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P-Permutation Equivalences Between Blocks Of Finite Groups

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P-PERMUTATION EQUIVALENCES BETWEEN BLOCKS OF FINITE GROUPS

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by

Philipp Perepelitsky

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Abstract

p-permutation equivalences between blocks of finite groups

by

Philipp Perepelitsky

Let G and H be finite groups. Let A be a block of FG and let B be a block of FH. A *p-permutation equivalence* between A and B is an element γ in the group of (A, B)-p-permutation bimodules with twisted diagonal vertices such that $\gamma \cdot_H \gamma^\circ = [A]$ and $\gamma^\circ \cdot_G \gamma = [B]$. A p-permutation equivalence lies between a splendid Rickard equivalence and an isotypy.

We introduce the notion of a γ-Brauer pair, which generalizes the notion of a Brauer pair for a p-block of a finite group. The γ-Brauer pairs satisfy an appropriate Sylow theorem. Furthermore, each maximal γ-Brauer pair identifies the defect groups, fusion systems and Külshammer-Puig classes of A and B. Additionally, the Brauer construction applied to γ induces a p-permutation equivalence at the local level, and a splendid Morita equivalence between the Brauer correspondents of A and B.

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Chapter 1

Introduction

Let G and H be finite groups. A subgroup U of $G \times H$ is **twisted diagonal** if there are isomorphic subgroups Q of G and R of H and an isomorphism $\varphi : R \to Q$ such that $U = \{(\varphi(r), r) : r \in R\}$.

Let A be a block of FG and let B be a block of FH. We denote by $T^{\Delta}(A, B)$ the Grothendieck group with respect to the direct sum relation of trivial source (A, B)-bimodules whose indecomposable direct summands have twisted diagonal vertices. Introduced in [3] was the notion of a p-**permutation equivalence** between A and B, which is an element $\gamma \in T^{\Delta}(A, B)$ such that $\gamma \otimes_B \gamma^o = [A]$ in $T^{\Delta}(A, A)$ and $\gamma^o \otimes_A \gamma = [B]$ in $T^{\Delta}(B, B)$. It was also shown in [3] that the existence of a splendid Rickard equivalence (introduced in [12]) between A and B implies the existence of a p-permutation equivalence between A and B. In this paper, we show that the existence of a p-permutation equivalence between A and B implies the existence of an isotypy (introduced in [5]) between A and B (see Theorem 14.5). Moreover, we show that a p-permutation equivalence between A and B determines an identification between many of the important invariants of A and B, including their defect groups, fusion systems, and Külshammer-Puig classes.

We introduce the notion of a Brauer pair for a virtual trivial source module (see Definition 10.4), which generalizes the notion of a Brauer pair for a p-block of a finite group introduced in [1]. We show that applying the Brauer construction to a p-permutation equivalence γ between A and B at a γ-Brauer pair yields a p-permutation equivalence between corresponding Brauer correspondents of A and B (see Theorem 12.2). An analogous result for basic Rickard equivalences was proved in [11], generalizing an earlier result for splendid Rickard equivalences that was obtained in [12] under certain additional assumptions. We determine the maximal Brauer pairs of a p-permutation equivalence γ between A and B, and show that they form a $G \times H$-conjugacy class (see Theorem 11.9). This generalizes the "Sylow theorem" for Brauer pairs of a p-block of a finite group proved in [1]. We show that a maximal γ-Brauer pair determines an isomorphism between the fusion systems of A and B respectively (see Theorem 12.1). This generalizes the analogous result for splendid Rickard equivalences proved in [10], where the result was in fact proved for basic Rickard equivalences. We also show that the γ-Brauer pairs identify the Külshammer-Puig classes of A and B respectively (see Theorem 12.5).

We show that every Brauer pair of an indecomposable (A, B)-bimodule with nonzero coefficient in γ is a γ-Brauer, but that there is a unique indecomposable (A, B)-bimodule M with nonzero coefficient in γ such that every γ-Brauer pair is an M-Brauer pair (see Theorem 12.9). Thus every Brauer pair of every other indecomposable (A, B)-bimodule with nonzero coefficient in γ is an M-Brauer pair, so we call M the **maximal module** of γ (see Definition 12.8). The coefficient of M in γ is 1 or -1. In Theorem 12.11, we show that the "Brauer correspondent" of M induces a splendid Morita equivalence between the Brauer correspondent of A and the Brauer correspondent of B, thus proving that the Brauer corresponents are Puig equivalent (see Theorem 12.12) by a result given in [14] and independently in [10] (see also [8] Theorem 4.1). In Theorem 13.2, we show that there are only finitely many p-permutation equivalences between A and B.

Chapter 2

Group theoretic preliminaries

Throughout this section, let G and H be finite groups.

Definition 2.1. *For $g \in G$, let $c_g : G \to G$ denote the automorphism of conjugation by g. For $h \in G$ and $\Omega \subseteq G$, we denote $c_g(h) = ghg^{-1}$ by ${}^g h$ and $c_g(\Omega)$ by ${}^g\Omega$. We denote by $Aut(G)$ the* ***automorphism group*** *of G, and we denote by 1_G the identity element of $Aut(G)$. If H is a normal subgroup of G, we denote by $Aut_G(H)$ the subgroup of $Aut(H)$ consisting of all automorphisms of the form c_g for $g \in G$. If H and K are subgroups of G, we write $H \leq_G K$ if H is G-conjugate to a subgroup of K, and we write $H =_G K$ if H is G-conjugate to K. Similarly, for elements $g, k \in G$, we write $g =_G k$ if g and k are G-conjugate.*

Definition 2.2. *For a prime p, we denote by $G_{p'}$ the* ***p'-elements*** *of G, which are the elements of G whose order is not divisible by p.*

Definition 2.3. *Let Γ be a set on which G acts. We denote by $\Gamma/\sim_G$ a complete set of representatives for the orbits of G on Γ. If $H \leq G$, we denote by Γ^H the subset of Γ consisting of the H-fixed points of Γ.*

Definition 2.4. *We denote by $p_1 : G \times H \to G$ the projection homomorphism of $G \times H$ onto G and by $p_2 : G \times H \to H$ the projection homomorphism of $G \times H$ onto H. For a subgroup U of $G \times H$, we denote by $k_1(U)$ the normal subgroup $\{g \in G : (g, 1) \in U\}$ of $p_1(U)$ and by $k_2(U)$ the normal subgroup $\{h \in H : (1, h) \in U\}$ of $p_2(U)$.*

Definition 2.5. *We say that a subgroup U of $G \times H$ is* ***twisted diagonal*** *if there are isomorphic subgroups Q of G and R of H and an isomorphism $\varphi : R \to Q$ such that $U =$*

$\{(\varphi(r), r) : r \in R\}$. *In this case, we denote the twisted diagonal subgroup* U *by* $\Delta(Q, \varphi, R)$. *Note that a subgroup* U *of* $G \times H$ *is twisted diagonal if and only if* $k_1(U) = k_2(U) = 1$. *For* $Q \leq G$, *we denote the subgroup* $\Delta(Q, 1_Q, Q)$ *of* $G \times G$ *by* $\Delta(Q)$.

