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Abstract

The Lighthill transformation gives the solution for two-dimensional diffusion layers at high Schmidt numbers if the shear-stress distribution is known. This diffusion-layer solution breaks down at the rear of the object, but it provides a basis for obtaining the solution in the rear region. The analysis is restricted to symmetric, two-dimensional cylinders with rounded back sides at Reynolds numbers so low that no eddies exist behind the cylinder. The local Nusselt number in the rear region is found to be proportional to the Schmidt number to the one-sixth power.

Introduction

Levich[1] obtained the diffusion-layer solution for mass transfer to a sphere in Stokes flow at high Péclet numbers, an analysis which is not valid near the rear of the sphere. Recently, it has been shown[2] how to treat the rear region and the wake. Such an analysis can be extended directly to other axisymmetric bodies and to somewhat higher Reynolds numbers (where eddies still are not formed), and a similar analysis can be applied to two-dimensional objects.

At high Schmidt numbers, the diffusion layer is very thin, even when compared to any hydrodynamic boundary layer which may be present. Under these conditions it is valid to approximate the velocity within the diffusion layer by the first term in an expansion in the normal distance y from the surface, for example,

$$v_x = y\beta(x) \qquad \text{where } \beta = \partial v_x/\partial y \text{ at } y = 0 \text{ .} \tag{1}$$

For a steady, two-dimensional diffusion layer, the Lighthill transformation[3] then provides a similarity solution for the concentration profile, as pointed out by Acrivos.[4] Thus the rate of mass transfer can be calculated from a knowledge only of $\beta(x)$, the velocity derivative at the wall.

The Lighthill method also breaks down at the rear of an object, but from earlier work[2] on mass transfer to the rear of a sphere it appears likely that the rate of mass transfer in the rear region can also be calculated correctly when only $\beta(x)$ is given. Figure 1 illustrates the types of two-dimensional objects considered in this work. The back sides of these are similar in that they are all rounded. It is assumed that the flow is symmetric about a plane parallel to the undisturbed flow and that there are no eddies behind the object. The Lighthill transformation

