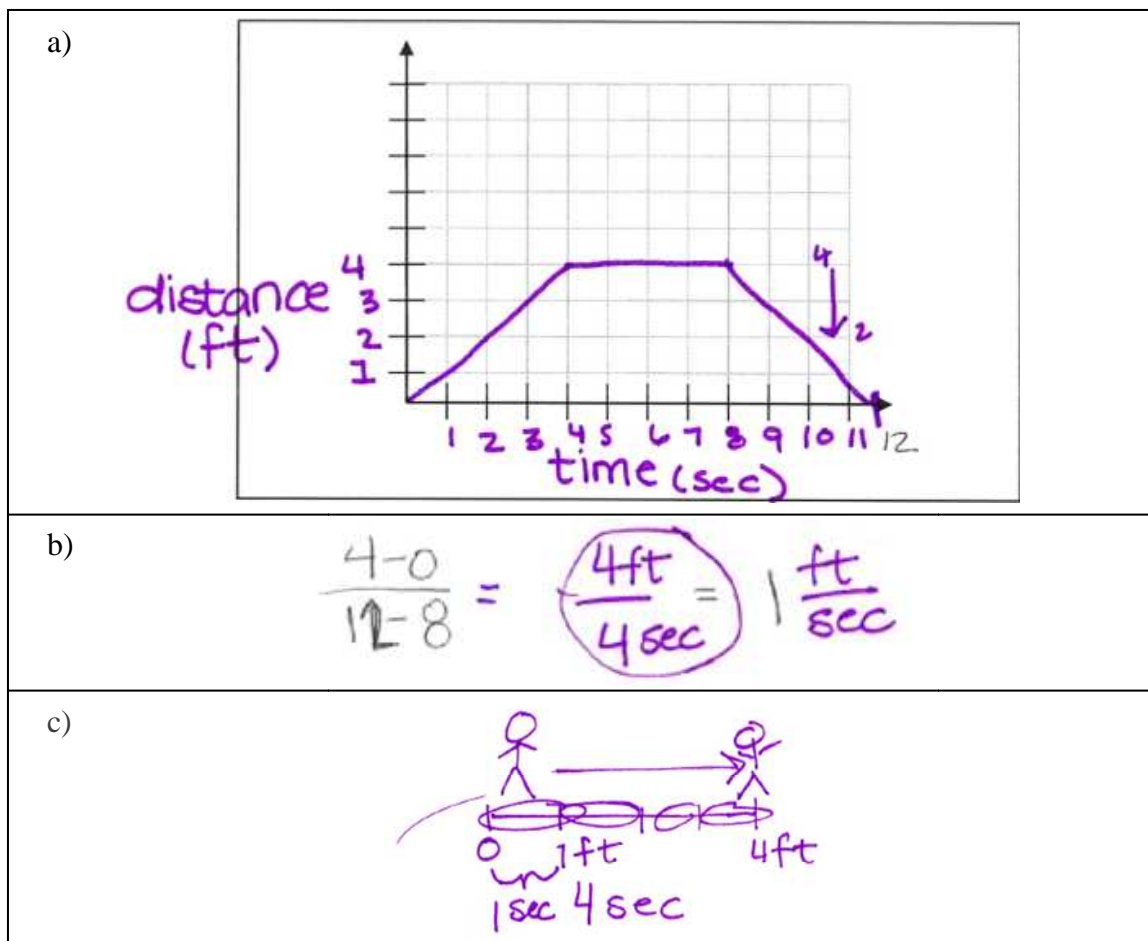
Figure 5.7: (a) Emma's graphically represented linear function; (b) Various ratios; (c) Emma's picture of the meaning of ⎯ ⎯.

Furthermore, Emma explained that students should come to interpret the division used in the slope formula as an operation that takes "two changing quantities and sort of reconcile[s] them and sort of tr[ies] to describe how they're *both* changing using *one* value." She elaborated on this statement both verbally and in writing, saying that students should view the division in the slope formula as:

> Almost like a rate, like think of it as *per*, like what this division is saying is you go 4-0 feet over 12-8 [seconds] … do you mind if I make it just like that [wrote "=⎯" to the right of "⎯" (see Figure 5.7b)]? So, it's like we're going 4 feet for 4 seconds, for every 4 seconds … It's describing a rate … This 1 [pointed to the "1" in Figure 5.7b] tells

us his speed … I mean, it tells us how he is going. He is going 1 foot
for every 1 second [wrote "$\frac{ft}{sec}$" to the right of the "1" in Figure 5.7b].

Emma also drew the picture shown in Figure 5.7c, wherein 4 ft in 4 s were separated into

four 1 ft in 1 s sections, to illustrate how one should interpret the meaning of division in

the context of her story. That is, she emphasized a partitive meaning for division (cf.,

Carpenter, Fennema, Franke, Levi, & Empson, 1999), here sharing 4 ft equally over 4 s

to arrive at 1 ft every 1 s. The evidence also suggests that Emma wanted her students to

see division as an operation that produces a single quantity (in this case, 1 ft/s) to

describe how two quantities change in relation to one another. Finally, Emma explained

that students should come to interpret a numerical value for *slope* in the context of her

story as providing a measure of a new third quantity, namely the person's *speed* (i.e., a

slope of "1 tells us his speed … one foot for every one second").

There were several teaching actions Emma offered in support of the development

of these meanings. First, to support students in thinking about the quantities represented

in a graphical representation, she suggested (a) asking students to draw upon an

accompanying real-world context to interpret the meaning of various portions of the

graphical representation and (b) reminding students that the axes of the graphical

representation have quantities associated with them (e.g., remind students that "*time* is

here [on the *x*-axis]"). Second, to help ensure students conceive of the various quantities

represented in a graphical representation, she suggested attending to student explanations

to make sure they involve quantities and "not [just] the numbers." Third, to support

students in conceiving of the necessity and meaning of division in the slope formula, she

suggested giving students distance-time information for two different characters and

posing questions involving a new quantity like "How can you compare the characters'

movement" to figure out "who is going *faster*" or whether "they [are] going the same

speed." Fourth, to help students interpret the meaning of slope as a description of a new

third quantity, Emma suggested asking students to explain "what are we finding when we

find the slope; [for example] we find a change in distance over a change in time, like

what *is* that?" Fifth, to support students in interpreting the meaning of a numerical value

representing slope (e.g., a slope of 2) as description of the relationship between two

quantities, Emma suggested supporting a focus on unit rates, for example, by asking

students to explain why comparing unit rates makes it easier to figure out who is going

faster (e.g., "if we talk about this in terms of 'every second' … like the same amount of

time, they become easier to compare because we are talking about the same block of

time"). These moves show Emma's consistent emphasis on actions that support students

in developing an interpretation of the quantitative meaning of slope and its associated

formula.

The teaching actions offered by the other teachers in Category 2 included: (a)

using contextualized problem situations; (b) asking students to draw pictures showing the

meaning of slope given a real-world context; (c) not accepting explanations for the

meaning of slope that invoke formulas; (d) asking students to provide explanations for

the meaning of various graphical features given a real-world context (e.g., a coordinate

point); (e) labeling all numerical values with appropriate units; and (f) connecting aspects

of other conventional representations (e.g., the difference between certain entries in a

table) to their meanings in a given real-world context. Such actions are consistent with

supporting the development of a quantitative interpretation of the meaning of slope.

In sum, when explaining how students might come to develop meaning for slope and its associated formula, the teachers in this category continuously emphasized the development of quantitative interpretations of different elements of the slope formula (e.g., differences of the form x_2-x_1 and division). Moreover, the teachers in this category offered rich and varied set of teaching actions in support of said development.

Blake

When asked, in The Linear Situation Task, to graphically represent a linear function and to create an accompanying story, Blake drew the picture shown in Figure 5.8 and provided the following story: "A manufacturing company, every day they produce 1,000 cups." When talking about what students would have to understand in order to develop meaning for the slope formula, Blake was not consistent in his focus on students' quantitative reasoning. For example, when explaining one way in which students could come to interpret the meaning of the slope formula, Blake did not emphasize (a) quantities from the manufacturing situation, nor did he emphasize (b) a measurement or partitive interpretation of the meaning of division as Emma had. Rather, Blake said students should "start by looking at" several numerator-denominator combinations, for example, "2,000 over 2 versus 5,000 over 5," "reduce" them," and notice that "every time, you get 1,000 over 1." Thus, Blake provided an explanation of students developing an interpretation of the meaning of slope wherein students identify multiple number combinations, interpret division as the physical placement of one number "over" another number wherein the resulting fraction is reduced (presumably via calculation), and interpret slope as the reduced form of the fraction. In another explanation of how students could come to interpret the meaning of slope in the

manufacturing situation, Blake emphasized students' recognition of a particular

numerical pattern, namely that the numerator and the denominator of the fractions

referenced above (i.e., $\frac{5,000}{5}$, $\frac{2,000}{2}$, and $\frac{1,000}{1}$) are the same except that the numerator has

"three zeros after [it]." In this way, Blake's explanations did not consistently emphasize

the development of students' quantitative interpretations of slope but also emphasized the
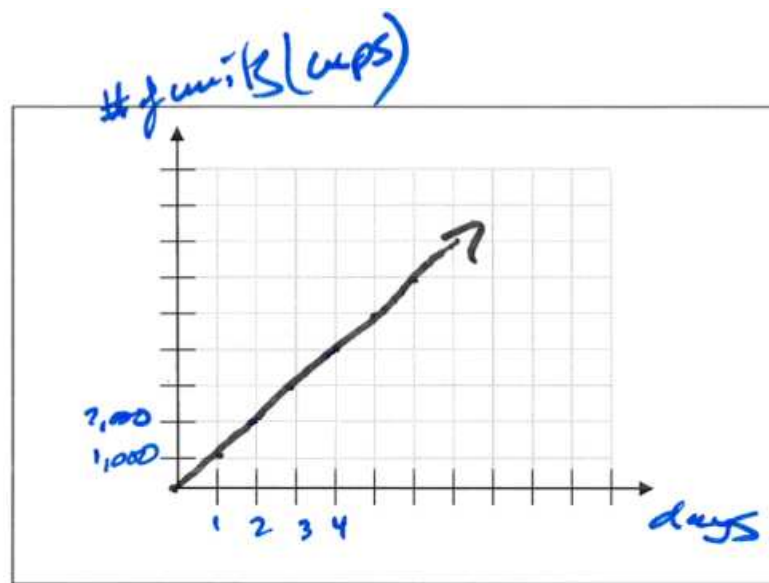
development of numeric interpretations of slope.

Figure 5.8: Blake's graphically represented linear function.

There were two instances in which Blake pointed to students' quantitative

reasoning as being an important step in the development of students' interpretation of the

meaning of slope. First, Blake reported that students would have to interpret the meaning

of differences of the form y_2-y_1 (e.g., 8,000-5,000) and x_2-x_1 (e.g., 8-5) in terms of the

quantities units of production and number of days, respectively. Moreover, he said

students would have to coordinate these quantities by thinking, for example, "What does

that [units of production] mean in terms of days worth of production?" However, Blake

did not explain, as Emma did, how students could come to interpret the relationship between these quantities quantitatively, for example, how one goes from coordinating one pair of quantities (e.g., 5,000 cups in 5 days) to finding another pair of quantities representing the same relationship (e.g., 1,000 cups in 1 day). Secondly, in a third description of how students should interpret the meaning of slope in the manufacturing situation, Blake said "as the rate at which units are produced on a daily basis." However, his description of *how* students could come to interpret the meaning of slope in this way was vague and involved students' use of the word "per."

There were four teaching actions Blake offered in support of the development of students' interpretation of the meaning of slope and its associated formula only two of which seemed to have a quantitative component. First, to support students in interpreting the quantitative meaning of numerical differences in the slope formula, he suggested asking students to provide descriptions of the meaning of those differences (e.g., "Asking them 'what does 5,000-3,000 *mean*?'"). Second, to support students in coordinating numerical differences, he suggested using a graphical representation as a tool to support students in finding the value that corresponds to a given numerical difference. For instance, he said, "I might just be able to say '5,000-3,000' and then say 'What does that mean in terms of your *other* axis?'" It is unclear whether this move supports students in reasoning quantitatively as students could follow the lines of a coordinate grid from a number on one axis (e.g., 5,000 or 3,000) to a corresponding number on the other axis. Third, to support students in developing an interpretation of the meaning of slope as rate at which two quantities change in relation to one another, Blake suggested posing a series

of questions to students. For example, in relation to the manufacturing situation, Blake

suggested the following questions:

> How many units do you see being produced for 4 days? How many
> units do you see being produced over 10 days? How many units do
> you see being produced over 15 days? And *then* can you tell me how
> many units are being produced *per* day?

The problem with this series of questions in terms of supporting the development of an

interpretation of slope as rate is that they do not require students to conceive of how the

quantities number of units produced and number of days are changing together as

students can answer each question with a single quantity, namely number of units

produced. Finally, to support students in finding the numerical value associated with

slope, Blake suggested supporting students in recognizing numerical patterns like the one

above wherein students notice that to find the value for slope in the manufacturing

situation one can put the same number in the numerator as the denominator and then add

three 0s to the number in the numerator. As these instructional supports show, Blake was

inconsistent with respect to supporting the development of a quantitative interpretation of

slope for as is evident from the list presented here, Blake also suggested moves in support

of the development of students' abilities to identify numbers and numerical patterns.

The Other Category 1 Teachers (Donna and Anne)

Unsurprisingly, neither Donna nor Anne were able to answer questions about how

students come to develop quantitative meaning for the slope formula or what they would

do to support their students in developing such meaning for the slope formula. This is

unsurprising because they did not exhibit evidence that they personally interpreted slope

as a ratio describing the relationship between two quantities; thus, it was unlikely that

they would expect such an interpretation from students or know what to do to support the development of such an interpretation. Rather, when asked specific questions about how students could develop meaning for slope or what they could do to support students in developing meaning for slope, Donna and Anne did not answer the questions directly but rather seemed to speak at length in generalities and without specificity. For example, when asked what she would do to support students in developing meaning for the slope of the linear situation she created during The Linear Situation Task (i.e., a dog eats ½ of a shirt every day), Anne spoke for 3 min and 3 s about an activity she uses in her classroom wherein students are presented with graphs, asked to write stories that could be represented by the graphs, told to give their stories to another student who subsequently draws a graph from the story, and then asked to compare the original graphs with student-created graphs. Anne never explained how this activity would help students derive meaning for slope nor did she ever talk specifically about what she could do to help students understand the meaning the slope of her linear function.

Discussion: Relationship between Teachers' Understanding of Two Other Components of MKT and Their Beliefs Regarding the Generalization of Students' Learning

When examining the way in which the seven teachers form Categories 1 and 2 in Chapter 4 seemed to personally understand slope, there was almost perfect alignment between (a) the teachers who emphasized the generalization of meaning (i.e., Category 2 teachers) and the teachers who understood slope as a ratio and (b) the teachers who emphasized the generalization of an association, procedure, or formula (i.e., Category 1 teachers) and the teachers who appeared to have limited or no understanding of ratio but

rather interpreted slope as the steepness of a line. Blake was the only teacher who emphasized the generalization of a formula while demonstrating a personal understanding of slope as ratio.

Examination of the second two components of MKT (i.e., teachers' understanding of the ways in which students come to develop meaning for slope and teachers' understanding of the actions that could be taken in support of the development of students' slope understanding) revealed some interesting differences between the teachers from Category 2 in Chapter 4 and Blake. When the topic of conversation shifted from teachers' personal understandings of slope to students' understandings of slope, all of the teachers from Category 2 *consistently* emphasized the development of quantitative interpretations of the meaning of slope while Blake did not. Rather, it seemed the shift in conversation served to illuminate another interpretation of slope that Blake found important for *students*, namely a numeric interpretation of slope.

Moreover, the teachers from Category 2 seemed to have a richer understanding of ways in which they might support students in developing a quantitative interpretation of slope. These actions ranged from providing students with distance-time information for two characters and asking them to figure out how to compare the characters' relative speeds to labeling all numerical values with appropriate units to requiring students explanations be quantitative in nature. Blake, on the other hand, seemed to have a limited understanding of how he could support students in developing a quantitative interpretation of slope. The pedagogical action he described that most clearly supported students' quantitative reasoning was to ask students what each difference in the slope formula (y_2-y_1 and x_2-x_1) means in a particular context. However, in contrast to the

teachers from Category 2, Blake also described pedagogical actions that could be seen as sending a contradictory message to students, for example, accepting their identifications of numerical patterns, without asking for a quantitative explanation, and accepting the calculation of division without asking for an explanation that demonstrated some meaning for division (such as the partitive or measurement meanings for division).

