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Distributed decision-making of networked multi-agent systems in complex environments

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(CP-4) M. Zhu and S. Martínez, "On the convergence time of distributed quantized averaging algorithms", *The* 47$^{\text{th}}$ *IEEE Conference on Decision and Control*, pages 3971 – 3976, Cancun, Mexico, Dec. 2008.

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The results presented in Chapters 4 and 5 are published or to appear in the following papers:

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(JP-4) M. Zhu and S. Martínez, "On distributed convex optimization under inequality and equality constraints", *IEEE Transactions on Automatic Control*, 2011, to appear.

(CP-10) M. Zhu and S. Martínez, "An approximate dual sugbradient algorithm for multi-agent nonconvex optimization", *The* 49$^{\text{th}}$ *IEEE Conference on Decision and Control*, pages 7487 – 7492, Atlanta, USA, Dec. 2010.

(CP-8) M. Zhu and S. Martínez, "On distributed optimization under inequality constraints via Lagrangian primal-dual subgradient methods", *The* 29$^{\text{th}}$ *American Control Conference*, pages 4863 – 4868, Baltimore, USA, Jun. 2010.

(CP-7) M. Zhu and S. Martínez, "On distributed optimization under inequality and equality constraints via penalty primal-dual subgradient methods", *The* 29$^{\text{th}}$ *American Control Conference*, pages 2434 – 2439, Baltimore, USA, Jun. 2010.

The following papers summarize the results in Chapters 6 and 7:

(JP-8) M. Zhu and S. Martínez, "On attack-resilient distributed formation control in operator-vehicle networks", *SIAM Journal on Control and Optimization*, 2011, submitted.

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(CP-12) M. Zhu and S. Martínez, "Attack-resilient distributed formation control via online adaptation", *The* 50^{th} *IEEE Conference on Decision and Control and European Control Conference*, Orlando, USA, Dec. 2011, to appear.

(CP-6) M. Zhu and S. Martínez, "Distributed coverage games for mobile visual sensor networks (II): Reaching the set of global optima", *The Joint* 48^{th} *IEEE Conference on Decision and Control and* 28^{th} *Chinese Control Conference*, pages $175 - 180$, Shanghai, China, Dec. 2009.

(CP-5) M. Zhu and S. Martínez, "Distributed coverage games for mobile visual sensor networks (I): Reaching the set of Nash equilibria", *The Joint* 48^{th} *IEEE Conference on Decision and Control and* 28^{th} *Chinese Control Conference*, pages $169 - 174$, Shanghai, China Dec. 2009.

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PUBLICATIONS

Journal Publications

(JP-10) T. Sadikhov, M. Zhu and S. Martínez, "On optimal target assignment of multiple vehicles in an unknown environment", preprint.

(JP-9) M. Zhu and S. Martínez, "On distributed resilient consensus against replay attacks in adversarial networks", preprint.

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(JP-2) M. Zhu and S. Martínez, "On discrete-time dynamic average consensus", *Automatica*, 46(2), pages 322 – 329, 2010.

(JP-1) M. Zhu and J. Huang, "Small-gain theorems with restrictions for time-varying nonlinear systems", *Communication in Information and Systems*, 6(2), pages 115 – 136, 2006.

Conference Proceedings

(CP-13) M. Zhu and S. Martínez, "On distributed resilient consensus against replay attacks in adversarial networks", *The* 31^{th} *American Control Conference*, Montreal, Canada, preprint.

(CP-12) M. Zhu and S. Martínez, "Attack-resilient distributed formation control via online adaptation", *The* 50^{th} *IEEE Conference on Decision and Control and European Control Conference*, Orlando, USA, Dec. 2011, to appear.

(CP-11) M. Zhu and S. Martínez, "Stackelberg-game analysis of correlated attacks in cyber-physical systems", *The* 30^{th} *American Control Conference*, pages 4063 – 4068, San Francisco, USA, Jun. 2011.

(CP-10) M. Zhu and S. Martínez, "An approximate dual sugbradient algorithm for multi-agent nonconvex optimization", *The* 49^{th} *IEEE Conference on Decision and Control*, pages 7487 – 7492, Atlanta, USA, Dec. 2010.

(CP-9) T. Sadikhov, M. Zhu and S. Martínez, "A distributed joint learning and auction algorithm for target assignment", *The* 49^{th} *IEEE Conference on Decision and Control*, pages 5450 – 5455, Atlanta, USA, Dec. 2010.

(CP-8) M. Zhu and S. Martínez, "On distributed optimization under inequality constraints via Lagrangian primal-dual subgradient methods", *The* 29^{th} *American Control Conference*, pages 4863 – 4868, Baltimore, USA, Jun. 2010.

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(CP-3) M. Zhu and S. Martínez, "Dynamic average consensus on synchronous communication networks", *The* 27^{th} *American Control Conference*, pages $4382 - 4387$, Seattle, USA, Jun. 2008.

(CP-2) M. Zhu and J. Huang, "Global robust stabilization for a class of feedforward systems with dynamic uncertainties", *The* 6^{th} *World Congress on Intelligent Control and Automation*, pages $159 - 163$, Dalian, China, Jun. 2006.

(CP-1) M. Zhu and J. Huang, "Semi-global robust stabilization for a class of feedforward systems", *The* 25^{th} *American Control Conference*, pages $97 - 102$, Minneapolis, USA, Jun. 2006.

ABSTRACT OF THE DISSERTATION

Distributed decision-making of networked multi-agent systems in complex environments

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This dissertation is concerned with *distributed decision making* in networked multi-agent systems; that is, developing practical mechanisms which agents can utilize to autonomously coordinate their actions/decisions through local message exchanges and successfully achieve a system level goal with a satisfactory performance guarantee. In particular, this dissertation is divided into three parts and each one focuses on the following three classes of problems: *(1) distributed average consensus; (2) distributed cooperative constrained optimization; (3) distributed online learning based coordination.*

Part I: Distributed average consensus. This dissertation starts from the fundamental problem of distributed average consensus. In Part I, we first propose

a class of dynamic average consensus algorithms and show that these algorithms allow agents to asymptotically track the average of a class of time-varying individual reference inputs. We then come up with a class of gossip-based algorithms which agents can use to achieve approximate average consensus via exchanging quantized information.

Part II: Distributed cooperative constrained optimization. Part II is concerned with a class of general multi-agent optimization problems. In particular, each agent is associated with a local objective function and a local constrained set. There is a pair of inequality and equality constraints known to all the agents. We first present a class of distributed primal-dual subgradient algorithms to solve the case when all the ingredients are convex. We then introduce a distributed approximate dual subgradient algorithm to address the non-convex counterpart.

Part III: Distributed online learning based coordination. Part III studies distributed coordination schemes with online learning. The first problem considered is to optimally deploy a group of visual mobile sensors where the environmental distribution is unknown *a priori*. We formulate the problem as a non-cooperative game and come with up two distributed learning algorithms which allow sensors converge to the set of Nash equilibria and global optimum with probability one, respectively. The second problem is distributed formation control against a class of deception attacks. We propose a class of algorithms which allow vehicles adapt their strategies online and achieve the desired formation in the presence of deception attacks.

Chapter 1

Introduction

The last decades have witnessed a radical evolution in the domains of computation, communication and sensing. Information technology advances have fostered the emergence of a new generation of large-scale networked systems, namely *networked multi-agent systems*. Examples include mobile multi-vehicle networks, power grids, air/ground transportation networks, water/gas distribution networks, the Internet and social networks. These networked systems have now become part of the fabric of society, and even some of them have been deployed in infrastructures of vital significance.

In spite of their great importance and promise in engineering applications, there is a lack of scientific and systematic methodologies to understand and control multi-agent systems. This hampers the optimal management of these networked systems and the realization of their full potential. Although their applications cut across different domains, multi-agent systems share a remarkable characteristics: they consist of large collections of agents which are capable of sensing, computation, communication and/or actuation which allows their interaction and the ability to solve problems beyond their individual capabilities. The central theme of controlling multi-agent systems is *distributed decision-making*; that is, *developing practical mechanisms which agents can utilize to autonomously coordinate their actions/decisions through local message exchanges and successfully achieve a system level goal with satisfactory performance guarantees.*

This non-classical decision-making paradigm offers a number of competitive

advantages over traditional centralized management. Firstly, message passing is only necessary between neighboring agents. The feature of local communication, on the one hand, allows for scalability to the network expansion; on the other hand, enhances network-wide robustness to unpredicted component failure (especially the centralized authority). Secondly, the need of transmitting data to a central center is eliminated, and computational burden is shared among parallel agents. As a result, concurrent decision-making performed by agents greatly improves computational efficiency and reduces the demand of communication bandwidth. Thirdly, decisions are made by individual agents autonomously. This capability enables agents reason about rapid changes in the environment, and the networks demonstrate the capability of self-adaptation to environmental evolution.

At the same time, the task of distributed decision-making is significantly challenged by two factors: (1) inherent complexities of multi-agent systems; (2) dynamic, uncertain and adversarial elements in the operating environments.

Inherent system complexities. Multi-agent systems are strikingly complex systems. Firstly, they are *de facto* systems of systems, and can have a highly structural complexity. More specifically, multi-agent systems are composed of a large number of interacting agents in which each agent is associated with its own dynamics or decision-making process and coupled with others through physical and informational layers. Secondly, multi-agent systems can comprise heterogeneous agents. Heterogeneity may arise from different dynamics, distinct capabilities, the conflicts in pursuing interests, and the deviation of common possessed information.

Complex operating environments. Multi-agent systems are interacting in a dynamic, uncertain and adversarial environment. Firstly, the environment can undergo unanticipated changes, and these can disrupt the pre-determined strategies. Secondly, agents may lack of knowledge on prior information about the environment, and these uncertainties can degrade the network performance. Thirdly, information technology systems are inherently susceptible to cyber attacks, and cyber-vulnerability poses a significant risk to the success of mission completion.

The seminal work [14], provides a first unified framework for parallel and distributed decision making among agents. Since then, there has been substantial

research effort on distributed decision making of networked multi-agent systems in complex environments; e.g., see [25] [26] [59] [73] [80] [129]. This dissertation aims at contributing to this broad field. In particular, we will investigate three classes of problems: *distributed average consensus, distributed cooperative constrained optimization*, and *distributed online learning based coordination*. Next, we will provide literature review for each class of problems, and then summarize our contributions.

1.1 Part I: Distributed average consensus

Consensus addresses the question how agents can agree upon a quantity of interest via local communication and computation actions. Consensus is a fundamental problem in multi-agent systems: on the one hand, its study is beneficial for understanding information constraints to achieve a network-wise objective; on the other hand, consensus algorithms serve as building blocks of more sophisticated protocols for; e.g., distributed estimation, optimization and task assignment. A special case of particular interest, namely distributed average consensus (or distributed averaging), consists of computing the average of the values generated by different agents.

Literature review

Consensus roots in Computer Science and plays a fundamental role in parallel and distributed computation [14]. The first consensus algorithm was proposed in [41]. Recently, the emergence of multi-agent systems has attracted researchers from various engineering and scientific disciplines and this has yielded substantial generalizations of the basic consensus algorithm. It is hard to provide a complete literature review given the vast volume of papers devoted to consensus algorithms. Here we only list some representative papers concerned with different aspects of consensus. In particular, the papers [47] [108] study continuous-time consensus, and the papers [19] [65] [93] instead focus on discrete-time consensus. In [88], the authors discuss the asynchronous implementation of consensus algorithms. The convergence rate of consensus algorithms is e.g., characterized in [111] [112] [147].

Gossip-based consensus algorithms are investigated in [20] [44], and the paper [102] treats the problem of reaching the consensus state when it is constrained in some given convex set. Conditions on consensus algorithms to achieve different consensus values are provided in [34]. The papers [35] [135] address how to achieve consensus within a finite time.

New algorithms have significantly extended the application scope of consensus. Some interesting examples include, to name a few, attitude alignment [121], clock synchronization [30], coverage control [54], opinion formation [153], oscillator synchronization [31, 45], parameter estimation [148], social learning [66], and multi-vehicle formation control [122].

Summary of main contributions

In multi-agent systems, nodes are expected to share real-time information and adaptively react to unanticipated dynamic changes in the environment. This motivates us to investigate the extension of classical consensus problems to a dynamic setting where agents seek to track the average of individually measured time-varying signals. In Chapter 2, we propose the first class of discrete-time dynamic average consensus algorithms and show that the proposed algorithms are able to guarantee zero or sufficiently small steady-state error provided that the maximum deviations between n^{th}-order differences of individual signals are uniformly bounded.

From a practical point of view, real-valued average consensus algorithms are not feasible due to finite capacities of communication channels, finite memory capacities of agents and finite precision of the computations. These constraints motivate the problem of *quantized averaging* where agents are only allowed to exchange quantized information to reach average consensus. In Chapter 3, we come up with a class of distributed quantized averaging algorithms on asynchronous communication networks characterized by fixed, switching and random topologies. The algorithms are shown to asymptotically reach quantized average consensus in probability. Furthermore, we utilize meeting times of two random walks on graphs as a unified approach to derive polynomial upper bounds on the expected

convergence times of our presented algorithms.

Chapters 2 and 3 are based on the following published papers:

(JP-3) M. Zhu and S. Martínez, "On the convergence time of asynchronous distributed quantized averaging algorithms", *IEEE Transactions on Automatic Control*, 56(2), pages 386 – 390, 2011.

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(CP-3) M. Zhu and S. Martínez, "Dynamic average consensus on synchronous communication networks", *The* 27^{th} *American Control Conference*, pages 4382 – 4387, Seattle, USA, Jun. 2008.

1.2 Part II: Distributed cooperative constrained optimization

In order to allow the network perform at an optimal level, decision makers face the problem of choosing the best option among a set of candidates. Distributed optimization provides a holistic and mathematically rigorous framework to entail network-wise optimal decision making. In particular, a wealth of engineering problems can be formulated as a distributed optimization problem where agents possess different objective functions and are required to obey inhomogeneous constraints. For the purpose of preserving information privacy, agents are not willing to reveal their own components, but they share the common goal of collectively finding a decision vector which minimizes the sum of local objective functions and simultaneously enforces all the individual constraints. The lack of global information prevents individual agents to solve the problem on their own,

and it becomes necessary for agents to negotiate their estimates with neighbors in order to determine a global optimal solution.

Literature review

The seminal work [14] provides a framework to tackle optimizing a global objective function among different processors where each processor knows the global objective function. In the context of multi-agent systems, the paper [118] presents an algorithm using average consensus algorithms to minimize a sum of continuously differentiable local objective functions. These results are significantly extended in the paper [100] to nonsmooth objective functions and time-varying topologies via subgradient methods. Using projection in the algorithm of [100], the authors in [102] further address a more general scenario that takes local state constraint sets into account. The paper [79] focuses on the scenario where communication is Markovian with respect to the states of agents; i.e., the probability each link becomes available depends upon the states of agents. The paper [68] addresses a special case of [102] where the network topology is fixed and all the local constraint sets are identical.

In the robotics and control communities, convex optimization has been exploited to design algorithms coordinating mobile robotic networks. In [40], in order to increase the connectivity of a multi-agent system, a distributed supergradient-based algorithm is proposed to maximize the second smallest eigenvalue of the Laplacian matrix of the state dependent proximity graph of agents. In [42], optimal shape changes of mobile robots are achieved through second-order cone programming techniques. In [43], a target tracking problem is addressed by means of a generic semi-definite program where the constraints of network connectivity and full target coverage are articulated as linear-matrix inequalities. In [94], in order to attain the highest possible positioning accuracy for mobile robots, the authors express the covariance matrix of the pose errors as a functional relation of measurement frequencies, and then formulate an optimal sensing problem as a convex programming of measurement frequencies.

Summary of main contributions

In this dissertation, we investigate a general class of distributed optimization problems in Chapters 4 and 5, where agents are to collectively minimize a global objective function subject to a global inequality constraint, a global equality constraint, and a global constraint set. The global objective function, representing the network objective, is a sum of local objective functions. This multi-agent optimization problem is characterized by the distinct feature that all the functions depend on a global decision vector.

In Chapter 4, we study the convex case of the aforementioned multi-agent optimization problem; i.e., all the ingredients of the problem are convex. In particular, we study two cases: one in which the equality constraint is absent, and the other in which the local constraint sets are identical. We propose two distributed algorithms based on the (Lagrangian and penalty) saddle-point characterization of primal-dual solutions. These algorithms allow agents to asymptotically agree upon a global optimal solution and the optimal value under the standard Slater's condition. These algorithms can be implemented over the dynamically changing topologies which satisfy some standard connectivity assumption.

Chapter 5 aims to relax the convexity assumption in Chapter 4; i.e., the objective and constraint functions as well as the state-constraint set could be non-convex. We first introduce an approximation of the problem of interest where the exact consensus is slightly relaxed. We propose a distributed dual subgradient algorithm to solve the approximate problem where the update rule for local dual estimates combines a dual subgradient scheme with average consensus algorithms and local primal estimates are generated from local dual optimal solution sets. This algorithm is shown to asymptotically converge to a pair of primal-dual solutions to the approximate problem provided that: firstly, the dual optimal solution set is singleton; secondly, dynamically changing network topologies satisfy some standard connectivity condition.

The results presented in Chapters 4 and 5 are published or to appear in the following papers:

(JP-6) M. Zhu and S. Martínez, "An approximate dual subgradient algorithm for

distributed non-convex constrained optimization", *IEEE Transactions on Automatic Control*, 2011, provisionally accepted.

(JP-4) M. Zhu and S. Martínez, "On distributed convex optimization under inequality and equality constraints", *IEEE Transactions on Automatic Control*, 2011, to appear.

(CP-10) M. Zhu and S. Martínez, "An approximate dual sugbradient algorithm for multi-agent nonconvex optimization", *The* 49$^{\text{th}}$ *IEEE Conference on Decision and Control*, pages 7487 − 7492, Atlanta, USA, Dec. 2010.

(CP-8) M. Zhu and S. Martínez, "On distributed optimization under inequality constraints via Lagrangian primal-dual subgradient methods", *The* 29$^{\text{th}}$ *American Control Conference*, pages 4863 − 4868, Baltimore, USA, Jun. 2010.

(CP-7) M. Zhu and S. Martínez, "On distributed optimization under inequality and equality constraints via penalty primal-dual subgradient methods", *The* 29$^{\text{th}}$ *American Control Conference*, pages 2434 − 2439, Baltimore, USA, Jun. 2010.

1.3 Part III: Distributed online learning based coordination

The multi-agent systems are expected to remain capable of performing the given missions when facing environmental uncertainties and even interference of adversaries. A robust solution simply takes into account the worst-case of the uncertainties induced by the physical environment or human adversaries. Because they do not utilize observations to refine the underlying model poorly known in advance, robust solutions are usually over-conservative, greatly degrading the quality of service provided by the network. In contrast, online learning and adaptation enable agents to explore the (partially) unknown environment via continuous interaction, and adapt their behavior by exploiting past experiences. The process of exploration-exploitation realizes that agents are able to autonomously adjust

themselves to compensate the negative effects caused by uncertain and malicious components. In this part, we integrate online learning and adaptation into distributed decision making, and address two different cooperative control problems: seeking equilibrium of optimal coverage games in an unknown environment, and distributed formation control against cyber attacks in an operator-vehicle network.

Literature review

Game-theoretic coordination

Non-cooperative game theory addresses the question how multiple players independently reason about each other to achieve their own goal. Because of the inherent feature of decision making as a distributed process, non-cooperative game theory becomes compelling in distributed coordination.

In the areas of networking and communication, non-cooperative game theory and learning schemes have been widely used to analyze a variety of fundamental problems; e.g., [3] on power control, [126] on routing, and [139] on flow control. Its usage to address the coordination of multiple vehicles is relatively new. Recently, the paper [82] establishes a link between cooperative control problems (in particular, consensus problems), and games (in particular, potential games and weakly acyclic games). In [81], the authors present a game-theoretic analysis of a coverage optimization problem for static sensor networks. This problem is equivalent to the weapon-target assignment problem in [95] which is NP complete. In general, the solution to assignment problems is hard from a combinatorial optimization viewpoint. In the paper [9], a game-theoretic learning algorithm is proposed to minimize the assumption on inter-agent communication in dynamic vehicle routing. In the recent papers [50, 78, 133], extremum seeking is novelly applied to find Nash equilibrium when the game model is not available in advance.

Cyber security of cyber-physical systems

Nowadays, control systems have grown into *cyber-physical systems* where a number of information technologies (e.g., wireless communication and embed-

ded computation) have been integrated to monitor and control physical processes. Information infrastructures, on the one hand, introduce new and competitive functions into control systems; on the other hand, impose significant security risks on control systems due to their inherent vulnerability to cyber attacks.

In information technology networks, either reactive or protective mechanisms has been exploited to prevent cyber attacks. Non-cooperative game theory [51] is advocated as a mathematical framework to model the interdependency between attackers and administrators, and predict the behavior of attackers; see an incomplete list of references [2, 57, 127, 141]. These findings can help us assess network vulnerability; however, the papers aforementioned do not consider how to maintain the operational function of networked systems in the presence of malicious attacks.

There has been a considerable research effort on investigating networked control systems in which the effects of imperfect communication channels on remote control are analyzed and compensated. Most of the existing papers focus on; e.g., band-limited channels [77, 97], quantization [24, 96], packet dropout [58, 128], delay [22, 145], and sampling [103]. The paper [60] presents an excellent survey on recent advances in networked control systems.

Very recently, cyber-security of the emerging cyber-physical systems has drawn mounting attention in the control society. Denial-of-service attacks, affecting the data availability in control systems, are entailed in recent papers [4, 6, 13, 57]. Another important class of cyber attacks, namely false data injection, compromises the data integrity of state estimation and is attracting considerable effort; an incomplete reference list includes [90, 115, 140, 149]. In [17, 18], the authors exploit pursuit-evasion games to compute optimal evasion strategies for mobile agents in the face of jamming attacks. Other relevant papers include [5] examining the stability of a SCADA water management system under a class of switching attacks. In the paper [67], a class of trust based distributed Kalman filters is proposed for power systems to prevent data disseminated by untrusted phase measurement units.

Summary of main contributions

In Chapter 6, we investigate an optimal coverage problem for a group of mobile visual sensors where the environmental distribution function is unaccessible in advance. The problem of interest is formulated as a non-cooperative constrained game between agents where the utility function is unknown *a prior* to each player. To solve this model-free game, we extend the use of the payoff-based learning dynamics, and come up with two distributed learning algorithms. In particular, the first one allows for the convergence in probability to the set of (constrained) Nash equilibria, from which no agent is willing to unilaterally deviate. The second algorithm is shown to be convergent in probability to the set of global maxima of a coverage performance metric.

In Chapter 7, we study a formation control problem for an operator-vehicle network in which each vehicle is remotely controlled by an operator. Each operator-vehicle pair is attacked by an adversary, who corrupts the control commands sent to the vehicle. The adversaries are modeled as rational decision makers and their strategies are linearly parameterized by some (potentially time-varying) matrices which are unknown to operators in advance. We investigate two plausible scenarios depending on the learning capabilities of adversaries. The first scenario involves *unilateral learning*, where adversaries possess (potentially incorrect) private information of operators in advance, but do not update such information during the attacking course. The second scenario assumes *bilateral learning*, where adversaries are intelligent and attempt to infer some private information of operators through their observations. We propose a class of novel distributed attack-resilient formation control algorithms each consisting of two feedback-connected blocks: a formation control block and an online learning block. We show how each proposed algorithm guarantees that vehicles achieve asymptotically the desired formation from any initial vehicle configuration and any initial estimates of adversaries. For each proposed algorithm, the sequence of the distances to the desired formation is shown to be square summable.

The following papers summarize the results in Chapters 6 and 7:

(JP-8) M. Zhu and S. Martínez, "On attack-resilient distributed formation control

in operator-vehicle networks", *SIAM Journal on Control and Optimization*, 2011, submitted.

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(CP-12) M. Zhu and S. Martínez, "Attack-resilient distributed formation control via online adaptation", *The* 50^{th} *IEEE Conference on Decision and Control and European Control Conference*, Orlando, USA, Dec. 2011, to appear.

(CP-6) M. Zhu and S. Martínez, "Distributed coverage games for mobile visual sensor networks (II): Reaching the set of global optima", *The Joint* 48^{th} *IEEE Conference on Decision and Control and* 28^{th} *Chinese Control Conference*, pages $175 - 180$, Shanghai, China, Dec. 2009.

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Part I

Distributed average consensus

Chapter 2

Distributed dynamic average consensus

2.1 Introduction

We consider the problem in which a set of autonomous agents aims to track the average of individually measured time-varying signals by local communication with neighbors. This problem is referred to as dynamic average consensus in opposition to the more studied static consensus. The dynamic average consensus problem arises in different contexts, such as formation control [152], sensor fusion [105][109][132], distributed estimation [87] and distributed tracking [151]. These tasks require that all agents agree on the average of time-varying signals and thus the consensus on a static average value, e.g., the initial states of the agents, is insufficient.

Literature review. The distributed static consensus problem was introduced in the literature of parallel processors in [142] and has attracted significant attention in the controls community. A necessarily incomplete list of references includes [47][108] for continuous-time consensus, [19][65][93] for discrete-time consensus, [14][88] discuss asynchronous consensus, and [20][69][138][102] treat randomized consensus, quantized consensus, consensus over random graphs and constrained consensus, respectively. The convergence rate of consensus algorithms is

e.g., discussed in [111][147], consensus propagation is considered in [91], and conditions on consensus algorithms to achieve different consensus values is discussed in [34]. Consensus algorithms find application in a variety of areas such as load balancing [39][150], formation control [47][152], and, as we have mentioned, sensor fusion [87][105][109][132], distributed tracking [106][151] and consensus-based belief propagation in Bayesian networks [107].

The dynamic average consensus problem in continuous-time is studied in [48] [109][120][132]. By using standard frequency-domain techniques, the authors in [132] showed that their algorithm was able to track the average of ramp reference inputs with zero steady-state error. In the context of input-to-state stability, [48] showed that proportional dynamic average consensus algorithm could track with bounded steady-state error the average of bounded reference inputs with bounded derivatives. On the other hand, [48] showed that proportional-integral dynamic average consensus algorithm could track the average of constant reference inputs with sufficiently small steady-state error. The authors in [109] proposed a dynamic consensus algorithm and applied it to the design of consensus filters. The algorithm in [109] can track with some bounded steady-state error the average of a common reference input with a bounded derivative. The problem studied in [120] is similar to that in [109], and consensus of agents is over a common time-varying reference signal. However, the algorithm in [120] assumes that agents know the nonlinear model which generates the time-varying reference function. The problem studied in the present chapter is close to those in [48] and [132] and includes those in [109] and [120] as special cases.

Statement of contributions. In this chapter, we propose a class of discrete-time dynamic average consensus algorithms and analyze their convergence properties. This chapter contributes to the problem of dynamic average consensus in the following aspects: The continuous-time communication assumption for dynamic average consensus in [48] and [132] is relaxed, and we consider more realistic discrete-time synchronous communication models. This allows us to obtain a direct relation between the frequency of inter-agent communication and the differences of reference signals. Our dynamic average consensus algorithms are able

to track the average of a larger class of time-varying reference inputs than [48] and [132] with zero or sufficiently small steady-state error. This includes polynomials, logarithmic-type functions, periodic functions and other functions whose n^{th}-order differences are bounded, for $n \geq 1$. We can also handle the case where the difference of the common part, that appears in all the individual reference inputs, explodes. Furthermore, the algorithms proposed are robust to the dynamic change of communication topologies as well as the joining and leaving (or failure) of nodes. The convergence analysis for our dynamic average consensus algorithms relies upon the input-to-output stability property of discrete-time static average consensus algorithms in the presence of external disturbances.

2.2 Preliminaries and problem statement

In this section, we introduce the notation to be employed along the chapter and state the problem of dynamic average consensus.

The positive real number h is the time discretization unit and the update time instants $k \in \mathbb{R}$ (or s, τ) will be of the form $k = ph$ (or $s = ph$, $\tau = qh$) for $p, q \in \mathbb{Z}$.

We will consider a network of N nodes or agents, labeled by $i \in V = \{1, \cdots, N\}$, interacting over a communication network. The topology of the network at time k will be represented by a directed graph $\mathcal{G}(k) = (V,\ E(t))$ with an edge set $E(k) \subset V \times V$. We consider that $(i, j) \in E(k)$ if and only if node i communicates to node j at time k. The in-neighbors of node i at time k are denoted by $\mathcal{N}_i(k) = \{j \in V : (j, i) \in E(k) \text{ and } j \neq i\}$. The matrix $A(k) = [a_j^i(k)] \in \mathbb{R}^{N\times N}$ represents the adjacency matrix of $\mathcal{G}(k)$ where $a_j^i(k) \neq 0$ if edge $(j, i) \in E(k)$. Finally, $\mathbf{1} \in \mathbb{R}^N$ is the column vector whose entries are all ones.

At each time instant k, every node synchronously measures the local continuous physical process $r_i : \mathbb{R} \to \mathbb{R}$, communicates with its neighbors and updates the state of its consensus algorithm. We ignore the delays induced by the communication and computation process. In the remainder of this chapter, the sample $r_i(k)$ is referred to as the reference signal (or input) of node i at time k. Denote

by $\bar{r}(k) = \frac{1}{N}\sum_{i=1}^{N} r_i(k)$ the average of the reference inputs at time k.

Our objective is to design an n^{th}-order dynamic average consensus algorithm that the nodes can utilize to asymptotically achieve the average of the reference inputs if the maximum relative deviation between the n^{th}-order differences of any two reference inputs is bounded for some integer $n \geq 1$. We denote by $x_i(k) = (x_i^{[1]}(k), \cdots, x_i^{[n]}(k)) \in \mathbb{R}^n$ the consensus state of node i at time k. The quantity of $\max_{i\in V} \limsup_{t\to\infty} |x_i^{[n]}(k) - \bar{r}(k-h)|$ is referred to as the steady-state error of n^{th}-order dynamic average consensus algorithm. This can be interpreted as a measurement of how far the components of the consensus state $(x_1^{[n]}(k), \cdots, x_N^{[n]}(k))$ are from achieving the dynamic average consensus. Our algorithms will reach the dynamic average consensus with either a zero steady-state error or rendering the steady-state error smaller than (or equal to) any given bound.

2.3 First-order dynamic average consensus algorithm

In this section, we present first-order algorithms to achieve dynamic average consensus. Main references include [14][19], and [111]. We define:

$$\begin{aligned} &M(k) = \max_{i\in V} x_i(k), && m(k) = \min_{i\in V} x_i(k), \\ &D(k) = M(k) - m(k), && \Delta r_i(k) = r_i(k) - r_i(k-h), \\ &\Delta r_{\max}(k) = \max_{i\in V} \Delta r_i(k), && \Delta r_{\min}(k) = \min_{i\in V} \Delta r_i(k). \end{aligned}$$

By induction, the n^{th}-order difference of $r_i(k)$ is given by

$$\Delta^{(n)} r_i(k) = \Delta^{(n-1)} r_i(k) - \Delta^{(n-1)} r_i(k-h)$$

for $n \geq 2$ where $\Delta^{(1)} r_i(k) = \Delta r_i(k)$. We will use the notations $\Delta^{(n)} r_{\max}(k) = \max_{i\in V} \Delta^{(n)} r_i(k)$ and $\Delta^{(n)} r_{\min}(k) = \min_{i\in V} \Delta^{(n)} r_i(k)$ for $n \geq 2$.

In what follows, we will make use of the following assumption on $\mathcal{G}(k)$ that was proposed in [65] and also used in [19][111].

Assumption 2.3.1 (Periodical strong connectivity) There is some positive integer $B \geq 1$ such that, for all time instant $k \geq 0$, the directed graph $(V, E(k) \cup E(k+h) \cup \cdots \cup E(k+(B-1)h))$ is strongly connected.

Assumption 2.3.2 (Relatively bounded first-order differences) For any $h > 0$, there exists a time invariant constant $\theta > 0$ such that

$$\Delta R(k) := \Delta r_{\max}(k) - \Delta r_{\min}(k) \leq h\theta, \quad \forall k \geq 0. \tag{2.1}$$

Remark 2.3.1 Inequality (2.1) becomes $\max_{i \in V} \dot{r}_i(k) - \min_{i \in V} \dot{r}_i(k) \leq \theta$ as $h \to 0$. Hence, Assumption 2.3.2 can be viewed as the discretized version of the property $\max_{i \in V} \dot{r}_i(k) - \min_{i \in V} \dot{r}_i(k) \leq \theta$ for some fixed $\theta \geq 0$ and all time instant $k \geq 0$. •

We propose the *First-Order Dynamic Average Consensus algorithm* (the FODAC algorithm for short) below to reach dynamic average consensus:

$$x_i(k+h) = x_i(k) + \sum_{j \neq i} a_j^i(k)(x_j(k) - x_i(k)) + \Delta r_i(k), \tag{2.2}$$

when the reference input $r(k)$ satisfies Assumption 2.3.2.

Remark 2.3.2 The FODAC algorithm can be rewritten as:

$$[x_i(k+h) - x_i(k)]/h = \delta \sum_{j \neq i} a_j^i(k)(x_j(k) - x_i(k)) + [r_i(k) - r_i(k-h)]/h, \tag{2.3}$$

where the parameters δ and h satisfy $h\delta = 1$. Observe that (2.3) is close to the discretized version of the continuous-time dynamic consensus algorithm in [132] but is not exactly the same. If $h \to 0$, then $\delta = \frac{1}{h} \to \infty$, and thus the right-hand side of (2.3) is not well-defined. •

We will further suppose that the coefficients $a_j^i(k)$ in the FODAC algorithm satisfy the following two assumptions.

Assumption 2.3.3 ([19], Non-degeneracy) There exists a constant $\alpha > 0$ such that $a_i^i(k) = 1 - \sum_{j \neq i} a_j^i(k) \geq \alpha$, and $a_j^i(k)$ $(i \neq j)$ satisfies $a_j^i(k) \in \{0\} \cup [\alpha,\ 1]$, $\forall k \geq 0$.

Assumption 2.3.4 ([108], Balanced Communication) There hold for all $k \geq 0$ that $\mathbf{1}^T A(k) = \mathbf{1}^T$ and $A(k)\mathbf{1} = \mathbf{1}$.

Equivalently, the matrix $A(k)$ is referred to as doubly stochastic, each of whose rows and columns sums to 1. Assumption 2.3.4 renders the conservation property $\sum_{i=1}^{N} x_i(k+h) = \sum_{i=1}^{N} r_i(k)$ which is essential to reach the dynamic average consensus. It plays a similar role in achieving the static average consensus [108].

We now proceed to analyze the FODAC algorithm. Let us fix $\kappa \in V$ for every s and define $\mathcal{D}_0 = \{\kappa\}$. By Assumption 2.3.1, there is a non-empty set $\mathcal{D}_1 \subset V \setminus \{\kappa\}$ of nodes such that for all $i \in \mathcal{D}_1$, node κ communicates to node i at least once during the time frame $[s, s + (B-1)h]$. By induction, a set $\mathcal{D}_{\ell+1} \subset V \setminus (\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\ell)$ can be defined by considering those nodes j to which some $i \in \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\ell$ communicates at least once during the time frame $[s + \ell Bh, s + ((\ell+1)B - 1)h]$. By Assumption 2.3.1, $\mathcal{D}_{\ell+1} \neq \emptyset$ as long as $V \setminus (\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\ell) \neq \emptyset$. Thus, there exists $\mathcal{L} \leq N - 1$ such that the collection of $\mathcal{D}_0, \ldots, \mathcal{D}_{\mathcal{L}}$ is a partition of V.

Lemma 2.3.1 *Consider the* FODAC *algorithm and suppose that Assumptions 2.3.1 and 2.3.3 hold. Let $s \geq 0$ and $\kappa \in V$ be fixed and consider the associated $\mathcal{D}_0, \ldots, \mathcal{D}_{\mathcal{L}}$. Then for every $\ell \in \{1, \ldots, \mathcal{L}\}$, there exists a real number $\eta_\ell > 0$ such that for every integer $p \in [\ell B,\ (\mathcal{L}B + B - 1)]$, and $i \in \mathcal{D}_\ell$, it holds that for $k = s + ph$*

$$x_i(k) \geq m(s) + \sum_{q=0}^{p-1} \Delta r_{\min}(s + qh) + \eta_\ell(x_\kappa(s) - m(s)), \tag{2.4}$$

$$x_i(k) \leq M(s) + \sum_{q=0}^{p-1} \Delta r_{\max}(s + qh) - \eta_\ell(M(s) - x_\kappa(s)). \tag{2.5}$$

Proof: Without loss of generality, we only consider the case where $s = 0$, being the proof for a general s identical. Fix some i, it holds that

$$\begin{aligned} x_i(h) &= x_i(0) + \sum_{j\neq i} a_j^i(0)(x_j(0) - x_i(0)) + \Delta r_i(0) \\ &= (1 - \sum_{j\neq i} a_j^i(0))x_i(0) + \sum_{j\neq i} a_j^i(0)x_j(0) + \Delta r_i(0) \\ &\geq (1 - \sum_{j\neq i} a_j^i(0))m(0) + \sum_{j\neq i} a_j^i(s)m(0) + \Delta r_{\min}(0) \\ &= m(0) + \Delta r_{\min}(0). \end{aligned} \tag{2.6}$$

Since (2.6) holds for all i, we have

$$m(h) \geq m(0) + \Delta r_{\min}(0). \tag{2.7}$$

Repeatedly applying (2.7) gives that

$$m(k) \geq m(0) + \sum_{p=0}^{\frac{k}{h}-1} \Delta r_{\min}(ph). \tag{2.8}$$

Since $\sum_{j=1}^{N} a_j^\kappa(k) = 1$ at every $k \geq 0$, we have that

$$\begin{aligned} & x_\kappa(k+h) - m(0) - \sum_{p=0}^{\frac{k}{h}} \Delta r_{\min}(ph) \\ &= \sum_{j=1}^{N} a_j^\kappa(k)(x_j(k) - m(0) - \sum_{p=0}^{\frac{k}{h}-1} \Delta r_{\min}(ph)) + \Delta r_\kappa(k) - \Delta r_{\min}(k) \\ &\geq a_{\kappa\kappa}(t)(x_\kappa(t) - m(0) - \sum_{p=0}^{\frac{k}{h}-1} \Delta r_{\min}(ph)) \\ &\geq \alpha(x_\kappa(t) - m(0) - \sum_{p=0}^{\frac{k}{h}-1} \Delta r_{\min}(ph)), \end{aligned} \tag{2.9}$$

where we are using the property of (2.8) in the last two inequalities. Applying repeatedly (2.9) we have that, for any integer $p \in [0, (\mathcal{L}B + B - 1)]$, the following

holds for $k = ph$

$$\begin{aligned} & x_\kappa(k) - m(0) - \sum_{q=0}^{p-1} \Delta r_{\min}(qh) \\ & \geq \alpha^{p-1}(x_\kappa(h) - m(0) - \Delta r_{\min}(0)) \\ & \geq \alpha^p(x_\kappa(0) - m(0)) \geq \eta_0(x_\kappa(0) - m(0)) \,, \end{aligned}$$

where $\eta_0 = \alpha^{NB-1}$ and we are using the properties of (2.8) and $x_\kappa(0) - m(0) \geq 0$. This proves inequality (2.4) for the nodes in $\mathcal{D}_0 = \{\kappa\}$ and for any integer $p \in [0, (\mathcal{L}B + B - 1)]$.

Now we proceed by induction on ℓ. Suppose that (2.4) holds for some $0 \leq \ell < \mathcal{L}$; then we should show (2.4) for $i \in \mathcal{D}_{\ell+1}$. It follows from the construction of the sets of $\{\mathcal{D}_0, \cdots, \mathcal{D}_\mathcal{L}\}$ that there exists some time $k' \in [\ell Bh,\ (\ell B + B - 1)h]$ such that $a_j^i(k') \neq 0$ for some $j \in \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\ell$ and $i \in \mathcal{D}_{\ell+1}$. By the induction hypothesis, we have that for all integers $p \in [\ell B,\ (\mathcal{L}B + B - 1)]$, there exists some $\eta_\ell > 0$ such that the following holds for $k = ph$

$$x_j(k) - m(0) - \sum_{\tau=0}^{p-1} \Delta r_{\min}(qh) \geq \eta_\ell(x_\kappa(0) - m(0)).$$

Consequently, as in (2.9), we have

$$\begin{aligned} & x_i(k' + h) - m(0) - \sum_{q=0}^{\frac{k'}{h}} \Delta r_{\min}(qh) \\ & \geq a_j^i(k')(x_j(k') - m(0) - \sum_{q=0}^{\frac{k'}{h}-1} \Delta r_{\min}(qh)) \geq \alpha\eta_\ell(x_\kappa(0) - m(0)). \end{aligned}$$

Following along the same lines as in (2.9), we have that

$$x_i(k + h) - m(0) - \sum_{q=0}^{p} \Delta r_{\min}(qh) \geq \eta_{\ell+1}(x_\kappa(0) - m(0)) \,,$$

holds for all $p \in [(\ell + 1)B,\ (\mathcal{L}B + B - 1)]$, where $\eta_{\ell+1} = \alpha^{(N-\ell)B}\eta_\ell$ and $k = ph$. This establishes (2.4) for $i \in \mathcal{D}_{\ell+1}$. By induction, we have shown that (2.4) holds. The proof for (2.5) is analogous. ■

The following theorem is the main result in this section and shows the convergence properties of the FODAC algorithm.

Theorem 2.3.1 *Let δ_1 be a positive constant and $h_1 = \frac{\delta_1 \alpha^{\frac{1}{2}N(N+1)B+1}}{4\theta(NB-2)}$. Under Assumptions 2.3.1, 2.3.3, 2.3.4 and 2.3.2, the implementation of the* FODAC *algorithm with $h \in (0, h_1]$ and initial conditions $x_i(0) = r_i(-h)$, $i \in \{1, \ldots, N\}$, achieves dynamic average consensus with a nonzero steady-state error upper bounded by δ_1.*

Proof: Let $\eta = \alpha^{\frac{1}{2}N(N+1)B-1}$, then $\eta \leq \eta_\ell$ for any $\ell \in \{1, \ldots, N-1\}$. By replacing s and k in (2.4) with k and $k_1 = k + (\mathcal{L}B + B - 1)h$ respectively, we have that for every $k \geq 0$, there holds that

$$\begin{aligned} m(k_1) &= \min_{\ell \in \{0, \cdots, \mathcal{L}\}} \min_{i \in \mathcal{D}_\ell} x_i(k_1) \\ &\geq m(k) + \sum_{q=\frac{k}{h}}^{\frac{k_1}{h}-1} \Delta r_{\min}(qh) + \min_{\ell} \eta_\ell (x_\kappa(k) - m(k)) \\ &\geq m(k) + \sum_{q=\frac{k}{h}}^{\frac{k_1}{h}-1} \Delta r_{\min}(qh) + \eta (x_\kappa(k) - m(k)). \end{aligned}$$

Similarly, we have

$$M(k_1) \leq M(k) + \sum_{q=\frac{k}{h}}^{\frac{k_1}{h}-1} \Delta r_{\max}(qh) - \eta (M(k) - x_\kappa(k)).$$

Combining the above two inequalities gives that

$$D(k_1) \leq (1-\eta) D(k) + \sum_{q=\frac{k}{h}}^{\frac{k_1}{h}-1} \Delta R(qh). \tag{2.10}$$

Let us denote $T_\nu = \nu(NB-1)h$ for an integer $\nu \geq 1$. From (2.7), we know that $D(k+h) \leq D(k) + \Delta R(k)$. Thus we have

$$D(T_1) \leq (1-\eta) D(0) + \sum_{q=0}^{\frac{T_1}{h}-1} \Delta R(qh). \tag{2.11}$$

and thus, $D(T_n) \leq (1-\eta)^n D(0) + \Omega(n)$, where

$$\Omega(n) = (1-\eta)^{n-1} \sum_{q=0}^{\frac{T_1}{h}-1} \Delta R(qh) + \cdots + \sum_{q=\frac{T_{n-1}}{h}}^{\frac{T_n}{h}-1} \Delta R(qh).$$

For any $k \geq 0$, let ℓ_k to be the largest integer such that $\ell_k(NB-1)h \leq k$, and $\bar{\Omega}(k) := \Omega(\ell_k) + \sum_{q=\frac{T_{\ell_k}}{h}}^{\frac{k}{h}-1} \Delta R(qh)$. Thus for all $k \geq 0$ it follows that

$$\begin{aligned} D(k) &\leq D(T_{\ell_k}) + \sum_{q=\frac{T_{\ell_k}}{h}}^{\frac{k}{h}-1} \Delta R(qh) \\ &\leq (1-\eta)^{\ell_t} D(0) + \bar{\Omega}(k) \leq (1-\eta)^{\frac{t}{(NB-1)h}-1} D(0) + \bar{\Omega}(k). \end{aligned} \tag{2.12}$$

Since $\Delta R(k) \leq h\theta$, $D(k)$ is input-to-output stable with ultimate bound $\Xi \leq 4h\theta(NB-2)\frac{1}{\eta} \leq 4h\theta(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1}$, i.e., there exist $\Gamma > 0$ and $0 < \lambda < 1$ such that

$$D(k) \leq \max\{\Gamma\lambda^{\frac{k}{h}}, \Xi\}, \qquad \forall\, k \geq 0. \tag{2.13}$$

Choose as initial state $x_i(0) = r_i(-h)$ for all $i \in \{1, \ldots, N\}$. By Assumption 2.3.4, the following conservation property is satisfied for all $k \geq 0$:

$$\begin{aligned} \sum_{i=1}^{N} x_i(k+h) &= \sum_{i=1}^{N} x_i(k) + \sum_{i=1}^{N} \Delta r_i(k) \\ &= \sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N} \sum_{q=0}^{\frac{t}{h}} \Delta r_i(qh) \\ &= \sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N} (r_i(k) - r_i(-h)) = \sum_{i=1}^{N} r_i(k), \end{aligned} \tag{2.14}$$

where we have used the induction in Line 2 of the above expressions.

It follows from (2.14) that $m(k+h) \leq \frac{1}{N}\sum_{i=1}^{N} r_i(k) \leq M(k+h)$ and thus

$$\max_{i \in V} \limsup_{k\to\infty} |x_i(k) - \frac{1}{N}\sum_{i=1}^{N} r_i(k-h)| \leq \limsup_{k\to\infty} D(k) \leq \Xi.$$

Hence, for any given $\delta_1 > 0$, choosing $h \leq h_1$ gives an steady-state error $\Xi \leq \delta_1$. In other words, choosing a step of size h induces at least an error of order $4\theta(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1}$. ■

The following corollary states an interesting special case of Theorem 7.3.2 when $\lim_{k\to\infty} \Delta R(k) = 0$ for any $h > 0$.

Corollary 2.3.1 *Suppose Assumptions 2.3.1, 2.3.3, 2.3.4 hold for any $h > 0$. If $\lim_{k\to\infty} \Delta R(k) = 0$, then the implementation of the* FODAC *algorithm with any $h > 0$ and initial state $x_i(0) = r_i(-h)$, $i \in \{1, \ldots, N\}$, achieves the dynamic average consensus with a zero steady-state error.*

2.4 Higher-order algorithms for dynamic average consensus

In this section, we present n^{th}-order algorithms for dynamic average consensus where $n \geq 2$. First of all, let us consider the case of $n = 2$. We will assume that the reference inputs satisfy the following condition weaker than Assumption 2.3.2.

Assumption 2.4.1 (Relatively bounded second-order differences) For any $h > 0$, there exists a time invariant constant $\theta_2 > 0$ such that

$$\Delta^{(2)} r_{\max}(k) - \Delta^{(2)} r_{\min}(k) \leq h\theta_2, \quad t \geq 0.$$

Correspondingly, we propose the following *Second-Order Dynamic Average Consensus* algorithm (the SODAC algorithm for short)

$$\begin{aligned} x_i^{[2]}(k+h) &= x_i^{[2]}(k) + \sum_{j\neq i} a_j^i(t)(x_j^{[2]}(k) - x_i^{[2]}(k)) + x_i^{[1]}(k+h), \\ x_i^{[1]}(k+h) &= x_i^{[1]}(k) + \sum_{j\neq i} a_j^i(t)(x_j^{[1]}(k) - x_i^{[1]}(k)) + \Delta^{(2)} r_i(k), \end{aligned} \tag{2.15}$$

and its convergence properties are described in the following theorem and corollary.

Theorem 2.4.1 *Let δ_2 be a positive constant and $h_2 = \frac{\delta_2 \alpha^{N(N+1)B+2}}{16\theta_2(NB-2)^2}$. Under Assumptions 2.3.1, 2.3.3, 2.3.4 and 2.4.1, the implementation of the* SODAC *algorithm with $h \in (0, h_2]$, and initial states $x_i^{[1]}(0) = \Delta r_i(-h)$, $x_i^{[2]}(0) = r_i(-h)$, $i \in \{1, \ldots, N\}$, achieves dynamic average consensus with a nonzero steady-state error upper bounded by δ_2.*

Proof: Note that the algorithm for $x_i^{[1]}(k)$ in the SODAC algorithm has the same form as the FODAC algorithm, and can be obtained from this by replacing $\Delta r_i(k)$ with $\Delta^{(2)} r_i(k)$. Since Assumption 2.4.1 holds, it follows from Theorem 7.3.2 that by choosing the initial state as $x_i^{[1]}(0) = \Delta r_i(-h)$ we can find $\Gamma_1 > 0$ and $0 < \lambda_1 < 1$ such that for all $k \geq 0$ and all $i \in \{1, \ldots, N\}$, there holds that

$$|x_i^{[1]}(k) - \frac{1}{N}\sum_{i=1}^{N} \Delta r_i(k-h)| \leq D^{[1]}(k) \leq \max\{\Gamma_1 \lambda_1^{\frac{k}{h}}, \Xi_1\},$$

where $D^{[1]}(k) = \max_{i\in V} x_i^{[1]}(k) - \min_{i\in V} x_i^{[1]}(k)$ and $\Xi_1 \leq 4h\theta_2(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1}$.

Hence, there exists a finite $\bar{k} \geq 0$ such that $\Gamma_1 \lambda_1^k \leq \Xi_1$ for all $k \geq \bar{k}$. Then for $k \geq \bar{k}$, $x_i^{[2]}(k)$ in the SODAC algorithm can be written in the following way:

$$x_i^{[2]}(k+h) = x_i^{[2]}(k) + \sum_{j\neq i} a_j^i(k)(x_j^{[2]}(k) - x_i^{[2]}(k)) + d_i(k), \tag{2.16}$$

with a reference input $d_i(k) = \frac{1}{N}\sum_{i=1}^{N} \Delta r_i(k) + \vartheta_i(k)$ and $|\vartheta_i(k)| \leq \Xi_1$. Note that for all $k \geq \bar{k}$, there holds that

$$\max_{i\in V} d_i(k) - \min_{i\in V} d_i(k) \leq 2\Xi_1 \leq 4h\theta_2(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1}.$$

Hence, (2.16) has the same form as the FODAC algorithm, and can be obtained from it by replacing $\Delta r_i(k)$ with $d_i(k)$ where $\theta = 4\theta_2(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1}$ in Assumption 2.3.2.

Furthermore, consider as initial states $x_i^{[2]}(0) = r_i(-h)$ for all $i \in \{1, \ldots, N\}$. Similarly to (2.14) with $\Delta r_i(k)$ instead of $r_i(k)$, we can obtain the following conservation property of the SODAC algorithm for every $k \geq 0$

$$\sum_{i=1}^{N} x_i^{[1]}(k+h) = \sum_{i=1}^{N} \Delta r_i(k), \ \sum_{i=1}^{N} x_i^{[2]}(k+h) = \sum_{i=1}^{N} r_i(k).$$

By using similar arguments to those employed in Theorem 7.3.2, we have that there exist $\Gamma_2 > 0$ and $0 < \lambda_2 < 1$ such that for all $k \geq \bar{k}$ and all $i \in V$, there holds

$$|x_i^{[2]}(k) - \frac{1}{N}\sum_{i=1}^{N} r_i(k-h)| \leq D^{[2]}(k) \leq \max\{\Gamma_2 \lambda_2^{\frac{k-\bar{k}}{h}}, \Xi_2\},$$

where $D^{[2]}(k) = \max_{i\in V} x_i^{[2]}(k) - \min_{i\in V} x_i^{[2]}(k)$ and $\Xi_2 = 4h\theta(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1} = 16h\theta_2(NB-2)^2\alpha^{-N(N+1)B+2}$. For any given $\delta_2 > 0$, choosing $h \le h_2$ leads to the property of $\Xi_2 \le \delta$. ■

Corollary 2.4.1 *Suppose Assumptions 2.3.1, 2.3.3, 2.3.4 hold for any $h > 0$. If $\lim_{k\to\infty}(\Delta^{(2)}r_{\max}(k) - \Delta^{(2)}r_{\min}(k)) = 0$, then the implementation of the* SODAC *algorithm with any $h > 0$ and initial states $x_i^{[1]}(0) = \Delta r_i(-h)$, $x_i^{[2]}(0) = r_i(-h)$ for all $i \in \{1,\dots,N\}$ achieves dynamic average consensus with a zero steady-state error.*

Now, let us consider the following general n^{th}*-Order Dynamic Average Consensus* algorithm (the NODAC algorithm for short).

$$\begin{aligned} x_i^{[\ell]}(k+h) &= x_i^{[\ell]}(k) + \sum_{j\neq i} a_j^i(k)(x_j^{[\ell]}(k) - x_i^{[\ell]}(k)) + x_i^{[\ell-1]}(k+h)\,, \\ x_i^{[1]}(k+h) &= x_i^{[1]}(k) + \sum_{j\neq i} a_j^i(k)(x_j^{[1]}(k) - x_i^{[1]}(k)) + \Delta^{(n)}r_i(k), \qquad \ell \in \{2,\dots,n\}\,. \end{aligned} \tag{2.17}$$

Remark 2.4.1 In [123], the authors propose a continuous-time higher-order consensus algorithm to allow higher-order derivatives converge to common values. While related, the problem statement of [123] is different from ours. •

The previous algorithm is the cascade of n FODAC algorithms and can be compactly rewritten in the following vector form

$$\begin{aligned} x^{[\ell]}(k+h) &= A(k)x^{[\ell]}(k) + x^{[\ell-1]}(k+h)\,, \\ x^{[1]}(k+h) &= A(k)x^{[1]}(k) + \Delta^{(n)}r(k), \quad \ell \in \{2,\dots,n\}. \end{aligned}$$

The above NODAC algorithm is able to track the average of reference inputs which satisfy the following condition under which Theorem 2.4.2 holds.

Assumption 2.4.2 (Relatively bounded n^{th}-order differences) For any $h > 0$, there exists a time invariant constant $\theta_n > 0$ such that

$$\Delta^{(n)}r_{\max}(k) - \Delta^{(n)}r_{\min}(k) \le h\theta_n\,, \qquad \forall\, t \ge 0.$$

The following theorem and corollary show the convergence properties of n^{th}-order dynamic average consensus algorithm.

Theorem 2.4.2 *Let δ_n be a positive constant and $h_n = \frac{\delta_n \alpha^{n(\frac{1}{2}N(N+1)B+1)}}{2^{2n}\theta_n(NB-2)^n}$. Under the Assumptions 2.3.1, 2.3.3, 2.3.4 and 2.4.2, the implementation of the* NODAC *algorithm with $h \in (0, h_n]$ and initial states $x_i^{[\ell]}(0) = \Delta^{(n-\ell)} r_i(-h)$ $(\ell = 1, \cdots, n-1)$, $x_i^{[n]}(0) = r_i(-h)$ for all $i \in \{1, \ldots, N\}$, achieves the dynamic average consensus with a nonzero steady-state error upper bounded by δ_n.*

Proof: The case of $n = 1$ has been proven in Theorem 7.3.2. By using similar arguments in Theorem 2.4.1, we can finish the proof in an inductive way. ■

Corollary 2.4.2 *Suppose Assumptions 2.3.1, 2.3.3, 2.3.4 hold for any $h > 0$. If $\lim_{t\to\infty}(\Delta^{(n)} r_{\max}(t) - \Delta^{(n)} r_{\min}(t)) = 0$, then the implementation of the* NODAC *algorithm for any $h > 0$ and initial states $x_i^{[\ell]}(0) = \Delta^{(n-\ell)} r_i(-h)$ $(\ell = 1, \cdots, n-1)$, $x_i^{[n]}(0) = r_i(-h)$ for all $i \in \{1, \ldots, N\}$, achieves dynamic average consensus with a zero steady-state error.*

2.5 Extensions

This section includes some remarks about the possible extension of the presented results.

2.5.1 Discussion on the choice of the order for the dynamic average consensus algorithm

If Assumption 2.4.2 holds, m^{th}-order dynamic consensus algorithm can reach the dynamic average consensus for any $m > n$. However, we need a smaller h than the NODAC algorithm to render the steady-state error smaller than the given bound. Then there is no advantage to use m^{th}-order average dynamic consensus algorithm when Assumption 2.4.2 is satisfied.

2.5.2 Discussion on Assumption 2.4.2

It can be shown that for any n^{th}-order polynomial $f(k) = \sum_{i=0}^{n} a_i k^i$, there holds that $\Delta^{(n)} f(k) = a_n n! h$. Hence, any set of n^{th}-order polynomials satisfies Assumption 2.4.2 with $\theta_{n+1} = 0$.

If the reference inputs $r_i(k)$ take the form of $r_i(k) = v(k) + \tilde{r}_i(k)$, $i \in V$, and the function $\tilde{r}_i(k)$ is a linear combination of polynomials, the logarithmic function, periodic functions and other functions whose n^{th}-order differences are bounded, then Assumption 2.4.2 also holds for any common $v(k)$ even when n^{th}-order difference of $v(k)$ explodes, e.g., like the exponential function. It is worth mentioning that it is unnecessary for Assumption 2.4.2 to hold that $\Delta^{(n)} r_i(k)$ be bounded for all i, $k \geq 0$.

2.5.3 Discussion on Assumption 2.3.1

In the case that the communication is symmetric; i.e., when $(i, j) \in E(k)$ if and only if $(j, i) \in E(k)$, then Assumption 2.3.1 in Corollary 2.4.2 can be weakened into the following one:

Assumption 2.5.1 (Eventual strong connectivity) For any time instant $k \geq 0$, the directed graph $(V, \cup_{s \geq k} E(s))$ is strongly connected.

Corollary 2.5.1 *Suppose Assumptions 2.5.1, 2.3.3, 2.3.4 and the relation of* $\lim_{t \to \infty} (\Delta^{(n)} r_{\max}(k) - \Delta^{(n)} r_{\min}(k)) = 0$ *hold for any* $h > 0$. *If* $\mathcal{G}(k)$ *is undirected, the implementation of the* NODAC *algorithm with any* $h > 0$ *and initial states* $x_i^{[\ell]}(0) = \Delta^{(n-\ell)} r_i(-h)$ *(*$\ell = 1, \cdots, n-1$*),* $x_i^{[n]}(0) = r_i(-h)$ *for all* $i \in \{1, \ldots, N\}$*, achieves dynamic average consensus with a zero steady-state error.*

If the communication is symmetric, Assumption 2.3.1 in Corollary 2.4.2 can also be replaced with the assumption in Proposition 2 of [93]; i.e., for any time instant $k \geq 0$, there is a node connected to all other nodes in the undirected graph $(V, \cup_{s \geq k} E(s))$. It is interesting to further think about the weaker assumption in Proposition 1 of [93]; i.e., there exists an integer $B \geq 1$ such that for any time instant $k \geq 0$, there is a node connected to all other nodes in the directed graph $(V, E(k) \cup E(k+h) \cup \cdots \cup E(k+(B-1)h))$.

2.5.4 The robustness to joining and leaving of nodes

If some nodes join the network at some time $t_0 > 0$ during the implementation of the NODAC algorithm, all the nodes in the new network are able to reach the new dynamic average consensus as long as the joining nodes choose their "initial" states at time t_0 according to the rules in Theorem 2.4.2 and Assumptions 2.3.1, 2.3.3, 2.3.4 and 2.4.2 are satisfied for the new network.

To make the NODAC algorithm adaptive to the departure of some nodes, we slightly modify its implementation. Assume node κ wants to leave the network at some time k_0 and $(\kappa, i) \in E(k_0)$ for some node i. Node κ sends the value of $x_\kappa^{[n]}(k_0) - r_\kappa(k_0)$ to node i, and then node i updates its values according to the NODAC algorithm by replacing the top equation in the NODAC algorithm with the following

$$\begin{aligned} x_i^{[n]}(k_0+h) &= x_i^{[n]}(t_0) + \sum_{j\neq i} a_j^i(k_0)(x_j^{[n]}(k_0) - x_i^{[n]}(k_0)) \\ &+ x_i^{[n-1]}(k_0+h) + (x_\kappa^{[n]}(k_0) - r_\kappa(k_0))\,. \end{aligned} \tag{2.18}$$

All other remaining nodes update their values according to the NODAC algorithm at time k_0. After time k_0, the remaining nodes in the network update their values according to the NODAC algorithm, and then the dynamic average consensus is reached if Assumptions 2.3.1, 2.3.3, 2.3.4 and 2.4.2 are satisfied for the new network. The update law (2.18) ensures the following conservation property:

$$\sum_{j\neq\kappa} x_j^{[n]}(k_0+h) = \sum_{j\neq\kappa} r_j(k_0).$$

2.5.5 Asynchronous first-order dynamic average consensus algorithm

In this part, the asynchronism is incorporated into the FODAC algorithm. First, let us define the following notations: a set T_i of time instants when node i measures $r_i(k)$; a variable $\tau_i(k)$ which denotes the latest time instant before time k in T_i. We adopt the partial asynchronism time model adapted from [14]; i.e., there exists a positive integer B_a such that $k - B_a h \leq \tau_i(k) \leq k$ for each $i \in V$ and each $k \geq 0$.

Asynchronous First-Order Dynamic Average Consensus algorithm (asynchronous FODAC algorithm for short) is given by: if $k \in T_i$, node i measures $r_i(k)$ and updates its value according to the following rule:

$$x_i(k+h) = x_i(k) + \sum_{j \neq i} a_j^i(k)(x_j(k) - x_i(k)) + (r_i(k) - r_i(\tau_i(k))); \tag{2.19}$$

otherwise, node i sets $x_i(k+h) = x_i(k)$.

Assumption 2.5.2 (Bounded first-order differences) For any $h > 0$, there exist time invariant constants $\rho_1 > 0$ and $\rho_2 < 0$ such that $\Delta r_{\max}(k) \leq h\rho_1$ and $\Delta r_{\min}(k) \geq h\rho_2$ hold for all $k \geq 0$.

Theorem 2.5.1 *Let $\tilde{\delta}_1$ be a positive constant and*

$$\tilde{h}_1 = \frac{\tilde{\delta}_1}{4B_a(\rho_1 - \rho_2)(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1} + \max\{\rho_1, |\rho_2|\}(B_a - 1)}.$$

Under Assumptions 2.3.1, 2.3.3, 2.3.4 and 2.5.2, the implementation of the asynchronous NODAC *algorithm with the partial asynchronism time model, $h \in (0, \bar{h}_1]$ and initial conditions $x_i(0) = r_i(-h)$, $i \in \{1, \ldots, N\}$, achieves dynamic average consensus with a nonzero steady-state error upper bounded by $\tilde{\delta}_1$.*

Proof: Here we only provide a sketch of the proof. Following the same lines in Lemma 2.3.1 and Theorem 7.3.2, we utilize Assumption 2.5.2 to have that there exist $\Gamma > 0$ and $\lambda > 0$ such that

$$D(k) \leq \max\{\Gamma\lambda^{\frac{t}{h}}, \Xi\}, \qquad \forall\, t \geq 0 \tag{2.20}$$

where $\Xi \leq 4B_a h(\rho_1 - \rho_2)(NT-2)\frac{1}{\eta} \leq 4B_a h(\rho_1 - \rho_2)(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1}$.

Since $x_i(0) = r_i(-h)$ for all $i \in \{1, \ldots, N\}$, the conservation property of $\sum_{i=1}^N x_i(k) = \sum_{i=1}^N r_i(\tau_i(k))$ holds. From Assumption 2.5.2 and the property of $t - \tau_i(k) < B_a h$, it can be shown that

$$\frac{1}{N}\sum_{i=1}^N r_i(k-h) - (B_a - 1)h\rho_1 \leq \frac{1}{N}\sum_{i=1}^N x_i(k) \leq \frac{1}{N}\sum_{i=1}^N r_i(k-h) - (B_a - 1)h\rho_2. \tag{2.21}$$

Combining (2.20) and (2.21) gives the following estimate for the steady-state error

$$\max_{i\in V} \limsup_{k\to\infty} |x_i(k) - \frac{1}{N}\sum_{i=1}^{N} r_i(k-h)|$$
$$\leq 4B_a h(\rho_1 - \rho_2)(NB-2)\alpha^{-\frac{1}{2}N(N+1)B+1} + \max\{\rho_1, |\rho_2|\}(B_a - 1)h.$$

■

2.6 Simulations

In this section, we present several examples with their simulations to demonstrate the effectiveness of our theoretical results.

Example 1

We first illustrate the conclusion of Corollary 2.3.1 with a simulation. Let us consider a network consisting of four nodes, labeled 1 through 4. Suppose that the graph $\mathcal{G}(k)$ satisfies Assumption 2.3.1 with $B = 4$. The reference inputs are given by:

$$r_1(k) = 5\sin k + \frac{10}{k+2} + 1, \quad r_2(k) = 5\sin k + \frac{10}{(k+2)^2} + 2,$$
$$r_3(k) = 5\sin k + \frac{10}{(k+2)^3} + 3, \quad r_4(k) = 5\sin k + 10e^{-k} + 4.$$

Figure 2.1 shows that the tracking errors of the nodes asymptotically converge to zero.