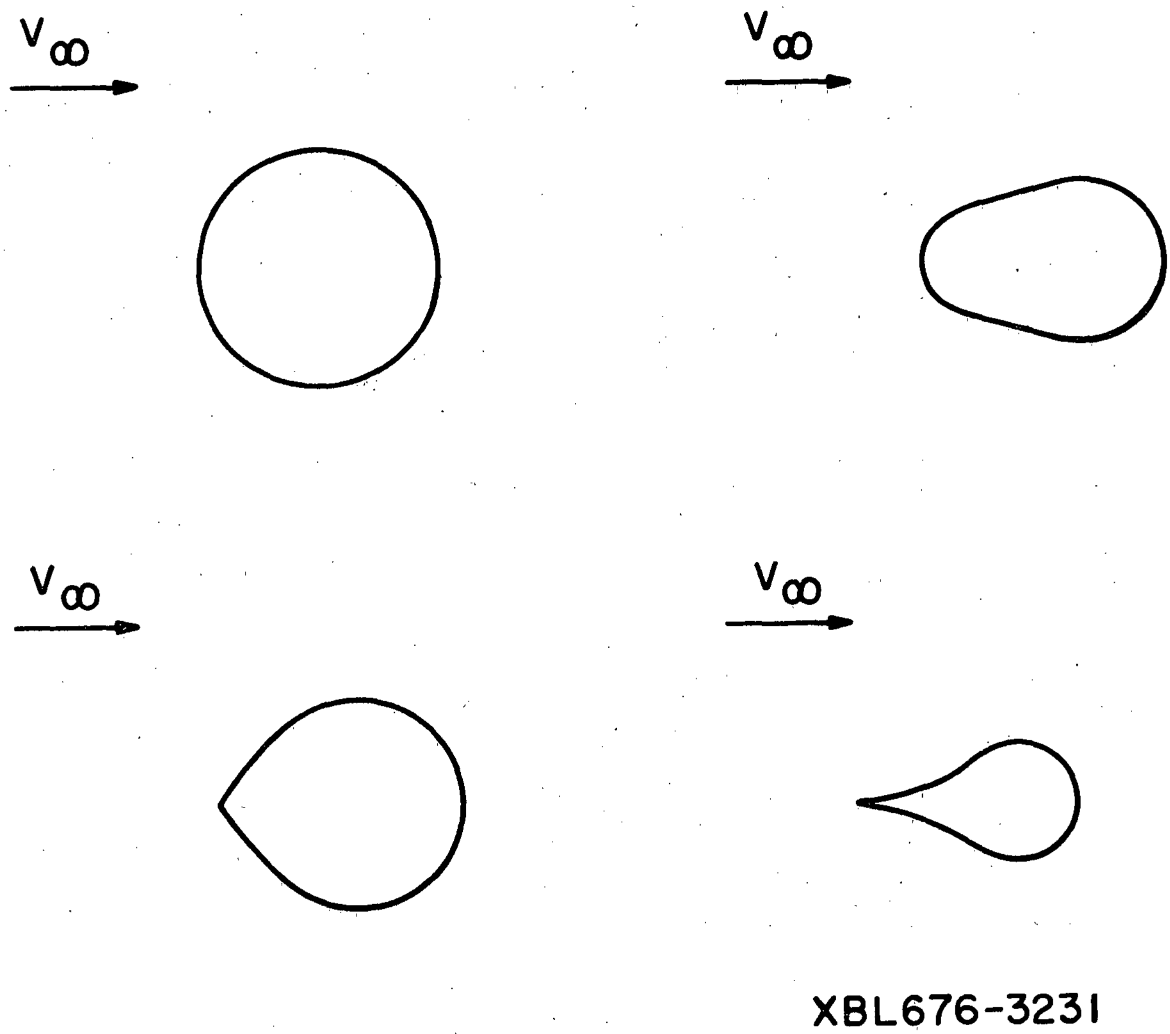

Figure 1. Symmetric cylinders with rounded back sides.

provides the diffusion-layer solutions at high Schmidt numbers for the front part of these objects, as well as for a wider class of problems.

The two-dimensional problem is treated first. Later it is shown in detail how the earlier results[2] for the sphere are applicable to other axisymmetric problems. The rate of mass transfer to the surface can be calculated from $\beta(x)$, but treatment of the concentration profiles in the wake would require additional information about the velocity distribution.

Analysis

Near the solid surface, the tangential component of the velocity can be approximated by equation (1), and for two-dimensional objects the normal component is

$$v_y = -\frac{1}{2} y^2 \beta'(x) . \tag{2}$$

For a constant concentration at the surface, the Lighthill transformation yields the solution in the diffusion layer

$$\Theta_d = \frac{1}{\Gamma(4/3)} \int_0^{\eta} e^{-x^3} dx , \tag{3}$$

where

$$\Theta = \frac{c_i - c_o}{c_\infty - c_o} \quad \text{and} \quad \eta = \frac{y\sqrt{\beta}}{\left[9D \int_0^x \sqrt{\beta}\, dx\right]^{1/3}} . \tag{4}$$

The subscript d denotes the diffusion-layer solution; x is measured along the surface from the front, and y is the normal distance from the surface.

