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CHARACTERIZATION OF DAMPED LINEAR DYNAMICAL SYSTEMS IN FREE MOTION

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Abstract. It is well known that the free motion of a single-degree-of-freedom damped linear dynamical system can be characterized as overdamped, underdamped, or critically damped. Using the methodology of phase synchronization, which transforms any system of linear second-order differential equations into independent second-order equations, this characterization of free motion is generalized to multi-degree-of-freedom damped linear systems. A real scalar function, termed the viscous damping function, is introduced as an extension of the classical damping ratio. It is demonstrated that the free motion of a multi-degree-of-freedom system is characterized by its viscous damping function, and sometimes the characterization may be conducted with ease by examining the extrema of the viscous damping function.

1. **Introduction.**

We consider the set of homogeneous linear second-order equations

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{0} \ , \tag{1}$$

with initial conditions $\mathbf{q}(0) = \mathbf{q}_0$ and $\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0$. All quantities in (1) are real and the superposed dots denote derivatives with respect to the independent variable $t \geq 0$ (time). The coefficients $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric positive definite (SPD) $n \times n$ matrices, and $\mathbf{q}(t)$ is an n-dimensional column vector. Equation (1) is a cornerstone in vibration theory and, for example, models the motion of particles around their equilibrium positions, or the currents and voltages in electrical networks [13, 14, 16, 18, 22, 27]. Adopting vibration terminology, we refer to (1) as an n-degree-of-freedom linear system, or simply a system, for short. The response of the system (1) can exhibit oscillations, i.e., the components of the solution $\mathbf{q}(t)$ can cross zero infinitely often before settling to zero as $t \to \infty$. The decay of these vibrations

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is controlled by the term $\mathbf{C}\dot{\mathbf{q}}(t)$, and the matrix $\mathbf{C}$ is referred to as the damping matrix. If the damping is very strong, no oscillatory behavior can be observed, and the components of the solution $\mathbf{q}(t)$ cross zero at most once before approaching zero as $t \to \infty$.

In many applications, it is important to determine the effect of damping on the solution of (1), i.e., to find out whether system (1) exhibits oscillatory or non-oscillatory behaviors. For example, in engineering design applications one needs to know how oscillations can be suppressed by varying certain system parameters; a very slow decay of the oscillations is often desirable in electrical networks. Characterization of the free motion of (1) is well understood in single-degree-of-freedom systems (i.e., $n = 1$ for which the coefficient matrices are simply positive real numbers m, c and k). In this case, the nature of damped free motion can be determined by inspection of the viscous damping ratio, which is a scalar defined by (see, e.g., [22])

$$\zeta = \frac{c}{2\sqrt{km}} \,. \tag{2}$$

In vibration terminology, the system is termed underdamped if $\zeta < 1$; it is critically damped if $\zeta = 1$ and overdamped if $\zeta > 1$. Oscillatory behaviors can be observed in underdamped systems, while the free response of an overdamped system decays exponentially without oscillations. Critical damping represents the boundary between oscillatory and non-oscillatory behaviors.

The situation is less clear in multi-degree-of-freedom systems. In principle, one could determine whether or not the free response of system (1) is oscillatory by inspection of its solution. However this approach is impractical because the system may exhibit oscillations for one set of initial conditions, while the response is non-oscillatory for another set. Searching for oscillatory and non-oscillatory behaviors within the space of initial conditions is unfeasible for large systems, so it is desirable to study the effects of viscous damping based upon solution of algebraic equations, rather than by studying the differential equation (1). Various criteria for determining the response characteristics of (1) have been presented in the literature. For example, a sufficient condition for non-oscillatory behavior is Duffins overdamping condition [9]

$$(\mathbf{x}^T\mathbf{C}\mathbf{x})^2 > 4(\mathbf{x}^T\mathbf{M}\mathbf{x})(\mathbf{x}^T\mathbf{K}\mathbf{x}), \tag{3}$$

for all real column vectors $\mathbf{x} \neq \mathbf{0}$. However, this condition and others reported in [1, 3, 4] are rather difficult to verify and have not really found their ways into applications. Most other approaches to this problem [2, 12, 15, 24, 25, 26, 27] rely upon simultaneous diagonalization of the coefficient matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ by linear coordinate transformations. However, it is well known that three matrices cannot be diagonalized by linear transformations unless certain restrictive conditions apply [5]. Systems that can be diagonalized by linear coordinate transformations are termed classically damped. The techniques in [2, 12, 15, 24, 25, 26, 27] apply to classically damped systems only and are not applicable in general.

The purpose of this paper is to study the response characteristics of multi-degree-of-freedom, second-order linear systems under viscous damping. Unlike previous attempts, no restriction is placed on the damping matrix $\mathbf{C}$. In Section 2, we briefly review a methodology and algorithm for transforming (1) into independent scalar second-order equations using a time-shifting map. This process is termed "decoupling." Decoupling simplifies the response characterization of system (1) to studying the response characteristics of n independent single-degree-of-freedom systems. It is shown that the decoupling of (1) represents a complete solution to this problem, but it requires the solution of a quadratic eigenvalue problem. In Section 3, we define a real scalar function and show that the effects of viscous damping can be determined by finding the global extrema of this function, without solving any eigenvalue problems. In Section 4, several examples are given to highlight advantages and disadvantages of the decoupling and minimization approaches. The paper concludes with a summary of major findings in Section 5.

2. **The decoupling of second-order linear systems.** It is well known that two SPD matrices $\mathbf{M}$ and $\mathbf{K}$ can be simultaneously diagonalized by a congruence transformation [22]. The same congruence transformation that diagonalizes $\mathbf{M}$ and $\mathbf{K}$ also diagonalizes a SPD matrix $\mathbf{C}$ if and only if [5]

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \ . \tag{4}$$

It follows that system (1) generally cannot be decoupled into a set of mutually independent, real, scalar, second-order equations by a linear mapping $\mathbf{q}(t) \to \mathbf{L}\mathbf{p}(t)$, with the linear operator $\mathbf{L}$ independent of t. However, it was recently shown that *any* system can be decoupled if one utilizes time-dependent transformations [17, 20, 21, 23]. Here, we follow [23] closely to review how system (1) is decoupled.