Example 2

Now, we provide an example to illustrate the robustness of the NODAC algorithm. Consider a network with five nodes. The graph $\mathcal{G}$ is fixed when no node joins or leaves the network. The reference inputs are given by:

$$r_1(k) = k + 1 + 5\sin k, \quad r_2(k) = k - 1 + 5\sin k,$$
$$r_3(k) = k + 5\sin k, \quad r_4(k) = k + 50 + 5\sin k, \quad r_5(k) = k - 50 + 5\sin k.$$

It can be readily verified that Assumption 2.3.2 holds with $\theta = 0$. Thus we choose the FODAC algorithm with $h = 1$. During the simulation, node 5 leaves the network at time 50 and joins the network at time 100 again. Figure 2.2 provides the consensus states in comparison with the average of the reference inputs.

Example 3

In this part, we present an example of the asynchronous FODAC algorithm. The network consists of five nodes. The topology is switching and satisfies Assumption 2.3.1 with $B = 3$. The reference inputs for the nodes are as follows

$$r_1(k) = -k - 2, \quad r_2(k) = -2k - 2,$$
$$r_3(k) = 5k + 0.5, \quad r_4(k) = 2k + 1.5, \quad r_5(k) = k + 2.$$

Then Assumption 2.5.2 holds with $\rho_1 = 5$ and $\rho_2 = -2$. For the given $\tilde{\delta}_1 = 1$, we choose $h = 0.01$ for the algorithm. The consensus states with the average of the reference inputs are shown in Figure 2.3. The oscillation in the evolution is induced by the changes of network topologies. And the steady-state error is upper bounded by 0.9.

2.7 Conclusions

We have proposed a class of discrete-time dynamic average consensus algorithms and analyze their convergence properties. Due to slow convergence rates of the algorithms, tracking is shown at the expense of frequent communication and higher throughput. This chapter is based on the following published papers:

(JP-2) M. Zhu and S. Martínez, "On discrete-time dynamic average consensus", *Automatica*, 46(2), pages $322 - 329$, 2010.

(CP-3) M. Zhu and S. Martínez, "Dynamic average consensus on synchronous communication networks", *The* 27^{th} *American Control Conference*, pages $4382 - 4387$, Seattle, USA, Jun. 2008.

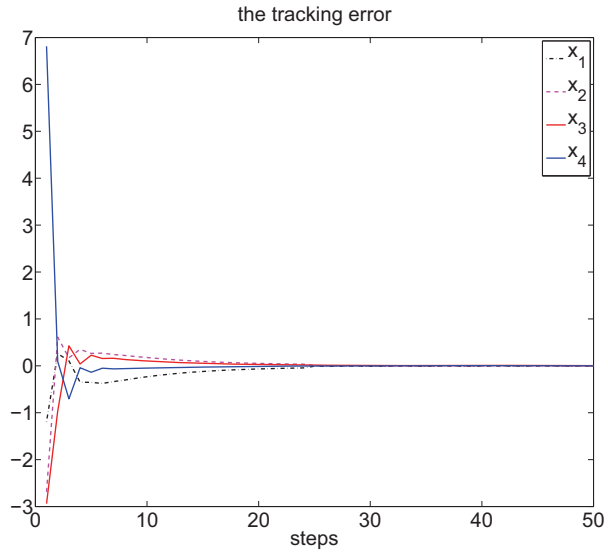

Figure 2.1: The tracking errors of the FODAC algorithm

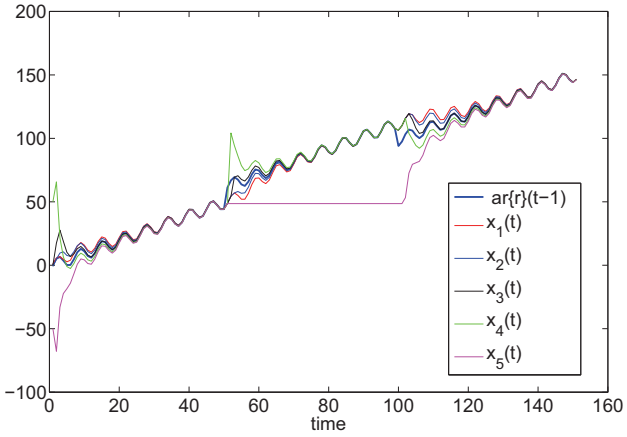

Figure 2.2: Evolution of the states of the FODAC algorithm in comparison with the average of the inputs with the joining and leaving nodes

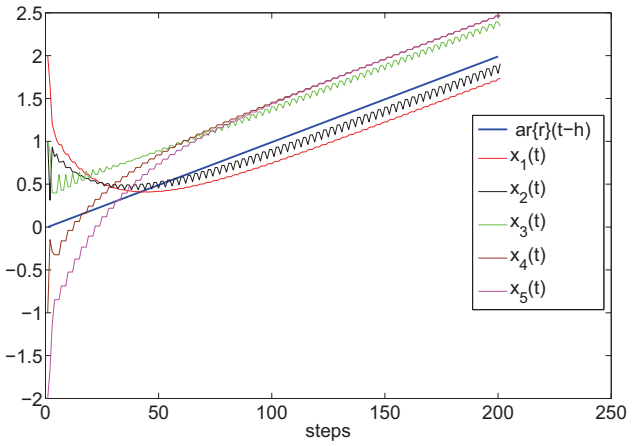

Figure 2.3: Evolution of the states of the asynchronous first-order dynamic average consensus algorithm in comparison with the average of the inputs

Chapter 3

Distributed quantized average consensus

3.1 Introduction

In real-world communication networks, the capacities of communication channels and the memory capacities of agents are finite. Furthermore, the computations can only be carried out with finite precision. From a practical point of view, real-valued averaging algorithms are not feasible and these realistic constraints motivate the problem of average consensus via quantized information. Another motivation for distributed quantized averaging is load balancing with indivisible tasks. Prior work on distributed quantized averaging over fixed graphs includes [12, 28, 29, 69]. Recently, [98] examines quantization effects on distributed averaging algorithms over time-varying topologies. As in [69], we focus on quantized averaging algorithms preserving the sum of the state values at each iteration. This setup has the following properties of interest: the sum cannot be changed in some situations, such as load balancing; and the constant sum leads to a small steady-state error with respect to the average of individual initial states. This error is equal to either one quantization step size or zero (when the average of the initial states is located at one of the quantization levels) and thus is independent of N. This is in contrast to the setup in [98] where the sum of the states is not

maintained, resulting in a steady-state error of the order $O(N^3 \log N)$.

The convergence time is typically utilized to quantify the performance of distributed averaging algorithms. The authors in [20, 111] study the convergence time of real-valued averaging, while [69, 98] discuss the case of quantized averaging. The polynomial bounds of the expected convergence time on fixed complete and linear graphs are derived in [69]. Recently, the authors in [98] give a polynomial bound on the convergence time of a class of quantized averaging algorithms over switching topologies. Among the papers aforementioned, [20, 98, 111] require global synchronization, and [69] needs some global information (e.g, a centralized entity or the total number of the edges) to explicitly bound the expected convergence times. However, real-world communication networks are inherently asynchronous environment and lack of centralized coordination.

Statement of contributions. The present chapter proposes a class of distributed quantized averaging algorithms on asynchronous communication networks with fixed, switching and random topologies. The algorithms are shown to asymptotically reach quantized average consensus in probability. Furthermore, we utilize meeting times of two random walks on graphs as a unified approach to derive polynomial bounds on the expected convergence times of our presented algorithms. To the best of our knowledge, this note is the first step toward characterizing the expected convergence times of completely distributed quantized averaging algorithms over asynchronous communication networks.

3.2 Preliminaries and problem statement

In this section, we present the problem formulation along with some notation and terminology.

Asynchronous time model. In this note, we will employ the asynchronous time model proposed in [20]. More precisely, consider a network of N nodes, labeled 1 through N. Each node has a clock which ticks according to a rate 1 Poisson process. Hence, the inter-tick times at each node are random variables with rate 1 exponential distribution, independent across nodes and independent over time.

By the superposition theorem for Poisson processes, this setup is equivalent to a single global clock modeled as a rate N Poisson process ticking at times $\{Z_t\}_{t\geq 0}$. By the orderliness property of Poisson processes, the clock ticks do not occur simultaneously. The inter-agent communication and the update of consensus states only occur at $\{Z_t\}_{t\geq 0}$. In the reminder of this chapter, the time instant k will be discretized according to $\{Z_t\}_{t\geq 0}$ and defined in terms of the number of clock ticks.

Network model. We will employ the undirected graph $\mathcal{G}(k) = (V, E(k))$ to model the network. Here $V := \{1, \cdots, N\}$ is the vertex set, and an edge $(j, i) \in E(k)$ if and only if node j can receive the message from node i (e.g., node j is within the communication range of node i) at time t. The neighbors of node i at time t are denoted by $\mathcal{N}_i(k) = \{j \in V \mid (j, i) \in E(k) \text{ and } j \neq i\}$. The state of node i at time t is denoted by $x_i(k) \in \mathbb{R}$ and the network state is denoted by $x(k) = (x_1(k), \cdots, x_N(k))^T$. Suppose the initial states $x_i(0) \in [U_{\min}, U_{\max}]$ for all $i \in V$ and some real numbers $U_{\min}$ and $U_{\max}$.

Quantization scheme. Let R denote the number of bits per sample. The total number of quantization levels can be represented by $L = 2^R$ and the step size is $\Delta = (U_{\max} - U_{\min})/2^R$. The set of quantization levels, $\{\omega_1, \cdots, \omega_L\}$, is a strictly increasing sequence in $\mathbb{R}$ and the levels are uniformly spaced in the sense that $\omega_{i+1} - \omega_i = \Delta$. A quantizer $\mathcal{Q} : [U_{\min}, U_{\max}] \to \{\omega_1, \cdots, \omega_L\}$ is adopted to quantize the message $u \in [U_{\min}, U_{\max}]$ in such a way that $\mathcal{Q}(u) = \omega_i$ if $u \in [\omega_i, \omega_{i+1})$ for some $i \in \{1, \cdots, L-1\}$. Assume that the initial states $x_i(0)$ for all $i \in V$ are multiples of Δ.

Problem statement. The problem of interest in this chapter is to design distributed averaging algorithms which the nodes can utilize to update their states by communicating with neighbors via quantized messages in an asynchronous setting. Ultimately, quantized average consensus is reached in probability; i.e., for any initial state $x(0)$, there holds that $\lim_{k\to\infty} \mathbb{P}(x(k) \in \mathcal{W}(x(0))) = 1$. The set $\mathcal{W}(x(0))$ is dependent on initial state $x(0) \in \mathbb{R}^N$ and defined as follows. If $\bar{x}(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$ is not a multiple of Δ, then $\mathcal{W}(x(0)) = \{x \in \mathbb{R}^N \mid x_i \in \{\mathcal{Q}(\bar{x}(0)), \mathcal{Q}(\bar{x}(0)) + \Delta\}\}$; otherwise, $\mathcal{W}(x(0)) = \{x \in \mathbb{R}^N \mid x_i = \bar{x}(0)\}$. Now it is clear that the steady-state error is at most Δ after quantized average consensus is reached.

Notions of random walks on graphs. In this chapter, random walks on graphs play an important role in characterizing the convergence properties of our quantized averaging algorithms. The following definitions are generalized from those defined for fixed graphs in [23, 33].

Definition 3.2.1 (Random walks) A random walk on the graph $\mathcal{G}(k)$ under the transition matrix $P(k) = (p_{ij}(k))$, starting from node v at time s, is a stochastic process $\{X(k)\}_{k\geq s}$ such that $X(s) = v$ and $\mathbb{P}(X(k+1) = j \mid X(k) = i) = p_{ij}(k)$. A random walk is said to be simple if for any $i \in V$, $p_{ii}(k) = 0$ for all $k \geq 0$; otherwise, it is said to be natural. •

Definition 3.2.2 (Hitting time) Consider a random walk on the graph $\mathcal{G}(k)$, beginning from node i at time s and evolving under the transition matrix $P(k)$. The hitting time from node i to the set $\Lambda \subseteq V$, denoted as $H_{(\mathcal{G}(k),P(k),s)}(i,\Lambda)$, is the expected time it takes this random walk to reach the set Λ for the first time. We denote $H_{(\mathcal{G}(k),P(k))}(\Lambda) = \sup_{s\geq 0} \max_{i\in V} H_{(\mathcal{G}(k),P(k),s)}(i,\Lambda)$ as the hitting time to reach the set Λ. The hitting time of the pair i, j, denoted as $H_{(\mathcal{G}(k),P(k),s)}(i,j)$, is the expected time it takes this random walk to reach node j for the first time. Denote $H_{(\mathcal{G}(k),P(k))} = \sup_{s\geq 0} \max_{i,j\in V} H_{(\mathcal{G}(k),P(k),s)}(i,j)$ as the hitting time of going between any pair of nodes. •

Definition 3.2.3 (Meeting time) Consider two random walks on the graph $\mathcal{G}(k)$ under the transition matrix $P(k)$, starting at time s from node i and node j respectively. The meeting time $M_{(\mathcal{G}(k),P(k),s)}(i,j)$ of these two random walks is the expected time it takes them to meet at some node for the first time. The meeting time on the graph $\mathcal{G}(k)$ is defined as $M_{(\mathcal{G}(k),P(k))} = \sup_{s\geq 0} \max_{i,j\in V} M_{(\mathcal{G}(k),P(k),s)}(i,j)$. •

For the ease of notation, we will drop the subscript s in the hitting time and meeting time notions for fixed graphs. The following notion is only defined for fixed graphs.

Definition 3.2.4 ***(Irreducibility and reversibility)*** A random walk on the graph $\mathcal{G}$ is irreducible if it is possible to get to any other node from any node. An irreducible random walk with stationary distribution π is called reversible if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in V$. •

Notations. For $\alpha \in \mathbb{R}$, define $V_\alpha : \mathbb{R}^N \to \mathbb{R}$ as $V_\alpha(x) = \sum_{i=1}^N (x_i - \alpha)^2$. We define $J : \mathbb{R}^N \to \mathbb{R}$ as $J(x) = (\max_{i\in V} x_i - \min_{i\in V} x_i)/\Delta$. Denote the set $\Theta = \{(\kappa, \kappa) \mid \kappa \in V\}$. The *distribution* of a vector $x \in \mathbb{R}^N$ is defined to be the list $\{(q_1, m_1), (q_2, m_2), \cdots, (q_k, m_\kappa)\}$ for some $\kappa \in V$ where $\sum_{\ell=1}^{\kappa} m_\ell = N$, $q_i \neq q_j$ for $i \neq j$ and m_ℓ is the cardinality of the set $\{i \in V \mid x_i = q_\ell\}$. The cardinality of the set M is denoted by $|M|$.

3.3 Asynchronous distributed quantized averaging on fixed graphs

In this section, we propose and analyze an asynchronous distributed quantized averaging algorithm on the fixed and connected graph $\mathcal{G}$. Main references are [69] on quantized gossip algorithms and [23] on the meeting time of two simple random walks on fixed graphs.

3.3.1 Proposed algorithm

The *asynchronous distributed quantized averaging algorithm on the fixed and connected graph* $\mathcal{G}$ (AF, for short) is described as follows. Suppose node i's clock ticks at time t. Node i randomly chooses one of its neighbors, say node j, with equal probability. Node i and j then execute the following local computation. If $x_i(k) \geq x_j(k)$, then

$$x_i(k+1) = x_i(k) - \delta, \quad x_j(k+1) = x_j(k) + \delta; \tag{3.1}$$

otherwise,

$$x_i(k+1) = x_i(k) + \delta, \quad x_j(k+1) = x_j(k) - \delta, \tag{3.2}$$

where $\delta = \frac{1}{2}(x_i(k) - x_j(k))$ if $\frac{x_i(k)-x_j(k)}{2\Delta}$ is an integer; otherwise, $\delta = \mathcal{Q}(\frac{1}{2}(x_i(k) - x_j(k))) + \Delta$. Every other node $k \in V \setminus \{i, j\}$ preserves its current state; i.e., $x_k(k+1) = x_k(k)$.

Remark 3.3.1 The precision $\frac{\Delta}{2}$ is sufficient for the computation of δ and thus the update laws (3.1) and (3.2). It is easy to verify that $x_i(k) \in [U_{\min}, U_{\max}]$ and $x_i(k)$

are multiples of Δ for all $i \in V$ and $k \geq 0$. Furthermore, the sum of the state values is preserved at each iteration.

If $|x_i(k) - x_j(k)| = \Delta$, the update laws (3.1) and (3.2) become $x_i(k+1) = x_j(k)$ and $x_j(k+1) = x_i(k)$. Such update is referred to as a *trivial average* in [69]. If $|x_i(k) - x_j(k)| > \Delta$, then (3.1) or (3.2) is referred to as a *non-trivial average*. Although it does not directly contribute to reaching quantized average consensus, trivial average helps the information flow over the network. •

3.3.2 The meeting time of two natural random walks on the fixed graph $\mathcal{G}$

To analyze the convergence properties of AF, we first study a variation of the problem in [33], namely, *the meeting time of two natural random walks on the fixed graph* $\mathcal{G}$. More precisely, assume that the fixed graph $\mathcal{G}$ be undirected and connected. Initially, two tokens are placed on the graph $\mathcal{G}$; at each time, one of the tokens is chosen with probability $\frac{1}{N}$ and the chosen token moves to one of the neighboring nodes with equal probability. What is the meeting time for the tokens?

The tokens move as two natural random walks with the transition matrix $P_{\rm AF}$ on the graph $\mathcal{G}$. The matrix $P_{\rm AF} = (\tilde{p}_{ij}) \in \mathbb{R}^{N\times N}$ is given by $\tilde{p}_{ii} = 1 - \frac{1}{N}$ for $i \in V$, $\tilde{p}_{ij} = \frac{1}{N|\mathcal{N}_i|}$ for $(i,j) \in E$. The meeting time of these two natural random walks is denoted as $M_{(\mathcal{G},P_{\rm AF})}$. Denote any of these two natural random walks as $X_{\mathcal{N}}$. Correspondingly, we construct a simple random walk, say $X_{\mathcal{S}}$, with the transition matrix $P_{\rm SF}$ on the graph $\mathcal{G}$ where the matrix $P_{\rm SF} = (p_{ij}) \in \mathbb{R}^{N\times N}$ is given by $p_{ii} = 0$ and $p_{ij} = \frac{1}{|\mathcal{N}_i|}$ if $(i,j) \in E$. The hitting times of the random walks $X_{\mathcal{S}}$ and $X_{\mathcal{N}}$ are denoted as $H_{(\mathcal{G},P_{\rm SF})}$ and $H_{(\mathcal{G},P_{\rm AF})}$, respectively.

Proposition 3.3.1 *The meeting time of two natural random walks with transition matrices* $P_{\rm AF}$ *on the fixed graph* $\mathcal{G}$ *satisfies that* $M_{(\mathcal{G},P_{\rm AF})} \leq 2NH_{(\mathcal{G},P_{\rm SF})} - N$.

Proof: Since the fixed graph $\mathcal{G}$ is undirected and connected, the random walks $X_{\mathcal{N}}$ and $X_{\mathcal{S}}$ are irreducible. The reminder of the proof is based on the following claims:

(i) It holds that $H_{(\mathcal{G},P_{\rm AF})} \geq N$.

(ii) For any pair $i, j \in V$ with $i \neq j$, we have $H_{(\mathcal{G},P_{\rm AF})}(i,j) = NH_{(\mathcal{G},P_{\rm SF})}(i,j)$.

(iii) For any $i, j, \iota \in V$, the following equality holds:

$$\begin{aligned} &H_{(\mathcal{G},P_{\rm AF})}(i,j) + H_{(\mathcal{G},P_{\rm AF})}(j,\iota) + H_{(\mathcal{G},P_{\rm AF})}(\iota,i) \\ &= H_{(\mathcal{G},P_{\rm AF})}(i,\iota) + H_{(\mathcal{G},P_{\rm AF})}(\iota,j) + H_{(\mathcal{G},P_{\rm AF})}(j,i). \end{aligned}$$

(iv) There holds that $M_{(\mathcal{G},P_{\rm AF})} \leq 2H_{(\mathcal{G},P_{\rm AF})} - N$.

Now, let us prove each of the above claims.

(i) The quantity $H_{(\mathcal{G},P_{\rm AF})}(i,j)$ reaches the minimum when $\mathcal{N}_i = \{j\}$. We now consider the graph $\mathcal{G}$ with $\mathcal{N}_i = \{j\}$ and compute $H_{(\mathcal{G},P_{\rm AF})}(i,j)$. The probability that $X_{\mathcal{N}}$ stays up with node i before time ℓ and moves to node j at time ℓ is $\frac{1}{N}(1-\frac{1}{N})^{\ell-1}$. Then, we have $H_{(\mathcal{G},P_{\rm AF})}(i,j) = \sum_{\ell=1}^{+\infty} \ell \frac{1}{N}(1-\frac{1}{N})^{\ell-1} = N$ and Claim (i) holds.

(ii) For any pair $i, j \in V$ with $i \neq j$, it holds that

$$H_{(\mathcal{G},P_{\rm AF})}(i,j) = \sum_{\iota \in \mathcal{N}_i} \frac{1}{N|\mathcal{N}_i|}(H_{(\mathcal{G},P_{\rm AF})}(\iota,j)+1) + (1-\frac{1}{N})(H_{(\mathcal{G},P_{\rm AF})}(i,j)+1).$$

Hence, we have that $H_{(\mathcal{G},P_{\rm AF})}(i,j) = N + \sum_{\iota \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} H_{(\mathcal{G},P_{\rm AF})}(\iota,j)$. Furthermore, $H_{(\mathcal{G},P_{\rm SF})}(i,j) = \sum_{\iota \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|}(H_{(\mathcal{G},P_{\rm SF})}(\iota,j)+1) = 1 + \sum_{\iota \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_i|} H_{(\mathcal{G},P_{\rm SF})}(\iota,j)$. Hence, Claim (ii) holds.

(iii) Denote by $\pi_i = |\mathcal{N}_i|/\mathcal{N}_{\max}$ and $\pi = (\pi_1, \cdots, \pi_N)^T$ where $\mathcal{N}_{\max} = \max_{i \in V}\{|\mathcal{N}_i|\}$. Since $P_{\rm AF}^T \pi = \pi$, then π is the stationary distribution of the random walk $X_{\mathcal{N}}$. Furthermore, for any pair $i, j \in V$, we have $\pi_i \tilde{p}_{ij} = \frac{|\mathcal{N}_i|}{\mathcal{N}_{\max}} \frac{1}{N|\mathcal{N}_i|} = \frac{1}{N\mathcal{N}_{\max}} = \pi_j \tilde{p}_{ji} = \frac{|\mathcal{N}_j|}{\mathcal{N}_{\max}} \frac{1}{N|\mathcal{N}_j|}$ and thus the random walk $X_{\mathcal{N}}$ is reversible. From Lemma 2 of [33] it follows that Claim (iii) holds.

(iv) Claim (iv) is an extension of Theorem 2 in [33]. An immediate result of Claim (iii) gives a node-relation on V; i.e., $i \leq j$ if and only if $H_{(\mathcal{G},P_{\rm AF})}(i,j) \leq H_{(\mathcal{G},P_{\rm AF})}(j,i)$. This relation is transitive and constitutes a pre-order on V. Then there exists a node u satisfying $H_{(\mathcal{G},P_{\rm AF})}(v,u) \geq H_{(\mathcal{G},P_{\rm AF})}(u,v)$ for any other node $v \in V$. Such a node u is called *hidden*. As in [33], we define a potential function Φ by $\Phi(i,j) = H_{(\mathcal{G},P_{\rm AF})}(i,j) + H_{(\mathcal{G},P_{\rm AF})}(j,u) - H_{(\mathcal{G},P_{\rm AF})}(u,j)$.

Define the functions $\Phi(\bar{i}, j)$ and $M_{(\mathcal{G},P_{\mathrm{AF}})}(\bar{i}, j)$ below, the averages of the functions Φ and $M_{(\mathcal{G},P_{\mathrm{AF}})}$ over the neighbors of node i and j, respectively:

$$\Phi(\bar{i}, j) = \frac{1}{|\mathcal{N}_i|}\sum_{\iota\in\mathcal{N}_i}\Phi(\iota, j) = \frac{1}{|\mathcal{N}_i|}\sum_{\iota\in\mathcal{N}_i} H_{(\mathcal{G},P_{\mathrm{AF}})}(\iota, j) + H_{(\mathcal{G},P_{\mathrm{AF}})}(j, u) - H_{(\mathcal{G},P_{\mathrm{AF}})}(u, j),$$

$$M_{(\mathcal{G},P_{\mathrm{AF}})}(\bar{i}, j) = \frac{1}{|\mathcal{N}_i|}\sum_{\iota\in\mathcal{N}_i} M_{(\mathcal{G},P_{\mathrm{AF}})}(\iota, j).$$

In Claim (ii), we have shown that $H_{(\mathcal{G},P_{\mathrm{AF}})}(i, j) = \sum_{\iota\in\mathcal{N}_i}\frac{1}{|\mathcal{N}_i|}H_{(\mathcal{G},P_{\mathrm{AF}})}(\iota, j) + N$. Thus, $\Phi(\bar{i}, j) + N = \Phi(i, j)$. Similarly, $M_{(\mathcal{G},P_{\mathrm{AF}})}(\bar{i}, j) + N = M_{(\mathcal{G},P_{\mathrm{AF}})}(i, j)$.

We are now in a position to show that for any pair $i, j \in V$, it holds that

$$M_{(\mathcal{G},P_{\mathrm{AF}})}(i, j) \leq \Phi(i, j). \tag{3.3}$$

Assume that (3.3) does not hold. Let ϕ be $\phi = \max_{w,v\in V}(M_{(\mathcal{G},P_{\mathrm{AF}})}(w, v) - \Phi(w, v)) > 0$. Choose a pair of i, j with minimum distance among the set $\Xi = \{(w, v) \in V \times V \mid M_{(\mathcal{G},P_{\mathrm{AF}})}(w, v) - \Phi(w, v) = \phi\}$. Toward this end, consider the following two cases:

(1) $j \in \mathcal{N}_i$. Observe that the following holds:

$$\Phi(j, j) = H_{(\mathcal{G},P_{\mathrm{AF}})}(j, j) + H_{(\mathcal{G},P_{\mathrm{AF}})}(j, u) - H_{(\mathcal{G},P_{\mathrm{AF}})}(u, j) \geq 0 = M_{(\mathcal{G},P_{\mathrm{AF}})}(j, j).$$

We have $\Phi(\bar{i}, j) + \phi > M_{(\mathcal{G},P_{\mathrm{AF}})}(\bar{i}, j)$ and thus

$$\begin{aligned} M_{(\mathcal{G},P_{\mathrm{AF}})}(i, j) &= \Phi(i, j) + \phi = N + \Phi(\bar{i}, j) + \phi \\ &> N + M_{(\mathcal{G},P_{\mathrm{AF}})}(\bar{i}, j) = M_{(\mathcal{G},P_{\mathrm{AF}})}(i, j). \end{aligned} \tag{3.4}$$

(2) $j \notin \mathcal{N}_i$. There exists node $\iota \in \mathcal{N}_i$ such that node ι is closer to node j than node i. Since the pair of i, j has the minimum distance in the set Ξ, we have $M_{(\mathcal{G},P_{\mathrm{AF}})}(\iota, j) - \Phi(\iota, j) < \phi$. It yields that $\Phi(\bar{i}, j) + \phi > M_{(\mathcal{G},P_{\mathrm{AF}})}(\bar{i}, j)$, and thus (3.4) holds.

In both cases, we get to the contradiction $M_{(\mathcal{G},P_{\mathrm{AF}})}(i, j) > M_{(\mathcal{G},P_{\mathrm{AF}})}(i, j)$, and thus (3.3) holds.

Combining Claims (i), (ii) and inequality (3.3) gives the desired result of $M_{(\mathcal{G},P_{\mathrm{AF}})} \leq 2NH_{(\mathcal{G},P_{\mathrm{SF}})} - N$.■

3.3.3 Convergence analysis

We now proceed to analyze the convergence properties of AF. The convergence time of AF is a random variable defined as follows: $T_{\text{con}}(x(0)) = \inf\{k \mid x(k) \in \mathcal{W}(x(0))\}$, where $x(k)$ starts from $x(0)$ and evolves under AF. Choose $V_{\bar{x}(0)}(x) = \sum_{i=1}^{N}(x_i - \bar{x}(0))^2$ as a Lyapunov function candidate for AF. One can readily see that $V_{\bar{x}(0)}(x(k+1)) = V_{\bar{x}(0)}(x(k))$ when a trivial average occurs and $V_{\bar{x}(0)}(x)$ reduces at least $2\Delta^2$ when a non-trivial average occurs. Hence, $V_{\bar{x}(0)}(x)$ is non-increasing along the trajectories, and the number of non-trivial averages is at most $\frac{1}{2\Delta^2}V_{\bar{x}(0)}(x(0))$. Define the set $\Psi = \{x \in \mathbb{R}^N \mid \text{the distribution of } x \text{ is } \{(0,1),(\Delta,N-2),(2\Delta,1)\}\}$ and denote $\mathbb{E}[T_\Psi] = \max_{x(0)\in\Psi}\mathbb{E}[T_{\text{con}}(x(0))]$. It is clear that the expected time between any two consecutive non-trivial averages is not larger than $\mathbb{E}[T_\Psi]$. Then we have the following estimates on $\mathbb{E}[T_{\text{con}}(x(0))]$:

$$\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{1}{2\Delta^2}V_{\bar{x}(0)}(x(0))\mathbb{E}[T_\Psi] \leq \frac{NJ(x(0))^2}{8}\mathbb{E}[T_\Psi], \tag{3.5}$$

where the second inequality is a direct result of Lemma 4 in [69].

Theorem 3.3.1 *For any initial state $x(0) \notin \mathcal{W}(x(0))$, the expected convergence time $\mathbb{E}[T_{\text{con}}(x(0))]$ of* AF *is upper bounded by $\frac{N^2J(x(0))^2}{8}(\frac{8}{27}N^3 - 1)$.*

Proof: By (3.5), it suffices to bound $\mathbb{E}[T_\Psi]$. Assume that $x(0) \in \Psi$. Before they meet for the first time, the values 0 and 2Δ move as two natural random walks which are identical to $X_{\mathcal{N}}$ in Proposition 3.3.1. At their meeting for the first time, the values of 0 and 2Δ average and quantized average consensus is reached. Hence, $\mathbb{E}[T_\Psi] = M_{(\mathcal{G},P_{\text{AF}})}$ and thus inequality (3.5) becomes

$$\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{NJ(x(0))^2}{8}M_{(\mathcal{G},P_{\text{AF}})} \leq \frac{NJ(x(0))^2}{8}(2NH_{(\mathcal{G},P_{\text{SF}})} - N), \tag{3.6}$$

where we use Proposition 3.3.1 in the second inequality. By letting $M = 0$ in the theorem of Page 265 in [23], we can obtain the upper bound $\frac{4}{27}N^3$ on $H_{(\mathcal{G},P_{\text{SF}})}$. Substituting this upper bound into inequality (3.6) gives the desired upper bound on $\mathbb{E}[T_{\text{con}}(x(0))]$.■

Theorem 3.3.2 *Let $x(0) \in \mathbb{R}^N$ and suppose $x(0) \notin \mathcal{W}(x(0))$. Under* AF*, almost any evolution $x(k)$ starting from $x(0)$ reaches quantized average consensus.*

Proof: Denote $\tilde{T} = \frac{N^2 J(x(0))^2}{4}(\frac{8}{27}N^3 - 1)$, and consider the first $\tilde{T}$ clock ticks of evolution of AF starting from $x(0)$. From Markov's inequality, we have the following estimate:

$$\mathbb{P}(T_{\text{con}}(x(0)) > \tilde{T} \mid x(0) \notin \mathcal{W}(x(0))) \leq \frac{\mathbb{E}[T_{\text{con}}(x(0))]}{\tilde{T}} \leq \frac{1}{2},$$

that is, the probability that after $\tilde{T}$ clock ticks AF has not reached quantized average consensus is less than $\frac{1}{2}$. Starting from $x(\tilde{T})$, let us consider the posterior evolution of $x(k)$ in the next $\tilde{T}$ clock ticks. We have

$$\mathbb{P}(T_{\text{con}}(x(\tilde{T})) > \tilde{T} \mid x(\tilde{T}) \notin \mathcal{W}(x(0))) \leq \frac{\mathbb{E}[T_{\text{con}}(x(\tilde{T}))]}{\tilde{T}} \leq \frac{1}{2}.$$

That is, the probability that after $2\tilde{T}$ clock ticks $x(k)$ has not reached quantized average consensus is at most $(\frac{1}{2})^2$. By induction, it follows that after $n\tilde{T}$ clock ticks the probability $x(k)$ not reaching quantized average consensus is at most $(\frac{1}{2})^n$. Since the set $\mathcal{W}(x(0))$ is absorbing, we have $\lim_{k\to\infty} \mathbb{P}(x(k) \notin \mathcal{W}(x(0))) = 0$. This completes the proof.■

3.4 Asynchronous distributed quantized averaging on switching graphs

We now turn our attention to the more challenging scenario where the communication graphs are undirected but dynamically changing. We will propose and analyze an *asynchronous distributed quantized averaging algorithm on switching graphs* (AS, for short). The convergence rate of distributed real-valued averaging algorithms on switching graphs in [98] will be employed to characterize the hitting time of random walks on switching graphs.

3.4.1 Proposed algorithm

The main steps of AS can be summarized as follows. At time k, let node i's clock tick. If $|\mathcal{N}_i(k)| \neq 0$, node i randomly chooses one of its neighbors, say node j, with probability $\frac{1}{\max\{|\mathcal{N}_i(k)|,|\mathcal{N}_j(k)|\}}$. Then, node i and j execute the computation

(3.1) or (3.2) and every other node $k \in V\backslash\{i,j\}$ preserves its current state. If $|\mathcal{N}_i(k)| = 0$, all the nodes do nothing at this time.

Here, we assume that the communication graph $\mathcal{G}(k)$ be undirected and satisfies the following connectivity assumption also used in [19, 65, 98, 111].

Assumption 3.4.1 (Periodical connectivity) There exists some $B \in \mathbb{N}_{>0}$ such that, for all $k \geq 0$, the undirected graph $(V, E(k) \cup E(k+1) \cup \cdots \cup E(k+B-1))$ is connected.

Remark 3.4.1 In the AS, the probability that node i chooses a neighbor j is $\frac{1}{\max\{|\mathcal{N}_i(k)|,|\mathcal{N}_j(k)|\}}$. Thus, this information should be available to node i. In this way, the matrix $P_{\rm AS}(k)$ defined later is symmetric and double stochastic. •

3.4.2 The meeting time of two natural random walks on the time-varying graph $\mathcal{G}(k)$

Before analyzing AS, we consider the following problem which generalizes the problem in Section 3.3.2 to the case of dynamically changing graphs.

The meeting time of two natural random walks on the time-varying graph $\mathcal{G}(k)$. Assume that $\mathcal{G}(k)$ be undirected and satisfies Assumption 3.4.1. Initially, two tokens are placed on $\mathcal{G}(0)$. At each time, one of the tokens is chosen with probability $\frac{1}{N}$. The chosen token at some node, say i, moves to one of the neighbors, say node j, with probability $\frac{1}{\max\{|\mathcal{N}_i(k)|,|\mathcal{N}_j(k)|\}}$ if $|\mathcal{N}_i(k)| \neq 0$; otherwise, it will stay up with node i. What is the meeting time for these two tokens?

Clearly, the movements of two tokens are two natural random walks, say X_1 and X_2, on the switching graph $\mathcal{G}(k)$. Their meeting time is denoted as $M_{(\mathcal{G}(k),P_{\rm AS}(k))}$ where the transition matrix $P_{\rm AS}(k) = (\bar{p}_{ij}(k))$ is given as follows: if $|\mathcal{N}_i(k)| \neq 0$, then $\bar{p}_{ij}(k) = \frac{1}{N\max\{|\mathcal{N}_i(k)|,|\mathcal{N}_j(k)|\}}$ for $(i,j) \in E(k)$ and $\bar{p}_{ii}(k) = 1 - \sum_{(i,j)\in E(k)} \frac{1}{N\max\{|\mathcal{N}_i(k)|,|\mathcal{N}_j(k)|\}}$; if $|\mathcal{N}_i(k)| = 0$, then $\bar{p}_{ii}(k) = 1$. One can easily verify that the matrix $P_{\rm AS}(k)$ is symmetric and doubly stochastic. The natural random walks X_1 and X_2 on the graph $\mathcal{G}(k)$ are equivalent to a single natural random walk, say X_M, on the product graph $\mathcal{G}(k) \times \mathcal{G}(k)$. That is, X_M moving from node $(i_1, i_2) \in V \times V$ to node $(j_1, j_2) \in V \times V$ on the graph $\mathcal{G}(k) \times \mathcal{G}(k)$ at

time t, is equivalent to X_1 moving from i_1 to j_1 and X_2 moving from i_2 to j_2 on the graph $\mathcal{G}(k)$ at time t. Denote the transition matrix of the random walk X_M as $Q(k) = (q_{(i_1,i_2)(j_1,j_2)}(k)) \in \mathbb{R}^{N^2\times N^2}$.

In the following lemma, we will consider the random walk $\bar{X}_M$ on the graph $\mathcal{G}(k) \times \mathcal{G}(k)$ with the absorbing set Θ and the transition matrix $\bar{Q}(k) \in \mathbb{R}^{N^2\times N^2}$. Denote $e_{(\ell_1,\ell_2)}$ by the row corresponding to $(\ell_1, \ell_2) \in V\times V$ in a $N^2\times N^2$ identity matrix. The transition matrix $\bar{Q}(k)$ is defined by replacing the row associated with the absorbing state $(\ell_1, \ell_2) \in \Theta$ in $Q(k)$ with $e_{(\ell_1,\ell_2)}$. Define $\vartheta_{(\ell_1,\ell_2)}(k) = \mathbb{P}(X_M(k) = (\ell_1, \ell_2))$, $\vartheta(k) = \mathrm{col}\{\vartheta_{(\ell_1,\ell_2)}(k)\} \in \mathbb{R}^{N^2}$, $\vartheta_\Theta(k) = \sum_{(\ell_1,\ell_2)\in\Theta} \vartheta_{(\ell_1,\ell_2)}(k)$ for the random walk X_M, and $\bar{\vartheta}_{(\ell_1,\ell_2)}(k) = \mathbb{P}(\bar{X}_M(k) = (\ell_1, \ell_2))$, $\bar{\vartheta}(k) = \mathrm{col}\{\bar{\vartheta}_{(\ell_1,\ell_2)}(k)\} \in \mathbb{R}^{N^2}$, $\bar{\vartheta}_\Theta(k) = \sum_{(\ell_1,\ell_2)\in\Theta} \bar{\vartheta}_{(\ell_1,\ell_2)}(k)$ for the random walk $\bar{X}_M$.

Lemma 3.4.1 *Consider a network of N nodes whose communication graph $\mathcal{G}(k)$ be undirected and satisfies Assumption 3.4.1. Let $(i_1, i_2) \in V \times V$ be a given node and suppose that the random walks X_M and $\bar{X}_M$ start from node (i_1, i_2) at time 0. Then it holds that $\bar{\vartheta}_\Theta(k) \geq \vartheta_\Theta(k) \geq \frac{1}{2N}$ for $k \geq t_1$ where t_1 is the smallest integer which is larger than $B(8N^6 \log(\sqrt{2}N) + 1)$.*

Proof: It is not difficult to check that $\mathcal{G}(k)\times\mathcal{G}(k)$ is undirected and satisfies Assumption 3.4.1 with period B. The minimum of nonzero entries in $Q(k)$ is lower bounded by $\frac{1}{N(N-1)}$, and $Q(k)$ is symmetric. Observe that for any $(i_1, i_2) \in V \times V$ and any $k \geq 0$, it holds that

$$\begin{aligned}\sum_{(j_1,j_2)\in V\times V} q_{(i_1,i_2)(j_1,j_2)}(k) &= \sum_{(j_1,j_2)\in V\times V} \bar{p}_{i_1j_1}(k)\bar{p}_{i_2j_2}(k)\\ &= \sum_{j_1\in V} \bar{p}_{i_1j_1}(k) \times \sum_{j_2\in V} \bar{p}_{i_2j_2}(k) = 1\end{aligned}$$

where we use the fact that the matrix $P_{\mathrm{AS}}(k)$ is doubly stochastic. Hence, the matrix $Q(k)$ is doubly stochastic.

The evolution of $\vartheta(k)$ is governed by the equation $\vartheta(k+1) = Q^T(k)\vartheta(k)$ with initial state $\vartheta(0) = e^T_{(i_1,i_2)}$. Consider the Lyapunov function $V_{\frac{1}{N^2}}(\vartheta) = \sum_{i=1}^{N^2}(\vartheta_i - \frac{1}{N^2})^2$ with $V_{\frac{1}{N^2}}(\vartheta(0)) = 1 - \frac{1}{N^2}$. It follows from Lemma 5 in [98] that

$$V_{\frac{1}{N^2}}(\vartheta((k+1)B)) \leq (1 - \frac{1}{2N^5(N-1)})V_{\frac{1}{N^2}}(\vartheta(kB)) \tag{3.7}$$

for $k \in \mathbb{N}_0$. Denote $\mathbf{1} \in \mathbb{R}^{N^2}$ as the vector of N^2 ones and note that

$$\begin{aligned}
&V_{\frac{1}{N^2}}(\vartheta(k)) - V_{\frac{1}{N^2}}(\vartheta(k+1)) \\
&= V_{\frac{1}{N^2}}(\vartheta(k)) - V_{\frac{1}{N^2}}(Q^T(k)\vartheta(k)) \\
&= (\vartheta(k) - \frac{1}{N}\mathbf{1})^T(\vartheta(k) - \frac{1}{N}\mathbf{1}) - (P^T(k)\vartheta(k) - \frac{1}{N}\mathbf{1})^T(P^T(k)\vartheta(k) - \frac{1}{N}\mathbf{1}) \\
&= (\vartheta(k) - \frac{1}{N}\mathbf{1})^T(\vartheta(k) - \frac{1}{N}\mathbf{1}) \\
&- (P^T(k)\vartheta(k) - \frac{1}{N}P^T(k)\mathbf{1})^T(P^T(k)\vartheta(k) - \frac{1}{N}P^T(k)\mathbf{1}) \\
&= (\vartheta(k) - \frac{1}{N^2}\mathbf{1})^T(I - Q(k)Q^T(k))(\vartheta(k) - \frac{1}{N^2}\mathbf{1}).
\end{aligned}$$

Since $Q(k)$ is doubly stochastic, so is $Q(k)Q^T(k)$. Hence, the diagonal entries of the matrix $\Gamma(k) = I - Q(k)Q^T(k) = (\gamma_{ij}(k)) \in \mathbb{R}^{N^2 \times N^2}$ are dominant in the sense of $\gamma_{ii}(k) = \sum_{j \neq i} \gamma_{ij}(k)$. According to Gershgorin theorem in [62], all eigenvalues of $\Gamma(k)$ lie in a closed disk centered at $\max_{i \in \{1,\cdots,N^2\}} \gamma_{ii}(k)$ with a radius $\max_{i \in \{1,\cdots,N^2\}} \gamma_{ii}(k)$. Hence, $\Gamma(k)$ is positive semi-definite. Consequently, $V_{\frac{1}{N^2}}(\vartheta(k)) - V_{\frac{1}{N^2}}(\vartheta(k+1)) \geq 0$ and thus $V_{\frac{1}{N^2}}(\vartheta(k))$ is non-increasing along the trajectory of $\vartheta(k)$. Combining (3.7) with the non-increasing property of $V_{\frac{1}{N^2}}(\vartheta(k))$ gives that

$$\begin{aligned}
V_{\frac{1}{N^2}}(\vartheta(k)) &\leq V_{\frac{1}{N^2}}(\vartheta(0))(1 - \frac{1}{2N^5(N-1)})^{\frac{t}{B}-1} \\
&= \frac{N^2-1}{N^2}(1 - \frac{1}{2N^5(N-1)})^{\frac{t}{B}-1}. \qquad (3.8)
\end{aligned}$$

Since $\vartheta(k)^T\mathbf{1} = 1$, then $\vartheta_{\min}(k) := \min_{(\ell_1,\ell_2) \in V \times V} \vartheta_{(\ell_1,\ell_2)}(k) \leq \frac{1}{N^2}$. Since $V_{\frac{1}{N^2}}(\vartheta(k)) \geq (\vartheta_{\min}(k) - \frac{1}{N^2})^2$, inequality (3.8) gives that

$$\vartheta_{\min}(k) \geq \frac{1}{N^2} - (\frac{N^2-1}{N^2}(1 - \frac{1}{2N^5(N-1)})^{\frac{t}{B}-1})^{\frac{1}{2}}.$$

Therefore, it holds that

$$\vartheta_{\min}(k) \geq \frac{1}{2N^2}, \forall k \geq B(\frac{\log(4N^2(N^2-1))}{-\log(1 - \frac{1}{2N^5(N-1)})} + 1).$$

Since $\log x \leq x - 1$, there holds

$$\frac{1}{-\log(1 - \frac{1}{2N^5(N-1)})} \leq 2N^5(N-1) \leq 2N^6.$$

Hence, we have that $\vartheta_{\min}(k) \geq \frac{1}{2N^2}$ and thus $\vartheta_\Theta(k) \geq \frac{1}{2N}$ for $k \geq t_1$.

Note that the evolution of $\bar{\vartheta}(k)$ is governed by the equation $\bar{\vartheta}(k+1) = \bar{Q}(k)^T\bar{\vartheta}(k)$ with $\bar{\vartheta}(0) = e_{(i_1,i_2)}$. Since the set Θ is absorbing, $\bar{\vartheta}_\Theta(k) \geq \vartheta_\Theta(k)$ for all $k \geq 0$ and thus the desired result follows.■

Proposition 3.4.1 *The meeting time of two natural random walks with transition matrix $P_{\rm AS}(k)$ on the time-varying graph $\mathcal{G}(k)$ satisfies that $M_{(\mathcal{G}(k),P_{\rm AS}(k))} \leq 4Nt_1$.*

Proof: Denote by $H_{(\mathcal{G}(k)\times\mathcal{G}(k),Q(k))}(\Theta)$ the hitting time of the random walk X_M to reach the set of Θ. Observe that $M_{(\mathcal{G}(k),P_{\rm AS}(k))} = H_{(\mathcal{G}(k)\times\mathcal{G}(k),Q(k))}(\Theta)$. To find an upper bound on $H_{(\mathcal{G}(k)\times\mathcal{G}(k),Q(k))}(\Theta)$, we construct the random walk $X_M^{(i_1,i_2)}$ in such a way that $X_M^{(i_1,i_2)}$ starts from (i_1,i_2) at time 0 with $i_1 \neq i_2$ and the set Θ is the absorbing set of $X_M^{(i_1,i_2)}$. The transition matrix of $X_M^{(i_1,i_2)}$ is $\bar{Q}(k)$ defined before Lemma 3.4.1. Define $\vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(k) = \mathbb{P}(X_M^{(i_1,i_2)}(k) = (\ell_1,\ell_2))$, and $\vartheta^{(i_1,i_2)}(k) = {\rm col}\{\vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(k)\} \in \mathbb{R}^{N^2}$. The dynamics of $\vartheta^{(i_1,i_2)}(k)$ is given by $\vartheta^{(i_1,i_2)}(k+1) = \bar{Q}(k)^T\vartheta^{(i_1,i_2)}(k)$ with the initial state $\vartheta^{(i_1,i_2)}(0) = e_{(i_1,i_2)}^T$.

Define the function $\mu_{(\ell_1,\ell_2)}^{(i_1,i_2)} : \mathbb{N}_0 \to \{0,1\}$ in such a way that $\mu_{(\ell_1,\ell_2)}^{(i_1,i_2)} = 1$ if $X_M^{(i_1,i_2)}(k) = (\ell_1,\ell_2)$; otherwise, $\mu_{(\ell_1,\ell_2)}^{(i_1,i_2)}(k) = 0$. Define $n_{(\ell_1,\ell_2)}^{(i_1,i_2)} = \sum_{\tau=0}^{+\infty}\mu_{(\ell_1,\ell_2)}^{(i_1,i_2)}(\tau)$ which is the total times that the random walk $X_M^{(i_1,i_2)}$ is at node (ℓ_1,ℓ_2). Then, the hitting time $H_{(\mathcal{G}(k)\times\mathcal{G}(k),Q(k),0)}((i_1,i_2),\Theta)$ of $X_M^{(i_1,i_2)}$ equals the expected time that $X_M^{(i_1,i_2)}$ stays up with the nodes in $V\times V\backslash\Theta$, that is,

$$H_{(\mathcal{G}(k)\times\mathcal{G}(k),Q(k),0)}((i_1,i_2),\Theta) = \sum_{(\ell_1,\ell_2)\notin\Theta}\mathbb{E}[n_{(\ell_1,\ell_2)}^{(i_1,i_2)}] = \sum_{(\ell_1,\ell_2)\notin\Theta}\mathbb{E}[\sum_{\tau=0}^{+\infty}\mu_{(\ell_1,\ell_2)}^{(i_1,i_2)}(\tau)]$$
$$= \sum_{(\ell_1,\ell_2)\notin\Theta}\sum_{\tau=0}^{+\infty}\mathbb{E}[\mu_{(\ell_1,\ell_2)}^{(i_1,i_2)}(\tau)] = \sum_{\tau=0}^{+\infty}\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(\tau). \tag{3.9}$$

It follows from Lemma 3.4.1 that $\vartheta_\Theta^{(i_1,i_2)}(k) \geq \frac{1}{2N}$ for $k \geq t_1$. With that, the fact of $\vartheta^{(i_1,i_2)}(k)^T\mathbf{1} = 1$ implies that

$$\sum_{(\ell_1,\ell_2)\notin\Theta}\vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(t_1) \leq 1 - \frac{1}{2N}. \tag{3.10}$$

For each $(k_1,k_2) \notin \Theta$, we construct the random walk $\tilde{X}_M^{(k_1,k_2)}$ in such a way that $\tilde{X}_M^{(k_1,k_2)}$ starts from (k_1,k_2) at time t_1 and the set Θ is the absorbing

set of $\tilde{X}_M^{(k_1,k_2)}$. The transition matrix of $\tilde{X}_M^{(k_1,k_2)}$ is $\bar{Q}(k)$. Define $\tilde{\vartheta}_{(\ell_1,\ell_2)}^{(k_1,k_2)}(k) = \mathbb{P}(\tilde{X}_M^{(k_1,k_2)}(k) = (\ell_1,\ell_2))$. Following the forgoing arguments for $X_M^{(i_1,i_2)}$, we have

$$\sum_{(\ell_1,\ell_2)\notin\Theta} \tilde{\vartheta}_{(\ell_1,\ell_2)}^{(k_1,k_2)}(2t_1) \leq 1 - \frac{1}{2N}. \tag{3.11}$$

Combining (3.10) and (3.11) gives that

$$\begin{aligned}\sum_{(\ell_1,\ell_2)\notin\Theta} \vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(2t_1) &= \sum_{(\ell_1,\ell_2)\notin\Theta}\sum_{(k_1,k_2)\notin\Theta} \vartheta_{(k_1,k_2)}^{(i_1,i_2)}(t_1)\tilde{\vartheta}_{(\ell_1,\ell_2)}^{(k_1,k_2)}(2t_1)\\ &= \sum_{(k_1,k_2)\notin\Theta} \vartheta_{(k_1,k_2)}^{(i_1,i_2)}(t_1) \sum_{(\ell_1,\ell_2)\notin\Theta} \tilde{\vartheta}_{(\ell_1,\ell_2)}^{(k_1,k_2)}(2t_1) \leq (1-\frac{1}{2N})^2. \end{aligned}\tag{3.12}$$

By induction, we have $\sum_{(\ell_1,\ell_2)\notin\Theta} \vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(nt_1) \leq (1-\frac{1}{2N})^n$ and then obtain a strictly decreasing sequence $\sum_{(\ell_1,\ell_2)\notin\Theta} \vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(nt_1)$ with respect to $n \in \mathbb{Z}_0$. Since the set Θ is absorbing, then $\sum_{(\ell_1,\ell_2)\notin\Theta} \vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(k)$ is non-increasing with respect to $k \geq 0$. Therefore, we have the following estimate

$$\sum_{(\ell_1,\ell_2)\notin\Theta} \vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(k) \leq \sum_{(\ell_1,\ell_2)\notin\Theta} \vartheta_{(\ell_1,\ell_2)}^{(i_1,i_2)}(0)(1-\frac{1}{2N})^{\frac{t}{t_1}-1} = (1-\frac{1}{2N})^{\frac{t}{t_1}-1}. \tag{3.13}$$

Substituting (3.13) into (3.9) gives that

$$\begin{aligned} H_{(\mathcal{G}(k)\times\mathcal{G}(k),Q(k),0)}((i_1,i_2),\Theta) &\leq \sum_{\tau=0}^{+\infty}(1-\frac{1}{2N})^{\frac{\tau}{t_1}-1}\\ &= (1-\frac{1}{2N})^{-\frac{1}{t_1}} \cdot \frac{1}{1-(1-\frac{1}{2N})^{\frac{1}{t_1}}}. \end{aligned}\tag{3.14}$$

Since $t_1 > 1$, it holds that $(1-\frac{1}{2N})^{-\frac{1}{t_1}} \leq 2^{\frac{1}{t_1}} < 2$. It follows from Bernoulli's inequality that $(1-\frac{1}{2N})^{\frac{1}{t_1}} \leq 1 - \frac{1}{2Nt_1}$, and thus $\frac{1}{1-(1-\frac{1}{2N})^{\frac{1}{t_1}}} \leq 2Nt_1$. Inequality (3.14) becomes

$$H_{(\mathcal{G}(k)\times\mathcal{G}(k),Q(k),0)}((i_1,i_2),\Theta) \leq 4Nt_1. \tag{3.15}$$

Actually, inequality (3.15) holds for any starting time, any starting node (i_1,i_2). Thus it holds that $M_{(\mathcal{G}(k),P_{\mathrm{AS}}(k))} = H_{(\mathcal{G}(k)\times\mathcal{G}(k),Q(k),)}(\Theta) \leq 4Nt_1$. This completes the proof.■

3.4.3 Convergence analysis

We are now in the position to characterize the convergence properties of AS. The quantities $T_{\text{con}}(x(0))$ and T_Ψ for AS are defined in a similar way to those in Section 3.3.

Theorem 3.4.1 *Let $x(0) \in \mathbb{R}^N$ and suppose $x(0) \notin \mathcal{W}(x(0))$. Assume that $\mathcal{G}(k)$ be undirected and satisfies Assumption 3.4.1. Under* AS*, almost any evolution $x(k)$ starting from $x(0)$ reaches quantized average consensus. Furthermore, $\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{1}{2}BJ(x(0))^2N^2(16N^7+1)$.*

Proof: Note that inequality (3.5) also hold for AS. Similar to Theorem 3.3.1, we have $\mathbb{E}[T_\Psi] = M_{(\mathcal{G}(k),P_{\text{AS}}(k))}$. As a result, the following estimate on $\mathbb{E}[T_{\text{con}}(x(0))]$ holds:

$$\mathbb{E}[T_{\text{con}}(x(0))] \leq \frac{NJ(x(0))^2}{8} M_{(\mathcal{G}(k),P_{\text{AS}}(k))}. \tag{3.16}$$

Substituting the upper bound on $M_{(\mathcal{G}(k),P_{\text{AS}}(k))}$ in Proposition 3.4.1 into (3.16) and using $\log(\sqrt{2}N) \leq 2N$ gives the desired upper bound on $\mathbb{E}[T_{\text{con}}(x(0))]$ of AS. The reminder of the proof on the convergence to quantized average consensus is analogous to Theorem 3.3.2, and thus omitted.■

3.5 Discussion

3.5.1 Asynchronous distributed quantized averaging on random graphs

Random graphs have been widely used to model real-world networks such as Internet, transportation networks, communication networks, biological networks and social networks. The Erdős - Rényi model $\mathcal{G}(N,p)$ is the most commonly studied one, and constructed by randomly placing an edge between any two of N nodes with probability p.

At any time, the probability that the (directed) edge (i,j) is selected is

$$p_0 := \frac{1}{N}\sum_{m=0}^{N-2} \frac{p}{m+1} C_{N-2}^m p^m (1-p)^{N-2-m},$$

that is, node i is active, the edge (i,j) with other $m \in \{0,\cdots,N-2\}$ edges connecting node i are placed, and the edge (i,j) is selected by node i. To study the convergence properties of AF on $\mathcal{G}(N,p)$, it is equivalent to study AF on complete graphs with the transition matrix $P_{\rm AR} = (\hat{p}_{ij}) \in \mathbb{R}^{N\times N}$ where $\hat{p}_{ij} = p_0$ and $\hat{p}_{ii} = 1-(N-1)p_0$. The meeting time is denoted as $M_{(\mathcal{G}(N,p),P_{\rm AR})}$. The probability that the two tokens meet for the first time at time t is $2p_0$, that is, one of the tokens is chosen and simultaneously the edge between the two tokens is chosen. Hence, we have

$$M_{(\mathcal{G}(N,p),P_{\rm AR})} = \sum_{\ell=1}^{+\infty} \ell 2p_0(1-2p_0)^{\ell-1} = \frac{1}{2p_0}.$$

Observe that the following estimate holds for p_0:

$$\begin{aligned} p_0 &= \frac{p}{N}\sum_{m=0}^{N-2}\frac{1}{m+1}C_{N-2}^m p^m(1-p)^{N-2-m} \\ &\geq \frac{2p}{N(N-1)}\sum_{m=0}^{N-2} C_{N-2}^m p^m(1-p)^{N-2-m} = \frac{2p}{N(N-1)}. \end{aligned}$$

Like Theorem 3.3.1, we have

$$\begin{aligned} \mathbb{E}[T_{\rm con}(x(0))] &\leq \frac{NJ(x(0))^2}{8}\mathbb{E}[T_\Psi] = \frac{NJ(x(0))^2}{8}M_{(\mathcal{G}(N,p),P_{\rm AR})} \\ &= \frac{NJ(x(0))^2}{16p_0} \leq \frac{N^2(N-1)J(x(0))^2}{32p}. \end{aligned}$$

3.5.2 Discussion on the bounds obtained

Consider a fixed graph L_N^m with N vertices consists of a clique on m vertices, including vertex i, and a path of length $N-m$ with one end connected to one vertex $\iota \neq i$ of the clique, and the other end of the path being j. It was shown in [23] that $H_{(L_N^{m_0},P_{\rm SF})}$ is $O(N^3)$ where $m_0 = \lfloor \frac{2N+1}{3} \rfloor$. Let us consider the case that the algorithm AF is implemented on the graph $L_N^{m_0}$ and initial states $x_i(0) = 0$, $x_j(0) = 2$ and $x_\iota(0) = 1$ for all $\iota \neq i,j$. Observe that $\mathbb{E}[T_{\rm con}(x(0))] = M_{(L_N^{m_0},P_{\rm AF})}$. From Proposition 3.3.1, we have that $\mathbb{E}[T_{\rm con}(x(0))]$ is $O(N^4)$, that is one order less than the bound in Theorem 3.3.1.

Consider switching graphs $\mathcal{G}(k)$ where $\mathcal{G}(k)$ is the graph $L_N^{m_0}$ defined above when k is a multiple of B; otherwise, all the vertices in $\mathcal{G}(k)$ are isolated. Random walks on $\mathcal{G}(k)$ can be viewed as time-scaled versions of those on $L_N^{m_0}$, that is, random walks on $\mathcal{G}(k)$ only make the movements when k is a multiple of B. Let us consider the case that the algorithm AS is implemented on the graph $L_N^{m_0}$ and initial states $x_i(0) = 0$, $x_j(0) = 2$ and $x_\iota(0) = 1$ for all $\iota \neq i, j$. Following the same lines above, we have that the bound on $\mathbb{E}[T_{\text{con}}(x(0))]$ is $O(BN^4)$ which is $N^4 \log N$-order less than that in Theorem 3.3.2.

It can be directly computed that $H_{(\mathcal{G}_{\text{com}}, P_{\text{SF}})}$ is $O(N^2)$ where $\mathcal{G}_{\text{com}}$ is a complete graph with N vertices. Following the same lines in Theorem 3.3.1, we have that $\mathbb{E}[T_{\text{con}}(x(0))]$ is $O(N^3)$ when the algorithm AF is implemented on the graph $\mathcal{G}_{\text{com}}$. It implies that the convergence of AF on $\mathcal{G}_{\text{com}}$ is as fast as that on $\mathcal{G}(N, p)$ when p is independent of N. This is consistent with the fact that the underlining graph of $\mathcal{G}(N, p)$ is $\mathcal{G}_{\text{com}}$.

3.6 Simulations

This section presents a simulation of AS. Consider a network of 10 nodes. Assume that the quantization step size $\Delta = 1$ and the communication graph $\mathcal{G}(k)$ satisfies Assumption 3.4.1 with $B = 3$. Suppose the initial state $x(0) = (5, 6, 14, 17, 0, 11, 10, 21, 10, 6)^T$ with average $\bar{x}(0) = 10$. The worst-case upper bound on $\mathbb{E}[T_{\text{con}}(x(0))]$ in Theorem 3.4.1 is 10^{10} clock ticks. Figure 3.1 shows that all the consensus states agree on $\bar{x}(0)$ after about 70 clock ticks.

3.7 Conclusions

In this chapter, we have proposed a class of quantized average consensus algorithms, and characterized their expected convergence times. The results presented in this chapter are based on the following published papers:

(JP-3) M. Zhu and S. Martínez, "On the convergence time of asynchronous distributed quantized averaging algorithms", *IEEE Transactions on Automatic*

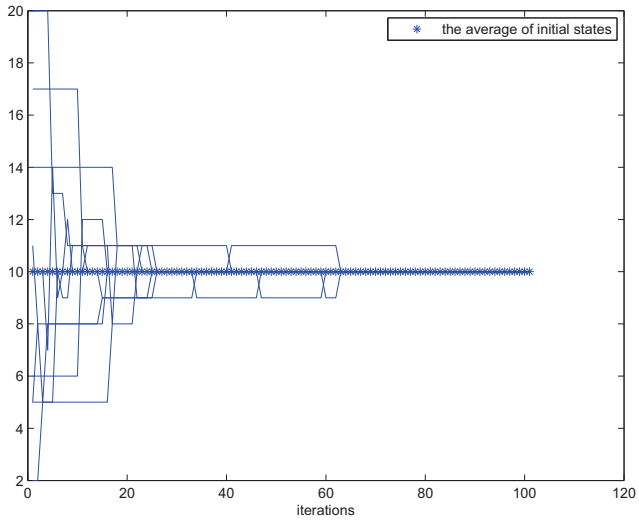

Figure 3.1: The states of asynchronous quantized averaging algorithm on switching graphs

Control, 56(2), pages 386 – 390, 2011.

(CP-4) M. Zhu and S. Martínez, "On the convergence time of distributed quantized averaging algorithms", *The* 47$^{\text{th}}$ *IEEE Conference on Decision and Control*, pages 3971 – 3976, Cancun, Mexico, Dec. 2008.

Part II

Distributed cooperative constrained optimization

Chapter 4

Distributed cooperative convex optimization

4.1 Introduction

Recent advances in sensing, communication and computation technologies are challenging the way in which control mechanisms are designed for their efficient exploitation in a coordinated manner. This has motivated a wealth of algorithms for information processing, cooperative control, and optimization of large-scale networked multi-agent systems performing a variety of tasks. Due to a lack of a centralized authority, the proposed algorithms aim to be executed by individual agents through local actions, with the main feature of being robust to dynamic changes of network topologies.

In this chapter, we consider a general multi-agent optimization problem where the goal is to minimize a global objective function, given as a sum of local objective functions, subject to global constraints, which include an inequality constraint, an equality constraint and a (state) constraint set. Each local objective function is convex and only known to one particular agent. On the other hand, the inequality (resp. equality) constraint is given by a convex (resp. affine) function and known by all agents. Each node has its own convex constraint set, and the global constraint set is defined as their intersection. This problem is

motivated by others in distributed estimation [104] [136], distributed source localization [117], network utility maximization [70], optimal flow control in power systems [110, 146] and optimal shape changes of mobile robots [42]. An important feature of the problem is that the objective and (or) constraint functions depend upon a global decision vector. This requires the design of distributed algorithms where, on the one hand, agents can align their decisions through a local information exchange and, on the other hand, the common decisions will coincide with an optimal solution and the optimal value.

Literature Review. In [14] and [142], the authors develop a general framework for parallel and distributed computation over a set of processors. Consensus problems, a class of canonical problems on networked multi-agent systems, have been intensively studied since then. A necessarily incomplete list of references includes [47, 108] tackling continuous-time consensus, [19, 65, 93] investigating discrete-time versions, and [88] where asynchronous implementation of consensus algorithms is discussed. The papers [20, 69, 138] treat randomized consensus via gossip communication, achieving consensus through quantized information and consensus over random graphs, respectively. The convergence rate of consensus algorithms is discussed, e.g., in [111, 147], and the author in [34] derives conditions to achieve different consensus values.

In robotics and control communities, convex optimization has been exploited to design algorithms coordinating mobile multi-agent systems. In [40], in order to increase the connectivity of a multi-agent system, a distributed supergradient based algorithm is proposed to maximize the second smallest eigenvalue of the Laplacian matrix of the state dependent proximity graph of agents. In [42], optimal shape changes of mobile robots are achieved through second-order cone programming techniques. In [43], a target tracking problem is addressed by means of a generic semidefinite program where the constraints of network connectivity and full target coverage are articulated as linear-matrix inequalities. In [94], in order to attain the highest possible positioning accuracy for mobile robots, the authors express the covariance matrix of the pose errors as a functional relation of measurement frequencies, and then formulate an optimal sensing problem as a

convex programming of measurement frequencies.