We suspect, on the basis of the treatment of mass transfer to the rear of a sphere in Stokes flow, that the region of interest at the rear of the cylinder is small in both the x and y directions and that, since x and y are essentially cartesian coördinates in this region, the equation of convective diffusion can be written

$$v_x \frac{\partial\Theta}{\partial x} + v_y \frac{\partial\Theta}{\partial y} = D\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\right) . \qquad (5)$$

It should still be possible further to approximate the velocity profiles by their values near the solid surface, but β should go to zero at the rear like

$$\beta \rightarrow A(x_o - x) \text{ as } x \rightarrow x_o , \qquad (6)$$

where x_o is the value of x at the rear of the cylinder. In the rear region the equation of convective diffusion thus becomes

$$Ay(x_o-x)\frac{\partial\Theta}{\partial x} + \frac{1}{2}Ay^2\frac{\partial\Theta}{\partial y} = D\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\right) . \qquad (7)$$

The diffusion coefficient and the constant A can be made to disappear by stretching the coördinates:

$$X = (x_o-x)(A/D)^{1/3}, \quad Y = y(A/D)^{1/3} , \qquad (8)$$

so that Θ satisfies the equation

$$\frac{1}{2}Y^2\frac{\partial\Theta}{\partial Y} - YX\frac{\partial\Theta}{\partial X} = \frac{\partial^2\Theta}{\partial X^2} + \frac{\partial^2\Theta}{\partial Y^2} . \qquad (9)$$

Near the rear region the similarity variable η of the diffusion layer can be expressed in these coördinates in the form

$$\eta = \frac{y\sqrt{\beta(x)}}{\left[9D\int_0^{x_o - X(D/A)^{1/3}} \sqrt{\beta}\, dx\right]^{1/3}} \approx \frac{y\sqrt{A(x_o-x)}}{\left[9D\int_0^{x_o}\sqrt{\beta}\, dx\right]^{1/3}} = Y\sqrt{X}\left[\frac{\sqrt{D}}{9\int_0^{x_o}\sqrt{\beta}\, dx}\right]^{1/3} , \qquad (10)$$

which is small because the diffusion coefficient is small. Hence, in order to match the solution in the rear region with the diffusion-layer solution, we must have

$$\Theta \to \frac{Y\sqrt{X}}{\Gamma(4/3)}\left[\frac{\sqrt{D}}{9\int_0^{x_0}\sqrt{\beta}\,dx}\right]^{1/3} \quad \text{as } X \to \infty . \tag{11}$$

The problem in the rear region can finally now be made independent of D by stretching the concentration. Let

$$\bar{\Theta} = \Theta\,\Gamma(4/3)\left[\frac{9}{\sqrt{D}}\int_0^{x_0}\sqrt{\beta}\,dx\right]^{1/3} . \tag{12}$$

This will also make the problem for $\bar{\Theta}$ independent of the Reynolds number and of the geometry of the front part of the cylinder.

The diffusion equation now becomes

$$\frac{1}{2}Y^2\frac{\partial\bar{\Theta}}{\partial Y} - YX\frac{\partial\bar{\Theta}}{\partial X} = \frac{\partial^2\bar{\Theta}}{\partial X^2} + \frac{\partial^2\bar{\Theta}}{\partial Y^2} , \tag{13}$$

with the boundary conditions

1. $\bar{\Theta} = 0$ at $Y = 0$, on the solid surface.
2. $\partial\bar{\Theta}/\partial X = 0$ at $X = 0$, on the plane of symmetry.
3. $\bar{\Theta} \to Y\sqrt{X}$ as $X \to \infty$, in order to match with the diffusion-layer solution.
4. As $Y \to \infty$, we suspect that $\partial^2\bar{\Theta}/\partial Y^2$ should become negligible, in order to match with regions downstream where diffusion in the direction normal to the surface should be negligible.