These differences between teachers from Category 2 and Blake from Category 1 suggest possible reasons for why a teacher who seems to personally interpret slope as a ratio would seem to have a belief about the generalization of students' learning that involved a formula for slope. It could be that Blake believes the generalization of quantitative interpretations of the meaning of mathematical topics like slope is important, but is unaware that an emphasis on numbers and numerical patterns could contradict and work against an emphasis on quantitative interpretations since, as was shown in Part 1 of this chapter, Blake is more than capable of reasoning about numbers quantitatively.

Alternately, it could be that Blake believes interpreting the meaning of slope quantitatively is an easy thing for students to do since they encounter multiple examples of slope in their daily lives. As Blake said, "I think they all have a preliminary understanding of rate; anyone can conceptualize walking at different rates or arriving at the same place at different times, [for example, in] relay races." In this way, Blake may believe it is more important to focus on the generalization of methods for finding or calculating slope than on a quantitative interpretation of the meaning of slope. Thus, Blake's beliefs regarding the generalization of students' learning will be examined more closely in Chapter 6.

CHAPTER 6:

Results on the Relationship between Teachers' Beliefs Regarding the Generalization

of Students' Learning and Teachers' Classroom Practices

In this chapter, I present findings to answer the third research question:

What is the relationship between teachers' beliefs regarding the generalization of students' learning and their classroom practices?

Recall that answering the question above involved analysis of data collected during Phase II of the research design (i.e., the classroom phase). In the classroom phase, teachers were observed as they taught the lesson they created according to the instructions listed in The Lesson Plan Activity. The Lesson Plan Activity asked teachers to develop (or adapt an existing) lesson on slope that implemented some of the ideas they had discussed during their Phase I interviews regarding the generalization of their students' learning. After the lesson was observed, teachers engaged in a debriefing in which they discussed the various teaching actions they enacted during the lesson and the ways in which those teaching actions related to their beliefs regarding the generalization of students' learning.

In this chapter, I present findings from analysis of Blake's observation and subsequent debriefing. He was selected for study for two reasons. First, as discussed in Chapter 4, Blake (unlike Sam, for example) was in multiple categories of belief. In fact, Blake appeared to hold three beliefs regarding the generalization of students' learning: students will productively generalize their learning to a novel situation (a) if the novel situation prompts them to make use of a learned association, procedure, or formula (Category 1); (b) if they develop confidence in their abilities to engage in mathematical activity (Category 4); and (c) if they develop the view that mathematics is relevant and

useful outside of the mathematics classroom (Category 5). Thus, it seemed likely there

would be evidence of at least one of his beliefs in his lesson. Second, Blake was the only

teacher who did not fit the alignment illustrated in Table 5.1 and discussed in Chapter 5.

This led me to wonder whether his beliefs regarding the generalization of learning as

espoused in and inferred from the Phase I interview data could be clarified or elaborated

via analysis of his classroom practices.

Due to pacing and sequencing issues, Blake could not teach his lesson on slope to

either of his Algebra 1 classes. Thus, Blake's lesson on slope took place in his Geometry

classes. Because the students engaging in Blake's lesson had already received instruction

on slope in their previous math class (i.e., Algebra 1), students were already aware of the

slope formula. Thus, the Category 1 belief involving associations, procedures, and

formulas was not an emphasis of Blake's lesson. Similarly, as Blake's Geometry classes

were dubbed "honors" classes, the Category 4 belief involving confidence in one's ability

to engage in mathematical activity was not an emphasis of his lesson. Instead, the

Category 5 belief of viewing mathematics as relevant and useful outside of the classroom

was the emphasis of Blake's lesson.

Interestingly, another belief regarding the generalization of students' learning

appeared during Blake's lesson that was not captured by the Phase I interview data

analyzed and presented in Chapter 4. This belief was indeed a content belief, but it was

not the Category 1 content belief. Rather, the Category 2 content belief regarding

mathematically valid interpretations of the meaning of mathematical topics surfaced

during Blake's lesson. In particular, Blake appeared to believe students would be able to

generalize their learning to novel situations if they develop an interpretation of the

meaning of slope as rate. Thus, I draw upon Blake's lesson to present findings regarding Blake's Category 2 and 5 beliefs.

I provide evidence to show that Blake's lesson and subsequent debriefing seemed to further elaborate the particular meanings the Category 2 and 5 beliefs held for Blake. The Category 2 belief appeared to mean (with respect to the topic of slope): students will be able to productively generalize their learning of slope if they develop an interpretation of slope as rate, and even more specifically slope as the rate at which two quantities change in relation to one another. The Category 5 belief appeared to mean: students will be able to productively generalize their learning if they view the use of a mathematical lens through which one sees the real world as important and/or necessary.

I show that in his lesson, Blake was able to successfully implement his Category 5 belief that students will be able to generalize their learning if they view the use of a mathematical lens through which one sees the real world as important and/or necessary. He selected a wide variety of contexts and demonstrated how mathematics could be found in everyday, real-world situations. Blake even created a group activity to engage students in mathematizing a specific real-world situation. His verbalizations and pedagogical actions communicated his meaning that it was important to use a mathematical lens to see mathematics in real-world situations. Students' verbalizations seemed to confirm that they were indeed able to make use of a mathematical lens to see mathematics in various real-world contexts.

I also show that Blake was less successful in implementing his Category 2 belief that students will be able to generalize their learning if they develop a mathematically valid interpretation of slope as the rate at which two quantities change in relation to one

another. While Blake selected a particular set of real-world contexts that implicitly involved rate and communicated to students that he wanted them to interpret that meaning of slope as the rate at which two quantities change in relation to one another, many of his pedagogical actions did not focus on such an interpretation of slope. Moreover, students' verbalizations seemed to lend support to the claim that students were not necessarily supported in generalizing the interpretation of slope that Blake had intended.

This chapter is separated into two sections mirroring the format of Blake's lesson, which involved a PowerPoint presentation and a cell-phone activity. In each section, I present findings from 4 different analyses. First, I present findings regarding the *form(s)* or ways in which the Category 2 and 5 beliefs seemed to manifest in Blake's lesson. Second, I present findings regarding the specific *meanings* the Category 2 and 5 beliefs appeared to take on for Blake. Third, I present findings regarding *what* Blake's pedagogical actions appeared to support the generalization of. Fourth, I draw upon student data to present findings regarding the *function* the Category 2 and 5 beliefs seemed to serve with respect to the generalization of students' learning. In other words, I present evidence in an attempt to how these beliefs functioned for students.

Part 1 of Blake's Lesson: The PowerPoint Presentation

Blake's lesson on slope began with a PowerPoint presentation. Since Blake's students had already received formal instruction on slope, the presentation (along with the accompanying lecture) did not involve an introduction to the topic of slope but rather a lecture about slope and various real-world contexts. The presentation contained 17

slides, the first 13[13] of which were accompanied by talk about the relevance of slope in the real world.

Form

There were four different forms or ways in which the Category 5 belief (i.e., a view that mathematics is relevant and useful outside of the classroom) seemed to manifest during Blake's PowerPoint presentation. The forms were: (a) multiple real-world contexts and visual images in Blake's PowerPoint slides; (b) supplementary real-world contexts referenced verbally; (c) verbalizations regarding what can be seen in various real-world contexts; and (d) verbalizations regarding the relevance and/or importance of the topic of slope given a particular real-world context. Interestingly and unexpectedly, the Category 2 belief (i.e., the generalization of students' learning is supported by mathematically valid interpretations of the meanings of topics like slope) also seemed to manifest during Blake's PowerPoint presentation. This belief seemed to manifest in the form of (a) particular real-world contexts Blake chose to present to students; (b) verbal statements telling students to think about slope as a rate; and (c) verbalizations regarding seeing rate in various real-world contexts. It should be noted that there were times during the PowerPoint presentation that forms of both beliefs manifested together. For example, a verbal statement regarding seeing rate in a real-world context was taken as a Category 5 form because it indicated that students should be able to see and identify mathematics outside of the classroom in a real-world situation and as a Category 2 form because it indicated an interpretation of slope as rate. Here, I draw upon the PowerPoint presentation to provide examples of each form.

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[13] The remaining 4 slides related to The Cell Phone Activity.

Blake opened the presentation by asking his students "What do you think of when I say *slope*?" On the board, Blake wrote down several student responses and then said: "I want us to be thinking about it as a [points to *rate of change* written on the board]; what is this? … A rate of change." Here, Blake selected rate of change from a list that included two slope formulas (i.e., "rise over run" and "$m = \frac{y_2 - y_1}{x_2 - x_1}$") and explicitly told students that he wanted them to think about slope as a rate of change. Thus, the Category 2 belief seemed to manifest in Blake' lesson in the form of a verbal statement telling students to think of slope as a rate of change.

Blake then illustrated (via an image on a PowerPoint slide) or verbally referenced 10 different real-world contexts during the PowerPoint presentation. The contexts, in the order in which they appeared during the lesson, were: speed, mountain ranges, a rolling ball, hiking, El Capitan in Yosemite, cityscapes, rooftops, business models, salaries, and bungee jumping. During Blake's debriefing, he explained that he wanted "to shower them [his students] with rates" so they would "realize that rates are everywhere." This seemed to indicate Blake believed that by making use of multiple real-world contexts (each of which implicitly involves rate) he was actually enabling students to see mathematics (and rate in particular) in the world and was thus supporting them in coming to view mathematics (and an interpretation of slope as rate in particular) as relevant and useful in the real world. In this way, the use of multiple real-world contexts was taken as a form of the Category 5 belief and the particular real-world contexts were taken as a form of the Category 2 belief.

When images of these contexts were projected on the board, Blake made explicit statements regarding what could be seen in those contexts. These statements were

generally of the form "When I look at a _____, I see _____." For example, when projecting

the image of a mountain range (see Figure 6.1a), Blake said, "When I look at a mountain,

I see slope. I see different steepnesses." Similarly, when projecting the image of a

cityscape (see Figure 6.1b), Blake said, "When I look at a city, I see math … I see slopes;

I see rates of change; that's what I'm looking at and everywhere in the world, that's what

I see." Moreover, after telling his students that he sees slope when looking at various

objects or situations in the real world, Blake asked his students to try to see it (i.e., slope)

too. For example, immediately following the excerpt above wherein Blake told his

students that he sees *slope* (i.e., "different steepnesses") when looking at mountain

ranges, Blake asked, "How many different slopes do *you* see on here?" Thus, the

Category 5 theme seemed to manifest in the form of verbalizations regarding seeing

mathematics in real-world contexts and the Category 2 theme appeared as a statement

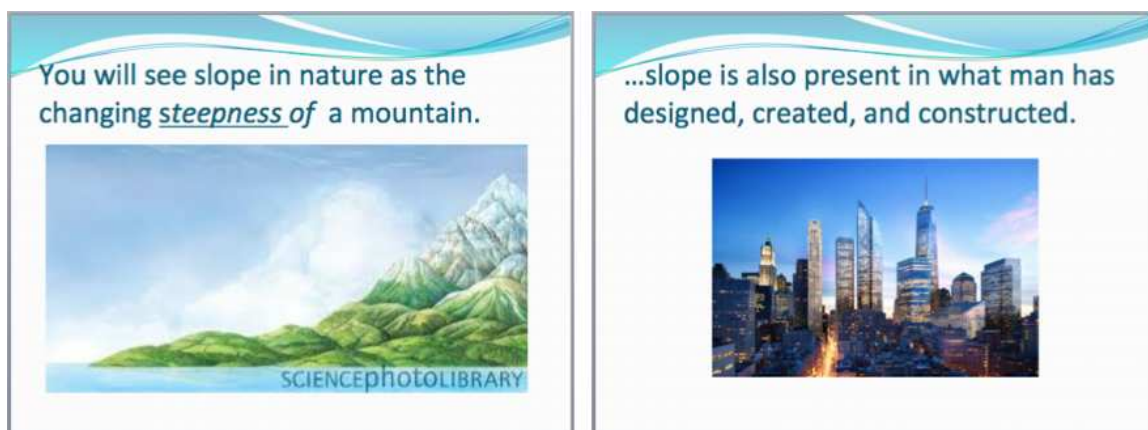involving an interpretation of slope as a rate of change.

Figure 6.1: (a) A mountain range; (b) A cityscape.

Finally, when discussing various contexts, Blake often made comments about the

relative importance of considering mathematics when looking at the real world. For

instance, after showing his students the image of a rooftop and telling them that the

mathematical topic of slope could be seen when looking at the rooftop, Blake asked his students to compare "the steepness" of rooftops in places where it snows (e.g., the mountains) with the steepness of rooftops in places where it does not snow (e.g., the valley). He then reiterated to his students that, in the valley, "you can have a flat roof because it's not going to snow and your roof is not going to cave in, but you can't have it [a flat roof] in the mountains." Blake then said, "It seems kind of important, right, maybe have some set of standards so when it snows, your roof doesn't cave in?" Likewise, Blake described a situation in which a bungee cord broke after a 250 lb man bungee jumped using the same bungee cord as a 100 lb woman (thereby stretching the cord past its limit as established by the spring constant—a ratio of force and mass—of the cord and the length of the unstretched cord). Blake then said, "So, it might be good to have these numbers accurate." Such statements were taken as a form of the Category 5 belief, specifically the "relevant and useful" portion of the theme (i.e., a view that mathematics is *relevant* and *useful* outside of the classroom).

Meaning

Above I presented three forms or ways in which the Category 2 belief (namely that the generalization of learning depends on mathematically-valid interpretations of the meaning of mathematical topics) appeared in Blake's PowerPoint presentation. Additionally, I presented four forms or ways in which the Category 5 belief (namely that the generalization of learning is mediated by students' view that mathematics is relevant and useful outside of the mathematics classroom) manifested in Blake's lesson. I now draw upon the PowerPoint presentation and subsequent debriefing to present findings to further flesh out the meaning these beliefs held for Blake.

After telling students he wanted them to think of slope as a rate of change (as described above), Blake went on to elaborate the meaning of rate of change for his students. He said: "A rate of change—it's a relationship between a couple quantities that describes change that's happening." Thus, it appeared that Blake not only wanted his students to think about slope as a rate of change but that he wanted them to interpret the meaning of rate of change in a particular way, namely as providing a description of how two quantities change in relation to one another. Moreover, in his debrief, Blake pointed to the fact that the particular real-world contexts he chose involved *rate* and that this would support the generalization of students' learning. Specifically, he explained that he wanted to "shower them [his students] with *rates … and then* maybe they can *apply it [slope] to another context* [emphasis added]." In other words, it seemed Blake believed that if students come to interpret the meaning of slope as rate and see rate in multiple contexts, then they will be enabled to generalize their learning to novel situations. Together this suggests that the Category 2 belief as it relates to the topic of slope seemed to hold the following meaning for Blake: *the generalization of slope depends on the mathematically valid interpretation of the meaning of slope as a description of the rate at which two quantities change in relation to one another.*

As described in the previous section, the first two forms in which the Category 5 belief appeared in Blake's lesson involved multiple real-world contexts (illustrated in a PowerPoint slide or with a verbal description). When choosing which real-world contexts to visually or verbally present to students, Blake did not merely select any contexts. Rather Blake chose contexts from the real world that he believed his students would have experience with. For instance, when explaining that one could see slope

when looking at a rooftop, Blake said "I'm trying to relate it [slope] to something you've seen in the *world* [emphasis added], which is, you've seen houses before." Such careful selection of real-world contexts coupled with the third form in which the Category 5 belief appeared (verbalizations regarding what can be seen in real-world contexts wherein, for example, Blake repeatedly made statements about the fact that *slope* could be seen in those contexts) seemed to suggest that, for Blake, the Category 5 belief more specifically meant a view that mathematics is relevant and useful in *the real world*. Furthermore, it seemed Blake drew upon such real-world contexts in order to make students aware of a *mathematical lens* through which said contexts could be viewed. In fact, Blake explained during the debrief that he was trying to show his students "that there is a different point of view" and, in particular, that one "can view it [the world] from a *mathematical* point of view." Blake went on to explain that he could have contrasted such a point of view with others:

> I could have contrasted that [mathematical point of view] with an artistic point of view or an architectural point of view or a mason's point of view or an engineer's point of view. I mean there's a lot of different ways you could look at anything, really.