The recent papers [100, 102] are the most relevant to our work. In [100], the authors solve a multi-agent unconstrained convex optimization problem through a novel combination of average consensus algorithms with subgradient methods. More recently, the paper [102] further takes local constraint sets into account. To deal with these constraints, the authors in [102] present an extension of their distributed subgradient algorithm, by projecting the original algorithm onto the local constraint sets. Two cases are solved in [102]: the first assumes that the network topologies can dynamically change and satisfy a periodic strong connectivity assumption (i.e., the union of the network topologies over a bounded period of time is strongly connected), but then the local constraint sets are identical; the second requires that the communication graphs are (fixed and) complete and then the local constraint sets can be different. Another related paper is [68] where a special case of [102], the network topology is fixed and all the local constraint sets are identical, is addressed.

Statement of Contributions. Building on the work [102], this chapter further incorporates global inequality and equality constraints. More precisely, we study two cases: one in which the equality constraint is absent, and the other in which the local constraint sets are identical. For the first case, we adopt a Lagrangian relaxation approach, define a Lagrangian dual problem and devise the distributed Lagrangian primal-dual subgradient algorithm (DLPDS, for short) based on the characterization of the primal-dual optimal solutions as the saddle points of the Lagrangian function. The DLPDS algorithm involves each agent updating its estimates of the saddle points via a combination of an average consensus step, a subgradient (or supgradient) step and a primal (or dual) projection step onto its local constraint set (or a compact set containing the dual optimal set). The DLPDS algorithm is shown to asymptotically converge to a pair of primal-dual optimal solutions under the Slater's condition and the periodic strong connectivity assumption. Furthermore, each agent asymptotically agrees on the optimal value by implementing a dynamic average consensus algorithm developed in Chapter 2, which allows a multi-agent system to track time-varying average values.

For the second case, to dispense with the additional equality constraint, we adopt a penalty relaxation approach, while defining a penalty dual problem and devising the distributed penalty primal-dual subgradient algorithm (DPPDS, for short). Unlike the first case, the dual optimal set of the second case may not be bounded, and thus the dual projection steps are not involved in the DPPDS algorithm. It renders that dual estimates and thus (primal) subgradients may not be uniformly bounded. This challenge is addressed by a more careful choice of step-sizes. We show that the DPPDS algorithm asymptotically converges to a primal optimal solution and the optimal value under the Slater's condition and the periodic strong connectivity assumption.

4.2 Problem formulation and assumptions

4.2.1 Problem formulation

Consider a network of agents labeled by $V := \{1, \ldots, N\}$ that can only interact with each other through local communication. The objective of the multi-agent group is to cooperatively solve the following optimization problem:

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} \sum_{i=1}^{N} f_i(x), \\ &\text{s.t. } g(x) \leq 0, \;\; h(x) = 0, \;\; x \in \cap_{i=1}^{N} X_i, \end{aligned} \tag{4.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is the convex objective function of agent i, $X_i \subseteq \mathbb{R}^n$ is the compact and convex constraint set of agent i, and x is a global decision vector. Here we assume that the projection onto the set X_i is easy to compute. Assume that f_i and X_i are only known by agent i, and probably different. The function $g : \mathbb{R}^n \to \mathbb{R}^m$ is known to all the agents with each component g_ℓ, for $\ell \in \{1, \ldots, m\}$, being convex. The inequality $g(x) \leq 0$ is understood component-wise; i.e., $g_\ell(x) \leq 0$, for all $\ell \in \{1, \ldots, m\}$, and represents a global inequality constraint. The function $h : \mathbb{R}^n \to \mathbb{R}^\nu$, defined as $h(x) := Ax - b$ with $A \in \mathbb{R}^{\nu\times n}$, represents a global equality constraint, and is known to all the agents. We denote $X := \cap_{i=1}^{N} X_i$, $f(x) := \sum_{i=1}^{N} f_i(x)$, and $Y := \{x \in \mathbb{R}^n \mid g(x) \leq 0, \;\; h(x) = 0\}$. We assume

that the set of feasible points is non-empty; i.e., $X \cap Y \neq \emptyset$. Since X is compact and Y is closed, then we can deduce that $X \cap Y$ is compact. The convexity of f_i implies that of f and thus f is continuous. In this way, the optimal value p^* of the problem (4.1) is finite and X^*, the set of primal optimal points, is non-empty. Throughout this chapter, we suppose the following Slater's condition holds:

Assumption 4.2.1 (Slater's Condition) There exists a vector $\bar{x} \in X$ such that $g(\bar{x}) < 0$ and $h(\bar{x}) = 0$. And there exists a relative interior point $\tilde{x}$ of X such that $h(\tilde{x}) = 0$ where $\tilde{x}$ is a relative interior point of X; i.e., $\tilde{x} \in X$ and there exists an open sphere S centered at $\tilde{x}$ such that $S \cap \text{aff}(X) \subset X$ with $\text{aff}(X)$ being the affine hull of X.

In this chapter, we will study two particular cases of problem (4.1): one in which the global equality constraint $h(x) = 0$ is not included, and the other in which all the local constraint sets are identical. For the case where the constraint $h(x) = 0$ is absent, the Slater's condition 4.2.1 reduces to the existence of a vector $\bar{x} \in X$ such that $g(\bar{x}) < 0$.

4.2.2 Network model

We will consider that the multi-agent network operates synchronously. The topology of the network at time $k \geq 0$ will be represented by a directed weighted graph $\mathcal{G}(k) = (V, E(k), A(k))$ where $A(k) := [a_j^i(k)] \in \mathbb{R}^{N \times N}$ is the adjacency matrix with $a_j^i(k) \geq 0$ being the weight assigned to the edge (j, i) and $E(k) \subset V \times V \setminus \text{diag}(V)$ is the set of edges with non-zero weights $a_j^i(k)$. The in-neighbors of node i at time k are denoted by $\mathcal{N}_i(k) = \{j \in V \mid (j, i) \in E(k) \text{ and } j \neq i\}$. We here make the following assumptions on the network communication graphs, which are standard in the analysis of average consensus algorithms; e.g., see [108], [111], and distributed optimization in [100], [102].

Assumption 4.2.2 (Non-degeneracy) There exists a constant $\alpha > 0$ such that $a_i^i(k) \geq \alpha$, and $a_j^i(k)$, for $i \neq j$, satisfies $a_j^i(k) \in \{0\} \cup [\alpha,\ 1]$, for all $k \geq 0$.

Assumption 4.2.3 (Balanced Communication) [1] It holds that $\sum_{j=1}^{N} a_j^i(k) = 1$ for all $i \in V$ and $k \geq 0$, and $\sum_{i=1}^{N} a_j^i(k) = 1$ for all $j \in V$ and $k \geq 0$.

Assumption 4.2.4 (Periodical Strong Connectivity) There is a positive integer B such that, for all $k_0 \geq 0$, the directed graph $(V, \bigcup_{k=0}^{B-1} E(k_0+k))$ is strongly connected.

4.2.3 Notion and notations

The following notion of saddle point plays a critical role in our chapter.

Definition 4.2.1 (Saddle point) Consider a function $\phi : X \times M \to \mathbb{R}$ where X and M are non-empty subsets of $\mathbb{R}^{\bar{n}}$ and $\mathbb{R}^{\bar{m}}$. A pair of vectors $(x^*, \mu^*) \in X \times M$ is called a saddle point of ϕ over $X \times M$ if $\phi(x^*, \mu) \leq \phi(x^*, \mu^*) \leq \phi(x, \mu^*)$ hold for all $(x, \mu) \in X \times M$.

Remark 4.2.1 Equivalently, (x^*, μ^*) is a saddle point of ϕ over $X \times M$ if and only if $(x^*, \mu^*) \in X \times M$, and $\sup_{\mu \in M} \phi(x^*, \mu) \leq \phi(x^*, \mu^*) \leq \inf_{x \in X} \phi(x, \mu^*)$. •

In this chapter, we do not assume the differentiability of f_i and g_ℓ. At the points where the function is not differentiable, the subgradient plays the role of the gradient. For a given convex function $F : \mathbb{R}^{\bar{n}} \to \mathbb{R}$ and a point $\tilde{x} \in \mathbb{R}^{\bar{n}}$, a *subgradient* of the function F at $\tilde{x}$ is a vector $\mathcal{D}F(\tilde{x}) \in \mathbb{R}^{\bar{n}}$ such that the following subgradient inequality holds for any $x \in \mathbb{R}^{\bar{n}}$:

$$\mathcal{D}F(\tilde{x})^T(x - \tilde{x}) \leq F(x) - F(\tilde{x}).$$

Similarly, for a given concave function $G : \mathbb{R}^{\bar{m}} \to \mathbb{R}$ and a point $\bar{\mu} \in \mathbb{R}^{\bar{m}}$, a *supgradient* of the function G at $\bar{\mu}$ is a vector $\mathcal{D}G(\bar{\mu}) \in \mathbb{R}^{\bar{m}}$ such that the following supgradient inequality holds for any $\mu \in \mathbb{R}^{\bar{m}}$:

$$\mathcal{D}G(\bar{\mu})^T(\mu - \bar{\mu}) \geq G(\mu) - G(\bar{\mu}).$$

Given a set S, we denote by $\mathrm{co}(S)$ its convex hull. We let the function $[\cdot]^+ : \mathbb{R}^{\bar{m}} \to \mathbb{R}^{\bar{m}}_{\geq 0}$ denote the projection operator onto the non-negative orthant in $\mathbb{R}^{\bar{m}}$. For any vector $c \in \mathbb{R}^{\bar{n}}$, we denote $|c| := (|c_1|, \cdots, |c_{\bar{n}}|)^T$, while $\|\cdot\|$ is the 2-norm in the Euclidean space.

[1] It is also referred to as double stochasticity.

4.3 Case (i): absence of equality constraint

In this section, we study the case of problem (4.1) where the equality constraint $h(x) = 0$ is absent; i.e.,

$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{N} f_i(x), \quad \text{s.t.} \quad g(x) \leq 0, \quad x \in \cap_{i=1}^{N} X_i. \tag{4.2}$$

We first provide some preliminaries, including a Lagrangian saddle-point characterization of the problem (4.2) and finding a superset containing the Lagrangian dual optimal set of the problem (4.2). After that, we present the distributed Lagrangian primal-dual subgradient algorithm and summarize its convergence properties.

4.3.1 Preliminaries

We here introduce some preliminary results which are essential to the development of the distributed Lagrangian primal-dual subgradient algorithm.

A Lagrangian saddle-point characterization

Firstly, the problem (4.2) is equivalent to

$$\min_{x\in\mathbb{R}^n} f(x), \quad \text{s.t.} \quad Ng(x) \leq 0, \quad x \in X,$$

with associated Lagrangian dual problem given by

$$\max_{\mu\in\mathbb{R}^m} q_L(\mu), \quad \text{s.t.} \quad \mu \geq 0.$$

Here, the Lagrangian dual function, $q_L : \mathbb{R}^m_{\geq 0} \to \mathbb{R}$, is defined as $q_L(\mu) := \inf_{x\in X} \mathcal{L}(x, \mu)$, where $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ is the *Lagrangian function* $\mathcal{L}(x, \mu) = f(x) + N\mu^T g(x)$. We denote the Lagrangian dual optimal value of the Lagrangian dual problem by d^*_L and the set of Lagrangian dual optimal points by D^*_L. As is well-known, under the Slater's condition 4.2.1, the property of strong duality holds; i.e., $p^* = d^*_L$, and $D^*_L \neq \emptyset$. The following theorem is a standard result on Lagrangian duality stating that the primal and Lagrangian dual optimal solutions can be characterized as the saddle points of the Lagrangian function.

Theorem 4.3.1 ***(Lagrangian Saddle-point Theorem [15])***
The pair of (x^*, μ^*) *is a saddle point of the Lagrangian function* $\mathcal{L}$ *over* $X \times \mathbb{R}^m_{\geq 0}$ *if and only if it is a pair of primal and Lagrangian dual optimal solutions and the following* Lagrangian minimax equality *holds:*

$$\sup_{\mu \in \mathbb{R}^m_{\geq 0}} \inf_{x \in X} \mathcal{L}(x, \mu) = \inf_{x \in X} \sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathcal{L}(x, \mu).$$

This following lemma presents some preliminary analysis of saddle points.

Lemma 4.3.1 (Preliminary results on saddle points) *Let* M *be any superset of* D^*_L.

(a) If (x^*, μ^*) *is a saddle point of* $\mathcal{L}$ *over* $X \times \mathbb{R}^m_{\geq 0}$, *then* (x^*, μ^*) *is also a saddle point of* $\mathcal{L}$ *over* $X \times M$.

(b) There is at least one saddle point of $\mathcal{L}$ *over* $X \times M$.

(c) If $(\check{x}, \check{\mu})$ *is a saddle point of* $\mathcal{L}$ *over* $X \times M$, *then* $\mathcal{L}(\check{x}, \check{\mu}) = p^*$ *and* $\check{\mu}$ *is Lagrangian dual optimal.*

Proof: (a) It just follows from the definition of saddle point of $\mathcal{L}$ over $X \times M$.

(b) Observe that

$$\sup_{\mu \in \mathbb{R}^m_{\geq 0}} \inf_{x \in X} \mathcal{L}(x, \mu) = \sup_{\mu \in \mathbb{R}^m_{\geq 0}} q_L(\mu) = d^*_L,$$
$$\inf_{x \in X} \sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathcal{L}(x, \mu) = \inf_{x \in X \cap Y} f(x) = p^*.$$

Since the Slater's condition 4.2.1 implies zero duality gap, the Lagrangian minimax equality holds. From Theorem 4.3.1 it follows that the set of saddle points of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$ is the Cartesian product $X^* \times D^*_L$. Recall that X^* and D^*_L are nonempty, so we can guarantee the existence of the saddle point of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$. Then by (a), we have that (b) holds.

(c) Pick any saddle point (x^*, μ^*) of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$. Since the Slater's condition 4.2.1 holds, from Theorem 4.3.1 one can deduce that (x^*, μ^*) is a pair of

primal and Lagrangian dual optimal solutions. This implies that

$$d_L^* = \inf_{x \in X} \mathcal{L}(x, \mu^*) \leq \mathcal{L}(x^*, \mu^*) \leq \sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathcal{L}(x^*, \mu) = p^*.$$

From Theorem 4.3.1, we have $d_L^* = p^*$. Hence, $\mathcal{L}(x^*, \mu^*) = p^*$. On the other hand, we pick any saddle point $(\check{x}, \check{\mu})$ of $\mathcal{L}$ over $X \times M$. Then for all $x \in X$ and $\mu \in M$, it holds that $\mathcal{L}(\check{x}, \mu) \leq \mathcal{L}(\check{x}, \check{\mu}) \leq \mathcal{L}(x, \check{\mu})$. By Theorem 4.3.1, then $\mu^* \in D_L^* \subseteq M$. Recall $x^* \in X$, and thus we have $\mathcal{L}(\check{x}, \mu^*) \leq \mathcal{L}(\check{x}, \check{\mu}) \leq \mathcal{L}(x^*, \check{\mu})$. Since $\check{x} \in X$ and $\check{\mu} \in \mathbb{R}^m_{\geq 0}$, we have $\mathcal{L}(x^*, \check{\mu}) \leq \mathcal{L}(x^*, \mu^*) \leq \mathcal{L}(\check{x}, \mu^*)$. Combining the above two relations gives that $\mathcal{L}(\check{x}, \check{\mu}) = \mathcal{L}(x^*, \mu^*) = p^*$. From Remark 4.2.1 we see that $\mathcal{L}(\check{x}, \check{\mu}) \leq \inf_{x \in X} \mathcal{L}(x, \check{\mu}) = q_L(\check{\mu})$. Since $\mathcal{L}(\check{x}, \check{\mu}) = p^* = d_L^* \geq q_L(\check{\mu})$, then $q_L(\check{\mu}) = d_L^*$ and thus $\check{\mu}$ is a Lagrangian dual optimal solution. ■

Remark 4.3.1 Despite that (c) holds, the reverse of (a) may not be true in general. In particular, x^* may be infeasible; i.e., $g_\ell(x^*) > 0$ for some $\ell \in \{1, \ldots, m\}$. •

A upper estimate of the Lagrangian dual optimal set

In what follows, we will find a compact superset of D_L^*. To do that, we define the following primal problem for each agent i:

$$\min_{x \in \mathbb{R}^n} f_i(x), \quad \text{s.t.} \quad g(x) \leq 0, \quad x \in X_i.$$

Due to the fact that X_i is compact and the f_i are continuous, the primal optimal value p_i^* of each agent's primal problem is finite and the set of its primal optimal solutions is non-empty. The associated dual problem is given by

$$\max_{\mu \in \mathbb{R}^m} q_i(\mu), \quad \text{s.t.} \quad \mu \geq 0.$$

Here, the dual function $q_i : \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ is defined by $q_i(\mu) := \inf_{x \in X_i} \mathcal{L}_i(x, \mu)$, where $\mathcal{L}_i : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ is the Lagrangian function of agent i and given by $\mathcal{L}_i(x, \mu) = f_i(x) + \mu^T g(x)$. The corresponding dual optimal value is denoted by d_i^*. In this way, $\mathcal{L}$ is decomposed into a sum of local Lagrangian functions; i.e., $\mathcal{L}(x, \mu) = \sum_{i=1}^N \mathcal{L}_i(x, \mu)$.

Define now the set-valued map $Q : \mathbb{R}^m_{\geq 0} \to 2^{(\mathbb{R}^m_{\geq 0})}$ by

$$Q(\tilde{\mu}) = \{\mu \in \mathbb{R}^m_{\geq 0} \mid q_L(\mu) \geq q_L(\tilde{\mu})\}.$$

Additionally, define a function $\gamma : X \to \mathbb{R}$ by $\gamma(x) = \min_{\ell \in \{1,\ldots,m\}}\{-g_\ell(x)\}$. Observe that if x is a Slater vector, then $\gamma(x) > 0$. The following lemma is a direct result of Lemma 1 in [99].

Lemma 4.3.2 (Boundedness of dual solution sets) *The set $Q(\tilde{\mu})$ is bounded for any $\tilde{\mu} \in \mathbb{R}^m_{\geq 0}$, and, in particular, for any Slater vector $\bar{x}$, it holds that*

$$\max_{\mu \in Q(\tilde{\mu})} \|\mu\| \leq \frac{1}{\gamma(\bar{x})}(f(\bar{x}) - q_L(\tilde{\mu})).$$

□

Notice that $D^*_L = \{\mu \in \mathbb{R}^m_{\geq 0} \mid q_L(\mu) \geq d^*_L\}$. Picking any Slater vector $\bar{x} \in X$, and letting $\tilde{\mu} = \mu^* \in D^*_L$ in Lemma 4.3.2 gives that

$$\max_{\mu^* \in D^*_L} \|\mu^*\| \leq \frac{1}{\gamma(\bar{x})}(f(\bar{x}) - d^*_L). \tag{4.3}$$

Define the function $r : X \times \mathbb{R}^m_{\geq 0} \to \mathbb{R} \cup \{+\infty\}$ by $r(x, \mu) := \frac{N}{\gamma(x)} \max_{i \in V}\{f_i(x) - q_i(\mu)\}$. By the property of weak duality, it holds that $d^*_i \leq p^*_i$ and thus $f_i(x) \geq q_i(\mu)$ for any $(x, \mu) \in X \times \mathbb{R}^m_{\geq 0}$. Since $\gamma(\bar{x}) > 0$, thus $r(\bar{x}, \mu) \geq 0$ for any $\mu \in \mathbb{R}^m_{\geq 0}$. With this observation, we pick any $\tilde{\mu} \in \mathbb{R}^m_{\geq 0}$ and the following set is well-defined: $\bar{M}_i(\bar{x}, \tilde{\mu}) := \{\mu \in \mathbb{R}^m_{\geq 0} \mid \|\mu\| \leq r(\bar{x}, \tilde{\mu}) + \theta_i\}$ for some $\theta_i \in \mathbb{R}_{>0}$. Observe that for all $\mu \in \mathbb{R}^m_{\geq 0}$:

$$\begin{aligned} q_L(\mu) &= \inf_{x \in \cap_{i=1}^m X_i} \sum_{i=1}^N (f_i(x) + \mu^T g(x)) \\ &\geq \sum_{i=1}^N \inf_{x \in X_i} (f_i(x) + \mu^T g(x)) = \sum_{i=1}^N q_i(\mu). \end{aligned} \tag{4.4}$$

Since $d^*_L \geq q_L(\tilde{\mu})$, it follows from (4.3) and (4.4) that

$$\begin{aligned} \max_{\mu^* \in D^*_L} \|\mu^*\| &\leq \frac{1}{\gamma(\bar{x})}(f(\bar{x}) - q_L(\tilde{\mu})) \leq \frac{1}{\gamma(\bar{x})}(f(\bar{x}) - \sum_{i=1}^N q_i(\tilde{\mu})) \\ &\leq \frac{N}{\gamma(\bar{x})} \max_{i \in V}\{f_i(\bar{x}) - q_i(\tilde{\mu})\} = r(\bar{x}, \tilde{\mu}). \end{aligned}$$

Hence, we have $D_L^* \subseteq \bar{M}_i(\bar{x}, \tilde{\mu})$ for all $i \in V$.

Note that in order to compute $\bar{M}_i(\bar{x}, \tilde{\mu})$, all the agents have to agree on a common Slater vector $\bar{x} \in \cap_{i=1}^N X_i$ which should be obtained in a distributed fashion. To handle this difficulty, we now propose a distributed algorithm, namely *Distributed Slater-vector Computation Algorithm*, which allows each agent i to compute a superset of $\bar{M}_i(\bar{x}, \tilde{\mu})$.

Initially, each agent i chooses a common value $\tilde{\mu} \in \mathbb{R}^m_{\geq 0}$; e.g., $\tilde{\mu} = 0$, and computes two positive constants $b_i(0)$ and $c_i(0)$ such that $b_i(0) \geq \sup_{x\in J_i}\{f_i(x) - q_i(\tilde{\mu})\}$ and $c_i(0) \leq \min_{1\leq \ell \leq m} \inf_{x\in J_i}\{-g_\ell(x)\}$ where $J_i := \{x \in X_i \mid g(x) < 0\}$.

At every time $k \geq 0$, each agent i updates its estimates by using the following rules:

$$b_i(k+1) = \max_{j\in \mathcal{N}_i(k)\cup\{i\}} b_j(k), \quad c_i(k+1) = \min_{j\in \mathcal{N}_i(k)\cup\{i\}} c_j(k).$$

We denote $b^* := \max_{j\in V} b_j(0)$,$c^* := \min_{j\in V} c_j(0)$ for all $k \geq (N-1)B$, and $M^{[i]}(\tilde{\mu}) := \{\mu \in \mathbb{R}^m_{\geq 0} \mid \|\mu\| \leq \frac{Nb^*}{c^*} + \theta_i\}$, $J := \{x \in X \mid g(x) < 0\}$.

Lemma 4.3.3 ***(Convergence properties of the distributed Slater-vector Computation Algorithm):*** *Assume that the periodical strong connectivity assumption 4.2.4 holds. Consider the sequences of $\{b_i(k)\}$ and $\{c_i(k)\}$ generated by the Distributed Slater-vector Computation Algorithm. It holds that after at most $(N-1)B$ steps, all the agents reach the consensus, i.e., $b_i(k) = b^*$ and $c_i(k) = c^*$ for all $k \geq (N-1)B$. Furthermore, we have $M^{[i]}(\tilde{\mu}) \supseteq \bar{M}_i(\bar{x}, \tilde{\mu})$ for $i \in V$.*

Proof: It is not difficult to verify achieving max-consensus by using the periodical strong connectivity assumption 4.2.4. Note that $J \subseteq J_i$, $\forall i \in V$. Hence, we have

$$\max_{i\in V} \sup_{x\in J}\{f_i(x) - q_i(\tilde{\mu})\} \leq \max_{i\in V} \sup_{x\in J_i}\{f_i(x) - q_i(\tilde{\mu})\} \leq b^*,$$
$$\inf_{x\in J} \min_{1\leq \ell\leq m}\{-g_\ell(x)\} \geq \min_{i\in V} \inf_{x\in J_i} \min_{1\leq \ell\leq m}\{-g_\ell(x)\} \geq c^*.$$

Since $\bar{x} \in J$, then the following estimate on $r(\bar{x}, \tilde{\mu})$ holds:

$$r(\bar{x}, \tilde{\mu}) \leq \frac{N \sup_{x\in J} \max_{i\in V}\{f_i(x) - q_i(\tilde{\mu})\}}{\inf_{x\in J} \min_{1\leq \ell\leq m}\{-g_\ell(x)\}} \leq \frac{Nb^*}{c^*}.$$

The desired result immediately follows. ■

From Lemma 4.3.3 and the fact that $D_L^* \subseteq \bar{M}_i(\bar{x}, \tilde{\mu})$, we can see that the set of $M(\tilde{\mu}) := \cap_{i=1}^N M^{[i]}(\tilde{\mu})$ contains D_L^*. In addition, $M^{[i]}(\tilde{\mu})$ and $M(\tilde{\mu})$ are nonempty, compact and convex. To simplify the notations, we will use the shorthands $M_i := M^{[i]}(\tilde{\mu})$ and $M := M(\tilde{\mu})$.

Convexity of $\mathcal{L}$

For each $\mu \in \mathbb{R}^m_{\geq 0}$, we define the function $\mathcal{L}_{i\mu} : \mathbb{R}^n \to \mathbb{R}$ as $\mathcal{L}_{i\mu}(x) := \mathcal{L}_i(x, \mu)$. Note that $\mathcal{L}_{i\mu}$ is convex since it is a nonnegative weighted sum of convex functions. For each $x \in \mathbb{R}^n$, we define the function $\mathcal{L}_{ix} : \mathbb{R}^m_{\geq 0} \to \mathbb{R}$ as $\mathcal{L}_{ix}(\mu) := \mathcal{L}_i(x, \mu)$. It is easy to check that $\mathcal{L}_{ix}$ is a concave (actually affine) function. Then the Lagrangian function $\mathcal{L}$ is the sum of a collection of convex-concave local functions. This property motivates us to significantly extend primal-dual subgradient methods in [8, 101] to the networked multi-agent scenario.

4.3.2 Distributed Lagrangian primal-dual subgradient algorithm

Here, we introduce the *Distributed Lagrangian Primal-Dual Subgradient* Algorithm (DLPDS, for short) to find a saddle point of the Lagrangian function $\mathcal{L}$ over $X \times M$ and the optimal value. This saddle point will coincide with a pair of primal and Lagrangian dual optimal solutions which is not always the case; see Remark 4.3.1.

Through the algorithm, at each time k, each agent i maintains the estimate of $(x^i(k), \mu^i(k))$ to the saddle point of the Lagrangian function $\mathcal{L}$ over $X \times M$ and the estimate of $y^i(k)$ to p^*. To produce $x^i(k+1)$ (resp. $\mu^i(k+1)$), agent i takes a convex combination $v_x^i(k)$ (resp. $v_\mu^i(k)$) of its estimate $x^i(k)$ (resp. $\mu^i(k)$) with the estimates sent from its neighboring agents at time k, makes a subgradient (resp. supgradient) step to minimize (resp. maximize) the local Lagrangian function $\mathcal{L}_i$, and takes a primal (resp. dual) projection onto the local constraint X_i (resp. M_i). Furthermore, agent i generates the estimate $y^i(k+1)$ by taking a convex com-

bination $v_y^i(k)$ of its estimate $y^i(k)$ with the estimates of its neighbors at time k and taking one step to track the variation of the local objective function f_i. More precisely, the DLPDS algorithm is described as follows:

Initially, each agent i picks a common $\tilde{\mu} \in \mathbb{R}^m_{\geq 0}$ and computes the set M_i with some $\theta_i > 0$ by using the Distributed Slater-vector Computation Algorithm. Agent i then chooses any initial state $x^i(0) \in X_i$, $\mu^i(0) \in \mathbb{R}^m_{\geq 0}$, and $y^i(1) = Nf_i(x^i(0))$.

At every $k \geq 0$, each agent i generates $x^i(k+1)$, $\mu^i(k+1)$ and $y^i(k+1)$ according to the following rules:

$$
\begin{aligned}
&v_x^i(k) = \sum_{j=1}^N a_j^i(k) x^j(k), \quad v_\mu^i(k) = \sum_{j=1}^N a_j^i(k) \mu^j(k), \quad v_y^i(k) = \sum_{j=1}^N a_j^i(k) y^j(k), \\
&x^i(k+1) = P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}_x^i(k)], \\
&\mu^i(k+1) = P_{M_i}[v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)], \\
&y^i(k+1) = v_y^i(k) + N(f_i(x^i(k)) - f_i(x^i(k-1))),
\end{aligned}
$$

where P_{X_i} (resp. P_{M_i}) is the projection operator onto the set X_i (resp. M_i), the scalars $a_j^i(k)$ are non-negative weights and the scalars $\alpha(k) > 0$ are step-sizes[2]. We use the shorthands $\mathcal{D}_x^i(k) \equiv \mathcal{DL}_{iv_\mu^i(k)}(v_x^i(k))$, and $\mathcal{D}_\mu^i(k) \equiv \mathcal{DL}_{iv_x^i(k)}(v_\mu^i(k))$.

The following theorem summarizes the convergence properties of the DLPDS algorithm where it is guaranteed that agents asymptotically agree upon a pair of primal-dual optimal solutions.

Theorem 4.3.2 ***(Convergence properties of the DLPDS algorithm):*** *Consider the optimization problem* (4.2)*. Let the non-degeneracy assumption 4.2.2, the balanced communication assumption 4.2.3 and the periodic strong connectivity assumptions 4.2.4 hold. Consider the sequences of* $\{x^i(k)\}$, $\{\mu^i(k)\}$ *and* $\{y^i(k)\}$ *of the distributed Lagrangian primal-dual subgradient algorithm with the step-sizes* $\{\alpha(k)\}$ *satisfying* $\lim_{k\to+\infty} \alpha(k) = 0$, $\sum_{k=0}^{+\infty} \alpha(k) = +\infty$, *and* $\sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$. *Then, there is a pair of primal and Lagrangian dual optimal solutions* $(x^*, \mu^*) \in X^* \times D_L^*$

[2]Each agent i executes the update law of $y^i(k)$ for $k \geq 1$.

such that $\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0$ *and* $\lim_{k\to+\infty} \|\mu^i(k) - \mu^*\| = 0$ *for all* $i \in V$. *Furthermore, we have that* $\lim_{k\to+\infty} \|y^i(k) - p^*\| = 0$ *for all* $i \in V$.

Remark 4.3.2 For a convex-concave function, continuous-time gradient-based methods are proved in [8] to converge globally towards the saddle-point. Recently, [101] presents (discrete-time) primal-dual subgradient methods which relax the differentiability in [8] and further incorporate state constraints. The method in [8] is adopted by [86] and [119] to study a distributed optimization problem on fixed graphs where objective functions are separable.

The DLPDS algorithm is a generalization of primal-dual subgradient methods in [101] to the networked multi-agent scenario. It is also an extension of the distributed projected subgradient algorithm in [102] to solve multi-agent convex optimization problems with inequality constraints. Additionally, the DLPDS algorithm enables agents to find the optimal value. Furthermore, the DLPDS algorithm objective is that of reaching a saddle point of the Lagrangian function in contrast to achieving a (primal) optimal solution in [102]. •

4.4 Case (ii): identical local constraint sets

In last section, we study the case where the equality constraint is absent in problem (4.1). In this section, we turn our attention to another case of problem (4.1) where $h(x) = 0$ is taken into account but we require that local constraint sets are identical; i.e., $X_i = X$ for all $i \in V$. We first adopt a penalty relaxation and provide a penalty saddle-point characterization of the primal problem (4.1) with $X_i = X$. We then introduce the distributed penalty primal-dual subgradient algorithm, followed by its convergence properties.

4.4.1 Preliminaries

Some preliminary results are developed in this section, and these results are essential to the design of the distributed penalty primal-dual subgradient algorithm in the sequel.

A penalty saddle-point characterization

Note that the primal problem (4.1) with $X_i = X$ is trivially equivalent to the following:

$$\min_{x\in\mathbb{R}^n} f(x), \quad \text{s.t.} \quad Ng(x) \leq 0, \quad Nh(x) = 0, \quad x \in X, \tag{4.5}$$

with associated penalty dual problem given by

$$\max_{\mu\in\mathbb{R}^m,\lambda\in\mathbb{R}^\nu} q_P(\mu,\lambda), \quad \text{s.t.} \quad \mu \geq 0, \quad \lambda \geq 0. \tag{4.6}$$

Here, the penalty dual function, $q_P : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0} \to \mathbb{R}$, is defined by

$$q_P(\mu,\lambda) := \inf_{x\in X} \mathcal{H}(x,\mu,\lambda),$$

where $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0} \to \mathbb{R}$ is the *penalty function* given by $\mathcal{H}(x,\mu,\lambda) = f(x) + N\mu^T[g(x)]^+ + N\lambda^T|h(x)|$. We denote the penalty dual optimal value by d^*_P and the set of penalty dual optimal solutions by D^*_P. We define the penalty function $\mathcal{H}_i(x,\mu,\lambda) : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0} \to \mathbb{R}$ for each agent i as follows: $\mathcal{H}_i(x,\mu,\lambda) = f_i(x) + \mu^T[g(x)]^+ + \lambda^T|h(x)|$. In this way, we have that $\mathcal{H}(x,\mu,\lambda) = \sum_{i=1}^N \mathcal{H}_i(x,\mu,\lambda)$. As proven in the next lemma, the Slater's condition 4.2.1 ensures zero duality gap and the existence of penalty dual optimal solutions.

Lemma 4.4.1 ***(Strong duality and non-emptyness of the penalty dual optimal set):*** *The values of p^* and d^*_P coincide, and D^*_P is non-empty.*

Proof: Consider the auxiliary Lagrangian function $\mathcal{L}_a : \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu \to \mathbb{R}$ given by $\mathcal{L}_a(x,\mu,\lambda) = f(x) + N\mu^T g(x) + N\lambda^T h(x)$, with the associated dual problem defined by

$$\max_{\mu\in\mathbb{R}^m,\lambda\in\mathbb{R}^\nu} q_a(\mu,\lambda), \quad \text{s.t.} \quad \mu \geq 0. \tag{4.7}$$

Here, the dual function, $q_a : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu \to \mathbb{R}$, is defined by

$$q_a(\mu,\lambda) := \inf_{x\in X} \mathcal{L}_a(x,\mu,\lambda).$$

The dual optimal value of problem (4.7) is denoted by d^*_a and the set of dual optimal solutions is denoted D^*_a. Since X is convex, f and g_ℓ, for $\ell \in \{1,\ldots,m\}$, are convex,

p^* is finite and the Slater's condition 4.2.1 holds, it follows from Proposition 5.3.5 in [15] that $p^* = d_a^*$ and $D_a^* \neq \emptyset$. We now proceed to characterize d_P^* and D_P^*. Pick any $(\mu^*, \lambda^*) \in D_a^*$. Since $\mu^* \geq 0$, then

$$\begin{aligned} d_a^* = q_a(\mu^*, \lambda^*) &= \inf_{x \in X} \{f(x) + N(\mu^*)^T g(x) + N(\lambda^*)^T h(x)\} \\ &\leq \inf_{x \in X} \{f(x) + N(\mu^*)^T [g(x)]^+ + N|\lambda^*|^T |h(x)|\} \\ &= q_P(\mu^*, |\lambda^*|) \leq d_P^*. \end{aligned} \tag{4.8}$$

On the other hand, pick any $x^* \in X^*$. Then x^* is feasible, i.e., $x^* \in X$, $[g(x^*)]^+ = 0$ and $|h(x^*)| = 0$. It implies that $q_P(\mu, \lambda) \leq \mathcal{H}(x^*, \mu, \lambda) = f(x^*) = p^*$ holds for any $\mu \in \mathbb{R}^m_{\geq 0}$ and $\lambda \in \mathbb{R}^\nu_{\geq 0}$, and thus $d_P^* = \sup_{\mu \in \mathbb{R}^m_{\geq 0}, \lambda \in \mathbb{R}^\nu_{\geq 0}} q_P(\mu, \lambda) \leq p^* = d_a^*$. Therefore, we have $d_P^* = p^*$.

To prove the emptyness of D_P^*, we pick any $(\mu^*, \lambda^*) \in D_a^*$. From (4.8) and $d_a^* = d_P^*$, we can see that $(\mu^*, |\lambda^*|) \in D_P^*$ and thus $D_P^* \neq \emptyset$. ■

The following is a slight extension of Theorem 4.3.1 to penalty functions.

Theorem 4.4.1 (Penalty Saddle-point Theorem) *The pair of (x^*, μ^*, λ^*) is a saddle point of the penalty function $\mathcal{H}$ over $X \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}$ if and only if it is a pair of primal and penalty dual optimal solutions and the following* penalty minimax equality *holds:*

$$\sup_{(\mu,\lambda) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}} \inf_{x \in X} \mathcal{H}(x, \mu, \lambda) = \inf_{x \in X} \sup_{(\mu,\lambda) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}} \mathcal{H}(x, \mu, \lambda).$$

Proof: The proof is analogous to that of Proposition 6.2.4 in [16], and for the sake of completeness, we provide the details here. It follows from Proposition 2.6.1 in [16] that (x^*, μ^*, λ^*) is a saddle point of $\mathcal{H}$ over $X \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}$ if and only if the penalty minimax equality holds and the following conditions are satisfied:

$$\sup_{(\mu,\lambda) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}} \mathcal{H}(x^*, \mu, \lambda) = \min_{x \in X} \{ \sup_{(\mu,\lambda) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}} \mathcal{H}(x, \mu, \lambda)\}, \tag{4.9}$$

$$\inf_{x \in X} \mathcal{H}(x, \mu^*, \lambda^*) = \max_{(\mu,\lambda) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}} \{ \inf_{x \in X} \mathcal{H}(x, \mu, \lambda)\}. \tag{4.10}$$

Notice that $\inf_{x\in X}\mathcal{H}(x,\mu,\lambda) = q_P(\mu,\lambda)$; and if $x \in Y$, then the following holds:

$$\sup_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \mathcal{H}(x,\mu,\lambda) = f(x),$$

otherwise, $\sup_{(\mu,\lambda)\in\mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}} \mathcal{H}(x,\mu,\lambda) = +\infty$. Hence, the penalty minimax equality is equivalent to $d^*_P = p^*$. Condition (4.9) is equivalent to the fact that x^* is primal optimal, and condition (4.10) is equivalent to (μ^*,λ^*) being a penalty dual optimal solution. ■

Convexity of $\mathcal{H}$

Since g_ℓ is convex and $[\cdot]^+$ is convex and non-decreasing, thus $[g_\ell(x)]^+$ is convex in x for each $\ell \in \{1,\ldots,m\}$. Denote $A := (a_1^T,\cdots,a_\nu^T)^T$. Since $|\cdot|$ is convex and $a_\ell^T x - b_\ell$ is an affine mapping, then $|a_\ell^T x - b_\ell|$ is convex in x for each $\ell \in \{1,\ldots,\nu\}$.

We denote $w := (\mu,\lambda)$. For each $w \in \mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0}$, we define the function $\mathcal{H}_{iw} : \mathbb{R}^n \to \mathbb{R}$ as $\mathcal{H}_{iw}(x) := \mathcal{H}_i(x,w)$. Note that $\mathcal{H}_{iw}(x)$ is convex in x by using the fact that a nonnegative weighted sum of convex functions is convex. For each $x \in \mathbb{R}^n$, we define the function $\mathcal{H}_{ix} : \mathbb{R}^m_{\geq 0}\times\mathbb{R}^\nu_{\geq 0} \to \mathbb{R}$ as $\mathcal{H}_{ix}(w) := \mathcal{H}_i(x,w)$. It is easy to check that $\mathcal{H}_{ix}(w)$ is concave (actually affine) in w. Then the penalty function $\mathcal{H}(x,w)$ is the sum of convex-concave local functions.

Remark 4.4.1 The Lagrangian relaxation does not fit to our approach here since the Lagrangian function is not convex in x by allowing λ entries to be negative. •

4.4.2 Distributed penalty primal-dual subgradient algorithm

We now proceed to devise the *Distributed Penalty Primal-Dual Subgradient* Algorithm (DPPDS, for short), that is based on the penalty saddle-point theorem 4.4.1, to find the optimal value and a primal optimal solution to the primal problem (4.1) with $X_i = X$. The main steps of the DPPDS algorithm are described as follow.

Initially, agent i chooses any initial state $x^i(0) \in X$, $\mu^i(0) \in \mathbb{R}^m_{\geq 0}$, $\lambda^i(0) \in \mathbb{R}^\nu_{\geq 0}$, and $y^i(1) = Nf_i(x^i(0))$. At every time $k \geq 0$, each agent i computes the

following convex combinations:

$$v_x^i(k) = \sum_{j=1}^{N} a_j^i(k) x^j(k), \quad v_y^i(k) = \sum_{j=1}^{N} a_j^i(k) y^j(k),$$
$$v_\mu^i(k) = \sum_{j=1}^{N} a_j^i(k) \mu^j(k), \quad v_\lambda^i(k) = \sum_{j=1}^{N} a_j^i(k) \lambda^j(k),$$

and generates $x^i(k+1)$, $y^i(k+1)$, $\mu^i(k+1)$ and $\lambda^i(k+1)$ according to the following:

$$\begin{aligned}
x^i(k+1) &= P_X[v_x^i(k) - \alpha(k)\mathcal{S}_x^i(k)], \\
y^i(k+1) &= v_y^i(k) + N(f_i(x^i(k)) - f_i(x^i(k-1))), \\
\mu^i(k+1) &= v_\mu^i(k) + \alpha(k)[g(v_x^i(k))]^+, \\
\lambda^i(k+1) &= v_\lambda^i(k) + \alpha(k)|h(v_x^i(k))|,
\end{aligned}$$

where P_X is the projection operator onto the set X, the scalars $a_j^i(k)$ are non-negative weights and the positive scalars $\{\alpha(k)\}$ are step-sizes[3]. The vector

$$\mathcal{S}_x^i(k) := \mathcal{D}f_i(v_x^i(k)) + \sum_{\ell=1}^{m} v_\mu^i(k)_\ell \mathcal{D}[g_\ell(v_x^i(k))]^+ + \sum_{\ell=1}^{\nu} v_\lambda^i(k)_\ell \mathcal{D}|h_\ell|(v_x^i(k))$$

is a subgradient of $\mathcal{H}_{iw^i(k)}(x)$ at $x = v_x^i(k)$ where $w^i(k) := (v_\mu^i(k), v_\lambda^i(k))$.

Given a step-size sequence $\{\alpha(k)\}$, we define the sum over $[0,k]$ by $s(k) := \sum_{\ell=0}^{k} \alpha(\ell)$ and assume that:

Assumption 4.4.1 (Step-size assumption) *The step-sizes satisfy*

$$\lim_{k\to+\infty} \alpha(k) = 0, \sum_{k=0}^{+\infty} \alpha(k) = +\infty, \sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty,$$
$$\lim_{k\to+\infty} \alpha(k+1)s(k) = 0, \sum_{k=0}^{+\infty} \alpha(k+1)^2 s(k) < +\infty, \sum_{k=0}^{+\infty} \alpha(k+1)^2 s(k)^2 < +\infty.$$

The following theorem is the main result of this section, characterizing the convergence of the DPPDS algorithm where a optimal solution and the optimal value are asymptotically achieved.

[3]Each agent i executes the update law of $y^i(k)$ for $k \geq 1$.

Theorem 4.4.2 ***(Convergence properties of the DPPDS algorithm):*** *Consider the problem* (4.1) *with* $X_i = X$*. Let the non-degeneracy assumption 4.2.2, the balanced communication assumption 4.2.3 and the periodic strong connectivity assumption 4.2.4 hold. Consider the sequences of* $\{x^i(k)\}$ *and* $\{y^i(k)\}$ *of the distributed penalty primal-dual subgradient algorithm where the step-sizes* $\{\alpha(k)\}$ *satisfy the step-size assumption 4.4.1. Then there exists a primal optimal solution* $\tilde{x} \in X^*$ *such that* $\lim_{k\to+\infty} \|x^i(k) - \tilde{x}\| = 0$ *for all* $i \in V$*. Furthermore, we have* $\lim_{k\to+\infty} \|y^i(k) - p^*\| = 0$ *for all* $i \in V$*.*

Remark 4.4.2 As the primal-dual subgradient algorithm in [8, 101], the DPPDS algorithm produces a pair of primal and dual estimates at each step. Main differences include: firstly, the DPPDS algorithm extends the primal-dual subgradient algorithm in [101] to the multi-agent scenario; secondly, it further takes the equality constraint into account. The presence of the equality constraint can make D_P^* unbounded. Therefore, unlike the DLPDS algorithm, the DPPDS algorithm does not involve the dual projection steps onto compact sets. This may cause the subgradient $\mathcal{S}_x^i(k)$ not to be uniformly bounded, while the boundedness of subgradients is a standard assumption in the analysis of subgradient methods, e.g., see [15, 16, 99, 100, 101, 102]. This difficulty will be addressed by a more careful choice of the step-size policy; i.e, assumption 4.4.1, which is stronger than the more standard diminishing step-size scheme, e.g., in the DLPDS algorithm and [102]. We require this condition in order to prove, in the absence of the boundedness of $\{\mathcal{S}_x^i(k)\}$, the existence of a number of limits and summability of expansions toward Theorem 4.4.2. Thirdly, the DPPDS algorithm adopts the penalty relaxation instead of the Lagrangian relaxation in [101]. •

Remark 4.4.3 Observe that $\mu^i(k) \geq 0$, $\lambda^i(k) \geq 0$ and $v_x^i(k) \in X$ (due to the fact that X is convex). Furthermore, $([g(v_x^i(k))]^+, |h(v_x^i(k))|)$ is a supgradient of $\mathcal{H}_{iv_x^i(k)}(w^i(k))$; i.e. the following *penalty supgradient inequality* holds for any $\mu \in \mathbb{R}^m_{\geq 0}$ and $\lambda \in \mathbb{R}^\nu_{\geq 0}$:

$$\begin{aligned} &([g(v_x^i(k))]^+)^T(\mu - v_\mu^i(k)) + |h(v_x^i(k))|^T(\lambda - v_\lambda^i(k)) \\ &\geq \mathcal{H}_i(v_x^i(k), \mu, \lambda) - \mathcal{H}_i(v_x^i(k), v_\mu^i(k), v_\lambda^i(k)). \end{aligned} \tag{4.11}$$

•

Remark 4.4.4 A step-size sequence that satisfies the step-size assumption 4.4.1 is the harmonic series $\{\alpha(k) = \frac{1}{k+1}\}_{k\in\mathbb{Z}_{\geq 0}}$. It is obvious that $\lim_{k\to+\infty} \frac{1}{k+1} = 0$, and well-known that $\sum_{k=0}^{+\infty} \frac{1}{k+1} = +\infty$ and $\sum_{k=0}^{+\infty} \frac{1}{(k+1)^2} < +\infty$. We now proceed to check the property of $\lim_{k\to+\infty} \alpha(k+1)s(k) = 0$. For any $k \geq 1$, there is an integer $n \geq 1$ such that $2^{n-1} \leq k < 2^n$. It holds that

$$\begin{aligned} s(k) &\leq s(2^n) = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + \cdots + (\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}) \\ &\leq 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{3}) + \cdots + (\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^{n-1}+1}) \\ &\leq 1 + 1 + 1 + \cdots + 1 = n \leq \log_2 k + 1. \end{aligned}$$

Then we have $\limsup_{k\to+\infty} \frac{s(k)}{k+2} \leq \lim_{k\to+\infty} \frac{\log_2 k + 1}{k+2} = 0$. Obviously, $\liminf_{k\to+\infty} \frac{s(k)}{k+2} \geq 0$. Then we have the property of $\lim_{k\to+\infty} \alpha(k+1)s(k) = 0$. Since $\log_2 k \leq (\log_2 k)^2 < (k+2)^{\frac{1}{2}}$, then

$$\begin{aligned} \sum_{k=0}^{+\infty} \alpha(k+1)^2 s(k)^2 &\leq \sum_{k=0}^{+\infty} \frac{(\log_2 k + 1)^2}{(k+2)^2} \\ &= \sum_{k=0}^{+\infty} (\frac{(\log_2 k)^2}{(k+2)^2} + \frac{2\log_2 k}{(k+2)^2} + \frac{1}{(k+2)^2}) \\ &\leq \sum_{k=0}^{+\infty} \frac{1}{(k+2)^{\frac{3}{2}}} + \sum_{k=0}^{+\infty} \frac{2}{(k+2)^{\frac{3}{2}}} + \sum_{k=0}^{+\infty} \frac{1}{(k+2)^2} < +\infty. \end{aligned}$$

Additionally, we have $\sum_{k=0}^{+\infty} \alpha(k+1)^2 s(k) \leq \sum_{k=0}^{+\infty} \alpha(k+1)^2 s(k)^2 < +\infty$. •

4.5 Convergence analysis

We next provide the proofs for the main results, Theorem 4.3.2 and 4.4.2, of this chapter. We start our analysis by providing some useful properties of the sequences weighted by $\{\alpha(k)\}$.

Lemma 4.5.1 ***(Convergence properties of weighted sequences):*** *Let $K \geq 0$. Consider the sequence $\{\delta(k)\}$ defined by $\delta(k) := \frac{\sum_{\ell=K}^{k-1} \alpha(\ell)\rho(\ell)}{\sum_{\ell=K}^{k-1} \alpha(\ell)}$ where $k \geq K+1$, $\alpha(k) > 0$ and $\sum_{k=K}^{+\infty} \alpha(k) = +\infty$.*

(a) If $\lim_{k\to+\infty}\rho(k)=+\infty$, *then* $\lim_{k\to+\infty}\delta(k)=+\infty$.
(b) If $\lim_{k\to+\infty}\rho(k)=\rho^*$, *then* $\lim_{k\to+\infty}\delta(k)=\rho^*$.

Proof: (a) For any $\Pi>0$, there exists $k_1\geq K$ such that $\rho(k)\geq\Pi$ for all $k\geq k_1$. Then the following holds for all $k\geq k_1+1$:

$$\begin{aligned}\delta(k)&\geq\frac{1}{\sum_{\ell=K}^{k-1}\alpha(\ell)}(\sum_{\ell=K}^{k_1-1}\alpha(\ell)\rho(\ell)+\sum_{\ell=k_1}^{k-1}\alpha(\ell)\Pi)\\&=\Pi+\frac{1}{\sum_{\ell=K}^{k-1}\alpha(\ell)}(\sum_{\ell=K}^{k_1-1}\alpha(\ell)\rho(\ell)-\sum_{\ell=K}^{k_1-1}\alpha(\ell)\Pi).\end{aligned}$$

Take the limit on k in the above estimate and we have $\liminf_{k\to+\infty}\delta(k)\geq\Pi$. Since Π is arbitrary, then $\lim_{k\to+\infty}\delta(k)=+\infty$.

(b) For any $\epsilon>0$, there exists $k_2\geq K$ such that $\|\rho(k)-\rho^*\|\leq\epsilon$ for all $k\geq k_2+1$. Then we have

$$\begin{aligned}\|\delta(k)-\rho^*\|&=\|\frac{\sum_{\tau=K}^{k-1}\alpha(\tau)(\rho(\tau)-\rho^*)}{\sum_{\tau=K}^{k-1}\alpha(\tau)}\|\\&\leq\frac{1}{\sum_{\tau=K}^{k-1}\alpha(\tau)}(\sum_{\tau=K}^{k_2-1}\alpha(\tau)\|\rho(\tau)-\rho^*\|+\sum_{\tau=k_2}^{k-1}\alpha(\tau)\epsilon)\\&\leq\frac{\sum_{\tau=K}^{k_2-1}\alpha(\tau)\|\rho(\tau)-\rho^*\|}{\sum_{\tau=K}^{k-1}\alpha(\tau)}+\epsilon.\end{aligned}$$

Take the limit on k in the above estimate and we have $\limsup_{k\to+\infty}\|\delta(k)-\rho^*\|\leq\epsilon$. Since ϵ is arbitrary, then $\lim_{k\to+\infty}\|\delta(k)-\rho^*\|=0$. ■

4.5.1 Proofs of Theorem 4.3.2

We now proceed to show Theorem 4.3.2. To do that, we first rewrite the DLPDS algorithm into the following form:

$$\begin{aligned}&x^i(k+1)=v_x^i(k)+e_x^i(k),\quad \mu^i(k+1)=v_\mu^i(k)+e_\mu^i(k),\\&y^i(k+1)=v_y^i(k)+u^i(k),\end{aligned}$$

where $e_x^i(k)$ and $e_\mu^i(k)$ are projection errors described by

$$\begin{aligned}e_x^i(k)&:=P_{X_i}[v_x^i(k)-\alpha(k)\mathcal{D}_x^i(k)]-v_x^i(k),\\e_\mu^i(k)&:=P_{M_i}[v_\mu^i(k)+\alpha(k)\mathcal{D}_\mu^i(k)]-v_\mu^i(k),\end{aligned}$$

and $u^i(k) := N(f_i(x^i(k)) - f_i(x^i(k-1)))$ is the local input which allows agent i to track the variation of the local objective function f_i. In this manner, the update law of each estimate is decomposed in two parts: a convex sum to fuse the information of each agent with those of its neighbors, plus some local error or input. With this decomposition, all the update laws are put into the same form as the dynamic average consensus algorithm in the Appendix. This observation allows us to divide the analysis of the DLPDS algorithm in two steps. Firstly, we show all the estimates asymptotically achieve consensus by utilizing the property that the local errors and inputs are diminishing. Secondly, we further show that the consensus vectors coincide with a pair of primal and Lagrangian dual optimal solutions and the optimal value.

Lemma 4.5.2 (Lipschitz continuity of $\mathcal{L}_i$) *Consider $\mathcal{L}_{i\mu}$ and $\mathcal{L}_{ix}$. Then there are $L > 0$ and $R > 0$ such that $\|\mathcal{D}\mathcal{L}_{i\mu}(x)\| \leq L$ and $\|\mathcal{D}\mathcal{L}_{ix}(\mu)\| \leq R$ for each pair of $x \in \text{co}(\cup_{i=1}^N X_i)$ and $\mu \in \text{co}(\cup_{i=1}^N M_i)$. Furthermore, for each $\mu \in \text{co}(\cup_{i=1}^N M_i)$, the function $\mathcal{L}_{i\mu}$ is Lipschitz continuous with Lipschitz constant L over $\text{co}(\cup_{i=1}^N X_i)$, and for each $x \in \text{co}(\cup_{i=1}^N X_i)$, the function $\mathcal{L}_{ix}$ is Lipschitz continuous with Lipschitz constant R over $\text{co}(\cup_{i=1}^N M_i)$.*

Proof: Observe that $\mathcal{D}\mathcal{L}_{i\mu} = \mathcal{D}f_i + \mu^T \mathcal{D}g$ and $\mathcal{D}\mathcal{L}_{ix} = g$. Since f_i and g_ℓ are convex, it follows from Proposition 5.4.2 in [15] that ∂f_i and ∂g_ℓ are bounded over the compact $\text{co}(\cup_{i=1}^N X_i)$. Since $\text{co}(\cup_{i=1}^N M_i)$ is bounded, so is $\partial\mathcal{L}_{i\mu}$, i.e., for any $\mu \in \text{co}(\cup_{i=1}^N M_i)$, there exists $L > 0$ such that $\|\partial\mathcal{L}_{i\mu}(x)\| \leq L$ for all $x \in \text{co}(\cup_{i=1}^N X_i)$. Since g_ℓ is continuous (due to its convexity) and $\text{co}(\cup_{i=1}^N X_i)$ is bounded, then g and thus $\partial\mathcal{L}_{ix}$ are bounded, i.e., for any $x \in \text{co}(\cup_{i=1}^N X_i)$, there exists $R > 0$ such that $\|\partial\mathcal{L}_{ix}(\mu)\| \leq R$ for all $\mu \in \text{co}(\cup_{i=1}^N M_i)$.

It follows from the Lagrangian subgradient inequality that

$$\mathcal{D}\mathcal{L}_{i\mu}(x)^T(x' - x) \leq \mathcal{L}_{i\mu}(x') - \mathcal{L}_{i\mu}(x),$$
$$\mathcal{D}\mathcal{L}_{i\mu}(x')^T(x - x') \leq \mathcal{L}_{i\mu}(x) - \mathcal{L}_{i\mu}(x'),$$

for any $x, x' \in \text{co}(\cup_{i=1}^N X_i)$. By using the boundedness of the subdifferentials, the above two inequalities give that $-L\|x - x'\| \leq \mathcal{L}_{i\mu}(x) - \mathcal{L}_{i\mu}(x') \leq L\|x - x'\|$. This

implies that $\|\mathcal{L}_{i\mu}(x) - \mathcal{L}_{i\mu}(x')\| \leq L\|x - x'\|$ for any $x, x' \in \mathrm{co}(\cup_{i=1}^{m} X_i)$. The proof for the Lipschitz continuity of $\mathcal{L}_{ix}$ is analogous by using the Lagrangian supgradient inequality. ■

The following lemma provides a basic iteration relation used in the convergence proof for the DLPDS algorithm.

Lemma 4.5.3 (Basic iteration relation) *Let the balanced communication assumption 4.2.3 and the periodic strong connectivity assumption 4.2.4 hold. For any $x \in X$, any $\mu \in M$ and all $k \geq 0$, the following estimates hold:*

$$\begin{aligned}
&\sum_{i=1}^{N} \|e_x^i(k) + \alpha(k)\mathcal{D}_x^i(k)\|^2 \\
&\leq \sum_{i=1}^{N} \alpha(k)^2 \|\mathcal{D}_x^i(k)\|^2 + \sum_{i=1}^{N} \{\|x^i(k) - x\|^2 - \|x^i(k+1) - x\|^2\} \\
&- \sum_{i=1}^{N} 2\alpha(k)(\mathcal{L}_i(v_x^i(k), v_\mu^i(k)) - \mathcal{L}_i(x, v_\mu^i(k))), \qquad (4.12) \\
&\sum_{i=1}^{N} \|e_\mu^i(k) - \alpha(k)\mathcal{D}_\mu^i(k)\|^2 \\
&\leq \sum_{i=1}^{N} \alpha(k)^2 \|\mathcal{D}_\mu^i(k)\|^2 + \sum_{i=1}^{N} \{\|\mu^i(k) - \mu\|^2 - \|\mu^i(k+1) - \mu\|^2\} \\
&+ \sum_{i=1}^{N} 2\alpha(k)(\mathcal{L}_i(v_x^i(k), v_\mu^i(k)) - \mathcal{L}_i(v_x^i(k), \mu)). \qquad (4.13)
\end{aligned}$$

Proof: By Lemma 4.8.1 with $Z = M_i$, $z = v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)$ and $y = \mu \in M$, we have that for all $k \geq 0$

$$\begin{aligned}
&\sum_{i=1}^{N} \|e_\mu^i(k) - \alpha(k)\mathcal{D}_\mu^i(k)\|^2 \\
&\leq \sum_{i=1}^{N} \|v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k) - \mu\|^2 - \sum_{i=1}^{N} \|\mu^i(k+1) - \mu\|^2 \\
&= \sum_{i=1}^{N} \|v_\mu^i(k) - \mu\|^2 + \sum_{i=1}^{N} \alpha(k)^2 \|\mathcal{D}_\mu^i(k)\|^2 \\
&+ \sum_{i=1}^{N} 2\alpha(k)\mathcal{D}_\mu^i(k)^T(v_\mu^i(k) - \mu) - \sum_{i=1}^{N} \|\mu^i(k+1) - \mu\|^2 \\
&\leq \sum_{i=1}^{N} \alpha(k)^2 \|\mathcal{D}_\mu^i(k)\|^2 + \sum_{i=1}^{N} 2\alpha(k)\mathcal{D}_\mu^i(k)^T(v_\mu^i(k) - \mu) \\
&+ \sum_{i=1}^{N} \|\mu^i(k) - \mu\|^2 - \sum_{i=1}^{N} \|\mu^i(k+1) - \mu\|^2. \qquad (4.14)
\end{aligned}$$

One can show (4.13) by substituting the following Lagrangian supgradient inequality into (4.14):

$$\mathcal{D}_\mu^i(k)^T(\mu - v_\mu^i(k)) \geq \mathcal{L}_i(v_x^i(k), \mu) - \mathcal{L}_i(v_x^i(k), v_\mu^i(k)).$$

Similarly, the equality (4.12) can be shown by using the following Lagrangian subgradient inequality: $\mathcal{D}_x^i(k)^T(x - v_x^i(k)) \leq \mathcal{L}_i(x, v_\mu^i(k)) - \mathcal{L}_i(v_x^i(k), v_\mu^i(k))$. ■

The following lemma shows that the consensus is asymptotically reached.

Lemma 4.5.4 (Achieving consensus) *Let the non-degeneracy assumption 4.2.2, the balanced communication assumption 4.2.3 and the periodic strong connectivity assumption 4.2.4 hold. Consider the sequences of* $\{x^i(k)\}$, $\{\mu^i(k)\}$ *and* $\{y^i(k)\}$ *of the DLPDS algorithm with the step-size sequence* $\{\alpha(k)\}$ *satisfying* $\lim_{k\to+\infty} \alpha(k) = 0$. *Then there exist* $x^* \in X$ *and* $\mu^* \in M$ *such that*

$$\begin{aligned}
&\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0, \quad \lim_{k\to+\infty} \|\mu^i(k) - \mu^*\| = 0, \quad \forall i \in V, \\
&\lim_{k\to+\infty} \|y^i(k) - y^j(k)\| = 0, \quad \forall i, j \in V.
\end{aligned}$$

Proof: Observe that $v_x^i(k) \in \text{co}(\cup_{i=1}^N X_i)$ and $v_\mu^i(k) \in \text{co}(\cup_{i=1}^N M_i)$. Then it follows from Lemma 4.5.2 that $\|\mathcal{D}_x^i(k)\| \leq L$. From Lemma 4.5.3 it follows that

$$\sum_{i=1}^N \|x^i(k+1) - x\|^2 \leq \sum_{i=1}^N \|x^i(k) - x\|^2 + \sum_{i=1}^N \alpha(k)^2 L^2$$
$$+ \sum_{i=1}^N 2\alpha(k)(\|\mathcal{L}_i(v_x^i(k), v_\mu^i(k))\| + \|\mathcal{L}_i(x, v_\mu^i(k))\|). \tag{4.15}$$

Notice that $v_x^i(k) \in \text{co}(\cup_{i=1}^N X_i)$, $v_\mu^i(k) \in \text{co}(\cup_{i=1}^N M_i)$ and $x \in X$ are bounded. Since $\mathcal{L}_i$ is continuous, then $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ and $\mathcal{L}_i(x, v_\mu^i(k))$ are bounded. Since $\lim_{k\to+\infty} \alpha(k) = 0$, one can verify that $\lim_{k\to+\infty} \sum_{i=1}^N \|x^i(k) - x\|^2$ exists for any $x \in X$. On the other hand, taking limits on both sides of (4.12) we obtain

$$\lim_{k\to+\infty} \sum_{i=1}^N \|e_x^i(k) + \alpha(k)\mathcal{D}_x^i(k)\|^2 = 0,$$

and therefore we deduce that $\lim_{k\to+\infty} \|e_x^i(k)\| = 0$ for all $i \in V$. It follows from Corollary 2.3.1 in Chapter 2 that $\lim_{k\to+\infty} \|x^i(k) - x^j(k)\| = 0$ for all $i, j \in V$. Combining this with the property that $\lim_{k\to+\infty} \|x^i(k) - x\|$ exists for any $x \in X$, we deduce that there exists $x^* \in \mathbb{R}^n$ such that $\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0$ for all $i \in V$. Since $x^i(k) \in X_i$ and X_i is closed, it implies that $x^* \in X_i$ for all $i \in V$ and thus $x^* \in X$. Similarly, one can show that there is $\mu^* \in M$ such that $\lim_{k\to+\infty} \|\mu^i(k) - \mu^*\| = 0$ for all $i \in V$.

Since $\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0$ and f_i is continuous, then $\lim_{k\to+\infty} \|u^i(k)\| = 0$. It follows from Corollary 2.3.1 in Chapter 2 that $\lim_{k\to+\infty} \|y^i(k) - y^j(k)\| = 0$ for all $i, j \in V$. ■

From Lemma 4.5.4, we know that the sequences of $\{x^i(k)\}$ and $\{\mu^i(k)\}$ of the DLPDS algorithm asymptotically agree on to some point in X and some point in M, respectively. Denote by $\Theta \subseteq X \times M$ the set of such limit points. Denote by the average of primal and dual estimates $\hat{x}(k) := \frac{1}{N}\sum_{i=1}^N x^i(k)$ and $\hat{\mu}(k) := \frac{1}{N}\sum_{i=1}^N \mu^i(k)$, respectively. The following lemma further characterizes that the points in Θ are saddle points of the Lagrangian function $\mathcal{L}$ over $X \times M$.

Lemma 4.5.5 (Saddle-point characterization of Θ) *Each point in Θ is a saddle point of the Lagrangian function $\mathcal{L}$ over $X \times M$.*

Proof: Denote by the maximum deviation of primal estimates $\Delta_x(k) := \max_{i,j\in V} \|x^j(k) - x^i(k)\|$. Notice that

$$\begin{aligned}\|v_x^i(k) - \hat{x}(k)\| &= \|\sum_{j=1}^N a_j^i(k)x^j(k) - \sum_{j=1}^N \frac{1}{N}x^j(k)\| \\ &= \|\sum_{j\neq i} a_j^i(k)(x^j(k) - x^i(k)) - \sum_{j\neq i}\frac{1}{N}(x^j(k) - x^i(k))\| \\ &\leq \sum_{j\neq i} a_j^i(k)\|x^j(k) - x^i(k)\| + \sum_{j\neq i}\frac{1}{N}\|x^j(k) - x^i(k)\| \leq 2\Delta_x(k).\end{aligned}$$

Denote by the maximum deviation of dual estimates $\Delta_\mu(k) := \max_{i,j\in V} \|\mu^j(k) - \mu^i(k)\|$. Similarly, we have $\|v_\mu^i(k) - \hat{\mu}(k)\| \leq 2\Delta_\mu(k)$.