The last boundary condition can be stated more explicitly by finding the asymptotic solution as $Y \to \infty$. Then $\bar{\Theta}$ should satisfy the equation

$$\frac{1}{2}Y^2\frac{\partial\bar{\Theta}}{\partial Y} - YX\frac{\partial\bar{\Theta}}{\partial X} = \frac{\partial^2\bar{\Theta}}{\partial X^2} \tag{14}$$

with the solution

$$\overline{\Theta} \rightarrow Y^{3/4} \; F(\xi) \text{ where } \xi = X\sqrt{Y} \tag{15}$$

and where F satisfies the ordinary differential equation

$$F'' + \frac{3}{4}\xi F' - \frac{3}{8} F = 0 \tag{16}$$

with the boundary conditions

$$F' = 0 \text{ at } \xi = 0 \text{ and } F \rightarrow \sqrt{\xi} \text{ as } \xi \rightarrow \infty \, .$$

Results

The problem posed in equation (13) was solved by finite-difference methods using successive overrelaxation on a high-speed digital computer, the method being similar to that used for mass transfer to the rear of a sphere.[2] The rate of mass transfer to the cylinder in the rear region can be calculated from the values of $\partial\overline{\Theta}/\partial Y$ at $Y = 0$, which are plotted in figure 2. The value at the rear itself is 0.819. From these values and the diffusion-layer solution, one can construct a composite expansion for the mass-transfer rate if desired. The local Nusselt number is of order $Sc^{1/6}$ near the rear of the cylinder.

It is interesting that the solution in the rear region approaches the diffusion-layer solution from below rather than from above as $X \rightarrow \infty$. That this is really the case can be verified by obtaining the asymptotic solution for $\overline{\Theta}$ as $X \rightarrow \infty$. Let

$$\overline{\Theta} = Y\sqrt{X} + T. \tag{17}$$

Substitution into equation (13) gives a differential equation for T:

$$\frac{1}{2} Y^2 \frac{\partial T}{\partial Y} - YX \frac{\partial T}{\partial X} = - \frac{Y}{4X^{3/2}} + \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \, . \tag{18}$$

The term $- Y/4X^{3/2}$ acts as a negative source in this equation. By neglecting