Definition 2.6. *Let* K *be a finite group and let* U *and* V *be subgroups of* $G \times H$ *and* $H \times K$ *respectively. We denote by* U^o *the subgroup of* $H \times G$ *consisting of all elements* $(h, g) \in H \times G$ *such that* $(g, h) \in U$. *We denote by* $U * V$ *the subgroup of* $G \times K$ *consisting of all elements* $(g, k) \in G \times K$ *such that there is an* $h \in H$ *such that* $(g, h) \in U$ *and* $(h, k) \in V$.

Definition 2.7. *Let* Q *and* R *be isomorphic subgroup of* G *and* H *respectively, and let* $\varphi : R \to Q$ *be an isomorphism. Let* $C_G(Q) \leq I \leq N_G(Q)$ *and let* $C_H(R) \leq J \leq N_H(R)$. *We denote by* $N_{(\varphi,J,I)}$ *the subgroup of* J *consisting of all elements* $h \in J$ *such that there exists* $g \in I$ *such that* $c_g \circ \varphi = \varphi \circ c_h$ *as isomorphisms from* R *to* Q.

Lemma 2.8. *Let* Q *be a subgroup of* G *and let* R *be a subgroup of* H *such that there is an isomorphism* $\varphi : R \to Q$.
(1) For $(g, h) \in G \times H$, ${}^{(g,h)}\Delta(Q, \varphi, R) = \Delta({}^{g}Q, c_g \varphi c_h^{-1}, {}^{h}R)$.
(2) We have that $N_{G \times G}(\Delta(Q)) = \Delta(N_G(Q))(C_G(Q) \times 1)$.

Proof. This is straightforward. □

Lemma 2.9. *Let* Q *be a subgroup of* G *and let* R *be a subgroup of* H *such that there is an isomorphism* $\varphi : R \to Q$. *Let* $C_G(Q) \leq I \leq N_G(Q)$ *and let* $C_H(R) \leq J \leq N_H(R)$.
(1) We have that $p_1(N_{I \times J}(\Delta(Q, \varphi, R))) = N_{(\varphi^{-1},I,J)}$ *and* $p_2(N_{I \times J}(\Delta(Q, \varphi, R))) = N_{(\varphi,J,I)}$.
(2) $k_1((N_{I \times J}(\Delta(Q, \varphi, R)))) = C_G(Q)$ *and* $k_2((N_{I \times J}(\Delta(Q, \varphi, R)))) = C_H(R)$.

Proof. This follows immediately from lemma 2.8.1. □

Lemma 2.10. *For a subgroup* X *of* $G \times H, X * X^o = \Delta(p_1(X))(k_1(X) \times 1)$.

Proof. It is easy to see that $\Delta(p_1(X))(k_1(X) \times 1) \leq X * X^o$, so it remains to prove that $X * X^o \leq \Delta(p_1(X))(k_1(X) \times 1)$. Let $(g, r) \in X * X^o$. There exists $h \in H$ such that $(g, h), (r, h) \in X$, so $(r^{-1}g, 1) \in X$ and hence $g \in rk_1(X)$. Therefore, $(g, r) \in (r, r)(k_1(X) \times 1) \subseteq \Delta(p_1(X))(k_1(X) \times 1)$. This shows that $X * X^o \leq \Delta(p_1(X))(k_1(X) \times 1)$, so the lemma holds. □

Lemma 2.11. *For a subgroup* X *of* $G \times H$ *and for any* $p_1(X) \leq I \leq N_G(k_1(X))$, $(\Delta(I)(k_1(X) \times 1)) * X = X$.

Proof. It is easy to see that $X \leq (\Delta(I)(k_1(X) \times 1)) * X$, so it suffices to show that $(\Delta(I)(k_1(X) \times 1)) * X \leq X$. Let $(g, s) \in (\Delta(I)(k_1(X) \times 1)) * X$, and let $r \in G$ such that $(g, r) \in \Delta(I)(k_1(X) \times 1)$ and $(r, s) \in X$. As $(g, r) \in \Delta(I)(k_1(X) \times 1), g \in rk_1(X)$, so $(g, s) \in (r, s)(k_1(X) \times 1) \subseteq X$. This shows that $(\Delta(I)(k_1(X) \times 1)) * X \leq X$, so the lemma holds. □

Lemma 2.12. *For a subgroup X of $G \times H, X * X^o * X = X$.*

Proof. By lemma 2.10, $X * X^o = \Delta(p_1(X))(k_1(X) \times 1)$, so the result follows from lemma 2.11. □

Chapter 3

Block theoretic preliminaries

The following notation will be used throughout the paper:
Let F be a field. For an F-algebra A, we denote by $pi(A)$ the set of primitive idempotents of A and we denote by $A^\times$ the unit group of A. For a set X, we denote by FX the F-vector space with basis X. For a finite group G, we denote by FG the group algebra of G with coefficients in the field F. For $\alpha = \sum_{g\in G} a_g g \in FG$ and $\beta = \sum_{h\in H} b_h h \in FH$, let $\alpha^o = \sum_{g\in G} a_g g^{-1} \in FG$, and let $\alpha \otimes \beta = \sum_{(g,h)\in G\times H} a_g b_h (g,h) \in F[G\times H]$. For $\Omega \subseteq FG$ and $\Lambda \subseteq FH$, let $\Omega^o = \{\alpha^o : \alpha \in \Omega\}$, and let $\Omega \otimes \Lambda = \{\alpha\otimes\beta : \alpha\in\Omega, \beta\in\Lambda\}$. We denote by $bli(FG)$ the set of block idempotents of FG, and by $Bl(FG)$ the set of blocks of FG.

Throughout this section, let F be a field of characteristic $p > 0$ and let G be a finite group.

Definition 3.1. *Let H be a normal subgroup of G and let e be a block idempotent of FH. The* ***inertial group*** *of FHe in G is the point stabilizer I of the block idempotent e in G under the conjugation action of G on FH. The* ***inertial quotient*** *of FHe in G is the factor group I/H.*

Definition 3.2. *Let P be a p-subgroup of G, and consider the conjugation action of P on FG. We define the map $Br_P : (FG)^P \to FC_G(P)$ by $\sum_{g\in G} a_g g \mapsto \sum_{g\in C_G(P)} a_g g$. By [6] (Proposition 2.2), Br_P is a surjective F-algebra homomorphism. The homomorphism Br_P is called the* ***Brauer homomorphism*** *of FG with respect to P.*

Definition 3.3. *A* ***Brauer pair*** *of FG is a pair (P, e), where is P is a p-subgroup of G and e is a block idempotent of $FC_G(P)$. We denote by $N_G((P, e))$ the inertial group of $FC_G(P)e$*

*in $N_G(P)$. Let (P, e) and (Q, f) be Brauer pairs of FG. Then $(Q, f) \leq (P, e)$ if $Q \leq P$ and for every primitive idempotent $i \in (FG)^P$ such that $Br_P(i)e \neq 0, Br_Q(i)f = Br_Q(i)$. Also, $(Q, f) \trianglelefteq (P, e)$ if $(Q, f) \leq (P, e)$ and $Q \trianglelefteq P$. Let $B = FGe_B$ be a block of FG. We say that a Brauer pair (P, e) of FG is a B-**Brauer pair** if $(1, e_B) \leq (P, e)$, or equivalently, $Br_P(e_B)e = e$.*

Lemma 3.4. *(1) For a Brauer pair (P, e) of FG and $Q \leq P$, there is a unique block idempotent f of $FC_G(Q)$ such that $(Q, f) \leq (P, e)$.*
(2) Let B be a block of FG. The maximal B-Brauer pairs are precisely the B-Brauer pairs (P, e), where P is a defect group of B. Furthermore, all the maximal B-Brauer pairs are G-conjugate.