Associated with (1) is the regular quadratic eigenvalue problem [11, 18, 19, 30]

$$(\mathbf{M}\lambda_j^2 + \mathbf{C}\lambda_j + \mathbf{K})\mathbf{v}_j = \mathbf{0} \ , \tag{5}$$

where λ_j is termed an eigenvalue and $\mathbf{v}_j$ is the corresponding eigenvector. There are $2n$ eigenpairs $\{\lambda_j, \mathbf{v}_j\}$ $(j = 1, \ldots, 2n)$ that are complex in general. Because $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real matrices, the complex conjugate of the eigenpair $\{\lambda_j, \mathbf{v}_j\}$ is also an eigenpair. The system is non-defective if the eigenvectors associated with repeated eigenvalues are linearly independent [18, 19, 30]. Because a damped linear system selected at random is non-defective with probability one [20], we assume for the reminder of this paper that (1) is non-defective. However, this assumption can be relaxed [17].

Upon solving the quadratic eigenvalue problem (5), the solution of the differential equation (1) can be written in terms of the solution of the algebraic equation (5):

$$\mathbf{q}(t) = \sum_{j=1}^{2n} \mathbf{v}_j e^{\lambda_j t} a_j \ , \tag{6}$$

where a_j are constants determined by the initial conditions. If all eigenvalues are complex, the solution becomes

$$\mathbf{q}(t) = \sum_{j=1}^{n} \mathbf{v}_j e^{\lambda_j t} a_j + \overline{\mathbf{v}}_j e^{\overline{\lambda}_j t} \overline{a}_j \ . \tag{7}$$

Every summand in the above equations is real, and we refer to a summand

$$\mathbf{s}_j(t) = \mathbf{v}_j e^{\lambda_j t} a_j + \overline{\mathbf{v}}_j e^{\overline{\lambda}_j t} \overline{a}_j \tag{8}$$

as a mode. These modes $\mathbf{s}_j(t)$ $(j = 1, \ldots, n)$ may be written in the form

$$\mathbf{s}_j(t) = a_j \mathbf{v}_j e^{\lambda_j t} + \overline{a}_j \overline{\mathbf{v}}_j e^{\overline{\lambda}_j t} = C_j e^{\alpha_j t} \begin{bmatrix} r_{j1} \cos(\omega_j t - \theta_j - \varphi_{j1}) \\ \vdots \\ r_{jn} \cos(\omega_j t - \theta_j - \varphi_{jn}) \end{bmatrix} \ , \tag{9}$$

where θ_j and C_j depend on the initial conditions, α_j and ω_j denote the real and imaginary part of the eigenvalue λ_j, respectively, and where r_{jk} and φ_{jk} are the absolute value and phase angle, respectively, of the elements of the eigenvector $\mathbf{v}_j$. When the modes are written as in (9), it becomes clear that each component of (1) performs exponentially decaying harmonic motion with the same frequency and the same exponential decay when vibrating in a mode. However, there is a constant phase difference between any two system components. The key idea to decoupling (1) is to synchronize all modes $\mathbf{s}_j(t)$ by evaluating each component at a different, but fixed, time lag. This process is called phase synchronization [20, 21, 23]. Upon phase synchronization, we obtain a synchronized vector

$$\mathbf{y}_j(t) = p_j(t)\mathbf{z}_j \ , \tag{10}$$

where

$$p_j(t) = C_j e^{\alpha_j t}\cos(\omega_j t - \theta_j) \ , \qquad \mathbf{z}_j = \begin{bmatrix} e^{\alpha_j \varphi_{j1}/\omega_j} & \cdots & e^{\alpha_j \varphi_{jn}/\omega_j} \end{bmatrix}^T \ . \tag{11}$$

To invert the synchronization operation, we can apply the time-shifting operation

$$\mathbf{s}_j(t) = \begin{bmatrix} s_{j1}(t) \\ \vdots \\ s_{jn}(t) \end{bmatrix} = \begin{bmatrix} y_{j1}(t - \varphi_{j1}/\omega_j) \\ \vdots \\ y_{jn}(t - \varphi_{jn}/\omega_j) \end{bmatrix} \tag{12}$$

and derive a formula for the mode $\mathbf{s}_j(t)$ in terms of the scalar functions $p_j(t)$:

$$\mathbf{s}_j(t) = \operatorname{diag}\left[p_j(t - \varphi_{j1}/\omega_j) \ , \quad \ldots \ , \quad p_j(t - \varphi_{jn}/\omega_j)\right]\mathbf{z}_j \ . \tag{13}$$

Because the homogeneous solution $\mathbf{q}(t)$ is the superposition of the n modes, we obtain

$$\mathbf{q}(t) = \sum_{j=1}^{n} \operatorname{diag}\left[p_j(t - \varphi_{j1}/\omega_j) \ , \quad \ldots \ , \quad p_j(t - \varphi_{jn}/\omega_j)\right]\mathbf{z}_j \ . \tag{14}$$

The above equation represents a mapping from the set of mutually independent functions $p_j(t)$ to the homogeneous solution $\mathbf{q}(t)$ of system (1). Straightforward calculations show that the functions $p_j(t)$ satisfy the system of second-order differential equations

$$\ddot{\mathbf{p}} + \mathbf{D}_1\dot{\mathbf{p}} + \boldsymbol{\Omega}_1\mathbf{p} = \mathbf{0} \ , \tag{15}$$

where

$$\mathbf{p}(t) = \begin{bmatrix} p_1(t) & \cdots & p_n(t) \end{bmatrix}^T \ , \tag{16}$$

$$\mathbf{D}_1 = -\operatorname{diag}\left[\lambda_1 + \overline{\lambda}_1 \ , \quad \ldots \ , \quad \lambda_n + \overline{\lambda}_n\right] \ , \tag{17}$$

$$\boldsymbol{\Omega}_1 = \operatorname{diag}\left[\lambda_1\overline{\lambda}_1 \ , \quad \ldots \ , \quad \lambda_n\overline{\lambda}_n\right] \ . \tag{18}$$

Note that the coefficients in (15) are real and diagonal, so (15) represents a real, decoupled system into which (1) is transformed. The decoupling process is complete if we connect the initial conditions of (1) with those of (15). It can be shown [20, 23] that

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda} & \overline{\boldsymbol{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\boldsymbol{\Lambda} & \overline{\mathbf{V}\boldsymbol{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{bmatrix} \ , \tag{19}$$

where $\mathbf{I}$ is an $n \times n$ identity matrix, $\boldsymbol{\Lambda}$ is a diagonal matrix of the n eigenvalues λ_j (that constitute the n complex conjugate pairs) and $\mathbf{V}$ is an $n \times n$ matrix of the n corresponding eigenvectors $\mathbf{v}_j$.