These excerpts from Blake's debrief seem to support the claim that, for Blake, the Category 5 belief meant: students should be aware that *there is a mathematical lens through which one can see the real world*.

The fourth form in which the Category 5 belief appeared (verbalizations regarding the relevance and/or importance of considering mathematical when looking at the real world) wherein Blake made comments about, for instance, the importance of building codes for rooftops (to prevent their caving in) and the importance of correctly choosing

an appropriate bungee cord (to avoid breakage) seemed to indicate that, for Blake, the Category 5 belief additionally meant: students should be aware of *the importance and/or necessity of using a mathematical lens to see the real world*. Taken together, the Category 5 belief seemed to hold a particular meaning for Blake: *students should view the use of a mathematical lens through which one sees the real world as important and/or necessary*.

What

With respect to the Category 5 belief, Blake's presentation appeared successful. As discussed above, Blake presented (both visually and verbally) a wide variety of real-world contexts to his students to demonstrate for them that a mathematical lens can be used to see mathematics outside of the classroom in common, everyday real-world situations. However, with respect to the Category 2 belief, Blake's presentation did not appear successful. While Blake stated that *what* he wanted students to see in the real world and subsequently generalize was an interpretation of the meaning of slope as the rate at which two quantities change in relation to one another, what Blake focused on through his pedagogical actions was unclear as it varied throughout the presentation. In other words, the analysis of *what* Blake seemed to want his students to generalize had two levels. At a general level, Blake seemed to want his students to apply a mathematical lens to the real world in order to see mathematics in it. At a more fine-grained level, Blake seemed to want his students to "see" particular mathematics in the real world, namely slope as a rate of change. However, in identifying the nature of the mathematics Blake wanted his students to see, different and somewhat problematic interpretations of rate began to appear. As the evidence below will demonstrate, Blake focused on a

multitude of ideas (likely unintentionally) as what was to be seen and generalized in future situations. These included work, mechanical advantage, the steepness of an object, acceleration, and either the steepness of an object at a point or the average steepness of an object.

In one real-world example, Blake asked his students which "hill" they would "rather hike up," showed them the lines illustrated in Figure 6.2, and then said "depends on how much of a challenge you like." The possible confusion with such a statement is that it is not clear which attribute Blake wanted his students to see or conceive of. For instance, if he wanted his students to see or conceive of the amount of *work* (i.e., the total amount of energy associated with the position and motion of an object) it requires to move up inclined planes with different steepnesses, then there is no difference (given the vertical rises of the inclined planes are the same) since
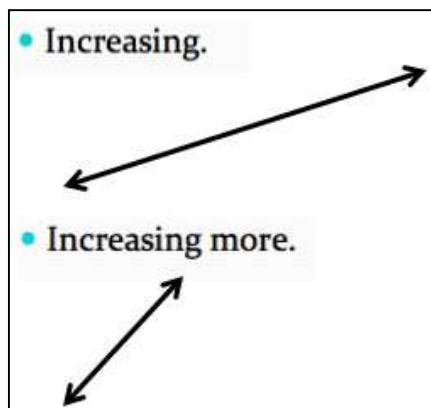
.

Figure 6.2: Using a mathematical lens to see the amount of work required to hike up two inclined planes with different steepnesses.

If, however, Blake wanted his students to see or conceive of the *mechanical advantage* (i.e., the ratio of the slant height to the vertical height of an inclined plane; see Figure 6.3) of using one inclined plane versus another, then it is optimal to use a less

steep incline because there is more of a mechanical advantage. However, one must be

careful here. While mechanical advantage does indeed provide a legitimate measure of

the steepness of an inclined plane, greater steepnesses are associated with less mechanical

advantage. Thus, there is an inverse relation between the steepness of a physical object

and mechanical advantage. This means that when answering Blake's question regarding

which hill one would rather hike up, one would choose a steeper incline if one wants

more of a challenge (since the mechanical advantage is less) and a less steep incline if

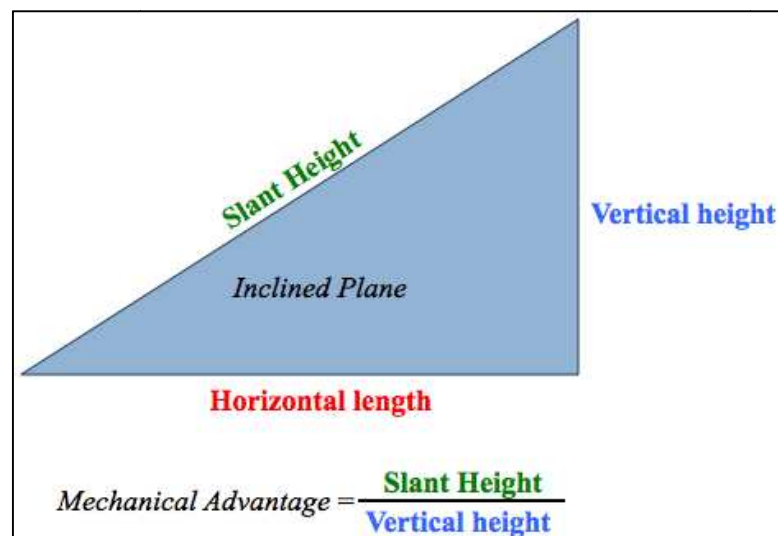one wants less of a challenge (since the mechanical advantage is greater).

Figure 6.3: Using a mathematical lens to see the mechanical advantage of using an inclined plane.

In another example, Blake showed his students the two lines shown in Figure 6.4

and asked his students to compare the speeds at which a ball (placed at the top of each

line and then released) would roll down the lines. Specifically, Blake asked his students

to think about what would happen if he were:

> To take a ball up to the top of this line [pointed to the first line shown
> in Figure 6.4] and let go of it; how would that be different than if I did
> it for this line [pointed to the second line shown in Figure 6.4]."

In this scenario, it is important to note that the ball is being placed at the top of the lines

and then let go; in other words, there are no additional forces acting on the ball. Blake

provided the following answer to his students: "The first one would go faster …
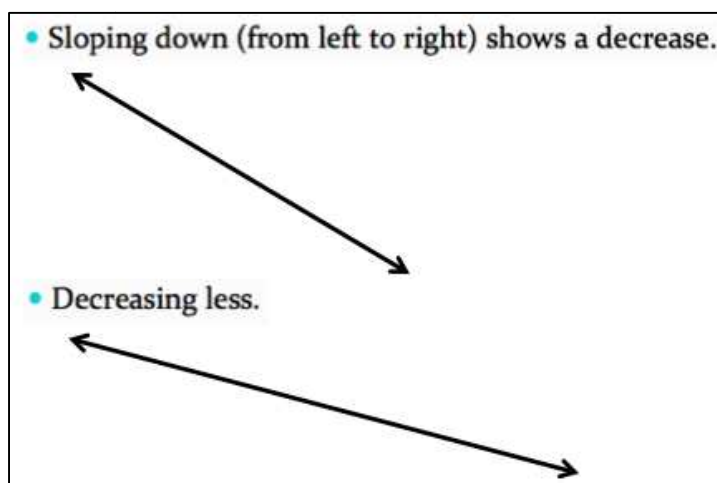
[because] it's steeper."

Figure 6.4: Using a mathematical lens to see the speed at which a ball would roll down inclined planes with different steepnesses.

In the subsequent debrief, Blake provided his rationale for inserting this example

into the PowerPoint presentation. He said:

> I tried to give them the example of the ball and how fast the ball would
> move because I figure that would help tie in rate because the ball is
> going to roll at a different rate. The steeper the line the faster the rate
> is.

While using the example of a ball moving down an incline is a creative way to support

students in linking "rate" with visual images typically associated with graphical images

of linear functions, it could create confusion mathematically. If one focuses on the image

of the lines, and considers Blake's example of a ball rolling down inclined planes with

two different steepnesses, the lines seem to depict surfaces (e.g., ramps, hills, or rooftops)

whose *steepnesses* can be modeled with linear functions relating the quantities *height* and

length of the respective surfaces. Here, slope can be conceived of (and seen in the images) as a measure of the *steepness* of each surface, or the constant "rate" at which the height and the length of each surface change in relation to one another. The slope, or steepness, of each of the lines is constant. Moreover, the slope (i.e., steepness) of the first surface is indeed greater than the slope (i.e., steepness) of the second surface since it has a greater increase in height per unit of length.

If, instead, one conceives of the speed of a ball rolling down the two inclined planes, the "rate" at which "the ball would move" when rolling down each incline increases over time and could be modeled with linear[14] functions relating the *ball's speed* and the *amount of time since the ball starting rolling*. In each case, the speed of the ball does not necessarily depend on the steepness of the incline but rather on how long the ball has been rolling down the incline. Thus, if one measures the ball's speed towards the top of the steeper incline (where the amount of time since the ball started rolling is small), the speed could be less than if one measures the ball's speed towards the bottom of the less steep incline (where the amount of time since the ball started rolling is large). Moreover, slope in this case does not measure the steepness of the incline but rather the *acceleration* of the ball, or the rate at which the ball's speed changes over time. In this way, Blake's attempt to support his students in seeing the rate or the speed at which a ball rolls down an incline as a real-world manifestation of slope by making a statement regarding the relationship between the rate at which a ball moves and the steepness of the

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[14] The speed of the ball can be measured with a *linear* function in each case since the speed of the ball would increase at a *constant* rate over time given the shape of the incline and the acceleration of the ball due to gravity.

incline it moves down is misleading as the speed of the ball increases over time, thus, steeper does not (as Blake said) necessarily mean faster.

In yet another example, when discussing the idea that slope can be seen in the real world as the steepness of a mountain, Blake placed lines at the edges of various surfaces of the mountain range shown in Figure 6.1a (see Figure 6.5). In the image, the slope (i.e., steepness) of the mountain is not constant and the lines shown in Figure 6.5 therefore measure steepnesses at particular instants or locations (or average steepnesses) on the surface of the mountain. In other words, the steepness of the mountain range depends on precisely where one is on the mountain's surface and, unlike the steepnesses of the inclined planes referenced and shown in Figure 6.4, differs at various locations. Here, slope (i.e., steepness) is a measure of the *instantaneous* (or *average*) rate at which the vertical height and the horizontal length of the mountain range change in relation to one another as opposed to a measure of the *constant* rate at which the vertical height and horizontal length change in relation to one another.
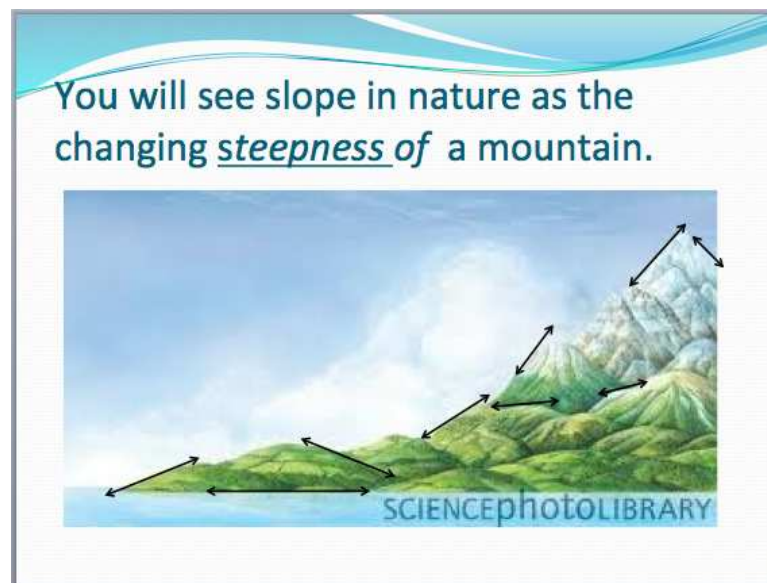

Figure 6.5: Using a mathematical lens to see steepness in a mountain range.

In sum, Blake referenced multiple real-world contexts and attempted to support his students in using a mathematical lens to see mathematics in those real-world contexts. In this way, it seemed Blake's actions seemed to serve the generalization of a mathematical lens. However, Blake focused on the following range of attributes in the real-world contexts he referenced: work, mechanical advantage, the steepness of an inclined plane, acceleration, and either the steepness of an object at a point or the average steepness of an object. Thus, the specific interpretation Blake's pedagogical actions supported his students in seeing and subsequently generalizing is unclear as he focused on attributes both related and unrelated to slope and on constant and instantaneous (or average) rates of change.

Function

In this section, I present my findings for function. In particular, I draw upon student data from the PowerPoint presentation to illustrate the role that the Category 2 and Category 5 beliefs seemed to play in the generalization of students' learning. In particular, I show that the Category 5 belief seemed to support students in using a mathematical lens to see the world so that when looking at a novel real-world context, students were supported in seeing mathematics in that context. It was, however, unclear how the Category 2 belief supported students in seeing specific mathematics, namely rate of change, in the world. It should be noted that there were limited opportunities for students to participate verbally during the presentation. Thus, the data available from which to infer function was limited.

Throughout Blake's lesson, he posed questions that supported his students in looking at and conceiving of real-world situations using a mathematical lens (as opposed

to an artistic lens, for instance). For example, after showing his students the image of the mountain range shown in Figure 6.1a and telling them that he sees slope (i.e., different steepnesses) when he looks at mountains, he placed several lines on the mountain range (see Figure 6.5). Blake then asked his students how many different slopes *they* could see. Two student responses to this question were "a lot" and "too many." Blake then asked, "Are they the same?" to which several students responded, "No." Later in the presentation when students were shown a different real-world context (i.e., rooftops) and asked to compare the *slopes* (i.e., the steepness) of rooftops in places where it snows with the slopes of rooftops in places where it does not, a student said, "Their roofs are like this [gestured with an arm indicating a very steep rooftop] because when it snows, so it doesn't collapse in." Thus, it seemed students were supported in using a mathematical lens to see mathematics when looking at a mountain range. At least one student was then enabled to compare the relative steepnesses in a new rooftop context. In this way, it seemed the Category 5 belief functioned during the PowerPoint presentation to support students in using a mathematical lens to see mathematics in the real world.

As there were so many different real-world contexts used during the PowerPoint presentation and so few opportunities for students to talk, it was unclear how Blake's Category 2 belief functioned for students during the PowerPoint presentation to support them in generalizing a mathematically valid interpretation of slope as the rate at which two quantities change in relation to one another. The limited data that did emerge did not point to a particular interpretation of slope. For example, in a real-world example that Blake introduced by saying "I want you to be thinking of slope as a rate of change," Blake verbally described the context of salaries, saying "the more time you work, the

more money you make even if you're on salary … let's say you make 50,000 a year."

After describing the context in this way, Blake asked the class, "What's the rate?" to

which one student replied "50,000" and another replied "50,000 per year." As students

were not asked to provide explanations for their responses, it is unclear how students

interpreted the meaning of rate as responses like "50,000" could indicate a student who

responded by identifying the numerical value in the description of the context. In another

instance, Blake asked students to provide examples of rates. Several students responded

with "miles per hour." Again, it was unclear how students interpreted the meaning of

rate in this instance since no further elaboration was given or asked for and responses like

"miles per hour" could merely be labels for students.

Part 2 of Blake's Lesson: The Cell-Phone Activity

After the PowerPoint presentation, Blake introduced another real-world context—

cell-phone charges—with a sentence of text on a PowerPoint slide. He then asked his

students general questions regarding their cell-phone plans, played an audio file in which

a customer argued with his cell-phone carrier about how much he was being charged

based on the amount of data he had used, asked his students how much they trusted their

cell-phone carriers, and then asked his students to get into small groups to engage in an

activity involving the context of cell-phone charges. In the activity, students were shown

the table in Figure 6.6, asked to fill in the missing entries and to find the unit rate.

Number of international minutes	Cell phone charges (dollars)
0	
10	51
50	55
100	60
150	65
200	
	95

Figure 6.6: Table used in the group activity involving the context of cell-phone charges.