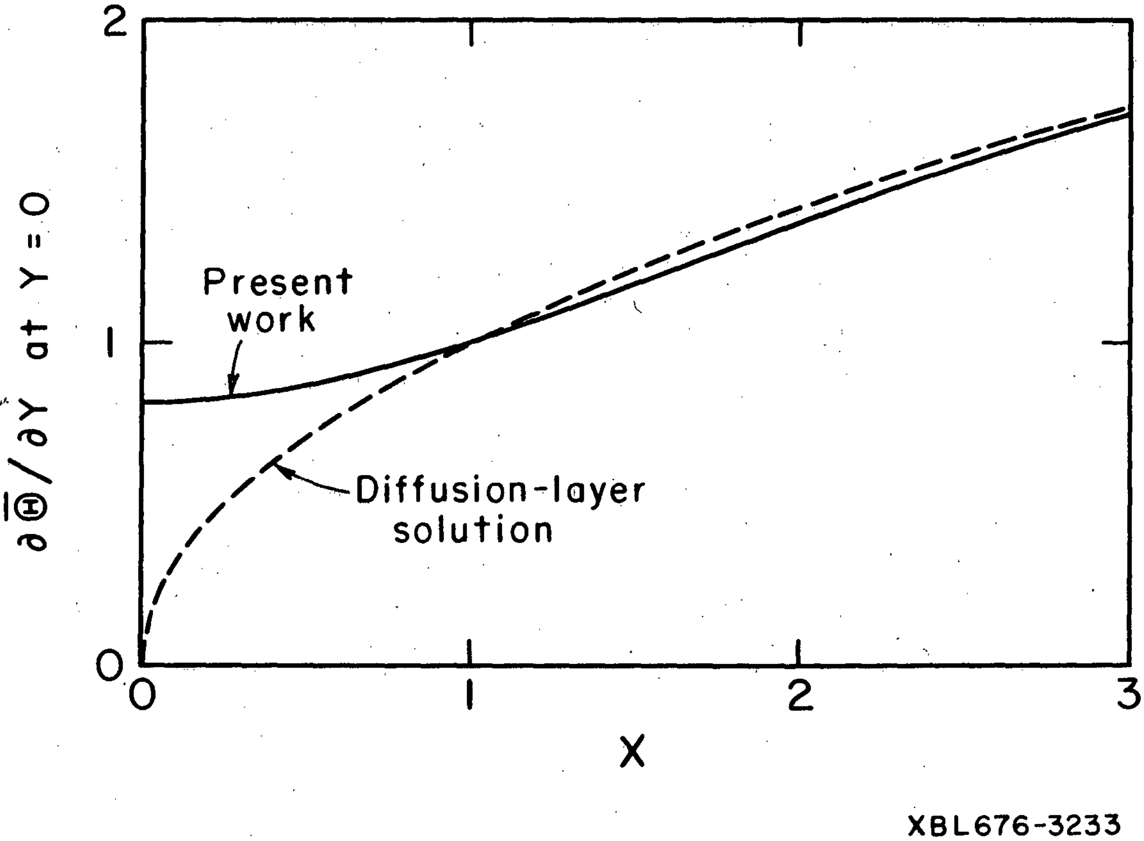

Figure 2. Rate of mass transfer near the rear of a bluff two-dimensional body at large Schmidt numbers.

the term $\partial^2 T/\partial X^2$ as $X \to \infty$, one obtains the solution for T as $X \to \infty$:

$$T = G(Y)/X^{3/2} , \tag{19}$$

where G satisfies the ordinary differential equation

$$4G'' = Y + 2Y^2 G' + 6YG \tag{20}$$

and the boundary conditions

$$G = 0 \text{ at } Y = 0 \quad \text{and } G \to -1/6 \text{ as } Y \to \infty .$$

Thus the concentration derivative can be expressed for large X as

$$\left.\frac{\partial \bar{\Theta}}{\partial Y}\right|_{Y=0} \to \sqrt{X} + \frac{G'(0)}{X^{3/2}} = \sqrt{X} - \frac{0.1242}{X^{3/2}} \text{ as } X \to \infty . \tag{21}$$

Here $\sqrt{X}$ represents the diffusion-layer solution as plotted in figure 2, and the correction term $G'(0)/X^{3/2}$ is negative.

The Axisymmetric Problem

In this section we should like to show explicitly how the results for mass transfer to the rear of a sphere in Stokes flow[2] can be applied to other axisymmetric problems as long as the back side of the object is rounded and the Reynolds number is so low that there are no eddies behind the object. For an axisymmetric body the continuity equation within the diffusion layer has the form

$$\frac{\partial \mathcal{R} v_x}{\partial x} + \mathcal{R} \frac{\partial v_y}{\partial y} = 0 , \tag{22}$$

where $\mathcal{R}(x)$ is the normal distance of the surface from the axis of symmetry. Correspondingly, when v_x is approximated by equation (1), the appropriate form for v_y within the diffusion layer is

$$v_y = -\frac{1}{2} y^2 \frac{(\mathcal{R}\beta)'}{\mathcal{R}} . \tag{23}$$

The solution in the diffusion layer is then given by equation (3) where

$$\eta = \frac{y\sqrt{\mathcal{R}\beta}}{\left[9D\int_0^x \mathcal{R}\sqrt{\mathcal{R}\beta}\,dx\right]^{1/3}} \,. \tag{24}$$

Near the rear of the body,

$$\beta \rightarrow A(x_o - x) \quad \text{and } \mathcal{R} \rightarrow x_o - x \text{ as } x \rightarrow x_o \,. \tag{25}$$

Since y and $\mathcal{R}$ are like cylindrical coördinates in the small region at the rear, the equation of convective diffusion becomes

$$v_x \frac{\partial\Theta}{\partial x} + v_y \frac{\partial\Theta}{\partial y} = A\mathcal{R}y \frac{\partial\Theta}{\partial x} + Ay^2 \frac{\partial\Theta}{\partial y} = D\left(\frac{\partial^2\Theta}{\partial y^2} + \frac{1}{\mathcal{R}}\frac{\partial}{\partial x}\mathcal{R}\frac{\partial\Theta}{\partial x}\right) . \tag{26}$$