Although we have presented the decoupling procedure only for the case of complex eigenvalues, the decoupling method and formulas remain valid if some or all eigenvalues are real (see [21, 23] for details). It can be checked that the quadratic eigenvalue problems associated with (1) and (15) yield the same eigenvalues. Similar transformations decouple systems of the form (1) with non-symmetric coefficients and with a non-zero right-hand-side [21, 23]. Finally, the decoupling transformation generated by phase synchronization is the only transformation (unique up to an equivalence class) that decouples system (1) while keeping its eigenvalues invariant [21, 23]. For this reason, decoupling by phase synchronization brings about a unique characterization of system (1).

Upon decoupling (1), it is clear that the solution $\mathbf{q}(t)$ is oscillatory if at least one of the decoupled coordinates, say, $p_k(t)$ is oscillatory and if, in addition, $p_k(0) \neq 0$. The solution $\mathbf{q}(t)$ is oscillatory for every set of initial conditions if all decoupled coordinates $p_j(t)$ are oscillatory. Similarly, $\mathbf{q}(t)$ is non-oscillatory for every set of initial conditions if all decoupled coordinates $p_j(t)$ are non-oscillatory. Thus, decoupling (1) reduces the problem of studying the response characteristics of (1) to studying the response characteristics of the n independent, single-degree-of-freedom oscillators of (15). To determine whether all or some decoupled coordinates $p_j(t)$ are oscillatory or not, we can use the viscous damping ratio (2). When system (1) is non-defective, the number of overdamped, underdamped, or critically damped coordinates is an essential and invariant characteristic of system (1) [17]. The decoupling of (1) by phase synchronization thus represents a complete solution to the problem of determining the response characteristics of the linear system (1). To decouple the system, the quadratic eigenvalue problem (5) needs to be solved, and hence the complexity of determining the response characteristics of (1) by decoupling is on the order of n^3 [8, 20, 21]. A flowchart for decoupling (1) under real and complex eigenvalues is given in [21, 23].

3. **The viscous damping function.**

Recall that the viscous damping ratio (2) provides a straightforward means to determine if the free response of a single-degree-of-freedom system is oscillatory or not. How can the viscous damping ratio (2) be generalized to apply to multi-degree-of-freedom systems? Is there a way to determine the response characteristics of (1) that does not involve decoupling the equations of motion?

Define a viscous damping function for any complex vector $\mathbf{x} \neq \mathbf{0}$ by

$$\zeta(\mathbf{x}) = \frac{\mathbf{x}^*\mathbf{C}\mathbf{x}}{2\sqrt{(\mathbf{x}^*\mathbf{M}\mathbf{x})\,(\mathbf{x}^*\mathbf{K}\mathbf{x})}}\ , \tag{20}$$

where $\mathbf{x}^*$ denotes the complex conjugate transpose of $\mathbf{x}$. Note that the viscous damping function is a nonlinear function of $\mathbf{x}$ and that it is real because $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are SPD. If $\mathbf{v}_j$ is a complex eigenvector, then $\zeta(\mathbf{v}_j)$ yields the viscous damping ratio of one of the decoupled equations. Furthermore, the viscous damping function, when evaluated at any eigenvector $\mathbf{v}_j$, determines if the corresponding eigenvalue λ_j is real or complex. To demonstrate this statement, pre-multiply (5) by $\mathbf{v}_j^*$ to obtain

$$\lambda_j = \frac{-\mathbf{v}_j^*\mathbf{C}\mathbf{v}_j \pm \sqrt{\left(\mathbf{v}_j^*\mathbf{C}\mathbf{v}_j\right)^2 - 4\left(\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j\right)\left(\mathbf{v}_j^*\mathbf{K}\mathbf{v}_j\right)}}{2\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j}\ . \tag{21}$$

In terms of the viscous damping function, (21) becomes

$$\lambda_j = \left(-\zeta(\mathbf{v}_j) \pm \sqrt{\zeta^2(\mathbf{v}_j) - 1}\right)\sqrt{\frac{\mathbf{v}_j^*\mathbf{K}\mathbf{v}_j}{\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j}}\ . \tag{22}$$

Evidently, the eigenvalue λ_j is complex if and only if $\zeta(\mathbf{v}_j) < 1$ and it is real if and only if $\zeta(\mathbf{v}_j) > 1$. Moreover, it can be shown [17, 20, 21, 23] that all decoupled coordinates of (1) are oscillatory if and only if all eigenvalues are complex and all decoupled coordinates are non-oscillatory if and only if all eigenvalues are real. Because the decoupled coordinates determine the response characteristics of (1), we have the following theorem.

Theorem 3.1. *The free response of* (1) *is oscillatory for every set of initial conditions if and only if* $\max_j \zeta(\mathbf{v}_j) < 1$*. The free response of* (1) *is non-oscillatory for every set of initial conditions if and only if* $\min_j \zeta(\mathbf{v}_j) > 1$.

To utilize Theorem 3.1, one needs to compute the eigenvectors of (1). To decouple (1) by phase synchronization, only the eigenvalues need to be determined. The eigenvectors are only needed for the transformation from $\mathbf{p}$ back to $\mathbf{q}$, but not to construct the decoupled equations. With respect to numerical efficiency, it makes more sense to decouple system (1) by phase synchronization and to compute the damping ratios of the n independent equations of motion, rather than applying Theorem 3.1. However, we can simplify Theorem 3.1 by realizing that

$$\min_{\mathbf{x}\in C^n} \zeta(\mathbf{x}) \le \min_j \zeta(\mathbf{v}_j) \le \max_j \zeta(\mathbf{v}_j) \le \max_{\mathbf{x}\in C^n} \zeta(\mathbf{x}) \ . \tag{23}$$

The following corollary is a simple consequence of this inequality.