Form

There were four forms or ways in which the Category 5 belief (i.e., students will be able to generalize their learning if they develop a view that mathematics is relevant and useful outside of the classroom) seemed to manifest during the cell-phone activity: (a) the real-world context of cell-phone charges (illustrated via text on a PowerPoint slide, an audio file, and a group activity); (b) inquiries regarding students' cell-phone plans; (c) verbalizations regarding amount of trust in the accuracy of cell-phone bills; and (d) verbalizations regarding what can be seen in the real world and heard in the audio file. Interestingly, as was the case in Blake's PowerPoint presentation, the Category 2 belief (i.e., the generalization of students' learning is supported by mathematically valid interpretations of the meanings of topics like slope) seemed to manifest in Blake's lesson despite the fact it did not come up during Blake's Phase I interviews. The forms in which the Category 2 belief appeared were: (a) verbalizations regarding listening for or hearing "unit rate" in the audio file; (b) verbalizations and written inscriptions regarding what rate looks like; (c) verbalizations instructing students that their goal is to find "unit rate"

in the group activity; (d) the table in which the group activity is represented; and (e) verbalizations involving the numerical patterns that can be seen in the table. As was the case in the previous section forms of both beliefs often manifested together. For example, a verbal statement instructing students to listen for a unit rate in the audio file was taken as a Category 5 form because it indicated that students should be able to hear and identify mathematics given an audio file from a real-world interaction and as a Category 2 form because it indicated an interpretation of the meaning of slope as rate. Here, I draw upon the cell-phone activity to provide examples of each form.

This portion of Blake's lesson began where Blake's PowerPoint presentation ended. As noted above, Blake's PowerPoint presentation involved a lecture about slope and many different real-world contexts. The cell-phone activity was introduced at the end of the presentation on a PowerPoint slide via the following text: "There are other ways to think about slope … [for example,] monthly charges for cell phone usage." In this way, a real-world context that was not discussed during the PowerPoint presentation was introduced via text on a slide and linked to a particular mathematical topic, namely slope. It was therefore taken as a form of the Category 5 belief since it indicated a view that mathematics is relevant and useful outside of the classroom. Moreover, Blake followed the introduction of this real-world instantiation of a mathematical topic by inquiring about *students'* cell-phone plans with questions like "You don't get charged for minutes do you? You have a flat rate; it's rollover? It's unlimited? But if you leave the country, is that true?" Despite the fact these questions did not lead to a discussion but rather to a peppering of "Yes" and "No" responses from various students in the classroom, they were taken as a form of the Category 5 belief since they seemed to

indicate a view that mathematics is relevant and useful outside of the classroom, namely in situations involving *their* cell-phone plans.

Blake then played an audio file wherein a customer argued with his cell-phone carrier about being charged $71.79 for data usage when he had been quoted 0.002 *cents* per kilobyte as opposed to 0.002 *dollars* per kilobyte. Before playing the file, Blake said, "I want you to pay attention to the unit rate; I want you to listen for the unit rate." The audio file and accompanying instructions were taken as forms of the Category 5 belief (i.e., the real-world context of cell-phone charges and verbalizations regarding what can be heard in the audio file) because they indicated students should be able to identify mathematics in real-world interactions as well as a form of the Category 2 belief (i.e., verbalizations regarding listening for "the unit rate" in the audio file) since they emphasized a particular interpretation of the mathematical topic of slope, namely slope as rate.

After the audio file was played, Blake polled the class to see how many of the students used the same cell-phone carrier as the customer in the audio file and asked, "How comfortable are you with your cell phone bill?" These verbalizations were taken as forms of the Category 5 belief (i.e., inquiries regarding students' cell-phone plans and verbalizations regarding amount of trust in the accuracy of cell-phone bills) because they seemed to suggest that considering mathematics while outside the classroom, for example when thinking about their cell-phone bills, is relevant and useful (since cell-phone carriers can and do make mistakes).

Blake then asked the class if they "picked up on what the unit rate [in the audio file] was." Several students began to answer when Blake said, "It was .002 cents per

what" to which the class responded "Kilobyte!" Blake repeated, "Per kilobyte; per 1

kilobyte of usage" and wrote "$\frac{.002\ cents}{1\ kb}$" on the board saying, "*That* is a rate." In this

exchange we see a form of the Category 5 belief (i.e., verbalizations regarding what can

be heard in the audio file) because as before Blake's verbalizations seemed to indicate

that mathematics could be heard in the recorded interaction. We also see forms of the

Category 2 belief (i.e., verbalizations regarding hearing unit rate in the audio file and

verbalizations/inscriptions regarding what rate looks like) since Blake's interactions

seemed to emphasize a particular interpretation of slope while simultaneously illustrating

for students what rate looks like.

Blake went on to provide a second way of writing the unit rate from the audio file

and to compare the rates seen in the real world with the rates seen in the mathematics

classroom:

> You can write it like this .002 cents per kilobyte [wrote ".002 cents per
> kb" on the board]. We're used to seeing, when we look at the world,
> we're used to seeing things like this [circled ".002 cents per kb" on the
> board]. We're not used to seeing them like this [pointed to "$\frac{.002\ cents}{1\ kb}$"
> on the board] and in math, in your algebra classes, you learn it more
> like this [pointed to "$\frac{.002\ cents}{1\ kb}$" on the board] and you usually don't
> even have any units attached to it. You just have a number and it's a
> fraction.

Here, Blake seemed to draw a connection between something that is typically referred to

as a "rate" in math class (i.e., a unit-less number or fraction) and the way in which one

would typically read, write, or hear that rate in the real world (e.g., as a statement of the

form "____ per ____"). This excerpt was therefore taken as a form of the Category 5 belief

(i.e., verbalizations regarding what can be seen in the real world and heard in the audio

file) and form of the Category 2 belief (i.e., verbalizations/inscriptions regarding what rate looks like both in and out of the classroom).

Finally, Blake told his students to get into groups, presented the table shown in Figure 6.6 via a PowerPoint slide, and told students to fill in the missing entries of the table and to find the unit rate. Blake's instructions "to find a unit rate" during the group activity was taken as a form of the Category 2 belief since it indicated a particular interpretation of slope. Blake then supported students in finding the missing entries of the table by focusing their attention on the numerical patterns in each column of the table (as will be discussed in further detail later). Similarly, he supported students in finding the unit rate by detailing how they could use a supplementary table to complete the activity (as will be discussed in further detail later). In the debrief that followed Blake's lesson, Blake explicitly highlighted the table as well as the patterns contained in the original and supplementary tables as being crucial with respect to the generalization of students' learning because, as Blake stated, "in the pattern [of the table] lies the answers to their problems." Given the fact that one of his students' "problems" during the group activity was to find unit rate and the fact that Blake emphasized (on multiple occasions both during and after the group activity) the importance of the table in which data for the group activity were represented as well as the associated patterns therein in the generalization of students' interpretation of rate, they were taken as forms of the Category 2 belief.

Meaning

Above I presented five forms or ways in which the Category 2 belief (namely that the generalization of learning depends on mathematically valid interpretations of the

meaning of mathematical topics like slope) appeared during the cell-phone activity. I also presented four forms or manifestations of the Category 5 belief (namely that the generalization of learning is facilitated by students' view that mathematics is relevant and useful outside of the mathematics classroom). I now draw upon the cell-phone activity and subsequent debriefing to present findings regarding the meaning these beliefs seemed to hold for Blake. The findings presented here are consistent with those presented and discussed in the previous meaning section.

With respect to the Category 2 belief, examination of the cell-phone activity and the portion of the debriefing relating to this activity seemed to provide further elaboration of the mathematically valid interpretation of slope Blake hoped his students would generalize from the activity and, thus, further elaboration of the Category 2 belief as it relates to the topic of slope. As noted above, during the debrief following Blake's lesson, Blake identified the way in which he chose to present the cell-phone data (i.e., via a table) as being critical with respect to supporting his students in generalizing an interpretation of slope as rate. He explained that he "really wanted them [students] to look at it [the table] and get it down to a unit … the unit rate." Then, Blake provided further elaboration regarding this interpretation of slope (as it relates to the table used in the cell-phone activity):

> There is a relationship happening between the two values [pointed to corresponding entries in each column of the table] and I think that's *what* they're seeing; it's contextualized; they can see that there is an input value and an output value and I think that they can find the pattern and the relationship between those two ... I am hoping they would extend this table out and find the relationship between these two values, like there has to be some fixed amount between them that exists and if they can find that fixed amount, they can subdivide it probably by 5s or even by 1s.

In this explanation, Blake provided rationale regarding why the table was so important in the generalization of students' learning—because it supported students in "seeing" slope in a particular way, namely as a relationship between the values of the quantities represented in the table (e.g., a person is being charged $5 for every 50 international minutes). In addition, it seemed Blake viewed the table as important because it assists students in seeing another relationship, namely the relationship between a rate and a unit rate (i.e., a rate can be subdivided to find a unit rate of, in this case, $0.10 for every 1 minute). Thus, it appeared that for Blake the Category 2 belief held the following meaning: *the generalization of slope in a linear situation depends on the mathematically-valid interpretation of the meaning of slope as a description of the constant rate at which two quantities change in relation to one another.*

With respect to the Category 5 belief, Blake provided little elaboration on the meaning of this belief as it manifested during the cell-phone activity. The only explanation he gave was that in using the cell-phone context, his goal for students was "for them to realize that they might not be thinking about it [slope], but it's there." Thus, the meaning of the Category 5 belief for Blake seemed to be: *the generalization of students' learning is facilitated by students' view that a mathematical lens can be used to see mathematics in the real world*, which is consistent with findings regarding meaning that were presented in the previous meaning section.

What

As illustrated in the previous sections, Blake seemed to want his students to generalize a mathematical lens *and* an interpretation of slope as rate. During the PowerPoint presentation, Blake hoped his students would be supported in using a

mathematical lens to see mathematics in many different real-world situations. Then, it seemed he wanted his students to make use of this mathematical lens during the cell-phone activity to mathematize a particular real-world context. To support students in using a mathematical lens to mathematize the context of cell-phone charges, Blake went to great lengths choosing an audio file, playing it for his students, asking his students to attend to the mathematics in the file, inquiring about students' personal cell-phone plans, creating a group activity around the context, asking students to identify mathematics in the group activity, and supporting students in identifying mathematics in the group activity. Thus, the cell-phone activity seemed to serve the generalization of a mathematical lens.

At a more specific level, Blake wanted his students to see particular mathematics in the cell-phone activity, namely rate and unit rate. Moreover, Blake hoped his group activity involving the context of cell-phone charges would support students in generalizing the interpretation of slope in a linear situation as the constant rate at which two quantities change in relation to one another. However, as will be demonstrated below, he was unable to manifest such an interpretation of slope during the group activity. Rather, the actions Blake took seemed to support his students in seeing two separate patterns or uncoordinated sequences in the table (i.e., 50, 100, 150, 200 and 55, 60, 65) as well as associated (and also separate) patterns of differences (i.e., a constant difference of 50 and a constant difference of 5).

Recall that in the group activity, students were provided with a table in which there was unequal spacing between the *x* values (or the values representing *number of international minutes*; see Figure 6.6). The table provided to students represented a

linear relation between the number of minutes spent on international calls and cell-phone charges. The students were asked to fill in missing entries of the table and, subsequently, to find the amount being charged per minute of international-talk time.

As the students worked to complete the table, Blake repeated the same set of actions for several individual students, small groups, and finally for the class as a whole. In this set of actions, Blake focused students' attention on the four rows of the table in which there *was* equal spacing between the *x* values (see Figure 6.7). To do this, Blake either physically covered everything but those four rows or made verbal statements to his students instructing them to "ignore everything" but "these 4 rows." Then, while looking at this subsection of the table, Blake focused on changes between consecutive entries in each column of the table. He did this by pointing to the highlighted portion of the "number of international minutes" column and asking, "This is going by what?" After a student provided an answer (e.g., by saying "by 50s"), Blake pointed to entries in the "cell phone charges" column and asked, "And these are going by?" After a student provided an answer (e.g., by saying "by 5s"), Blake asked what the missing entry in the highlighted portion of the table was (i.e., 70).

Number of international minutes	Cell phone charges (dollars)
0	
10	51
50	55
100	60
150	65
200	
	95

Figure 6.7: Focusing on a portion of the table to find a missing entry.

Blake's instruction to look at a subsection of the table coupled with his gestures and questions about individual columns of the table focused on two separate sequences of values (i.e., 50, 100, 150, 200 and 55, 60, 65, ___). Moreover, gestures highlighting a single column of the table coupled with questions about what each column or set of entries was "going by" seemed to support students in seeing the constant difference of 50 pattern in the "number of international minutes" column of the table and the constant difference of 5 pattern in the "cell phone charges" column of the table.

Blake's focus on two separate and uncoordinated sequences of values was seen again during Blake's instruction regarding how to fill in the other two entries (i.e., the entries outside of the box shown in Figure 6.7). To support his class in filling in the first entry in the "cell phone charges" column (i.e., the missing entry located *above* the box shown in Figure 6.7), Blake drew a supplementary table wherein additional rows of data were added between the first four rows from the original table (see Figure 6.8a for the author's recreation of this supplementary table and how it relates to the original table). In this supplementary table, Blake added entries to the original "number of international

minutes" column so that in the supplementary table it contained entries at evenly spaced intervals of 10. He then asked his class, "Any of these we know?" and subsequently filled in three entries of the "d" column (see Figure 6.8b) while saying, for example, "10 is 51." When a student shouted, "oh, it's just by 1s," Blake went down the "d" column and filled in the remaining entries. Blake enacted a similar set of steps to support his students in filling in the last entry in the "number of international minutes" column (i.e., the missing entry located *below* the box shown in Figure 6.7).

Number of international minutes	Cell phone charges (dollars)		(a)	m	d		(b)	m	d
				0				0	
0				10				10	51
10	51			20				20	
50	55			30				30	
100	60			40				40	
150	65			50				50	55
200				60				60	
	95			70				70	
				80				80	
				90				90	
				100				100	60

Figure 6.8: (a) A supplementary table created from the original table; (b) Filling in 3 missing entries.

In this episode, Blake capitalized on a student comment that the "d" column goes down by 1s (see Figure 6.9a) and filled in the entries by going down the column (see Figure 6.9b). Such a move focused on using the "by 1s" pattern to fill in the second column of the table. Furthermore, due to Blake's choice to use equal spacing for the "m" column, students did not necessarily have to look at both columns to complete the table. Rather, students could focus on a single sequence with 11 entries (i.e., __, 51, __, __, __,

55, __, __, __, __, 60) and simply fill in the blanks "by 1s" making sure the known

numbers fit the pattern. As there was no further discussion of the table or the relationship

between the values of the quantities in each column of the table, it is possible that Blake's

actions supported the seeing of only one sequence in the table and the pattern of constant

differences (i.e., going by 1s) therein.

(a)	m	d		(b)	m	d
	0				0	50
	10	51			10	51
	20				20	52
	30				30	53
	40				40	54
	50	55			50	55
	60				60	56
	70				70	57
	80				80	58
	90				90	59
	100	60			100	60

Figure 6.9: (a) The column goes down by 1s; (b) Using the "by 1s" pattern to fill in missing entries of the table.

Moreover, Blake's actions did not appear to support the partitioning or

subdivision of a relationship between the two quantities represented in the table as he had

intended. Recall that Blake stated during the debriefing that he believed the table would

support his students in seeing a relationship between values of the quantities represented

in the table (e.g., $1 for every 10 international minutes) that could subsequently be

subdivided by 1s, for example, to find a unit rate (e.g., $1 for every 10 minutes can be

split into ten groups of $0.10 for every 1 minute). However, during the time students

worked in groups to find the unit rate, Blake never asked a student to explain his

reasoning nor did Blake ever offer an explanation that went beyond the statement of a

particular (non-unit) ratio. Moreover, the whole-class discussion surrounding this unit

rate consisted of a query to the class: "What did you come up with for the rate," a

restatement of the answer: "10 cents per minute," and a poll regarding how many people

got the aforementioned answer. When a student subsequently asked Blake for further

clarification regarding how to find the unit rate, Blake's response was "Do you agree that

for every 10 minutes, it's a dollar? That's how I got it [10 cents per minute]." Thus,

Blake's actions did not specify how he wanted his students to see or conceive of a unit

rate in the situation. It is possible that the actions observed and reported here rendered

unit rate a simple calculation for students wherein one divides one number (e.g., 1) by

another number (e.g., 10).

In sum, Blake said that he wanted his students to see a relationship between two

quantities, specifically, a rate at which the quantities change in relation to one another.

However, his actions during the group activity focused on two separate and

uncoordinated sequences of numbers, which previous research has demonstrated does not

constitute a ratio or rate for students (Lobato et al., 2003). Furthermore, as no

explanation or justification was offered regarding unit rates, Blake's actions left it unclear

as to how he wanted students to think about unit rate. Thus, there was an apparent

mismatch between what Blake said wanted his students to see and what his actions

focused on. Consequently, it seems more likely that Blake's actions served the

generalization of an interpretation of slope as two separate and uncoordinated sequences

(and the differences contained therein) than an interpretation of slope as rate.