Proof. See [1](Theorem 3.4 and Theorem 3.10) □

Definition 3.5. *Let (P, e) be a Brauer pair of FG such that $Z(P)$ is the defect group of $FC_G(P)e$. Let I be the inertial group of $FPC_G(P)e$ in $N_G(P)$ and let $\overline{I} = I/PC_G(P)$ be the inertial quotient of $FPC_G(P)e$ in $N_G(P)$. By [9] (Theorems 5.8.10 and 5.8.11), P is the defect group of $FPC_G(P)e$ and hence by [9] (Lemma 5.8.12), $FPC_G(P)e$ has a unique simple module V. It follows that I is the inertial group of V in $N_G(P)$, and hence by [9] (Theorem 3.5.7), there is a unique 2-cohomology class $[\overline{\theta}] \in H^2(\overline{I}, F^\times)$ such that V admits an $F_\theta I$-module structure which extends its $FPC_G(P)$-module structure. The 2-cohomology class $[\overline{\theta}]$ is called the **Külshammer-Puig class** associated with (P, e).*

Definition 3.6. *Let B be a block of FG and let (P, e) be a maximal B-Brauer pair. For $Q \leq P$, we denote by e_Q the unique block idempotent of $FC_G(Q)$ such that $(Q, e_Q) \leq (P, e)$, which exists by Lemma 3.4.1. We define a category $\mathcal{F}$ as follows: The objects of $\mathcal{F}$ are the subgroups of P, and for $Q, R \leq P, Hom_{\mathcal{F}}(Q, R)$ is the set of all group homomorphisms of the form $c_g : Q \to R$, where $g \in G$ such that ${}^g(Q, e_Q) \leq (R, e_R)$. By Theorem 3.9 in [6], $\mathcal{F}$ is a (saturated) fusion system. The category $\mathcal{F}$ is called the **fusion system** associated with (P, e).*

Definition 3.7. *An **interior G-algebra** is an F-algebra A endowed with a group homomorphism $\psi : G \to A^\times$. For $g \in G$ and $a \in A$, we denote the element $\psi(g)a$ of A by ga and the element $a\psi(g)$ of A by ag. If A and B are interior G-algebras, an **interior G-algebra homomorphism** (respectively isomorphism) is an F-algebra homomorphism (respectively isomorphism) $\sigma : A \to B$ such that $\sigma(ga) = g\sigma(a)$ and $\sigma(ag) = \sigma(a)g$ for $a \in A$ and $g \in G$.*

Definition 3.8. *Let B be a block of FG. A* ***source idempotent*** *of B is a primitive idempotent i of B^P such that $Br_P(i)e = Br_P(i) \neq 0$ for some maximal B-Brauer pair (P, e). We say that the source idempotent i is* ***associated with*** *(P, e). A* ***source algebra*** *of B is an interior P-algebra of the form iBi endowed with the group homomorphism $P \to (iBi)^\times$ defined by $u \mapsto ui$ for $u \in P$, where (P, e) is a maximal B-Brauer pair and $i \in pi(B^P)$ is a source idempotent of B associated with (P, e).*

Lemma 3.9. *For a block B of FG and a source idempotent i of $B, B = BiB$ and B is Morita equivalent to iBi.*

Proof. This holds by Proposition 38.2(a) and Theorem 9.9 in [16]. □

Chapter 4

Module theoretic preliminaries

Throughout this paper, all modules are finitely generated.

Throughout this section, $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal (π) and residue field $F = \mathcal{O}/(\pi)$ of characteristic $p > 0$, and G is a finite group. For $\alpha \in \mathcal{O}$, let α^* denote the image of α under the canonical $\mathcal{O}$-algebra homomorphism from $\mathcal{O}$ onto F.

For $\mathcal{O}G$-modules M and N, we write $M|N$ if M is isomorphic to a direct summand of N.

Lemma 4.1. *Let H be a subgroup of G, and let e be an idempotent in $(\mathcal{O}H)^G$. For an $\mathcal{O}H$-module $N, Ind_H^G(eN) \cong eInd_H^G(N)$.*

Proof. This follows from the equality $\mathcal{O}G \otimes_{\mathcal{O}H} eN = e(\mathcal{O}G \otimes_{\mathcal{O}H} N)$. □

Definition 4.2. *(1) For an $\mathcal{O}G$-module M, the **dual module** of M, denoted by M^o, is the $\mathcal{O}G$-module consisting of the $\mathcal{O}$-vector space $Hom_{\mathcal{O}}(M, \mathcal{O})$ endowed with the G-action defined by $g\sigma(m) = \sigma(g^{-1}m)$ for $g \in G, m \in M$, and $\sigma \in Hom_{\mathcal{O}}(M, \mathcal{O})$.*
(2) If H is a finite group, $X \leq G \times H$, and M is an $\mathcal{O}X$-module, we view M^o as an $\mathcal{O}X^o$-module by transporting the $\mathcal{O}X$-module structure of M^o onto an $\mathcal{O}X^o$-module structure via the isomorphism $X^o \to X$ defined by $(h, g) \mapsto (g, h)$ for $(h, g) \in X^o$.

Definition 4.3. *For an FG-module M, the **head** of M, denote by $hd(M)$, is the largest semisimple quotient of M.*

Definition 4.4. *The **trivial module** of $\mathcal{O}G$, denoted by $\mathcal{O}_G$, is the $\mathcal{O}G$-module consisting of the $\mathcal{O}$-module $\mathcal{O}$ endowed with the trivial G-action.*

Definition 4.5. *Let H be a subgroup of G and let M be an $\mathcal{O}H$-module. For $g \in G$, we denote by ${}^gM$ the $\mathcal{O}({}^gH)$-module obtained by transporting the $\mathcal{O}H$-module structure of M along the isomorphism $c_g^{-1} : {}^gH \to H$.*

Definition 4.6. *Let M be an $\mathcal{O}G$-module and let H be a subgroup of G. We define the $\mathcal{O}$-linear map $tr_H^G : M^H \to M^G$ by $m \mapsto \sum_{g \in G/H} gm$ for $m \in M^H$. The map tr_H^G is called the **trace** map. Note that $\sum_{Q<H} tr_Q^H(M^Q) + \pi M^H$ is an $\mathcal{O}N_G(H)$-submodule of M^H. We denote by $M(H)$ the $FN_G(H)$-module $M^H/(\sum_{Q<H} tr_Q^H(M^Q) + \pi M^H)$. The $FN_G(H)$-module $M(H)$ is called the **Brauer construction** of M with respect to H.*

Remark 4.7. *Let H be a finite group. For an $(\mathcal{O}G, \mathcal{O}H)$-bimodule M, we may view M as an $\mathcal{O}[G \times H]$-module via $(g,h)m = gmh^{-1}$ for $m \in M, g \in G$ and $h \in H$. Conversely, for an $\mathcal{O}[G \times H]$-module M, we may view M as an $(\mathcal{O}G, \mathcal{O}H)$-bimodule via $gmh = (g, h^{-1})m$ for $m \in M, g \in G$ and $h \in H$. In this way, we identify $(\mathcal{O}G, \mathcal{O}H)$-bimodules with $\mathcal{O}[G \times H]$-modules.*

Definition 4.8. *An $\mathcal{O}G$-module M is called a **trivial source module** if M is isomorphic to a direct summand of a permutation module.*

Lemma 4.9. *For an $\mathcal{O}G$-module M, the following are equivalent:*
(1) M is a trivial source module
(2) The trivial module is the source of every indecomposable direct summand of M
(3) For every p-subgroup P of $G, Res_P^G(M)$ is a permutation module.