Corollary 1. *If* $\max_{\mathbf{x}\in C^n} \zeta(\mathbf{x}) < 1$*, then the response of* (1) *is oscillatory for every set of initial conditions. If* $\min_{\mathbf{x}\in C^n} \zeta(\mathbf{x}) > 1$*, then the response of* (1) *is non-oscillatory for every set of initial conditions.*

We can further reduce the computational costs of checking Corollary 1 by recognizing that

$$\max_{\mathbf{x}\in C^n} \zeta(\mathbf{x}) = \max_{\mathbf{x}\in S^n_{R+}} \zeta(\mathbf{x}), \qquad \min_{\mathbf{x}\in C^n} \zeta(\mathbf{x}) = \min_{\mathbf{x}\in S^n_{R+}} \zeta(\mathbf{x}) \ , \tag{24}$$

where $S^n_{R+} = \{\mathbf{x} \in R^n : ||\,\mathbf{x}\,|| = 1, x_j \ge 0 \text{ for all } j\}$ denotes half of the real unit sphere $S^n_R = \{\mathbf{x} \in R^n : ||\,\mathbf{x}\,|| = 1\}$. To prove (24), first observe that $\zeta(\alpha\mathbf{x}) = \zeta(\mathbf{x})$ for any $\alpha \neq 0$ and that $\zeta(\bar{\mathbf{x}}) = \zeta(\mathbf{x})$. Thus,

$$\max_{\mathbf{x}\in C^n} \zeta(\mathbf{x}) = \max_{\mathbf{x}\in S^n_{C+}} \zeta(\mathbf{x}), \qquad \min_{\mathbf{x}\in C^n} \zeta(\mathbf{x}) = \min_{\mathbf{x}\in S^n_{C+}} \zeta(\mathbf{x}) \ , \tag{25}$$

i.e., it is sufficient to search for the extrema of $\zeta(\mathbf{x})$ only on that part of the complex unit sphere represented by $S^n_{C+} = \{\mathbf{x} \in C^n : ||\,\mathbf{x}\,|| = 1, \mathrm{Im}(x_j) \ge 0 \text{ for all } j\}$. It remains to show that the viscous damping function attains its extrema at real $\mathbf{x}$. We first note that $\zeta(\mathbf{x})$ is continuously differentiable so that the global maximum and minimum of the viscous damping function are attained at a critical point (a local maximum, minimum or saddle point). Moreover, it can be shown that

$$\left(\frac{2\mathbf{C}}{\hat{\mathbf{x}}^*\mathbf{C}\hat{\mathbf{x}}} - \frac{\mathbf{K}}{\hat{\mathbf{x}}^*\mathbf{K}\hat{\mathbf{x}}} - \frac{\mathbf{M}}{\hat{\mathbf{x}}^*\mathbf{M}\hat{\mathbf{x}}}\right)\hat{\mathbf{x}} = \mathbf{0} \tag{26}$$

if and only if $\hat{\mathbf{x}}$ is a critical point (i.e., the gradient of $\zeta(\mathbf{x})$ with respect to the real and imaginary part of its argument equals zero). Let $\hat{\mathbf{x}}$ be a critical point of $\zeta(\mathbf{x})$, and let $\mathbf{U}$ be a real $n \times n$ matrix that defines a congruence transformation to diagonalize $\mathbf{M}$ and $\mathbf{K}$ simultaneously: $\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{I}$ and $\mathbf{U}^T\mathbf{K}\mathbf{U} = \mathbf{\Omega}$, where $\mathbf{\Omega}$ (the spectral matrix) is diagonal with positive diagonal elements. Then $\hat{\mathbf{p}} = \mathbf{U}^{-1}\hat{\mathbf{x}}$ is a critical point of

$$\hat{\zeta}(\mathbf{p}) = \frac{\mathbf{p}^*\mathbf{D}\mathbf{p}}{2\sqrt{(\mathbf{p}^*\mathbf{p})\,(\mathbf{p}^*\boldsymbol{\Omega}\mathbf{p})}} \;, \tag{27}$$

where $\mathbf{D} = \mathbf{U}^T\mathbf{C}\mathbf{U}$ (the modal damping matrix) is SPD. Because $\hat{\mathbf{p}}$ is a critical point of $\hat{\zeta}$,

$$\left(\frac{2\mathbf{D}}{\hat{\mathbf{p}}^*\mathbf{D}\hat{\mathbf{p}}} - \frac{\boldsymbol{\Omega}}{\hat{\mathbf{p}}^*\boldsymbol{\Omega}\hat{\mathbf{p}}} - \frac{\mathbf{I}}{\hat{\mathbf{p}}^*\hat{\mathbf{p}}}\right)\hat{\mathbf{p}} = \mathbf{0} \;. \tag{28}$$

The above equation implies that

$$\mathbf{D}\hat{\mathbf{p}} = \frac{\hat{\mathbf{p}}^*\mathbf{D}\hat{\mathbf{p}}}{2}\left(\frac{\boldsymbol{\Omega}}{\hat{\mathbf{p}}^*\boldsymbol{\Omega}\hat{\mathbf{p}}} + \frac{\mathbf{I}}{\hat{\mathbf{p}}^*\hat{\mathbf{p}}}\right)\hat{\mathbf{p}} \tag{29}$$

so that

$$\mathbf{E}\mathbf{D}\hat{\mathbf{p}} = \mathbf{D}\mathbf{E}\hat{\mathbf{p}} \tag{30}$$

for any diagonal matrix $\mathbf{E}$. If $\mathbf{E}$ is diagonal and unitary, then $\hat{\mathbf{y}} = \mathbf{E}\hat{\mathbf{p}}$ is a critical point of $\hat{\zeta}$ because

$$\begin{aligned}\frac{\hat{\mathbf{y}}^*\mathbf{D}\hat{\mathbf{y}}}{2}\left(\frac{\boldsymbol{\Omega}}{\hat{\mathbf{y}}^*\boldsymbol{\Omega}\hat{\mathbf{y}}} + \frac{\mathbf{I}}{\hat{\mathbf{y}}^*\hat{\mathbf{y}}}\right)\hat{\mathbf{y}} &= \mathbf{E}\frac{\hat{\mathbf{p}}^*\mathbf{D}\hat{\mathbf{p}}}{2}\left(\frac{\boldsymbol{\Omega}}{\hat{\mathbf{p}}^*\boldsymbol{\Omega}\hat{\mathbf{p}}} + \frac{\mathbf{I}}{\hat{\mathbf{p}}^*\hat{\mathbf{p}}}\right)\hat{\mathbf{p}} \\ &= \mathbf{E}\mathbf{D}\hat{\mathbf{p}} = \mathbf{D}\mathbf{E}\hat{\mathbf{p}} = \mathbf{D}\hat{\mathbf{y}} \;.\end{aligned} \tag{31}$$

Pick $\mathbf{E}$ to rotate the critical point $\hat{\mathbf{p}}$ towards the real axis, i.e.,

$$\mathbf{E} = \operatorname{diag}\left[e^{-i\theta_1}, \ldots, e^{-i\theta_n}\right] \;, \tag{32}$$

where $\theta_1, \ldots, \theta_n$ are the phase angles of the elements of $\hat{\mathbf{p}}$. This matrix $\mathbf{E}$ is unitary and, therefore, $\hat{\mathbf{y}}$ is a real critical point of $\hat{\zeta}$. Thus, $\hat{\mathbf{x}} = \mathbf{U}\hat{\mathbf{y}}$ is a real critical point of ζ. Equation (24) now follows from the continuity and symmetry of ζ. We thus have derived the following theorem.