Function

In this section, I draw upon student data from the cell-phone activity to illustrate the way in which Blake's Category 2 and 5 beliefs functioned to support students in generalizing their learning. As this portion of the lesson involved students' verbalizations regarding the mathematics associated with a group activity involving the real-world context of cell-phone charges (e.g., "I came up with $50 for 0 minutes," "50.10," and "It goes up by 5"), it appeared that Blake's Category 5 belief functioned to support students in making use of mathematical lens to see mathematics (e.g., numbers and patterns of differences) in real-world situations.

Student data suggested that Blake's Category 2 belief was unsuccessful in supporting the generalization of a mathematically valid interpretation of the meaning of slope. Rather, student verbalizations lend support to the claim that rather than seeing rate in the tables shown in Figure 6.6 and Figure 6.8a, students came to see patterns of numerical differences. For instance, student comments like "it's going by 5s," "it's going by 50s," and "it's going by 1s" suggested students saw individual patterns of differences in the table. These verbalizations coupled with student questions like "I'm confused; I don't like 51; why did you put 51?" and "How did you get 10 cents per minute?" further indicated that students did not interpret the meaning of slope as the rate at which the two quantities represented in the table change together.

Discussion

Two beliefs regarding the generalization of students' learning manifested in Blake's lesson, only one of which was identified by analysis of the Phase I interview data presented in Chapter 4 (i.e., Blake's Category 5 belief that students generalize their

learning if they develop the view that mathematics is relevant and useful outside of the

mathematics classroom). Analysis of Blake's lesson and subsequent debriefing showed

that for Blake, the Category 5 belief seemed to mean something in particular, namely that

students will be able to generalize their learning to novel situations if they view the use of

a mathematical lens through which one sees the real world as important and/or necessary.

Interestingly, a new content belief manifested for Blake during his lesson, namely the

Category 2 belief that students will generalize their learning if they develop

mathematically valid interpretations of the meaning of mathematical topics like slope.

Specifically, Blake appeared to believe that students will be able to generalize their

learning of slope if they interpret slope as the rate at which two quantities change in

relation to one another.

In this chapter, I showed that while Blake was able to successfully implement his

Category 5 belief, he was unable to successfully implement his Category 2 belief. Blake

selected and illustrated a wide variety of contexts (via PowerPoint slides, verbal

descriptions, and an audio file) to his students to demonstrate that a mathematical lens

can be used to see mathematics in the real world. He even designed and implemented an

activity centered around a particular context that asked students to engage in the

mathematization of that context. However, when Blake dove into the particulars of those

contexts and into the solving of the activity, he was unable to manifest the particular

mathematics he intended. During the PowerPoint presentation, Blake apparently

intended to emphasize attributes of real-world situations associated with constant rates of

change (such as the steepness of a rooftop) but additionally (and likely unintentionally)

focused on a variety of attributes including work, mechanical advantage, acceleration,

and average steepness or steepness as a point for an object whose steepness changes. During the cell-phone activity, Blake apparently intended to emphasize unit rate, but instead focused on two uncoordinated sequences of numbers in the table of data (and corresponding patterns of differences), which research indicates does not support the development of an interpretation of slope as rate for students (Lobato et al., 2003).

By drawing upon the evidence presented in Chapter 5, I investigate possible relationships between Blake's mathematical knowledge for teaching and the mismatches between his intent to emphasize slope as a rate and his pedagogical actions. In Chapter 5, I claimed Blake demonstrated strong evidence of ratio reasoning. This meant Blake appeared able to (a) coordinate and join two quantities to form a unit composed of two quantities, (b) iterate or partition a unit composed of two quantities to form an equivalent unit, (c) reason with a set of equivalent ratios, and (d) interpret the meaning of ratio in a variety of situations. Furthermore, I showed that Blake rejected the notion that slope be defined in terms of the steepness of a line and instead preferred a definition of slope that involved the rate at which two quantities change in relation to one another. These analyses of Blake's interview (presented in Chapter 5) suggested Blake had some understanding of slope as ratio. However, the lesson presented in this chapter suggests that either he was unable to implement his personal understanding of slope as a ratio in the lesson or that there are some problematic aspects of his personal understanding of slope, which came to light in the lesson (but not in the interview tasks), namely his understanding of ratio-as-measure (Simon & Blume, 1994).

Moreover, Blake's limited understandings of (a) how *students* come to develop an understanding of slope and (b) the actions he could take to support students' development

of an understanding of slope, provided insight into the pedagogical actions he took during the cell-phone activity. Recall from Chapter 5 that despite the fact Blake seemed to interpret slope as a ratio, he did not consistently emphasize the development of students' quantitative meanings and suggested teacher actions that could support a numerical interpretation of slope (e.g., accepting students' identifications of numerical patterns, without asking for a quantitative explanations). The evidence presented in Chapter 5 thus point to a possible reason for Blake's pedagogical actions during the cell-phone activity.

In particular, Blake's lack of awareness of the potential roadblocks students encounter when developing an understanding of ratio (as evidenced during his interviews and presented in Chapter 5) is consistent with his pedagogical actions during the cell-phone activity. Research has indicated that it is difficult for students to look at a table of data and coordinate the values of the two quantities represented in the table in a way that preserves a multiplicative relationship and that it is common for students to instead engage in univariate (recursive) reasoning or additive reasoning (Lobato et al., 2003). Blake played into this by pointing, in turn, to columns of data in a table and asking questions like "What is this going by?" In this way, Blake did not focus on the coordination of the values represented both columns of the table. Rather he treated the columns as two separate and uncoordinated sequences of numbers and focused on the pattern of differences contained therein (Lobato et al., 2003).

Similarly, Blake's other goal was to support students in finding a unit rate (e.g., $0.10 per minute). To accomplish this, Blake told the students to work in groups and then took a poll regarding the number of students who arrived at a particular answer. He did not ask for explanations or justifications nor did he provide any. Accepting a

numerical answer alone has been linked with the development of procedural skills rather than conceptual understanding (e.g., Kazemi & Stipek, 2001). Furthermore, studies have shown how asking students to explain why two pairs of quantities represent the same attribute (e.g., why $5 for 50 minutes is the same cell-phone rate as $0.10 for 1 min) can promote the formation of ratio (Lobato & Ellis, 2010; Lobato et al., in press; Lobato & Siebert, 2002). In this way, Blake's unawareness of pedagogical actions to support students in the formation of slope as a ratio (e.g., asking them to explain why two pairs of quantities form equivalent ratios), as demonstrated in his interview and presented in Chapter 5, manifested in the pedagogical actions he enacted during in his lesson. In other words, Blake's mathematical knowledge for teaching (MKT) seemed to have affect on the way in which his Category 2 belief regarding the generalization of students' learning was enacted in his classroom.

CHAPTER 7:

Conclusion

In this chapter, I present a summary of the findings of this dissertation study. The findings are organized around the 3 research questions this study sought to answer and are presented in turn. Then I discuss the significance of this study, methodological considerations, and constraints, challenges, and future research directions.

Summary of Findings

Research Question 1

One goal of this dissertation study was to answer the following two-part research question: What are teachers' beliefs regarding (a) the generalization of students' learning and (b) how to support the generalization of students' learning? In answering this question, 5 categories (fitting into 3 supercategories) of teachers' beliefs regarding the generalization of students' learning were identified. A summary of these 5 categories of beliefs is provided below. In addition, 2-3 beliefs regarding how to support the generalization of students' learning were identified for each of these 5 categories of beliefs. (A summary of the 13 corresponding categories of beliefs regarding how to support the generalization of students' learning can be found in Table 4.1.)

The first two categories of teachers' beliefs regarding the generalization of students' learning involved the role of mathematical content. In particular, 3 of the 8 practicing teachers who participated in this study seemed to believe that students productively generalize their learning to novel situations if those novel situations prompt students to make use of a learned association, procedure, or formula (Category 1). Another 4 participants seemed to believe that students productively generalize their

learning to novel situations if they develop mathematically valid interpretations of the meaning of mathematical topics like slope, for example, slope is a ratio which provides a description of the multiplicative relationship between two quantities (Category 2). These content beliefs seemed specific in nature in the sense that as a mechanism for supporting the generalization of students' learning, mathematical content (e.g., a mathematically valid meaning or a particular association) appeared to function in a particular way in teachers' beliefs. For example, one teacher seemed to believe that if a novel problem prompted a learned association (e.g., a link between a graphical image of positive slope and the word "away") then a student would generalize her learning by making use of that specific association.

Two other categories of teachers' beliefs regarding the generalization of students' learning involved the role of students' affect. In particular, 6 of the 8 practicing teachers who participated in this study appeared to believe that the productive generalization of students' learning is dependent upon students' confidence in their ability to engage in mathematical activity (Category 4). Two of these 6 participants plus one other participant also believed that the productive generalization of students' learning is dependent upon a view of mathematics as relevant and useful outside of the mathematics classroom (Category 5). These beliefs involving student affect seemed less specific in nature than the content beliefs in the sense that as a mechanism for supporting the generalization of students' learning, the way in which students' affect appeared to function in each teacher's belief was vague. For example, it could have been that students' confidence, for instance, acted as a key unlocking a door to students' engagement with novel tasks thereby allowing students' to make use of topic-specific

learning or it could have been that students' confidence acted at a more general level allowing students to engage with novel situations that transcend students' learning of a particular topic.

Finally, 3 of the 8 practicing-teacher participants emphasized the role of disposition in their beliefs regarding the generalization of students' learning (Category 3). In other words, they seemed to believe that the generalization of students' learning is dependent upon students' personal orientation towards or outlook on problem solving.

Research Question 2

A second goal of this dissertation study was to answer the following research question: What is the relationship between teachers' beliefs regarding the generalization of students' learning and teachers' mathematical knowledge for teaching (MKT; as defined by Silverman and Thompson (2008))[15]? In answering the second research question, 3 categories of the first component of MKT—teachers' personal understanding of the topic of slope—were identified: (a) no evidence of ratio reasoning, along with the conception of slope as steepness; (b) limited evidence of ratio reasoning that is disconnected from slope, along with the conception of slope as steepness; and (c) strong evidence of ratio reasoning connected to slope, along with steepness emphasized only secondarily or not at all. With respect to the seven teachers in Categories 1 and 2 in the previous section, there was alignment (with the exception of one teacher) between teachers' personal understanding of slope and their beliefs about the generalization of students' learning. Teachers who exhibited little or no ratio reasoning and appeared to understand slope in terms of steepness were the teachers who appeared to believe that

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[15] A summary of the findings related to this research question can be found in Table 5.1.

productive generalizing is supported by learned associations, procedures, or formulas (Category 1). Teachers who exhibited strong ratio reasoning and further seemed to have that reasoning connected to the mathematical topic of slope were the teachers who appeared to believe that productive generalizing is supported by mathematically valid interpretations of the meanings of topics like slope (Category 2). The exception was Blake who demonstrated strong evidence of ratio reasoning and further seemed to have that reasoning connected to slope but held the Category 1 belief regarding the generalization of students' learning. Thus, having a personal understanding of slope as ratio, with a de-emphasis on slope as steepness seemed to be a necessary but not sufficient condition for membership in Category 2 regarding the generalization of students' learning.

Examination of the second two components of MKT—teachers' understanding of the ways in which students come to develop meaning for slope and teachers' understanding of the actions that could be taken in support of the development of students' slope understanding—revealed some interesting differences between the teachers from Category 2 regarding the generalization of students' learning and Blake. When the topic of conversation shifted from teachers' personal understandings of slope to students' understandings of slope, all of the teachers from Category 2 *consistently* emphasized the development of quantitative interpretations of the meaning of slope while Blake did not; rather, it seemed the shift in conversation served to illuminate another interpretation of slope that Blake found important for *students*, namely a numeric interpretation. Moreover, the teachers from Category 2 seemed to have a richer understanding of ways in which they might support students in developing a quantitative

interpretation of slope as they offered a long and varied list of instructional actions in support of such development. Blake, on the other hand, seemed to have a limited understanding of how he could support students in developing a quantitative interpretation of the meaning of slope as he only offered one pedagogical action in clear support of students' quantitative reasoning and other actions that could be seen as supporting students' numerical reasoning (e.g., accepting students' identification of numerical patterns and accepting students' calculational answers).

Thus, examining the second two components of MKT led to possible explanations for why a teacher who seemed to personally interpret the meaning of slope as a ratio would hold a belief about the generalization of students' learning that involved a formula for slope. It could have been that Blake believed the generalization of quantitative interpretations of the meaning of mathematical topics like slope is important, but was unaware that an emphasis on numbers and numerical patterns could work against an emphasis on quantitative interpretations since Blake was himself more than capable of reasoning about numbers quantitatively. Alternately, it could have been that Blake viewed the quantitative interpretation of the meaning of slope as a given since students encounter multiple examples of slope as rate in their daily lives; as Blake said, "I think they [students] all have a preliminary understanding of rate; anyone can conceptualize walking at different rates or arriving at the same place at different times, [for example, in] relay races;" in this way, Blake may have believed it more important to focus on the generalization of methods for finding or calculating slope than on a quantitative interpretation of the meaning of slope.

Research Question 3

The third goal of this dissertation study was to answer the following research question: What is the relationship between teachers' beliefs regarding the generalization of students' learning and teachers' classroom practices? In answering this question, I observed three teachers as they taught lessons with the specific intention of supporting students in generalizing their learning of slope. In this dissertation, I presented findings from analysis of the observation of Blake's lesson and subsequent debriefing (rationale for the selection of this teacher is given in Chapter 6).

Analysis of the first research question indicated that Blake held three beliefs regarding the generalization of students' learning: students will productively generalize their learning to a novel situation (a) if the novel situation prompts them to make use of a learned association, procedure, or formula (Category 1); (b) if they develop confidence in their abilities to engage in mathematical activity (Category 4); and (c) if they develop the view that mathematics is relevant and useful outside of the mathematics classroom (Category 5). Only one of these beliefs, namely the Category 5 belief, manifested in Blake's lesson. Interestingly and surprisingly, another belief regarding the generalization of students' learning appeared during Blake's lesson that was not captured by the Phase I interview data analyzed and presented in Chapter 4. This belief was indeed a content belief, but it was not the Category 1 content belief. Rather, it was the Category 2 content belief regarding mathematically valid interpretations of the meaning of mathematical topics. In particular, analysis of Blake's lesson and subsequent debriefing indicated that he believed students would be able to generalize their learning to novel situations if they developed an interpretation of the meaning of slope as rate.

Analysis of Blake's lesson and subsequent debriefing provided further elaboration of the meanings the Category 2 and 5 beliefs held for Blake. The Category 2 belief involving mathematically valid interpretations of the meanings of mathematical topics seemed to mean (with respect to the topic of slope): students will be able to productively generalize their learning of slope if they develop an interpretation of slope as rate, and even more specifically an interpretation of slope as the rate at which two quantities change in relation to one another. The Category 5 belief involving a view of mathematics as relevant and useful outside of the mathematics classroom seemed to mean: students will be able to productively generalize their learning if they view the use of a mathematical lens through which one sees the real world as important and/or necessary.

I also showed that while Blake was able to successfully implement his Category 5 belief, he was unable to successfully implement his Category 2 belief. Blake selected and illustrated a wide variety of real-world contexts (via PowerPoint slides, verbal descriptions, and an audio file) and demonstrated how mathematics could be found in those contexts. Blake even created a group activity to engage students in mathematizing a specific real-world context. His verbalizations and pedagogical actions communicated the meaning that it was important to use a mathematical lens to see mathematics in real-world situations. Students' verbalizations seemed to confirm that they were indeed able to make use of a mathematical lens to see mathematics in various real-world contexts.

However, when Blake dove into the particulars of the real-world contexts he chose and into the solving of the activity, he was unable to manifest the particular mathematics he intended. During the PowerPoint presentation, Blake apparently intended to emphasize attributes of real-world situations associated with constant rates of

change (such as the steepness of a rooftop) but additionally (and likely unintentionally) focused on a variety of attributes including work, mechanical advantage, acceleration, and average steepness or steepness as a point for an object whose steepness changes. During the cell-phone activity, Blake apparently intended to emphasize unit rate, but instead focused on two uncoordinated sequences of numbers in the table of data (and associated patterns of differences), which research indicates does not support the development of an interpretation of slope as rate for students (Lobato et al., 2003). Moreover, students' verbalizations seemed to lend support to the claim that students were not necessarily supported in generalizing the particular interpretation of slope that Blake had intended.