In order to obtain the same form as for the sphere, let us stretch the coördinates as follows:

$$Y = y(2A/3D)^{1/3}, \quad S = \mathcal{R}(2A/3D)^{1/3} = (x_o - x)(2A/3D)^{1/3} \,. \tag{27}$$

The mass-transfer equation becomes

$$\frac{3}{2} Y^2 \frac{\partial\Theta}{\partial Y} - \frac{3}{2} YS \frac{\partial\Theta}{\partial S} = \frac{\partial^2\Theta}{\partial Y^2} + \frac{\partial^2\Theta}{\partial S^2} + \frac{1}{S}\frac{\partial\Theta}{\partial S} \,. \tag{28}$$

In the coördinates Y and S, the similarity variable η of the diffusion-layer solution is

$$\eta = \frac{y\sqrt{\mathcal{R}\beta}}{\left[9D\int_0^{x_o} \mathcal{R}\sqrt{\mathcal{R}\beta}\,dx\right]^{1/3}} = \frac{YS\,D^{1/3}}{\left[4\sqrt{A}\int_0^{x_o} \mathcal{R}\sqrt{\mathcal{R}\beta}\,dx\right]^{1/3}} \,. \tag{29}$$

Consequently, in order to match with the diffusion-layer solution, the solution in the rear region must satisfy the condition

$$\Theta \rightarrow \frac{\eta}{\Gamma(4/3)} = \frac{YS}{\Gamma(4/3)} \frac{D^{1/3}}{\left[4\sqrt{A}\int_0^{x_o} \mathcal{R}\sqrt{\mathcal{R}\beta}\,dx\right]^{1/3}} \quad \text{as } S \rightarrow \infty \,. \tag{30}$$

With a new dimensionless concentration

$$\overline{\Theta} = \Theta \frac{\Gamma(4/3)}{D^{1/3}} \left[4\sqrt{A} \int_0^{x_0} \mathcal{R}\sqrt{\mathcal{R}\beta}\, dx \right]^{1/3} , \qquad (31)$$

the problem becomes independent of the conditions on the front part of the body. Now $\overline{\Theta}$ satisfies

$$\frac{3}{2} Y^2 \frac{\partial\overline{\Theta}}{\partial Y} - \frac{3}{2} YS \frac{\partial\overline{\Theta}}{\partial S} = \frac{\partial^2\overline{\Theta}}{\partial Y^2} + \frac{\partial^2\overline{\Theta}}{\partial S^2} + \frac{1}{S}\frac{\partial\overline{\Theta}}{\partial S} , \qquad (32)$$

subject to the boundary conditions

1. $\overline{\Theta} = 0$ at $Y = 0$, on the solid surface.
2. $\partial\overline{\Theta}/\partial S = 0$ at $S = 0$, on the rear axis.
3. $\overline{\Theta} \to YS$ as $S \to \infty$, in order to match with the diffusion-layer solution.
4. As $Y \to \infty$, the term $\partial^2\overline{\Theta}/\partial Y^2$ should become negligible.

A similar problem has been solved[2] for mass transfer to the rear of a sphere in Stokes flow. The dimensionless mass-transfer rate $\partial\overline{\Theta}/\partial Y$ at $Y = 0$ is plotted in figure 3. For large S,

$$\left.\frac{\partial\overline{\Theta}}{\partial Y}\right|_{Y=0} \to S + \frac{0.406}{S} \quad \text{as } S \to \infty , \qquad (33)$$

while at the rear itself the value is 1.124.