Theorem 3.2. *The response characteristics of the linear dynamical system* (1) *can be determined by inspection of its viscous damping function* $\zeta(\mathbf{x})$.

(1) If $\min_{\mathbf{x}\in S^n_{R+}} \zeta > 1$, *the response is non-oscillatory for every set of initial conditions.*

(2) If $\max_{\mathbf{x}\in S^n_{R+}} \zeta < 1$, *the response is oscillatory for every set of initial conditions.*

(3) At least one eigenvalue is real and defective if $\zeta(\hat{\mathbf{x}}) = 1$ *and* $\nabla\zeta(\hat{\mathbf{x}}) = \mathbf{0}$ *(i.e., the gradient of* $\zeta(\mathbf{x})$ *is zero at* $\hat{\mathbf{x}}$*) for at least one real* $\hat{\mathbf{x}} \neq \mathbf{0}$.

Proof of Part (3) of Theorem 3.2. To prove Part (3) of Theorem 3.2, suppose that $\hat{\mathbf{x}} \neq \mathbf{0}$ satisfies the equations $\zeta(\hat{\mathbf{x}}) = 1$ and $\nabla\zeta(\hat{\mathbf{x}}) = \mathbf{0}$. From the definition of the viscous damping function given in (20), $\zeta(\hat{\mathbf{x}}) = 1$ implies that

$$\hat{\mathbf{x}}^*\mathbf{K}\hat{\mathbf{x}} = \frac{(\hat{\mathbf{x}}^*\mathbf{C}\hat{\mathbf{x}})^2}{4\hat{\mathbf{x}}^*\mathbf{M}\hat{\mathbf{x}}} \;. \tag{33}$$

It is easy to see that $\nabla\zeta(\hat{\mathbf{x}}) = \mathbf{0}$ if and only if

$$\left(\frac{2\mathbf{C}}{\hat{\mathbf{x}}^*\mathbf{C}\hat{\mathbf{x}}} - \frac{\mathbf{K}}{\hat{\mathbf{x}}^*\mathbf{K}\hat{\mathbf{x}}} - \frac{\mathbf{M}}{\hat{\mathbf{x}}^*\mathbf{M}\hat{\mathbf{x}}}\right)\hat{\mathbf{x}} = \mathbf{0} \;. \tag{34}$$

Equations (33) and (34) imply that

$$\left(\mathbf{M}\left(-\frac{\hat{\mathbf{x}}^*\mathbf{C}\hat{\mathbf{x}}}{2\hat{\mathbf{x}}^*\mathbf{M}\hat{\mathbf{x}}}\right)^2 + \left(-\frac{\hat{\mathbf{x}}^*\mathbf{C}\hat{\mathbf{x}}}{2\hat{\mathbf{x}}^*\mathbf{M}\hat{\mathbf{x}}}\right)\mathbf{C} + \mathbf{K}\right)\hat{\mathbf{x}} = \mathbf{0} \;. \tag{35}$$

Thus, $-(\hat{\mathbf{x}}^*\mathbf{C}\hat{\mathbf{x}})\,/\,(2\hat{\mathbf{x}}^*\mathbf{M}\hat{\mathbf{x}})$ is an eigenvalue of (5) with eigenvector $\hat{\mathbf{x}}$. By Theorem 4.2 in [18], this eigenvalue is real and defective. Therefore, at least one eigenvalue of system (1) is real and defective. □

The significance of (24) in terms of computations is evident. To determine the response characteristics of system (1), it is sufficient to compute the maximum and minimum of the real viscous damping function over half of the real unit sphere S^n_{R+} (instead of over the complex vector space C^n). Eigenvalues need never be computed. If $n = 1$, the viscous damping function $\zeta(\mathbf{x})$ in (20) becomes the viscous damping ratio ζ in (2) and Theorem 3.2 gives the familiar definitions of overdamping, underdamping and critical damping of single-degree-of-freedom systems. It is in this sense that the viscous damping function (20) represents a direct extension of the viscous damping ratio ζ.

Simpler sufficient conditions can be obtained in terms of upper and lower bounds of the viscous damping function. Sharp upper and lower bounds are

$$\frac{\underline{\sigma}(\mathbf{C})}{2\sqrt{\bar{\sigma}(\mathbf{M})\bar{\sigma}(\mathbf{K})}} \le \zeta(\mathbf{x}) \le \frac{\bar{\sigma}(\mathbf{C})}{2\sqrt{\underline{\sigma}(\mathbf{M})\underline{\sigma}(\mathbf{K})}}\ . \tag{36}$$

In the above inequality, $\underline{\sigma}$ and $\bar{\sigma}$ denote the smallest and largest eigenvalue, respectively. To show that the bounds given in (36) are sharp, simply check that the bounds are achieved by a system with $\mathbf{M} = \operatorname{diag}[1, 2]$, $\mathbf{C} = \operatorname{diag}[0.5, 0.1]$ and $\mathbf{K} = \operatorname{diag}[1, 3]$. We thus have the following corollary:

Corollary 2. *If* $\bar{\sigma}(\mathbf{C})/2\sqrt{\underline{\sigma}(\mathbf{M})\underline{\sigma}(\mathbf{K})} < 1$*, the response of the linear dynamical system* (1) *is oscillatory for every set of initial conditions. If* $\underline{\sigma}(\mathbf{C})/2\sqrt{\bar{\sigma}(\mathbf{M})\bar{\sigma}(\mathbf{K})} > 1$*, the response of the linear dynamical system* (1) *is non-oscillatory for every set of initial conditions.*

4. **Examples and further discussion.** To illustrate the decoupling theory of Section 2 as well as the use of the viscous damping function (20) and Theorem 3.2, several numerical examples are provided. Inherent limitations in the optimization of the viscous damping function are then discussed, followed by a comparison of the relative merits of the two strategies developed for system characterization: optimization of the viscous damping function and decoupling by phase synchronization.

4.1. **Example 1.** Consider a simplified, quarter-car model as shown in Fig. 1. The

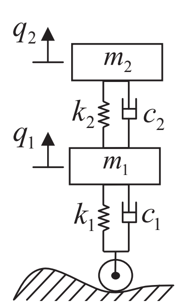

FIGURE 1. Quarter-car suspension model of Example 1.

unsprung and sprung masses are denoted by m_1 and m_2, respectively. The tire is modeled by a linear spring k_1 in parallel with a viscous damper c_1. The suspension is modeled by a linear spring k_2 in parallel with a viscous damper c_2. Typical parameters [6, 31] of a lightly damped passenger car are listed in Table 1.