Significance

The findings reported in this study show that teachers can and do engage in thinking about transfer. In fact, this dissertation study highlighted the importance of bringing practicing teachers into the ongoing conversation about transfer. By engaging teachers in conversations using artifacts from their own teaching and asking questions about transfer using the terminology of "generalization of students' learning," new beliefs regarding the generalization of students' learning were identified. In other words, talking to teachers led to the identification of beliefs not found in the transfer literature. As noted above, the role of students' affect was present in all but one of the practicing-teacher participants' beliefs regarding the generalization of students' learning despite the fact that it is absent in the transfer literature. This finding indicates that while transfer researchers have not identified affect as playing an important role in the transfer of students' learning experiences, teachers have. Moreover, the practicing-teacher participants offered new

ideas regarding *how to support* the generalization of students' learning that were consistent with extant beliefs regarding the generalization of students' learning offered by transfer researchers (e.g., the actor-oriented approach to transfer). That is, in talking to teachers, researchers operating within, for example, an actor-oriented approach to transfer were imparted with new ideas regarding how to support transfer. In addition, in talking to teachers, an extant belief regarding the generalization of students' learning (i.e., the dispositional approach to transfer) was extended into the domain of mathematics.

This is not to say that there was no overlap between teachers' beliefs and transfer-researchers' beliefs regarding the generalization of students' learning. For example, one teacher seemed to believe in the Thorndikean idea of transfer being mediated by common associations. However, the pedagogical actions this teacher offered in support of the generalization of students' learning seemed to entail practice-based differences. For instance, this teacher believed that drawing upon multiple reform-oriented and constructivist-inspired activities would support the generalization of students' learning. In this way, her instructional beliefs suggested practice-based decision-making that transcends a Thorndikean approach to curricula.

Interestingly, the fact that all of the teachers but one were in multiple categories of beliefs suggests that *in practice* multiple beliefs regarding the generalization of students' learning may function together. Together, these findings suggest that research on transfer could benefit from a new approach—investigating the phenomenon through the eyes of practicing teachers.

Findings for the second and third research questions point to the importance of teachers' understanding of *how students come to develop* conceptually meaningful

interpretations of mathematical topics (i.e., the second component of MKT) as this understanding may support teachers in being able to productively implement their content beliefs regarding the generalization of students' learning. Such findings could provide insight regarding teachers' professional development. As was shown in this dissertation study, a personal interpretation of the meaning of slope as rate without a strong understanding of how students come to develop that interpretation or the roadblocks students typically encounter in that development hindered a teacher from being able to implement his belief that the productive generalization of students' learning is dependent upon the mathematically valid interpretation of slope as rate. This suggests that professional development that does not focus on the development of teachers' understanding of developmental trajectories associated with the learning of various mathematical topics may fail to support teachers in the classroom even if they are successful in developing productive teacher beliefs and teachers' personal understandings of the mathematics.

Methodological Considerations

The teaching artifacts and inferred tasks were crucial in generating data regarding how aspects of a teacher's beliefs regarding the generalization of students' learning functioned as a mechanism to support transfer. For example, some teachers brought teaching artifacts that included, what transfer researchers would call, learning tasks. In discussing the ways in which teachers believed such tasks (and accompanying instruction) supported the generalization of students' learning, it started to become clear *what* mathematics teachers believed facilitated productive generalization from learning tasks to novel, future situations (e.g., the Category 2 teachers focused on particular

interpretations of the meaning of slope such as a ratio describing the relationship between two quantities). The subsequent inferred tasks then helped to explore those beliefs in greater detail. For instance, when the researcher provided teachers with a novel, transfer task involving slope and asked them if they believed their learning task (and accompanying instruction) would support students in being able to generalize their learning to *that* transfer task, certain features of teachers' beliefs were highlighted (e.g., the Category 2 teachers seemed to believe that supporting the generalization of learning involves an examination of ways in which *students* come to interpret the meaning of slope during initial learning). In this way, teachers' artifacts combined with the inferred tasks led to interesting conclusions regarding teachers' beliefs. For example, the Category 2 teachers seemed to be actor (i.e., learner) oriented; however, I was unable to attribute an actor-oriented transfer perspective to these teachers. To do so, it would have been helpful to include a hypothetical student task in which a student generalizes their learning but has an element of mathematical incorrectness in the solution and ask teachers if they count this as generalization of students' learning or not.

Moreover, I was not expecting students' affect to appear in teachers' beliefs about the generalization of students' learning. Thus, these categories of belief relied heavily on tasks designed to get at teachers' espoused beliefs. It was, therefore, unclear how aspects of these beliefs functioned to support the generalization of students' learning. Future studies would benefit from tasks designed to generate data that would allow more detailed inferences to be made about these beliefs. For example, in addition to including hypothetical student solutions in the inferred tasks, it would have been helpful to see whether and how information regarding the hypothetical students themselves (e.g., their

confidence in their mathematical abilities, view of mathematics, and view of learning) would have contributed to teachers' predictions and explanations regarding the nature of the tasks these students would be able to generalize their learning to.

Constraints, Challenges, and Future Research

This dissertation study brought together three related themes: teachers' beliefs about the generalization of students' learning and how to support it, teachers' mathematical knowledge for teaching, and teachers' practices related to the generalization of students' learning. As a tradeoff to this breadth, I was unable to go in depth with any one teacher to examine how his or her beliefs about the generalization of students' learning might evolve over time. Thus, a future study could examine the evolution of a small group (e.g., 5-8) of teachers' beliefs about the generalization of students' learning and how to support it instructionally. Participants in this study could be practicing teachers with access to classroom environments where their beliefs could be implemented and examined both by the teacher (in several clinical interviews spread out over a significant period of time) and by the group of teacher participants (in several focus-group discussions spread out over the same period of time). Focus-group discussions about teachers' ideas regarding the specific tasks, discussion topics, small-group activities, and so on that contribute to the generalization of their students' learning could provide a means by which teachers' beliefs become elaborated and reshaped.

Furthermore, to support the level of detail of such discussions, participants for this study could be selected on the basis of the mathematical topics they teach. As the present study was constrained to the generalization of students' learning experiences with a particular topic (i.e., slope), selecting teachers who have opportunities to teach a different

topic could serve as a way to validate the categorization of beliefs about the generalization of students' learning presented here in Chapter 4.

In this dissertation study, I observed 3 teachers as they implemented a lesson that was created or adapted to support their students in generalizing their learning. A limitation of my research design was that data regarding how these teachers' lessons actually supported their students in generalizing their learning were limited. Thus, it would have been difficult to establish strong linkages between the ways in which teachers' lessons and associated pedagogical actions supported their students' in generalizing their learning. A future study could investigate the relationship between teachers' beliefs about the generalization of learning as they manifest in the teaching of several lessons and the nature of their students' transfer of learning experiences. Such a study could involve pre-instruction clinical interviews wherein teachers discuss their goals (with respect to the generalization of students' learning) for a 10-15 hour unit on a particular mathematical topic as well as classroom observations of the unit coupled with short follow-up stimulated-recall interviews to get at how teachers view particular actions taken during instruction in relation to the generalization of students' learning. Analysis of these data would then be coordinated with data collected from post-instruction semi-structured transfer interviews with the teachers' students who were present for and engaged in the 10-15 hour instructional unit. These student interviews would be designed to get at the ways in which students actually transferred their learning experiences. In other words, this study could speak to the relationship among teachers' beliefs about how to support the generalization of students' learning as enacted in their classroom practices and their students' subsequent reasoning during transfer tasks.

Lastly, as the present study involved a two-phase design (an interview phase followed by a classroom observation phase), I was unable to investigate the ways in which the two phases might interact over time. Thus, it would be interesting to bring the two studies outlined above together to see what would happen to the evolution of teachers' beliefs about the generalization of students' learning and how to support it (first study) when they are shown data collected during student interviews (second study). If, for a moment, we assume that the teacher participants are the same across the two studies, what would happen to these teachers' beliefs as a result of seeing how their students actually ended up generalizing their learning experiences (after the 10-15 hour instructional unit).

Appendix A

The Teaching Item Activity: Helping Students to Generalize Their Understanding of Slope to Future Situations

I am interested in studying the ways in which teachers think about supporting their students in being able to *generalize* or apply their learning to new problems or situations. Throughout the school year you teach many mathematical topics where you might expect your students to be able to generalize their knowledge in order to successfully engage with new and novel problems. But in the subsequent activities and discussions I'd like to narrow the focus to the topic of slope and linear functions so that we can talk about specific examples instead of hypothetical situations. In the context of slope and linear functions, I am interested in finding out how you think about the activities of teaching (e.g., structuring a lesson, sequencing a sequence of lessons, assigning homework, creating tests, or organizing discussions) in relation to the idea that students should be able to generalize their learning to successfully engage in new tasks and/or activities as a consequence of the learning that emerges and develops during the unit on slope and linear functions.

I know that there are many considerations to be made when designing and carrying out a lesson (e.g., timing, standards, testing, or student motivation). Thus, the idea of structuring class activities so that students are enabled to successfully engage in novel situations may not always be an explicit goal of instruction. This is OK. I am interested in the times in teachers' teaching lives when they *do* think about helping students to be able to generalize their learning to new tasks and/or activities.

Therefore, I am asking you to do the following:

1) Reflect on the last time you taught a unit on slope and linear functions.
2) Go through your teaching materials for this unit on slope and linear functions.
3) Choose something (e.g., a lesson plan, a homework assignment, a test, notes about how to run a discussion) that you think shows an instance in which you were thinking about *supporting* your students so that they would able to generalize their understanding of slope to a new task, activity, or situation.
4) Bring the artifact that you chose to the first interview. We will discuss it.

Question Set 1: Questions associated with The Teaching Item Activity

1. Tell me a little bit about the artifact you chose and how you think it shows that you were thinking about helping students to make future use of their learning. (…helping students to generalize their learning to new situations).

2. What in particular were you doing to help your students so that they would be enabled to successfully engage with new tasks, activities, or situations?

3. What else do you do during your unit on slope and linear functions that helps enable students to successfully engage with new tasks, activities, or situations?

4. Of all the actions you identified above, which actions do you think are the most important in helping students to generalize or make future use of their learning?

5. Do you think that as a consequence of your students' engagement with your teaching item and all that happens during that engagement (e.g., discussions, actions that the teacher named in #2, 3, and 4), your students would be enabled to successfully engage with The Water Pump Task[16]? Why or why not? How do you think they would approach the task and why (i.e., how does the action of ___ prepare them for this task)?
   a. If a teacher says no, ask the teacher what he/she could do to support students' engagement with that task.
   b. If a teacher says yes, ask the teacher to explain what in particular about the teaching item and the teaching actions taken in association with that item he/she thinks prepared students for The Water Pump Task.
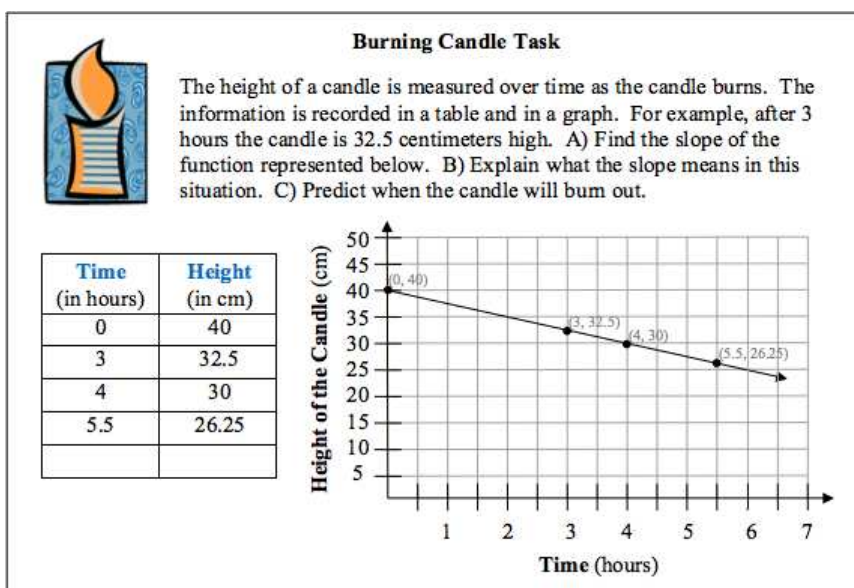
Water Pump Task

Water is being pumped through a hose into a large swimming pool. The table below shows the total amount of water in the pool to begin with as well as the total amount of water in the pool after 3, 5, and 9 minutes. The amount of water is measured in gallons. The time is measured in minutes.

Time (minutes)	Amount of Water (gallons)
0	0
3	4
5	$6\frac{2}{3}$
9	12

Does the data above represent a water pump that is pumping equally fast over time? If so, find the rate at which the pump is pumping. If not, explain what is happening (i.e., the pump is speeding up; the pump is slowing down; the pump is speeding up and then slowing down).

[16] The Water Pump Task was developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.
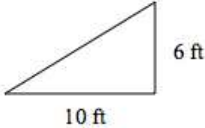
6. Do you think that as a consequence of your students' engagement with your teaching item and all that happens during that engagement (e.g., discussions, actions that the teacher named in #2, 3, and 4), your students would be enabled to successfully engage with The Burning Candle Task[17]? Why or why not? How do you think they would approach the task and why (i.e., how does the action of ___ prepare them for this task)?
   a. If a teacher says no, ask the teacher what he/she could do to support students' engagement with that task.
   b. If a teacher says yes, ask the teacher to explain what in particular about the teaching item and the teaching actions taken in association with that item he/she thinks prepared students for The Burning Candle Task.

Burning Candle Task

The height of a candle is measured over time as the candle burns. The information is recorded in a table and in a graph. For example, after 3 hours the candle is 32.5 centimeters high. A) Find the slope of the function represented below. B) Explain what the slope means in this situation. C) Predict when the candle will burn out.

Time (in hours)	Height (in cm)
0	40
3	32.5
4	30
5.5	26.25

7. Do you think that as a consequence of your students' engagement with your teaching item and all that happens during that engagement (e.g., discussions, actions that the teacher named in #2, 3, and 4), your students would be enabled to successfully engage with The Ice Cream Task[18]? Why or why not? How do you think they would approach the task and why (i.e., how does the action of ___ prepare them for this task)?
   a. If a teacher says no, ask the teacher what he/she could do to support students' engagement with that task.
   b. If a teacher says yes, ask the teacher to explain what in particular about the teaching item and the teaching actions taken in association with that item he/she thinks prepared students for The Ice Cream Task.

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[17] The Burning Candle Task was developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.
[18] The Ice Cream Task was adapted from a task developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.

Ice Cream Task

The ratio $\frac{3}{5}$ provides a measure of the *steepness* or *slantiness* of the hill below.

6 ft

10 ft

What does $\frac{2}{5}$ provide a measure of in the following situation:

Chocolate ice cream is made in the ratio of 2 parts chocolate to 5 parts cream.

8. Could you create a lesson that implements some of the ideas you have discussed here about supporting students to generalize their understanding of slope? Could you create a lesson that includes a task in it that students haven't been taught directly how to do?

Hypothetical Students' Work[19]
<u>**Molly's Work on a Slope Task**</u>

Suppose you gave one of your students, Molly, the following problem on slope. What do you find noteworthy about Molly's answer and associated explanation (Molly created the graph below as part of her explanation)?

Slope Problem: *A leaky bucket drips water into a measuring cup. The water level in the measuring cup is recorded every minute. The table below shows the water level in the cup over time. The cup started out with some water in it before the dripping began. Plot these points on a graph and then find the slope of the line.*

Elapsed Time (minutes)	Amount of Water in the Cup (ounces)
0	4
3	6
6	8
12	12

<u>Molly's Answer:</u> The slope is 2/3.

<u>Molly's Explanation (and Graph):</u> First you graph the points and then connect them to make a line. Now I pick two points (3,6) and (6,8). Three minutes passed between 3 minutes and 6 minutes. During that time, the water rose in the cup 2 oz. Two oz in 3 minutes is the same as 2/3 oz in 1 minute. So the slope is 2/3, which is how much the water rises every minute.