We will show this lemma by contradiction. Suppose that there is $(x^*, \mu^*) \in \Theta$ which is not a saddle point of $\mathcal{L}$ over $X \times M$. Then at least one of the following equalities holds:

$$\exists x \in X \quad \text{s.t.} \quad \mathcal{L}(x^*, \mu^*) > \mathcal{L}(x, \mu^*), \tag{4.16}$$

$$\exists \mu \in M \quad \text{s.t.} \quad \mathcal{L}(x^*, \mu) > \mathcal{L}(x^*, \mu^*). \tag{4.17}$$

Suppose first that (4.16) holds. Then, there exists $\varsigma > 0$ such that $\mathcal{L}(x^*, \mu^*) = \mathcal{L}(x, \mu^*) + \varsigma$. Consider the sequences of $\{x^i(k)\}$ and $\{\mu^i(k)\}$ which converge respectively to x^* and μ^* defined above. The estimate (4.12) leads to

$$\begin{aligned}\sum_{i=1}^N \|x^i(k+1) - x\|^2 \leq \sum_{i=1}^N \|x^i(k) - x\|^2 + \alpha(k)^2 \sum_{i=1}^N \|\mathcal{D}_x^i(k)\|^2 - 2\alpha(k) \\ \times \sum_{i=1}^N (A_i(k) + B_i(k) + C_i(k) + D_i(k) + E_i(k) + F_i(k)),\end{aligned}$$

where the notations are given by:

$$\begin{aligned}A_i(k) &= \mathcal{L}_i(v_x^i(k), v_\mu^i(k)) - \mathcal{L}_i(\hat{x}(k), v_\mu^i(k)), \\ B_i(k) &= \mathcal{L}_i(\hat{x}(k), v_\mu^i(k)) - \mathcal{L}_i(\hat{x}(k), \hat{\mu}(k)), \\ C_i(k) &= \mathcal{L}_i(\hat{x}(k), \hat{\mu}(k)) - \mathcal{L}_i(x^*, \hat{\mu}(k)), \\ D_i(k) &= \mathcal{L}_i(x^*, \hat{\mu}(k)) - \mathcal{L}_i(x^*, \mu^*), \\ E_i(k) &= \mathcal{L}_i(x^*, \mu^*) - \mathcal{L}_i(x, \mu^*), \\ F_i(k) &= \mathcal{L}_i(x, \mu^*) - \mathcal{L}_i(x, v_\mu^i(k)).\end{aligned}$$

It follows from the Lipschitz continuity property of $\mathcal{L}_i$; see Lemma 4.5.2, that

$$\begin{aligned}
&\|A_i(k)\| \leq L\|v_x^i(k) - \hat{x}(k)\| \leq 2L\Delta_x(k),\\
&\|B_i(k)\| \leq R\|v_\mu^i(k) - \hat{\mu}(k)\| \leq 2R\Delta_\mu(k),\\
&\|C_i(k)\| \leq L\|\hat{x}(k) - x^*\| \leq \frac{L}{N}\sum_{i=1}^{N}\|x^i(k) - x^*\|,\\
&\|D_i(k)\| \leq R\|\hat{\mu}(k) - \mu^*\| \leq \frac{R}{N}\sum_{i=1}^{N}\|\mu^i(k) - \mu^*\|,\\
&\|F_i(k)\| \leq R\|\mu^* - v_\mu^i(k)\| \leq R\|\mu^* - \hat{\mu}(k)\|\\
&+ R\|\hat{\mu}(k) - v_\mu^i(k)\| \leq \frac{R}{N}\sum_{i=1}^{N}\|\mu^*(k) - \mu^i(k)\| + 2R\Delta_\mu(k).
\end{aligned}$$

Since $\lim_{k\to+\infty}\|x^i(k) - x^*\| = 0$, $\lim_{k\to+\infty}\|\mu^i(k) - \mu^*\| = 0$, $\lim_{k\to+\infty}\Delta_x(k) = 0$ and $\lim_{k\to+\infty}\Delta_\mu(k) = 0$, then all $A_i(k), B_i(k), C_i(k), D_i(k), F_i(k)$ converge to zero as $k \to +\infty$. Then there exists $k_0 \geq 0$ such that for all $k \geq k_0$, it holds that

$$\sum_{i=1}^{N}\|x^i(k+1) - x\|^2 \leq \sum_{i=1}^{N}\|x^i(k) - x\|^2 + N\alpha(k)^2L^2 - \varsigma\alpha(k).$$

Following a recursive argument, we have that for all $k \geq k_0$, it holds that

$$\sum_{i=1}^{N}\|x^i(k+1) - x\|^2 \leq \sum_{i=1}^{N}\|x^i(k_0) - x\|^2 + NL^2\sum_{\tau=k_0}^{k}\alpha(\tau)^2 - \varsigma\sum_{\tau=k_0}^{k}\alpha(\tau).$$

Since $\sum_{k=k_0}^{+\infty}\alpha(k) = +\infty$ and $\sum_{k=k_0}^{+\infty}\alpha(k)^2 < +\infty$ and $x^i(k_0) \in X_i$, $x \in X$ are bounded, the above estimate yields a contradiction by taking k sufficiently large. In other words, (4.16) cannot hold. Following a parallel argument, one can show that (4.17) cannot hold either. This ensures that each $(x^*, \mu^*) \in \Theta$ is a saddle point of $\mathcal{L}$ over $X \times M$. ■

The combination of (c) in Lemmas 4.3.1 and Lemma 4.5.5 gives that, for each $(x^*, \mu^*) \in \Theta$, we have that $\mathcal{L}(x^*, \mu^*) = p^*$ and μ^* is Lagrangian dual optimal. We still need to verify that x^* is a primal optimal solution. We are now in the position to show Theorem 4.3.2 based on two claims.

Proofs of Theorem 4.3.2:

Claim 1: Each point $(x^*, \mu^*) \in \Theta$ is a point in $X^* \times D_L^*$.

Proof: The Lagrangian dual optimality of μ^* follows from (c) in Lemma 4.3.1 and Lemma 4.5.5. To characterize the primal optimality of x^*, we define an auxiliary sequence $\{z(k)\}$ by $z(k) := \frac{\sum_{\tau=0}^{k-1}\alpha(\tau)\hat{x}(\tau)}{\sum_{\tau=0}^{k-1}\alpha(\tau)}$ which is a weighted version of the average of primal estimates. Since $\lim_{k\to+\infty}\hat{x}(k) = x^*$, it follows from Lemma 5.4 (b) that $\lim_{k\to+\infty} z(k) = x^*$.

Since (x^*, μ^*) is a saddle point of $\mathcal{L}$ over $X \times M$, then $\mathcal{L}(x^*, \mu) \le \mathcal{L}(x^*, \mu^*)$ for any $\mu \in M$; i.e., the following relation holds for any $\mu \in M$:

$$g(x^*)^T(\mu - \mu^*) \le 0. \tag{4.18}$$

Choose $\mu_a = \mu^* + \min_{i\in V}\theta_i\frac{\mu^*}{\|\mu^*\|}$ where $\theta_i > 0$ is given in the definition of M_i. Then $\mu_a \ge 0$ and $\|\mu_a\| \le \|\mu^*\| + \min_{i\in V}\theta_i$ implying $\mu_a \in M$. Letting $\mu = \mu_a$ in (4.18) gives that

$$\frac{\min_{i\in V}\theta_i}{\|\mu^*\|}g(x^*)^T\mu^* \le 0.$$

Since $\theta_i > 0$, we have $g(x^*)^T\mu^* \le 0$. On the other hand, we choose $\mu_b = \frac{1}{2}\mu^*$ and then $\mu_b \in M$. Letting $\mu = \mu_b$ in (4.18) gives that $-\frac{1}{2}g(x^*)^T\mu^* \le 0$ and thus $g(x^*)^T\mu^* \ge 0$. The combination of the above two estimates guarantees the property of $g(x^*)^T\mu^* = 0$.

We now proceed to show $g(x^*) \le 0$ by contradiction. Assume that $g(x^*) \le 0$ does not hold. Denote $J^+(x^*) := \{1 \le \ell \le m \mid g_\ell(x^*) > 0\} \ne \emptyset$ and $\eta := \min_{\ell\in J^+(x^*)}\{g_\ell(x^*)\}$. Then $\eta > 0$. Since g is continuous and $v_x^i(k)$ converges to x^*, there exists $K \ge 0$ such that $g_\ell(v_x^i(k)) \ge \frac{\eta}{2}$ for all $k \ge K$ and all $\ell \in J^+(x^*)$. Since $v_\mu^i(k)$ converges to μ^*, without loss of generality, we say that $\|v_\mu^i(k) - \mu^*\| \le \frac{1}{2}\min_{i\in V}\theta_i$ for all $k \ge K$. Choose $\hat{\mu}$ such that $\hat{\mu}_\ell = \mu_\ell^*$ for $\ell \notin J^+(x^*)$ and $\hat{\mu}_\ell = \mu_\ell^* + \frac{1}{\sqrt{m}}\min_{i\in V}\theta_i$ for $\ell \in J^+(x^*)$. Since $\mu^* \ge 0$ and $\theta_i > 0$, thus $\hat{\mu} \ge 0$. Furthermore, $\|\hat{\mu}\| \le \|\mu^*\| + \min_{i\in V}\theta_i$, then $\hat{\mu} \in M$. Equating μ to $\hat{\mu}$ and letting

$\mathcal{D}_\mu^i(k) = g(v_x^i(k))$ in the estimate (4.14), the following holds for $k \geq K$:

$$\begin{aligned} N|J^+(x^*)|\eta \min_{i\in V}\theta_i\alpha(k) &\leq 2\alpha(k)\sum_{i=1}^N \sum_{\ell\in J^+(x^*)} g_\ell(v_x^i(k))(\hat{\mu}-v_\mu^i(k))_\ell \\ &\leq \sum_{i=1}^N \|\mu^i(k)-\hat{\mu}\|^2 - \sum_{i=1}^N \|\mu^i(k+1)-\hat{\mu}\|^2 + NR^2\alpha(k)^2 \\ &- 2\alpha(k)\sum_{i=1}^N \sum_{\ell\notin J^+(x^*)} g_\ell(v_x^i(k))(\hat{\mu}-v_\mu^i(k))_\ell. \end{aligned} \tag{4.19}$$

Summing (4.19) over $[K, k-1]$ with $k \geq K+1$, dividing by $\sum_{\tau=K}^{k-1}\alpha(\tau)$ on both sides, and using $-\sum_{i=1}^N \|\mu^i(k)-\hat{\mu}\|^2 \leq 0$, we obtain

$$\begin{aligned} &N|J^+(x^*)|\eta \min_{i\in V}\theta_i \\ &\leq \frac{1}{\sum_{\tau=K}^{k-1}\alpha(\tau)}\{\sum_{i=1}^N \|\mu^i(K)-\hat{\mu}\|^2 + NR^2\sum_{\tau=K}^{k-1}\alpha(\tau)^2 \\ &- \sum_{\tau=K}^{k-1} 2\alpha(\tau)\sum_{i=1}^N \sum_{\ell\notin J^+(x^*)} g_\ell(v_x^i(\tau))(\hat{\mu}-v_\mu^i(\tau))_\ell\}. \end{aligned} \tag{4.20}$$

Since $\mu^i(K) \in M_i$, $\hat{\mu} \in M$ are bounded and $\sum_{\tau=K}^{+\infty}\alpha(\tau) = +\infty$, then the limit of the first term on the right hand side of (4.20) is zero as $k \to +\infty$. Since $\sum_{\tau=K}^{+\infty}\alpha(\tau)^2 < +\infty$, then the limit of the second term is zero as $k \to +\infty$. Since $\lim_{k\to+\infty} v_x^i(k) = x^*$ and $\lim_{k\to+\infty} v_\mu^i(k) = \mu^*$, thus the following holds:

$$\lim_{k\to+\infty} 2\sum_{i=1}^N \sum_{\ell\notin J^+(x^*)} g_\ell(v_x^i(k))(\hat{\mu}-v_\mu^i(k))_\ell = 0.$$

Then it follows from Lemma 5.4 (b) that then the limit of the third term is zero as $k \to +\infty$. Then we have $N|J^+(x^*)|\eta \min_{i\in V}\theta_i \leq 0$. Recall that $|J^+(x^*)| > 0$, $\eta > 0$ and $\theta_i > 0$. Then we reach a contradiction, implying that $g(x^*) \leq 0$.

Since $x^* \in X$ and $g(x^*) \leq 0$, then x^* is a feasible solution and thus $f(x^*) \geq p^*$. On the other hand, since $z(k)$ is a convex combination of $\hat{x}(0), \cdots, \hat{x}(k-1)$

and f is convex, thus we have the following estimate:

$$f(z(k)) \leq \frac{\sum_{\tau=0}^{k-1}\alpha(\tau)f(\hat{x}(\tau))}{\sum_{\tau=0}^{k-1}\alpha(\tau)}$$
$$= \frac{1}{\sum_{\tau=0}^{k-1}\alpha(\tau)}\{\sum_{\tau=0}^{k-1}\alpha(\tau)\mathcal{L}(\hat{x}(\tau),\hat{\mu}(\tau)) - \sum_{\tau=0}^{k-1}N\alpha(\tau)\hat{\mu}(\tau)^T g(\hat{x}(\tau))\}.$$

Recall the following convergence properties:

$$\lim_{k\to+\infty} z(k) = x^*, \quad \lim_{k\to+\infty}\mathcal{L}(\hat{x}(k),\hat{\mu}(k)) = \mathcal{L}(x^*,\mu^*) = p^*,$$
$$\lim_{k\to+\infty}\hat{\mu}(k)^T g(\hat{x}(k)) = g(x^*)^T\mu^* = 0.$$

It follows from Lemma 5.4 (b) that $f(x^*) \leq p^*$. Therefore, we have $f(x^*) = p^*$, and thus x^* is a primal optimal point. ■

Claim 2: It holds that $\lim_{k\to+\infty}\|y^i(k) - p^*\| = 0$**.**

Proof: The following can be proven by induction on k for a fixed $k' \geq 1$:

$$\sum_{i=1}^{N} y^i(k+1) = \sum_{i=1}^{N} y^i(k') + N\sum_{\ell=k'}^{k}\sum_{i=1}^{N}(f_i(x^i(\ell)) - f_i(x^i(\ell-1))). \tag{4.21}$$

Let $k' = 1$ in (4.21) and recall that initial state $y^i(1) = Nf_i(x^i(0))$ for all $i \in V$. Then we have

$$\sum_{i=1}^{N} y^i(k+1) = \sum_{i=1}^{N} y^i(1) + N\sum_{i=1}^{N}(f_i(x^i(k)) - f_i(x^i(0))) = N\sum_{i=1}^{N} f_i(x^i(k)). \tag{4.22}$$

The combination of (4.22) with $\lim_{k\to+\infty}\|y^i(k) - y^j(k)\| = 0$ gives the desired result. ■

4.5.2 Proofs of Theorem 4.4.2

In order to analyze the DPPDS algorithm, we first rewrite it into the following form:

$$\mu^i(k+1) = v_\mu^i(k) + u_\mu^i(k), \quad \lambda^i(k+1) = v_\lambda^i(k) + u_\lambda^i(k),$$
$$x^i(k+1) = v_x^i(k) + e_x^i(k), \quad y^i(k+1) = v_y^i(k) + u_y^i(k),$$

where $e_x^i(k)$ is projection error described by

$$e_x^i(k) := P_X[v_x^i(k) - \alpha(k)\mathcal{S}_x^i(k)] - v_x^i(k),$$

and $u_\mu^i(k) := \alpha(k)[g(v_x^i(k))]^+$, $u_\lambda^i(k) := \alpha(k)|h(v_x^i(k))|$, $u_y^i(k) = N(f_i(x^i(k)) - f_i(x^i(k-1)))$ are some local inputs. Denote by the maximum deviations of dual estimates $M_\mu(k) := \max_{i\in V} \|\mu^i(k)\|$ and $M_\lambda(k) := \max_{i\in V} \|\lambda^i(k)\|$. Before showing Lemma 4.5.6, we present some useful facts. Since X is compact, and f_i, $[g(\cdot)]^+$ and h are continuous, there exist $F, G^+, H > 0$ such that for all $x \in X$, it holds that $\|f_i(x)\| \le F$ for all $i \in V$, $\|[g(x)]^+\| \le G^+$ and $\|h(x)\| \le H$. Since X is a compact set and f_i, $[g_\ell(\cdot)]^+$, $|h_\ell(\cdot)|$ are convex, then it follows from Proposition 5.4.2 in [15] that there exist $D_F, D_{G^+}, D_H > 0$ such that for all $x \in X$, it holds that $\|\mathcal{D}f_i(x)\| \le D_F$ $(i \in V)$, $m\|\mathcal{D}[g_\ell(x)]^+\| \le D_{G^+}$ $(1 \le \ell \le m)$ and $\nu\|\mathcal{D}|h_\ell|(x)\| \le D_H$ $(1 \le \ell \le \nu)$. Denote by the averages of primal and dual estimates $\hat{x}(k) := \frac{1}{N}\sum_{i=1}^N x^i(k)$, $\hat{\mu}(k) := \frac{1}{N}\sum_{i=1}^N \mu^i(k)$ and $\hat{\lambda}(k) := \frac{1}{N}\sum_{i=1}^N \lambda^i(k)$.

Lemma 4.5.6 ***(Diminishing and summable properties)*** *Suppose the balanced communication assumption 4.2.3 and the step-size assumption 4.4.1 hold.*

(a) The following holds:

$$\lim_{k\to+\infty} \alpha(k)M_\mu(k) = 0, \quad \lim_{k\to+\infty} \alpha(k)M_\lambda(k) = 0, \quad \lim_{k\to+\infty} \alpha(k)\|\mathcal{S}_x^i(k)\| = 0.$$

Furthermore, the sequences of $\{\alpha(k)^2 M_\mu^2(k)\}$, $\{\alpha(k)^2 M_\lambda^2(k)\}$ *and* $\{\alpha(k)^2\|\mathcal{S}_x^i(k)\|^2\}$ *are summable.*

(b) The following sequences are summable:

$$\{\alpha(k)\|\hat{\mu}(k) - v_\mu^i(k)\|\}, \{\alpha(k)\|\hat{\lambda}(k) - v_\lambda^i(k)\|\}, \{\alpha(k)M_\mu(k)\|\hat{x}(k) - v_x^i(k)\|\},$$
$$\{\alpha(k)M_\lambda(k)\|\hat{x}(k) - v_x^i(k)\|\}, \{\alpha(k)\|\hat{x}(k) - v_x^i(k)\|\}.$$

Proof: (a) Notice that

$$\|v_\mu^i(k)\| = \|\sum_{j=1}^N a_j^i(k)\mu^j(k)\| \le \sum_{j=1}^N a_j^i(k)\|\mu^j(k)\| \le \sum_{j=1}^N a_j^i(k)M_\mu(k) = M_\mu(k),$$

where in the last equality we use the balanced communication assumption 4.2.3. Recall that $v_x^i(k) \in X$. This implies that the following holds for all $k \geq 0$:

$$\begin{aligned}\|\mu^i(k+1)\| &\leq \|v_\mu^i(k) + \alpha(k)[g(v_x^i(k))]^+\| \\ &\leq \|v_\mu^i(k)\| + G^+\alpha(k) \leq M_\mu(k) + G^+\alpha(k).\end{aligned}$$

From here, then we deduce the following recursive estimate on $M_\mu(k+1)$: $M_\mu(k+1) \leq M_\mu(k) + G^+\alpha(k)$. Repeatedly applying the above estimates yields that

$$M_\mu(k+1) \leq M_\mu(0) + G^+ s(k). \tag{4.23}$$

Similar arguments can be employed to show that

$$M_\lambda(k+1) \leq M_\lambda(0) + Hs(k). \tag{4.24}$$

Since $\lim_{k\to+\infty} \alpha(k+1)s(k) = 0$ and $\lim_{k\to+\infty} \alpha(k) = 0$, then we know that

$$\lim_{k\to+\infty} \alpha(k+1)M_\mu(k+1) = 0, \quad \lim_{k\to+\infty} \alpha(k+1)M_\lambda(k+1) = 0.$$

Notice that the following estimate on $\mathcal{S}_x^i(k)$ holds:

$$\|\mathcal{S}_x^i(k)\| \leq D_F + D_{G^+}M_\mu(k) + D_H M_\lambda(k). \tag{4.25}$$

Recall that $\lim_{k\to+\infty} \alpha(k) = 0$, $\lim_{k\to+\infty} \alpha(k)M_\mu(k) = 0$ and $\lim_{k\to+\infty} \alpha(k)M_\lambda(k) = 0$. Then the result of $\lim_{k\to+\infty} \alpha(k)\|\mathcal{S}_x^i(k)\| = 0$ follows. By (4.23), we have

$$\sum_{k=0}^{+\infty} \alpha(k)^2 M_\mu^2(k) \leq \alpha(0)^2 M_\mu^2(0) + \sum_{k=1}^{+\infty} \alpha(k)^2 (M_\mu(0) + G^+ s(k-1))^2.$$

It follows from the step-size assumption 4.4.1 that $\sum_{k=0}^{+\infty} \alpha(k)^2 M_\mu^2(k) < +\infty$. Similarly, one can show that $\sum_{k=0}^{+\infty} \alpha(k)^2 M_\lambda^2(k) < +\infty$. By using (4.23), (4.24) and (4.25), we have the following estimate:

$$\begin{aligned}&\sum_{k=0}^{+\infty} \alpha(k)^2 \|\mathcal{S}_x^i(k)\|^2 \leq \alpha(0)^2 (D_F + D_{G^+}M_\mu(0) + D_H M_\lambda(0))^2 \\ &+ \sum_{k=1}^{+\infty} \alpha(k)^2 (D_F + D_{G^+}(M_\mu(0) + G^+ s(k-1)) + D_H(M_\lambda(0) + Hs(k-1)))^2.\end{aligned}$$

Then the summability of $\{\alpha(k)^2\}$, $\{\alpha(k+1)^2 s(k)\}$ and $\{\alpha(k+1)^2 s(k)^2\}$ verifies that of $\{\alpha(k)^2 \|\mathcal{S}_x^i(k)\|^2\}$.

(b) Consider the dynamics of $\mu^i(k)$ which is in the same form as the distributed projected subgradient algorithm in [102]. Recall that $\{[g(v_x^i(k))]^+\}$ is uniformly bounded. Then following from Lemma 4.8.2 in the Appendix with $Z = \mathbb{R}^m_{\geq 0}$ and $d_i(k) = -[g(v_x^i(k))]^+$, we have the summability of $\{\alpha(k) \max_{i\in V} \|\hat{\mu}(k) - \mu^i(k)\|\}$. Then $\{\alpha(k)\|\hat{\mu}(k) - v_\mu^i(k)\|\}$ is summable by using the following set of inequalities:

$$\|\hat{\mu}(k) - v_\mu^i(k)\| \leq \sum_{j=1}^N a_j^i(k)\|\hat{\mu}(k) - \mu^j(k)\| \leq \max_{i\in V} \|\hat{\mu}(k) - \mu^i(k)\|, \tag{4.26}$$

where we use $\sum_{j=1}^N a_j^i(k) = 1$. Similarly, it holds that $\sum_{k=0}^{+\infty} \alpha(k)\|\hat{\lambda}(k) - v_\lambda^i(k)\| < +\infty$.

We now consider the evolution of $x^i(k)$. Recall that $v_x^i(k) \in X$. By Lemma 4.8.1 with $Z = X$, $z = v_x^i(k) - \alpha(k)\mathcal{S}_x^i(k)$ and $y = v_x^i(k)$, we have

$$\begin{aligned}\|x^i(k+1) - v_x^i(k)\|^2 &\leq \|v_x^i(k) - \alpha(k)\mathcal{S}_x^i(k) - v_x^i(k)\|^2 \\ &- \|x^i(k+1) - (v_x^i(k) - \alpha(k)\mathcal{S}_x^i(k))\|^2,\end{aligned}$$

and thus $\|e_x^i(k) + \alpha(k)\mathcal{S}_x^i(k)\| \leq \alpha(k)\|\mathcal{S}_x^i(k)\|$. With this relation, from Lemma 4.8.2 with $Z = X$ and $d_i(k) = \mathcal{S}_x^i(k)$, the following holds for some $\gamma > 0$ and $0 < \beta < 1$:

$$\|x^i(k) - \hat{x}(k)\| \leq N\gamma\beta^{k-1} \sum_{i=0}^N \|x^i(0)\| + 2N\gamma \sum_{\tau=0}^{k-1} \beta^{k-\tau}\alpha(\tau)\|\mathcal{S}_x^i(\tau)\|. \tag{4.27}$$

Multiplying both sides of (4.27) by $\alpha(k)M_\mu(k)$ and using (4.25), we obtain

$$\begin{aligned}\alpha(k)M_\mu(k)\|x^i(k) - \hat{x}(k)\| \leq N\gamma \sum_{i=0}^N \|x^i(0)\|\alpha(k)M_\mu(k)\beta^{k-1} + 2N\gamma\alpha(k)M_\mu(k) \\ \times \sum_{\tau=0}^{k-1} \beta^{k-\tau}\alpha(\tau)(D_F + D_{G^+}M_\mu(\tau) + D_H M_\lambda(\tau)).\end{aligned}$$

Notice that the above inequalities hold for all $i \in V$. Then by employing the relation of $ab \leq \frac{1}{2}(a^2 + b^2)$ and regrouping similar terms, we obtain

$$\begin{aligned}\alpha(k)M_\mu(k)\max_{i\in V}\|x^i(k)-\hat{x}(k)\| \leq N\gamma\big(\frac{1}{2}\sum_{i=0}^{N}\|x^i(0)\| + (D_F + D_{G^+} + D_H)\sum_{\tau=0}^{k-1}\beta^{k-\tau}\big)\\ \times \alpha(k)^2M_\mu^2(k) + \frac{1}{2}N\gamma\sum_{i=0}^{N}\|x^i(0)\|\beta^{2(k-1)}\\ + N\gamma\sum_{\tau=0}^{k-1}\beta^{k-\tau}\alpha(\tau)^2(D_F + D_{G^+}M_\mu^2(\tau) + D_HM_\lambda^2(\tau)).\end{aligned}$$

Part (a) gives that $\{\alpha(k)^2M_\mu^2(k)\}$ is summable. Combining this fact with $\sum_{\tau=0}^{k-1}\beta^{k-\tau} \leq \sum_{k=0}^{+\infty}\beta^k = \frac{1}{1-\beta}$, then we can say that the first term on the right-hand side in the above estimate is summable. It is easy to check that the second term is also summable. It follows from Part (a) that

$$\lim_{k\to+\infty}\alpha(k)^2(D_F + D_{G^+}M_\mu^2(k) + D_HM_\lambda^2(k)) = 0,$$

and thus $\{\alpha(k)^2(D_F + D_{G^+}M_\mu^2(k) + D_HM_\lambda^2(k))\}$ is summable. Then Lemma 7 in [102] with $\gamma_\ell = N\gamma\alpha(\ell)^2(D_F + D_{G^+}M_\mu^2(\ell) + D_HM_\lambda^2(\ell))$ ensures that the third term is summable. Therefore, the summability of $\{\alpha(k)M_\mu(k)\max_{i\in V}\|x^i(k) - \hat{x}(k)\|\}$ is guaranteed. Following the same lines in (4.26), one can show the summability of $\{\alpha(k)M_\mu(k)\|v_x^i(k)-\hat{x}(k)\|\}$. Following analogous arguments, we have that $\{\alpha(k)M_\lambda(k)\|v_x^i(k) - \hat{x}(k)\|\}$ and $\{\alpha(k)\|v_x^i(k) - \hat{x}(k)\|\}$ are summable. ■

Remark 4.5.1 *In Lemma 4.5.6, the assumption of all local constraint sets being identical is utilized to find an upper bound of the convergence rate of $\|\hat{x}(k) - v_x^i(k)\|$ to zero. This property is crucial to establish the summability of expansions pertaining to $\|\hat{x}(k) - v_x^i(k)\|$ in part (b).* •

The following is a basic iteration relation of the DPPDS algorithm.

Lemma 4.5.7 (Basic iteration relation) *The following estimates hold for any*

$x \in X$ *and* $(\mu, \lambda) \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^\nu_{\geq 0}$*:*

$$\sum_{i=1}^N \|e^i_x(k) + \alpha(k)\mathcal{S}^i_x(k)\|^2 \leq \sum_{i=1}^N \alpha(k)^2 \|\mathcal{S}^i_x(k)\|^2$$
$$- \sum_{i=1}^N 2\alpha(k)(\mathcal{H}_i(v^i_x(k), v^i_\mu(k), v^i_\lambda(k)) - \mathcal{H}_i(x, v^i_\mu(k), v^i_\lambda(k)))$$
$$+ \sum_{i=1}^N (\|x^i(k) - x\|^2 - \|x^i(k+1) - x\|^2), \tag{4.28}$$
$$0 \leq \sum_{i=1}^N (\|\mu^i(k) - \mu\|^2 - \|\mu^i(k+1) - \mu\|^2)$$
$$+ \sum_{i=1}^N (\|\lambda^i(k) - \lambda\|^2 - \|\lambda^i(k+1) - \lambda\|^2) +$$
$$\sum_{i=1}^N 2\alpha(k)(\mathcal{H}_i(v^i_x(k), v^i_\mu(k), v^i_\lambda(k)) - \mathcal{H}_i(v^i_x(k), \mu, \lambda))$$
$$+ \sum_{i=1}^N \alpha(k)^2 (\|[g(v^i_x(k))]^+\|^2 + \|h(v^i_x(k))\|^2). \tag{4.29}$$

Proof: One can finish the proof by following analogous arguments in Lemma 4.5.3. ■

Lemma 4.5.8 (Achieving consensus) *Let us suppose that the non-degeneracy assumption 4.2.2, the balanced communication assumption 4.2.3 and the periodical strong connectivity assumption 4.2.4 hold. Consider the sequences of* $\{x^i(k)\}$*,* $\{\mu^i(k)\}$*,* $\{\lambda^i(k)\}$ *and* $\{y^i(k)\}$ *of the distributed penalty primal-dual subgradient algorithm with the step-size sequence* $\{\alpha(k)\}$ *and the associated* $\{s(k)\}$ *satisfying* $\lim_{k\to+\infty} \alpha(k) = 0$ *and* $\lim_{k\to+\infty} \alpha(k+1)s(k) = 0$*. Then there exists* $\tilde{x} \in X$ *such that* $\lim_{k\to+\infty} \|x^i(k) - \tilde{x}\| = 0$ *for all* $i \in V$*. Furthermore,* $\lim_{k\to+\infty} \|\mu^i(k) - \mu^j(k)\| = 0$*,* $\lim_{k\to+\infty} \|\lambda^i(k) - \lambda^j(k)\| = 0$ *and* $\lim_{k\to+\infty} \|y^i(k) - y^j(k)\| = 0$ *for all* $i, j \in V$*.*

Proof: Similar to (4.14), we have

$$\sum_{i=1}^N \|x^i(k+1) - x\|^2 \leq \sum_{i=1}^N \|x^i(k) - x\|^2$$
$$+ \sum_{i=1}^N \alpha(k)^2 \|\mathcal{S}^i_x(k)\|^2 + \sum_{i=1}^N 2\alpha(k)\|\mathcal{S}^i_x(k)\|\|v^i_x(k) - x\|.$$

Since $\lim_{k\to+\infty}\alpha(k)\|\mathcal{S}_x^i(k)\|=0$, the proofs of $\lim_{k\to+\infty}\|x^i(k)-\tilde{x}\|=0$ for all $i\in V$ are analogous to those in Lemma 4.5.4. The remainder of the proofs can be finished by Corollary 2.3.1 with the properties of $\lim_{k\to+\infty}u_\mu^i(k)=0$, $\lim_{k\to+\infty}u_\lambda^i(k)=0$ and $\lim_{k\to+\infty}u_y^i(k)=0$ (due to $\lim_{k\to+\infty}x^i(k)=\tilde{x}$ and f_i is continuous). ■

We now proceed to show Theorem 4.4.2 based on five claims.

Proof of Theorem 4.4.2:

Claim 1: For any $x^*\in X^*$ and $(\mu^*,\lambda^*)\in D_P^*$, the following sequences are summable:

$$\{\alpha(k)\big[\sum_{i=1}^N\mathcal{H}_i(x^*,v_\mu^i(k),v_\lambda^i(k))-\mathcal{H}(x^*,\hat{\mu}(k),\hat{\lambda}(k))\big]\},$$
$$\{\alpha(k)\big[\sum_{i=1}^N\mathcal{H}_i(v_x^i(k),\mu^*,\lambda^*)-\mathcal{H}(\hat{x}(k),\mu^*,\lambda^*)\big]\}$$

Proof: Observe that

$$\begin{aligned}&\|\mathcal{H}_i(x^*,v_\mu^i(k),v_\lambda^i(k))-\mathcal{H}_i(x^*,\hat{\mu}(k),\hat{\lambda}(k))\|\\&\le\|v_\mu^i(k)-\hat{\mu}(k)\|\|[g(x^*)]^+\|+\|v_\lambda^i(k)-\hat{\lambda}(k)\|\|h(x^*)\|\\&\le G^+\|v_\mu^i(k)-\hat{\mu}(k)\|+H\|v_\lambda^i(k)-\hat{\lambda}(k)\|.\end{aligned}\tag{4.30}$$

By using the summability of $\{\alpha(k)\|\hat{\mu}(k)-v_\mu^i(k)\|\}$ and $\{\alpha(k)\|\hat{\lambda}(k)-v_\lambda^i(k)\|\}$ in Part (b) of Lemma 4.5.6, we have that the following are summable:

$$\{\alpha(k)\sum_{i=1}^N\|\mathcal{H}_i(x^*,v_\mu^i(k),v_\lambda^i(k))-\mathcal{H}_i(x^*,\hat{\mu}(k),\hat{\lambda}(k))\|\},$$
$$\{\alpha(k)\big[\sum_{i=1}^N\big(\mathcal{H}_i(x^*,v_\mu^i(k),v_\lambda^i(k))-\mathcal{H}_i(x^*,\hat{\mu}(k),\hat{\lambda}(k))\big)\big]\}.$$

Similarly, the following estimates hold:

$$\begin{aligned}&\|\mathcal{H}_i(v_x^i(k),\mu^*,\lambda^*)-\mathcal{H}_i(\hat{x}(k),\mu^*,\lambda^*)\|\\&\le\|f_i(v_x^i(k))-f_i(\hat{x}(k))\|+\|(\mu^*)^T([g(v_x^i(k))]^+-[g(\hat{x}(k))]^+)\|\\&+\|(\lambda^*)^T(|h(v_x^i(k))|-|h(\hat{x}(k))|)\|\\&\le(D_F+D_{G^+}\|\mu^*\|+D_H\|\lambda^*\|)\|v_x^i(k)-\hat{x}(k)\|.\end{aligned}$$

Then the property of $\sum_{k=0}^{+\infty}\alpha(k)\|\hat{x}(k)-v_x^i(k)\|<+\infty$ in Part (b) of Lemma 4.5.6 implies the summability of the following sequences:

$$\{\alpha(k)\sum_{i=1}^{N}\|\mathcal{H}_i(v_x^i(k),\mu^*,\lambda^*)-\mathcal{H}_i(\hat{x}(k),\mu^*,\lambda^*)\|\},$$
$$\{\alpha(k)\sum_{i=1}^{N}\big(\mathcal{H}_i(v_x^i(k),\mu^*,\lambda^*)-\mathcal{H}_i(\hat{x}(k),\mu^*,\lambda^*)\big)\}.$$

■

Claim 2: Denote the weighted version of $\mathcal{H}_i$ as

$$\hat{\mathcal{H}}_i(k):=\frac{1}{s(k-1)}\sum_{\ell=0}^{k-1}\alpha(\ell)\mathcal{H}_i(v_x^i(\ell),v_\mu^i(\ell),v_\lambda^i(\ell)).$$

The following property holds: $\lim_{k\to+\infty}\sum_{i=1}^{N}\hat{\mathcal{H}}_i(k)=p^*$.

Proof: Summing (4.28) over $[0,k-1]$ and replacing x by $x^*\in X^*$ leads to

$$\begin{aligned}&\sum_{\ell=0}^{k-1}\alpha(\ell)\sum_{i=1}^{N}(\mathcal{H}_i(v_x^i(\ell),v_\mu^i(\ell),v_\lambda^i(\ell))-\mathcal{H}_i(x^*,v_\mu^i(\ell),v_\lambda^i(\ell)))\\&\le\sum_{i=1}^{N}\|x^i(0)-x^*\|^2+\sum_{\ell=0}^{k-1}\sum_{i=1}^{N}\alpha(\ell)^2\|\mathcal{S}_x^i(\ell)\|^2.\end{aligned}\tag{4.31}$$

The summability of $\{\alpha(k)^2\|\mathcal{S}_x^i(k)\|^2\}$ in Part (b) of Lemma 4.5.6 implies that the right-hand side of (4.31) is finite as $k\to+\infty$, and thus

$$\limsup_{k\to\infty}\frac{1}{s(k-1)}\sum_{\ell=0}^{k-1}\alpha(\ell)\big[\sum_{i=1}^{N}\big(\mathcal{H}_i(v_x^i(\ell),v_\mu^i(\ell),v_\lambda^i(\ell))-\mathcal{H}_i(x^*,v_\mu^i(\ell),v_\lambda^i(\ell))\big)\big]\le 0.\tag{4.32}$$

Pick any $(\mu^*,\lambda^*)\in D_P^*$. It follows from Theorem 4.4.1 that (x^*,μ^*,λ^*) is a saddle point of $\mathcal{H}$ over $X\times\mathbb{R}_{\ge0}^m\times\mathbb{R}_{\ge0}^\nu$. Since $(\hat{\mu}(k),\hat{\lambda}(k))\in\mathbb{R}_{\ge0}^m\times\mathbb{R}_{\ge0}^\nu$, then we have $\mathcal{H}(x^*,\hat{\mu}(k),\hat{\lambda}(k))\le\mathcal{H}(x^*,\mu^*,\lambda^*)=p^*$. Combining this relation, Claim 1

and (4.32) renders that

$$\begin{aligned}
&\limsup_{k\to+\infty}\frac{1}{s(k-1)}\sum_{\ell=0}^{k-1}\alpha(\ell)\big[\sum_{i=1}^{N}\mathcal{H}_i(v_x^i(\ell),v_\mu^i(\ell),v_\lambda^i(\ell))-p^*\big]\\
&\le\limsup_{k\to+\infty}\frac{1}{s(k-1)}\sum_{\ell=0}^{k-1}\alpha(\ell)\big[\sum_{i=1}^{N}\big(\mathcal{H}_i(v_x^i(\ell),v_\mu^i(\ell),v_\lambda^i(\ell))-\mathcal{H}_i(x^*,v_\mu^i(\ell),v_\lambda^i(\ell))\big)\big]\\
&+\limsup_{k\to+\infty}\frac{1}{s(k-1)}\sum_{\ell=0}^{k-1}\alpha(\ell)\big[\sum_{i=1}^{N}\mathcal{H}_i(x^*,v_\mu^i(\ell),v_\lambda^i(\ell))-\mathcal{H}(x^*,\hat{\mu}(\ell),\hat{\lambda}(\ell))\big]\\
&+\limsup_{k\to+\infty}\frac{1}{s(k-1)}\sum_{\ell=0}^{k-1}(\mathcal{H}(x^*,\hat{\mu}(\ell),\hat{\lambda}(\ell))-p^*)\le 0,
\end{aligned}$$

and thus $\limsup_{k\to+\infty}\sum_{i=1}^{N}\hat{\mathcal{H}}_i(k)\le p^*$.

On the other hand, $\hat{x}(k)\in X$ (due to the fact that X is convex) implies that $\mathcal{H}(\hat{x}(k),\mu^*,\lambda^*)\ge\mathcal{H}(x^*,\mu^*,\lambda^*)=p^*$. Along similar lines, by using (4.29) with $\mu=\mu^*$, $\lambda=\lambda^*$, and Claim 1, we have the following estimate: $\liminf_{k\to+\infty}\sum_{i=1}^{N}\hat{\mathcal{H}}_i(k)\ge p^*$. Then we have the desired relation. ■

Claim 3: Denote by $\pi(k):=\sum_{i=1}^{N}\mathcal{H}_i(v_x^i(k),v_\mu^i(k),v_\lambda^i(k))-\mathcal{H}(\hat{x}(k),\hat{\mu}(k),\hat{\lambda}(k))$. And we denote the weighted version of $\mathcal{H}$ as

$$\Gamma(k):=\frac{1}{s(k-1)}\sum_{\ell=0}^{k-1}\alpha(\ell)\mathcal{H}(\hat{x}(\ell),\hat{\mu}(\ell),\hat{\lambda}(\ell)).$$

The following property holds: $\lim_{k\to+\infty}\Gamma(k)=p^*$.

Proof: Notice that

$$\begin{aligned}
\pi(k)&=\sum_{i=1}^{N}(f_i(v_x^i(k))-f_i(\hat{x}(k)))\\
&+\sum_{i=1}^{N}\big(v_\mu^i(k)^T[g(v_x^i(k))]^+-v_\mu^i(k)^T[g(\hat{x}(k))]^+\big)\\
&+\sum_{i=1}^{N}\big(v_\mu^i(k)^T[g(\hat{x}(k))]^+-\hat{\mu}(k)^T[g(\hat{x}(k))]^+\big)\\
&+\sum_{i=1}^{N}\big(v_\lambda^i(k)^T|h(v_x^i(k))|-v_\lambda^i(k)^T|h(\hat{x}(k))|\big)\\
&+\sum_{i=1}^{N}\big(v_\lambda^i(k)^T|h(\hat{x}(k))|-\hat{\lambda}(k)^T|h(\hat{x}(k))|\big).
\end{aligned}\tag{4.33}$$

By using the boundedness of subdifferentials and the primal estimates, it follows from (4.33) that

$$\|\pi(k)\| \leq (D_F + D_{G^+}M_\mu(k) + D_H M_\lambda(k)) \times \sum_{i=1}^{N} \|v_x^i(k) - \hat{x}(k)\|$$
$$+ G^+ \sum_{i=1}^{N} \|v_\mu^i(k) - \hat{\mu}(k)\| + H \sum_{i=1}^{N} \|v_\lambda^i(k) - \hat{\lambda}(k)\|. \tag{4.34}$$

Then it follows from (b) in Lemma 4.5.6 that $\{\alpha(k)\|\pi(k)\|\}$ is summable. Notice that $\|\Gamma(k) - \sum_{i=1}^{N} \hat{\mathcal{H}}_i(k)\| \leq \frac{\sum_{\ell=0}^{k-1} \alpha(\ell)\|\pi(\ell)\|}{s(k-1)}$, and thus $\lim_{k\to+\infty} \|\Gamma(k) - \sum_{i=1}^{N} \hat{\mathcal{H}}_i(k)\| = 0$. The desired result immediately follows from Claim 2. ■

Claim 4: The limit point $\tilde{x}$ in Lemma 4.5.8 is a primal optimal solution.

Proof: Let $\hat{\mu}(k) = (\hat{\mu}_1(k), \cdots, \hat{\mu}_m(k))^T \in \mathbb{R}^m_{\geq 0}$. By the balanced communication assumption 4.2.3, we obtain

$$\sum_{i=1}^{N} \mu^i(k+1) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_j^i(k)\mu^j(k) + \alpha(k) \sum_{i=1}^{N} [g(v_x^i(k))]^+$$
$$= \sum_{j=1}^{N} \mu^j(k) + \alpha(k) \sum_{i=1}^{N} [g(v_x^i(k))]^+.$$

This implies that the sequence $\{\hat{\mu}_\ell(k)\}$ is non-decreasing in $\mathbb{R}_{\geq 0}$. Observe that $\{\hat{\mu}_\ell(k)\}$ is lower bounded by zero. In this way, we distinguish the following two cases:

Case 1: The sequence $\{\hat{\mu}_\ell(k)\}$ is upper bounded. Then $\{\hat{\mu}_\ell(k)\}$ is convergent in $\mathbb{R}_{\geq 0}$. Recall that $\lim_{k\to+\infty} \|\mu^i(k) - \mu^j(k)\| = 0$ for all $i, j \in V$. This implies that there exists $\mu_\ell^* \in \mathbb{R}_{\geq 0}$ such that $\lim_{k\to+\infty} \|\mu_\ell^i(k) - \mu_\ell^*\| = 0$ for all $i \in V$. Observe that $\sum_{i=1}^{N} \mu^i(k+1) = \sum_{i=1}^{N} \mu^i(0) + \sum_{\tau=0}^{k} \alpha(\tau) \sum_{i=1}^{N} [g(v_x^i(\tau))]^+$. Thus, we have $\sum_{k=0}^{+\infty} \alpha(k) \sum_{i=1}^{N} [g_\ell(v_x^i(k))]^+ < +\infty$, implying that $\liminf_{k\to+\infty} [g_\ell(v_x^i(k))]^+ = 0$. Since $\lim_{k\to+\infty} \|x^i(k) - \tilde{x}\| = 0$ for all $i \in V$, then $\lim_{k\to+\infty} \|v_x^i(k) - \tilde{x}\| = 0$, and thus $[g_\ell(\tilde{x})]^+ = 0$.

Case 2: The sequence $\{\hat{\mu}_\ell(k)\}$ is not upper bounded. Since $\{\hat{\mu}_\ell(k)\}$ is non-decreasing, then $\hat{\mu}_\ell(k) \to +\infty$. It follows from Claim 3 and (a) in Lemma 5.4 that it is impossible that $\mathcal{H}(\hat{x}(k), \hat{\mu}(k), \hat{\lambda}(k)) \to +\infty$. Assume that $[g_\ell(\tilde{x})]^+ > 0$. Then

we have

$$\begin{aligned}\mathcal{H}(\hat{x}(k),\hat{\mu}(k),\hat{\lambda}(k)) &= f(\hat{x}(k)) + N\hat{\mu}(k)^T[g(\hat{x}(k))]^+ + N\lambda(k)^T|h(\hat{x}(k))| \\ &\geq f(\hat{x}(k)) + \hat{\mu}_\ell(k)[g_\ell(\hat{x}(k))]^+. \end{aligned} \tag{4.35}$$

Taking limits on both sides of (4.35) and we obtain:

$$\liminf_{k\to+\infty} \mathcal{H}(\hat{x}(k),\hat{\mu}(k),\hat{\lambda}(k)) \geq \limsup_{k\to+\infty}(f(\hat{x}(k)) + \hat{\mu}_\ell(k)[g_\ell(\hat{x}(k))]^+) = +\infty.$$

Then we reach a contradiction, implying that $[g_\ell(\tilde{x})]^+ = 0$.

In both cases, we have $[g_\ell(\tilde{x})]^+ = 0$ for any $1 \leq \ell \leq m$. By utilizing similar arguments, we can further prove that $|h(\tilde{x})| = 0$. Since $\tilde{x} \in X$, then $\tilde{x}$ is feasible and thus $f(\tilde{x}) \geq p^*$. On the other hand, since $\frac{\sum_{\ell=0}^{k-1}\alpha(\ell)\hat{x}(\ell)}{\sum_{\ell=0}^{k-1}\alpha(\ell)}$ is a convex combination of $\hat{x}(0),\cdots,\hat{x}(k-1)$ and $\lim_{k\to+\infty}\hat{x}(k) = \tilde{x}$, then Claim 3 and (b) in Lemma 5.4 implies that

$$\begin{aligned} p^* &= \lim_{k\to+\infty}\Gamma(k) = \lim_{k\to+\infty}\frac{\sum_{\ell=0}^{k-1}\alpha(\ell)\mathcal{H}(\hat{x}(\ell),\hat{\mu}(\ell),\hat{\lambda}(\ell))}{\sum_{\ell=0}^{k-1}\alpha(\ell)} \\ &\geq \lim_{k\to+\infty} f(\frac{\sum_{\ell=0}^{k-1}\alpha(\ell)\hat{x}(\ell)}{\sum_{\ell=0}^{k-1}\alpha(\ell)}) = f(\tilde{x}). \end{aligned}$$

Hence, we have $f(\tilde{x}) = p^*$ and thus $\tilde{x} \in X^*$. ■

Claim 5: It holds that $\lim_{k\to+\infty}\|y^i(k) - p^*\| = 0$.

Proof: The proof follows the same lines in Claim 2 of Theorem 4.3.2 and thus omitted here. ■

4.6 Discussion

In this section, we present some possible extensions and interesting special cases.

4.6.1 Discussion on the periodic strong connectivity assumption in Theorem 4.3.2

In the case that $\mathcal{G}(k)$ is undirected, then the periodic strong connectivity assumption 4.2.4 in Theorem 4.3.2 can be weakened into:

Assumption 4.6.1 (Eventual strong connectivity) *The undirected graph* $(V, \cup_{k\geq s} E(k))$ *is connected for all time instant* $s \geq 0$.

If $\mathcal{G}(k)$ is undirected, the periodic connectivity assumption 4.2.4 in Theorem 4.3.2 can also be replaced with the assumption in Proposition 2 of [93]; i.e., for any time instant $k \geq 0$, there is an agent connected to all other agents in the undirected graph $(V, \cup_{k\geq s} E(k))$.

4.6.2 A generalized step-size scheme

The step-size scheme in the DLPDS algorithm can be slightly generalized the case that the maximum deviation of step-sizes of agents at each time is not large. It is formally stated as follows: $\lim_{k\to+\infty} \alpha^i(k) = 0$, $\sum_{k=0}^{+\infty} \alpha^i(k) = +\infty$, $\sum_{k=0}^{+\infty} \alpha^i(k)^2 < +\infty$, $\min_{i\in V} \alpha^i(k) \geq C_\alpha \max_{i\in V} \alpha^i(k)$, where $\alpha^i(k)$ is the step-size of agent i at time k and $C_\alpha \in (0, 1]$.

4.6.3 Discussion on the Slater's condition in Theorem 4.4.2

If g_ℓ $(1 \leq \ell \leq m)$ is linear, then the Slater's condition 4.2.1 can be weakened to the following: there exists a relative interior point $\bar{x}$ of X such that $h(\bar{x}) = 0$ and $g(\bar{x}) \leq 0$. For this case, the strong duality and the non-emptyness of the penalty dual optimal set can be ensured by replacing Proposition 5.3.5 [15] with Proposition 5.3.4 [15] in the proofs of Lemma 4.4.1. In this way, the convergence results of the DPPDS algorithm still hold for the case of linear g_ℓ.

4.6.4 The special case in the absence of inequality and equality constraints

The following special case of problem (4.1) is studied in [102]:

$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{N} f_i(x), \quad \text{s.t.} \quad x \in \cap_{i=1}^{N} X_i. \tag{4.36}$$

The following *Distributed Primal Subgradient* Algorithm is a special case of the DLPDS algorithm, and can be utilized to solve the problem (4.36):

$$x^i(k+1) = P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}f_i(v_x^i(k))].$$

Corollary 4.6.1 ***(Convergence properties of the distributed primal subgradient algorithm):*** *Consider the problem* (4.36)*, and let the non-degeneracy assumption 4.2.2, the balanced communication assumption 4.2.3 and the periodic strong connectivity assumption 4.2.4 hold. Consider the sequence* $\{x^i(k)\}$ *of the distributed primal subgradient algorithm with initial states* $x^i(0) \in X_i$ *and the step-sizes satisfying* $\lim_{k\to+\infty} \alpha(k) = 0$, $\sum_{k=0}^{+\infty} \alpha(k) = +\infty$, *and* $\sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$. *Then there exists an optimal solution* x^* *such that* $\lim_{k\to+\infty} \|x^i(k) - x^*\| = 0$ *all* $i \in V$.

Proof: The result is an immediate consequence of Theorem 4.3.2 with $g(x) \equiv 0$. ■

4.7 Conclusions

We have studied a multi-agent optimization problem where the agents aim to minimize a sum of local objective functions subject to a global inequality constraint, a global equality constraint and a global constraint set defined as the intersection of local constraint sets. We have considered two cases: the first one in the absence of the equality constraint and the second one with identical local constraint sets. To address these cases, we have introduced two distributed subgradient algorithms which are based on Lagrangian and penalty primal-dual methods, respectively. These two algorithms were shown to asymptotically converge to primal solutions and optimal values. The results presented are published or to appear in the following papers:

(JP-4) M. Zhu and S. Martínez, "On distributed convex optimization under inequality and equality constraints", *IEEE Transactions on Automatic Control*, 2011, to appear.

(CP-8) M. Zhu and S. Martínez, "On distributed optimization under inequality constraints via Lagrangian primal-dual subgradient methods", *The* 29th *American Control Conference*, pages 4863 – 4868, Baltimore, USA, Jun. 2010.

(CP-7) M. Zhu and S. Martínez, "On distributed optimization under inequality and equality constraints via penalty primal-dual subgradient methods", *The* 29th *American Control Conference*, pages 2434 – 2439, Baltimore, USA, Jun. 2010.

4.8 Appendix

4.8.1 A property of projection operators

The proof of the following lemma can be found in [15], [16] and [102].

Lemma 4.8.1 *Let Z be a non-empty, closed and convex set in $\mathbb{R}^n$. For any $z \in \mathbb{R}^n$, the following holds for any $y \in Z$: $\|P_Z[z] - y\|^2 \leq \|z - y\|^2 - \|P_Z[z] - z\|^2$.*

4.8.2 Some properties of the distributed projected subgradient algorithm in [102]

Consider the following distributed projected subgradient algorithm proposed in [102]: $x^i(k+1) = P_Z[v_x^i(k) - \alpha(k)d_i(k)]$. Denote by $e^i(k) := P_Z[v_x^i(k) - \alpha(k)d_i(k)] - v_x^i(k)$. The following is a slight modification of Lemma 8 and its proof in [102].

Lemma 4.8.2 *Let the non-degeneracy assumption 4.2.2, the balanced communication assumption 4.2.3 and the periodic strong connectivity assumption 4.2.4 hold. Suppose $Z \in \mathbb{R}^n$ is a closed and convex set. Then there exist $\gamma > 0$ and $\beta \in (0,1)$ such that*

$$\|x^i(k) - \hat{x}(k)\| \leq N\gamma \sum_{\tau=0}^{k-1} \beta^{k-\tau} \{\alpha(\tau)\|d_i(\tau)\| + \|e^i(\tau) + \alpha(\tau)d_i(\tau)\|\} + N\gamma\beta^{k-1} \sum_{i=0}^{N} \|x^i(0)\|.$$

Suppose $\{d_i(k)\}$ *is uniformly bounded for each* $i \in V$*, and* $\sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$*, then we have* $\sum_{k=0}^{+\infty} \alpha(k) \max_{i \in V} \|x^i(k) - \hat{x}(k)\| < +\infty$.

Chapter 5

Distributed cooperative non-convex optimization

5.1 Introduction

The focus of the current chapter is to relax the convexity assumption in Chapter 4. Our method will integrate Lagrangian dualization, subgradient schemes and average consensus algorithms. The techniques of Lagrangian dualization and subgradient schemes have been popular and efficient approaches to solve large-scale, structured convex optimization problems, e.g., [15, 16]. Numerous approaches have been designed to construct primal solutions to convex programs; e.g., by removing the nonsmoothness [124], by employing ascent approaches [72], and generating ergodic sequences [75, 99].

Statement of Contributions. This chapter investigates a multi-agent optimization problem where agents desire to agree upon a global decision vector which minimizes a sum of local objective functions in the presence of a global inequality constraint and a global state constraint set. The objective and constraint functions as well as the state-constraint set could be non-convex. We first introduce an approximation of the problem of interest where the exact consensus is slightly relaxed. We propose a distributed dual subgradient algorithm to solve the approximate problem where the update rule for local dual estimates combines a

dual subgradient scheme with average consensus algorithms and local primal estimates are generated from local dual optimal solution sets. This algorithm is shown to asymptotically converge to a pair of primal-dual solutions to the approximate problem provided that: firstly, the dual optimal solution set is singleton; secondly, dynamically changing network topologies satisfying some standard connectivity condition.

5.2 Problem formulation and preliminaries

Consider a networked multi-agent system where agents are labeled by $i \in V := \{1, \ldots, N\}$. The multi-agent system operates in a synchronous way at time instants $k \in \mathbb{N} \cup \{0\}$, and its topology will be represented by a directed weighted graph $\mathcal{G}(k) = (V, E(k), A(k))$, for $k \geq 0$. Here, $A(k) := [a_j^i(k)] \in \mathbb{R}^{N\times N}$ is the adjacency matrix, where the scalar $a_j^i(k) \geq 0$ is the weight assigned to the edge (j, i) pointing from agent j to agent i, and $E(k) \subseteq V \times V \setminus \mathrm{diag}(V)$ is the set of edges with non-zero weights. The set of in-neighbors of agent i at time k is denoted by $\mathcal{N}_i(k) = \{j \in V \mid (j, i) \in E(k) \text{ and } j \neq i\}$. Similarly, we define the set of out-neighbors of agent i at time k as $\mathcal{N}_i^{\text{out}}(k) = \{j \in V \mid (i, j) \in E(k) \text{ and } j \neq i\}$. We here make the following assumptions on network communication graphs:

Assumption 5.2.1 (Non-degeneracy) There exists a constant $\alpha > 0$ such that $a_i^i(k) \geq \alpha$, and $a_j^i(k)$, for $i \neq j$, satisfies $a_j^i(k) \in \{0\} \cup [\alpha,\ 1]$, for all $k \geq 0$.

Assumption 5.2.2 (Balanced Communication) [1]It holds that $\sum_{j\in V} a_j^i(k) = 1$ for all $i \in V$ and $k \geq 0$, and $\sum_{i\in V} a_j^i(k) = 1$ for all $j \in V$ and $k \geq 0$.

Assumption 5.2.3 (Periodical Strong Connectivity) There is a positive integer B such that, for all $k_0 \geq 0$, the directed graph $(V, \bigcup_{k=0}^{B-1} E(k_0 + k))$ is strongly connected.

The above network model is standard to characterize a networked multi-agent system, and has been widely used in the analysis of average consensus algorithms; e.g., see [108, 111], and distributed optimization in [102]. Recently, an

[1]It is also referred to as double stochasticity of the adjacency matrix $A(k)$.

algorithm is given in [55] which allows agents to construct a balanced graph out of a non-balanced one under certain assumptions.

The objective of the agents is to cooperatively solve the following primal problem (P):

$$\min_{z\in\mathbb{R}^n} \sum_{i\in V} f_i(z), \quad \text{s.t.} \quad g(z) \leq 0, \quad z \in X, \tag{5.1}$$

where $z \in \mathbb{R}^n$ is the global decision vector. The function $f_i : \mathbb{R}^n \to \mathbb{R}$ is only known to agent i, continuous, and referred to as the objective function of agent i. The set $X \subseteq \mathbb{R}^n$, the state constraint set, is compact. The function $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuous, and the inequality $g(z) \leq 0$ is understood component-wise; i.e., $g_\ell(z) \leq 0$, for all $\ell \in \{1, \ldots, m\}$, and represents a global inequality constraint. We will denote $f(z) := \sum_{i\in V} f_i(z)$ and $Y := \{z \in \mathbb{R}^n \mid g(z) \leq 0\}$. We will assume that the set of feasible points is non-empty; i.e., $X \cap Y \neq \emptyset$. Since X is compact and Y is closed, then we can deduce that $X \cap Y$ is compact. The continuity of f follows from that of f_i. In this way, the optimal value p^* of the problem (P) is finite and X^*, the set of primal optimal points, is non-empty. Throughout this chapter, we suppose the following Slater's condition holds:

Assumption 5.2.4 (Slater's Condition) There exists a vector $\bar{z} \in X$ such that $g(\bar{z}) < 0$. Such $\bar{z}$ is referred to as a Slater vector of the problem (P).

Remark 5.2.1 All the agents can agree upon a common Slater vector $\bar{z}$ through a maximum-consensus scheme. This can be easily implemented as part of an initialization step, and thus the assumption that the Slater vector is known to all agents does not limit the applicability of our algorithm. Specifically, the maximum-consensus algorithm is described as follows:

Initially, each agent i chooses a Slater vector $z_i(0) \in X$ such that $g(z_i(0)) < 0$. At every time $k \geq 0$, each agent i updates its estimates by using the rule of $z_i(k+1) = \max_{j\in\mathcal{N}_i(k)\cup\{i\}} z_j(k)$, where we use the following relation for vectors: for $a, b \in \mathbb{R}^n$, $a < b$ if and only if there is some $\ell \in \{1, \ldots, n-1\}$ such that $a_\kappa = b_\kappa$ for all $\kappa < \ell$ and $a_\ell < b_\ell$.

The periodical strong connectivity assumption 5.2.3 ensures that after at most $(N-1)B$ steps, all the agents reach the consensus; i.e., $z_i(k) = \max_{j\in V} z_j(0)$

for all $k \geq (N-1)B$. In the remainder of this chapter, we assume that the Slater vector $\bar{z}$ is known to all the agents. •

In Chapter 4, in order to solve the convex case of the problem (P) (i.e.; f_i and g are convex functions and X is a convex set), we propose two distributed primal-dual subgradient algorithms where primal (resp. dual) estimates move along subgradients (resp. supgradients) and are projected onto convex sets. The absence of convexity impedes the use of the algorithms in Chapter 4 since, on the one hand, (primal) gradient-based algorithms are easily trapped in local minima.; on the other hand, projection maps may not be well-defined when (primal) state constraint sets are non-convex. In the sequel, we will employ Lagrangian dualization, subgradient methods and average consensus schemes to design a distributed algorithm which is able to find an approximate solution to the problem (P).

Towards this end, we construct a directed cyclic graph $\mathcal{G}_{\text{cyc}} := (V, E_{\text{cyc}})$ where $|E_{\text{cyc}}| = N$. We assume that each agent has a unique in-neighbor (and out-neighbor). The out-neighbor (resp. in-neighbor) of agent i is denoted by i_D (resp. i_U). With the graph $\mathcal{G}_{\text{cyc}}$, we will study the following approximate problem of problem (P):

$$\begin{aligned} &\min_{(x_i)\in\mathbb{R}^{nN}} \sum_{i\in V} f_i(x_i), \\ &\text{s.t.} \quad g(x_i) \leq 0, \quad -x_i + x_{i_D} - \Delta \leq 0, \quad x_i - x_{i_D} - \Delta \leq 0, \quad x_i \in X, \quad \forall i \in V, \end{aligned} \tag{5.2}$$

where $\Delta := \delta \mathbf{1}$, with δ a small positive scalar, and $\mathbf{1}$ is the column vector of n ones. The problem (5.2) provides an approximation of the problem (P), and will be referred to as problem (P_Δ). In particular, the approximate problem (5.2) reduces to the problem (P) when $\delta = 0$. Its optimal value and the set of optimal solutions will be denoted by p^*_Δ and X^*_Δ, respectively. Similarly to the problem (P), p^*_Δ is finite and $X^*_\Delta \neq \emptyset$.

Remark 5.2.2 The cyclic graph $\mathcal{G}_{\text{cyc}}$ can be replaced by any strongly connected graph. Each agent i is endowed with two inequality constraints: $x_i - x_j - \Delta \leq 0$ and $-x_i + x_j - \Delta \leq 0$, for each out-neighbor j. For notational simplicity, we

will use the cyclic graph $\mathcal{G}_{\text{cyc}}$, which has a minimum number of constraints, as the initial graph. •

5.2.1 Dual problems

Before introducing dual problems, let us denote by $\Xi' := \mathbb{R}^m_{\geq 0} \times \mathbb{R}^{nN}_{\geq 0} \times \mathbb{R}^{nN}_{\geq 0}$, $\Xi := \mathbb{R}^{mN}_{\geq 0} \times \mathbb{R}^{nN}_{\geq 0} \times \mathbb{R}^{nN}_{\geq 0}$, $\xi_i := (\mu_i, \lambda, w) \in \Xi'$, $\xi := (\mu, \lambda, w) \in \Xi$ and $x := (x_i) \in X^N$. The dual problem (D_Δ) associated with (P_Δ) is given by

$$\max_{\mu,\lambda,w} Q(\mu, \lambda, w), \quad \text{s.t.} \quad \mu, \lambda, w \geq 0, \tag{5.3}$$

where $\mu := (\mu_i) \in \mathbb{R}^{mN}$, $\lambda := (\lambda_i) \in \mathbb{R}^{nN}$ and $w := (w_i) \in \mathbb{R}^{nN}$. Here, the dual function $Q : \Xi \to \mathbb{R}$ is given as $Q(\xi) \equiv Q(\mu, \lambda, w) := \inf_{x \in X^N} \mathcal{L}(x, \mu, \lambda, w)$, where $\mathcal{L} : \mathbb{R}^{nN} \times \Xi \to \mathbb{R}$ is the Lagrangian function

$$\begin{aligned}
&\mathcal{L}(x, \xi) \equiv \mathcal{L}(x, \mu, \lambda, w) \\
&:= \sum_{i \in V} \big(f_i(x_i) + \langle \mu_i, g(x_i) \rangle + \langle \lambda_i, -x_i + x_{i_D} - \Delta \rangle + \langle w_i, x_i - x_{i_D} - \Delta \rangle \big).
\end{aligned}$$

We denote the dual optimal value of the problem (D_Δ) by d^*_Δ and the set of dual optimal solutions by D^*_Δ. We endow each agent i with the local Lagrangian function $\mathcal{L}_i : \mathbb{R}^n \times \Xi' \to \mathbb{R}$ and the local dual function $Q_i : \Xi' \to \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{L}_i(x_i, \xi_i) &:= f_i(x_i) + \langle \mu_i, g(x_i) \rangle + \langle -\lambda_i + \lambda_{i_U}, x_i \rangle \\
&\quad + \langle w_i - w_{i_U}, x_i \rangle - \langle \lambda_i, \Delta \rangle - \langle w_i, \Delta \rangle, \\
Q_i(\xi_i) &:= \inf_{x_i \in X} \mathcal{L}_i(x_i, \xi_i).
\end{aligned}$$

In the approximate problem (P_Δ), the introduction of $-\Delta \leq x_i - x_{i_D} \leq \Delta$, $i \in V$, renders the f_i and g separable. As a result, the global dual function Q can be decomposed into a simple sum of the local dual functions Q_i. More precisely, the following holds:

$$Q(\xi) = \inf_{x \in X^N} \sum_{i \in V} \big(f_i(x_i) + \langle \mu_i, g(x_i) \rangle + \langle \lambda_i, -x_i + x_{i_D} - \Delta \rangle + \langle w_i, x_i - x_{i_D} - \Delta \rangle \big).$$

Notice that in the sum of $\sum_{i \in V} \langle \lambda_i, -x_i + x_{i_D} - \Delta \rangle$, each x_i for any $i \in V$ appears in two terms: one is $\langle \lambda_i, -x_i + x_{i_D} - \Delta \rangle$, and the other is $\langle \lambda_{i_U}, -x_{i_U} + x_i - \Delta \rangle$.