TABLE 1. Parameters of the quarter-car suspension model in Fig. 1.

Sprung mass	m_1	36 kg
Unsprung mass	m_2	240 kg
Damping coefficient of tire	c_1	10 N-s/m
Damping coefficient of suspension	c_2	980 N-s/m
Stiffness of tire	k_1	160,000 N/m
Stiffness of suspension	k_2	16,000 N/m

The system is governed by (1) with mass, damping and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ , \quad \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \ , \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \ . \tag{37}$$

The maximum of the viscous damping function of this system is computed to be $\zeta_{\max} = 0.27$ at $\hat{\mathbf{x}} = \mathbf{x}_{\max} = [0.15,\ -0.99]^T$. The viscous damping function takes on its minimum at $\hat{\mathbf{x}} = \mathbf{x}_{\min} = [0.71,\ 0.71]^T$ with $\zeta_{\min} = 7.51 \times 10^{-4}$. By Theorem 3.2, the response of this system is oscillatory for every set of initial conditions. To verify this result, the system is decoupled by phase synchronization and the viscous damping ratio for each decoupled degree of freedom is computed according to (2). It is found that both degrees of freedom, $p_1(t)$ and $p_2(t)$, are underdamped with viscous damping ratios 0.22 and 0.20, respectively, confirming the results we obtained by using the viscous damping function.

In some applications, damping in the tire is neglected [6, 31] so that $c_1 = 0$. The viscous damping matrix $\mathbf{C}$ is now symmetric positive semi-definite. Upon solution of the quadratic eigenvalue problem (5) with $c_1 = 0$ and the remaining parameters as in Table 1, it is found that all decoupled degrees of freedom associated with (1) are underdamped with viscous damping ratios 0.22 and 0.20, respectively. Thus, the system response is oscillatory for every set of initial conditions, but no undamped motion can be observed because the system is pervasively damped [7, 12, 13]. The viscous damping function is applicable to pervasively damped systems with a symmetric positive semi-definite viscous damping matrix. Note, however, that $\zeta_{\min} = 0$ does not imply that at least one degree of freedom is undamped. If system (1) is not pervasively damped, undamped coordinates should be removed by classical modal analysis [22]. The viscous damping function should be applied to the remaining pervasively damped subsystem. In applications, viscous damping can be expected to be pervasive because energy is always dissipated and no truly undamped motion exists.

4.2. **Example 2.** This example is taken from [7], where it is used to show that the overdamping condition proposed in [15] can be misleading. The system is governed by (1) with

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \quad \mathbf{C} = \begin{bmatrix} 7/4 & 0 \\ 0 & 21/2 \end{bmatrix} \ , \quad \mathbf{K} = \begin{bmatrix} 1/2 & 1 \\ 1 & 7 \end{bmatrix} \ . \tag{38}$$

Using the Euclidean norm, the viscous damping function in (20) is evaluated on the unit circle $||\,\mathbf{x}\,|| = 1$ and plotted in Fig. 2.

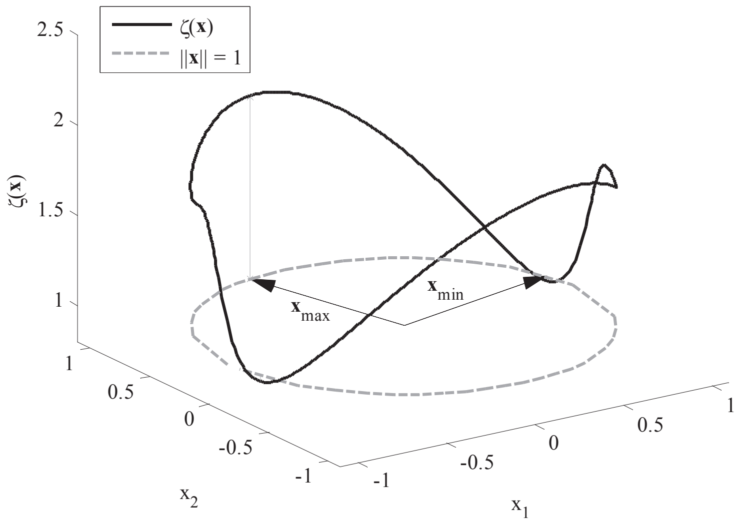

FIGURE 2. Viscous damping function $\zeta(\mathbf{x})$ of Example 2.

We observe that $\zeta_{\min} < 1$ and that $\zeta_{\max} > 1$ and, thus, Theorem 3.2 is inconclusive. Upon decoupling this system by phase synchronization, it is found that one decoupled degree of freedom is underdamped (with viscous damping ratio 0.99) and the other is overdamped (with viscous damping ratio 1.42). The system may exhibit oscillatory behaviors for some initial conditions and non-oscillatory behaviors for others.

4.3. **Example 3.** A two-degree-of-freedom system is defined by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \mathbf{C} = \begin{bmatrix} 4 & \sqrt{41 - 24\sqrt{2}} \\ \sqrt{41 - 24\sqrt{2}} & 8 \end{bmatrix} , \quad \mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} . \tag{39}$$

The viscous damping function of this system is evaluated on the unit circle $||\,\mathbf{x}\,|| = 1$ and plotted in Fig. 3. By inspection, $\zeta(\mathbf{x}) \geq 1$ for all real $\mathbf{x}$. By Theorem 3.2, the free response of this system is non-oscillatory for any set of initial conditions. It can be checked that $\zeta(\hat{\mathbf{x}}) = \zeta_{\min} = 1$ and $\nabla\zeta(\hat{\mathbf{x}}) = \mathbf{0}$ for $\hat{\mathbf{x}} = \mathbf{x}_1 = [0.82,\ -0.58]^T$. This system possesses at least one real, defective eigenvalue. Decoupling of the system by phase synchronization confirms that one of the decoupled degrees of freedom is critically damped, which implies that one eigenvalue is real and defective [17]. The other degree of freedom is overdamped with viscous damping ratio 3.24. Thus, the response of this system is non-oscillatory for every set of initial conditions.