Proof. See [4] ((0.4)). □

Lemma 4.10. *(1) If M and N are trivial source $\mathcal{O}G$-modules, then $M \oplus N$ and $M \otimes N$ are trivial source $\mathcal{O}G$-modules.*
(2) If M is a trivial source $\mathcal{O}G$-module, then so is M^o.
(3) If M is a trivial source $\mathcal{O}G$-module, then $M/\pi M$ is a trivial source FG-module.
(4) If H is a subgroup of G and M is a trivial source $\mathcal{O}G$-module, then $Res_H^G(M)$ is a trivial source $\mathcal{O}H$-module.

(5) If H is a subgroup of G and M is a trivial source $\mathcal{O}H$-module, then ${}^gM$ is a trivial source $\mathcal{O}({}^gH)$-module and $Ind_H^G(M)$ is a trivial source $\mathcal{O}G$-module.

Proof. This follows easily from lemma 4.9. □

Lemma 4.11. *Let M be a trivial source $\mathcal{O}G$-module, let P be a p-subgroup of M, and let X be a P-invariant $\mathcal{O}$-basis of M, which exists by lemma 4.9. Define the $\mathcal{O}$-linear map $Br_X : M \to FX^P$ by $\sum_{x\in X} \alpha_x x \mapsto \sum_{x\in X^P} \alpha_x^* x$. The $FN_G(P)$-module $M(P)$ is isomorphic to the $FN_G(P)$-module consisting of the F-vector space FX^P endowed with the $N_G(P)$-action defined by $g \cdot x = Br_X(gx)$ for $x \in X^P$ and $g \in N_G(P)$.*

Proof. See [16] (Proposition 27.6(a)). □

Lemma 4.12. *Let M be a trivial source $\mathcal{O}G$-module and let P be a p-subgroup of G.*
(1) We have that $M(P) \cong (M/\pi M)(P)$ as $FN_G(P)$-modules.
(2) We have that $M^o(P) \cong (M(P))^o$ as $FN_G(P)$-modules.

Proof. Note that (1) follows immediately from lemma 4.11. For (2), see the proof to lemma (2.4) in [4]. □

Lemma 4.13. *For an indecomposable trivial source $\mathcal{O}G$-module M and a normal p-subgroup Q of G, Q is contained in a vertex of M if and only if Q acts trivially on M.*

Proof. If Q acts trivially on M, then Q is contained in the vertex of M by [9] (Theorem 4.7.8). Conversely, suppose that Q is contained in a vertex P of M. As M is an indecomposable trivial source $\mathcal{O}G$-module with vertex $P, M | Ind_P^G(\mathcal{O}_P)$, so the result follows from the Mackey formula. □

Lemma 4.14. *Let M be a trivial source FG-module and let P be a p-subgroup of G. By the Krull-Schmidt theorem, there exist unique $FN_G(P)$-modules L and N such that P acts trivially on L, P does not act trivially on any indecomposable direct summand of N, and $Res^G_{N_G(P)}(M) \cong L \oplus N$. We have that $M(P) \cong L$ as $FN_G(P)$-modules.*

Proof. As P acts trivially on $L, L(P) \cong L$ by lemma 4.11. As P does not act trivially on any indecomposable direct summand of N, no indecomposable direct summand of N has vertex containing P by lemma 4.13, and hence $N(P) = 0$ by [4] ((1.3)). Thus as $M(P) \cong (Res^G_{N_G(P)}(M))(P) \cong L(P) \oplus N(P)$, the lemma holds. □

Lemma 4.15. *(1) Let M be an indecomposable trivial source $\mathcal{O}G$-module with vertex P, and let Q be a subgroup of P. The $FN_G(Q)$-module $M(Q)$ is a trivial source module and every vertex of every indecomposable direct summand of $M(Q)$ is contained in a G-conjugate of P.*
(2) Let H be a finite group and let M be an indecomposable trivial source $(\mathcal{O}G, \mathcal{O}H)$-bimodule with twisted diagonal vertex. For any subgroup U of $G\times H$ contained in a vertex of $M, M(U)$ is a trivial source $FN_{G\times H}(U)$-module and every indecomposable direct summand of $M(U)$ has twisted diagonal vertex.

Proof. It suffices to prove (1), as then (2) follows. As M is an indecomposable trivial source $\mathcal{O}G$-module with vertex $P, M|Ind_P^G(\mathcal{O}_P)$, so by the Mackey formula,
$Res^G_{N_G(Q)}(M/\pi M)| \bigoplus_{t\in N_G(Q)\backslash G/P} Ind^{N_G(Q)}_{N_{{}^tP}(Q)}(F_{N_{{}^tP}(Q)})$. Now as $M(Q)\cong (M/\pi M)(Q)$ by lemma 4.12, $M(Q)$ is a direct summand of $Res^G_{N_G(Q)}(M/\pi M)$ by lemma 4.14, so the lemma follows. □

Lemma 4.16. *For an indecomposable trivial source FG-module M with vertex $P, M(P)$ is the Green correspondent of M.*

Proof. This is an immediate consequence of lemma 4.13 and lemma 4.14. □

Lemma 4.17. *Let Q be a p-subgroup of G and let R be a p-subgroup of $N_G(Q)$. For a trivial source $\mathcal{O}G$-module $M, (M(Q))(R)\cong M(QR)$ as $F(N_G(Q)\cap N_G(R))$-modules.*

Proof. This follows from lemma 4.11. □

Lemma 4.18. *Let M be a trivial source FG-module, let Q be a p-subgroup of G, and let i be an idempotent in $(FG)^Q$. If $iM = M$, then $Br_Q(i)M(Q) = M(Q)$.*

Proof. This is well-known and it is straightforward. □

Lemma 4.19. *Let H be a finite group and let $\Delta(Q,\varphi,R)$ be a twisted diagonal p-subgroup of $G\times H$ such that $Q\trianglelefteq G$ and $R\trianglelefteq H$. If $\Delta(D,\varphi,E)$ is a twisted diagonal p-subgroup of $G\times H$ containing $\Delta(Q,\varphi,R)$ and M is an indecomposable trivial source $\mathcal{O}N_{G\times H}(\Delta(Q,\varphi,R))$-module with twisted diagonal vertex containing $\Delta(Q,\varphi,R)$, then*
$(Ind^{G\times H}_{N_{G\times H}(\Delta(Q,\varphi,R))}(M))(\Delta(D,\varphi,E))\cong M(\Delta(D,\varphi,E))$ as $FN_{G\times H}(\Delta(D,\varphi,E))$-modules.