With this observation, we regroup the terms in the summation in terms of x_i, and have the following:

$$\begin{aligned} Q(\xi) &= \inf_{x\in X^N} \sum_{i\in V} \big(f_i(x_i) + \langle \mu_i, g(x_i)\rangle + \langle -\lambda_i + \lambda_{i_U}, x_i\rangle \\ &\quad + \langle w_i - w_{i_U}, x_i\rangle - \langle \lambda_i, \Delta\rangle - \langle w_i, \Delta\rangle \big) \\ &= \sum_{i\in V} \inf_{x_i\in X} \big(f_i(x_i) + \langle \mu_i, g(x_i)\rangle + \langle -\lambda_i + \lambda_{i_U}, x_i\rangle \\ &\quad + \langle w_i - w_{i_U}, x_i\rangle - \langle \lambda_i, \Delta\rangle - \langle w_i, \Delta\rangle \big) \\ &= \sum_{i\in V} Q_i(\xi_i). \end{aligned} \tag{5.4}$$

It is worth mentioning that $\sum_{i\in V} Q_i(\xi_i)$ is not separable since Q_i depends upon neighbor's multipliers λ_{i_U} and w_{i_U}.

5.2.2 Dual solution sets

The Slater's condition ensures the boundedness of dual solution sets for convex optimization; e.g., [61, 99]. We will shortly see that the Slater's condition plays the same role in non-convex optimization. To achieve this, we define the function $\hat{Q}_i : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ as follows:

$$\begin{aligned} \hat{Q}_i(\mu_i, \lambda_i, w_i) = \inf_{x_i\in X, x_{i_D}\in X} \big(& f_i(x_i) + \langle \mu_i, g(x_i)\rangle \\ & + \langle \lambda_i, -x_i + x_{i_D} - \Delta\rangle + \langle w_i, x_i - x_{i_D} - \Delta\rangle \big). \end{aligned}$$

Let $\bar{z}$ be a Slater vector for problem (P). Then $\bar{x} = (\bar{x}_i) \in X^N$ with $\bar{x}_i = \bar{z}$ is a Slater vector of the problem (P_Δ). Similarly to (3) and (4) in Chapter 4, we have that for any $\mu_i, \lambda_i, w_i \geq 0$, it holds that

$$\max_{\xi\in D^*_\Delta} \|\xi\| \leq N \max_{i\in V} \frac{f_i(\bar{z}) - \hat{Q}_i(\mu_i, \lambda_i, w_i)}{\beta(\bar{z})}, \tag{5.5}$$

where $\beta(\bar{z}) := \min\{\min_{\ell\in\{1,\ldots,m\}} -g_\ell(\bar{z}), \delta\}$. Let μ_i, λ_i and w_i be zero in (5.5), and it leads to the following upper bound on D^*_Δ:

$$\max_{\xi\in D^*_\Delta} \|\xi\| \leq N \max_{i\in V} \frac{f_i(\bar{z}) - \hat{Q}_i(0,0,0)}{\beta(\bar{z})}, \tag{5.6}$$

where $\hat{Q}_i(0,0,0) = \inf_{x_i \in X} f_i(x_i)$ and it can be computed locally. We denote

$$\gamma_i(\bar{z}) := \frac{f_i(\bar{z}) - \hat{Q}_i(0,0,0)}{\beta(\bar{z})}. \tag{5.7}$$

Since f_i and g are continuous and X is compact, it is known that Q_i is continuous; e.g., see Theorem 1.4.16 in [11]. Similarly, Q is continuous. Since D^*_Δ is also bounded, then we have that $D^*_\Delta \neq \emptyset$.

Remark 5.2.3 The requirement of exact agreement on z in the problem P is slightly relaxed in the problem P_Δ by introducing a small positive scalar δ. In this way, on the one hand, the global dual function Q is a sum of the local dual functions Q_i, as in (5.4); on the other hand, D^*_Δ is non-empty and uniformly bounded. These two properties play important roles in the devise of our subsequent algorithm. •

5.2.3 Other notation

Define the set-valued map $\Omega_i : \Xi' \to 2^X$ as $\Omega_i(\xi_i) := \operatorname{argmin}_{x_i \in X} \mathcal{L}_i(x_i, \xi_i)$; i.e., given ξ_i, the set $\Omega_i(\xi_i)$ is the collection of solutions to the following local optimization problem:

$$\min_{x_i \in X} \mathcal{L}_i(x_i, \xi_i). \tag{5.8}$$

Here, Ω_i is referred to as the *marginal map* of agent i. Since X is compact and f_i, g are continuous, then $\Omega_i(\xi_i) \neq \emptyset$ in (5.8) for any $\xi_i \in \Xi'$. In the algorithm we will develop in next section, each agent is required to obtain *one* (globally) optimal solution and the optimal value the local optimization problem (5.8) at each iterate. We assume that this can be easily solved, and this is the case for problems of $n = 1$, or f_i and g being smooth (the extremum candidates are the critical points of the objective function and isolated corners of the boundaries of the constraint regions) or having some specific structure which allows the use of global optimization methods such as branch and bound algorithms.

In the space $\mathbb{R}^n$, we define the distance between a point $z \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$ as $\operatorname{dist}(z, A) := \inf_{y \in A} \|z - y\|$, and the Hausdorff distance between two sets $A, B \subset \mathbb{R}^n$ as $\operatorname{dist}(A, B) := \max\{\sup_{z \in A} \operatorname{dist}(z, B), \sup_{y \in B} \operatorname{dist}(A, y)\}$. We denote

by $B_{\mathcal{U}}(A,r) := \{u \in \mathcal{U} \mid \text{dist}(u,A) \leq r\}$ and $B_{2^{\mathcal{U}}}(A,r) := \{U \in 2^{\mathcal{U}} \mid \text{dist}(U,A) \leq r\}$ where $\mathcal{U} \subset \mathbb{R}^n$.

5.3 Distributed approximate dual subgradient algorithm

In this section, we devise a distributed approximate dual subgradient algorithm which aims to find a pair of primal-dual solutions to the approximate problem (P_Δ). Its convergence properties are also summarized.

For each agent i, let $x_i(k) \in \mathbb{R}^n$ be the estimate of the primal solution x_i to the approximate problem (P_Δ) at time $k \geq 0$, $\mu_i(k) \in \mathbb{R}^m_{\geq 0}$ be the estimate of the multiplier on the inequality constraint $g(x_i) \leq 0$, $\lambda^i(k) \in \mathbb{R}^{nN}_{\geq 0}$ (resp. $w^i(k) \in \mathbb{R}^{nN}_{\geq 0}$)[2] be the estimate of the multiplier associated with the collection of the local inequality constraints $-x_j + x_{j_D} - \Delta \leq 0$ (resp. $x_j - x_{j_D} - \Delta \leq 0$), for all $j \in V$. We let $\xi_i(k) := (\mu_i(k)^T, \lambda^i(k)^T, w^i(k)^T)^T \in \Xi'$, for $i \in V$ to be the collection of dual estimates of agent i. And denote $v_i(k) := (\mu_i(k)^T, v^i_\lambda(k)^T, v^i_w(k)^T)^T \in \Xi'$ where $v^i_\lambda(k) := \sum_{j\in V} a^i_j(k)\lambda^j(k) \in \mathbb{R}^{nN}_{\geq 0}$ and $v^i_w(k) := \sum_{j\in V} a^i_j(k) w^j(k) \in \mathbb{R}^{nN}_{\geq 0}$ are convex combinations of dual estimates of agent i and its neighbors at time k.

At time k, we associate each agent i a supgradient vector $\mathcal{D}_i(k)$ defined as $\mathcal{D}_i(k) := (\mathcal{D}^i_\mu(k)^T, \mathcal{D}^i_\lambda(k)^T, \mathcal{D}^i_w(k)^T)^T$, where $\mathcal{D}^i_\mu(k) := g(x_i(k)) \in \mathbb{R}^m$, $\mathcal{D}^i_\lambda(k)$ has components $\mathcal{D}^i_\lambda(k)_i := -\Delta - x_i(k) \in \mathbb{R}^n$, $\mathcal{D}^i_\lambda(k)_{i_U} := x_i(k) \in \mathbb{R}^n$, and $\mathcal{D}^i_\lambda(k)_j = 0 \in \mathbb{R}^n$ for $j \in V \backslash \{i, i_U\}$, while the components of $\mathcal{D}^i_w(k)$ are given by: $\mathcal{D}^i_w(k)_i := -\Delta + x_i(k) \in \mathbb{R}^n$, $\mathcal{D}^i_w(k)_{i_U} := -x_i(k) \in \mathbb{R}^n$, and $\mathcal{D}^i_w(k)_j = 0 \in \mathbb{R}^n$, for $j \in V \setminus \{i, i_U\}$. For each agent i, we define the set $M_i := \{\xi_i \in \Xi' \mid \|\xi_i\| \leq \gamma + \theta_i\}$ for some $\theta_i > 0$. Let P_{M_i} to be the projection onto the set M_i. It is easy to check that M_i is closed and convex, and thus the projection map P_{M_i} is well-defined.

The *Distributed Approximate Dual Subgradient* (DADS, for short) Algorithm is described in Table 1.

Remark 5.3.1 The DADS algorithm is an extension of the classical dual algo-

[2]We will use the superscript i to indicate that $\lambda^i(k)$ and $w^i(k)$ are estimates of some global variables.

Algorithm 1 The Distributed Approximate Dual Subgradient Algorithm

Require: Initially, all the agents agree upon some $\delta > 0$ in the approximate problem (P_Δ). Each agent i chooses a common Slater vector $\bar{z}$, computes $\gamma_i(\bar{z})$ and obtains $\gamma := N \max_{i\in V} \gamma_i(\bar{z})$ through a max-consensus algorithm where $\gamma_i(\bar{z})$ is given in (5.7). After that, each agent i chooses initial states $x_i(0) \in X$ and $\xi_i(0) \in \Xi'$.

Ensure: At each time k, each agent i executes the following steps:

1: For each $k \geq 1$, given $v_i(k)$, solve the local optimization problem (5.8), obtain a solution $x_i(k) \in \Omega_i(v_i(k))$ and the dual optimal value $Q_i(v_i(k))$.

2: For each $k \geq 0$, generate the dual estimate $\xi_i(k+1)$ according to the following rule:

$$\xi_i(k+1) = P_{M_i}[v_i(k) + \alpha(k)\mathcal{D}_i(k)], \tag{5.9}$$

where the scalar $\alpha(k) \geq 0$ is a step-size.

3: Repeat for $k = k + 1$.

rithm, e.g., in [116] and [15] to the multi-agent setting and non-convex case. In the initialization of the DADS algorithm, the value γ serves as an upper bound on D^*_Δ. In Step 1, *one solution* in $\Omega_i(v_i(k))$ is needed, and it is unnecessary to compute the whole set $\Omega_i(v_i(k))$. •

In the remainder of the chapter, we further assume that $\Omega_i(\xi^*_i)$ is singleton given any $\xi^* \in D^*_\Delta$:

Assumption 5.3.1 ***(Singleton optimal dual solution sets)*** Given any dual optimal solution $\xi^* \in D^*_\Delta$, the set of $\Omega_i(\xi^*_i)$ is singleton where $\xi^*_i = (\mu^*_i, \lambda^*, w^*)$ for each $i \in V$.

Recall that γ provides an upper bound of D^*_Δ. Then it suffices to verify that $\Omega_i(\xi_i)$ is singleton for any $\|\xi_i\| \leq \gamma$. The above assumption is easy to check when x_i is a scalar. Furthermore, it follows from the second equality in (5.4) that Assumption 5.3.1 is equivalent to the following:

Assumption 5.3.2 Given any $\xi^* \in D_\Delta^*$, there is a unique solution to $\min_{x \in X^N} \mathcal{L}(x, \xi^*)$.

The primal and dual estimates in the DADS algorithm will be shown to asymptotically converge to a pair of primal-dual solutions to the approximate problem (P_Δ). We formally state this in the following theorem:

Theorem 5.3.1 (Convergence of the DADS algorithm) *Consider the problem (P) and the corresponding approximate problem (P_Δ) with some $\delta > 0$. We let the non-degeneracy assumption 5.2.1, the balanced communication assumption 5.2.2 and the periodic strong connectivity assumption 5.2.3 hold. In addition, suppose the Slater's condition 5.2.4 holds for the problem (P) and the assumption 5.3.1 holds for the approximate problem (P_Δ). Consider the dual sequences of $\{\mu_i(k)\}$, $\{\lambda^i(k)\}$, $\{w^i(k)\}$ and the primal sequence of $\{x_i(k)\}$ of the distributed approximate dual subgradient algorithm with $\{\alpha(k)\}$ satisfying the following:*

$$\lim_{k \to +\infty} \alpha(k) = 0, \quad \sum_{k=0}^{+\infty} \alpha(k) = +\infty, \quad \sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty.$$

Then, there exists a pair of primal-dual solution $(x^*, \xi^*) \in X_\Delta^* \times D_\Delta^*$ where $\xi^* := (\mu^*, \lambda^*, w^*)$ with $\mu^* := (\mu_i^*)$ and $x^* := (x_i^*)$ such that the following holds for all $i \in V$:

$$\lim_{k \to +\infty} \|\mu_i(k) - \mu_i^*\| = 0, \quad \lim_{k \to +\infty} \|\lambda^i(k) - \lambda^*\| = 0,$$
$$\lim_{k \to +\infty} \|w^i(k) - w^*\| = 0, \quad \lim_{k \to +\infty} \|x_i(k) - x_i^*\| = 0.$$

5.4 Convergence analysis

This section provides the complete analysis of Theorem 5.3.1. Recall that g is continuous and X is compact. Then there are $G, H > 0$ such that $\|g(x)\| \leq G$ and $\|x\| \leq H$ for all $x \in X$. We start our analysis from the computation of supgradients of Q_i.

Lemma 5.4.1 (Supgradient computation) *If $\bar{x}_i \in \Omega_i(\bar{\xi}_i)$, then $\big(g(\bar{x}_i)^T, (-\Delta - \bar{x}_i)^T, \bar{x}_i^T, (\bar{x}_i - \Delta)^T, -\bar{x}_i^T)^T$ is a supgradient of Q_i at $\bar{\xi}_i$; i.e., the following holds for any $\xi_i \in \Xi'$:*

$$Q_i(\xi_i) - Q_i(\bar{\xi}_i) \leq \langle g(\bar{x}_i), \mu_i - \bar{\mu}_i \rangle + \langle -\Delta - \bar{x}_i, \lambda_i - \bar{\lambda}_i \rangle + \langle \bar{x}_i, \lambda_{i_U} - \bar{\lambda}_{i_U} \rangle + \langle \bar{x}_i - \Delta, w_i - \bar{w}_i \rangle + \langle -\bar{x}_i, w_{i_U} - \bar{w}_{i_U} \rangle. \quad (5.10)$$

Proof: The proof is based on the computation of dual subgradients, e.g., in [15, 16]. ■

It follows from Lemma 5.4.1 that $(g(x_i(k))^T, (-\Delta - x_i(k))^T, x_i(k)^T, (x_i(k) - \Delta)^T, -x_i(k)^T)$ is a supgradient of Q_i at $v_i(k)$; i.e., the following supgradient inequality holds for any $\xi_i \in \Xi'$:

$$Q_i(\xi_i) - Q_i(v_i(k)) \leq \langle g(x_i(k)), \mu_i - \mu_i(k) \rangle + \langle -\Delta - x_i(k), \lambda_i - v_\lambda^i(k)_i \rangle + \langle x_i(k), \lambda_{i_U} - v_\lambda^i(k)_{i_U} \rangle + \langle x_i(k) - \Delta, w_i - v_w^i(k)_i \rangle + \langle -x_i(k), w_{i_U} - v_w^i(k)_{i_U} \rangle. \quad (5.11)$$

Now we can see that the update rule (5.9) of dual estimates in the DADS algorithm is a combination of a dual subgradient scheme and average consensus algorithms. The following establishes that Q_i is Lipschitz continuous with some Lipschitz constant L.

Lemma 5.4.2 (Lipschitz continuity of Q_i) *There is a constant $L > 0$ such that for any $\xi_i, \bar{\xi}_i \in \Xi'$, it holds that $\|Q_i(\xi_i) - Q_i(\bar{\xi}_i)\| \leq L\|\xi_i - \bar{\xi}_i\|$.*

Proof: Similarly to Lemma 5.4.1, one can show that if $\bar{x}_i \in \Omega_i(\bar{\xi}_i)$, then $(g(\bar{x}_i)^T, (-\Delta - \bar{x}_i)^T, \bar{x}_i^T, (\bar{x}_i - \Delta)^T, -\bar{x}_i^T)^T$ is a supgradient of Q_i at $\bar{\xi}_i$; i.e., the following holds for any $\xi_i \in \Xi'$:

$$Q_i(\xi_i) - Q_i(\bar{\xi}_i) \leq \langle g(\bar{x}_i), \mu_i - \bar{\mu}_i \rangle + \langle -\Delta - \bar{x}_i, \lambda_i - \bar{\lambda}_i \rangle + \langle \bar{x}_i, \lambda_{i_U} - \bar{\lambda}_{i_U} \rangle + \langle \bar{x}_i - \Delta, w_i - \bar{w}_i \rangle + \langle -\bar{x}_i, w_{i_U} - \bar{w}_{i_U} \rangle.$$

Since $\|g(\bar{x}_i)\| \leq G$ and $\|\bar{x}_i\| \leq H$, there is $L > 0$ such that $Q_i(\xi_i) - Q_i(\bar{\xi}_i) \leq L\|\xi_i - \bar{\xi}_i\|$. Similarly, $Q_i(\bar{\xi}_i) - Q_i(\xi_i) \leq L\|\xi_i - \bar{\xi}_i\|$. We then reach the desired result. ■

In the DADS algorithm, the error induced by the projection map P_{M_i} is given by:

$$e_i(k) := P_{M_i}[v_i(k) + \alpha(k)\mathcal{D}_i(k)] - v_i(k).$$

We next provide a basic iterate relation of dual estimates in the DADS algorithm.

Lemma 5.4.3 (Basic iterate relation) *Under the assumptions in Theorem 5.3.1, for any $((\mu_i), \lambda, w) \in \Xi$ with $(\mu_i, \lambda, w) \in M_i$ for all $i \in V$, the following estimate holds for all $k \geq 0$:*

$$\begin{aligned}
&\sum_{i\in V} \|e_i(k) - \alpha(k)\mathcal{D}_i(k)\|^2 \\
&\leq \alpha(k)^2 \sum_{i\in V} \|\mathcal{D}_i(k)\|^2 + \sum_{i\in V}(\|\xi_i(k) - \xi_i\|^2 - \|\xi_i(k+1) - \xi_i\|^2) \\
&+ 2\alpha(k) \sum_{i\in V}\{\langle g(x_i(k)), \mu_i(k) - \mu_i\rangle + \langle -\Delta - x_i(k), v^i_\lambda(k)_i - \lambda_i\rangle \\
&+ \langle x_i(k), v^i_\lambda(k)_{i_U} - \lambda_{i_U}\rangle + \langle x_i(k) - \Delta, v^i_w(k)_i - w_i\rangle + \langle -x_i(k), v^i_w(k)_{i_U} - w_{i_U}\rangle\}.
\end{aligned} \tag{5.12}$$

Proof: Recall that M_i is closed and convex. The proof is a combination of the nonexpansion property of projection operators in [16] and the property of balanced graphs. ■

The lemma below shows the asymptotic convergence of dual estimates.

Lemma 5.4.4 (Dual estimate convergence) *Under the assumptions in Theorem 5.3.1, there exists a dual optimal solution $\xi^* := ((\mu_i^*), \lambda^*, w^*) \in D^*_\Delta$ such that $\lim_{k\to+\infty} \|\mu_i(k) - \mu_i^*\| = 0$, $\lim_{k\to+\infty} \|\lambda^i(k) - \lambda^*\| = 0$, and $\lim_{k\to+\infty} \|w^i(k) - w^*\| = 0$.*

Proof: By the dual decomposition property (5.4) and the boundedness of dual optimal solution sets, the dual problem (D_Δ) is equivalent to the following:

$$\max_{(\xi_i)} \sum_{i\in V} Q_i(\xi_i), \quad \text{s.t.} \quad \xi_i \in M_i. \tag{5.13}$$

Note that Q_i is affine and M_i is convex, implying that the problem (5.13) is a constrained convex programming where the global objective function is a simple

sum of local ones and the local state constraints are convex and compact. The rest of the proofs can be finished by following similar lines in Chapter 4, and thus omitted. ■

The remainder of this section is dedicated to characterizing the convergence properties of primal estimates. Toward this end, we present some properties of Ω_i.

Lemma 5.4.5 (Properties of marginal maps) *The set-valued marginal map Ω_i is closed. In addition, it is upper semicontinuous at $\xi_i \in \Xi'$; i.e., for any $\epsilon' > 0$, there is $\delta' > 0$ such that for any $\tilde{\xi}_i \in B_{\Xi'}(\xi_i, \delta')$, it holds that $\Omega_i(\tilde{\xi}_i) \subset B_{2^X}(\Omega_i(\xi_i), \epsilon')$.*

Proof: Consider sequences $\{x_i(k)\}$ and $\{\xi_i(k)\}$ satisfying $\lim_{k\to+\infty} \xi_i(k) = \bar{\xi}_i$, $x_i(k) \in \Omega_i(\xi_i(k))$ and $\lim_{k\to+\infty} x_i(k) = \bar{x}_i$. Since $\mathcal{L}_i$ is continuous, then we have

$$\mathcal{L}_i(\bar{x}_i, \bar{\xi}_i) = \lim_{k\to+\infty} \mathcal{L}_i(x_i(k), \xi_i(k)) \leq \lim_{k\to+\infty} (Q_i(\xi_i(k))) = Q_i(\bar{\xi}_i),$$

where in the inequality we use the property of $x_i(k) \in \Omega_i(\xi_i(k))$, and in the last equality we use the continuity of Q_i. Then $\bar{x}_i \in \Omega_i(\bar{\xi}_i)$ and the closedness of Ω_i follows.

Note that $\Omega_i(\xi_i) = \Omega_i(\xi_i) \cap X$. Recall that Ω_i is closed and X is compact. Then it is a result of Proposition 1.4.9 in [11] that $\Omega_i(\xi_i)$ is upper semicontinuous at $\xi_i \in \Xi'$; i.e, for any neighborhood $\mathcal{U}$ in 2^X of $\Omega_i(\xi_i)$, there is $\delta' > 0$ such that $\forall \tilde{\xi}_i \in B_{\Xi'}(\xi_i, \delta')$, it holds that $\Omega_i(\tilde{\xi}_i) \subset \mathcal{U}$. Let $\mathcal{U} = B_{2^X}(\Omega_i(\xi_i), \epsilon')$, and we obtain upper semicontinuity at ξ_i. ■

With the above results, we are ready to show the convergence of primal estimates.

Lemma 5.4.6 (Primal estimate convergence) *Under the assumptions in Theorem 5.3.1, for each $i \in V$, there is $\tilde{x}_i \in \Omega_i(\xi_i^*)$ such that $\lim_{k\to+\infty} x_i(k) = \tilde{x}_i$.*

Proof: The combination of upper semicontinuity of Ω_i in Lemma 5.4.6 and $\lim_{k\to+\infty} \xi_i(k) = \xi_i^*$ with ξ_i^* given in Lemma 5.4.4 ensures that each accumulation point of $\{x_i(k)\}$ is a point in the set $\Omega_i(\xi_i^*)$; i.e., the convergence of $\{x_i(k)\}$ to

the set $\Omega_i(\xi_i^*)$ can be guaranteed. By Assumption 5.3.1, we notice that $\Omega_i(\xi_i^*)$ is singleton and then let $\tilde{x}_i = \Omega_i(\xi_i^*)$. We arrive in the desired result. ■

Now we are ready to show the main result of this chapter, Theorem 5.3.1. In particular, we will show complementary slackness, primal feasibility of $\tilde{x}$, and its primal optimality, respectively.

Proof for Theorem 5.3.1:

Claim 1: $\langle -\Delta - \tilde{x}_i + \tilde{x}_{i_D}, \lambda_i^* \rangle = 0$, $\langle -\Delta + \tilde{x}_i - \tilde{x}_{i_D}, w_i^* \rangle = 0$ and $\langle g(\tilde{x}_i), \mu_i^* \rangle = 0$.

Proof: Rearranging the terms related to λ in (5.12) leads to the following inequality holding for any $((\mu_i), \lambda, w) \in \Xi$ with $(\mu_i, \lambda, w) \in M$ for all $i \in V$:

$$
\begin{aligned}
&- \sum_{i \in V} 2\alpha(k)(\langle -\Delta - x_i(k), v_\lambda^i(k)_i - \lambda_i \rangle + \langle x_{i_D}(k), v_\lambda^{i_D}(k)_i - \lambda_i \rangle) \\
&\leq \alpha(k)^2 \sum_{i \in V} \|\mathcal{D}_i(k)\|^2 + \sum_{i \in V} (\|\xi_i(k) - \xi_i\|^2 - \|\xi_i(k+1) - \xi_i\|^2) \\
&+ 2\alpha(k) \sum_{i \in V} \{\langle -x_i(k), v_w^i(k)_{i_U} - w_{i_U} \rangle + \langle x_i(k) - \Delta, v_w^i(k)_i - w_i \rangle \\
&+ \langle g(x_i(k)), \mu_i(k) - \mu_i \rangle\}. \qquad (5.14)
\end{aligned}
$$

Sum (5.14) over $[0, K]$, divide by $s(K) := \sum_{k=0}^K \alpha(k)$, and we have

$$
\begin{aligned}
&\frac{1}{s(K)} \sum_{k=0}^K \alpha(k) \sum_{i \in V} 2(\langle \Delta + x_i(k), v_\lambda^i(k)_i - \lambda_i \rangle + \langle -x_{i_D}(k), v_\lambda^{i_D}(k)_i - \lambda_i \rangle) \\
&\leq \frac{1}{s(K)} \sum_{k=0}^K \alpha(k)^2 \sum_{i \in V} \|\mathcal{D}_i(k)\|^2 + \frac{1}{s(K)} \{\sum_{i \in V} (\|\xi_i(0) - \xi_i\|^2 - \|\xi_i(K+1) - \xi_i\|^2) \\
&+ \sum_{k=0}^K 2\alpha(k) \sum_{i \in V} (\langle g(x_i(k)), \mu_i(k) - \mu_i \rangle + \langle x_i(k) - \Delta, v_w^i(k)_i - w_i \rangle \\
&+ \langle -x_i(k), v_w^i(k)_{i_U} - w_{i_U} \rangle)\}. \qquad (5.15)
\end{aligned}
$$

We now proceed to show $\langle -\Delta - \tilde{x}_i + \tilde{x}_{i_D}, \lambda_i^* \rangle \geq 0$ for each $i \in V$. Notice that we have shown that $\lim_{k \to +\infty} \|x_i(k) - \tilde{x}_i\| = 0$ for all $i \in V$, and it also holds that $\lim_{k \to +\infty} \|\xi_i(k) - \xi_i^*\| = 0$ for all $i \in V$. Let $\lambda_i = \frac{1}{2}\lambda_i^*$, $\lambda_j = \lambda_j^*$ for $j \neq i$ and $\mu_i = \mu_i^*$, $w = w^*$ in (5.15). Recall that $\{\alpha(k)\}$ is not summable but square summable, and $\{\mathcal{D}_i(k)\}$ is uniformly bounded. Take $K \to +\infty$, and then it follows from Lemma in

Chapter 4 that:

$$\langle \Delta + \tilde{x}_i - \tilde{x}_{i_D}, \lambda_i^* \rangle \leq 0. \tag{5.16}$$

On the other hand, since $\xi^* \in D_\Delta^*$, we have $\|\xi^*\| \leq \gamma$ given the fact that γ is an upper bound of D_Δ^*. Let $\xi := (\mu, \lambda, w)$ where $\xi_i := (\mu_i, \lambda, w)$. Then we could choose a sufficiently small $\delta' > 0$ and $\xi \in \Xi$ in (5.15) such that $\|\xi_i\| \leq \gamma + \theta_i$ where θ_i is given in the definition of M_i and ξ is given by: $\lambda_i = (1+\delta')\lambda_i^*$, $\lambda_j = \lambda_j^*$ for $j \neq i$, $w = w^*$, $\mu = \mu^*$. Following the same lines toward (5.16), it gives that $-\delta\langle \Delta + \tilde{x}_i - \tilde{x}_{i_D}, \lambda_i^* \rangle \leq 0$. Hence, it holds that $\langle -\Delta - \tilde{x}_i + \tilde{x}_{i_D}, \lambda_i^* \rangle = 0$. The rest of the proof is analogous and thus omitted. ■

Claim 2: $\tilde{x}$ is primal feasible to the approximate problem (P_Δ).

Proof: We have known that $\tilde{x}_i \in X$. We proceed to show $-\Delta - \tilde{x}_i + \tilde{x}_{i_D} \leq 0$ by contradiction. Since $\|\xi^*\| \leq \gamma$, we could choose a sufficiently small $\delta' > 0$ and $\xi := (\mu, \lambda, w)$ where $\xi_i := (\mu_i, \lambda, w)$ and $\|\xi_i\| \leq \gamma + \theta_i$ in (5.15) as follows: if $(-\Delta - \tilde{x}_i + \tilde{x}_{i_D})_\ell > 0$, then $(\lambda_i)_\ell = (\lambda_i^*)_\ell + \delta'$; otherwise, $(\lambda_i)_\ell = (\lambda_i^*)_\ell$, and $w = w^*$, $\mu = \mu^*$. The rest of the proofs is analogous to Claim 1.

Similarly, one can show $g(\tilde{x}_i) \leq 0$ and $-\Delta + \tilde{x}_i - \tilde{x}_{i_D} \leq 0$ by applying analogous arguments. We conclude that $\tilde{x}$ is primal feasible to the approximate problem (P_Δ). ■

Claim 3: $\tilde{x}$ is a primal solution to the problem (P_Δ).

Proof: Since $\tilde{x}$ is primal feasible to the approximate problem (P_Δ), then $\sum_{i \in V} f_i(\tilde{x}_i) \geq p_\Delta^*$. On the other hand, it follows from Claim 1 that

$$\sum_{i \in V} f_i(\tilde{x}_i) = \sum_{i \in V} \mathcal{L}_i(\tilde{x}_i, \xi_i^*) \leq \sum_{i \in V} Q_i(\xi_i^*) \leq p_\Delta^*.$$

We then conclude that $\sum_{i \in V} f_i(\tilde{x}_i) = p_\Delta^*$. In conjunction with the feasibility of $\tilde{x}$, this further ensures that $\tilde{x}$ is primal optimal to the problem (P_Δ^*). This completes the proofs for Theorem 5.3.1. ■

5.5 Conclusions

We have studied a multi-agent optimization problem where the goal of agents is to minimize a sum of local objective functions in the presence of a global

inequality constraint and a global state constraint set. The optimization problem under consideration is not necessarily convex. We have presented the distributed approximate dual subgradient algorithm which allow agents to asymptotically converge to a pair of primal-dual solutions to the approximate problem. This chapter is based on the following papers:

(JP-6) M. Zhu and S. Martínez, "An approximate dual subgradient algorithm for distributed non-convex constrained optimization", *IEEE Transactions on Automatic Control*, 2011, provisionally accepted.

(CP-10) M. Zhu and S. Martínez, "An approximate dual sugbradient algorithm for multi-agent nonconvex optimization", *The* 49^{th} *IEEE Conference on Decision and Control*, pages $7487 - 7492$, Atlanta, USA, Dec. 2010.

Part III

Distributed online learning based coordination

Chapter 6

Game-theoretic optimal coverage of visual mobile sensors

6.1 Introduction

There is a widespread belief that continuous and pervasive monitoring will be possible in the near future with large numbers of networked, mobile, and wireless sensors. Thus, we are witnessing an intense research activity that focuses on the design of efficient control mechanisms for these systems. In particular, decentralized algorithms would allow sensor networks to react autonomously to changes in the environment with minimal human supervision.

A substantial body of research on sensor networks has concentrated on simple sensors that can collect scalar data; e.g., temperature, humidity or pressure data. Here, a main objective is the design of algorithms that can lead to optimal collective sensing through efficient motion control and communication schemes. However, scalar measurements can be insufficient in many situations; e.g., in automated surveillance or traffic monitoring applications. In contrast, data-intensive sensors such as cameras can collect visual data that are rich in information, thus having tremendous potential for monitoring applications, but at the cost of a higher processing overhead.

Precisely, this chapter aims to solve a coverage optimization problem tak-

ing into account part of the sensing/processing trade-off. Coverage optimization problems have mainly been formulated as cooperative problems where each sensor benefits from sensing the environment as a member of a group. However, sensing may also require expenditure; e.g., the energy consumed or the time spent by image processing algorithms in visual networks. Because of this, we endow each sensor with a utility function that quantifies this trade-off, formulating a coverage problem as a variation of congestion games in [125].

Literature review. In broad terms, the problem studied here is related to a bevy of sensor location and planning problems in the Computational Geometry, Geometric Optimization, and Robotics literature. For example, different variations on the (combinatorial) Art Gallery problem include [113][130][143]. The objective here is how to find the optimum number of guards in a non-convex environment so that each point is visible from at least one guard. A related set of references for the deployment of mobile robots with omnidirectional cameras includes [53][52]. Unlike the Art Gallery classic algorithms, the latter papers assume that robots have local knowledge of the environment and no recollection of the past. Other related references on robot deployment in convex environments include [37][76] for anisotropic and circular footprints.

The paper [1] is an excellent survey on multimedia sensor networks where the state of the art in algorithms, protocols, and hardware is surveyed, and open research issues are discussed in detail. As observed in [38], multimedia sensor networks enhance traditional surveillance systems by enlarging, enhancing, and enabling multi-resolution views. The investigation of coverage problems for static visual sensor networks is conducted in [32][63][144].

Another set of relevant references to this chapter comprise those on the use of game-theoretic tools to (i) solve static target assignment problems, and (ii) devise efficient and secure algorithms for communication networks. In [81], the authors present a game-theoretic analysis of a coverage optimization problem for static sensor networks. This problem is equivalent to the weapon-target assignment problem in [95] which is NP complete. In general, the solution to assignment problems is hard from a combinatorial optimization viewpoint.

Game Theory and Learning in Games are used to analyze a variety of fundamental problems in; e.g., wireless communication networks and the Internet. An incomplete list of references includes [3] on power control, [126] on routing, and [139] on flow control. However, there has been limited research on how to employ Learning in Games to develop distributed algorithms for mobile sensor networks. One exception is the paper [82] where the authors establish a link between cooperative control problems (in particular, consensus problems), and games (in particular, potential games and weakly acyclic games).

Statement of contributions. The contributions of this chapter pertain to both coverage optimization problems and Learning in Games. Compared with [74] and [76], this chapter employs a more accurate sensing model and the results can be easily extended to include non-convex environments. Contrary to [74], we do not consider energy expenditure from sensor motions.

Regarding Learning in Games, we extend the use of the payoff-based learning dynamics first novelly proposed in [83][84]. The coverage game we consider here is shown to be a (constrained) exact potential game. A number of learning rules; e.g., better (or best) reply dynamics and adaptive play, have been proposed to reach Nash equilibria in potential games. In these algorithms, each player must have access to the utility values induced by alternative actions. In our problem set-up; however, *this information is unaccessible* because of the information constraints caused by unknown rewards, motion and sensing limitations. To tackle this challenge, we develop two distributed payoff-based learning algorithms where each sensor only remembers its own utility values and actions played during the last two plays.

In the first algorithm, at each time step, each sensor repeatedly updates its action synchronously, either trying some new action in the state-dependent feasible action set or selecting the action which corresponds to a higher utility value in the most recent two time steps. As the algorithm for the special identical interest games in [84], the first algorithm employs a diminishing exploration rate. The dynamically changing exploration rate renders the algorithm a time-inhomogeneous Markov chain, and allows for the convergence in probability to the set of (constrained)

Nash equilibria, from which no agent is willing to unilaterally deviate.

The second algorithm is asynchronous. At each time step, only one sensor is active and updates its state by either trying some new action in the state-dependent feasible action set or selecting an action according to a Gibbs-like distribution from those played in last two time steps when it was active. The algorithm is shown to be convergent in probability to the set of global maxima of a coverage performance metric. Rather than maximizing the associated potential function in [83], the second algorithm optimizes the sum of local utility functions which captures better a global trade-off between the overall network benefit from sensing and the total energy the network consumes. By employing a diminishing exploration rate, our algorithm is guaranteed to have stronger convergence properties that the ones in [83].

6.2 Problem formulation

Here, we first review some basic game-theoretic concepts; see, for example [51]. This will allow us to formulate subsequently an optimal coverage problem for mobile visual sensor networks as a repeated multi-player game. We then introduce notations used throughout the chapter.

6.2.1 Background in Game Theory

A strategic game $\Gamma := \langle V, A, U \rangle$ has three components:

1. A set V enumerating players $i \in V := \{1, \cdots, N\}$.
2. An action set $A := \prod_{i=1}^{N} A_i$ is the space of all actions vectors, where $s_i \in A_i$ is the action of player i and a (multi-player) action $s \in A$ has components $s_1, \ldots, s_N$.
3. The collection of utility functions U, where the utility function $u_i : A \to \mathbb{R}$ models player i's preferences over action profiles.

Denote by s_{-i} the action profile of all players other than i, and by $A_{-i} = \prod_{j \neq i} A_j$ the set of action profiles for all players except i. The concept of (pure)

Nash equilibrium (NE, for short) is the most important one in Non-cooperative Game Theory [51] and is defined as follows.

Definition 6.2.1 (Nash equilibrium [51]) Consider the strategic game Γ. An action profile $s^* := (s_i^*, s_{-i}^*)$ is a (pure) NE of the game Γ if $\forall i \in V$ and $\forall s_i \in A_i$, it holds that $u_i(s^*) \geq u_i(s_i, s_{-i}^*)$.

An action profile corresponding to an NE represents a scenario where no player has incentive to unilaterally deviate. Exact potential games form an important class of strategic games where the change in a player's utility caused by a unilateral deviation can be exactly measured by a potential function.

Definition 6.2.2 (Exact potential game [92]) The strategic game Γ is an exact potential game with potential function $\phi : A \to \mathbb{R}$ if for every $i \in V$, for every $s_{-i} \in A_{-i}$, and for every $s_i, s_i' \in A_i$, it holds that

$$\phi(s_i, s_{-i}) - \phi(s_i', s_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}). \tag{6.1}$$

In conventional Non-cooperative Game Theory, all the actions in A_i always can be selected by player i in response to other players' actions. However, in the context of motion coordination, the actions available to player i will often be constrained to a state-dependent subset of A_i. In particular, we denote by $F_i(s_i, s_{-i}) \subseteq A_i$ the set of feasible actions of player i when the action profile is $s := (s_i, s_{-i})$. We assume that $s_i \in F_i(s_i, s_{-i})$ for any $s \in A$ throughout this chapter. Denote $F(s) := \prod_{i \in V} F_i(s) \subseteq A$, $\forall s \in A$ and $F := \cup\{F(s) \mid s \in A\}$. The introduction of F leads naturally to the notion of constrained strategic game $\Gamma_{\text{res}} := \langle V, A, U, F \rangle$, and the following associated concepts.

Definition 6.2.3 (Constrained Nash equilibrium) Consider the constrained strategic game Γ_{res}. An action profile s^* is a constrained (pure) NE of the game Γ_{res} if $\forall i \in V$ and $\forall s_i \in F_i(s_i^*, s_{-i}^*)$, it holds that $u_i(s^*) \geq u_i(s_i, s_{-i}^*)$.

Definition 6.2.4 (Constrained exact potential game) The game Γ_{res} is a constrained exact potential game with potential function $\phi(s)$ if for every $i \in V$, every $s_{-i} \in A_{-i}$, and every $s_i \in A_i$, the equality (6.1) holds for every $s_i' \in F_i(s_i, s_{-i})$.

With the assumption of $s_i \in F_i(s_i, s_{-i})$ for any $s \in A$, we observe that if s^* is an NE of the strategic game Γ, then it is also a constrained NE of the constrained strategic game Γ_{res}. For any given strategic game, NE may not exist. However, the existence of NE in exact potential games is guaranteed [92]. Hence, any constrained exact potential game with the assumption of $s_i \in F_i(s_i, s_{-i})$ for any $s \in A$ has at least one constrained NE.

6.2.2 Coverage problem formulation

Mission space

We consider a convex 2-D mission space that is discretized into a (squared) lattice. We assume that each square of the lattice has unit dimensions. Each square will be labeled with the coordinate of its center $q = (q_x, q_y)$, where $q_x \in [q_{x_{\min}}, q_{x_{\max}}]$ and $q_y \in [q_{y_{\min}}, q_{y_{\max}}]$, for some integers $q_{x_{\min}},\ q_{y_{\min}},\ q_{x_{\max}},\ q_{y_{\max}}$. Denote by $\mathcal{Q}$ the collection of all squares of the lattice.

We now define an associated location graph $\mathcal{G}_{\text{loc}} := (\mathcal{Q}, E_{\text{loc}})$ where $((q_x, q_y), (q_{x'}, q_{y'})) \in E_{\text{loc}}$ if and only if $|q_x - q_{x'}| + |q_y - q_{y'}| = 1$ for $(q_x, q_y), (q_{x'}, q_{y'}) \in \mathcal{Q}$. Note that the graph $\mathcal{G}_{\text{loc}}$ is undirected; i.e., $(q, q') \in E_{\text{loc}}$ if and only if $(q', q) \in E_{\text{loc}}$. The set of neighbors of q in E_{loc} is given by $\mathcal{N}_q^{\text{loc}} := \{q' \in \mathcal{Q} \setminus \{q\} \mid (q, q') \in E_{\text{loc}}\}$. We assume that the location graph $\mathcal{G}_{\text{loc}}$ is fixed and connected, and denote its diameter by D.

Agents are deployed in $\mathcal{Q}$ to detect certain events of interest. As agents move in $\mathcal{Q}$ and process measurements, they will assign a numerical value $W_q \geq 0$ to the events in each square with center $q \in \mathcal{Q}$. If $W_q = 0$, then there is no significant event at the square with center q. The larger the value of W_q is, the more of interest the set of events at the square with center q is of. Later, the amount W_q will be identified with a benefit of observing the point q. In this set-up, we assume the values W_q to be constant in time. Furthermore, W_q is not a prior knowledge to the agents, but the agents can measure this value through sensing the point q.

Modeling of the visual sensor nodes

Each mobile agent i is modeled as a point mass in $\mathcal{Q}$, with location $a_i := (x_i, y_i) \in \mathcal{Q}$. Each agent has mounted a pan-tilt-zoom camera, and can adjust its orientation and focal length.

The visual sensing range of a camera is directional, limited-range, and has a finite angle of view. Following a geometric simplification, we model the visual sensing region of agent i as an annulus sector in the 2-D plane; see Figure 6.1.

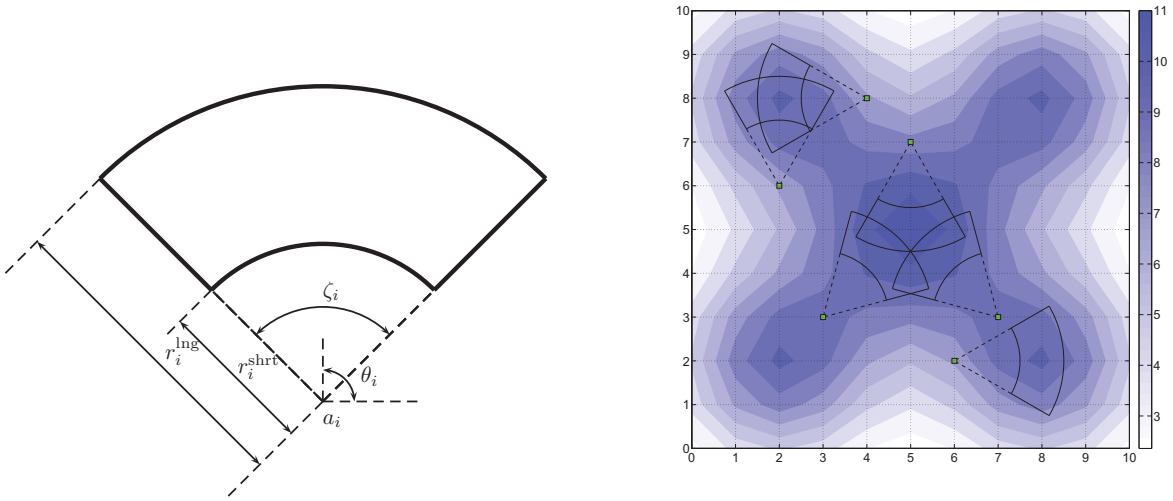

Figure 6.1: Visual sensor footprint and a configuration of the mobile sensor network

The visual sensor footprint is completely characterized by the following parameters: the position of agent i, $a_i \in \mathcal{Q}$, the camera orientation, $\theta_i \in [0, 2\pi)$, the camera angle of view, $\zeta_i \in [\alpha_{\min}, \alpha_{\max}]$, and the shortest range (resp. longest range) between agent i and the nearest (resp. farthest) object that can be recognized from the image, $r_i^{\text{shrt}} \in [r_{\min}, r_{\max}]$ (resp. $r_i^{\text{lng}} \in [r_{\min}, r_{\max}]$). The parameters r_i^{shrt}, r_i^{lng}, ζ_i can be tuned by changing the focal length FL_i of agent i's camera. In this way, $c_i := (\text{FL}_i, \theta_i) \in [0, \text{FL}_{\max}] \times [0, 2\pi)$ is the camera control vector of agent i. In what follows, we will assume that c_i takes values in a finite subset $\mathcal{C} \subset [0, \text{FL}_{\max}] \times [0, 2\pi)$. An agent action is thus a vector $s_i := (a_i, c_i) \in \mathcal{A}_i := \mathcal{Q} \times \mathcal{C}$, and a multi-agent action is denoted by $s = (s_1, \ldots, s_N) \in \mathcal{A} := \Pi_{i=1}^N \mathcal{A}_i$.

Let $\mathcal{D}(a_i, c_i)$ be the visual sensor footprint of agent i. Now we can define a proximity sensing graph[1] $\mathcal{G}_{\text{sen}}(s) := (V, E_{\text{sen}}(s))$ as follows: the set of neighbors of agent i, $\mathcal{N}_i^{\text{sen}}(s)$, is given as $\mathcal{N}_i^{\text{sen}}(s) := \{j \in V \backslash \{i\} \mid \mathcal{D}(a_i, c_i) \cap \mathcal{D}(a_j, c_j) \cap \mathcal{Q} \neq \emptyset\}$.

Each agent is able to communicate with others to exchange information. We assume that the communication range of agents is $2r_{\max}$. This induces a $2r_{\max}$-disk communication graph $\mathcal{G}_{\text{comm}}(s) := (V, E_{\text{comm}}(s))$ as follows: the set of neighbors of agent i is given by $\mathcal{N}_i^{\text{comm}}(s) := \{j \in V \backslash \{i\} |\ (x_i - x_j)^2 + (y_i - y_j)^2 \leq (2r_{\max})^2\}$. Note that $\mathcal{G}_{\text{comm}}(s)$ is undirected and that $\mathcal{G}_{\text{sen}}(s) \subseteq \mathcal{G}_{\text{comm}}(s)$.

The motion of agents will be limited to a neighboring point in $\mathcal{G}_{\text{loc}}$ at each time step. Thus, an agent feasible action set will be given by $\mathcal{F}_i(a_i) := (\{a_i\} \cup \mathcal{N}_{a_i}^{\text{loc}}) \times \mathcal{C}$.

Coverage game

We now proceed to formulate a coverage optimization problem as a constrained strategic game. For each $q \in \mathcal{Q}$, we denote $n_q(s)$ as the cardinality of the set $\{k \in V \mid q \in \mathcal{D}(a_k, c_k) \cap \mathcal{Q}\}$; i.e., the number of agents which can observe the point q. The "profit" given by W_q will be equally shared by agents that can observe the point q. The benefit that agent i obtains through sensing is thus defined by $\sum_{q \in \mathcal{D}(a_i, c_i) \cap \mathcal{Q}} \frac{W_q}{n_q(s)}$.

On the other hand, and as argued in [85], the processing of visual data can incur a higher cost than that of communication. This is in contrast with scalar sensor networks, where the communication cost dominates. With this observation, we model the energy consumption of agent i by $f_i(c_i) := \frac{1}{2}\zeta_i((r_i^{\text{lng}})^2 - (r_i^{\text{shrt}})^2)$. A similar energy model is used in [144] and references therein. This measure corresponds to the area of the visual sensor footprint and can serve to approximate the energy consumption or the cost incurred by image processing algorithms.

We will endow each agent with a utility function that aims to capture the above sensing/processing trade-off. In this way, we define a utility function for

[1] See [25] for a definition of proximity graph.

agent i by

$$u_i(s) = \sum_{q \in \mathcal{D}(a_i, c_i) \cap \mathcal{Q}} \frac{W_q}{n_q(s)} - f_i(c_i).$$

Note that the utility function u_i is local over the visual sensing graph $\mathcal{G}_{\text{sen}}(s)$; i.e., u_i is only dependent on the actions of $\{i\} \cup \mathcal{N}_i^{\text{sen}}(s)$. With the set of utility functions $U_{\text{cov}} = \{u_i\}_{i \in V}$, and feasible action set $\mathcal{F}_{\text{cov}} = \Pi_{i=1}^N \bigcup_{a_i \in \mathcal{A}_i} \mathcal{F}_i(a_i)$, we now have all the ingredients to introduce the coverage game $\Gamma_{\text{cov}} := \langle V, \mathcal{A}, U_{\text{cov}}, \mathcal{F}_{\text{cov}} \rangle$. This game is a variation of the congestion games introduced in [125].

Lemma 6.2.1 *The coverage game Γ_{cov} is a constrained exact potential game with potential function*

$$\phi(s) = \sum_{q \in \mathcal{Q}} \sum_{\ell=1}^{n_q(s)} \frac{W_q}{\ell} - \sum_{i=1}^{N} f_i(c_i).$$

Proof: The proof is a slight variation of that in [125]. Consider any $s := (s_i, s_{-i}) \in \mathcal{A}$ where $s_i := (a_i, c_i)$. We fix $i \in V$ and pick any $s_i' = (a_i', c_i')$ from $\mathcal{F}_i(a_i)$. Denote $s' := (s_i', s_{-i})$, $\Omega_1 := (\mathcal{D}(a_i, c_i) \backslash \mathcal{D}(a_i', c_i')) \cap \mathcal{Q}$ and $\Omega_2 := (\mathcal{D}(a_i', c_i') \backslash \mathcal{D}(a_i, c_i)) \cap \mathcal{Q}$. Observe that

$$\begin{aligned}
&\phi(s_i, s_{-i}) - \phi(s_i', s_{-i}) \\
&= \sum_{q \in \Omega_1} \Big(\sum_{\ell=1}^{n_q(s)} \frac{W_q}{\ell} - \sum_{\ell=1}^{n_q(s')} \frac{W_q}{\ell} \Big) + \sum_{q \in \Omega_2} \Big(- \sum_{\ell=1}^{n_q(s)} \frac{W_q}{\ell} + \sum_{\ell=1}^{n_q(s')} \frac{W_q}{\ell} \Big) - f_i(c_i) + f_i(c_i') \\
&= \sum_{q \in \Omega_1} \frac{W_q}{n_q(s)} - \sum_{q \in \Omega_2} \frac{W_q}{n_q(s')} - f_i(c_i) + f_i(c_i') \\
&= u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})
\end{aligned}$$

where in the second equality we utilize the fact that for each $q \in \Omega_1$, $n_q(s) = n_q(s') + 1$, and each $q \in \Omega_2$, $n_q(s') = n_q(s) + 1$. ■

We denote by $\mathcal{E}(\Gamma_{\text{cov}})$ the set of constrained NEs of Γ_{cov}. It is worth mentioning that $\mathcal{E}(\Gamma_{\text{cov}}) \neq \emptyset$ due to the fact that Γ_{cov} is a constrained exact potential game.

Remark 6.2.1 The assumptions of our problem formulation admit several extensions. For example, it is straightforward to extend our results to non-convex 3-D spaces. This is because the results that follow can also handle other shapes of the sensor footprint; e.g., a complete disk, a subset of the annulus sector. On the other hand, note that the coverage problem can be interpreted as a target assignment problem—here, the value $W_q \geq 0$ would be associated with the value of a target located at the point q. •

6.2.3 Notations

In the following, we will use the Landau symbol, O, as in $O(\epsilon^\iota)$, for some $\iota \geq 0$. This implies that $0 < \lim_{\epsilon \to 0^+} \frac{O(\epsilon^\iota)}{\epsilon^\iota} < +\infty$. We denote by $\text{diag}(\mathcal{A}) := \{(s,s) \in \mathcal{A}^2 |\ s \in \mathcal{A}\}$ and $\text{diag}(\mathcal{E}(\Gamma_{\text{cov}})) := \{(s,s) \in \mathcal{A}^2 |\ s \in \mathcal{E}(\Gamma_{\text{cov}})\}$.

Consider $a, a' \in \mathcal{Q}^N$ where $a_i \neq a'_i$ and $a_{-i} = a'_{-i}$ for some $i \in V$. The transition $a \to a'$ is feasible if and only if $(a_i, a'_i) \in E_{\text{loc}}$. A feasible path from a to a' consisting of multiple feasible transitions is denoted by $a \Rightarrow a'$. Let $\diamond a := \{a' \in \mathcal{Q} |\ a \Rightarrow a'\}$ be the reachable set from a.

Let $s = (a, c), s' = (a', c') \in \mathcal{A}$ where $a_i \neq a'_i$ and $a_{-i} = a'_{-i}$ for some $i \in V$. The transition $s \to s'$ is feasible if and only if $s'_i \in \mathcal{F}_i(a)$. A feasible path from s to s' consisting of multiple feasible transitions is denoted by $s \Rightarrow s'$. Finally, $\diamond s := \{s' \in \mathcal{A} |\ s \Rightarrow s'\}$ will be the reachable set from s.

6.3 Preliminaries

For the sake of a self-contained exposition, we include here some background in Markov chains [64] and the Theory of Resistance Trees [154].

Background in Markov chains

A *discrete-time Markov chain* is a discrete-time stochastic process on a finite (or countable) state space and satisfies the Markov property (i.e., the future state depends on its present state, but not the past states). A discrete-time Markov

chain is said to be *time-homogeneous* if the probability of going from one state to another is independent of the time when the step is taken. Otherwise, the Markov chain is said to be *time-inhomogeneous*.

Since time-inhomogeneous Markov chains include time-homogeneous ones as special cases, we will restrict our attention to the former in the remainder of this section. The evolution of a time-inhomogeneous Markov chain $\{\mathcal{P}_k\}$ can be described by the transition matrix $P(k)$ which gives the probability of traversing from one state to another at each time k.

Consider a Markov chain $\{\mathcal{P}_k\}$ with time-dependent transition matrix $P(k)$ on a finite state space X. Denote by $P(m,n) := \prod_{t=m}^{n-1} P(k)$, $0 \leq m < n$.

Definition 6.3.1 (Strong ergodicity [64]) The Markov chain $\{\mathcal{P}_k\}$ is strongly ergodic if there exists a stochastic vector μ^* such that for any distribution μ on X and any $m \in \mathbb{Z}_+$, it holds that $\lim_{k\to+\infty} \mu^T P(m,k) = (\mu^*)^T$.

Strong ergodicity of $\{\mathcal{P}_k\}$ is equivalent to $\{\mathcal{P}_k\}$ being convergent in distribution and will be employed to characterize the long-run properties of our learning algorithm. The investigation of conditions under which strong ergodicity holds is aided by the introduction of the coefficient of ergodicity and weak ergodicity defined next.

Definition 6.3.2 (Coefficient of ergodicity [64]) For any $n \times n$ stochastic matrix P, its coefficient of ergodicity is defined as follows:

$$\lambda(P) := 1 - \min_{1\leq i,j\leq n} \sum_{\tau=1}^{n} \min(P_{i\tau} P_{j\tau}).$$

Definition 6.3.3 (Weak ergodicity [64]) The Markov chain $\{\mathcal{P}_k\}$ is weakly ergodic if $\forall x, y, z \in X$, $\forall m \in \mathbb{Z}_+$, it holds that $\lim_{k\to+\infty}(P_{xz}(m,k) - P_{yz}(m,k)) = 0$.

Weak ergodicity merely implies that $\{\mathcal{P}_k\}$ asymptotically forgets its initial state, but does not guarantee convergence. For a time-homogeneous Markov chain, there is no distinction between weak ergodicity and strong ergodicity. The following theorem provides the sufficient and necessary condition for $\{\mathcal{P}_k\}$ to be weakly ergodic.

Theorem 6.3.1 ([64]) *The Markov chain $\{\mathcal{P}_k\}$ is weakly ergodic if and only if there is a strictly increasing sequence of positive numbers k_i, $i \in \mathbb{Z}_+$ such that $\sum_{i=0}^{+\infty}(1-\lambda(P(k_i,k_{i+1}))=+\infty$.*

We are now ready to present the sufficient conditions for strong ergodicity of the Markov chain $\{\mathcal{P}_t\}$.

Theorem 6.3.2 ([64]) *A Markov chain $\{\mathcal{P}_k\}$ is strongly ergodic if the following conditions hold:*

(B1) The Markov chain $\{\mathcal{P}_k\}$ is weakly ergodic.

(B2) For each k, there exists a stochastic vector μ^k on X such that μ^k is the left eigenvector of the transition matrix $P(k)$ with eigenvalue 1.

(B3) The eigenvectors μ^k in (B2) satisfy $\sum_{k=0}^{+\infty}\sum_{z\in X}|\mu_z^k-\mu_z^{k+1}|<+\infty$.

Moreover, if $\mu^*=\lim_{k\to+\infty}\mu^k$, then μ^* is the vector in Definition 6.3.1.

Background in the Theory of Resistance Trees

Let P^0 be the transition matrix of the time-homogeneous Markov chain $\{\mathcal{P}_k^0\}$ on a finite state space X. Furthermore, let P^ϵ be the transition matrix of a *perturbed Markov chain*, say $\{\mathcal{P}_k^\epsilon\}$. With probability $1-\epsilon$, the process $\{\mathcal{P}_k^\epsilon\}$ evolves according to P^0, while with probability ϵ, the transitions do not follow P^0.

A family of stochastic processes $\{\mathcal{P}_k^\epsilon\}$ is called a *regular perturbation* of $\{\mathcal{P}_k^0\}$ if the following holds $\forall x,y\in X$:

(A1) For some $\varsigma>0$, the Markov chain $\{\mathcal{P}_t^\epsilon\}$ is irreducible and aperiodic for all $\epsilon\in(0,\varsigma]$.

(A2) $\lim_{\epsilon\to0^+}P_{xy}^\epsilon=P_{xy}^0$.

(A3) If $P_{xy}^\epsilon>0$ for some ϵ, then there exists a real number $\chi(x\to y)\geq0$ such that $\lim_{\epsilon\to0^+}P_{xy}^\epsilon/\epsilon^{\chi(x\to y)}\in(0,+\infty)$.

In (A3), $\chi(x\to y)$ is called the resistance of the transition from x to y.

Let $H_1,H_2,\cdots,H_J$ be the recurrent communication classes of the Markov chain $\{\mathcal{P}_k^0\}$. Note that within each class H_ℓ, there is a path of zero resistance from every state to every other. Given any two distinct recurrence classes H_ℓ and H_s,

consider all paths which start from H_ℓ and end at H_s. Denote by $\chi_{\ell s}$ the least resistance among all such paths.

Now define a complete directed graph $\mathcal{G}$ where there is one vertex ℓ for each recurrent class H_ℓ, and the resistance on the edge (ℓ, s) is $\chi_{\ell s}$. An *ℓ-tree* on $\mathcal{G}$ is a spanning tree such that from every vertex $s \neq \ell$, there is a unique path from s to ℓ. Denote by $G(\ell)$ the set of all *ℓ-trees* on $\mathcal{G}$. The resistance of an *ℓ-tree* is the sum of the resistances of its edges. The *stochastic potential* of the recurrent class H_ℓ is the least resistance among all *ℓ-trees* in $G(\ell)$.

Theorem 6.3.3 ([154]) *Let $\{\mathcal{P}_k^\epsilon\}$ be a regular perturbation of $\{\mathcal{P}_k^0\}$, and for each $\epsilon > 0$, let $\mu(\epsilon)$ be the unique stationary distribution of $\{\mathcal{P}_k^\epsilon\}$. Then $\lim_{\epsilon \to 0^+} \mu(\epsilon)$ exists and the limiting distribution $\mu(0)$ is a stationary distribution of $\{\mathcal{P}_k^0\}$. The stochastically stable states (i.e., the support of $\mu(0)$) are precisely those states contained in the recurrence classes with minimum stochastic potential.*

6.4 Distributed learning algorithms and convergence results

In our coverage problem, we assume that W_q is unknown to all the sensors in advance. Furthermore, due to the restrictions of motion and sensing, each agent is unable to obtain the information of W_q if the point q is outside its sensing range. In addition, the utility of each agent depends upon the group strategy. These information constraints render that each agent is unable to access the utility values induced by alternative actions. Thus the action-based learning algorithms; e.g., better (or best) reply learning algorithm and adaptive play learning algorithm can not be employed to solve our coverage games. It motivates us to design distributed learning algorithms which only require the payoff received.

In this section, we come up with two distributed payoff-based learning algorithms, say *Distributed Inhomogeneous Synchronous Coverage Learning Algorithm* (DISCL, for short) and *Distributed Inhomogeneous Asynchronous Coverage Learning Algorithm* (DIACL, for short). We then present their convergence properties. Relevant algorithms include payoff-based learning algorithms proposed in [83][84].

6.4.1 Distributed Inhomogeneous Synchronous Coverage Learning Algorithm

For each $k \geq 1$ and $i \in V$, we define $\tau_i(k)$ as follows: $\tau_i(k) = k$ if $u_i(s(k)) \geq u_i(s(k-1))$, otherwise, $\tau_i(k) = k-1$. Here, $s_i(\tau_i(k))$ is the more successful action of agent i in last two steps. The DISCL algorithm is formally stated in the following table:

1: [Initialization:] At $k = 0$, all agents are uniformly placed in $\mathcal{Q}$. Each agent i uniformly chooses its camera control vector c_i from the set $\mathcal{C}$, communicates with agents in $\mathcal{N}_i^{\text{sen}}(s(0))$, and computes $u_i(s(0))$. At $k = 1$, all the agents keep their actions.

2: [Update:] At each time $k \geq 2$, each agent i updates its state according to the following rules:

- Agent i chooses the exploration rate $\epsilon(k) = t^{-\frac{1}{N(D+1)}}$ with D being the diameter of the location graph $\mathcal{G}_{\text{loc}}$, and computes $s_i(\tau_i(k))$.
- With probability $\epsilon(k)$, agent i experiments, and chooses the temporary action $s_i^{\text{tp}} := (a_i^{\text{tp}}, c_i^{\text{tp}})$ uniformly from the set $\mathcal{F}_i(a_i(k)) \setminus \{s_i(\tau_i(k))\}$.
- With probability $1 - \epsilon(k)$, agent i does not experiment, and sets $s_i^{\text{tp}} = s_i(\tau_i(k))$.
- After s_i^{tp} is chosen, agent i moves to the position a_i^{tp} and sets the camera control vector to c_i^{tp}.

3: [Communication and computation:] At position a_i^{tp}, each agent i sends the information $\mathcal{D}(a_i^{\text{tp}}, c_i^{\text{tp}}) \cap \mathcal{Q}$ to agents in $\mathcal{N}_i^{\text{sen}}(s_i^{\text{tp}}, s_{-i}^{\text{tp}})$. After that, each agent i identifies the quantity $n_q(s^{\text{tp}})$, for each $q \in \mathcal{D}(a_i^{\text{tp}}, c_i^{\text{tp}}) \cap \mathcal{Q}$, computes the utility $u_i(s_i^{\text{tp}}, s_{-i}^{\text{tp}})$ and the feasible action set of $\mathcal{F}_i(a_i^{\text{tp}})$.

4: Repeat Steps 2 and 3.

Remark 6.4.1 A variation of the DISCL algorithm corresponds to $\epsilon(k) = \epsilon \in (0, \frac{1}{2}]$ constant for all $k \geq 2$. If this is the case, we will refer to the algorithm as *Distributed Homogeneous Synchronous Coverage Learning Algorithm* (DHSCL, for

short). Later, the convergence analysis of the DISCL algorithm will be based on the analysis of the DHSCL algorithm. •

Denote the space $\mathcal{B} := \{(s, s') \in \mathcal{A} \times \mathcal{A} |\ s'_i \in \mathcal{F}_i(a_i),\ \forall i \in V\}$. Observe that $z(k) := (s(k-1), s(k))$ in the DISCL algorithm constitutes a time-inhomogeneous Markov chain $\{\mathcal{P}_k\}$ on the space $\mathcal{B}$. The following theorem implies that the DISCL algorithm asymptotically converges to the set of $\mathcal{E}(\Gamma_{\text{cov}})$ in probability.