Water Level in Ounces

Time in Minutes
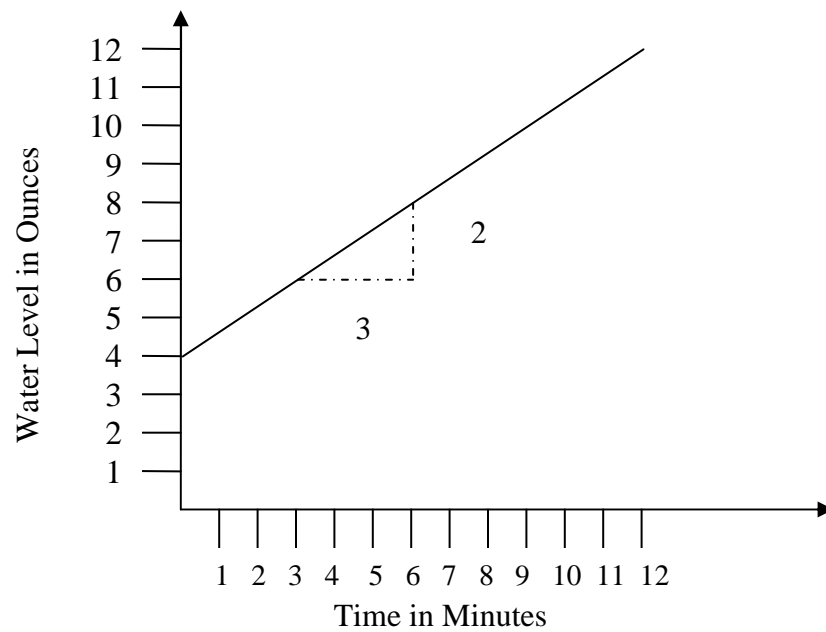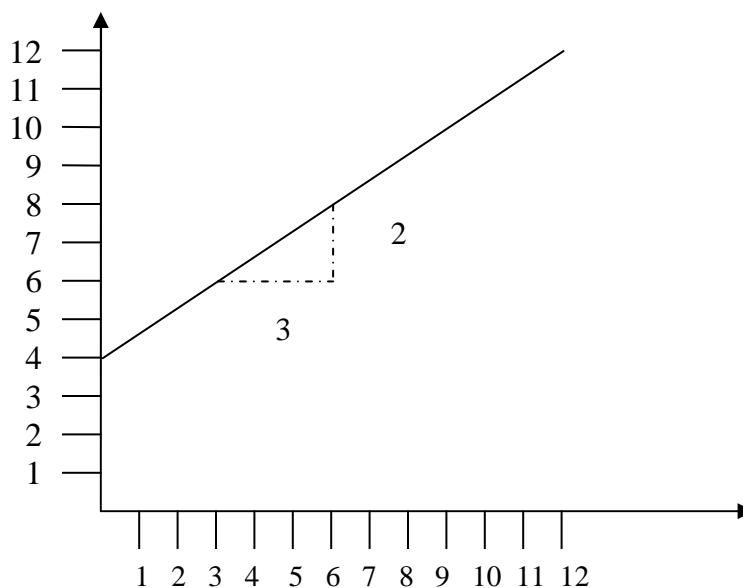
Lucy's Work on a Slope Task

Suppose you gave Lucy, another one of your students, the same slope problem. What do you find noteworthy about Lucy's answer and associated explanation (Lucy created the graph below as part of her explanation)?

Slope Problem: *A leaky bucket drips water into a measuring cup. The water level in the measuring cup is recorded every minute. The table below shows the water level in the cup over time. The cup started out with some water in it before the dripping began. Plot these points on a graph and then find the slope of the line.*

Elapsed Time (minutes)	Amount of Water in the Cup (ounces)
0	4
3	6
6	8
12	12

Lucy's Answer: The slope is 2/3.

Lucy's Explanation (and Graph): I plotted the points and drew a line. Then chose two points and drew a triangle. I need to find the rise and run. I found the rise by subtracting 8 - 6. Then I found the run by subtracting 6 - 3. So, the rise is 2 and the run is 3. That makes the slope 2 over 3.

Question Set 2: Questions associated with hypothetical students' work

The teacher will be shown two students' work on a slope task – Molly and Lucy. Molly provided a correct answer for the slope question. In addition, her explanation demonstrated an understanding of slope as ratio. Lucy also provided a correct answer for the slope question; however, her explanation did not illustrate an understanding of slope as ratio. Rather her explanation focused on a procedure and was not connected to the context of the leaky bucket. Give the teacher a moment to look over Molly's work and then ask questions 1-4 below. When finished, repeat the process with Lucy's work:

1. What do you find noteworthy about Molly's answer and associated explanation? Why?
2. Given Molly's response, what kinds of activities do you think Molly will be able to successfully engage with? Why? Ask teachers *what* they think Molly is generalizing from the Leaky Bucket Task to whatever tasks they bring up.
   - If the teacher only mentions tasks that involve tables and graphs or the students' ability to translate given information into a table or a graph, ask the teacher whether he or she thinks the student would still be successful if she was not able to (e.g., via time constraints) construct an associated graph or table or ask the teacher if there are other kinds of tasks that the students would be able to successfully engage with.
3. Given Molly's response, what kinds of activities do you think will be too difficult for Molly? Why?
   - If the teachers only mention tasks involving negative slope, ask whether and how their answers would change if I could provide similar work from Molly and Lucy showing their successful engagement with tasks involving negative slope.
   - If the teacher thinks that Molly will be successful interacting with all 3 tasks, as the teacher what he or she thinks Molly's teacher did to help support her in being able to generalize her learning?
4. What could you do with Molly during The Leaky Bucket Task to support her engagement with the more difficult tasks? Be specific. What specific activities, examples, etc. would you use?

Now present the teacher with the three tasks (i.e., The Line Task, The Fish Task[20], and The Orange Juice Task[21]) below and ask the following questions:
5. Which task(s) do you think Molly will be able to complete successfully? Why? What about Lucy?
6. Which task(s) do you think will be more difficult for Molly? Why? What about Lucy?

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[20] The Fish Task was adapted from a task developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.
[21] The Orange Juice Task was adapted from a task developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.

7. What could you do as a teacher to support Molly's engagement with the more difficult tasks? What about to support Lucy?
8. Suppose that Molly and Lucy had different teachers. What do you think each teacher did to support Molly and Lucy in being able to generalize their understanding of slope?

After the above set of questions have been asked with respect to each student's work, ask the following question:

1. Which student's work do you prefer? Why?

The Line Task

Find the slope of the line that goes through the points (0, 1) and (2, 5).

The Fish Task

The fish is swimming away from home. The table shows the distance she is from home every second. How fast is the fish swimming?

Time (in seconds)	Distance from Home (in feet)
1	2
3	6
8	16
10	20
12	24

The Orange Juice Task

To mix orange juice, you mix cans of orange juice from concentrate with glasses of water. You made two different batches. Batch A is made from two cans of orange juice and five glasses of water. Batch B is made from six cans of orange juice and fifteen glasses of water. Which batch will taste more "orangey" or will they taste equally "orangey"? Explain your reasoning.
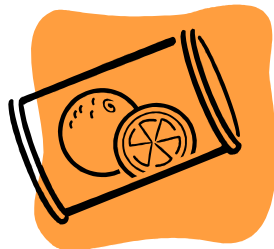
Hypothetical Teachers' Activities

Teacher A's Activity

<u>Instructions</u>: Examine 1, 2, and 3 below. Discuss any similarities and differences you notice.

1. The pitch of a roof is its slope. The roof shown below increases 6 feet vertically for every 10 feet it spans horizontally; thus, the roof has a pitch of 3/5.

2. The grade of a road is its slope written as a percent. The road shown below increases 2 meters vertically for every 16 meters it spans horizontally; thus, its grade is 12.5%.


3. The rate at which the number of cell phone users increased per year is the slope. In 2007, there were 20 million more cell phone users in the U.S. than in 2003. Thus, the rate at which the number of cell phone users increased was 5 million per year.
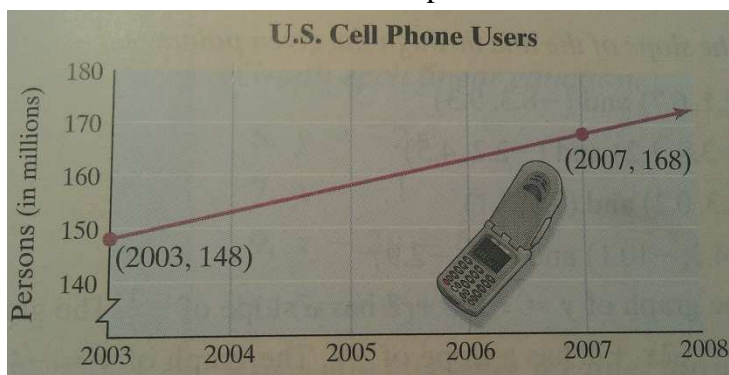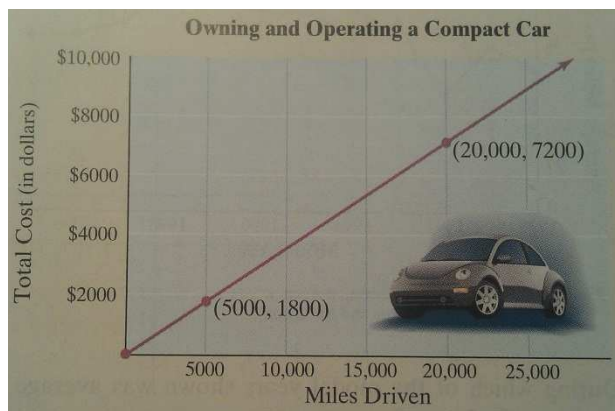
Teacher B's Activity

<u>Instructions:</u> The slope m of the line through (x_1, y_1) and (x_2, y_2) is given by the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$. For example, the slope of the line through $(0, 2)$ and $(5, 4)$ is $m = \frac{(4)-(2)}{(5)-(0)} = \frac{2}{5}$. Use this equation to complete the problems 1-3 below.
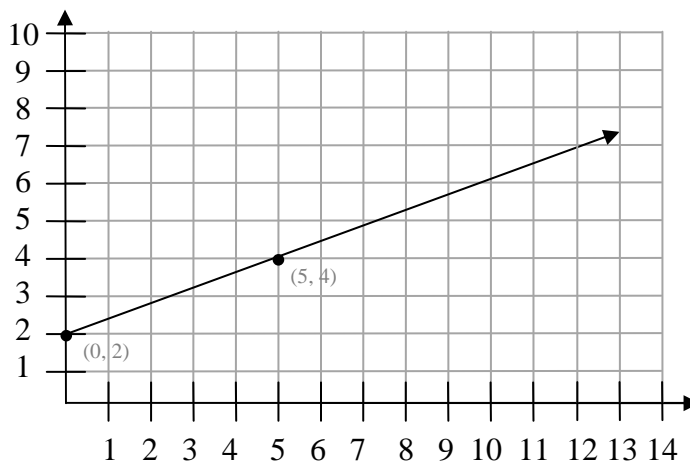
1. Plot each pair of points below and find the slope of the line that goes through them.
   a) (0, 0) and (7, 8)
   b) (-1, 5) and (6, -2)
   c) (1, 4) and (5, 3)
   d) (-4, 3) and (-2, 3)

2. Parallel lines have the same slope. Find the slope of the line parallel to the line through the following pairs of pints.
   a) (-3, -3) and (0, 0)
   b) (6, -2) and (1, 4)
   c) (-8, -4) and (3, 5)

3. The graph below shows the total cost y (in dollars) of owning and operating a compact car where x is the number of miles driven. Find the slope of the line shown in the graph.

Owning and Operating a Compact Car

(20,000, 7200)

(5000, 1800)

Total Cost (in dollars)

$10,000
$8000
$6000
$4000
$2000

5000 10,000 15,000 20,000 25,000

Miles Driven

Teacher C's Activity[22]

Cassandra decided to see how fast her brother is running a marathon. To do this, she watches the marathon on TV and periodically marks how far her brother is from where he started the race. The times and total distances from the starting line are recorded in the table below.
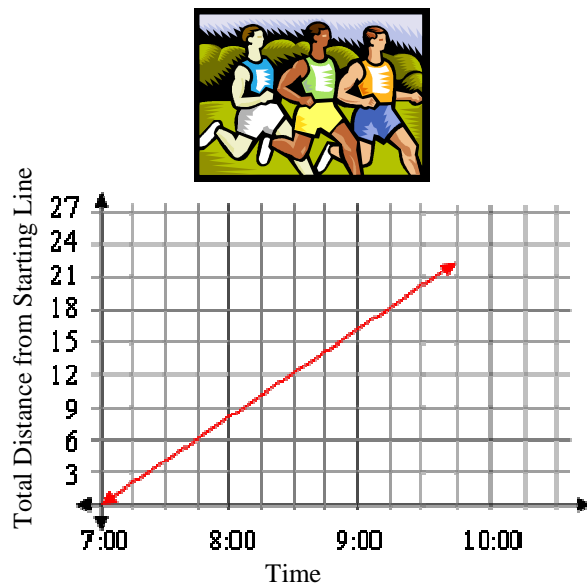
Time	Total Distance from Starting Line
7:00 a.m.	0 miles
7:15 a.m.	2 miles
7:45 a.m.	6 miles
8:15 a.m.	10 miles
9:00 a.m.	16 miles
9:30 a.m.	20 miles
9:45 a.m.	22 miles

1. Is Cassandra's brother running steadily throughout the race or is he speeding up and slowing down? Explain your reasoning?

2. How fast is Cassandra's brother running?

3. How far from the starting line do you think Cassandra's brother will be at 10: 15 a.m.?

Teacher C's Activity (Alternate Version)

Cassandra decided to see how fast her brother is running a marathon. To do this, she watches the marathon on TV and periodically marks how far her brother is from where he started the race. The times and total distances from the starting line are recorded in the table below.

Time	Total Distance from Starting Line
7:00 a.m.	0 miles
7:15 a.m.	2 miles
7:45 a.m.	6 miles
8:15 a.m.	10 miles
9:00 a.m.	16 miles
9:30 a.m.	20 miles
9:45 a.m.	22 miles

1. Is Cassandra's brother running steadily throughout the race or is he speeding up and slowing down? Explain your reasoning?

2. How fast is Cassandra's brother running?

3. How far from the starting line do you think Cassandra's brother will be at 10: 15 a.m.?

Question Set 2: Questions associated with hypothetical teachers' activities

1. Present the three hypothetical activities to teachers. Tell the teachers that these are activities developed by other teachers and ask them if they would use any of the activities for the explicit purpose of helping their students to generalize their understanding of slope.

   - If the teacher identifies an activity as one that she *would* use, ask the teacher *what* she hopes the students would get out of the lesson (e.g., An understanding of where the idea of slope might be useful? An understanding of the meaning of slope?) Ask the teacher *when* she would use the task in order to help support the generalization of students learning (e.g., Would it come at the beginning of the unit? The end?) Ask the teacher *how* she would use the lesson (e.g., Group activity? Individual activity?) Ask what *function* the activity would serve in helping students to generalize their understanding of slope.

   - If the teacher identifies an activity as one she would *not* use to support the generalization of her students' learning, ask why not? Ask how it inhibits the generalization of students' understanding of slope. Ask if there is anything the teacher could change about the task to make it so that it *could* be used to help support the generalization of students' learning.

   - If a teacher says that Teacher A's activity could be used because there are so many real world contexts being used, ask if it would still be helpful in supporting the generalization of learning if only one context was used.

Appendix C

The Lesson Plan Activity: A Lesson to Help Students Generalize their Understanding of Slope

Please create a lesson on slope for an Algebra 1 class that implements some of the ideas you discussed today regarding how to support the generalization of your students' learning. Feel free to adapt an existing lesson or to create something completely new. Whatever you choose, think of this as an opportunity to create a lesson explicitly geared towards helping your students to be able to generalize their understanding of slope to future situations. The lesson you create can be an introductory lesson or it can be a lesson that would take place after students have already been exposed to the concept of slope. The lesson can be a single-period or multi-period lesson.

Please include the following three components in your lesson plan:

1. *Mathematical Goals*. Describe what you want students to be able do and understand about slope as a result of your lesson. Include the specific mathematical ideas you would expect students to comprehend and the type of tasks you would expect them to perform. For example, if your lesson had been about coordinates and/or graphs, one of your goals might be for students to understand the meaning of the origin as a reference and be able to locate other coordinates in relation to the origin. Your list of goals for the lesson will be a very helpful resource for me to understand the mathematical ideas and procedures that you consider important.