Proof. By applying the Mackey formula to the restriction of $Ind^{G\times H}_{N_{G\times H}(\Delta(Q,\varphi,R))}(M)$ to $N_{G\times H}(\Delta(D,\varphi,E))$, it suffices to show that if $(g,h)\in G\times H$ such that $(Ind^{N_{G\times H}(\Delta(D,\varphi,E))}_{N_{G\times H}(\Delta(D,\varphi,E))\cap {}^{(g,h)}N_{G\times H}(\Delta(Q,\varphi,R))}({}^{(g,h)}M))(\Delta(D,\varphi,E))\neq 0$, then $(g,h)\in N_{G\times H}(\Delta(Q,\varphi,R))$.
As $(Ind^{N_{G\times H}(\Delta(D,\varphi,E))}_{N_{G\times H}(\Delta(D,\varphi,E))\cap {}^{(g,h)}N_{G\times H}(\Delta(Q,\varphi,R))}({}^{(g,h)}M))(\Delta(D,\varphi,E))\neq 0$, it follows that $\Delta(D,\varphi,E)$ is contained in a vertex of ${}^{(g,h)}M$, and hence $\Delta(Q,\varphi,R)$ is contained in a vertex of ${}^{(g,h)}M$. Thus as ${}^{(g,h)}M$ has twisted diagonal vertex containing ${}^{(g,h)}\Delta(Q,\varphi,R)$, it follows that $(g,h)\in N_{G\times H}(\Delta(Q,\varphi,R))$, so the lemma holds. □

Lemma 4.20. *For a trivial source FG-module M, there is a unique trivial source $\mathcal{O}G$-module N such that $N/\pi N\cong M$ as FG-modules.*

Proof. This holds by [9] (Theorem 4.8.9(iii)). □

Definition 4.21. *Let M be a trivial source FG-module. We denote by $M^{\mathcal{O}}$ the unique trivial source $\mathcal{O}G$-module such that $M^{\mathcal{O}}/\pi M^{\mathcal{O}}\cong M$ as FG-modules. If $\mathcal{O}$ has characteristic zero and K is the field of fractions of $\mathcal{O}$, we denote the KG-module $K\otimes_{\mathcal{O}} M^{\mathcal{O}}$ by M^K.*

Lemma 4.22. *Let M and N be trivial source FG-modules.*
(1) We have that $dim_F(Hom_{FG}(M,N)) = dim_{\mathcal{O}}(Hom_{\mathcal{O}G}(M^{\mathcal{O}},N^{\mathcal{O}}))$.
(2) The FG-module M is indecomposable if and only if the $\mathcal{O}G$-module $M^{\mathcal{O}}$ is indecomposable, and in this case, M and $M^{\mathcal{O}}$ have the same vertex.

Proof. This holds by [9] (Theorem 4.8.9 and Theorem 1.11.12) □

For the following lemma, we assume that $\mathcal{O}$ has characteristic zero and contains a primitive $|G|^{th}$ root of unity.

Lemma 4.23. *Let $B = FGe_B$ be a block of FG with defect group P such that $P\leq Z(G)$. For any subgroup Q of P, there is a unique indecomposable trivial source B-module with vertex Q.*

Proof. Let $\overline{G} = G/Q$. By the Green correspondence, it suffices to show that there is a unique projective indecomposable $\overline{B}$-module. By [9] (Theorem 5.8.10 and Theorem 5.8.11), $\overline{B}$ is a block of $F\overline{G}$ with defect group $\overline{P}$, so as $\overline{P}\leq Z(\overline{G})$, we may assume that $Q=1$. Thus the lemma holds as B has a unique projective indecomposable module by [16] (Proposition 39.2(b)). □

Chapter 5

Blocks with normal defect groups

Throughout this section, let F be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. Let $B = FGe_B$ be a block of FG with maximal B-Brauer (P, e) such that $P \trianglelefteq G$, and let I be the inertial group of $FC_G(P)e$ in $N_G(P)$.

Lemma 5.1. *Let i be a primitive idempotent of $FC_G(P)e$. Then i is a source idempotent of B and FIe and $iFGi = iBi = iFIi$.*

Proof. By Proposition 2.3 in [6], $ker(Br_P)$ is a nilpotent ideal of B^P, so as $i = Br_P(i)$ is a primitive idempotent of $Br_P(B^P) = FC_G(P)Br_P(e_B), i$ is a primitive idempotent of B^P, and hence i is a source idempotent of B. Thus as (P, e) is a maximal FIe-Brauer pair and $P \trianglelefteq I, i$ is a source idempotent of FIe. As $i \in B, iFGi = iBi$. To show that $iFGi = iFIi$, note that for $g \in G - I, igi = i\,{}^{g}ig = 0$ as i and ${}^{g}i$ lie in the distinct blocks $FC_G(P)e$ and $FC_G(P)\,{}^{g}e$ of $FC_G(P)$ respectively. □

Lemma 5.2. *Let $C_G(P) \le S \le I$ and let i be a primitive idempotent of $FC_G(P)e$.*
(1) Any block of the S-algebra FSe has defect group $S \cap P$.
(2) The $(FSe, iFSi)$-bimodule FSi induces a Morita equivalence between the F-algebras FSe and $iFSi$.
(3) If $P \le S$, then FSe is a block of FS with defect group P and i is a source idempotent of FSe.

Proof. (1) Let Λ be a block of FSe and let D be a defect group of Λ. Then as $S \cap P \trianglelefteq S, S \cap P \le D$. Now as $e \in FC_G(P) \subseteq FS$ and $S \le I$, it follows that $FIe \cong \bigoplus_{t \in S\backslash I/S} F[StS]e$ as (FS, FS)-bimodules, and hence FSe is an (FS, FS)-bimodule direct summand of FIe.

Thus as Λ is a block of FSe, Λ is an (FS, FS)-bimodule direct summand of FIe. Therefore, as $\Delta(D)$ is a vertex of the indecomposable (FS, FS)-bimodule Λ and $\Delta(P)$ is a vertex of the indecomposable (FI, FI)-bimodule $FIe, \Delta(D) \leq_{I\times I} \Delta(P)$, so $D \leq S \cap P$ and hence $D = S \cap P$. Thus (1) holds.

(2) As $(Z(P), e)$ is a maximal $FC_G(P)e$-Brauer pair by (1), i is a source idempotent of $FC_G(P)e$ by lemma 5.1. Therefore, $e \in (FC_G(P)e)i(FC_G(P)e) \subseteq (FSe)i(FSe)$ by lemma 3.9, so $FSe = (FSe)i(FSe)$ and hence (2) holds.

(3) As $P \trianglelefteq S$, any block idempotent of FS must lie in $FC_G(P)$. Thus as $FC_G(P)e$ is a block of $FC_G(P), FSe$ is a block of FS. By (1), P is the defect group of FSe, so (3) holds by lemma 5.1. □

Chapter 6

Brauer pairs for trivial source modules

Throughout this section, $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal (π) and residue field $F = \mathcal{O}/(\pi)$ of characteristic $p > 0$, and G is a finite group.