Theorem 6.4.1 *Consider the Markov chain* $\{\mathcal{P}_k\}$ *induced by the* DISCL *Algorithm. It holds that* $\lim_{k\to+\infty} \mathbb{P}(z(k) \in \text{diag}(\mathcal{E}(\Gamma_{\text{cov}}))) = 1$.

The proofs of Theorem 6.4.1 are provided in Section 6.5.

Remark 6.4.2 An algorithm is proposed for the general class of weakly acyclic games (including potential games as special cases) in [84], and is able to find an NE with an arbitrarily high probability by choosing an arbitrarily small and fixed exploration rate ϵ in advance. However, it is difficult to derive an analytic relation between the convergent probability and the exploration rate. For the special case of identical interest games (all players share an identical utility function), the authors in [84] exploit a diminishing exploration rate and obtain a stronger result of convergence in probability. This motivates us to utilize a diminishing exploration rate in the DISCL algorithm which allows for the convergence to the set of NEs in probability. In the algorithm for weakly acyclic games in [84], each player may execute the baseline action which depends on all the past plays. As as result, the algorithm for weakly acyclic games in [84] cannot be utilized to solve our problem because the baseline action may not be feasible when the state-dependent constraints are present. It is worth mentioning that the paper [84] also investigates a case where the utility values are corrupted by noises. •

6.4.2 Distributed Inhomogeneous Asynchronous Coverage Learning Algorithm

Lemma 6.2.1 shows that the coverage game Γ_{cov} is a constrained exact potential game with potential function $\phi(s)$. However, this potential function is not a

straightforward measure of the network coverage performance. On the other hand, the objective function $U_g(s) := \sum_{i\in V} u_i(s)$ captures the trade-off between the overall network benefit from sensing and the total energy the network consumes, and thus can be perceived as a more natural coverage performance metric. Denote by $S^* := \{s|\ \text{argmax}_{s\in\mathcal{A}} U_g(s)\}$ as the set of global maximizers of $U_g(s)$. In this part, we present the DIACL algorithm which is convergent in probability to the set S^*.

Before that , we first introduce some notations for the DIACL algorithm. Denote by the space $\mathcal{B}'$ as follows:

$$\mathcal{B}' := \left\{(s, s') \in \mathcal{A}\times\mathcal{A}|\ \ s_{-i} = s'_{-i},\ s'_i \in \mathcal{F}_i(a_i) \text{ for some } i \in V\right\}.$$

For any $s^0, s^1 \in \mathcal{A}$ with $s^0_{-i} = s^1_{-i}$ for some $i \in V$, we denote

$$\Delta_i(s^1, s^0) := \frac{1}{2}\sum_{q\in\Omega_1}\frac{W_q}{n_q(s^1)} - \frac{1}{2}\sum_{q\in\Omega_2}\frac{W_q}{n_q(s^0)},$$

where $\Omega_1 := \mathcal{D}(a^1_i, c^1_i)\backslash\mathcal{D}(a^0_i, c^0_i)\cap\mathcal{Q}$ and $\Omega_2 := \mathcal{D}(a^0_i, c^0_i)\backslash\mathcal{D}(a^1_i, c^1_i)\cap\mathcal{Q}$, and

$$\begin{aligned}
&\rho_i(s^0, s^1) := u_i(s^1) - \Delta_i(s^1, s^0) - u_i(s^0) + \Delta_i(s^0, s^1),\\
&\Psi_i(s^0, s^1) := \max\{u_i(s^0) - \Delta_i(s^0, s^1), u_i(s^1) - \Delta_i(s^1, s^0)\},\\
&m^* := \max_{(s^0,s^1)\in\mathcal{B}, s^0_i\neq s^1_i}\{\Psi_i(s^0, s^1) - (u_i(s^0) - \Delta_i(s^0, s^1)), \frac{1}{2}\}.
\end{aligned}$$

It is easy to check that $\Delta_i(s^1, s^0) = -\Delta_i(s^0, s^1)$ and $\Psi_i(s^0, s^1) = \Psi_i(s^1, s^0)$. Assume that at each time instant, one of agents becomes active with equal probability. This can be realized by employing the asynchronous time model proposed in [20] where each node has a clock which ticks according to a rate 1 Poisson process. For this reason, we will refer the following algorithm to be asynchronous. Denote by $\gamma_i(k)$ the last time instant before t when agent i was active. We then denote $\gamma^{(2)}_i(k) := \gamma_i(\gamma_i(k))$. The main steps of the DIACL algorithm are described in the following.

1: [Initialization:] At $k = 0$, all agents are uniformly placed in $\mathcal{Q}$. Each agent i uniformly chooses the camera control vector c_i from the set $\mathcal{C}$, and then communicates with agents in $\mathcal{N}^{\text{sen}}_i(s(0))$ and computes $u_i(s(0))$. Furthermore, each agent i chooses $m_i \in (2m^*, Km^*]$ for some $K \geq 2$. At $k = 1$, all the sensors keep their actions.

2: [Update:] Assume that agent i is active at time $k \geq 2$. Then agent i updates its state according to the following rules:

• Agent i chooses the exploration rate $\epsilon(k) = k^{-\frac{1}{(D+1)(K+1)m^*}}$.

• With probability $\epsilon(k)^{m_i}$, agent i experiments and uniformly chooses $s_i^{\text{tp}} := (a_i^{\text{tp}}, c_i^{\text{tp}})$ from the action set $\mathcal{F}_i(a_i(k)) \setminus \{s_i(k), s_i(\gamma_i^{(2)}(k)+1)\}$.

• With probability $1-\epsilon(k)^{m_i}$, agent i does not experiment and chooses s_i^{tp} according to the following probability distribution:

$$\mathbb{P}(s_i^{\text{tp}} = s_i(k)) = \frac{1}{1+\epsilon(k)^{\rho_i(s_i(\gamma_i^{(2)}(k)+1), s_i(k))}},$$

$$\mathbb{P}(s_i^{\text{tp}} = s_i(\gamma_i^{(2)}(k)+1)) = \frac{\epsilon(k)^{\rho_i(s_i(\gamma_i^{(2)}(k)+1), s_i(k))}}{1+\epsilon(k)^{\rho_i(s_i(\gamma_i^{(2)}(k)+1), s_i(k))}}.$$

• After s_i^{tp} is chosen, agent i moves to the position a_i^{tp} and sets its camera control vector to be c_i^{tp}.

3: [Communication and computation:] At position a_i^{tp}, the active agent i initiates a message to agents in $\mathcal{N}_i^{\text{sen}}(s_i^{\text{tp}}, s_{-i}(k))$. Then each agent $j \in \mathcal{N}_i^{\text{sen}}(s_i^{\text{tp}}, s_{-i}(k))$ sends the information of $\mathcal{D}(a_j^{\text{tp}}, c_j^{\text{tp}}) \cap \mathcal{Q}$ to agent i. After receiving such information, agent i identifies the quantity $n_q(s_i^{\text{tp}}, s_{-i}(k))$ for each $q \in \mathcal{D}(a_i^{\text{tp}}, c_i^{\text{tp}}) \cap \mathcal{Q}$, computes the utility $u_i(s_i^{\text{tp}}, s_{-i}(k))$, $\Delta_i((s_i^{\text{tp}}, s_{-i}(k)), s(\gamma_i(k)+1))$, and the feasible action set of $\mathcal{F}_i(a_i^{\text{tp}})$.

4: Repeat Steps 2 and 3.

Remark 6.4.3 A variation of the DIACL algorithm corresponds to $\epsilon(k) = \epsilon \in (0, \frac{1}{2}]$ constant for all $k \geq 2$. If this is the case, we will refer to the algorithm as the *Distributed Homogeneous Asynchronous Coverage Learning Algorithm* (DHACL, for short). Later, we will base the convergence analysis of the DIACL algorithm on that of the DHACL algorithm. •

Like the DISCL algorithm, $z(k) := (s(t-1), s(k))$ in the DIACL algorithm constitutes a time-inhomogeneous Markov chain $\{\mathcal{P}_k\}$ on the space $\mathcal{B}'$. The following theorem implies that the DIACL algorithm asymptotically converges to the set of S^* with probability one.

Theorem 6.4.2 *Consider the Markov chain* $\{\mathcal{P}_k\}$ *induced by the* DIACL *algorithm for the game* Γ_{cov}. *Then it holds that* $\lim_{k\to+\infty} \mathbb{P}(z(k) \in \text{diag}(S^*)) = 1$.

The proofs of Theorem 6.4.2 are provided in Section 6.5.

Remark 6.4.4 A synchronous payoff-based, log-linear learning algorithm is proposed in [83] for potential games in which players aim to maximize the potential function of the game. As we mentioned before, the potential function is not suitable to act as a coverage performance metric. As opposed to [83], the DIACL algorithm instead seeks to optimize a different function $U_g(s)$ perceived as a natural network performance metric. Furthermore, the DIACL algorithm exploits a diminishing step-size, and this choice allows for convergence to the set of global optima in probability. On the other hand, convergence in [83] is to the set of NE with arbitrarily high probability. Theoretically, our result is stronger than that of [83] by choosing an arbitrarily small and fixed exploration rate in advance. •

6.5 Convergence Analysis

In this section, we prove Theorem 6.4.1 and 6.4.2 by appealing to the Theory of Resistance Trees in [154] and the results in strong ergodicity in [64]. Relevant papers include [83][84] where the Theory of Resistance Trees in [154] are novelly utilized to study the class of payoff-based learning algorithms, and [7][56][89] where the strong ergodicity theory is employed to characterize the convergence properties of time-inhomogeneous Markov chains.

6.5.1 Convergence analysis of the DISCL Algorithm

We first utilize Theorem 6.3.3 to characterize the convergence properties of the associated DHSCL algorithm. This is essential for the analysis of the DISCL algorithm.

Observe that $z(k) := (s(k-1), s(k))$ in the DHSCL algorithm constitutes a time-homogeneous Markov chain $\{\mathcal{P}_k^\epsilon\}$ on the space $\mathcal{B}$. Consider $z, z' \in \mathcal{B}$. A feasible path from z to z' consisting of multiple feasible transitions of $\{\mathcal{P}_t^\epsilon\}$ is denoted by $z \Rightarrow z'$. The reachable set from z is denoted as $\diamond z := \{z' \in \mathcal{B} \mid z \Rightarrow z'\}$.

Lemma 6.5.1 *$\{\mathcal{P}_k^\epsilon\}$ is a regular perturbation of $\{\mathcal{P}_k^0\}$.*

Proof: Consider a feasible transition $z^1 \to z^2$ with $z^1 := (s^0, s^1)$ and $z^2 := (s^1, s^2)$. Then we can define a partition of V as $\Lambda_1 := \left\{ i \in V |\ s_i^2 = s_i^{\tau_i(0,1)} \right\}$ and $\Lambda_2 := \left\{ i \in V |\ s_i^2 \in \mathcal{F}_i(a_i^1) \setminus \{ s_i^{\tau_i(0,1)} \} \right\}$. The corresponding probability is given by

$$P^{\epsilon}_{z^1 z^2} = \prod_{i \in \Lambda_1} (1 - \epsilon) \times \prod_{j \in \Lambda_2} \frac{\epsilon}{|\mathcal{F}_i(a_i^1)| - 1}. \tag{6.2}$$

Hence, the resistance of the transition $z^1 \to z^2$ is $|\Lambda_2| \in \{0, 1, \cdots, N\}$ since

$$0 < \lim_{\epsilon \to 0^+} \frac{P^{\epsilon}_{z^1 z^2}}{\epsilon^{|\Lambda_2|}} = \prod_{j \in \Lambda_2} \frac{1}{|\mathcal{F}_i(a_i^1)| - 1} < +\infty.$$

We have that (A3) in Section 6.3 holds. It is not difficult to see that (A2) holds, and we are now in a position to verify (A1). Since $\mathcal{G}_{\rm loc}$ is undirected and connected, and multiple sensors can stay in the same position, then $\diamond a^0 = \mathcal{Q}^N$ for any $a^0 \in \mathcal{Q}$. Since sensor i can choose any camera control vector from $\mathcal{C}$ at each time, then $\diamond s^0 = \mathcal{A}$ for any $s^0 \in \mathcal{A}$. It implies that $\diamond z^0 = \mathcal{B}$ for any $z^0 \in \mathcal{B}$, and thus the Markov chain $\{\mathcal{P}_k^{\epsilon}\}$ is irreducible on the space $\mathcal{B}$.

It is easy to see that any state in $\mathrm{diag}(\mathcal{A})$ has period 1. Pick any $(s^0, s^1) \in \mathcal{B} \setminus \mathrm{diag}(\mathcal{A})$. Since $\mathcal{G}_{\rm loc}$ is undirected, then $s_i^0 \in \mathcal{F}_i(a_i^1)$ if and only if $s_i^1 \in \mathcal{F}_i(a_i^0)$. Hence, the following two paths are both feasible:

$$(s^0, s^1) \to (s^1, s^0) \to (s^0, s^1)$$
$$(s^0, s^1) \to (s^1, s^1) \to (s^1, s^0) \to (s^0, s^1).$$

Hence, the period of the state (s^0, s^1) is 1. This proves aperiodicity of $\{\mathcal{P}_k^{\epsilon}\}$. Since $\{\mathcal{P}_k^{\epsilon}\}$ is irreducible and aperiodic, then (A1) holds. ■

Lemma 6.5.2 *For any $(s^0, s^0) \in \mathrm{diag}(\mathcal{A}) \setminus \mathrm{diag}(\mathcal{E}(\Gamma_{\rm cov}))$, there is a finite sequence of transitions from (s^0, s^0) to some $(s^*, s^*) \in \mathrm{diag}(\mathcal{E}(\Gamma_{\rm cov}))$ that satisfies*

$$\mathcal{L} := (s^0, s^0) \overset{O(\epsilon)}{\to} (s^0, s^1) \overset{O(1)}{\to} (s^1, s^1) \overset{O(\epsilon)}{\to} (s^1, s^2)$$
$$\overset{O(1)}{\to} (s^2, s^2) \overset{O(\epsilon)}{\to} \cdots \overset{O(\epsilon)}{\to} (s^{\tau-1}, s^{\tau}) \overset{O(1)}{\to} (s^{\tau}, s^{\tau})$$

where $(s^{\tau}, s^{\tau}) = (s^*, s^*)$ for some $\tau \geq 1$.

Proof: If $s^0 \notin \mathcal{E}(\Gamma_{\text{cov}})$, there exists a sensor i with a action $s_i^1 \in \mathcal{F}_i(a_i^0)$ such that $u_i(s^1) > u_i(s^0)$ where $s_{-i}^0 = s_{-i}^1$. The transition $(s^0, s^0) \to (s^0, s^1)$ happens when only sensor i experiments, and its corresponding probability is $(1-\epsilon)^{N-1} \times \frac{\epsilon}{|\mathcal{F}_i(a_i^0)|-1}$. Since the function ϕ is the potential function of the game Γ_{cov}, then we have that $\phi(s^1) - \phi(s^0) = u_i(s^1) - u_i(s^0)$ and thus $\phi(s^1) > \phi(s^0)$.

Since $u_i(s^1) > u_i(s^0)$ and $s_{-i}^0 = s_{-i}^1$, the transition $(s^0, s^1) \to (s^1, s^1)$ occurs when all sensors do not experiment, and the associated probability is $(1-\epsilon)^N$.

We repeat the above process and construct the path $\mathcal{L}$ with length $\tau \geq 1$. Since $\phi(s^i) > \phi(s^{i-1})$ for $i = \{1, \ldots, \tau\}$, then $s^i \neq s^j$ for $i \neq j$ and thus the path $\mathcal{L}$ has no loop. Since $\mathcal{A}$ is finite, then τ is finite and thus $s^\tau = s^* \in \mathcal{E}(\Gamma_{\text{cov}})$. ■

A direct result of Lemma 6.5.1 is that for each ϵ, there exists a unique stationary distribution of $\{\mathcal{P}_k^\epsilon\}$, say $\mu(\epsilon)$. We now proceed to utilize Theorem 6.3.3 to characterize $\lim_{\epsilon \to 0^+} \mu(\epsilon)$.

Proposition 6.5.1 *Consider the regular perturbation $\{\mathcal{P}_k^\epsilon\}$ of $\{\mathcal{P}_k^0\}$. Then $\lim_{\epsilon \to 0^+} \mu(\epsilon)$ exists and the limiting distribution $\mu(0)$ is a stationary distribution of $\{\mathcal{P}_k^0\}$. Furthermore, the stochastically stable states (i.e., the support of $\mu(0)$) are contained in the set* $\text{diag}(\mathcal{E}(\Gamma_{\text{cov}}))$.

Proof: Notice that the stochastically stable states are contained in the recurrent communication classes of the unperturbed Markov chain that corresponds to the DHSCL Algorithm with $\epsilon = 0$. Thus the stochastically stable states are included in the set $\text{diag}(\mathcal{A}) \subset \mathcal{B}$. Denote by $T_{\min}$ the minimum resistance tree and by h_v the root of $T_{\min}$. Each edge of $T_{\min}$ has resistance $0, 1, 2, \ldots$ corresponding to the transition probability $O(1), O(\epsilon), O(\epsilon^2), \ldots$. The state z' is the *successor* of the state z if and only if $(z, z') \in T_{\min}$. Like Theorem 3.2 in [84], our analysis will be slightly different from the presentation in 6.3. We will construct $T_{\min}$ over states in the set $\mathcal{B}$ (rather than $\text{diag}(\mathcal{A})$) with the restriction that all the edges leaving the states in $\mathcal{B} \setminus \text{diag}(\mathcal{A})$ have resistance 0. The stochastically stable states are not changed under this difference.

Claim 1 *For any $(s^0, s^1) \in \mathcal{B} \setminus \text{diag}(\mathcal{A})$, there is a finite path*

$$\mathcal{L}' := (s^0, s^1) \overset{O(1)}{\to} (s^1, s^2) \overset{O(1)}{\to} (s^2, s^2)$$

where $s_i^2 = s_i^{\tau_i(0,1)}$ *for all* $i \in V$.

Proof: These two transitions occur when all agents do not experiment. The corresponding probability of each transition is $(1-\epsilon)^N$. ■

Claim 2 *The root* h_v *belongs to the set* $\mathrm{diag}(\mathcal{A})$.

Proof: Suppose that $h_v = (s^0, s^1) \in \mathcal{B} \setminus \mathrm{diag}(\mathcal{A})$. By Claim 1, there is a finite path $\mathcal{L}' := (s^0, s^1) \overset{O(1)}{\to} (s^1, s^2) \overset{O(1)}{\to} (s^2, s^2)$. We now construct a new tree T' by adding the edges of the path $\mathcal{L}'$ into the tree $T_{\min}$ and removing the redundant edges. The total resistance of adding edges is 0. Observe that the resistance of the removed edge exiting from (s^2, s^2) in the tree $T_{\min}$ is at least 1. Hence, the resistance of T' is strictly lower than that of $T_{\min}$, and we get to a contradiction.■

Claim 3 *Pick any* $s^* \in \mathcal{E}(\Gamma_{\mathrm{cov}})$ *and consider* $z := (s^*, s^*)$, $z' := (s^*, \tilde{s})$ *where* $\tilde{s} \neq s^*$. *If* $(z, z') \in T_{\min}$, *then the resistance of the edge* (z, z') *is some* $\tau \geq 2$.

Proof: Suppose the deviator in the transition $z \to z'$ is unique, say i. Then the corresponding transition probability is $O(\epsilon)$. Since $s^* \in \mathcal{E}(\Gamma_{\mathrm{cov}})$ and $\tilde{s}_i \in \mathcal{F}_i(a_i^*)$, we have that $u_i(s_i^*, s_{-i}^*) \geq u_i(\tilde{s}_i, \tilde{s}_{-i})$, where $s_{-i}^* = \tilde{s}_{-i}$.

Since $z' \in \mathcal{B} \setminus \mathrm{diag}(\mathcal{A})$, it follows from Claim 2 that the state z' can not be the root of $T_{\min}$ and thus has a successor z''. Note that all the edges leaving the states in $\mathcal{B} \setminus \mathrm{diag}(\mathcal{A})$ have resistance 0. Then none experiments in the transition $z' \to z''$ and $z'' = (\tilde{s}, \hat{s})$ for some $\hat{s}$. Since $u_i(s_i^*, s_{-i}^*) \geq u_i(\tilde{s}_i, \tilde{s}_{-i})$ with $s_{-i}^* = \tilde{s}_{-i}$, we have $\hat{s} = s^*$ and thus $z'' = (\tilde{s}, s^*)$. Similarly, the state z'' must have a successor z''' and $z''' = z$. We then obtain a loop in $T_{\min}$ which contradicts that $T_{\min}$ is a tree.

It implies that at least two sensors experiment in the transition $z \to z'$. Thus the resistance of the edge (z, z') is at least 2. ■

Claim 4 *The root* h_v *belongs to the set* $\mathrm{diag}(\mathcal{E}(\Gamma_{\mathrm{cov}}))$.

Proof: Suppose that $h_v = (s^0, s^0) \notin \mathrm{diag}(\mathcal{E}(\Gamma_{\mathrm{cov}}))$. By Lemma 6.5.2, there is a finite path $\mathcal{L}$ connecting (s^0, s^0) and some $(s^*, s^*) \in \mathrm{diag}(\mathcal{E}(\Gamma_{\mathrm{cov}}))$. We now

construct a new tree T' by adding the edges of the path $\mathcal{L}$ into the tree $T_{\min}$ and removing the edges that leave the states in $\mathcal{L}$ in the tree $T_{\min}$. The total resistance of adding edges is τ. Observe that the resistance of the removed edge exiting from (s^i, s^i) in the tree $T_{\min}$ is at least 1 for $i \in \{1, \cdots, \tau-1\}$. By Claim 3, the resistance of the removed edge leaving from (s^*, s^*) in the tree $T_{\min}$ is at least 2. The total resistance of removing edges is at least $\tau+1$. Hence, the resistance of T' is strictly lower than that of $T_{\min}$, and we get to a contradiction. ■

It follows from Claim 4 that the states in $\text{diag}(\mathcal{E}(\Gamma_{\text{cov}}))$ have minimum stochastic potential. Since Lemma 6.5.1 shows that Markov chain $\{\mathcal{P}_\tau^\epsilon\}$ is a regularly perturbed Markov process, Proposition 6.5.1 is a direct result of Theorem 6.3.3. ■

We are now ready to show Theorem 6.4.1.

Proof of Theorem 6.4.1:

Claim 5 *Condition (B2) in Theorem 6.3.2 holds.*

Proof: For each $k \geq 0$ and each $z \in X$, we define the numbers

$$\sigma_z(\epsilon(k)) := \sum_{T \in G(z)} \prod_{(x,y)\in T} P_{xy}^{\epsilon(k)}, \quad \sigma_z^k = \sigma_z(\epsilon(k))$$
$$\mu_z(\epsilon(k)) := \frac{\sigma_z(\epsilon(k))}{\sum_{x\in X} \sigma_x(\epsilon(k))}, \quad \mu_z^k = \mu_z(\epsilon(k)).$$

Since $\{\mathcal{P}_k^\epsilon\}$ is a regular perturbation of $\{\mathcal{P}_k^0\}$, then it is irreducible and thus $\sigma_z^k > 0$. As Lemma 3.1 of Chapter 6 in [49], one can show that $(\mu^k)^T P^{\epsilon(k)} = (\mu^k)^T$. Therefore, condition (B2) in Theorem 6.3.2 holds. ■

Claim 6 *Condition (B3) in Theorem 6.3.2 holds.*

Proof: We now proceed to verify condition (B3) in Theorem 6.3.2. To do that, let us first fix k, denote $\epsilon = \epsilon(k)$ and study the monotonicity of $\mu_z(\epsilon)$ with respect to ϵ. We write $\sigma_z(\epsilon)$ in the following form

$$\sigma_z(\epsilon) = \sum_{T \in G(z)} \prod_{(x,y)\in T} P_{xy}^\epsilon = \sum_{T \in G(z)} \prod_{(x,y)\in T} \frac{\alpha_{xy}(\epsilon)}{\beta_{xy}(\epsilon)} = \frac{\alpha_z(\epsilon)}{\beta_z(\epsilon)} \tag{6.3}$$

for some polynomials $\alpha_z(\epsilon)$ and $\beta_z(\epsilon)$ in ϵ. With (6.3) in hand, we have that $\sum_{x\in X}\sigma_x(\epsilon)$ and thus $\mu_z(\epsilon)$ are ratios of two polynomials in ϵ; i.e., $\mu_z(\epsilon)=\frac{\varphi_z(\epsilon)}{\beta(\epsilon)}$ where $\varphi_z(\epsilon)$ and $\beta(\epsilon)$ are polynomials in ϵ. The derivative of $\mu_z(\epsilon)$ is given by

$$\frac{\partial\mu_z(\epsilon)}{\partial\epsilon}=\frac{1}{\beta(\epsilon)^2}(\frac{\partial\varphi_z(\epsilon)}{\partial\epsilon}\beta(\epsilon)-\varphi_z(\epsilon)\frac{\partial\beta(\epsilon)}{\partial\epsilon}).$$

Note that the numerator $\frac{\partial\varphi_z(\epsilon)}{\partial\epsilon}\beta(\epsilon)-\varphi_z(\epsilon)\frac{\partial\beta(\epsilon)}{\partial\epsilon}$ is a polynomial in ϵ. Denote by $\iota_z\neq 0$ the coefficient of the leading term of $\frac{\partial\varphi_z(\epsilon)}{\partial\epsilon}-\varphi_z(\epsilon)\frac{\partial\beta(\epsilon)}{\epsilon}$. The leading term dominates $\frac{\partial\varphi_z(\epsilon)}{\partial\epsilon}-\varphi_z(\epsilon)\frac{\partial\beta(\epsilon)}{\epsilon}$ when ϵ is sufficiently small. Thus there exists $\epsilon_z>0$ such that the sign of $\frac{\partial\mu_z(\epsilon)}{\partial\epsilon}$ is the sign of ι_z for all $0<\epsilon\leq\epsilon_z$. Let $\epsilon^*=\max_{z\in X}\epsilon_z$.

Since $\epsilon(k)$ strictly decreases to zero, then there is a unique finite time instant k^* such that $\epsilon(k^*)=\epsilon^*$ (if $\epsilon(0)<\epsilon^*$, then $k^*=0$). Since $\epsilon(k)$ is strictly decreasing, we can define a partition of X as follows:

$$\Xi_1:=\{z\in X\mid \mu_z(\epsilon(k))>\mu_z(\epsilon(t+1)),\quad \forall t\in[k^*,+\infty)\},$$
$$\Xi_2:=\{z\in X\mid \mu_z(\epsilon(k))<\mu_z(\epsilon(t+1)),\quad \forall t\in[k^*,+\infty)\}.$$

We are now ready to verify (B3) of Theorem 6.3.2. Since $\{\mathcal{P}_k^\epsilon\}$ is a regular perturbed Markov chain of $\{\mathcal{P}_t^0\}$, it follows from Theorem 6.3.3 that $\lim_{t\to+\infty}\mu_z(\epsilon(k))=\mu_z(0)$, and thus it holds that

$$\begin{aligned}
&\sum_{k=0}^{+\infty}\sum_{z\in X}\|\mu_z^k-\mu_z^{k+1}\|=\sum_{k=0}^{+\infty}\sum_{z\in X}|\mu_z(\epsilon(k))-\mu_z(\epsilon(k+1))|\\
&=\sum_{k=0}^{k^*}\sum_{z\in X}|\mu_z(\epsilon(k))-\mu_z(\epsilon(k+1))|+\sum_{k=k^*+1}^{+\infty}(\sum_{z\in\Xi_1}\mu_z(\epsilon(k))-\sum_{z\in\Xi_1}\mu_z(\epsilon(k+1)))\\
&+\sum_{k=k^*+1}^{+\infty}(1-\sum_{z\in\Xi_1}\mu_z(\epsilon(k+1))-(1-\sum_{z\in\Xi_1}\mu_z(\epsilon(k))))\\
&=\sum_{k=0}^{k^*}\sum_{z\in X}|\mu_z(\epsilon(k))-\mu_z(\epsilon(k+1))|+2\sum_{z\in\Xi_1}\mu_z(\epsilon(k^*+1))-2\sum_{z\in\Xi_1}\mu_z(0)<+\infty.
\end{aligned}$$

■

Claim 7 *Condition (B1) in Theorem 6.3.2 holds.*

Proof: Denote by $P^{\epsilon(k)}$ the transition matrix of $\{\mathcal{P}_k\}$. As in (6.2), the probability of the feasible transition $z^1 \to z^2$ is given by

$$P^{\epsilon(k)}_{z^1 z^2} = \prod_{i\in\Lambda_1}(1-\epsilon(k)) \times \prod_{j\in\Lambda_2} \frac{\epsilon(k)}{|\mathcal{F}_i(a_i^1)|-1}.$$

Observe that $|\mathcal{F}_i(a_i^1)| \le 5|\mathcal{C}|$. Since $\epsilon(k)$ is strictly decreasing, there is $t_0 \ge 1$ such that t_0 is the first time when $1-\epsilon(k) \ge \frac{\epsilon(k)}{5|\mathcal{C}|-1}$. Then for all $k \ge t_0$, it holds that

$$P^{\epsilon(k)}_{z^1 z^2} \ge (\frac{\epsilon(k)}{5|\mathcal{C}|-1})^N.$$

Denote $P(m,n) := \prod_{k=m}^{n-1} P^{\epsilon(k)}$, $0 \le m < n$. Pick any $z \in \mathcal{B}$ and let $u_z \in \mathcal{B}$ be such that $P_{u_z z}(k, k+D+1) = \min_{x\in\mathcal{B}} P_{xz}(k, k+D+1)$. Consequently, it follows that for all $k \ge t_0$,

$$\begin{aligned}
&\min_{x\in\mathcal{B}} P_{xz}(k,k+D+1) = \sum_{i_1\in\mathcal{B}} \cdots \sum_{i_D\in\in\mathcal{B}} P^{\epsilon(k)}_{u_z i_1} \cdots P^{\epsilon(k+D-1)}_{i_{D-1} i_D} P^{\epsilon(k+D)}_{i_D z} \\
&\ge P^{\epsilon(k)}_{u_z i_1} \cdots P^{\epsilon(k+D-1)}_{i_{D-1} i_D} P^{\epsilon(k+D)}_{i_D z} \ge \prod_{i=0}^{D} (\frac{\epsilon(k+i)}{5|\mathcal{C}|-1})^N \ge (\frac{\epsilon(k)}{5|\mathcal{C}|-1})^{(D+1)N}
\end{aligned}$$

where in the last inequality we use that $\epsilon(k)$ is strictly decreasing. Then we have

$$\begin{aligned}
&1-\lambda(P(k,k+D+1)) = \min_{x,y\in\mathcal{B}} \sum_{z\in\mathcal{B}} \min\{P_{xz}(k,k+D+1), P_{yz}(k,k+D+1)\} \\
&\ge \sum_{z\in\mathcal{B}} P_{u_z z}(k,k+D+1) \ge |\mathcal{B}| (\frac{\epsilon(k)}{5|\mathcal{C}|-1})^{(D+1)N}.
\end{aligned}$$

Choose $k_i := (D+1)i$ and let i_0 be the smallest integer such that $(D+1)i_0 \ge t_0$. Then, we have that:

$$\begin{aligned}
&\sum_{i=0}^{+\infty} (1-\lambda(P(k_i,k_{i+1}))) \ge |\mathcal{B}| \sum_{i=i_0}^{+\infty} (\frac{\epsilon((D+1)i)}{5|\mathcal{C}|-1})^{(D+1)N} \\
&= \frac{|\mathcal{B}|}{(5|\mathcal{C}|-1)^{(D+1)N}} \sum_{i=i_0}^{+\infty} \frac{1}{(D+1)i} = +\infty. \qquad (6.4)
\end{aligned}$$

Hence, the weak ergodicity property follows from Theorem 6.3.1. ■

All the conditions in Theorem 6.3.2 hold. Thus it follows from Theorem 6.3.2 that the limiting distribution is $\mu^* = \lim_{k\to+\infty} \mu^k$. Note that $\lim_{k\to+\infty} \mu^k =$

$\lim_{k\to+\infty}\mu(\epsilon(k)) = \mu(0)$ and Proposition 6.5.1 shows that the support of $\mu(0)$ is contained in the set $\mathrm{diag}(\mathcal{E}(\Gamma_{\mathrm{cov}}))$. Hence, the support of μ^* is contained in the set $\mathrm{diag}(\mathcal{E}(\Gamma_{\mathrm{cov}}))$, implying that $\lim_{t\to+\infty}\mathbb{P}(z(k) \in \mathrm{diag}(\mathcal{E}(\Gamma_{\mathrm{cov}}))) = 1$. It completes the proof.

6.5.2 Convergence analysis of the DIACL Algorithm

First of all, we employ Theorem 6.3.3 to study the convergence properties of the associated DHACL algorithm. This is essential to analyze the DIACL algorithm.

To simplify notations, we will use $s_i(k-1) := s_i(\gamma_i^{(2)}(k)+1)$ in the remainder of this section. Observe that $z(k) := (s(k-1), s(k))$ in the DHACL algorithm constitutes a Markov chain $\{\mathcal{P}_k^\epsilon\}$ on the space $\mathcal{B}'$.

Lemma 6.5.3 *The Markov chain $\{\mathcal{P}_k^\epsilon\}$ is a regular perturbation of $\{\mathcal{P}_k^0\}$.*

Proof: Pick any two states $z^1 := (s^0, s^1)$ and $z^2 := (s^1, s^2)$ with $z^1 \neq z^2$. We have that $P^\epsilon_{z^1z^2} > 0$ if and only if there is some $i \in V$ such that $s^1_{-i} = s^2_{-i}$ and one of the following occurs: $s_i^2 \in \mathcal{F}_i(a_i^1)\setminus\{s_i^0, s_i^1\}$, $s_i^2 = s_i^1$ or $s_i^2 = s_i^0$. In particular, the following holds:

$$P^\epsilon_{z^1z^2} = \begin{cases} \eta_1, & s_i^2 \in \mathcal{F}_i(a_i^1)\setminus\{s_i^0, s_i^1\}, \\ \eta_2, & s_i^2 = s_i^1, \\ \eta_3, & s_i^2 = s_i^0, \end{cases}$$

where

$$\eta_1 := \frac{\epsilon^{m_i}}{N|\mathcal{F}_i(a_i^1)\setminus\{s_i^0, s_i^1\}|}, \quad \eta_2 := \frac{1-\epsilon^{m_i}}{N(1+\epsilon^{\rho_i(s^0,s^1)})}, \quad \eta_3 := \frac{(1-\epsilon^{m_i})\times\epsilon^{\rho_i(s^0,s^1)}}{N(1+\epsilon^{\rho_i(s^0,s^1)})}.$$

Observe that $0 < \lim_{\epsilon\to0^+}\frac{\eta_1}{\epsilon^{m_i}} < +\infty$. Multiplying the numerator and denominator of η_2 by $\epsilon^{\Psi_i(s^1,s^0)-(u_i(s^1)-\Delta_i(s^1,s^0))}$, we obtain

$$\eta_2 = \frac{1-\epsilon^{m_i}}{N} \times \frac{\epsilon^{\Psi_i(s^0,s^1)-(u_i(s^1)-\Delta_i(s^1,s^0))}}{\eta_2'},$$

where $\eta_2' := \epsilon^{\Psi_i(s^0,s^1)-(u_i(s^1)-\Delta_i(s^1,s^0))} + \epsilon^{\Psi_i(s^0,s^1)-(u_i(s^0)-\Delta_i(s^0,s^1))}$. Use

$$\lim_{\epsilon\to 0^+} \epsilon^x = \begin{cases} 1, & x = 0, \\ 0, & x > 0, \end{cases}$$

and we have

$$\lim_{\epsilon\to 0^+} \frac{\eta_2}{\epsilon^{\Psi_i(s^0,s^1)-(u_i(s^1)-\Delta_i(s^1,s^0))}} = \begin{cases} \frac{1}{N}, & u_i(s^0) - \Delta_i(s^0,s^1) \neq u_i(s^1) - \Delta_i(s^1,s^0), \\ \frac{1}{2N}, & \text{otherwise.} \end{cases}$$

Similarly, it holds that

$$\lim_{\epsilon\to 0^+} \frac{\eta_3}{\epsilon^{\Psi_i(s^0,s^1)-(u_i(s^0)-\Delta_i(s^0,s^1))}} \in \{\frac{1}{2N}, \frac{1}{N}\}.$$

Hence, the resistance of the feasible transition $z^1 \to z^2$, with $z^1 \neq z^2$ and sensor i as the unilateral deviator, can be described as follows:

$$\chi(z^1 \to z^2) = \begin{cases} m_i, & s_i^2 \in \mathcal{F}_i(a^1) \setminus \{s_i^0, s_i^1\}, \\ \Psi_i(s^0,s^1) - (u_i(s^1) - \Delta_i(s^1,s^0)), & s_i^2 = s_i^1, \\ \Psi_i(s^0,s^1) - (u_i(s^0) - \Delta_i(s^0,s^1)), & s_i^2 = s_i^0. \end{cases}$$

Then (A3) in Section 6.3 holds. It is straightforward to verify that (A2) in Section 6.3 holds. We are now in a position to verify (A1). Since $\mathcal{G}_{\text{loc}}$ is undirected and connected, and multiple sensors can stay in the same position, then $\diamond a^0 = \mathcal{Q}^N$ for any $a^0 \in \mathcal{Q}$. Since sensor i can choose any camera control vector from $\mathcal{C}$ at each time, then $\diamond s^0 = \mathcal{A}$ for any $s^0 \in \mathcal{A}$. This implies that $\diamond z^0 = \mathcal{B}'$ for any $z^0 \in \mathcal{B}'$, and thus the Markov chain $\{\mathcal{P}_t^\epsilon\}$ is irreducible on the space $\mathcal{B}'$.

It is easy to see that any state in $\text{diag}(\mathcal{A})$ has period 1. Pick any $(s^0, s^1) \in \mathcal{B}' \setminus \text{diag}(\mathcal{A})$. Since $\mathcal{G}_{\text{loc}}$ is undirected, then $s_i^0 \in \mathcal{F}_i(a_i^1)$ if and only if $s_i^1 \in \mathcal{F}_i(a_i^0)$. Hence, the following two paths are both feasible:

$$(s^0,s^1) \to (s^1,s^0) \to (s^0,s^1)$$
$$(s^0,s^1) \to (s^1,s^1) \to (s^1,s^0) \to (s^0,s^1).$$

Hence, the period of the state (s^0, s^1) is 1. This proves aperiodicity of $\{\mathcal{P}_t^\epsilon\}$. Since $\{\mathcal{P}_t^\epsilon\}$ is irreducible and aperiodic, then (A1) holds. ■

A direct result of Lemma 6.5.3 is that for each $\epsilon > 0$, there exists a unique stationary distribution of $\{\mathcal{P}_k^\epsilon\}$, say $\mu(\epsilon)$. From the proof of Lemma 6.5.3, we can see that the resistance of an experiment is m_i if sensor i is the unilateral deviator. We now proceed to utilize Theorem 6.3.3 to characterize $\lim_{\epsilon\to 0^+} \mu(\epsilon)$.

Proposition 6.5.2 *Consider the regular perturbed Markov process* $\{\mathcal{P}_k^\epsilon\}$. *Then* $\lim_{\epsilon\to 0^+} \mu(\epsilon)$ *exists and the limiting distribution* $\mu(0)$ *is a stationary distribution of* $\{\mathcal{P}_t^0\}$. *Furthermore, the stochastically stable states (i.e., the support of* $\mu(0)$*) are contained in the set* $\mathrm{diag}(S^*)$.

Proof: The unperturbed Markov chain corresponds to the DHACL Algorithm with $\epsilon = 0$. Hence, the recurrent communication classes of the unperturbed Markov chain are contained in the set $\mathrm{diag}(\mathcal{A})$. We will construct resistance trees over vertices in the set $\mathrm{diag}(\mathcal{A})$. Denote $T_{\min}$ by the minimum resistance tree. The remainder of the proof is divided into the following four claims.

Claim 8 $\chi((s^0, s^0) \Rightarrow (s^1, s^1)) = m_i + \Psi_i(s^1, s^0) - (u_i(s^1) - \Delta_i(s^1, s^0))$ *where* $s^0 \neq s^1$ *and the transition* $s^0 \to s^1$ *is feasible with sensor* i *as the unilateral deviator.*

Proof: One feasible path for $(s^0, s^0) \Rightarrow (s^1, s^1)$ is $\mathcal{L} := (s^0, s^0) \to (s^0, s^1) \to (s^1, s^1)$ where sensor i experiments in the first transition and does not experiment in the second one. The total resistance of the path $\mathcal{L}$ is $m_i + \Psi_i(s^1, s^0) - (u_i(s^1) - \Delta_i(s^1, s^0))$ which is at most $m_i + m^*$.

Denote by $\mathcal{L}'$ the path with minimum resistance among all the feasible paths for $(s^0, s^0) \Rightarrow (s^1, s^1)$. Assume that the first transition in $\mathcal{L}'$ is $(s^0, s^0) \to (s^0, s^2)$ where node j experiments and $s^2 \neq s^1$. Observe that the resistance of $(s^0, s^0) \to (s^0, s^2)$ is m_j. No matter whether j is equal to i or not, the path $\mathcal{L}'$ must include at least one more experiment to introduce s_i^1. Hence the total resistance of the path $\mathcal{L}'$ is at least $m_i + m_j$. Since $m_i + m_j > m_i + 2m^*$, then the path $\mathcal{L}'$ has a strictly larger resistance than the path $\mathcal{L}$. To avoid a contradiction, the path $\mathcal{L}'$ must start from the transition $(s^0, s^0) \to (s^0, s^1)$. Similarly, the sequent transition (which is also the last one) in the path $\mathcal{L}'$ must be $(s^0, s^1) \to (s^1, s^1)$ and thus $\mathcal{L}' = \mathcal{L}$.

Hence, the resistance of the transition $(s^0, s^0) \Rightarrow (s^1, s^1)$ is the total resistance of the path $\mathcal{L}$; i.e., $m_i + \Psi_i(s^1, s^0) - (u_i(s^1) - \Delta_i(s^1, s^0))$. ■

Claim 9 *All the edges $((s, s), (s', s'))$ in $T_{\min}$ must consist of only one deviator; i.e., $s_i \neq s'_i$ and $s_{-i} = s'_{-i}$ for some $i \in V$.*

Proof: Assume that $(s, s) \Rightarrow (s', s')$ has at least two deviators. Suppose the path $\hat{\mathcal{L}}$ has the minimum resistance among all the paths from (s, s) to (s', s'). Then, $\ell \geq 2$ experiments are carried out along $\hat{\mathcal{L}}$. Denote by i_τ the unilateral deviator in the τ-th experiment $s^{\tau-1} \to s^\tau$ where $1 \leq \tau \leq \ell$, $s^0 = s$ and $s^\ell = s'$. Then the resistance of $\hat{\mathcal{L}}$ is at least $\sum_{\tau=1}^{\ell} m_{i_\tau}$; i.e., $\chi((s^0, s^0) \Rightarrow (s', s')) \geq \sum_{\tau=1}^{\ell} m_{i_\tau}$.

Let us consider the following path on $T_{\min}$:

$$\bar{\mathcal{L}} := (s^0, s^0) \Rightarrow (s^1, s^1) \Rightarrow \cdots \Rightarrow (s^\ell, s^\ell).$$

From Claim 1, we know that the total resistance of the path $\bar{\mathcal{L}}$ is at most $\sum_{\tau=1}^{\ell} m_{i_\tau} + \ell m^*$.

A new tree T' can be obtained by adding the edges of $\bar{\mathcal{L}}$ into $T_{\min}$ and removing the redundant edges. The removed resistance is *strictly* greater than $\sum_{\tau=1}^{\ell} m_{i_\tau} + 2(\ell - 1)m^*$ where $\sum_{\tau=1}^{\ell} m_{i_\tau}$ is the lower bound on the resistance on the edge from (s^0, s^0) to (s^ℓ, s^ℓ), and $2(\ell - 1)m^*$ is the strictly lower bound on the total resistances of leaving (s^τ, s^τ) for $\tau = 1, \cdots, \ell - 1$. The adding resistance is the total resistance of $\bar{\mathcal{L}}$ which is at most $\sum_{\tau=1}^{\ell} m_{i_\tau} + \ell m^*$. Since $\ell \geq 2$, we have that $2(\ell - 1)m^* \geq \ell m^*$ and thus T' has a strictly lower resistance than $T_{\min}$. This contradicts the fact that $T_{\min}$ is a minimum resistance tree. ■

Claim 10 *Given any edge $((s, s), (s', s'))$ in $T_{\min}$, denote by i the unilateral deviator between s and s'. Then the transition $s_i \to s'_i$ is feasible.*

Proof: Assume that the transition $s_i \to s'_i$ is infeasible. Suppose the path $\check{\mathcal{L}}$ has the minimum resistance among all the paths from (s, s) to (s', s'). Then, there are $\ell \geq 2$ experiments in $\check{\mathcal{L}}$. The remainder of the proof is similar to that of Claim 9. ■

Claim 11 *Let h_v be the root of $T_{\min}$. Then, $h_v \in \text{diag}(S^*)$.*

Proof: Assume that $h_v = (s^0, s^0) \notin \mathrm{diag}(S^*)$. Pick any $(s^*, s^*) \in \mathrm{diag}(S^*)$. By Claim 9 and 10, we have that there is a path from (s^*, s^*) to (s^0, s^0) in the tree $T_{\min}$ as follows:

$$\tilde{\mathcal{L}} := (s^\ell, s^\ell) \Rightarrow (s^{\ell-1}, s^{\ell-1}) \Rightarrow \cdots \Rightarrow (s^1, s^1) \Rightarrow (s^0, s^0)$$

for some $\ell \geq 1$. Here, $s^* = s^\ell$, there is only one deviator, say i_τ, from s^τ to $s^{\tau-1}$, and the transition $s^\tau \to s^{\tau-1}$ is feasible for $\tau = \ell, \ldots, 1$.

Since the transition $s^\tau \to s^{\tau+1}$ is also feasible for $\tau = 0, \ldots, \ell-1$, we obtain the reverse path $\tilde{\mathcal{L}}'$ of $\tilde{\mathcal{L}}$ as follows:

$$\tilde{\mathcal{L}}' := (s^0, s^0) \Rightarrow (s^1, s^1) \Rightarrow \cdots \Rightarrow (s^{\ell-1}, s^{\ell-1}) \Rightarrow (s^\ell, s^\ell).$$

By Claim 8, the total resistance of the path $\tilde{\mathcal{L}}$ is

$$\chi(\tilde{\mathcal{L}}) = \sum_{\tau=1}^{\ell} m_{i_\tau} + \sum_{\tau=1}^{\ell} \{\Psi_{i_\tau}(s^\tau, s^{\tau-1}) - (u_{i_\tau}(s^{\tau-1}) - \Delta_{i_\tau}(s^{\tau-1}, s^\tau))\},$$

and the total resistance of the path $\tilde{\mathcal{L}}'$ is

$$\chi(\tilde{\mathcal{L}}') = \sum_{k=1}^{\ell} m_{i_\tau} + \sum_{\tau=1}^{\ell} \Psi_{i_\tau}(s^{\tau-1}, s^\tau) - (u_{i_\tau}(s^\tau) - \Delta_{i_\tau}(s^\tau, s^{\tau-1})).$$

We make the following notations:

$$\Lambda_1' := (\mathcal{D}(a_{i_\tau}^\tau, r_{i_\tau}^\tau) \backslash \mathcal{D}(a_{i_{\tau-1}}^{\tau-1}, r_{i_{\tau-1}}^{\tau-1})) \cap \mathcal{Q}, \quad \Lambda_2' := (\mathcal{D}(a_{i_{\tau-1}}^{\tau-1}, r_{i_{\tau-1}}^{\tau-1}) \backslash \mathcal{D}(a_{i_\tau}^\tau, r_{i_\tau}^\tau)) \cap \mathcal{Q}.$$

Observe that

$$\begin{aligned}
&U_g(s^\tau) - U_g(s^{\tau-1}) \\
&= u_{i_\tau}(s^\tau) - u_{i_\tau}(s^{\tau-1}) - \sum_{q \in \Lambda_1'} W_q\left(\frac{n_q(s^{\tau-1})}{n_q(s^{\tau-1})} - \frac{n_q(s^{\tau-1})}{n_q(s^\tau)}\right) \\
&+ \sum_{q \in \Lambda_2'} W_q\left(\frac{n_q(s^\tau)}{n_q(s^\tau)} - \frac{n_q(s^\tau)}{n_q(s^{\tau-1})}\right) \\
&= (u_{i_\tau}(s^\tau) - \Delta_{i_\tau}(s^\tau, s^{\tau-1})) - (u_{i_\tau}(s^{\tau-1}) - \Delta_{i_\tau}(s^{\tau-1}, s^\tau)).
\end{aligned}$$

We now construct a new tree T' with the root (s^*, s^*) by adding the edges of $\tilde{\mathcal{L}}'$ to the tree $T_{\min}$ and removing the redundant edges $\tilde{\mathcal{L}}$. Since $\Psi_{i_\tau}(s^{\tau-1}, s^\tau) =$

$\Psi_{i_\tau}(s^\tau, s^{\tau-1})$, the difference in the total resistances across the trees $\chi(T')$ and $\chi(T_{\min})$ is given by

$$\begin{aligned}
&\chi(T') - \chi(T_{\min}) = \chi(\tilde{\mathcal{L}}') - \chi(\tilde{\mathcal{L}}) \\
&= \sum_{\tau=1}^{\ell} -(u_{i_\tau}(s^{\tau-1}) - \Delta_{i_\tau}(s^{\tau-1}, s^\tau)) - \sum_{\tau=1}^{\ell} -(u_{i_\tau}(s^\tau) - \Delta_{i_\tau}(s^\tau, s^{\tau-1})) \\
&= \sum_{\tau=1}^{\ell} (U_g(s^\tau) - U_g(s^{\tau-1})) = U_g(s^0) - U_g(s^*) < 0.
\end{aligned}$$

This contradicts that $T_{\min}$ is a minimum resistance tree. ■

It follows from Claim 4 that the state $h_v \in \text{diag}(S^*)$ has minimum stochastic potential. Then Proposition 6.5.2 is a direct result of Theorem 6.3.3. ■

We are now ready to show Theorem 6.4.2.

Proof of Theorem 6.4.1:

Claim 12 *Condition (B2) in Theorem 6.3.2 holds.*

Proof: The proof is analogous to Claim 5. ■

Claim 13 *Condition (B3) in Theorem 6.3.2 holds.*

Proof: Denote by $P^{\epsilon(k)}$ the transition matrix of $\{\mathcal{P}_k\}$. Consider the feasible transition $z^1 \to z^2$ with unilateral deviator i. The corresponding probability is given by

$$P^{\epsilon(k)}_{z^1 z^2} = \begin{cases} \eta_1, & s_i^2 \in \mathcal{F}_i(a_i^1) \setminus \{s_i^0, s_i^1\}, \\ \eta_2, & s_i^2 = s_i^1, \\ \eta_3, & s_i^2 = s_i^0, \end{cases}$$

where

$$\begin{aligned}
\eta_1 &:= \frac{\epsilon(k)^{m_i}}{N|\mathcal{F}_i(a_i^1) \setminus \{s_i^0, s_i^1\}|}, \quad \eta_2 := \frac{1 - \epsilon(k)^{m_i}}{N(1 + \epsilon(k)^{\rho_i(s^0, s^1)})}, \\
\eta_3 &:= \frac{(1 - \epsilon(k)^{m_i}) \times \epsilon(k)^{\rho_i(s^0, s^1)}}{N(1 + \epsilon(k)^{\rho_i(s^0, s^1)})}.
\end{aligned}$$

The remainder is analogous to Claim 6. ■

Claim 14 *Condition (B1) in Theorem 6.3.2 holds.*

Proof: Observe that $|\mathcal{F}_i(a_i^1)| \leq 5|\mathcal{C}|$. Since $\epsilon(k)$ is strictly decreasing, there is $t_0 \geq 1$ such that t_0 is the first time when $1-\epsilon(k)^{m_i} \geq \epsilon(k)^{m_i}$.

Observe that for all $t \geq 1$, it holds that

$$\eta_1 \geq \frac{\epsilon(k)^{m_i}}{N(5|\mathcal{C}|-1)} \geq \frac{\epsilon(k)^{m_i+m^*}}{N(5|\mathcal{C}|-1)}.$$

Denote $b := u_i(s^1) - \Delta_i(s^1, s^0)$ and $a := u_i(s^0) - \Delta_i(s^0, s^1)$. Then $\rho_i(s^0, s^1) = b - a$. Since $b - a \leq m^*$, then for $k \geq t_0$ it holds that

$$\begin{aligned}\eta_2 &= \frac{1-\epsilon(k)^{m_i}}{N(1+\epsilon(k)^{b-a})} = \frac{(1-\epsilon(k)^{m_i})\epsilon(k)^{\max\{a,b\}-b}}{N(\epsilon(k)^{\max\{a,b\}-b}+\epsilon(k)^{\max\{a,b\}-a})} \\ &\geq \frac{\epsilon(k)^{m_i}\epsilon(k)^{\max\{a,b\}-b}}{2N} \geq \frac{\epsilon(k)^{m_i+m^*}}{N(5|\mathcal{C}|-1)}.\end{aligned}$$

Similarly, for $k \geq t_0$, it holds that

$$\eta_3 = \frac{(1-\epsilon(k)^{m_i})\epsilon(k)^{\max\{a,b\}-a}}{N(\epsilon(k)^{\max\{a,b\}-b}+\epsilon(k)^{\max\{a,b\}-a})} \geq \frac{\epsilon(k)^{m_i+m^*}}{N(5|\mathcal{C}|-1)}.$$

Since $m_i \in (2m^*, Km^*]$ for all $i \in V$ and $Km^* > 1$, then for any feasible transition $z^1 \to z^2$ with $z^1 \neq z^2$, it holds that:

$$P^{\epsilon(k)}_{z^1 z^2} \geq \frac{\epsilon(k)^{(K+1)m^*}}{N(5|\mathcal{C}|-1)}$$

for all $k \geq t_0$. Furthermore, for all $k \geq t_0$ and all $z^1 \in \mathrm{diag}(\mathcal{A})$, we have that:

$$P^{\epsilon(k)}_{z^1 z^1} = 1 - \frac{1}{N}\sum_{i=1}^{N}\epsilon(k)^{m_i} = \frac{1}{N}\sum_{i=1}^{N}(1-\epsilon(k)^{m_i}) \geq \frac{1}{N}\sum_{i=1}^{N}\epsilon(k)^{m_i} \geq \frac{\epsilon(k)^{(K+1)m^*}}{N(5|\mathcal{C}|-1)}.$$

Choose $k_i := (D+1)i$ and let i_0 be the smallest integer such that $(D+1)i_0 \geq t_0$. Similar to (6.4), we can derive the following property

$$\sum_{\ell=0}^{+\infty}(1-\lambda(P(k_\ell, k_{\ell+1}))) \geq \frac{|\mathcal{B}|}{(N(5|\mathcal{C}|-1))^{(D+1)(K+1)m^*}}\sum_{i=i_0}^{+\infty}\frac{1}{(D+1)i} = +\infty.$$

Hence, the weak ergodicity of $\{\mathcal{P}_k\}$ follows from Theorem 6.3.1. ■

All the conditions in Theorem 6.3.2 hold. Thus it follows from Theorem 6.3.2 that the limiting distribution is $\mu^* = \lim_{k\to+\infty} \mu^k$. Notice the following relation

$$\lim_{k\to+\infty} \mu^k = \lim_{k\to+\infty} \mu(\epsilon(k)) = \mu(0),$$

and Proposition 6.5.2 shows that the support of $\mu(0)$ is contained in the set $\mathrm{diag}(S^*)$. Hence, the support of μ^* is contained in the set $\mathrm{diag}(S^*)$, implying that $\lim_{k\to+\infty} \mathbb{P}(z(k) \in \mathrm{diag}(S^*)) = 1$. It completes the proof.

6.6 Discussion and simulations

In this section, we present some remarks along with two numerical examples to illustrate the performance of our algorithms. All the figures of the numerical examples are provided at the end of the chapter.

Theorem 6.4.1 and 6.4.2 guarantees the asymptotic convergence in probability of the proposed algorithms. However, our theoretic results do not provide any estimate of the convergence rates, which could be very slow in practice. This is a consequence of the well known exploration-exploitation tradeoff termed in reinforcement learning; e.g., in [137]. Intuitively, each algorithm starts from a relatively large exploration rate and this allows the algorithm to explore the unknown environment quickly. As time processes, the exploration rate is decreased, allowing each algorithm to exploit the information collected and converge to some desired configuration. In order to avoid being locked-in some undesired configuration, each algorithm requires a very slow exploration decreasing rate. In the numerical examples below, we have chosen suitable exploration rates empirically.

A numerical example of the DISCL algorithm

Consider a 10×10 square and each grid is 1×1 and a group of 9 mobile visual sensors are deployed in this area. Note that, given arbitrary sensing range and distribution, it would be difficult to compute an NE. In order to avoid this

computational challenge and make our simulation results evident, we make the following assumptions:

1. All the sensors are identical, and each has a fixed sensing range which is a circle of radius 1.5.

2. Each point in this region is associated with an uniform value of 1.

With these two assumptions, it is not difficult to see that any configuration where sensing ranges of sensors do not overlap is an NE at which the global potential function is equal to 81.

In this example, the diameter of the location graph is 20 and $N = 9$. According to our theoretic result, we should choose an exploration rate of $\epsilon(k) = (\frac{1}{k})^{\frac{1}{189}}$. The exploration rate decreases extremely slowly and the algorithm requires an extremely long time to converge. Instead, we choose $\epsilon(k) = (\frac{1}{k+2^{10}})^{\frac{1}{2}}$ in the our simulation. Figure 6.2 shows the initial configuration of the group where all of the sensors start at the same position. Figure 6.3 presents the configuration at iterate 500 and it is evident that this configuration is an NE. Figure 6.4 is the evolution of the global potential function which eventually oscillates between 78 and the maximal value of 81. This verifies that the sensors approach the set of NEs.

As [83][84], we will use fixed exploration rates in the DISCL algorithm which then reduces to the DHSCL algorithm. Figures 6.5, 6.6 and 6.7 presents the evolution of the global potential functions for $\epsilon = 0.1, 0.01, 0.001$, respectively. When $\epsilon = 0.1$, the convergence to the neighborhood of the value 81 is the fastest, but its variation is largest. When $\epsilon = 0.001$, the convergence rate is slowest. The performance of $\epsilon = 0.01$ is similar to the diminishing step-size $\epsilon(k) = (\frac{1}{k+2^{10}})^{\frac{1}{2}}$. This comparison shows that, for both diminishing and fixed exploration rates, we have to empirically choose the exploration rate to obtain a good performance.

A numerical example of the DIACL algorithm

We consider a lattice of unit grids and each point is associated with a uniform weight 0.1. There are four identical sensors, and each of them has a fixed sensing range which is a circle of radius 1.5. The global optimal value of U_g is 36.

All the sensors start from the center of the region. We run the DIACL algorithm for 50000 iterates and sample the data every 5 iterates. Figure 6.9 to 6.12 show the evolution of the global function U_g for the following four cases, respectively: $\epsilon(k) = \frac{1}{4}(\frac{1}{k+1})^{\frac{1}{4}}$, $\epsilon = 0.1$, $\epsilon = 0.01$ and $\epsilon = 0.001$.

6.7 Conclusions

We have formulated a coverage optimization problem as a constrained exact potential game. We have proposed two payoff-based distributed learning algorithms for this coverage game and shown that these algorithms converge in probability to the set of constrained NEs and the set of global optima of certain coverage performance metric, respectively. The following papers summarize the results presented in this paper:

(JP-5) M. Zhu and S. Martínez, "Distributed coverage games for mobile visual sensor networks", *SIAM Journal on Control and Optimization*, 2011, revised.

(CP-6) M. Zhu and S. Martínez, "Distributed coverage games for mobile visual sensor networks (II): Reaching the set of global optima", *The Joint* 48^{th} *IEEE Conference on Decision and Control and* 28^{th} *Chinese Control Conference*, pages $175 - 180$, Shanghai, China, Dec. 2009.

(CP-5) M. Zhu and S. Martínez, "Distributed coverage games for mobile visual sensor networks (I): Reaching the set of Nash equilibria", *The Joint* 48^{th} *IEEE Conference on Decision and Control and* 28^{th} *Chinese Control Conference*, pages $169 - 174$, Shanghai, China Dec. 2009.

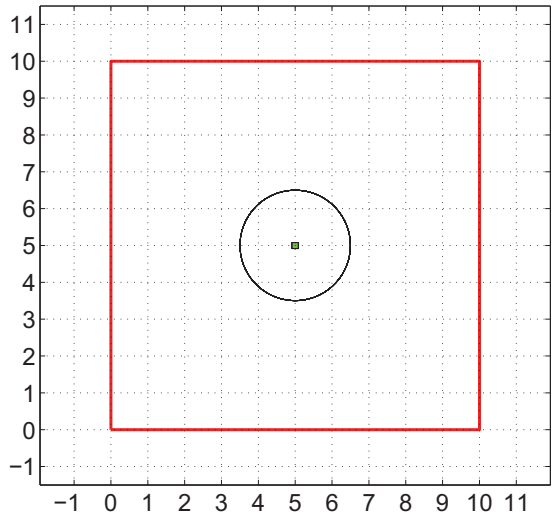

Figure 6.2: Initial configuration of the network

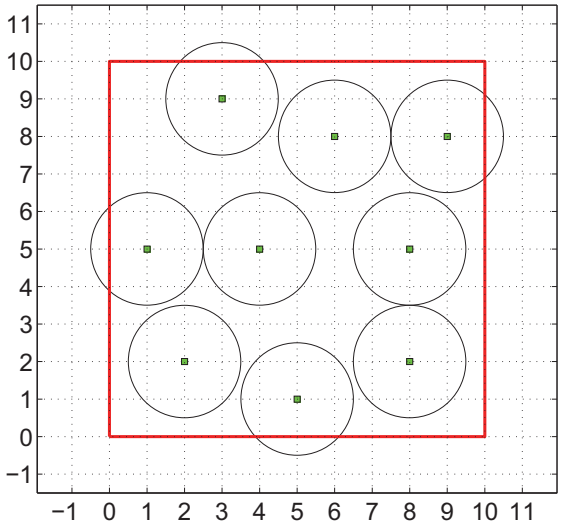

Figure 6.3: Final configuration of the network at iterate 5000 of the DISCL algorithm

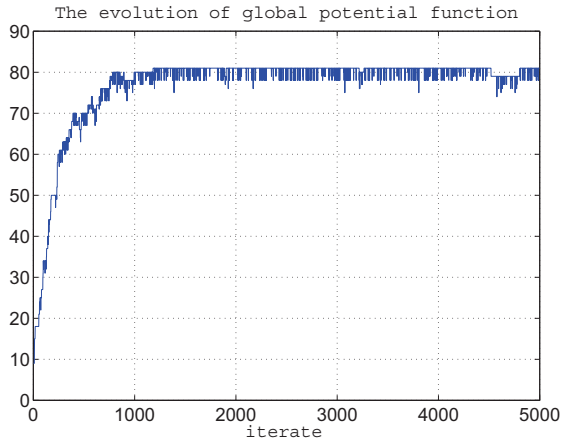

Figure 6.4: The evolution of the global potential function with a diminishing exploration rate for the DISCL algorithm.