4.4. **Example 4.** A three-degree-of-freedom system is given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} , \quad \mathbf{C} = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix} , \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} . \tag{40}$$

The viscous damping function $\zeta(\mathbf{x})$ of this system is evaluated on the unit sphere $||\,\mathbf{x}\,|| = 1$ and plotted in Fig. 4. The minimum and maximum of the viscous damping function are $\zeta_{\min} = 0.43$ and $\zeta_{\min} = 1.37$, respectively, and so Theorem 3.2 is inconclusive. Upon decoupling the system by phase synchronization, it can be

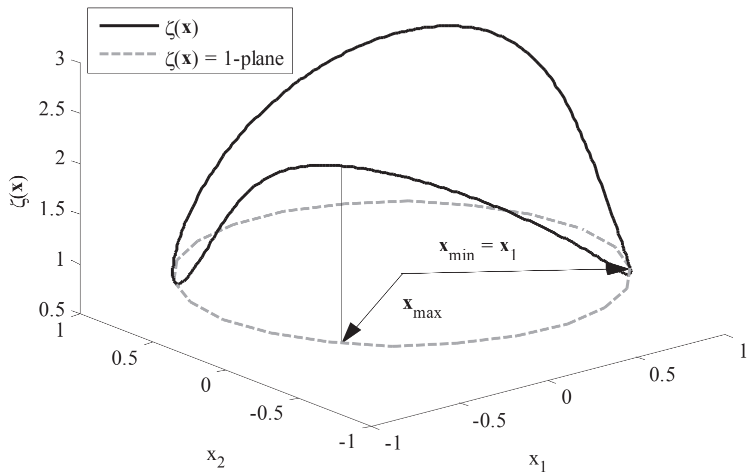

FIGURE 3. Viscous damping function $\zeta(\mathbf{x})$ of Example 3.

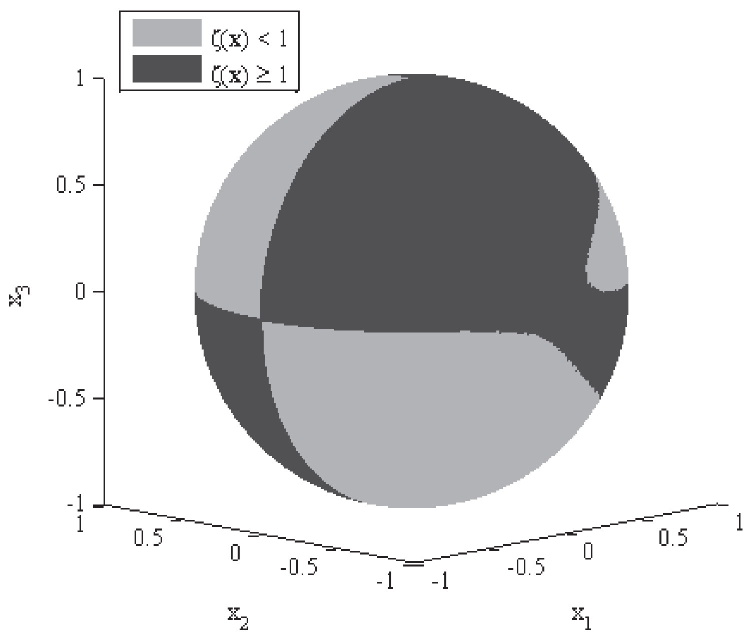

FIGURE 4. Viscous damping function $\zeta(\mathbf{x})$ of Example 4.

verified that one of the decoupled degrees of freedom is underdamped with viscous damping ratio 0.47 and the remaining two decoupled degrees of freedom are overdamped with viscous damping ratios 1.07 and 1.20, respectively. The system thus exhibits oscillations for some initial conditions and no oscillations for others.

4.5. **Discussion.** In the previous examples, minimization of the viscous damping function (20) was carried out by Newton's method, but other methods (such as quasi Newton methods, gradient descent, or conjugate gradient methods [10, 28]) may also be used. In any numerical minimization, an initial seed is required. In the above examples we initialized our search for the minimum using the eigenvector corresponding to the smallest eigenvalue (in magnitude) of the symmetric matrix

$$2\mathbf{C} - \bar{\sigma}_{\mathbf{C},\mathbf{K}}\mathbf{K} - \bar{\sigma}_{\mathbf{C},\mathbf{M}}\mathbf{M}\ , \tag{41}$$

where $\bar{\sigma}_{\mathbf{C},\mathbf{K}}$ and $\bar{\sigma}_{\mathbf{C},\mathbf{M}}$ denote, respectively, the largest eigenvalue (in magnitude) of the symmetric eigenvalue problems $\mathbf{Cu} = \sigma\mathbf{Ku}$ and $\mathbf{Cu} = \sigma\mathbf{Mu}$. We initialized the search for the maximum using the eigenvector corresponding to the largest eigenvalue (in magnitude) of the symmetric matrix

$$2\mathbf{C} - \underline{\sigma}_{\mathbf{C},\mathbf{K}}\mathbf{K} - \underline{\sigma}_{\mathbf{C},\mathbf{M}}\mathbf{M}\ , \tag{42}$$

where $\underline{\sigma}_{\mathbf{C},\mathbf{K}}$ and $\underline{\sigma}_{\mathbf{C},\mathbf{M}}$ denote the smallest eigenvalue (in magnitude) of the symmetric eigenvalue problems generated by $\mathbf{C}$ and $\mathbf{K}$ and $\mathbf{C}$ and $\mathbf{M}$, respectively. This initialization is motivated by the fact that, at a critical point, the corresponding vector $\hat{\mathbf{x}}$ is an eigenvector of the symmetric matrix

$$2\mathbf{C} - \frac{\hat{\mathbf{x}}^T\mathbf{C}\mathbf{x}}{\hat{\mathbf{x}}^T\mathbf{K}\mathbf{x}}\mathbf{K} - \frac{\hat{\mathbf{x}}^T\mathbf{C}\mathbf{x}}{\hat{\mathbf{x}}^T\mathbf{M}\mathbf{x}}\mathbf{M}\ . \tag{43}$$

This initialization worked well in the examples we considered. The computational costs of minimizing or maximizing the viscous damping function depends on the numerical minimization technique used. With Newton's method and a "good" initial guess, minimizing or maximizing the viscous damping function is comparable in complexity to decoupling (1) by phase synchronization. In all examples, we have confirmed the optimization results by "flooding the space," i.e., by evaluating the viscous damping function on a fine discretization of the unit sphere.