Definition 6.1. *For a trivial source $\mathcal{O}G$-module M, an M-**Brauer pair** is a Brauer pair (P, e) of FG such that $eM(P) \neq 0$.*

Remark 6.2. *Let B be a block of FG. The B-Brauer pairs are precisely the Brauer pairs of the form (P, e) such that $(\Delta(P), e \otimes e^o)$ is a B-Brauer pair of the indecomposable trivial source $F[G \times G]$-module B.*

Lemma 6.3. *Let M be a trivial source $\mathcal{O}G$-module. The set of M-Brauer pairs is closed under inclusion and G-conjugation.*

Proof. If (P, e) is an M-Brauer pair and $g \in G$, then ${}^g eM({}^gP) \cong {}^g(eM(P))$ as $FN_G(({}^gP, {}^ge))$-modules, so as $eM(P) \neq 0$, ${}^geM({}^gP) \neq 0$ and hence $({}^gP, {}^ge)$ is an M-Brauer pair. This shows that the set of M-Brauer pairs is closed under G-conjugation, so it remains to show that it is closed under inclusion.

It suffices to show that if (P, e) be an M-Brauer pair and (Q, f) is a Brauer pair of FG such that $(Q, f) \trianglelefteq (P, e)$, then (Q, f) is an M-Brauer pair. Assume the contrary. Let I be the inertial group of $FC_G(Q)f$ in $N_G(Q)$. As P is a subgroup of I by [16] (Theorem 40.4(b)), we may apply the Brauer construction at P to the FI-module $fM(Q)$. As (Q, f) is not an M-Brauer pair, $fM(Q) = 0$, and hence as $(fM(Q))(P) \cong Br_P(f)M(P)$ as $FC_G(P)$-modules by lemma 4.17 and lemma 4.18, $Br_P(f)M(P) = 0$. Thus as $eBr_P(f) = e$ by [16] (Theorem 40.4(b)), $eM(P) = 0$, which is a contradiction as (P, e) is an M-Brauer pair. Thus the lemma holds. □

The next lemma is a generalization of lemma 3.4.2 to M-Brauer pairs for an indecomposable trivial source FG-module M.

Lemma 6.4. *Let M be an indecomposable trivial source $\mathcal{O}G$-module. The maximal M-Brauer pairs are precisely the M-Brauer pairs (P, e), where P is a vertex of M. Furthermore, all maximal M-Brauer pairs are G-conjugate.*

Proof. This is a consequence of Theorem 2.5 in [15], but we shall give an independent proof.

By lemma 4.12 and lemma 4.22.2, we may assume that $\mathcal{O} = F$. First we show that the M-Brauer pairs of the form (P, e), where P is a vertex of M, are all G-conjugate. As the vertices of M are all G-conjugate, it suffices to show that if P is a vertex of M, then the M-Brauer pairs of the form (P, e) are all $N_G(P)$-conjugate. By lemma 4.16, $M(P)$ is the Green correspondent of M, and hence is an indecomposable $FN_G(P)$-module. Thus by [9] (Lemma 5.5.4), for a block idempotent e of $FC_G(P), (P, e)$ is an M-Brauer pair if and only if $FC_G(P)e$ is covered by the block of $FN_G(P)$ to which $M(P)$ belongs, so we obtain the desired result by [9] (Lemma 5.5.3). Therefore, it suffices to show that if (Q, f) is an M-Brauer pair, then (Q, f) is contained in an M-Brauer pair of the form (P, e), where P is a vertex of M.

As (Q, f) is an M-Brauer pair, $M(Q) \neq 0$, and hence Q is contained in a vertex of M by [4] ((1.3)). Thus we may proceed by induction on the index of Q in a vertex of M that contains Q. Let I be the inertial group of $FC_G(Q)f$ in $N_G(Q)$. As (Q, f) is an M-Brauer pair, $fM(Q) \neq 0$, and hence there is an indecomposable direct summand N of $M(Q)$ such that $fN \neq 0$. Assume that the FI-module fN has vertex Q. As $fN | Res_I^{N_G(Q)}(N)$ and $N \cong Ind_I^{N_G(Q)}(fN)$ by [9] (Lemma 5.5.4), it follows that N has vertex Q. Thus as $N | M(Q) | Res^G_{N_G(Q)}(M)$ by lemma 4.14, M has vertex Q by the Burry-Carlsson-Puig theorem, and hence the result holds in this case. Thus we may assume that fN has vertex R properly containing Q. As R is a vertex of $fN, (fN)(R) \neq 0$ by lemma 4.16, so as $fN | fM(Q), (fM(Q))(R) \neq 0$. Thus as $(fM(Q))(R) \cong Br_R(f)M(R)$ as $FC_G(R)$-modules by lemma 4.17 and 4.18, $Br_R(f)M(R) \neq 0$, and hence there is a block idempotent e of

$FC_G(R)$ such that $eBr_R(f) = e$ and $eM(R) \neq 0$. As $R \leq I$ and $eBr_R(f) = e, (Q, f) \trianglelefteq (R, e)$ by [16] (Theorem 40.4(b)), and as $eM(R) \neq 0, (R, e)$ is an M-Brauer pair, so as Q is properly contained in R, the result follows from our inductive hypothesis. □

Lemma 6.5. *Let M and N be indecomposable trivial source $\mathcal{O}G$-modules with a common maximal Brauer pair (P, e), and let I be the inertial group of $FC_G(P)e$ in $N_G(P)$. Then $M \cong N$ if and only if $eM(P) \cong eN(P)$ as FIe-modules.*

Proof. By lemma 4.12 and lemma 4.20, we may assume that $\mathcal{O} = F$. By lemma 6.4, P is a common vertex of M and N, so by lemma 4.16, $M(P)$ and $N(P)$ are the Green correspondents of M and N respectively, and hence $M(P) \cong N(P)$ as $FN_G(P)$-modules if and only if $M \cong N$. Let $C = (FC_G(P)e)^{N_G(P)}$ be the block of $FN_G(P)$ that covers $FC_G(P)e$. The $(FN_G(P), FI)$-bimodule $FN_G(P)e$ induces a Morita equivalence between C and FIe, so as $M(P)$ and $N(P)$ belong to $C, M(P) \cong N(P)$ as $FN_G(P)$-modules if and only if $eM(P) \cong eN(P)$ as FI-modules, so the lemma holds.

□

Chapter 7

A tensor product of modules

Throughout this section, let G, H and K be finite groups and let $\mathcal{O}$ be a commutative ring.