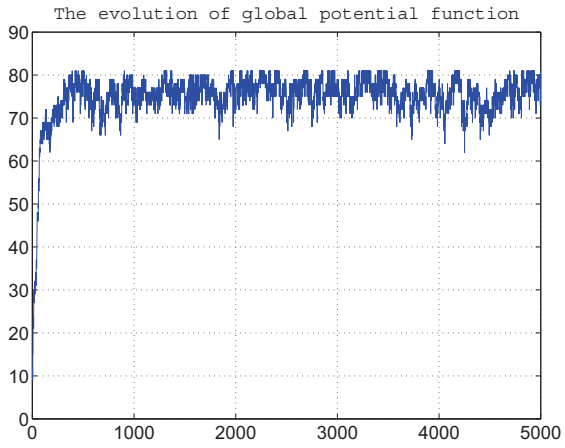

Figure 6.5: The evolution of the global potential function under DHSCL when $\epsilon = 0.1$

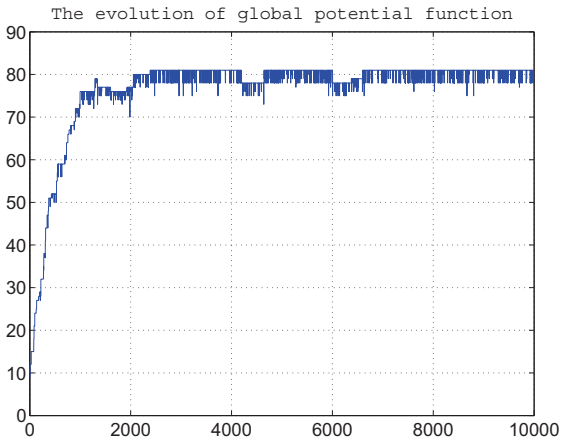

Figure 6.6: The evolution of the global potential function under DHSCL when $\epsilon = 0.01$

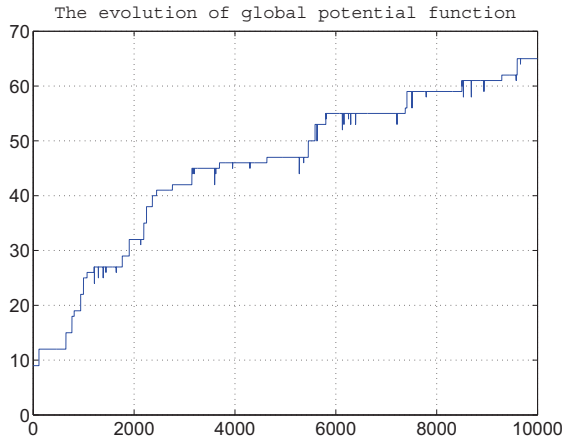

Figure 6.7: The evolution of the global potential function under DHSCL when $\epsilon = 0.001$

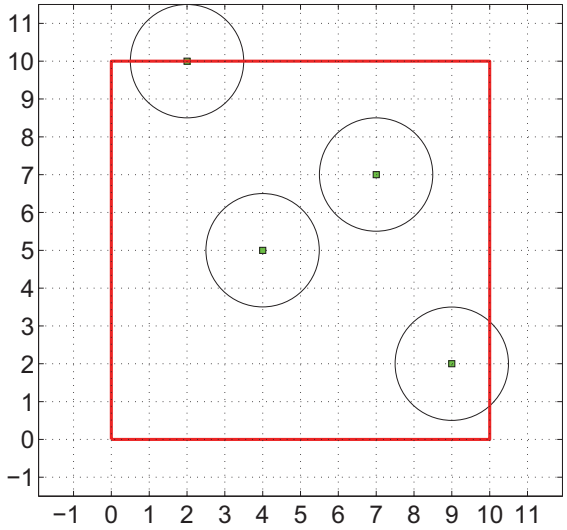

Figure 6.8: Final configuration of the network at iterate 50000 of the DIACL algorithm

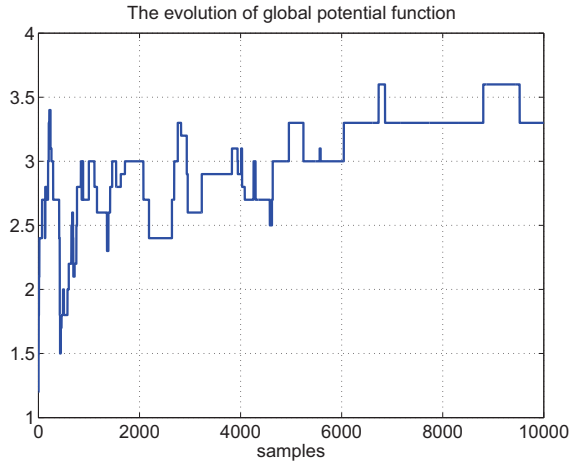

Figure 6.9: The evolution of the global potential function under the DIACL algorithm with a diminishing exploration rate

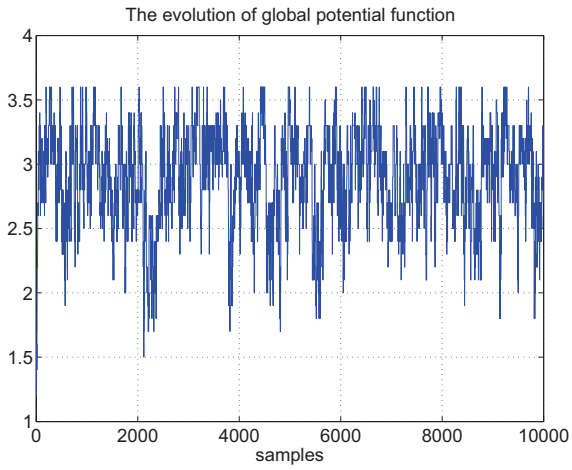

Figure 6.10: The evolution of the global potential function under the DIACL algorithm when $\epsilon = 0.1$ is kept fixed

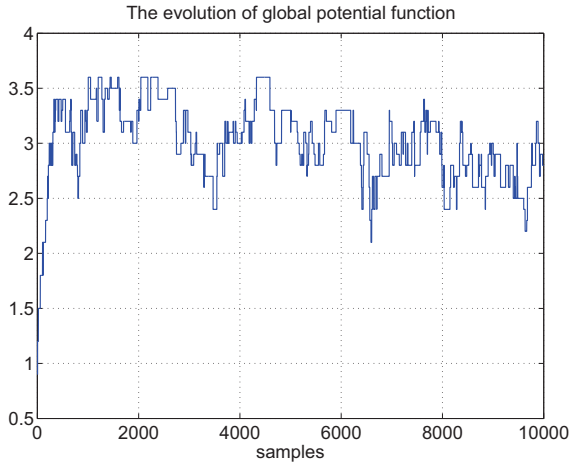

Figure 6.11: The evolution of the global potential function under the DIACL algorithm when $\epsilon = 0.01$ is kept fixed

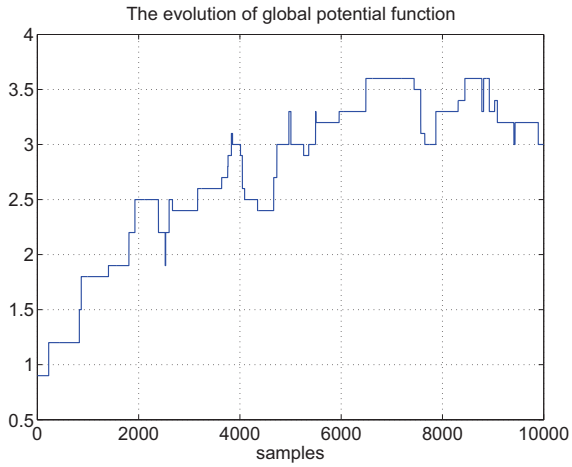

Figure 6.12: The evolution of the global potential function under the DIACL algorithm when $\epsilon = 0.001$ is kept fixed

Chapter 7

Distributed formation control against cyber-attacks

7.1 Introduction

Recent advances in communications, sensing and computation have made possible the development of highly sophisticated unmanned vehicles. Applications include, to name a few, border patrol, search and rescue, surveillance, and target identification operations. Unmanned vehicles operate without crew onboard, which lowers their deployment costs in scenarios that are hazardous to humans. More recently, the use of unmanned vehicles by (human) operators has been proposed to enhance information sharing and maintain situational awareness. However, this capability comes at the price of an increased vulnerability of information technology systems. Motivated by this, we consider a formation control problem for an operator-vehicle network where each unmanned vehicle is able to perform real-time coordination with operators (or ground stations) via sensor and communication interfaces. However, the operator-vehicle links can be attacked by adversaries, disrupting the overall network objective. Since we cannot rule out that adversaries are able to successfully amount attacks, it is of prominent importance to provide resilient solutions that assure mission completion despite the presence of security threats.

Literature review. In information technology networks, either reactive or protective mechanisms has been exploited to prevent cyber attacks. Non-cooperative game theory [51] is advocated as a mathematical framework to model the inter-dependency between attackers and administrators, and predict the behavior of attackers; see an incomplete list of references [2, 57, 127, 141]. These findings can help us assess network vulnerability; however, the papers mentioned do not consider how to maintain the operational function of networked systems in the presence of malicious attacks.

Another relevant field is networked control systems in which the effects of imperfect communication channels on remote control are analyzed and compensated. Most of the existing papers focus on; e.g., band-limited channels [77], quantization [24], packet dropout [131], delay [22], and sampling [103].

Very recently, cyber-security of the emerging cyber-physical systems has drawn mounting attention in the control society. *Denial-of-service attacks*, destroying the data availability in control systems, are entailed in recent papers [4, 6, 13, 57]. Another important class of cyber attacks, namely *false data injection*, compromises the data integrity of state estimation and is attracting considerable effort; an incomplete reference list includes [90, 115, 140, 149]. In [17, 18], the authors exploit pursuit-evasion games to compute optimal evasion strategies for mobile agents in the face of jamming attacks. Other relevant papers include [5] examining the stability of a SCADA water management system under a class of switching attacks, and our recent paper [155] studying a secure control problem of linear time-invariant systems through a receding-horizon Stackelberg game model. As [5, 155], the current chapter is devoted to studying *deception attacks* where attackers maliciously modify the transmitted data. In the paper [67], a class of trust based distributed Kalman filters is proposed for power systems to prevent data disseminated by untrusted phase measurement units.

Regarding malicious behavior in multi-agent systems, we distinguish [114, 134] as two representative references mostly relevant to this work. The paper [134] considers the problem of computing arbitrary functions of initial states in the presence of faulty or malicious agents, whereas [114] focuses on consensus problems. In

both settings, the faulty or malicious agents are part of the network and subject to unknown (arbitrarily non-zero) inputs. Their main objective is to determine conditions under which the misbehaving agents can (or cannot) be identified, and then devise algorithms to overcome the malicious behavior. This significantly departs from the problem formulation we consider here, where the attackers are external to the operator-vehicle network and can affect inter operator-vehicle connections. Additionally, we make use of a model of attackers as rational decision makers, who can make decisions in a real-time and feedback fashion. Here we aim to design completely distributed algorithms for the operator-vehicle network to maintain mission assurance under limited knowledge of teammates and opponents. Our objective is to determine an algorithm that is independent of the number of adversaries and robust to dynamical changes of communication graphs between operators.

Statement of contributions. The current chapter studies a formation control problem for an operator-vehicle network in which each vehicle is remotely controlled by an operator. Each operator-vehicle pair is attacked by an adversary, who corrupts the control commands sent to the vehicle. The adversaries are modeled as rational decision makers and their strategies are linearly parameterized by some (potentially time-varying) matrices which are unknown to operators in advance. We investigate two plausible scenarios depending on the learning capabilities of adversaries. The first scenario involves *unilateral learning*, where adversaries possess (potentially incorrect) private information of operators in advance, but do not update such information during the attacking course. The second scenario assumes *bilateral learning*, where adversaries are intelligent and attempt to infer some private information of operators through their observations. We propose a class of novel distributed attack-resilient formation control algorithms each consisting of two feedback-connected blocks: a formation control block and an online learning block. The online learning mechanism serves to collect information in a real-time fashion and update the estimates of adversaries through continuous contact with them. The formation control law of each operator is adapted online to minimize a local formation error function. To do this, each operator exploits the latest estimate of her opponent and locations of neighboring vehicles. We show

how each proposed algorithm guarantees that vehicles achieve asymptotically the desired formation from any initial vehicle configuration and any initial estimates of adversaries. For each proposed algorithm, the sequence of the distances to the desired formation is shown to be square summable. Two numerical examples are provided to verify the performance of the proposed algorithms. In the simulation, the convergence rates turn out to be exponential, which outperform the analytic results characterizing the worst-case convergence rates.

7.2 Problem formulation

In this section, we first articulate the layout of the operator-vehicle network and its formation control mission. Then, we present the adversary model that is used in the remainder of the current chapter. After this, we specify two scenarios investigated in the chapter.

7.2.1 Architecture and objective of the operator-vehicle network

Consider a group of vehicles in $\mathbb{R}^d$, labeled by $i \in V := \{1, \cdots, N\}$. The dynamics of each vehicle is governed by the following discrete-time and fully actuated system:

$$p_i(k+1) = p_i(k) + u_i(k), \tag{7.1}$$

where $p_i(k) \in \mathbb{R}^d$ is the position of vehicle i and $u_i(k) \in \mathbb{R}^d$ is its input. Each vehicle i is remotely maneuvered by an operator i, and this assignment to be one-to-one and fixed over time. For simplicity, we assume that vehicles communicate only with the associated operator and not with other vehicles. Moreover, each vehicle is able to identify its location and send this information to its operator. On the other hand, an operator can exchange information with neighboring operators and deliver control commands to her vehicle. We assume that the communications between operators, and from vehicle to operator are secure[1], while the communications

[1]Alternatively, it can be assumed that operators have access to vehicles' positions by an external and safe measurement system.

from operator to vehicle can be attacked. Other architectures are possible, and the

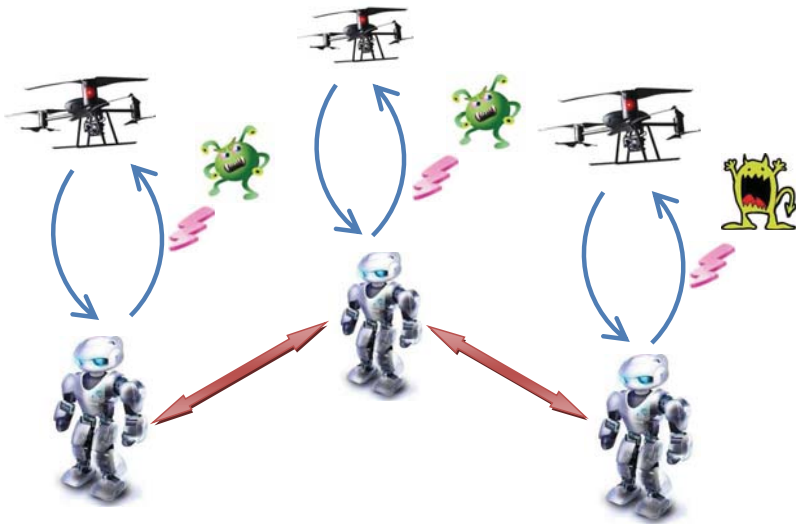

Figure 7.1: The architecture of the operator-vehicle network

present one is chosen as a first main class of operator-vehicle networked systems; see Figure 7.1.

The mission of the operator-vehicle network is to achieve some desired formation which is characterized by a formation digraph $\mathcal{G} := (V, \mathcal{E})$. Each edge $(j, i) \in \mathcal{E} \subseteq V \times V \setminus \text{diag}(V)$, starting from vehicle j and pointing to vehicle i, is associated with a vector $\nu_{ij} \in \mathbb{R}^d$. Denote by $\mathcal{N}_i := \{j \in V \mid (j, i) \in \mathcal{E}\}$ the set of in-neighbors of vehicle i in $\mathcal{G}$ and let n_i be the cardinality of $\mathcal{N}_i$; i.e., $n_i = |\mathcal{N}_i|$. The set of in-neighbors of agent i will be enumerated as $\mathcal{N}_i = \{i_1, \ldots, i_{n_i}\}$. Being a member of the team, each operator i is only aware of local formation constraints; i.e., ν_{ij} for $j \in \mathcal{N}_i$.

The multi-vehicle formation control mission can be formulated as a team optimization problem where the global optimum correspond to the desired formation of vehicles. In particular, we encode the formation control problem into the

following quadratic program:[2]

$$\min_{p} \big[J(p) := \sum_{(j,i)\in\mathcal{E}} \|p_i - p_j - \nu_{ij}\|^2_{P_{ij}} \big],$$

where the vector $p := [p_1^T, \cdots, p_N^T]^T \in \mathbb{R}^{Nd}$ is the collection of vehicles' locations. The matrix $P_{ij} \in \mathbb{R}^{d\times d}$ is a diagonal and positive-definite weight matrix and represents the preference of operator i on the link (j,i) with $j \in \mathcal{N}_i$. Observe that $J(p)$ is a convex function of p since $\|\cdot\|^2_{P_{ij}}$ is convex and $p_i - p_j - \nu_{ij}$ is affine [21]. Denote by the set of the (global) minimizers $X^* \subset \mathbb{R}^{Nd}$. We impose the following to ensure the desired formation is well-defined:

Assumption 7.2.1 *The digraph $\mathcal{G}$ is strongly connected. In addition, $X^* \neq \emptyset$ and $J(p^*) = 0$ for any $p^* \in X^*$.*

The objective function $J(p)$ can describe any shape in $\mathbb{R}^d$ by adjusting the formation vectors ν_{ij}. We assume that operators and vehicles are synchronized. The communication digraph between operators is assumed to be fixed and identical to $\mathcal{G}$. That is, each operator only receives information from in-neighbors in $\mathcal{N}_i$ at each time instant. We later discuss a possible extension to deal with time-varying communication digraphs; see Section 7.5.

Remark 7.2.1 Similar formation functions are used in [36, 46]. When $\nu_{ij} = 0$ for all $(i,j) \in \mathcal{E}$, then the formation control problem reduces to the special case of rendezvous which has received considerable attention [27, 65, 108]. •

7.2.2 Model of rational adversaries

A group of N adversaries aims to abort the mission of formation stabilization. To achieve this, an adversary is allocated to attack a specific operator-vehicle pair and this relation does not change over time. Thus, we identify adversary i with the operator-vehicle pair i. Each adversary is able to locate her target vehicle, and eavesdrop on incoming messages of her target operator. We further assume

[2]In this chapter, we denote by $\|x\|^2_A := x^T Ax$ the weighted norm of vector x for a matrix A with the proper dimensions.

that adversaries are able to collect some (potentially imperfect and dynamically changing) information of their opponents. Specifically, adversary i will have estimates $\nu_{ij}^a(k) \in \mathbb{R}^d$ of ν_{ij} at time k and $P_{ij}^a \in \mathbb{R}^{d\times d}$ of P_{ij}, for $j \in \mathcal{N}_i$. Here, the matrix P_{ij}^a is positive-definite and diagonal.

As [17, 18, 57], we assume that adversaries are rational decision makers, and they make real-time decisions based on the latest information available. In particular, at time k, adversary i identifies $p_i(k)$ of her target vehicle, eavesdrops $p_j(k)$ sent from operator $j \in \mathcal{N}_i$ to operator i, and intercepts $u_i(k)$ sent from operator i to vehicle i. The adversary then computes a command $v_i(k)$ which is added to $u_i(k)$ so that vehicle i receives and implements $u_i(k)+v_i(k)$ instead. The command $v_i(k)$ will be the solution to the following program:

$$\max_{v_i\in\mathbb{R}^d} \sum_{j\in\mathcal{N}_i} \|p_j(k) - (p_i(k) + u_i(k) + v_i) - \nu_{ij}^a(k)\|_{P_{ij}^a}^2 - \|v_i\|_{R_i}^2, \tag{7.2}$$

where $R_i \in \mathbb{R}^{d\times d}$ is diagonal and positive definite. The above optimization problem captures two partly conflicting objectives of adversary i. On the one hand, adversary i would like to destabilize the formation associated with vehicle i, and this malicious interest is encapsulated in the first term. On the other hand, adversary i would like to avoid a high attacking cost $\|v_i\|_{R_i}^2$. The attacking cost will be justified in the next part. We assume the following on the cost matrices of adversaries:

Assumption 7.2.2 *For each $i \in V$, it holds that* $\sum_{j\in\mathcal{N}_i} P_{ij}^a - R_i < 0$.

In this way, the objective function of the optimization problem (7.2) is strictly concave. This can be easily verified by noticing that the Hessian of $2\sum_{j\in\mathcal{N}_i} P_{ij}^a - 2R_i$ is negative definite. As a consequence, the optimization problem (7.2) is well defined, and its solution is uniquely determined by:

$$v_i(k) = -\sum_{j\in\mathcal{N}_i} L_{ij}(p_j(k) - (p_i(k) + u_i(k)) - \nu_{ij}^a(k)), \tag{7.3}$$

where the matrix of $L_{ij} := (R_i - \sum_{j\in\mathcal{N}_i} P_{ij}^a)^{-1} P_{ij}^a \in \mathbb{R}^{d\times d}$ is diagonal and positive definite.

7.2.3 Justification of attacking costs

Here we would like to justify the attacking cost $\|v_i\|_{R_i}^2$ in problem (7.2). At each time, adversary i has to spend some energy to successfully decode the message and deliver the wrong data to vehicle i. The energy consumption depends upon the security schemes; e.g., the cryptography and radio frequency, employed by operator i. A larger v_i alerts operator i that there is a greater risk to her vehicle, and consequently operator i raises the security level (e.g., the expansion of radio frequencies) of the link to vehicle i, increasing the subsequent costs paid by adversary i. The term $\|v_i(k)\|_{R_i}^2$ represents a consideration of adversary i for her *subsequent* energy consumption which is directly determined by $v_i(k)$. As a rational decision maker, adversary i is willing to reduce such cost.

In problem (7.2), the inclusion of the cost $\|v_i\|_{R_i}^2$ limits the actions of adversary i to some extent. If $\|v_i\|_{R_i}^2$ is removed or replaced by some constant, the problem (7.2) becomes ill-posed, and the optimal solution is trivially unbounded. In this way, the problem becomes trivial from the analysis point of view. It is noticed that, for denial-of-service attacks, the paper [57] limits the number of attacks, and the papers [4, 6, 13] then restrict the attacking strategies to follow some I.I.D. probability distributions. We argue that it is necessary to reasonably constrain the actions of adversaries when investigating attacking policies.

7.2.4 Information about opponents and online adaptation

In a hostile environment, it is not realistic to expect that decision makers have complete and perfect information of their opponents. On the other hand, information about opponents plays a vital role in defending or attacking a system. Throughout this chapter, we assume that operator i knows that adversary i is rational and makes decisions online based on the solution to the optimization problem (7.2). In particular, we will investigate the following two plausible attacking scenarios.

SCENARIO I - Unilateral learning

In the first scenario, adversary i does not update her estimates; i.e., $\nu_{ij}^a(k) =$

ν_{ij}^a for all $k \geq 0$ even though ν_{ij}^a and P_{ij}^a may be different from the true values of ν_{ij} and P_{ij}. On the other hand, operator i has no access to the values of R_i, P_{ij}^a and ν_{ij}^a which are some private information of adversary i. It would be hard for operators to gather this private information from adversaries *a priori*. In order to defeat them and enhance system resilience, operators can aim to identify the adversarial behavior. To do this, we will novelly exploit the ideas of reinforcement learning [137], and adaptive control [10], which operators can use to learn these parameters through continuous contact with adversaries.

SCENARIO II - Bilateral learning

Adversaries could be intelligent, attempting to learn some unknown information online as well. This motivates us to investigate a second scenario in which adversaries infer private information during the attacking course. For simplicity, we will assume that operator i and adversary i know the cost matrices of each other, and how each other makes real-time decisions on $v_i(k)$ and $u_i(k)$. However, adversary i is unaware of the formation vectors ν_{ij} associated with operator i, and thus attempts to identify these quantities online. In order to play against this class of intelligent adversaries, we show how operators can keep track of the dynamically changing and unmodeled estimates of adversaries, and in turn adapt their defense tactics.

7.2.5 Discussion

Informally speaking, we pose the formation control problem as a dynamic non-cooperative game between two teams of rational decision makers: operators and adversaries. In SCENARIO I (unilateral learning), adversaries does not adapt their strategies online, but they do in SCENARIO II (bilateral learning). In contrast to [17, 18], decision makers in our problem formulation do not aim to determine a Nash equilibrium, which is a widely used notion in non-cooperative game theory. From an operator's point of view, Nash strategies may not assure the mission of formation control despite malicious attacks. Instead, the main focus of the current chapter is to quantitatively analyze how online adaptation helps operators maintain system functions when they are facing vague and (potentially intelligent)

adversaries.

The papers [90, 115, 140, 149] focus on detection of false data injection attacks against state estimation. There, attackers could intelligently take advantage of channel noises and successfully bypass the detectors if they have perfect information of the system dynamics and detectors. The papers [114, 134] aim to detect malicious behavior in a multi-agent setting. Attack detection is a key security variable, and we should mention that this is trivial in the set-up of the current chapter. Since we assume communication channels are noise-free, then operators can verify whether their commands are corrupted by simply examining the locations of their associated vehicles. Here, our focus is network resilience to malicious attacks, which is another key security aspect. It is of interest to investigate attack detection in the setting of operator-vehicle networks and this is one of the future work.

Notations. In the sequel, we let tr be the trace operator of matrices, and let $\|A\|_F$ and $\|A\|$ denote the Frobenius norm and 2-norm of a real matrix $A \in \mathbb{R}^{m\times n}$, respectively. Recall that $\|A\|_F^2 = \mathrm{tr}(A^TA) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2$ and $\|A\| \leq \|A\|_F$. We will use the shorthand of $[B_{ij}]_{j\in\mathcal{N}_i} := [B_{ii_1}, \cdots, B_{ii_{n_i}}] \in \mathbb{R}^{n\times mn_i}$ where the dimensions of the given $B_{ij} \in \mathbb{R}^{n\times m}$ will be identical for all $j \in \mathcal{N}_i$. Consider the diagonal vector map, $\mathrm{diag}_{\mathrm{ve}} : \mathbb{R}^{d\times d} \to \mathbb{R}^d$, defined as $\mathrm{diag}_{\mathrm{ve}}(A) = v$, with $v_i = A_{ii}$, for all i. Similarly, define the diagonal matrix map, $\mathrm{diag}_{\mathrm{ma}} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, as $\mathrm{diag}_{\mathrm{ma}}(v) = D$, with $D_{ii} = v_i$, $D_{ij} = 0$, for all i, j and $j \neq i$. Let $\mathbb{P}_{\geq 0} : \mathbb{R}^d \to \mathbb{R}^d$ be the projection operator from $\mathbb{R}^d$ onto the non-negative orthant of $\mathbb{R}^d$. Now define the linear operator $\mathbb{P}_i : \mathbb{R}^{n_i(d+1)\times d} \to \mathbb{R}^{n_i(d+1)\times d}$ as follows. Given $\Lambda \in \mathbb{R}^{n_i(d+1)\times d}$, then $\mathbb{P}_i(\Lambda) = M \in \mathbb{R}^{n_i(d+1)\times d}$, defined block-wise as follows:

$$\begin{aligned} &\text{if } \Lambda^T := [[L_{ij}^T]_{j\in\mathcal{N}_i},\ [\eta_{ij}^T]_{j\in\mathcal{N}_i}], \quad \text{then } M^T := [[M_{ij}^T]_{j\in\mathcal{N}_i},\ [\mu_{ij}^T]_{j\in\mathcal{N}_i}], \text{ with} \\ &M_{ij}^T = \mathrm{diag}_{\mathrm{ma}}(\mathbb{P}_{\geq 0}(\mathrm{diag}_{\mathrm{ve}}(L_{ij}^T))),\ \ \mu_{ij}^T = \eta_{ij}^T, \quad j \in \mathcal{N}_i. \end{aligned} \tag{7.4}$$

The linear operator $\mathbb{P}_i$ will be used in the learning rule of the algorithm proposed for SCENARIO I (unilateral learning).

7.3 Attack-resilient distributed formation control with unilateral learning

In this section, we investigate SCENARIO I (unilateral learning) and propose a novel **attack-resilient distributed formation control algorithm under unilateral learning**, ARFCU for short, to guarantee the formation control mission under malicious attacks. It is worthy to recall that in this scenario adversary i does not update her estimates in this scenario; i.e., $\nu_{ij}^a(k) = \nu_{ij}^a$ for all $k \geq 0$.

7.3.1 A linearly parametric interpretation of attacking policies

Recall that operator i is aware that the decisions of adversary i are based on the solution to the optimization problem (7.2). This implies that operator i knows that $v_i(k)$ is in the form of (7.3), but does not have access to the real values of L_{ij} and ν_{ij}^a. A more compact expression for $v_i(k)$ is given in the following.

Lemma 7.3.1 *The vector $v_i(k)$ can be written in the following form:*

$$\begin{aligned} v_i(k) = \Theta_i^T \Phi_i(k) &= -\sum_{j\in\mathcal{N}_i} \{L_{ij}(p_j(k) - (p_i(k) + u_i(k)) - \nu_{ij}) + \eta_{ij}\} \\ &= -\sum_{j\in\mathcal{N}_i} L_{ij}\big((p_j(k) - (p_i(k) + u_i(k)) - \nu_{ij}) + (\nu_{ij} - \nu_{ij}^a)\big), \end{aligned}$$

where $\eta_{ij} := L_{ij}(\nu_{ij} - \nu_{ij}^a) \in \mathbb{R}^d$, and matrices $\Theta_i \in \mathbb{R}^{n_i(d+1)\times d}$, $\phi_i(k) \in \mathbb{R}^{n_i d}$, $\Phi_i(k) \in \mathbb{R}^{n_i(d+1)}$ are given by:

$$\begin{aligned} \phi_i(k) &:= -\begin{bmatrix} p_{i_1}(k) - (p_i(k) + u_i(k)) - \nu_{ii_1} \\ \vdots \\ p_{i_{n_i}}(k) - (p_i(k) + u_i(k)) - \nu_{ii_{n_i}} \end{bmatrix}, \\ \Theta_i^T &:= [[L_{ij}]_{j\in\mathcal{N}_i} \ \ [\eta_{ij}]_{j\in\mathcal{N}_i}], \quad \Phi_i(k) := -[\phi_i(k)^T \ 1\cdots 1]^T. \end{aligned} \tag{7.5}$$

Proof: This fact can be readily verified. ■

In the light of the above lemma, we will equivalently assume that operator i is aware of $v_i(k)$ being the product of Θ_i and $\Phi_i(k)$, where the unknown parameter Θ_i is referred to as the target parameter of operator i, and the vector $\Phi_i(k)$ is referred to as the regression vector of operator i at time k. In other words, from the point of view of operator i, the attacking strategy of adversary i is linearly parameterized by the unknown (but fixed) matrix Θ_i.

7.3.2 The ARFCU algorithm and its convergence properties

[*Informal description*] Overall, the ARFCU algorithm can be roughly described as follows. At each time instant, each operator first collects the current locations of neighboring operators' vehicles. Then, the operator computes a control command $u_i(k)$ minimizing a local formation error function by assuming that her neighboring vehicles do not move. This computation is based on the certainty equivalence principle; i.e., operator i exploits her latest estimate $\Theta_i(k)$ to predict that adversary i corrupts her command by adding $v_i^o(k) := \Theta_i(k)^T\Phi_i(k)$ as if $\Theta_i(k)$ were identical to Θ_i. After that, the operator sends the new command $u_i(k)$ to her associated vehicle. Adversary i then corrupts the command by adding the signal $v_i(k)$ linearly parameterized by Θ_i. Vehicle i receives, implements, and further sends back to operator i the new position $p_i(k+1)$. After that, operator i computes the new estimation error of Θ_i, and updates her estimate to minimize a local estimation error function.

We now formally state the interactions of the i^{th} group consisting of operator, vehicle and adversary i in Algorithm 2. The rule to compute $u_i(k)$, and the precise update law for $\Theta_i(k)$ can be found there. The notations used to describe the ARFCU algorithm are summarized in Table 7.1.

Remark 7.3.1 We denote $P_i := P_{ii} + \sum_{j\in\mathcal{N}_i} P_{ij}$ and let $\Theta_i(k)^T$ be partitioned in the form of $\Theta_i(k)^T = [[L_{ij}(k)]_{j\in\mathcal{N}_i} \quad [\eta_{ij}(k)]_{j\in\mathcal{N}_i}]$, where $L_{ij}(k) \in \mathbb{R}^{d\times d}$ and $\eta_{ij}(k) \in \mathbb{R}^d$, for $j \in \mathcal{N}_i = \{1, \cdots, n_i\}$. Then, the solution $u_i(k)$ to the quadratic

Algorithm 2 The ARFCU Algorithm for group i

Require: Operator i chooses any $\tilde{\Theta}_i \in \mathbb{R}^{n_i(d+1)\times d}$ and and lets $\Theta_i(0) = \mathbb{P}_i[\tilde{\Theta}_i]$ as the initial estimate of Θ_i.

Ensure: At each $k \geq 0$, adversary, operator, and vehicle i execute the following steps:

1: Operator i receives $p_j(k)$ from operator $j \in \mathcal{N}_i$, and solves the following quadratic program:

$$\min_{u_i(k)\in\mathbb{R}^d} \sum_{j\in\mathcal{N}_i} \|p_j(k) - p_i(k+1|k) - \nu_{ij}\|^2_{P_{ij}} + \|p_i(k) - p_i(k+1|k)\|^2_{P_{ii}},$$
$$\text{s.t. } p_i(k+1|k) = p_i(k) + u_i(k) + v_i^o(k), \tag{7.6}$$

to obtain the optimal solution $u_i(k)$ where $v_i^o(k) := \Theta_i(k)^T\Phi_i(k)$ and P_{ii} is a positive-definite and diagonal matrix.

2: Operator i sends $u_i(k)$ to vehicle i, and generates a prediction of $p_i(k+1)$ in such a way that $p_i(k+1|k) = p_i(k) + u_i(k) + v_i^o(k)$.

3: Adversary i identifies $p_i(k)$, eavesdrops on $p_j(k)$ sent from operator $j \in \mathcal{N}_i$ to operator i, and corrupts $u_i(k)$ by adding $v_i(k) = \Theta_i^T\Phi_i(k)$.

4: Vehicle i receives and implements the corrupted command $u_i(k) + v_i(k)$, and then sends back the new location $p_i(k+1) = p_i(k) + u_i(k) + v_i(k)$ to operator i.

5: Operator i computes the estimation error $e_i(k) = p_i(k+1) - p_i(k+1|k)$, and updates her parameter estimate as $\Theta_i(k+1) = \mathbb{P}_i[\Theta_i(k) + \frac{1}{m_i(k)^2}\Phi_i(k)e_i(k)^T]$, where $m_i(k) := \sqrt{1 + \|\Phi_i(k)\|^2}$.

6: Repeat for $k = k+1$.

Table 7.1: Notations used in the ARFCU algorithm

$p_i(k) \in \mathbb{R}^d$	the location of vehicle i at time k
$p_i(k+1\|k) \in \mathbb{R}^d$	the prediction of $p_i(k+1)$ produced by operator i at time k
$P_{ij} \in \mathbb{R}^{d\times d}$	the weight matrix assigned by operator i to the formation vector ν_{ij} for $j \in \mathcal{N}_i$
$P_{ii} \in \mathbb{R}^{d\times d}$	the weight matrix assigned by operator i to her own current location
$u_i(k) \in \mathbb{R}^d$	the control command of operator i at time k
$v_i(k) \in \mathbb{R}^d$	the command generated by adversary i at time k and given in (7.3)
$v_i^o(k) \in \mathbb{R}^d$	the prediction of $v_i(k)$ generated by operator i
$\Theta_i \in \mathbb{R}^{n_i(d+1)\times d}$	the target parameter of operator i given in (7.5)
$\Theta_i(k) \in \mathbb{R}^{n_i(d+1)\times d}$	the estimate of Θ_i produced by operator i at time k
$\Phi_i(k) \in \mathbb{R}^{n_i(d+1)}$	the regression vector of operator i at time k given in (7.5)
$m_i(k) := \sqrt{1+\|\Phi_i(k)\|^2}$	the normalized term of operator i
$\mathbb{P}_i$	a projection operator defined by (7.4)

program in Step 1 of the ARFCU algorithm can be explicitly computed as follows:

$$u_i(k) = \big(I + \sum_{j\in\mathcal{N}_i} L_{ij}(k)\big)^{-1} \times \big\{ \sum_{j\in\mathcal{N}_i} P_i^{-1}P_{ij}(p_j(k) - p_i(k) - \nu_{ij}) + \sum_{j\in\mathcal{N}_i} L_{ij}(k)(p_j(k) - p_i(k) - \nu_{ij}) + \sum_{j\in\mathcal{N}_i} \eta_{ij}(k)\big\}. \tag{7.7}$$

Hence, the program in Step 1 of the ARFCU algorithm is equivalent to the computation (7.7). In Step 5 of the ARFCU algorithm, operator i utilizes a projected parameter identifier to learn Θ_i online. This scheme extends the classic (vector) normalized gradient algorithm; e.g., in [10], to the matrix case and further incorporates the projection operator $\mathbb{P}_i$ to guarantee that $u_i(k)$ is well defined. That is, the introduction of $\mathbb{P}_i$ ensures that the estimate $L_{ij}(k)$ is positive definite, and that $I + \sum_{j\in\mathcal{N}_i} L_{ij}(k)$ is nonsingular. As in [10], the term $\frac{1}{m_i(k)^2}\Phi_i(k)e_i(k)^T$ in the update law of $\Theta_i(k)$ is to minimize the error cost $\frac{e_i(k)^T e_i(k)}{m_i(k)^2}$. Here, $e_i(k)$ is the position estimation error, and $m_i(k)$ is a normalizing factor. •

The following theorem guarantees that our proposed ARFCU algorithm is

attack-resilient and allows the multi-vehicle to achieve the desired formation in SCENARIO I (unilateral learning).

Theorem 7.3.1 ***(Convergence properties of the*** ARFCU ***algorithm):*** *Consider SCENARIO I (unilateral learning) with any initial configuration $p(0) \in \mathbb{R}^{Nd}$ of vehicles. If Assumptions 7.2.1 and 7.2.2 hold, then the* ARFCU *algorithm for every group i ensures that the vehicles asymptotically achieve the desired formation; i.e., $\lim_{k\to+\infty} \mathrm{dist}(p(k), X^*) = 0$. Furthermore, the convergence rate of the* ARFCU *algorithm ensures $\sum_{k=0}^{+\infty} \sum_{(i,j)\in\mathcal{E}} \|p_j(k) - p_i(k) - \nu_{ij}\|^2 < +\infty$.*

Proof of Theorem 7.3.1:

Proof: First of all note that, through the choice of $u_i(k)$, $p_i(k+1|k)$ is the minimizer of $\sum_{j\in\mathcal{N}_i} \|p_j(k) - p_i - \nu_{ij}\|^2_{P_{ij}} + \|p_i(k) - p_i\|^2_{P_{ii}}$ in p_i. Therefore,

$$p_i(k+1|k) = p_i(k) + \sum_{j\in\mathcal{N}_i} P_i^{-1} P_{ij}(p_j(k) - p_i(k) - \nu_{ij}),$$

where we use the fact that P_{ij} is diagonal and positive definite. Recall that $e_i(k) = p_i(k+1) - p_i(k+1|k)$. The above relation leads to:

$$p_i(k+1) = p_i(k+1|k) + e_i(k) = p_i(k) + \sum_{j\in\mathcal{N}_i} P_i^{-1} P_{ij}(p_j(k) - p_i(k) - \nu_{ij}) + e_i(k). \tag{7.8}$$

Pick any $p^* := [p_i^*]_{i\in V} \in X^*$. Then $p_j^* - p_i^* = \nu_{ij}$ for any $(j,i) \in \mathcal{E}$. Denote $y_i(k) = p_i(k) - p_i^*$, for $i \in V$. Subtracting p_i^* on both sides of (7.8) leads to:

$$\begin{aligned} y_i(k+1) &= y_i(k) + \sum_{j\in\mathcal{N}_i} P_i^{-1} P_{ij}\big((p_j(k) - p_j^*) - (p_i(k) - p_i^*)\big) \\ &\quad - \sum_{j\in\mathcal{N}_i} P_i^{-1} P_{ij}(-p_j^* + p_i^* + \nu_{ij}) + e_i(k) \\ &= y_i(k) + \sum_{j\in\mathcal{N}_i} P_i^{-1} P_{ij}(y_j(k) - y_i(k)) + e_i(k). \end{aligned} \tag{7.9}$$

Since the P_{ij} are diagonal and positive definite, system (7.9) can be viewed as d parallel first-order dynamic consensus algorithms in the variables $y_i(k)$ subject

to the time-varying signals $e_i(k)$. We can guarantee convergence of the vehicles to the desired formation if consensus in the $y_i(k)$ is achieved. In other words, $\lim_{k\to+\infty} \|y_i(k) - y_j(k)\| = 0$, for all $(i,j) \in \mathcal{E}$, is equivalent to $\lim_{k\to+\infty} \|p_i(k) - p_j(k) - (p_i^* - p_j^*)\| = 0$. Since $p_i^* - p_j^* = \nu_{ij}$, consensus on the $y_i(k)$ is equivalent to $\lim_{k\to+\infty} \|p_i(k) - p_j(k) - \nu_{ij}\| = 0$. The rest of the proof is devoted to verify this consensus property.

For each $\ell \in \{1, \cdots, d\}$, we denote the following:

$$g_\ell(k) := \max_{i,j\in V} \|e_{i\ell}(k) - e_{j\ell}(k)\|, \quad D_\ell(k) := \max_{i,j\in V} \|y_{i\ell}(k) - y_{j\ell}(k)\|.$$

Here, the quantity $D_\ell(k)$ represents the maximum disagreement of the ℓ^{th} consensus algorithm. The following claim characterizes the input-to-state stability properties of consensus algorithms, and it is based on the analysis of dynamic average consensus algorithms of Chapter 2:

Claim 1: There exist $D_\ell(0), \beta > 0$, and $\sigma \in (0,1)$, such that the following holds:

$$D_\ell(k+1) \leq \sigma^{k+1} D_\ell(0) + \beta \sum_{s=0}^{k} \sigma^{k-s} g_\ell(s). \tag{7.10}$$

Proof: Denote $T_k := k(N-1)$ and, for any integer $k \geq 0$, let ℓ_k be the largest integer such that $\ell_k(N-1) \leq k$. From (16) in the proof of Theorem in Chapter 2, we know that there exists some $\eta \in (0,1)$ such that

$$\begin{aligned} D_\ell(k) \leq (1-\eta)^{\ell_k} D_\ell(0) + (1-\eta)^{\ell_k - 1} \sum_{s=0}^{T_1-1} g_\ell(s) + \cdots \\ + (1-\eta) \sum_{s=T_{(\ell_k-2)}}^{T_{(\ell_k-1)}-1} g_\ell(s) + \sum_{s=T_{(\ell_k-1)}}^{T_{\ell_k}-1} g_\ell(s) + \sum_{s=T_{\ell_k}}^{k-1} g_\ell(s). \end{aligned}$$

This relation can be rewritten as follows:

$$D_\ell(k) \leq (1-\eta)^{\ell_k} D_\ell(0) + \sum_{s=0}^{k-1} (1-\eta)^{\ell_k - \ell_s} g_\ell(s). \tag{7.11}$$

Since $k \leq \ell_k(N-1)$ and $\frac{k-s}{N-1} - 1 \leq \ell_k - \ell_s$ for $k \geq s$, then it follows from (7.11) that

$$D_\ell(k) \leq (1-\eta)^{\frac{k}{N-1}} D_\ell(0) + \sum_{s=0}^{k-1} (1-\eta)^{\frac{k-s}{N-1}-1} g_\ell(s).$$

We get the desired result by letting $\sigma = (1-\eta)^{\frac{1}{N-1}}$ and $\beta = \frac{1}{1-\eta}$ in the above relation. ■

Define now an auxiliary scalar sequence $\{z(k)\}$:

$$z(k+1) = \sigma^{k+1} z(0) + \sum_{s=0}^{k} \sigma^{k-s} f(s), \quad k \geq 0, \tag{7.12}$$

where $z(0) = \max_{\ell \in \{1,\cdots,d\}} D_\ell(0)$, and $f(k) = \beta \max_{\ell \in \{1,\cdots,d\}} g_\ell(k)$. It is not difficult to verify that $\{z(k)\}$ is an upper bound of $\{D_\ell(k)\}$ in such a way that $0 \leq D_\ell(k) \leq z(k)$, for all $k \geq 0$ and $\ell \in \{1, \cdots, d\}$. In order to show the convergence of $\{D_\ell(k)\}$ to zero for any $i \in \{1, \cdots, d\}$, it suffices to show that $\{z(k)\}$ converges to zero. We do this in the following.

Observe that $\{z(k)\}$ satisfies the following recursion:

$$\begin{aligned} z(k+1) &= \sigma^{k+1} z(0) + \sum_{s=0}^{k} \sigma^{k-s} f(s) \\ &= \sigma(\sigma^k z(0) + \sum_{s=0}^{k-1} \sigma^{k-1-s} f(s)) + f(k) = \sigma z(k) + f(k). \end{aligned} \tag{7.13}$$

For any $\lambda > 0$, it follows from (7.13) that

$$z(k+1)^2 \leq (1+\lambda)\sigma^2 z(k)^2 + (1+\frac{1}{\lambda}) f(k)^2, \tag{7.14}$$

by noting that $2\sigma z(k) f(k) \leq \lambda \sigma^2 z(k)^2 + \frac{1}{\lambda} f(k)^2$. From the definition of $f(k)$, it is not difficult to see that $f(k)^2 \leq 4\beta^2 \sum_{i \in V} \|e_i(k)\|^2$. Therefore, we have the bound:

$$z(k+1)^2 \leq (1+\lambda)\sigma^2 z(k)^2 + 4(1+\frac{1}{\lambda})^2 \beta^2 \sum_{i \in V} \|e_i(k)\|^2.$$

In the sequel, we choose a (sufficiently small) $\lambda > 0$ such that $(1+\lambda)\sigma^2 < 1$. The following claim finds a bound for $\|e_i(k)\|$ in terms of $z(k)^2$, for each $i \in V$.

Claim 2: For each $i \in V$, there is a positive and summable sequence $\{\gamma_i(k)\}$, and positive constants λ_1, λ_2, such that the following holds:

$$\|e_i(k)\|^2 \leq \gamma_i(k)(1 + n_i + \lambda_1 z(k)^2 + \lambda_2). \tag{7.15}$$

Furthermore, $\{\|\Theta_i(k)\|\}$ is uniformly bounded.

Proof: Denote $\hat{\Theta}_i(k) := \Theta_i(k) + \frac{1}{m_i(k)^2}\Phi_i(k)e_i(k)^T$. Subtracting Θ_i on both sides leads to the following:

$$\hat{\Theta}_i(k) - \Theta_i = (\Theta_i(k) - \Theta_i) + \frac{1}{m_i(k)^2}\Phi_i(k)e_i(k)^T. \tag{7.16}$$

Recall that $\|A\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2$ for a matrix $A \in \mathbb{R}^{m\times n}$. Similarly to the vector normalized gradient algorithm in [10], one can compute $\frac{1}{2}\|\hat{\Theta}_i(k) - \Theta_i\|_F^2 = \frac{1}{2}\operatorname{tr}((\hat{\Theta}_i(k) - \Theta_i)^T(\hat{\Theta}_i(k) - \Theta_i))$, just plugging in (7.16), as follows:

$$\frac{1}{2}\|\hat{\Theta}_i(k) - \Theta_i\|_F^2 = \frac{1}{2}\|\Theta_i(k) - \Theta_i\|_F^2 - \frac{1}{2m_i(k)^2}\operatorname{tr}\left(e_i(k)(2 - \frac{\Phi_i(k)^T\Phi_i(k)}{m_i(k)^2})e_i(k)^T\right), \tag{7.17}$$

where we use the fact that tr is a linear operator, and that $e_i(k) = (\Theta_i - \Theta_i(k))^T\Phi_i(k)$. As a consequence, the difference of $\frac{1}{2}\|\hat{\Theta}_i(k) - \Theta_i\|_F^2 - \frac{1}{2}\|\Theta_i(k) - \Theta_i\|_F^2$ can be characterized in the following way:

$$\frac{1}{2}\|\hat{\Theta}_i(k) - \Theta_i\|_F^2 - \frac{1}{2}\|\Theta_i(k) - \Theta_i\|_F^2 \leq -\frac{1}{2m_i(k)^2}\operatorname{tr}\left(e_i(k)e_i(k)^T\right) = -\frac{\|e_i(k)\|^2}{2m_i(k)^2}, \tag{7.18}$$

where we have used that $2 - \frac{\Phi_i(k)^T\Phi_i(k)}{m_i(k)^2} \geq 1$, since $m_i(k)$ is a normalizing term. Since the projection operator $\mathbb{P}_i$ is applied block-wise, then $\|\Theta_i(k+1) - \Theta_i\|_F^2 \leq \|\hat{\Theta}_i(k) - \Theta_i\|_F^2$. Then from (7.18) we have:

$$\|\Theta_i(k+1) - \Theta_i\|_F^2 - \|\Theta_i(k) - \Theta_i\|_F^2 \leq -\frac{\|e_i(k)\|^2}{m_i(k)^2}. \tag{7.19}$$

This implies that $\{\|\Theta_i(k) - \Theta_i\|_F^2\}$ is non-increasing and uniformly bounded. Further, this ensures that $\{\|\Theta_i(k)\|\}$ is uniformly bounded by noting that:

$$\begin{aligned}\|\Theta_i(k)\|^2 = \|(\Theta_i(k) - \Theta_i) + \Theta_i\|^2 &\leq \|(\Theta_i(k) - \Theta_i) + \Theta_i\|_F^2 \\ &\leq 2\|\Theta_i(k) - \Theta_i\|_F^2 + 2\|\Theta_i\|_F^2.\end{aligned}$$

Denote $\gamma_i(k) := \|\Theta_i(k) - \Theta_i\|_F^2 - \|\Theta_i(k+1) - \Theta_i\|_F^2$. It is noted that

$$\sum_{k=0}^{K}\gamma_i(k) = \|\Theta_i(0) - \Theta_i\|_F^2 - \|\Theta_i(K+1) - \Theta_i\|_F^2.$$

The previous discussion implies that the sequence $\{\gamma_i(k)\}$ is non-negative, summable, and thus converges to zero by Lemma 7.8.1. Now, from (7.50) we obtain the following upper bound on $\|e_i(k)\|^2$ in terms of $\gamma_i(k)$:

$$\|e_i(k)\|^2 \leq \gamma_i(k)m_i(k)^2 = \gamma_i(k)(1+\|\Phi_i(k)\|^2) \leq \gamma_i(k)(1+n_i+\|\phi_i(k)\|^2). \quad (7.20)$$

We would like to find now a relation between $\|\phi_i(k)\|$ and $z(k)$. To do this, recover from (7.7) the expression for $u_i(k)$:

$$\begin{aligned} u_i(k) = (I+\sum_{j\in\mathcal{N}_i} L_{ij}(k))^{-1} \times \{\sum_{j\in\mathcal{N}_i} P_i^{-1}P_{ij}(y_j(k)-y_i(k)) \\ + \sum_{j\in\mathcal{N}_i} L_{ij}(k)(y_j(k)-y_i(k)) + \sum_{j\in\mathcal{N}_i}\eta_{ij}(k)\big\}. \end{aligned} \quad (7.21)$$

Recall that $L_{ij}(k)$ and P_{ij} are positive definite and diagonal. This gives us that $\|(I+\sum_{j\in\mathcal{N}_i} L_{ij}(k))^{-1}\| \leq 1$. Now, it follows from (7.21) that

$$\|u_i(k)\| \leq \sum_{j\in\mathcal{N}_i}\|P_i^{-1}P_{ij}\|\sqrt{d}z(k) + \sum_{j\in\mathcal{N}_i}\sqrt{d}\|L_{ij}(k)\|z(k) + \sum_{j\in\mathcal{N}_i}\|\eta_{ij}(k)\|.$$

Since $\{\|\Theta_i(k)\|\}$ is uniformly bounded, there exists some $\theta_1, \theta_2 > 0$ such that $\|u_i(k)\| \leq \theta_1 z(k) + \theta_2$, for all $k \geq 0$ and all $i \in V$. Notice that $\phi_i(k)$ can be rewritten as follows:

$$\phi_i(k) := \begin{bmatrix} y_{i_1}(k) - y_i(k) - u_i(k) \\ \vdots \\ y_{i_{n_i}}(k) - y_i(k) - u_i(k) \end{bmatrix}.$$

This implies that there exists some $\lambda_1, \lambda_2 > 0$ such that the following holds for all $k \geq 0$ and all $i \in V$:

$$\begin{aligned} &\|\phi_i(k)\|^2 \leq \lambda_1 z(k)^2 + \lambda_2, \\ &\|e_i(k)\|^2 \leq \gamma_i(k)(1+n_i+\|\phi_i(k)\|^2) \leq \gamma_i(k)(1+n_i+\lambda_1 z(k)^2+\lambda_2). \end{aligned}$$

■

Using the upper bound on the $\|e_i(k)\|^2$, and the uniform bound on the $\|\Theta_i(k)\|$, we can now obtain an inequality involving the $\{z(k)^2\}$ and other diminishing terms. This is used to determine the stability properties of $\{z(k)^2\}$.

Claim 3: The sequence $\{z(k)\}$ is square summable.

Proof: From the recursion for $z(k)$, we found that

$$z(k+1)^2 \leq (1+\lambda)\sigma^2 z(k)^2 + 4(1+\frac{1}{\lambda})\beta^2 \sum_{i\in V} \|e_i(k)\|^2, \tag{7.22}$$

where a (sufficiently small) $\lambda > 0$ is chosen such that $(1+\lambda)\sigma^2 \in (0,1)$. We now define $V(k) := z(k)^2 + \sum_{i\in V} \|e_i(k)\|^2$ to be a Lyapunov function candidate for the ARFCU algorithm, and have that:

$$\begin{aligned} V(k+1) - V(k) &= z(k+1)^2 + \sum_{i\in V} \|e_i(k+1)\|^2 - z(k)^2 - \sum_{i\in V} \|e_i(k)\|^2 \\ &\leq z(k+1)^2 + \sum_{i\in V} \|e_i(k+1)\|^2 - z(k)^2. \end{aligned}$$

Using now the bound for $\|e_i(k+1)\|^2$ in Claim 2, we obtain:

$$\begin{aligned} &V(k+1) - V(k) \leq \\ &(1+\lambda_1 \sum_{i\in V} \gamma_i(k+1)) z(k+1)^2 + \sum_{i\in V} \gamma_i(k+1)(1+n_i+\lambda_2) - z(k)^2. \end{aligned}$$

Finally, upper-bounding $z(k+1)^2$ as in (7.22), we get:

$$\begin{aligned} &V(k+1) - V(k) \leq (1+\lambda_1 \sum_{i\in V} \gamma_i(k+1))((1+\lambda)\sigma^2 z(k)^2 \\ &+ (1+\frac{1}{\lambda})4\beta^2 \sum_{i\in V} \|e_i(k)\|^2) - z(k)^2 + \sum_{i\in V} \gamma_i(k+1)(1+n_i+\lambda_2). \end{aligned} \tag{7.23}$$

Substituting the upper bound on $\|e_i(k)\|^2$ from (7.15) of Claim 2 into (7.23), we find that there exists $\alpha \in (0,1)$ and two scalar sequences $\{\pi_1(k)\}$ and $\{\pi_2(k)\}$ such that

$$V(k+1) - V(k) \leq (\alpha - 1 + \pi_1(k)) z(k)^2 + \pi_2(k), \tag{7.24}$$

where $\{\pi_1(k)\}$ is positive and diminishing and $\{\pi_2(k)\}$ is positive and summable by using that each sequence of $\{\gamma_i(k)\}$ is summable. There is a finite $K \geq 0$ such that $1 - \alpha - \pi_1(k) \leq 1 - \frac{\alpha}{2}$ for all $k \geq K$. Then, for $k \geq K$, we have the following relations for $z(k)$:

$$(1-\frac{\alpha}{2}) z(k)^2 \leq (1 - \alpha - \pi_1(k)) z(k)^2 \leq V(k) - V(k+1) + \pi_2(k).$$

This implies that

$$(1-\frac{\alpha}{2})\sum_{k=K}^{+\infty} z(k)^2 \leq V(K) + \sum_{k=K}^{+\infty} \pi_2(k). \tag{7.25}$$

Upper-bounding $\|e_i(k)\|^2$ by (7.15) from Claim 2 in the recursion (7.22), it can be found that $z(k)$ is finite for any finite k. As a consequence, $e_i(k)$, and thus $V(k)$, are finite for every finite time. In this way, $V(K)$ is finite in (7.25) and, since $\{\pi_2(k)\}$ is summable, so is $\{z(k)^2\}$. ■

Claim 3 guarantees that $\{z(k)\}$, and thus $\{D_\ell(k)\}$, for all $\ell \in \{1, \cdots, d\}$, converge to zero by Lemma 7.8.1. Therefore, $\{p(k)\}$ asymptotically converges to the set X^*. In order to estimate the convergence rate, note that

$$\begin{aligned}&\sum_{k=0}^{+\infty}\sum_{(i,j)\in\mathcal{E}} \|p_j(k)-p_i(k)-\nu_{ij}\|^2 = \sum_{k=0}^{+\infty}\sum_{(i,j)\in\mathcal{E}} \|y_j(k)-y_i(k)\|^2\\ &\leq d|\mathcal{E}|\sum_{k=0}^{+\infty} z(k)^2 < +\infty,\end{aligned}$$

where $|\mathcal{E}|$ is the cardinality of $\mathcal{E}$, and in the last inequality we use the summability of $\{z(k)^2\}$ from Claim 3. This completes the proof of Theorem 7.3.1. ■

7.4 Attack-resilient distributed formation control with bilateral learning

In this section, we investigate the more challenging SCENARIO II (bilateral learning) and we propose an **attack-resilient distributed formation control algorithm under bilateral learning**, ARFCB for short, to defeat a class of intelligent adversaries.

In SCENARIO II (bilateral learning), adversary i is aware of P_{ij} (i.e, $P_{ij}^a = P_{ij}$) and the policy of operator i to compute $u_i(k)$. However, adversary i has no access to the formation vectors of ν_{ij} for $j \in \mathcal{N}_i$ in advance. This motivates adversary i to learn ν_{ij} and the quantity $\nu_{ij}^a(k)$ is an estimate of ν_{ij} maintained by adversary i at time k. On the other hand, operator i is assumed to know R_i and the rule of adversary i making decisions without accessing the instantaneous

estimate $\nu_{ij}^a(k)$. In order to play against her opponent, operator i has to keep track of the time-varying quantity $\nu_{ij}^a(k)$. Operator i is completely unaware of the learning dynamics associated with the estimates $\nu_{ij}^a(k)$, and thus $\nu_{ij}^a(k)$ is totally unmodeled for operator i. The best operator i can do is to observe some quantity that depends on $\nu_{ij}^a(k)$ at time k, and generate a posterior estimate $\nu_{ij}^o(k+1)$ of $\nu_{ij}^a(k)$. Through the certainty equivalence principle, the actions of operator i and adversary i at time k employ the estimates of $\nu_{ij}^a(k)$ and $\nu_{ij}^o(k)$, respectively.

In the remainder of this section, the subscripts of a and o are used to indicate the target parameters of adversaries and operators, respectively, and the superscripts of a and o are employed to indicate the estimates of target parameters or other local variables of adversaries and operators, respectively. Towards this end, let us make the following notations: $\Omega_{a,i} = [[\nu_{ij}^T]_{j\in\mathcal{N}_i}]^T$ (resp. $\Psi_{o,i}(k) = \Omega_i^a(k)$) is the target parameter of adversary i (resp. operator i), and $\Omega_i^a(k) = [[\nu_{ij}^a(k)^T]_{j\in\mathcal{N}_i}]^T$ (resp. $\Psi_i^o(k) = [[\nu_{ij}^o(k)^T]_{j\in\mathcal{N}_i}]^T$) represents the estimate of $\Omega_{a,i}$ (resp. $\Psi_{o,i}(k-1)$) produced by adversary i (resp. operator i) at time k.

7.4.1 A linearly parametric interpretation of attacking policies and local formation control laws

In this part, we first find a linearly parametric interpretation of attacking policies from the point of view of operators. Then we devise a local formation control law for each operator.

Before doing that, we adopt the following notation[3]:

$$L_{ij} := (R_i - \sum_{j\in\mathcal{N}_i} P_{ij})^{-1} P_{ij}, \quad L_i := \sum_{j\in\mathcal{N}_i} L_{ij},$$
$$M_{ij} := (I + L_i)^{-1} P_i^{-1} P_{ij}, \quad M_i := \sum_{j\in\mathcal{N}_i} M_{ij}.$$

Throughout this section, we assume that the cost matrices of each operator are homogeneous, and this assumption is formally stated as follows:

Assumption 7.4.1 For each $i \in V$, there is a diagonal and positive-definite matrix $\bar{P}_i$ such that $P_{ij} = \frac{1}{n_i}\bar{P}_i$ for all $j \in \mathcal{N}_i$.

[3]Note that similar letters do not exactly match their meaning in the previous section.

With this assumption, it is easy to see that:

$$L_{ij} = \frac{1}{n_i}(R_i - \bar{P}_i)^{-1}\bar{P}_i, \quad L_{ij} = \frac{1}{n_i}L_i, \quad M_{ij} = \frac{1}{n_i}M_i.$$

Lemma 7.4.1 *The vector $v_i(k)$ can be written in the following way:*

$$v_i(k) = -\sum_{j\in\mathcal{N}_i} L_{ij}(p_j(k) - p_i(k) - u_i(k)) + (\Phi_i^o)^T\Psi_{o,i}(k), \tag{7.26}$$

where the matrices of Φ_i^o and $\Psi_{o,i}(k)$ are given by:

$$(\Phi_i^o)^T := [[L_{ij}]_{j\in\mathcal{N}_i}], \quad \Psi_{o,i}(k) := [[\nu_{ij}^a(k)^T]_{j\in\mathcal{N}_i}^T]^T \tag{7.27}$$

Proof: It is straightforward to verify this result. ■

In SCENARIO II (bilateral learning), operator i knows that adversary i bases her decisions on the solution to the optimization problem (7.2) which is parameterized by the unknown quantity $\nu_{ij}^a(k)$. Lemma 7.4.1 indicates that, from operator i's point of view, the attacking strategy of adversary i is linearly parameterized by the unknown and time-varying matrix $\Psi_{o,i}(k)$. The quantity Φ_i^o is referred to as the regression vector of operator i.

We are now in the position to devise a local formation control law for each operator. In particular, with $p_j(k)$ for $j \in \mathcal{N}_i$ at hand, operator i computes the control command $u_i(k)$ by solving the following quadratic program to minimize the local formation error:

$$\begin{aligned}&\min_{u_i(k)\in\mathbb{R}^d} \sum_{j\in\mathcal{N}_i} \|p_j(k) - p_i(k+1|k) - \nu_{ij}\|_{P_{ij}}^2 + \|p_i(k) - p_i(k+1|k)\|_{P_{ii}}^2,\\ &\text{s.t. } p_i(k+1|k) = p_i(k) + u_i(k) + v_i^o(k),\end{aligned} \tag{7.28}$$

where $v_i^o(k)$ is a prediction of $v_i(k)$ and defined as follows:

$$v_i^o(k) := -\sum_{j\in\mathcal{N}_i} L_{ij}(p_j(k) - (p_i(k) + u_i(k))) + (\Phi_i^o)^T\Psi_i^o(k). \tag{7.29}$$

The solution to (7.28) is uniquely determined by:

$$\begin{aligned}u_i(k) = (I + L_i)^{-1}\big\{ &\sum_{j\in\mathcal{N}_i} P_i^{-1}P_{ij}(p_j(k) - p_i(k) - \nu_{ij})\\ &+ \sum_{j\in\mathcal{N}_i} L_{ij}(p_j(k) - p_i(k) - \nu_{ij}^o(k))\big\}.\end{aligned} \tag{7.30}$$

7.4.2 A linearly parametric interpretation and estimates of formation control commands

In SCENARIO II (bilateral learning), adversary i, on the one hand, is unaware of the formation vector ν_{ij} for $j \in \mathcal{N}_i$; and on the other hand, is able to intercepts $u_i(k)$ produced by operator i. This motivates adversary i to infer ν_{ij} through the observation of $u_i(k)$. To achieve this, she generates the following estimate $u_i^a(k)$ of the control command $u_i(k)$ before receiving $u_i(k)$:

$$u_i^a(k) = (I + L_i)^{-1}\{\sum_{j\in\mathcal{N}_i} P_i^{-1}P_{ij}(p_j(k) - p_i(k) - \nu_{ij}^a(k)) + \sum_{j\in\mathcal{N}_i} L_{ij}(p_j(k) - p_i(k) - \nu_{ij}^a(k))\}, \tag{7.31}$$

and computes the estimation error $e_i^a(k) = u_i(k) - u_i^a(k)$ via the comparison with $u_i(k)$ and $u_i^a(k)$. In the next part, we will explain how adversary i updates her estimates of ν_{ij} based on $e_i^a(k)$.

7.4.3 The ARFCB algorithm and convergence properties

[*Informal description*] We informally describe the ARFCB algorithm as follows. At each time instant k, operator i, adversary i and vehicle i implement the following steps.

(1) Each operator first receives the information of $p_j(k)$ from neighboring operator j. The operator then computes a control command $u_i(k)$ to minimize a local formation error function by assuming that her neighboring vehicles do not move and the strategy of adversary i is linearly parameterized by $\Theta_i^o(k)$. After this computation, the operator sends the generated command $u_i(k)$ to vehicle i.

(2) Adversary i intercepts $p_j(k)$ for $j \in \mathcal{N}_i$ and $u_i(k)$, and further corrupts $u_i(k)$ by adding the signal $v_i(k)$ linearly parameterized by Θ_i^o. Adversary i maintains a scheduler T_i^a [4] which determines the collection of time instants to update her estimate $\Omega_i^a(k)$. In particular, if $k \in T_i^a$, then adversary i generates an estimate $u_i^a(k)$ of $u_i(k)$, identify her estimation error and then produces her estimate $\Omega_i^a(k)$ by minimizing some local estimation error function.

[4] Without loss of any generality, we assume that $0 \in T_i^a$.

(3) Vehicle i receives, implements, and further sends back to operator i the new position $p_i(k+1)$.

(4) After that, operator i determines the estimation error of $\Psi_{o,i}(k)$, and updates her estimate to minimize a local estimation error function.

We proceed to formally state the ARFCB algorithm in Algorithm 3. The notations used in Algorithm 3 are summarized in Table 7.2.