For a sphere in Stokes flow

$$\mathcal{R} = R \sin\frac{x}{R}, \quad \beta = \frac{3v_\infty}{2R}\sin\frac{x}{R}, \quad A = \frac{3v_\infty}{2R^2}, \quad Y = \frac{y}{R}\left(\frac{Rv_\infty}{D}\right)^{1/3} , \quad S = \theta\left(\frac{Rv_\infty}{D}\right)^{1/3} . \qquad (34)$$

Substitution into the defining equations shows that the local Nusselt number in the rear region is related to the derivative of $\overline{\Theta}$ by

$$Nu(\theta) = 2R \left.\frac{\partial\Theta}{\partial y}\right|_{y=0} = \frac{2}{\Gamma(4/3)(3\pi)^{1/3}} \left.\frac{\partial\overline{\Theta}}{\partial Y}\right|_{Y=0} = 1.060 \left.\frac{\partial\overline{\Theta}}{\partial Y}\right|_{Y=0} . \qquad (35)$$

This gives the value of the local Nusselt number at the rear of the sphere as 1.192, in agreement with the value reported[2] earlier.

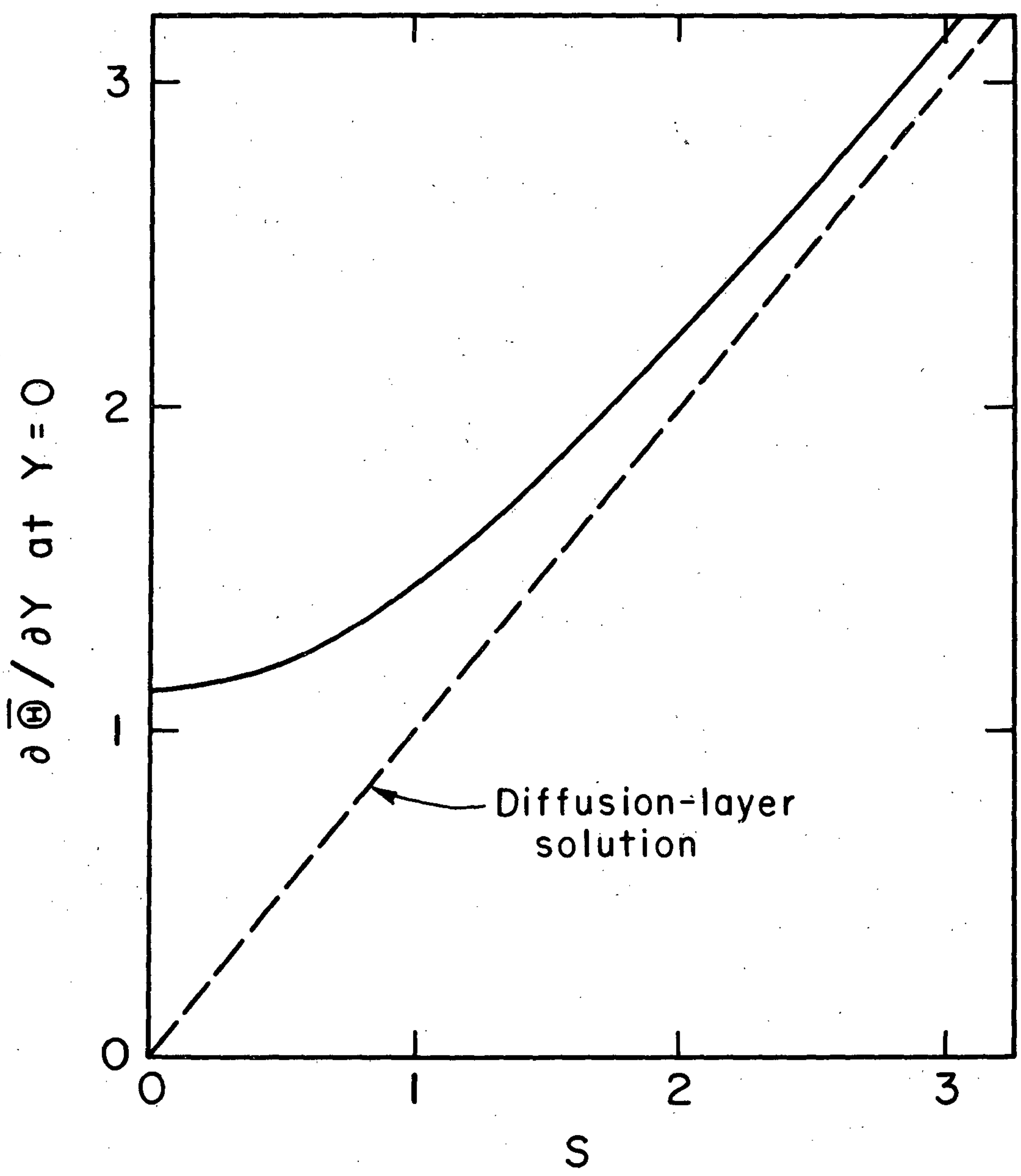

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Figure 3. Rate of mass transfer near the rear of a bluff axisymmetric body at large Schmidt numbers.

Conclusions

A diffusion-layer solution for velocity profiles is not valid near where the flow separates from the body. There are a few cases where such problems can be elucidated and where a valid description of the behavior near the point of separation can be obtained. For symmetric cylinders and axisymmetric bodies with rounded back sides, this separation occurs at the rear if no eddies are formed. Then, as shown in this work, the diffusion-layer solution for high Schmidt numbers can be used as a basis to obtain a valid approximation to the mass-transfer rate near the rear of the object.

The importance of this work lies mainly in providing this valid approximation to the mass-transfer rate near the rear, since the contribution of the rear region to the average Nusselt number is small. However, this contribution is considerably larger for the two-dimensional objects than for the axisymmetric bodies, as indicated in table 1. One should expect that, in the calculation of higher order contributions to the average Nusselt number by the diffusion-layer method, it will eventually be necessary to take into account the singular behavior at the rear.