Lemma 7.1. *Let X and Y be subgroups of $G \times H$ and $H \times K$ respectively, let M be an $\mathcal{O}X$-module and let N be an $\mathcal{O}Y$-module. The $(\mathcal{O}k_1(X), \mathcal{O}k_2(Y))$-bimodule $M \otimes_{\mathcal{O}(k_2(X) \cap k_1(Y))} N$ may be endowed with an $\mathcal{O}(X * Y)$-module structure which extends its $\mathcal{O}(k_1(X) \times k_2(Y))$-module structure as follows: For $(g, k) \in X * Y, m \in M$, and $n \in N$, let $(g, k)(m \otimes n) = (g, h)m \otimes (h, k)n$, for any $h \in H$ such that $(g, h) \in X$ and $(h, k) \in Y$.*

Proof. This is a straightforward verification. □

Lemma 7.2. *Let $S \leq X \leq G \times H$, let $Y \leq H \times K$, let V be an $\mathcal{O}S$-module and let W an $\mathcal{O}Y$-module. If $p_2(X) \leq p_1(Y)$, then $Ind_S^X(V) \otimes_{\mathcal{O}(k_2(X) \cap k_1(Y))} W \cong Ind_{S*Y}^{X*Y}(V \otimes_{\mathcal{O}(k_2(S) \cap k_1(Y))} W)$ as $\mathcal{O}(X * Y)$-modules.*

Proof. The map $\alpha : Ind_S^X(V) \otimes_{\mathcal{O}(k_2(X) \cap k_1(Y))} W \to Ind_{S*Y}^{X*Y}(V \otimes_{\mathcal{O}(k_2(S) \cap k_1(Y))} W)$ defined by $((g, h) \otimes v) \otimes w \mapsto (g, k) \otimes (v \otimes (h, k)^{-1} w)$ for $(g, h) \in X, v \in V$ and $w \in W$, where $k \in K$ such that $(h, k) \in Y$ is a well-defined $\mathcal{O}(X * Y)$-module isomorphism with inverse $\beta : Ind_{S*Y}^{X*Y}(V \otimes_{\mathcal{O}(k_2(S) \cap k_1(Y))} W) \to Ind_S^X(V) \otimes_{\mathcal{O}(k_2(X) \cap k_1(Y))} W$ defined by $(g, k) \otimes (v \otimes w) \mapsto ((g, h) \otimes v) \otimes (h, k)w$ for $(g, k) \in X * Y, v \in V$ and $w \in W$, where $h \in H$ such that $(g, h) \in X$ and $(h, k) \in Y$. □

Lemma 7.3. *Let $X \leq G \times H$, let $T \leq Y \leq H \times K$, let V be an $\mathcal{O}X$-module and W an $\mathcal{O}T$-module. If $p_1(Y) \leq p_2(X)$, then $V \otimes_{\mathcal{O}(k_2(X) \cap k_1(Y))} Ind_T^Y(W) \cong Ind_{X*T}^{X*Y}(V \otimes_{\mathcal{O}(k_2(X) \cap k_1(T))} W)$ as $\mathcal{O}(X * Y)$-modules.*

Proof. The map $\alpha : V \otimes_{\mathcal{O}(k_2(X)\cap k_1(Y))} Ind_T^Y(W) \to Ind_{X*T}^{X*Y}(V \otimes_{\mathcal{O}(k_2(X)\cap k_1(T))} W)$ defined by $v \otimes ((h,k)\otimes w) \mapsto (g,k)\otimes((g,h)^{-1}v \otimes w)$ for $(h,k) \in Y, v \in V$ and $w \in W$, where $g \in G$ such that $(g,h) \in X$ is an $\mathcal{O}(X*Y)$-module isomorphism with inverse $\beta : Ind_{X*T}^{X*Y}(V \otimes_{\mathcal{O}(k_2(X)\cap k_1(T))} W) \to V \otimes_{\mathcal{O}(k_2(X)\cap k_1(Y))} Ind_T^Y(W)$ defined by $(g,k)\otimes(v\otimes w) \mapsto (g,h)v \otimes ((h,k)\otimes w)$ for $(g,k) \in X*Y, v \in V$, and $w \in W$, where $h \in H$ such that $(g,h) \in X$ and $(h,k) \in Y$. □

For the remainder of the section we assume that $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal (π) and residue field $F = \mathcal{O}/(\pi)$ of characteristic $p > 0$.

Lemma 7.4. *Let X be a subgroup of $G \times H$ and let Y be a subgroup of $H \times K$.*
(1) If M is a trivial source $\mathcal{O}X$-module and N is a trivial source $\mathcal{O}Y$-module, then $M \otimes_{\mathcal{O}(k_2(X)\cap k_1(Y))} N$ is either the zero module or a trivial source $\mathcal{O}(X*Y)$-module.
(2) If M is an indecomposable $\mathcal{O}X$-module with twisted diagonal vertex and N is an indecomposable $\mathcal{O}Y$-module with twisted diagonal vertex, then every indecomposable direct summand of the $\mathcal{O}(X*Y)$-module $M \otimes_{\mathcal{O}(k_2(X)\cap k_1(Y))} N$ has twisted diagonal vertex.

Proof. (1) Let $X' = p_2^{-1}(p_1(Y))\cap X$. As $X'*Y = X*Y$ and $k_2(X)\cap k_1(Y) = k_2(X')\cap k_1(Y)$, it follows that $Res_{X'}^X(M) \otimes_{\mathcal{O}(k_2(X')\cap k_1(Y))} N \cong M \otimes_{\mathcal{O}(k_2(X)\cap k_1(Y))} N$ as $\mathcal{O}(X*Y)$-modules. Thus as $Res_{X'}^X(M)$ is a trivial source module by lemma 4.10.4, we may assume that $p_2(X) \le p_1(Y)$. We may also assume that $M = Ind_S^X(\mathcal{O}_S)$ for some subgroup S of X. Thus by lemma 7.2, $M \otimes_{\mathcal{O}(k_2(X)\cap k_1(Y))} N \cong Ind_{S*Y}^{X*Y}(\mathcal{O}_S \otimes_{\mathcal{O}(k_2(S)\cap k_1(Y))} N)$, so by lemma 4.10.5, it suffices to show that $\mathcal{O}_S \otimes_{\mathcal{O}(k_2(S)\cap k_1(Y))} N$ is a trivial source $\mathcal{O}(S*Y)$-module. Arguing as above, we may assume that $p_1(Y) \le p_2(S)$. We may also assume that $N = Ind_T^Y(\mathcal{O}_T)$ for some subgroup T of Y. Thus by lemma 7.3, $\mathcal{O}_S \otimes_{\mathcal{O}(k_2(S)\cap k_1(Y))} N \cong Ind_{S*T}^{S*Y}(\mathcal{O}_S \otimes_{\mathcal{O}(k_2(S)\cap k_1(T))} \mathcal{O}_T) \cong Ind_{S*T}^{S*Y}(\mathcal{O}_{S*T})$, so (1) holds.
(2) By the same argument as given in the proof of (1), we may assume that $p_2(X) \le p_1(Y)$. As M has twisted diagonal vertex, we may assume that $M = Ind_S^X(V)$ for some twisted diagonal subgroup S of X and some FS-module V. Therefore, by lemma 7.2, $M \otimes_{\mathcal{O}(k_2(X)\cap k_1(Y))} N \cong Ind_{S*Y}^{X*Y}(V \otimes_{\mathcal{O}(k_2(S)\cap k_1(Y))} N)$, and hence it suffices to show that every indecomposable direct summand of the $\mathcal{O}(S*Y)$-module $V \otimes_{\mathcal{O}(k_2(S)\cap k_1(Y))} N$ has twisted diagonal vertex. Arguing as in the proof of (1), we may assume that $p_1(Y) \le p_2(S)$. As N has twisted diagonal vertex, we may assume that $N = Ind_T^Y(W)$ for some twisted diagonal subgroup T of Y and some FT-module W. Thus by lemma 7.3, $V \otimes_{\mathcal{O}(k_2(S)\cap k_1(Y))} N \cong$