We can expect difficulties with the minimization of the viscous damping function for systems with a very large number of degrees of freedom. The minimization may get trapped in local extrema, and there is no guarantee that the global extrema of $\zeta(\mathbf{x})$ can be found, regardless of the minimization technique used. Optimization of the viscous damping function is thus feasible for problems of a relatively small dimension only, and the approach of studying the response characteristics of system (1) by decoupling via phase synchronization appears more applicable and more accurate. Moreover, Theorem 3.2 only addresses conditions of sufficiency and thus may be inconclusive in many applications (recall Examples 2 and 4), whereas necessary and sufficient conditions are obtained by decoupling (1) through phase synchronization.

5. **Concluding remarks.** Determining the response characteristics of a multi-degree-of-freedom, linear dynamical system in free motion is important in analysis and design. We have developed two strategies for such a task. Major findings are summarized in the following statements.

1. The response characteristics of a linear dynamical system can be determined by decoupling the equation of motion while keeping its eigenvalues invariant. The decoupled system can be obtained efficiently (at a cost of order n^3) and uniquely by the methodology of phase synchronization. Decoupling the system reduces the problem of determining the response characteristics of a

multi-degree-of-freedom system to studying the response characteristics of n independent, single-degree-of-freedom oscillators.

2. The effect of viscous damping on the free motion can also be determined by minimization and maximization of a viscous damping function, defined in (20). The viscous damping function represents a direct extension of the classical damping ratio and is applicable to multi-degree-of-freedom systems. In applications, optimization of the viscous damping function may be problematic because the iterations can get trapped around local extrema.

The methods presented herein are applicable to any damped linear dynamical system without the usual assumption of classical damping. This present paper thus extends and concludes the theory presented in [2, 12, 15, 24, 25, 26, 27]. Among other things, it is hoped that this paper helps to identify directions for further research. For example, stability analysis and response characterization of damped linear systems subjected to gyroscopic and circulatory forces now appears feasible and is worthwhile in a subsequent course of investigation.

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REFERENCES

[1] L. Barkwell and P. Lancaster, *Overdamped and gyroscopic vibrating systems*, ASME Journal of Applied Mechanics, **59** (1992), 176–181.

[2] A. Bhaskar, *Criticality of damping in multi-degree-of-freedom systems*, ASME Journal of Applied Mechanics, **64** (1997), 387–393.

[3] R. M. Bulatović, *Non-oscillatory damped multi-degree-of-freedom systems*, Acta Mechanica, **151** (2001), 235–244.

[4] R. M. Bulatović, *On the heavily damped response in viscously damped dynamic systems*, ASME Journal of Applied Mechanics, **71** (2004), 131–134.

[5] T. K. Caughey and M. E. J. Okelly, *Classical normal modes in damped linear dynamic systems*, ASME Journal of Applied Mechanics, **32** (1965), 583–588.

[6] R. M. Chalasani, *Ride performance potential of active suspension systems – part I: simplified analysis based on a quarter-car model*, in "ASME Symposium on Simulation and Control of Ground Vehicles and Transportation Systems," AMD-Vol. 80, DSC-Vol. 2, ASME, (1986), 187–204.

[7] G. M. Connell, *Asymptotic stability of second-order linear systems with semi-definite damping*, AIAA Journal, **7** (1969), 1185–1187.

[8] J. W. Demmel, "Applied Numerical Linear Algebra," Society for Industrial and Applied Mathematics, Philadelphia, 1997.

[9] R. J. Duffin, *A minimax theory for overdamped networks*, Journal of Rational Mechanics and Analysis, **4** (1955), 221–233.

[10] R. Fletscher, "Practical Methods of Optimization," 2^{nd} edition, Wiley, Hoboken, New Jersey, 2000.

[11] I. Gohberg, P. Lancaster and L. Rodman, "Matrix Polynomials," Academic Press, New York, 1982.

[12] A. J. Gray and A. N. Andry, *A simple calculation of the critical damping matrix of a linear multi-degree-of-freedom system*, Mechanics Research Communications, **9** (1982), 379–380.

[13] P. Hagedorn and S. Otterbein, "Technische Schwingungslehre," Springer, Berlin, Germany, 1987.

[14] K. Huseyin, "Vibrations and Stability of Multiple Parameter Systems," Noordhoff, Leiden, 1978.

[15] D. J. Inman and A. N. Andry, Jr., *Some results on the nature of eigenvalues of discrete damped linear systems*, ASME Journal of Applied Mechanics, **47** (1980), 927–930.

[16] D. J. Inman, "Vibration with Control," Wiley, Hoboken, New Jersey, 2006.

[17] D. T. Kawano, M. Morzfeld and F. Ma, *The decoupling of defective linear dynamical systems in free motion*, Journal of Sound and Vibration, **330** (2011), 5165–5183.

[18] P. Lancaster, "Lambda-Matrices and Vibrating Systems," Pergamon Press, Oxford, United Kingdom, 1966.

[19] P. Lancaster and M. Tismenetsky, "The Theory of Matrices," 2^{nd} edition, Academic Press, New York, 1985.

[20] F. Ma, A. Imam and M. Morzfeld, *The decoupling of damped linear systems in oscillatory free vibration*, Journal of Sound and Vibration, **324** (2009), 408–428.

[21] F. Ma, M. Morzfeld and A. Imam, *The decoupling of damped linear systems in free or forced vibration*, Journal of Sound and Vibration, **329** (2010), 3182–3202.

[22] L. Meirovitch, "Methods of Analytical Dynamics," McGraw-Hill, New York, 1970.

[23] M. Morzfeld, F. Ma and B. N. Parlett, *The transformation of second-order linear systems into independent equations*, SIAM Journal on Applied Mathematics, **71** (2011), 1026–1043.

[24] P. C. Müller, *Oscillatory damped linear systems*, Mechanics Research Communications, **6** (1979), 81–86.

[25] D. W. Nicholson, *Eigenvalue bounds for damped linear systems*, Mechanics Research Communications, **5** (1978), 147–152.

[26] D. W. Nicholson, *Eigenvalue bounds for linear mechanical systems with nonmodal damping*, Mechanics Research Communications, **14** (1978), 115–122.

[27] D. W. Nicholson, *Overdamping of a linear mechanical system*, Mechanics Research Communications, **10** (1983), 67–76.

[28] J. Nocedal and S. T. Wright, "Numerical Optimization," 2^{nd} edition, Springer, New York, 2006.

[29] J. W. Strutt (Lord Rayleigh), "The Theory of Sound, Vol. I," Dover, New York, 1945 (reprint of the 1894 edition).

[30] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Review, **43** (2001), 235–286.

[31] S. Türkay and H. Akçay, *A study of random vibration characteristics of a quarter car model*, Journal of Sound and Vibration, **282** (2005), 111–124.

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