Table 7.2: The notations of the ARFCB algorithm

$p_i(k) \in \mathbb{R}^d$	the location of vehicle i at time k
$p_i(k+1\|k) \in \mathbb{R}^d$	the prediction of $p_i(k+1)$ produced by operator i at time k
$u_i(k) \in \mathbb{R}^d$	the control command of operator i at time k
$u_i^a(k) \in \mathbb{R}^d$	the estimate of $u_i(k)$ maintained by adversary i at time k and given in (7.31)
$v_i(k) \in \mathbb{R}^d$	the command generated by adversary i at time k
$v_i^o(k) \in \mathbb{R}^d$	the prediction of $v_i(k)$ produced by operator i at time k and given by (7.29)
$\Omega_{a,i} = [[\nu_{ij}^T]_{j\in\mathcal{N}_i}]^T$	the target parameter of adversary i
$\Omega_i^a(k) = [[\nu_{ij}^a(k)^T]_{j\in\mathcal{N}_i}]^T$	the estimate of $\Omega_{a,i}$ produced by adversary i at time k
$\Psi_{o,i}(k) = \Omega_i^a(k)$	the target parameter of operator i
$\Psi_i^o(k) = [[\nu_{ij}^o(k)^T]_{j\in\mathcal{N}_i}]^T$	the posterior estimate of $\Psi_{o,i}(k-1)$ produced by operator i at time k
$(\Phi_i^a)^T := [[M_{ij}]_{j\in\mathcal{N}_i}]$	the regression vector of adversary i
$(\Phi_i^a)^T := [[L_{ij}]_{j\in\mathcal{N}_i}]$	the regression vector of operator i
$m_i := \sqrt{1+\|\Phi_i^o\|^2+\|\Phi_i^a\|^2}$	the normalized term of group i
$\mu_i^a \in (0,1]$	the step-size of adversary i
$\mu_i^o \in (0,1]$	the step-size of operator i
T_i^a	the scheduler of adversary i

We now set out to analyze the ARFCB algorithm. First of all, let us spell out the estimation errors $e_i^a(k)$ and $e_i^o(k)$ as follows:

$$\begin{aligned} e_i^o(k) &= p_i(k+1) - p_i(k+1|k) = (\Phi_i^o)^T(\Psi_{o,i}(k) - \Psi_i^o(k)), \\ e_i^a(k) &= u_i(k) - u_i^a(k) = r_i^a(k) + (I+L_i)^{-1}e_i^o(k), \end{aligned} \tag{7.34}$$

where $r_i^a(k) := (\Phi_i^a)^T(\Omega_{a,i} - \Omega_i^a(k))$. In addition, we notice that operator i is attempting to identify some time-varying quantities, and the evolution of her time-

Algorithm 3 The ARFCB Algorithm for group i

Require: Vehicle i informs operator i of its initial location $p_i(0) \in \mathbb{R}^d$. Operator i chooses initial estimate $\Psi_i^o(0)$, and adversary i chooses initial estimates of $\Omega_i^a(0)$.

Ensure: At each $k \geq 0$, adversary, operator, and vehicle i execute the following steps:

1: Operator i receives $p_j(k)$ from operator $j \in \mathcal{N}_i$, solves the quadratic program (7.28), and obtains the optimal solution $u_i(k)$. Operator i then sends $u_i(k)$ to vehicle i, and generates the prediction of $p_i(k+1|k) = p_i(k) + u_i(k) + v_i^o(k)$.

2: Adversary i identifies the location $p_i(k)$ of vehicle i, eavesdrops on $p_j(k)$ sent from operator $j \in \mathcal{N}_i$ to operator i, and corrupts $u_i(k)$ by adding $v_i(k)$ in (7.3). Adversary i produces an estimate $u_i^a(k)$ of $u_i(k)$ in the way of (7.31), and computes her estimation error $e_i^a(k) = u_i(k) - u_i^a(k)$. If $k \notin T_i^a$, then $\Omega_i^a(k+1) = \Omega_i^a(k)$; otherwise,

$$\Omega_i^a(k+1) = \Omega_i^a(k) + \frac{\mu_i^a}{m_i^2} \Phi_i^a e_i^a(k), \tag{7.32}$$

with the step-size $\mu_i^a \in (0, 1]$ and the normalized term $m_i := \sqrt{1 + \|\Phi_i^o\|^2 + \|\Phi_i^a\|^2}$.

3: Vehicle i receives and implements the corrupted command $u_i(k) + v_i(k)$, and then sends back its new location $p_i(k+1) = p_i(k) + u_i(k) + v_i(k)$ to operator i.

4: Operator i computes the estimation error $e_i^o(k) = p_i(k+1) - p_i(k+1|k)$, and updates her parameter estimate in the following manner:

$$\Psi_i^o(k+1) = \Psi_i^o(k) + \frac{\mu_i^o}{m_i^2} \Phi_i^o e_i^o(k), \tag{7.33}$$

with the step-size $\mu_i^o \in (0, 1]$.

5: Repeat for $k = k + 1$.

varying target parameters is given by:

$$\Psi_i^o(k+1) = \Psi_i^o(k) + \frac{\mu_i^a}{m_i^2}\Phi_i^a e_i^a(k), \tag{7.35}$$

which can be readily obtained from the update rules of (7.32) in Algorithm 3 by noting that $\Psi_i^o(k) = \Omega_{a,i}(k)$. The following lemma describes a linear relation between the regression vectors Φ_i^a and Φ_i^o. This fact will allow us to quantify the estimation errors of operators which are introduced by the variations of time-varying target parameters.

Lemma 7.4.2 *The regression vectors of the* i^{th} *adversary-operator pair satisfy* $\Phi_i^a = \Phi_i^o L_i^{-1} M_i$.

Proof: It follows from Assumption 7.4.1 and the non-singularity of corresponding matrices. ■

For each $k \geq 1$, we denote by $\tau_i^a(k)$ the largest time instant in T_i^a that satisfies $\tau_i^a(k) < k$. The following proposition summarizes the convergence properties of the estimation errors of the learning schemes in the ARFCB algorithm.

Proposition 7.4.1 *Consider SCENARIO II (bilateral learning) with any initial configuration* $p(0) \in \mathbb{R}^{Nd}$ *of vehicles and any initial estimates of* $\Omega_i^a(0)$ *and* $\Psi_i^o(0)$. *Suppose Assumptions 7.2.1, 7.2.2 and 7.4.1 hold and the following inequalities are satisfied:*

$$\begin{aligned} &-2\mu_i^a + 5(\mu_i^a)^2 + \big(\frac{\mu_i^a}{\mu_i^o}\big)^2 \|L_i^{-1}M_i\|^2 + \big(\frac{\mu_i^a}{\mu_i^o}\big)^2 \|(I+L_i)^{-1}\|^2 < 0, \\ &-2\mu_i^o + 4(\mu_i^o)^2 + 4(\mu_i^a)^2\|(I+L_i)^{-1}\|^2 + 2\mu_i^a\|L_i^{-1}M_i\|\|(I+L_i)^{-1}\| < 0. \end{aligned} \tag{7.36}$$

Then the following statements hold for the sequences generated by the ARFCB *algorithm:*

1. *The sequence of* $\{\Psi_i^a(k)\}$ *is uniformly bounded.*
2. *The sequence of* $\{e_i^o(k)\}$ *is square summable and the sequence of* $\{r_i^a(k)\}$ *is diminishing.*

3. *If there is an integer $T_B \geq 1$ such that $k - \tau_i^a(k) \leq T_B$ for any $k \geq 0$, then the sequence of $\{e_i^a(k)\}$ is square summable.*

Proof: We will divide the proof into several claims.

Claim 4: For adversary i, then the following relation holds when $k \in T_i^a$:

$$\|\Omega_i^a(k+1) - \Omega_{a,i}\|_F^2 - \|\Omega_i^a(k) - \Omega_{a,i}\|_F^2 \leq -2(\mu_i^a - (\mu_i^a)^2)\frac{\|r_i^a(k)\|^2}{m_i^2}$$
$$+ 2\mu_i^a\|(I+L_i)^{-1}\|\frac{\|r_i^a(k)\|\|e_i^o(k)\|}{m_i^2} + 2(\mu_i^a)^2\|(I+L_i)^{-1}\|^2\frac{\|e_i^o(k)\|^2}{m_i^2}. \quad (7.37)$$

If $k \notin T_i^a$, then the following holds:

$$\|\Omega_i^a(k+1) - \Omega_{a,i}\|_F^2 = \|\Omega_i^a(k) - \Omega_{a,i}\|_F^2. \quad (7.38)$$

Proof: First of all, we notice that, analogous to (7.17), the following holds for adversary i when $k \in T_i^a$:

$$\|\Omega_i^a(k+1) - \Omega_{a,i}\|_F^2 = \|\Omega_i^a(k) - \Omega_{a,i}\|_F^2 + \big(\frac{\mu_i^a}{m_i^2}\big)^2 \operatorname{tr}(e_i^a(k)^T(\Phi_i^a)^T\Phi_i^a e_i^a(k))$$
$$+ 2\operatorname{tr}\big((\Omega_i^a(k) - \Omega_{a,i})^T\frac{\mu_i^a}{m_i^2}\Phi_i^a e_i^a(k)\big). \quad (7.39)$$

For the last term on the right-hand side of the relation (7.39), we have

$$(\Omega_i^a(k) - \Omega_{a,i})^T\frac{\mu_i^a}{m_i^2}\Phi_i^a e_i^a(k) = (\Omega_i^a(k) - \Omega_{a,i})^T\frac{\mu_i^a}{m_i^2}\Phi_i^a(r_i^a(k) + (I+L_i)^{-1}e_i^o(k))$$
$$= -\frac{\mu_i^a}{m_i^2}r_i^a(k)^T r_i^a(k) - \frac{\mu_i^a}{m_i^2}r_i^a(k)^T(I+L_i)^{-1}e_i^o(k). \quad (7.40)$$

The trace of the second term in the last term of (7.40) can be upper bounded in the following way:

$$\frac{\mu_i^a}{m_i^2}\|\operatorname{tr}(r_i^a(k)^T(I+L_i)^{-1}e_i^o(k))\| \leq \frac{\|(I+L_i)^{-1}\|\mu_i^a}{m_i^2}\|r_i^a(k)\|\|e_i^o(k)\|. \quad (7.41)$$

Let us consider the second term on the right-hand side of the relation (7.39). Note that $(\Phi_i^a)^T\Phi_i^a = \operatorname{diag}(M_{ij}^2)$ is a diagonal matrix from the fact that M_{ij} is a diagonal matrix. Using the definition of m_i as a normalizing term and $e_i^a(k) = r_i^a(k) + (I+L_i)^{-1}e_i^o(k)$, we have

$$\big(\frac{\mu_i^a}{m_i^2}\big)^2\operatorname{tr}(e_i^a(k)^T(\Phi_i^a)^T\Phi_i^a e_i^a(k)) \leq \frac{(\mu_i^a)^2}{m_i^2}\|e_i^a(k)\|^2$$
$$\leq 2\frac{(\mu_i^a)^2}{m_i^2}(\|r_i^a(k)\|^2 + \|(I+L_i)^{-1}\|^2\|e_i^o(k)\|^2), \quad (7.42)$$

where in the last inequality we use the relations of $\|a+b\|^2 \leq 2(\|a\|^2+\|b\|^2)$ and $\|cd\| \leq \|c\|\|d\|$. Substitute the bounds of (7.41) and (7.42) into (7.39), and we have the desired relation (7.37) for $k \in T_i^a$ by using the fact that tr is a linear operator. The relation for $k \notin T_i^a$ is trivial to verify. ■

Claim 5: For operator i, the following relation holds when $k \in T_i^a$:

$$\begin{aligned}
&\|\Psi_i^o(k+1) - \Psi_{o,i}(k+1)\|_F^2 - \|\Psi_i^o(k) - \Psi_{o,i}(k)\|_F^2 \\
&\leq \big(-2\mu_i^o + (\mu_i^o)^2 + 2(\mu_i^a)^2\|(I+L_i)^{-1}\|^2 + 2\mu_i^a\|L_i^{-1}M_i\|\|(I+L_i)^{-1}\|\big)\frac{\|e_i^o(k)\|^2}{m_i^2} \\
&+ 2(\mu_i^a)^2\frac{\|r_i^a(k)\|^2}{m_i^2} + 2(\mu_i^a\|L_i^{-1}M_i\| + \mu_i^o\mu_i^a)\frac{\|r_i^a(k)\|\|e_i^o(k)\|}{m_i^2}.
\end{aligned} \tag{7.43}$$

If $k \notin T_i^a$, then the following holds:

$$\|\Psi_i^o(k+1) - \Psi_{o,i}(k+1)\|_F^2 - \|\Psi_i^o(k) - \Psi_{o,i}(k)\|_F^2 \leq \big(-2\mu_i^o + (\mu_i^o)^2\big)\frac{\|e_i^o(k)\|^2}{m_i^2}. \tag{7.44}$$

Proof: We first discuss the case when both adversary i and operator i update their estimates at time k. Note that the following holds for operator i:

$$\Psi_i^o(k+1) - \Psi_{o,i}(k+1) = \Psi_i^o(k) + \frac{\mu_i^o}{m_i^2}\Phi_i^o e_i^o(k) - \Psi_{o,i}(k) - \frac{\mu_i^a}{m_i^2}\Phi_i^a e_i^a(k). \tag{7.45}$$

Analogous to (7.17), it follows from (7.45) that

$$\begin{aligned}
&\|\Psi_i^o(k+1) - \Psi_{o,i}(k+1)\|_F^2 = \|\Psi_i^o(k) - \Psi_{o,i}(k)\|_F^2 \\
&+ \frac{\mu_i^o}{m_i^2}\operatorname{tr}\big((\Psi_i^o(k) - \Psi_{o,i}(k))^T\Phi_i^o e_i^o(k)\big) - \frac{\mu_i^a}{m_i^2}\operatorname{tr}\big((\Psi_i^o(k) - \Psi_{o,i}(k))^T\Phi_i^a e_i^a(k)\big).
\end{aligned} \tag{7.46}$$

One can verify that

$$(\Psi_i^o(k) - \Psi_{o,i}(k))^T\frac{\mu_i^o}{m_i^2}\Phi_i^o e_i^o(k) = -\frac{\mu_i^o}{m_i^2}e_i^o(k)^T e_i^o(k), \tag{7.47}$$

which produces the following upper bounds for the second term on the right-hand side of (7.46):

$$\|\operatorname{tr}(\Psi_i^o(k) - \Psi_{o,i}(k))^T\frac{\mu_i^o}{m_i^2}\Phi_i^o e_i^o(k)\| \leq \frac{\mu_i^o}{m_i^2}\|e_i^o(k)\|^2. \tag{7.48}$$

From (7.47), we can derive the following upper bounds for the third term on the right-hand side of (7.46):

$$\|(\Psi_i^o(k)-\Psi_{o,i}(k))^T\frac{\mu_i^a}{m_i^2}\Phi_i^a e_i^a(k)\| = \frac{\mu_i^a}{m_i^2}\|(\Psi_i^o(k)-\Psi_{o,i}(k))^T\Phi_i^o L_i^{-1}M_i e_i^a(k)\|$$
$$= \frac{\mu_i^a}{m_i^2}\|e_i^o(k)^T L_i^{-1}M_i e_i^a(k)\| \le \mu_i^a\|L_i^{-1}M_i\|\|\frac{e_i^o(k)}{m_i}\|\|\frac{e_i^a(k)}{m_i}\|$$
$$\le \mu_i^a\|L_i^{-1}M_i\|\|\frac{e_i^o(k)}{m_i}\|(\|\frac{r_i^a(k)}{m_i}\| + \|(I+L_i)^{-1}\|\|\frac{e_i^o(k)}{m_i}\|), \tag{7.49}$$

where in the first equality we use Lemma 7.4.2, in the second equality we use the definition of $e_i^o(k)$, and the third equality follows from the definition of $e_i^a(k)$. The combination of (7.46), (7.48) and (7.49) gives (7.43). When $k \notin T_i^a$, then $\Psi_i^o(k+1) = \Psi_i^o(k)$ and thus (7.43) reduces to (7.44). ■

We now denote the following quantity to characterize the estimation errors of the i^{th} group:

$$U_i(k) := \|\Omega_i^a(k)-\Omega_{a,i}\|_F^2 + \|\Psi_i^o(k)-\Psi_{o,i}(k)\|_F^2.$$

With the two claims just proved, one can characterize the difference $U_i(k+1)-U_i(k)$ as follows:

$$U_i(k+1)-U_i(k) \le \frac{1}{m_i^2}\begin{bmatrix} \|r_i^a(k)\| & \|e_i^o(k)\| \end{bmatrix}\Pi_i(k)\begin{bmatrix} \|r_i^a(k)\| \\ \|e_i^o(k)\| \end{bmatrix}, \tag{7.50}$$

where the time-varying matrix $\Pi(k)$ is given that: if $k \in T_i^a$, then

$$\Pi_i(k) = \Pi_i^{(1)} = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix},$$

with $\xi_1 := -2\mu_i^a + 5(\mu_i^a)^2 + \left(\frac{\mu_i^a\|L_i^{-1}M_i\|}{\mu_i^o}\right)^2 + \left(\frac{\|(I+L_i)^{-1}\|\mu_i^a}{\mu_i^o}\right)^2$ and $\xi_2 := -2\mu_i^o + 4(\mu_i^o)^2 + 4(\mu_i^a)^2\|(I+L_i)^{-1}\|^2 + 2\mu_i^a\|L_i^{-1}M_i\|\|(I+L_i)^{-1}\|$; otherwise,

$$\Pi_i(k) = \Pi_i^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & -2\mu_i^o + (\mu_i^o)^2 \end{bmatrix}.$$

Claim 6: The matrix $\Pi_i^{(2)}$ are negative semi-definite and $\Pi_i^{(1)}$ is negative definite.

Proof: Since $\mu_i^o \in (0,1)$, then it is easy to see that $\Pi_i^{(2)}$ is negative semi-definite. From (7.36), it can be seen that $\Pi_i^{(2)}$ is negative definite. ■

Claim 7: The sequence of $\{\Psi_i^o(k)\}$ is uniformly bounded. Furthermore, the sequence of $\{r_i^a(k)\}$ is diminishing, and the sequence of $\{e_i^o(k)\}$ is square summable.

Proof: It follows from Claim 6 and (7.50) that the sequence of $\{U_i(k)\}$ is non-increasing and uniformly bounded. Since $\|\Omega_i^a(k) - \Omega_{a,i}\|_F^2$ and $\|\Psi_i^o(k) - \Psi_{o,i}(k)\|_F^2$ are non-negative, they are uniformly bounded. Since Ω_i is constant, so $\{\Omega_i^a(k)\}$ and thus $\{\Psi_{o,i}(k)\}$ are uniformly bounded. It further implies that $\{\Psi_i^o(k)\}$ are uniformly bounded. Sum (7.50) over $[0, K]$, and we have the following relation:

$$-\frac{\lambda_{\max}(\Pi^{(2)})}{m_i^2}\sum_{0\le k\le K, k\in T_i^a}\left(\|e_i^o(k)\|^2 + \|r_i^a(k)\|^2\right) + \frac{2\mu_i^o - (\mu_i^o)^2}{m_i^2}\sum_{0\le k\le K, k\notin T_i^a}\|e_i^o(k)\|^2$$
$$\le U_i(0) - U_i(K+1) < +\infty.$$

This implies that the sequence of $\{\|e_i^o(k)\|^2\}$ and the subsequence $\{\|r_i^a(k)\|^2\}_{k\in T_i^a}$ are summable. Then the subsequence of $\{\|r_i^a(k)\|^2\}_{k\in T_i^a}$ diminishing by Lemma 7.8.1.

Notice that $r_i^a(s) = r_i^a(\tau_i^a(k)+1)$ for all $\tau_i^a(k)+1 \le s \le k$. We are now in the position to show the convergence of the whole sequence $\{e_i^a(k)\}$. Pick any $\epsilon > 0$, there is $K(\epsilon) \ge 0$ such that the following holds for any $k', k'' \ge K(\epsilon)$ with $k', k'' \in T_i^a$:

$$\|r_i^a(k') - r_i^a(k'')\| \le \epsilon. \tag{7.51}$$

Pick any $k_1, k_2 \ge K(\epsilon)$, the difference of $\|r_i^a(k_1) - r_i^a(k_2)\|$ can be characterized as follows:

$$\|r_i^a(k_1) - r_i^a(k_2)\| = \|r_i^a(\tau_i^a(k_1)) - r_i^a(\tau_i^a(k_2))\| \le \epsilon, \tag{7.52}$$

where the last inequality is a result of (7.51). As a result, the sequence of $\{r_i^a(k)\}_{k\ge 0}$ is a Cauchy sequence and thus converges. Since $\{r_i^a(k)\}_{k\in T_i^a}$ is a subsequence of $\{r_i^a(k)\}_{k\ge 0}$, it implies that $\{r_i^a(k)\}_{k\ge 0}$ has the same limit as $\{r_i^a(k)\}_{k\in T_i^a}$ and thus $\{r_i^a(k)\}_{k\ge 0}$ goes to zero. Since $e_i^a(k) = r_i^a(k) + (I + L_i)^{-1}e_i^o(k)$, this gives that $\{e_i^a(k)\}$ is diminishing. ■

Claim 8: If $k - \tau_i^a(k) \le T_B$, then the sequence of $\{e_i^a(k)\}$ is square summable.

Proof: Since $k - \tau_i^a(k) \leq T_B$, then we have

$$\sum_{k=0}^{+\infty} \|r_i^a(k)\|^2 \leq T_B \sum_{k \in T_i^a} \|r_i^a(k)\|^2 < +\infty. \tag{7.53}$$

Recall that $e_i^a(k) = r_i^a(k) + (I + L_i)^{-1} e_i^o(k)$. It follows from the square summability of $\{e_i^o(k)\}$ and $\{r_i^a(k)\}$ that $\{e_i^a(k)\}$ is square summable. ■ This completes the proof of Proposition 7.4.1. ■

Based on Proposition 7.4.1, we are able to characterize the asymptotic convergence properties of the ARFCB algorithm as follows.

Theorem 7.4.1 ***(Convergence properties of the*** ARFCB ***algorithm):*** *Consider SCENARIO II (bilateral learning) with any initial position $p(0) \in \mathbb{R}^{Nd}$ of vehicles and any initial estimates of $\Omega_i^a(0)$ and $\Psi_i^o(0)$. Suppose Assumptions 7.2.1, 7.2.2 and 7.4.1 and condition* (7.36) *hold. Then the* ARFCB *algorithm ensures that the vehicles asymptotically achieve the desired formation; i.e., the following relation holds:*

$$\lim_{k \to +\infty} \operatorname{dist}(p(k), X^*) = 0.$$

Furthermore, the convergence rate of the algorithm can be estimated in the following way:

$$\sum_{k=0}^{+\infty} \sum_{(i,j) \in \mathcal{E}} \|p_j(k) - p_i(k) - \nu_{ij}\|^2 < +\infty.$$

Proof: The proof is analogous to Theorem 7.3.1, and we only provide its sketch here. From Proposition 7.4.1, we know that $\{e_i^o(k)\}$ is square summable and $\{\Psi_i^o(k)\}$ is uniformly bounded. This result is the counterpart of Claim 2 in the proof of Theorem 7.3.1. The remainder of the proof can be finished by following analogous lines in Claim 1 and Claim 3 in Theorem 7.3.1. The details are omitted here. ■

Through the comparison of Theorem 7.4.1 and Theorem 7.3.1, it is not difficult to see that the ARFCB algorithm shares analogous convergence properties with the ARFCU algorithm, but requires an additional condition (7.36). The following provides a set of sufficient conditions that can ensure (7.36).

Lemma 7.4.3 *The following statements hold:*

1. *For any pair of step-sizes $\mu_i^a \in (0, \sqrt{\frac{2}{5}})$ and $\mu_i^o \in (0, \frac{1}{2})$, there is a $\bar{P}_i$ such that condition* (7.36) *holds.*

2. *For any given triple of $\mu_i^o \in (0, \frac{1}{2})$, $\|(I + L_i)^{-1}\|$ and $\|L_i^{-1} M_i\|$, then there is $\bar{\mu}_i^a \in (0, \sqrt{\frac{2}{5}})$ such that for any $\mu_i^a \in (0, \bar{\mu}_i^a]$, condition* (7.36) *holds.*

Proof: Let us investigate the first condition. If we take the limit on $R_i - \bar{P}_i$ to 0, then we have $\|(I + L_i)^{-1}\| \to 0$ and $\|L_i^{-1} M_i\| \to 0$. This means that operator i can always choose $\bar{P}_i$ such that $\|(I + L_i)^{-1}\|$ and $\|L_i^{-1} M_i\|$ are sufficiently small. As a result, operator i can always choose $\bar{P}_i$ to enforce condition (7.36). We now consider the second condition. When μ_i^a is sufficiently small, then $-2\mu_i^a$ and $-2\mu_i^o$ dominate in the two inequalities of condition (7.36), respectively. By continuity, there exists $\bar{\mu}_i^a \in (0, \sqrt{\frac{2}{5}})$ such that condition (7.36) holds for any $\mu_i^a \in (0, \bar{\mu}_i^a]$. ■

To conclude this section, we leverage singular perturbation theory (e.g., in [71]) to provide an informal interpretation of the conditions in Lemma 7.4.3. This will help us draw some insights from Proposition 7.4.1 and Theorem 7.4.1. From (7.34) and Lemma 7.4.2, we know the following:

$$e_i^a(k) = (\Phi_i^o L_i^{-1} M_i)^T (\Omega_{a,i} - \Omega_i^a(k)) + (I + L_i)^{-1} e_i^o(k).$$

The first condition in Lemma 7.4.3 renders that $\|(I + L_i)^{-1}\|$ and $\|L_i^{-1} M_i\|$, and thus $\|e_i^a(k)\|$, are sufficiently small. The second condition in Lemma 7.4.3 renders $\mu_i^a \approx 0$ and thus $\|e_i^a(k)\| \approx 0$ as well. Hence, under any condition in Lemma 7.4.3, the dynamics (7.32) approximates $\Omega_i^a(k+1) \approx \Omega_i^a(k)$; i.e., the learning dynamics of adversary i evolves on a slow manifold. On the other hand, for any fixed Ω_i^a, the update rule (7.33) becomes:

$$\Psi_i^o(k+1) = \Psi_i^o(k) + \frac{\mu_i^o}{m_i^2} \Phi_i^o (\Psi_i^o(k) - \Omega_i^a), \tag{7.54}$$

and the trajectories of (7.54) asymptotically reach the set of $\{\Psi_i^o \mid \|\Phi_i^o(\Psi_i^o - \Omega_i^a)\| = 0\}$ where the estimation error of operator i vanishes.

If we informally interpret μ_i^a and μ_i^o as learning rates of adversary i and operator i, respectively, then the second condition in Lemma 7.4.3 demonstrates

that operators can win the game if their learning rates are sufficiently faster than their opponents.

7.5 An extension to time-varying inter-operator communication digraphs

So far, we have only considered a fixed communication digraph of operators. The ARFCU algorithm, together with Theorem 7.3.1, can be extended to a simple case of time-varying inter-operator communication digraphs with some additional assumptions. Let $\mathcal{N}_i^C(k) \subseteq \mathcal{N}_i$ be the set of operators who can send information to operator i at time k. We define an operator communication digraph as $\mathcal{G}^C(k) := (V, \mathcal{E}^C(k))$ where $\mathcal{E}^C(k) := \{(j,i) \mid j \in \mathcal{N}_i^C(k)\}$. It can be seen that $\mathcal{G}^C(k)$ is a subgraph of $\mathcal{G}$. We slightly modify the ARFCU algorithm as follows. If $\mathcal{N}_i^C(k) \neq \mathcal{N}_i$, then operator i does nothing at this time instant. Since operator i does not send out any information, then adversary and vehicle i will have to keep idle at this time instant as well. If $\mathcal{N}_i^C(k) = \mathcal{N}_i$, then operator i, adversary i and vehicle i implement one iteration of the ARFCU algorithm. In other words, this situation models a type of asynchronous operator interactions under the assumption that vehicles can maintain their positions. To guarantee the convergence of the modified algorithm, we require that the frequency that the set $\mathcal{N}_i$ can be recovered by operators is high enough. Formally, we need the following to hold:

Assumption 7.5.1 *There is some integer $T \geq 1$ such that the event of $\mathcal{N}_i^C(k) = \mathcal{N}_i$ occurs at least once within any T consecutive steps.*

This assumption in conjunction with Assumption 7.2.1 ensures that for all $k_0 \geq 0$, the digraph $(V, \bigcup_{k=0}^{B-1} \mathcal{E}^C(k_0 + k))$ is strongly connected with the integer $B := NT$. The proof of Theorem 7.3.1 can be carried out almost exactly by only changing $T_k := k(NB - 1)$ in the proof of Claim 1 of the proofs for Theorem 7.3.1 in the appendix, as we did in Chapter 2. This extension applies to the ARFCB algorithm as well. The possible solution aforementioned allows for tolerating unexpected changes of communication digraphs between operators, but this robustness

comes with the expense of potentially slowing down the algorithms. An interesting future research problem is to maintain the convergence rates of algorithms under switching topologies.

7.6 Illustrative examples

Here we evaluate the performance of our proposed algorithms through some numerical examples. All the figures can be found at the end of the current chapter to facilitate the comparison.

7.6.1 A numerical example for the ARFCU algorithm

Consider a group of 15 vehicles which are initially randomly deployed over a square of 50×50 length units as shown in Figure 7.2. Figure 7.4 delineates the trajectory of each vehicle in the first 60 iterations of the algorithm. The configuration of the vehicles at the 60^{th} iteration of the ARFCU algorithm is given by Figure 7.3 and this one is identical to the desired formation. This fact can be verified by Figure 7.5, which shows the evolution of the formation errors of the ARFCU algorithm. Figure 7.5 also demonstrates that the convergence rate of the ARFCU algorithm in the simulation is exponential and this is faster than our analytical result in Theorem 7.3.1.

7.6.2 A numerical example for the ARFCB algorithm

In order to compare with the performance of the ARFCU algorithm, we consider the same problem where a group of 15 vehicles are initially randomly deployed over the square of 50×50. Figure 7.6 shows the initial configuration, and Figure 7.8 then presents the trajectory of each vehicle in the first 100 iterations of the algorithm. The group configuration at the 100^{th} iteration is provided in Figure 7.7. We can verify the fact that the desired formation is exponentially achieved from Figure 7.5 of the evolution of the formation errors of the ARFCB algorithm.

The simulations provide some insights of the algorithms. Comparing Figures 7.5 and 7.9, it can be seen that the ARFCU algorithm converges faster than the ARFCB algorithm. Figure 7.4 shows that vehicles stay close to the region where they start from while Figure 7.8 shows that vehicles drift significantly away from the starting area. These two facts verify the fact that the damage induced by intelligent adversaries is greater.

7.7 Conclusions

In this chapter, we have studied a distributed formation control problem for an operator-vehicle network which is threatened by a team of adversaries. We have proposed a class of novel attack-resilient distributed formation control algorithms and analyzed their asymptotic convergence properties. Our results have demonstrated the capability of online learning to enhance network resilience, and suggest a number of future research directions which we plan to investigate. For example, the current operator-vehicle architecture can be enlarged to allow for more complex interactions. Moreover, the types of malicious attacks can be broadened and the models of attackers can be further refined. In addition, it would be interesting to study the cyber-security of other cooperative control problems in the operator-vehicle setting. The following papers summarize the results in this chapter:

(JP-8) M. Zhu and S. Martínez, "On attack-resilient distributed formation control in operator-vehicle networks", *SIAM Journal on Control and Optimization*, submitted.

(CP-12) M. Zhu and S. Martínez, "Attack-resilient distributed formation control via online adaptation", *The* 50^{th} *IEEE Conference on Decision and Control and European Control Conference*, Orlando, USA, Dec. 2011, to appear.

7.8 Appendix

We give two instrumental facts as follows where the second one is a direct result of the first one.

Lemma 7.8.1 *The following statements hold:*

1. *Let* $\{a(k)\}$ *be a non-negative scalar sequence. If* $\{a(k)\}$ *is summable, then it converges to zero.*

2. *Consider non-negative scalar sequences of* $\{V(k)\}$ *and* $\{b(k)\}$ *such that* $V(k+1) - V(k) \leq -b(k)$. *Then it holds that* $\lim_{k \to +\infty} b(k) = 0$.

It is worthy to remark that the second fact in Lemma 7.8.1 is a discrete-time version of Barbalat's lemma (e.g., in [71]).

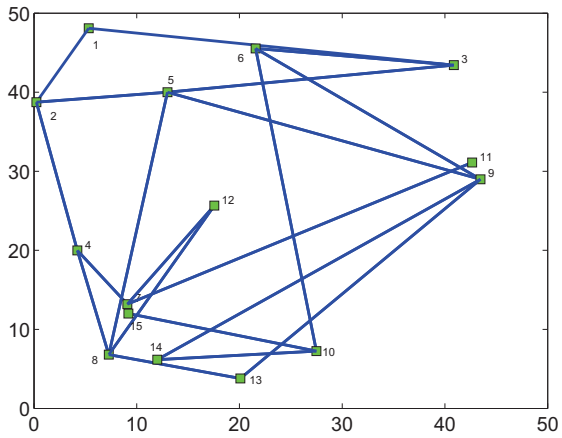

Figure 7.2: Initial configuration of vehicles for the ARFCU algorithm

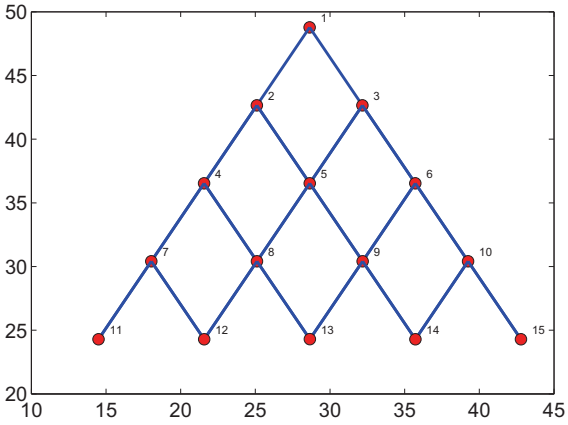

Figure 7.3: The configuration of vehicles at the 60^{th} iteration under the ARFCU algorithm

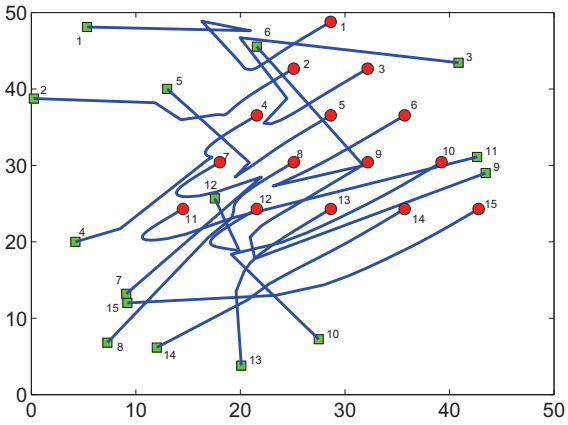

Figure 7.4: Trajectories of the vehicles during the first 60 iterations of the ARFCU algorithm. The green squares stand for initial locations and red circles represent final locations

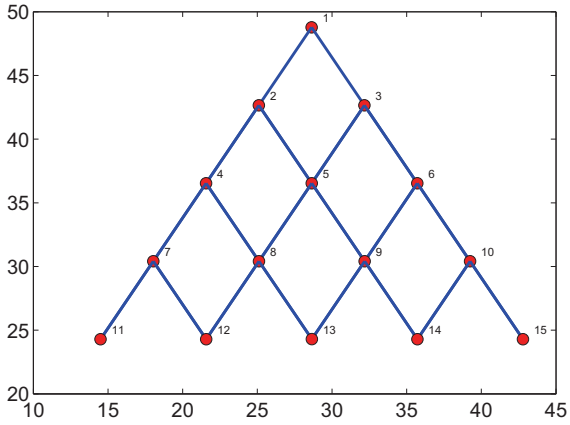

Figure 7.5: The evolution of formation errors during the first 60 iterations of the ARFCU algorithm

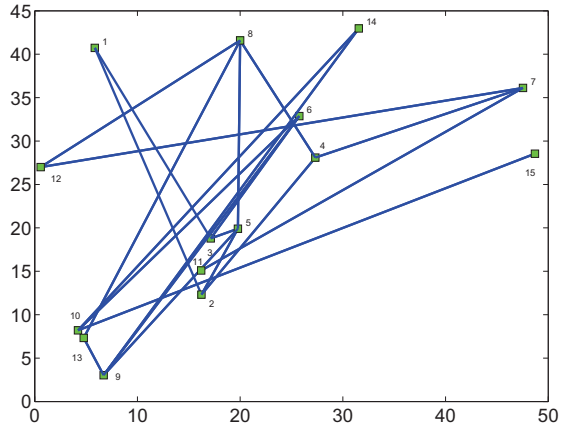

Figure 7.6: Initial configuration of vehicles for the ARFCB algorithm

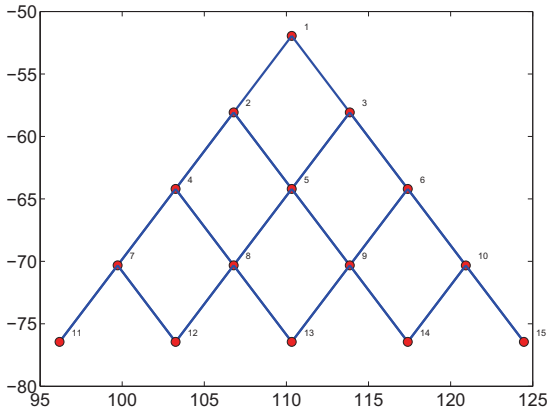

Figure 7.7: The configuration of vehicles at the 100^{th} iteration under the ARFCB algorithm

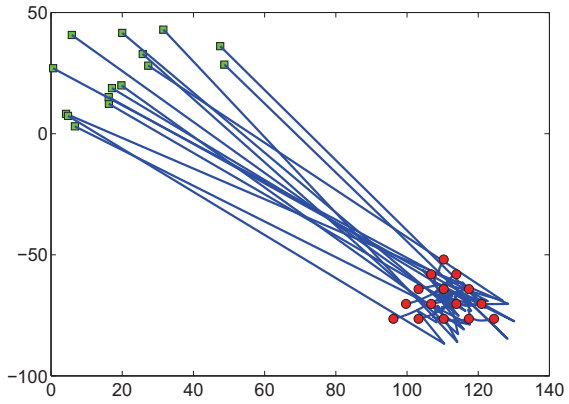

Figure 7.8: Trajectories of the vehicles during the first 100 iterations of the ARFCB algorithm. The green squares stand for initial locations and red circles represent final locations

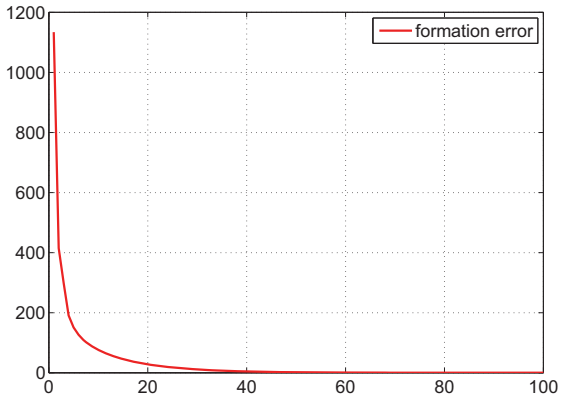

Figure 7.9: The evolution of formation errors during the first 100 iterations of the ARFCB algorithm

Chapter 8

Conclusions and Future Work

This dissertation investigates distributed decision making by networked multi-agent systems in complex environments. The objective of this dissertation is to design and analyze practical mechanisms which allow autonomous agents to coordinate their actions via local communication and perform given tasks with satisfactory performance guarantees. The environments where agents are deployed are dynamic, uncertain and adversarial. More specifically, three classes of networked decision making problems have been studied: (1) distributed average consensus; (2) distributed cooperative constrained optimization; and (3) distributed online learning based coordination. In the sequel, we discuss some possible future directions.

1. In Part II, we focus on the asymptotic convergence properties of the algorithms. It would be interesting to characterize their convergence rates and reveal the relation of convergence rates and network topologies.

   In addition, the algorithms proposed in Part II are subgradient-based. In the literature of optimization, it is already known that the (centralized) subgradient methods are easy to implement and suitable for large-scale optimization problems, but suffer from slow convergence. From a practical point of view, it would be of great interest to devise distributed algorithms whose convergence is faster.

   Furthermore, it is an interesting problem to investigate how imperfection of

communication channels (e.g., quantization, delay and channel noises) affects the performance of the algorithms developed in Part II.

2. In Chapter 6, the physical environment is discretized, and thus the action space of the induced game is finite. A possible future direction is to investigate the counterpart of the continuous state space.

   Moreover, in Chapter 6, the optimal coverage problem is posed as a non-cooperative game. The advantage of non-cooperative games is that decision making of players is distributed, robust and scalable. However, inefficiency of Nash equilibrium would degrade the system performance. In contrast, players in cooperative games aim to reach mutually beneficial agreements. One then can expect that cooperative games may induce more efficient network coalitions through proper negotiation mechanisms. It would be interesting to explore the application of cooperative games in distributed decision making.

3. In Chapter 6, our results have demonstrated that some desired configurations can still be expected in some completely unknown environment. In Chapter 7, operators are able to maintain system resilience if they have access to partial prior information on adversaries. This is achieved by the integration of online learning where it is key to efficiently exploit past observations and experiences. An interesting problem along the direction of Chapter 7 is that operators and adversaries have limited resources to defend or attack the networked control system, and what are their optimal strategies to adaptively reallocate their resources given the history of observations?

   In Chapter 7, we have investigated formation control mechanisms against deception attacks. One possible future direction is to address other cooperative control problems in the presence of different classes of cyber attacks; e.g., denial-of-service attacks and replay attacks.

Bibliography

[1] I.F. Akyildiz, T. Melodia, and K. Chowdhury. Wireless multimedia sensor networks: a survey. *IEEE Wireless Communications Magazine*, 14(6):32–39, 2007.

[2] T. Alpcan and T. Basar. *Network Security: A Decision And Game Theoretic Approach*. Cambridge University Press, 2011.

[3] T. Alpcan, T. Basar, and S. Dey. A power control game based on outage probabilities for multicell wireless data networks. *IEEE Transactions on Wireless Communications*, 5(4):890–899, 2006.

[4] S. Amin, A. Cardenas, and S. S. Sastry. Safe and secure networked control systems under denial-of-service attacks. In *Hybrid Systems: Computation and Control*, pages 31–45, 2009.

[5] S. Amin, X. Litrico, S. Sastry, and A. M. Bayen. Stealthy deception attacks on water SCADA systems. In *Proceedings of the 13th ACM International Conference on Hybrid systems: Computation and Control*, pages 161–170, Stockholm, Sweden, 2010.

[6] S. Amin, G. A. Schwartz, and S. S. Sastry. Security of interdependent and identical networked control systems. *Automatica*, July 2010. submitted.

[7] S. Anily and A. Federgruen. Ergodicity in parametric nonstationary Markov chains: an application to simulated annealing methods. *Operations Research*, 35(6):867–874, November 1987.

[8] K.J. Arrow, L. Hurwicz, and H. Uzawa. *Studies in linear and nonlinear programming*. Stanford University Press, 1958.

[9] A. Arsie, K. Savla, and E. Frazzoli. Efficient routing algorithms for multiple vehicles with no explicit communications. *IEEE Transactions on Automatic Control*, 54(10):2302–2317, 2009.

[10] K. J. Åström and B. Wittenmark. *Adaptive control*. Dover Publications, 2008.

[11] J.P. Aubin and H. Frankowska. *Set-valued analysis*. Birkhäuser, 1990.

[12] T. C. Aysal, M. Coates, and M. Rabbat. Distributed average consensus using probabilistic quantization. In *IEEE Workshop on Statistical Signal Processing*, pages 640–644, Maddison, Wisconsin, August 2007.

[13] G.K. Befekadu, V. Gupta, and P.J. Antsaklis. Risk-sensitive control under a class of denial-of-service attack models. In *American Control Conference*, pages 643–648, San Francisco, USA, June 2011.

[14] D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, 1997.

[15] D.P. Bertsekas. *Convex optimization theory*. Anthena Scietific, 2009.

[16] D.P. Bertsekas, A. Nedic, and A. Ozdaglar. *Convex analysis and optimization*. Anthena Scietific, 2003.

[17] S. Bhattacharya and T. Basar. Game-theoretic analysis of an aerial jamming attack on a UAV communication network. In *American Control Conference*, pages 818–823, Baltimore, USA, June 2010.

[18] S. Bhattacharya and T. Basar. Graph-theoretic approach for connectivity maintenance in mobile networks in the presence of a jammer. In *IEEE Conf. on Decision and Control*, pages 3560–3565, Atlanta, USA, December 2010.

[19] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. In *IEEE Conf. on Decision and Control and European Control Conference*, pages 2996–3000, Seville, Spain, December 2005.

[20] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *IEEE Transactions on Information Theory*, 52(6):2508–2530, 2006.

[21] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[22] M.S. Branicky, S.M. Phillips, and W. Zhang. Stability of networked control systems: explicit analysis of delay. In *Proceedings of American Control Conference*, pages 2352–2357, Chicago, USA, 2000.

[23] G. Brightwell and P. Winkler. Maximum hitting time for random walks on graphs. *Random Structures and Algorithms*, 1(3):263–276, 1990.

[24] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control*, 45(7):1279–1289, 2000.

[25] F. Bullo, J. Cortés, and S. Martínez. *Distributed Control of Robotic Networks*. Applied Mathematics Series. Princeton University Press, 2009. Available at http://www.coordinationbook.info.

[26] F. Bullo, J. Cortés, and B. Piccoli. Special issue on control and optimization in cooperative networks, 2009.

[27] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment - convergence rates, measurement delays and asynchronous events. *SIAM Journal on Control and Optimization*, 47(2):601–623, 2008.

[28] R. Carli, F. Bullo, and S. Zampieri. Quantized average consensus via dynamic coding/decoding schemes. *International Journal on Robust and Nonlinear Control*, 20(2):156–175, 2010.

[29] R. Carli, F. Fagnani, P. Frasca, T. Taylor, and S. Zampieri. Average consensus on networks with transmission noise or quantization. In *European Control Conference*, pages 1852–1857, Kos, Greece, June 2007.

[30] R. Carli and S. Zampieri. Networked clock synchronization based on second order linear consensus algorithms. In *IEEE Conf. on Decision and Control*, pages 7259–7264, Atlanta, USA, December 2010.

[31] N. Chopra and M. W. Spong. On exponential synchronization of Kuramoto oscillators. *IEEE Transactions on Automatic Control*, 54(2):353–357, 2009.

[32] K.Y. Chow, K.S. Lui, and E.Y. Lam. Maximizing angle coverage in visual sensor networks. In *Proc. of IEEE International Conference on Communications*, pages 3516–3521, June 2007.

[33] D. Coppersmith, P. Tetali, and P. Winkler. Collisions among random walks on a graph. *SIAM Journal on Discrete Mathematics*, 6(3):363–374, 1993.

[34] J. Cortés. Analysis and design of distributed algorithms for χ-consensus. In *IEEE Conf. on Decision and Control*, pages 3363–3368, San Diego, USA, December 2006.

[35] J. Cortés. Finite-time convergent gradient flows with applications to network consensus. *Automatica*, 42(11):1993–2000, 2006.

[36] J. Cortés. Global and robust formation-shape stabilization of relative sensing networks. *Automatica*, 45(12):2754–2762, 2009.

[37] J. Cortés, S. Martínez, and F. Bullo. Spatially-distributed coverage optimization and control with limited-range interactions. *ESAIM: Control, Optimisation & Calculus of Variations*, 11:691–719, 2005.

[38] R. Cucchiara. Multimedia surveillance systems. In *Proc. of the third ACM international workshop on Video surveillance and sensor networks*, pages 3–10, 2005.

[39] G. Cybenko. Dynamic load balancing for distributed memory multiprocessors. *Journal of Parallel and Distributed Computing*, 7(2):279–301, 1989.

[40] M. C. DeGennaro and A. Jadbabaie. Decentralized control of connectivity for multi-agent systems. In *IEEE Conf. on Decision and Control*, pages 3947–3952, San Diego, USA, Dec 2006.

[41] M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(35):118–121, 1974.

[42] J. Derenick and J. Spletzer. Convex optimization strategies for coordinating large-scale robot formations. *IEEE Transactions on Robotics*, 23(6):1252–1259, 2007.

[43] J. Derenick, J. Spletzer, and M. Ani Hsieh. An optimal approach to collaborative target tracking with performance guarantees. *Journal of Intelligent and Robotic Systems*, 56(1-2):47–67, 2009.

[44] A. G. Dimakis, S. Kar, J.M.F. Moura, M. G. Rabbat, and A. Scaglione. Gossip algorithms for distributed signal processing. *Proceedings of the IEEE*, 2011. to appear.

[45] F. Dörfler and F. Bullo. Synchronization and transient stability in power networks and non-uniform Kuramoto oscillators. *IEEE Transactions on Automatic Control*, January 2010. Submitted.

[46] W. B. Dunbar and R. M. Murray. Distributed receding horizon control for multi-vehicle formation stabilization. *Automatica*, 42(4):549–558, 2006.

[47] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, 2004.

[48] R. A. Freeman, P. Yang, and K. M. Lynch. Stability and convergence properties of dynamic average consensus estimators. In *IEEE Conf. on Decision and Control*, pages 398–403, San Diego, CA, December 2006.

[49] M. Freidlin and A. Wentzell. *Random perturbations of dynamical systems*. New York: Springer Verlag, 1984.

[50] P. Frihauf, M. Krstic, and T. Basar. Nash equilibrium seeking for games with non-quadratic payoffs. In *IEEE Conf. on Decision and Control*, pages 881–886, Atlanta, USA, December 2010.

[51] D. Fudenberg and J. Tirole. *Game theory*. The MIT press, 1991.

[52] A. Ganguli, J. Cortés, and F. Bullo. Visibility-based multi-agent deployment in orthogonal environments. In *American Control Conference*, pages 3426–3431, New York, July 2007.

[53] A. Ganguli, J. Cortés, and F. Bullo. Multirobot rendezvous with visibility sensors in nonconvex environments. *IEEE Transactions on Robotics*, 25(2):340–352, 2009.

[54] C. Gao, J. Cortés, and F. Bullo. Notes on averaging over acyclic digraphs and discrete coverage control. *Automatica*, 44(8), 2008. To appear.

[55] B. Gharesifard and J. Cortés. Distributed strategies for generating weight-balanced and doubly stochastic digraphs. *SIAM Journal on Control and Optimization*, October 2009. Submitted.

[56] B. Gidas. Nonstationary Markov chains and convergence of the annealing algorithm. *Journal of Statistical Physics*, 39(1):73–131, April 1985.

[57] A. Gupta, C. Langbort, and T. Basar. Optimal control in the presence of an intelligient jammer with limited actions. In *IEEE Conf. on Decision and Control*, pages 1096–1101, Atlanta, USA, December 2010.

[58] V. Gupta and N. Martins. On stability in the presence of analog erasure channels between controller and actuator. *IEEE Transactions on Automatic Control*, 55(1):175–179, 2010.

[59] C.N. Hadjicostis, C. Langbort, N.C. Martins, and S. Yüksel. Special issue on information processing and decision making in distributed control systems. *International Journal of Systems, Control and Communications*, 2(1-3), 2010.

[60] J. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proceddings of IEEE Special Issue on Technology of Networked Control Systems*, 95(1):138–162, 2007.

[61] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms*. Springer, 1996.

[62] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[63] E. Hörster and R. Lienhart. On the optimal placement of multiple visual sensors. In *Proc. of the third ACM international workshop on Video surveillance and sensor networks*, pages 111–120, 2006.

[64] D. Isaacson and R. Madsen. *Markov chains*. Wiley, 1976.

[65] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[66] A. Jadbabaie, A. Sandroni, and A. Tahbaz-Salehi. Non-bayesian social learning. PIER Working paper 10-005, February 2010.

[67] T. Jiang, I. Matei, and J. S. Baras. A trust based distributed Kalman filtering approach for mode estimation in power systems. In *Proceedings of The First Workshop on Secure Control Systems*, Stockholm, Sweden, April 2010.

[68] B. Johansson, T. Keviczky, M. Johansson, and K. H. Johansson. Subgradient methods and consensus algorithms for solving convex optimization problems. In *IEEE Conf. on Decision and Control*, pages 4185–4190, Cancun, Mexico, December 2008.

[69] A. Kashyap, T. Başar, and R. Srikant. Quantized consensus. *Automatica*, 43(7):1192–1203, 2007.

[70] F. P. Kelly, A. Maulloo, and D. Tan. Rate control in communication networks: Shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49(3):237–252, 1998.

[71] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 2 edition, 1995.

[72] K.C. Kiwiel. Approximations in bundle methods and decompostion of convex programs. *Journal of Optimization Theory and Applications*, 84:529–548, 1995.

[73] V. Kumar, S. Sastry, R. Zachery, F. Bullo, and D. Rus. Special issue on swarming in natural and engineered systems. *Proceedings of IEEE*, September 2011.

[74] A. Kwok and S. Martínez. Deployment algorithms for a power-constrained mobile sensor network. *International Journal of Robust and Nonlinear Control*, 20(7):725–842, 2010.

[75] T. Larsson, M. Patriksson, and A. Strömberg. Ergodic primal convergence in dual subgradient schemes for convex programming. *Mathematical Programming*, 86:283–312, 1999.

[76] K. Laventall and J. Cortés. Coverage control by multi-robot networks with limited-range anisotropic sensory. *International Journal of Control*, 82(6):1113–1121, 2009.

[77] D. Liberzon and J. P. Hespanha. Stabilization of nonlinear systems with limited information feedback. *IEEE Transactions on Automatic Control*, 50(6):910–915, 2005.

[78] S.J. Liu and M. Krstic. Stochastic Nash equilibrium seeking for games with general nonlinear payoffs. *SIAM Journal of Control and Optimization*, 49(1659-1679), 2011.

[79] I. Lobel, A. Ozdaglar, and D. Feijer. Distributed multi-agent optimization with state-dependent communication. *Mathematical Programming*, 2011. to appear.

[80] Z.Q. Luo, M. Gastpar, J. Liu, and A. Swami. Special issue on distributed signal processing in sensor networks. *IEEE Signal Processing Magazine*, July 2006.

[81] J. R. Marden and A. Wierman. Distributed welfare games. *Operations Research*, 2008. submitted.

[82] J.R. Marden, G. Arslan, and J.S. Shamma. Cooperative control and potential games. *IEEE Transactions on Systems, Man, and Cybernetics. Part B: Cybernetics*, 2009.

[83] J.R. Marden and J.S. Shamma. Revisiting log-linear learning: asynchrony, completeness, and payoff-based implementation. *Games and Economic Behavior*, 2008. submitted.

[84] J.R. Marden, H.P. Young, G. Arslan, and J.S. Shamma. Payoff based dynamics for multi-player weakly acyclic games. *SIAM Journal on Control and Optimization*, 48(1):373–396, Febuary 2009.

[85] C.B. Margi, V. Petkov, K. Obraczka, and R. Manduchi. Characterizing energy consumption in a visual sensor network testbed. In *Proc. of 2nd International Conference on Testbeds and Research Infrastructures for the Development of Networks and Communities*, pages 332–339, March 2006.

[86] P. Martin and M. Egerstedt. Optimization of multi-agent motion programs with applications to robotic marionettes. In *Hybrid Systems: Computation and Control*, April 2009.

[87] S. Martínez. Distributed representation of spatial fields through an adaptive interpolation scheme. In *2007 American Control Conference*, pages 2750–2755, 2007.

[88] M. Mehyar, D. Spanos, J. Pongsajapan, S. H. Low, and R. M. Murray. Asynchronous distributed averaging on communication networks. *IEEE/ACM Transactions on Networking*, 15(3):512–520, 2007.

[89] D. Mitra, F. Romeo, and A. Sangiovanni-Vincentelli. Convergence and finite-time behavior of simulated annealing. *Advances in Applied Probability*, 18(3):747–771, September 1986.

[90] Y. Mo, E. Garone, A. Casavola, and B. Sinopoli. False data injection attacks against state estimation in wireless sensor networks. In *IEEE Conf. on Decision and Control*, pages 5967–5972, Atlanta, USA, December 2010.

[91] C. C. Moallemi and B. Van Roy. Consensus propagation. *IEEE Transactions on Information Theory*, 52(11):4753–4766, 2006.

[92] D. Monderer and L. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.

[93] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, 2005.

[94] A. I. Mourikis and S. I. Roumeliotis. Optimal sensing strategies for mobile robot formations: Resource-constrained localization. In *Proceedings of Robotics: Science and Systems*, pages 281–288, Cambridge, USA, 2005.

[95] R. A. Murphey. Target-based weapon target assignment problems. In P. M. Pardalos and L. S. Pitsoulis, editors, *Nonlinear Assignment Problems: Algorithms and Applications*, pages 39–53. Kluwer Academic Publishers, 1999.

[96] G. N. Nair, R. J. Evans, I. M. Y. Mareels, and W. Moran. Topological feedback entropy and nonlinear stabilization. *IEEE Transactions on Automatic Control*, 49(9):1585–1597, 2004.

[97] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans. Feedback control under data rate constraints: an overview. *Proceddings of IEEE Special Issue on Technology of Networked Control Systems*, 95(1):108–137, 2007.

[98] A. Nedic, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis. On distributed averaging algorithms and quantization effects. Technical report, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, November 2007.

[99] A. Nedic and A. Ozdaglar. Approximate primal solutions and rate analysis for dual subgradient methods. *SIAM Journal on Optimization*, 19(4):1757–1780, 2009.

[100] A. Nedic and A. Ozdaglar. Distributed subgradient methods for multi-agent optimization. *IEEE Transactions on Automatic Control*, 54(1):48–61, 2009.

[101] A. Nedic and A. Ozdaglar. Subgradient methods for saddle-point problems. *Journal of Optimization Theory and Applications*, 142(1):205–228, 2009.

[102] A. Nedic, A. Ozdaglar, and P.A. Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 55(4):922–938, 2010.

[103] D. Nesic and A. Teel. Input-output stability properties of networked control systems. *IEEE Transactions on Automatic Control*, 49(10):1650–1667, 2004.

[104] R. D. Nowak. Distributed EM algorithms for density estimation and clustering in sensor networks. *IEEE Transactions on Signal Processing*, 51:2245–2253, 2003.

[105] R. Olfati-Saber. Distributed Kalman filter with embedded consensus filters. In *IEEE Conf. on Decision and Control and European Control Conference*, Seville, Spain, 2005. Submitted.

[106] R. Olfati-Saber. Distributed tracking for mobile sensor networks with information-driven mobility. In *American Control Conference*, pages 4606–4612, New York, USA, July 2007.

[107] R. Olfati-Saber, E. Franco, E. Frazzoli, and J. S. Shamma. Belief consensus and distributed hypothesis testing in sensor networks. In P.J. Antsaklis and P. Tabuada, editors, *Network Embedded Sensing and Control. (Proceedings of NESC'05 Worskhop)*, volume 331 of *Lecture Notes in Control and Information Sciences*, pages 169–182. Springer, 2006.

[108] R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[109] R. Olfati-Saber and J. S. Shamma. Consensus filters for sensor networks and distributed sensor fusion. In *IEEE Conf. on Decision and Control and European Control Conference*, pages 6698–6703, Seville, Spain, December 2005.

[110] A. Oliveira, S. Soares, and L. Nepomuceno. Optimal active power dispatch combining network flow and interior point approaches. *IEEE Transactions on Power Systems*, 18(4):1235–1240, 2003.

[111] A. Olshevsky and J. N. Tsitsiklis. Convergence speed in distributed consensus and averaging. *SIAM Journal on Control and Optimization*, 48(1):33–55, 2009.

[112] A. Olshevsky and J.N. Tsitsiklis. A lower bound on distributed averaging. *IEEE Transactions on Automatic Control*, 2011. provisionally accepted.

[113] J. O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987.

[114] F. Pasqualetti, A. Bicchi, and F. Bullo. Consensus computation in unreliable networks: A system theoretic approach. *IEEE Transactions on Automatic Control*, February 2010. To appear.

[115] F. Pasqualetti, R. Carli, and F. Bullo. Distributed estimation and false data detection with application to power networks. *Automatica*. submitted.

[116] B.T. Polyak. A general method for solving extremum problems. *Soviet Mathematics Doklady*, 3(8):593–597, 1967.

[117] M. G. Rabbat and R. D. Nowak. Decentralized source localization and tracking. In *IEEE Int. Conf. on Acoustics, Speech and Signal Processing*, pages 921–924, May 2004.

[118] M.G. Rabbat, R.D. Nowak, and J.A. Bucklew. Generalized consensus algorithms in networked systems with erasure links. In *the IEEE Workshop on Signal Processing Advances in Wireless Communications*, New York, USA, June 2005.

[119] A. Rantzer. Using game theory for distributed control engineering. In *Games 2008, 3rd World Congress of the Game Theory Society*, 2008.

[120] W. Ren. Consensus seeking in multi-vehicle systems with a time-varying reference state. In *American Control Conference*, pages 717–722, New York, USA, July 2007.

[121] W. Ren. Distributed cooperative attitude synchronization and tracking for multiple rigid bodies. *IEEE Transactions on Control Systems Technology*, 18(2):383–392, 2010.

[122] W. Ren and R. Beard. *Distributed Consensus in Multi-vehicle Cooperative Control*. Communications and Control Engineering. Springer-Verlag, London, 2008.

[123] W. Ren, K. Moore, and Y. Chen. High-order and model reference consensus algorithms in cooperative control of multi-vehicle systems. *ASME Journal of Dynamic Systems, Measurement, and Control*, 129(5):678–688, 2007.

[124] R. T. Rockafella. Augmented lagrangian and applications of the proximal point algorithm in convex programming. *Mathematics of Operations Research*, 1:97–116, 1976.

[125] R.W. Rosenthal. A class of games possessing pure strategy Nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.

[126] T. Roughgarden. *Selfish routing and the price of anarchy*. MIT press, 2005.

[127] S. Roy, C. Ellis, S. Shiva, D. Dasgupta, V. Shandilya, and Q. Wu. A survey of game theory as applied to network security. In *Proceedings of the 43rd Hawaii International Conference on System Sciences*, pages 1–10, Hawaii, USA, 2010.

[128] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry. Foundations of control and estimation over lossy networks. *Proceddings of IEEE Special Issue on Technology of Networked Control Systems*, 95(1):163–187, 2007.

[129] J.S. Shamma, editor. *Cooperative control of distributed multi-agent systems*. Wiley-Interscience, 2008.

[130] T. C. Shermer. Recent results in art galleries. *Proceedings of the IEEE*, 80(9):1384–1399, 1992.

[131] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry. Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Control*, 49(9):1453–63, 2004.

[132] D. P. Spanos, R. Olfati-Saber, and R. M. Murray. Dynamic consensus on mobile networks. In *IFAC World Congress*, Prague, Czech Republic, July 2005.

[133] M. S. Stankovic, K. H. Johansson, and D. M. Stipanovic. Distributed seeking of Nash equilibria with applications to mobile sensor networks. *IEEE Transactions on Automatic Control*. conditionally accepted.

[134] S. Sundaram and C. N. Hadjicostis. Distributed function calculation via linear iterative strategies in the presence of malicious agents. *IEEE Transactions on Automatic Control*. To appear.

[135] S. Sundaram and C. N. Hadjicostis. Finite-time distributed consensus in graphs with time-invariant topologies. In *American Control Conference*, pages 711–716, New York, 2007.

[136] S. Sundhar Ram, A. Nedic, and V. V. Veeravalli. Distributed and recursive parameter estimation in parametrized linear state-space models. *IEEE Transactions on Automatic Control*, 55(2):488–492, 2010.

[137] R. S. Sutton and A. G. Barto. *Reinforcement Learning: An Introduction*. MIT Press, 1998.

[138] A. Tahbaz-Salehi and A. Jadbabaie. Consensus over random networks. *IEEE Transactions on Automatic Control*, 53(3):791–795, 2008.

[139] A. Tang and L. Andrew. Game theory for heterogeneous flow control. In *Proc. of 42nd Annual Conference on Information Sciences and Systems*, pages 52–56, December 2008.

[140] A. Teixeira, S. Amin, H. Sandberg, K. H. Johansson, and S. S. Sastry. Cyber security analysis of state estimators in electric power systems. In *IEEE Conf. on Decision and Control*, pages 5991–5998, Atlanta, USA, December 2010.

[141] G. Theodorakopoulos and J. S. Baras. Game theoretic modeling of malicious users in collaborative networks. *IEEE Journal on Selected Areas in Communications*, 7:1317–1327, 2008.

[142] J. N. Tsitsiklis. *Problems in Decentralized Decision Making and Computation.* PhD thesis, Massachusetts Institute of Technology, November 1984. Technical Report LIDS-TH-1424. Available at `http://web.mit.edu/jnt/www/Papers/PhD-84-jnt.pdf`.

[143] J. Urrutia. Art gallery and illumination problems. In J. R. Sack and J. Urrutia, editors, *Handbook of Computational Geometry*, pages 973–1027. North-Holland, 2000.

[144] C. Vu. *Distributed energy-efficient solutions for area coverage problems in wireless sensor networks.* PhD thesis, Georgia State University, 2007.

[145] Z. Wang and F. Paganini. Global stability with time-delay in network congestion control. In *Proceedings of the 41st IEEE Conference on Decision and Control*, pages 3632–3637, Las Vegas, USA, 2002.

[146] H. Wei, H. Sasaki, J. Kubokawa, and R. Yokoyama. An interior point nonlinear programming for optimal power flow problems with a novel data structure. *IEEE Transactions on Power Systems*, 13:870–877, 1998.

[147] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53:65–78, 2004.

[148] L. Xiao, S. Boyd, and S.-J. Kim. Distributed average consensus with least-mean-square deviation. *Journal of Parallel and Distributed Computing*, 67(1):33–46, 2007.

[149] L. Xie, Y. Mo, and B. Sinopoli. False data injection attacks in electricity markets. In *2010 First International Conference on Smart Grid Communications*, pages 226–231, Gaithersburg, USA, October 2010.

[150] C. Xu and F. Lau. *Load balancing in parallel computers: theory and practice.* Kluwer Academic Publishers, 1997.

[151] P. Yang, R. A. Freeman, and K. M. Lynch. Distributed cooperative active sensing using consensus filters. In *IEEE International Conference on Robotics and Automation*, pages 405–410, Roma, Italy, April 2007.

[152] P. Yang, R. A. Freeman, and K. M. Lynch. Multi-agent coordination by decentralized estimation and control. *IEEE Transactions on Automatic Control*, 53(11):2480–2496, 2008.

[153] E. Yildiz, D. Acemoglu, A. Ozdaglar, A. Saberi, and A. Scaglione. Discrete opinion dynamics with stubborn agents. MIT LIDS 2858, 2011.

[154] H.P. Young. The evolution of conventions. *Econometrica*, 61:57–84, January 1993.

[155] M. Zhu and S. Martínez. Stackelberg game analysis of correlated attacks in cyber-physical systems. In *2011 American Control Conference*, 2011. To appear.