Table 1. Schmidt-number dependence of the Nusselt number at high Schmidt numbers.

	two-dimensional problem	axisymmetric problem
order of local Nusselt number in the rear region	$O(Sc^{1/6})$	$O(1)$
order of contribution of rear region to the average Nusselt number	$O(Sc^{-1/6})$	$O(Sc^{-2/3})$
order of average Nusselt number	$O(Sc^{1/3})$	$O(Sc^{1/3})$

Acknowledgment

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Nomenclature

A - shows how β goes to zero at the rear.

c_i - concentration of diffusing species.

c_o - concentration at the surface.

c_∞ - concentration far from the body.

D - diffusion coefficient.

F - expresses behavior of $\overline{\Theta}$ as $Y \to \infty$.

G - expresses behavior of $\overline{\Theta}$ as $X \to \infty$.

Nu - Nusselt number.

R - radius of sphere.

$\mathcal{R}$ - distance of surface from axis of symmetry.

S - dimensionless distance along surface from the rear.

Sc - Schmidt number, $Sc = \nu/D$.

T - correction to diffusion-layer solution as $X \to \infty$.

v_x - velocity in x-direction.

v_y - velocity in y-direction.

v_∞ - velocity far from the body.

x - distance measured along surface from the front.

x_o - value of x at the rear.

X - dimensionless distance along surface from the rear.

y - normal distance from the surface.

Y - dimensionless distance from the surface.

β - velocity derivative at the surface.

$\Gamma(4/3)$ - gamma function, $\Gamma(4/3) = 0.89298$.

η - similarity variable for diffusion layer.

θ - angle measured from the rear of the sphere.

Θ - dimensionless concentration.

$\overline{\Theta}$ - stretched dimensionless concentration appropriate to rear region.

ν - kinematic viscosity.

ξ - similarity variable for behavior as $Y \rightarrow \infty$.

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