2. *Lesson Outline*. Write a lesson plan that someone could use to teach your lesson. Include specific examples that will be used and specific problems for students to solve. If you are using any published material, please attach these to your lesson and describe which parts you will use and why and which parts you will not use and why. Be sure to mention the intended teaching methods (e.g., lecture, worked examples, students solving problems in small groups, etc.). Please provide sufficient detail for me to visualize what you would say and how you would expect your students to respond.

3. *A New Task or Activity.* After you have created your lesson, think of a task or activity that you think students will be able to generalize their learning to. In other words, think of a task that you think students will be able to successfully engage with as a consequence of your lesson and include it on a separate piece of paper. Then write a paragraph or two describing why you think your students would be able to successfully engage with the task despite the fact that they were not taught directly how to do so. The only constraint is that the task you include cannot be something that was taught directly during the lesson.

Question Set 3: Questions associated with The Lesson Plan Activity:

1. Describing the *lesson outline*. (a) Tell me a little bit about your lesson. (b) What is the most important part of the lesson? (c) What is the most important part of the lesson with respect the intended goal that you support the generalization of students' learning?

2. Preparing the lesson. (a) When designing your lesson, how did you think about the overall goal of preparing students to generalize their learning to novel tasks (i.e., tasks they have not been taught explicitly how to solve)? In other words, how did this goal shape your lesson, (for example, in a way that might be different than if you were just thinking about helping students to develop a solid understanding of slope without being concerned that they would be able to solve a new problem as a consequence of the lesson)?

3. *Mathematical goals.* (a) Tell me a little bit about the *Mathematical Goals* you described in your lesson plan? (b) If you were to rank these goals in order of importance with respect to supporting the generalization of students' learning, which one would be first on the list? Why?

4. *Preparing students.* (a) How do you think your lesson prepares students to generalize their learning (i.e., to tackle problems they have not explicitly been taught how to do)? (b) Can you identify specifically what you are doing during the lesson that is helping students to be able to generalize their learning?

5. *The task.* (a) Tell me a little bit about the task you included. (b) How do you see it as new or novel? (c) What in particular about your lesson prepares students to generalize their learning to this task? (In other words, can you identify what in particular you would be doing during the lesson that would be supporting students so that they are able to successfully engage with this particular novel task?) (d) What function or role do you see those actions playing in the generalization of your students' learning? (e) WHAT is it that you think your students would be generalizing from the lesson to the task?

Appendix D

Question Set 4: Questions about teachers' MKT

Teachers' Personal Understandings of Slope/Ratio: *
Teachers' Understanding of How Students Might Develop an Understanding of Slope: **
Teachers' Understanding of the Actions that Might Support Students in Developing an Understanding of Slope: ***

1*. (a) What is your personal understanding of slope? What does slope mean to you? (This is not asking how you normally teach slope.)
(b) Show me/draw me a picture that illustrates what you mean.

2*. What context or situation comes to mind when you think of slope?

3*. What does it mean to have a slope of ½? -1?
- If a teacher talks about going up or down for every unit walked horizontally, ask the teacher if there is anything she or he could say about slope of ½ or -1 given a flat surface?

4. Ask teachers to graphically represent a linear function and then to create a story that accompanies their representation. The pose questions (a) – (h) below.
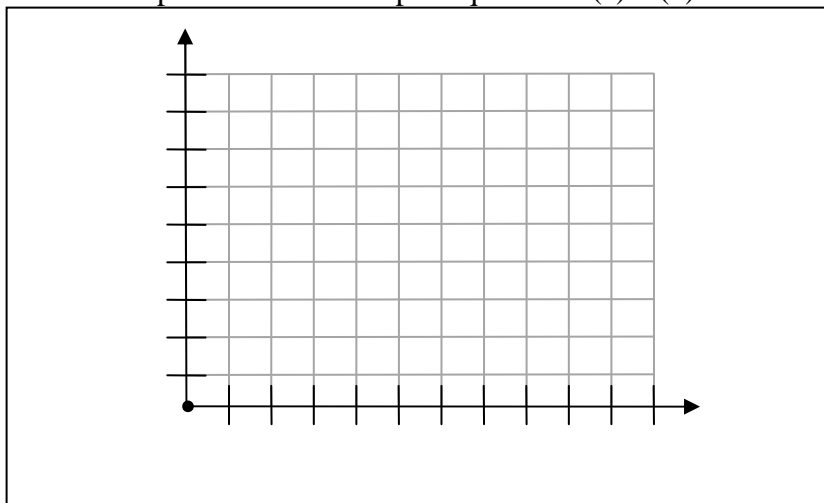

(a*) Explain the meaning of (pick two y-values represented by the graph the teacher draws) in the context of your story.
- If a teacher refers to the vertical distance between two points, ask if there is any other way that y_2-y_1 could be interpreted. Ask if the teacher can relate y_2-y_1 to anything in his or her story.

- If a teacher refers to the "rise" in the "rise over run" formula (or the y_2-y_1 in the slope formula $\frac{y_2 - y_1}{x_2 - x_1}$), ask if there is any other way y_2-y_1 could be interpreted. Ask if the teacher can relate y_2-y_1 to anything in his or her story.
- If the teacher has difficulty with this, point to the specific points on the line and ask if the teacher what it means.

(b)

(i**)　How might students come to develop that meaning for y_2-y_1?

(ii***) What would you do or say to help develop this meaning? Why?

(c*) Explain the meaning of (pick two x-values represented by the graph the teacher draws) in the context of your story.

- If a teacher refers to the horizontal distance between two points, ask if there is any other way that x_2-x_1 could be interpreted. Ask if the teacher can relate x_2-x_1 to anything in his or her story.
- If a teacher refers to the "run" in the "rise over run" formula (or the x_2-x_1 in the slope formula $\frac{y_2 - y_1}{x_2 - x_1}$), ask if there is any other way x_2-x_1 could be interpreted. Ask if the teacher can relate x_2-x_1 to anything in his or her story.

(d)

(i**)　How might students come to develop that meaning for x_2-x_1?

(ii***) What would you do or say to help develop this meaning? Why?

(e*) Explain what the *division* in the slope formula accomplishes in the context of your story. Draw a picture (not a graph) to go along with your explanation.

- If the teacher speaks purely in terms of computations, ask why one would divide rather than multiply, for example.
- Ask the teacher how he or she knows what to divide by.

(f)

(i**)　How might students come to develop this meaning for division?

(ii***) What would you do or say to help develop this meaning? Why?

(g*) Explain what *slope* represents in the context of your story.

- If the teacher mentions rise and run, ask what he or she would say to a student who was puzzled by the fact that that nothing was actually rising or running in the story the teacher relayed.
- If the teacher only mentions slant, steepness, tilt, or some other aspect of a line, ask the teacher again what the slope represents in the context of his or her story. This may mean that you have to remind the teacher of the story he or she relayed.

(h)

(i**)　How might students come to develop this meaning for slope?

(ii***) What would you do or say to help develop this meaning? Why?

5. [Problem adapted from Coe (2007)]

You provide both a table and a graph and ask a student to find how fast the function is changing between x=1.5 and x=4.

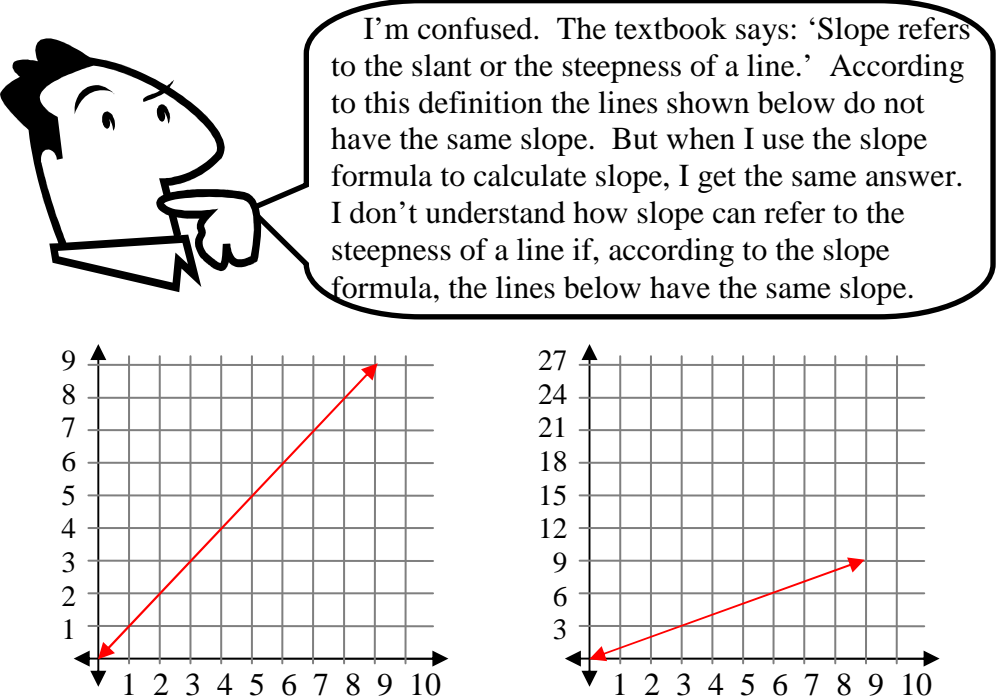| *x*	*f(x)*
0	9
1.5	15
2.5	19
4	25
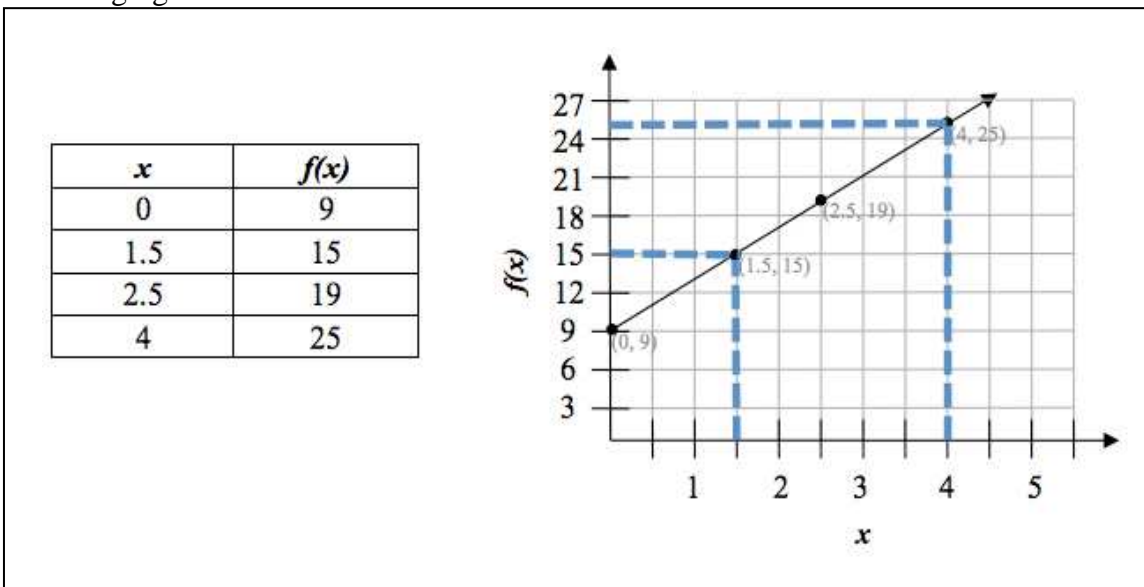
(a**)   What must a student understand to answer this question?
(b**)   What difficulties or sources of confusion might students have with this question?
(c**)   A student responds "10." What might this student have been thinking?
(d*/**)   What *should* the student have done?
(e***)   How would you initially respond to the student?
(f***)   What would you do or say to help guide the student? Why?

6. [Problem adapted from Coe (2007)]

Five students are discussing the meaning of slope in a linear context. One student says that slope is $\frac{y_2-y_1}{x_2-x_1}$. Another student says that slope is the angle of the line. A third student says that slope is rise over run. A fourth student says that slope is the rate of change of the line. A fifth student says that slope is the number *m*.

(a*/**) Which of the students is correct? [Explain.]
(b**)   What student understandings underlie these responses?
(c**)   What would you like students to understand about slope in a linear context?
(d**)   What might students say to indicate that they have developed an understanding of slope (as whatever was identified in (c))
  - Other teachers have said that forming an understanding of ratio is an important part of developing an understanding of slope. What might kids say to indicate that they had developed an understanding of slope as ratio?
(e**)   What difficulties or sources of confusion might arise for students when they are developing an understanding of slope as (whatever was identified in (c))?
  - What about that idea that some teachers raised regarding the formation of a ratio understanding? What sources of difficulty might arise for students when developing an understanding of ratio?
(f***) What would you say or do with each of these students to support them in developing a desirable understanding of slope? Why would you say or so those things?

7^{23}*. A new teacher comes to you and explains that he is experiencing some confusion regarding slope (as noted below). What would you say to him?
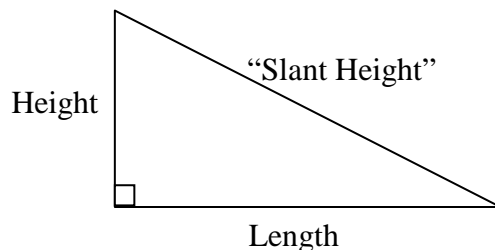
I'm confused. The textbook says: 'Slope refers to the slant or the steepness of a line.' According to this definition the lines shown below do not have the same slope. But when I use the slope formula to calculate slope, I get the same answer. I don't understand how slope can refer to the steepness of a line if, according to the slope formula, the lines below have the same slope.

8^{24}*. Consider the following challenge:

Can the ratio of "slant height" to length also be used as a measure of the steepness of a ramp (in addition to the ratio of height to length which is called slope)?

Let's give the new ratio a name (i.e., *ramp number*) so as to avoid confusing it with *slope* which is the ratio of height to length.

Examine several *ramp numbers* (e.g., 4:3; 5:4; 6:9). Test whether or not a *ramp number* is a good measure of steepness. This will involve thinking about what makes something a good measure. How can you test this measure to see if it's actually a measure of the steepness of the ramp? Think of a way and then test it.

DON'T USE ALGEBRAIC FORMULAS and DON'T USE YOUR KNOWLEDGE OF SLOPE!!!

Use only the kinds of tools (e.g., paper, pencil, straight edge) that students would have easy access to.

9. [Problem adapted from Coe (2007)]

Two students are having a debate about a function whose graph looks like the figure below. One student says that the function is "decreasing at a decreasing rate" while the other says it is "decreasing at an increasing rate"…
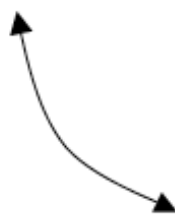
(a*/**) What, do you believe, is mathematically relevant to understand about language like "decreasing at a decreasing rate"?
(b**)   Which student, do you think, is correct?
(c**)   What makes understanding language like this complex?

(d***) What would you have the students focus on to support the development of their mathematical understandings? Why that?

(e*/**) What is the meaning of the use of "decreasing" in the given situation? What is the meaning of the first usage?

10. Here is an activity (Truck on a Track[25]).

(a**) What must students understand to answer this question?

(b**/***) What understanding of slope might students develop out of their engagement with this activity?

(c**) What understanding or way of reasoning do you think comes before and what understanding or way of reasoning do you think comes after (the understanding the teacher identified in the previous question)? In other words what understanding do you think precedes the understanding you identified and where do you think the understanding you identified leads?

(d*) How did you think through this problem?

(e*/**) What is hard about this question?

(f***) What would you say or do with students to support their understanding? Why?

Truck on a Track

A small battery operated truck travels on a track. Its journey consists of two legs. In the first leg of the journey, the truck travels 7cm in 4 seconds (at a constant speed). After the end of the second leg, the truck had traveled a total of 15 cm over 9 sec. Did the truck travel at the same speed during both legs of the journey or did it travel faster during one leg? Explain your reasoning.

15 cm

7 cm

4 sec   9 sec

11.

(a*) What do we mean when we speak of 'rate of change' (or 'ratio') in mathematics? [Use whatever language the teacher has been using]

(b**) What does it mean for a kid to have formed an understanding of slope as rate of change (or ratio)?

(c**) What would be evidence of this understanding?

(d**) How might they develop that understanding?

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[25] Truck on a Track was developed for the NSF-funded project, *Coordinating Social and Individual Aspects of Generalizing Activity* (DRL 0529502), Joanne Lobato, PI.

(e***) What could you do to support students in developing that understanding?

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