$Ind_{S*T}^{S*Y}(V \otimes_{\mathcal{O}(k_2(S)\cap k_1(T))} W)$. Therefore, $V \otimes_{\mathcal{O}(k_2(S)\cap k_1(Y))} N$ is $(S*T)$-projective, and as S and T are twisted diagonal, $S*T$ is twisted diagonal, so the lemma holds. $\square$

Lemma 7.5. *Let (Q,e) and (U,ϵ) be Brauer pairs of FG and FK respectively such that there is an isomorphism $\alpha : U \to Q$, let $C_G(Q) \le S \le N_G((Q,e))$, and let $C_K(U) \le T \le N_K((U,\epsilon))$. Let Γ denote the set of ordered quadruples (R,f,φ,ψ), such that (R,f) is a Brauer pair of FH and $\varphi : R \to Q$ and $\psi : U \to R$ are isomorphisms such that $\varphi \circ \psi = \alpha$. Note that $N_{G\times K}(\Delta(Q,\alpha,U)) \times H$ acts by conjugation on Γ via ${}^{((g,k),h)}(R,f,\varphi,\psi) = ({}^hR, {}^hf, c_g\varphi c_{h^{-1}}, c_h\psi c_{k^{-1}})$ for $(R,f,\varphi,\psi) \in \Gamma, (g,k) \in N_{G\times K}(\Delta(Q,\alpha,U))$, and $h \in H$. Let M be a trivial source $(\mathcal{O}G,\mathcal{O}H)$-bimodule such that every indecomposable direct summand of M has twisted diagonal vertex, and let N be a trivial source $(\mathcal{O}H,\mathcal{O}K)$-bimodule such that every indecomposable direct summand of N has twisted diagonal vertex.*

(1) We have that $e((M \otimes_{\mathcal{O}H} N)(\Delta(Q,\alpha,U)))\epsilon \cong$

$$\bigoplus_{(R,f,\varphi,\psi)\in\Gamma/\sim_H} eM(\Delta(Q,\varphi,R))f \otimes_{FC_H(R)} fN(\Delta(R,\psi,U))\epsilon$$

as $(FC_G(Q)e, FC_K(U)\epsilon)$-bimodules.

(2) We have that $e((M \otimes_{\mathcal{O}H} N)(\Delta(Q,\alpha,U)))\epsilon \cong$

$$\bigoplus_{(R,f,\varphi,\psi)\in\Gamma/\sim_{N_{S\times T}(\Delta(Q,\alpha,U))\times H}} Ind_{N_{S\times H}((\Delta(Q,\varphi,R),e\otimes f^o))*N_{H\times T}((\Delta(R,\psi,U),f\otimes\epsilon^o))}^{N_{S\times T}(\Delta(Q,\alpha,U))}(eM(\Delta(Q,\varphi,R))f$$

$\otimes_{FC_H(R)} fN(\Delta(R,\psi,U))\epsilon)$ *as $FN_{S\times T}(\Delta(Q,\alpha,U))(e\otimes\epsilon^o))$-modules.*

Proof. As $M \otimes_{\mathcal{O}H} N$ is a trivial source $(\mathcal{O}G,\mathcal{O}K)$-bimodule by lemma 7.4, we may assume that $\mathcal{O} = F$ by lemma 4.12. By Theorem 3.3 of [2], the map

$$\sigma : \bigoplus_{(R,f,\varphi,\psi)\in\Gamma/\sim_H} eM(\Delta(Q,\varphi,R))f \otimes_{FC_H(R)} fN(\Delta(R,\psi,U))\epsilon \to e((M\otimes_{FH}N)(\Delta(Q,\alpha,U)))\epsilon$$

defined by $\overline{m}\otimes\overline{n} \mapsto \overline{m\otimes n}$ for $(R,f,\varphi,\psi) \in \Gamma/\sim_H, m \in M^{\Delta(Q,\varphi,R)}$, and $n \in N^{\Delta(R,\psi,U)}$ is an isomorphism of $(FC_G(Q)e, FC_K(U)\epsilon)$-bimodules, so (1) holds. Furthermore, by transporting the $FN_{S\times T}(\Delta(Q,\alpha,U))(e\otimes\epsilon^o))$-module structure of $e((M\otimes_{FH}N)(\Delta(Q,\alpha,U)))\epsilon$ along the isomorphism σ, we see that for $(R,f,\varphi,\psi) \in \Gamma/\sim_H, (s,t)(eM(\Delta(Q,\varphi,R))f \otimes_{FC_H(R)} fN(\Delta(R,\psi,U))\epsilon) = {}^s eM(\Delta({}^sQ, c_s\varphi, R))f \otimes_{FC_H(R)} fN(\Delta(R,\psi c_{t^{-1}}, {}^tU))\,{}^t\epsilon$ for $(s,t) \in N_{S\times T}(\Delta(Q,\alpha,U))$, and that the $F(N_{S\times H}((\Delta(Q,\varphi,R),e\otimes f^o))*N_{H\times T}((\Delta(R,\psi,U),f\otimes\epsilon^o)))$-module structure of $eM(\Delta(Q,\varphi,R))f \otimes_{FC_H(R)} fN(\Delta(R,\psi,U))\epsilon$ is the one obtained by applying the construction given in lemma 7.1. Thus (2) follows. $\square$

Lemma 7.6. *Let (Q,e) be a Brauer pair of FG and let $C_G(Q) \le S \le N_G((Q,e))$. Let*

Γ denote the set of ordered triples (R, f, φ) such that (R, f) is a Brauer pair of FH and $\varphi : R \to Q$ is an isomorphism. Note that $S \times H$ acts by conjugation on Γ via ${}^{(g,h)}(R, f, \varphi) = ({}^hR, {}^hf, c_g\varphi c_{h^{-1}})$ for $(g, h) \in S \times H$ and $(R, f, \varphi) \in \Gamma$. Let M be a trivial source $(\mathcal{O}G, \mathcal{O}H)$-bimodule such that every indecomposable direct summand of M has twisted diagonal vertex, and let N be a trivial source $(\mathcal{O}H, \mathcal{O}G)$-bimodule such that every indecomposable direct summand of N has twisted diagonal vertex.

(1) $e((M \otimes_{\mathcal{O}H} N)(\Delta(Q)))e \cong$

$$\bigoplus_{(R,f,\varphi)\in\Gamma/\sim_H} e(M(\Delta(Q, \varphi, R)))f \otimes_{FC_H(R)} f(N(\Delta(R, \varphi^{-1}, Q)))e$$

as $(FC_G(Q)e, FC_G(Q)e)$-bimodules.

(2) $e((M \otimes_{\mathcal{O}H} N)(\Delta(Q)))e \cong$

$$\bigoplus_{(R,f,\varphi)\in\Gamma/\sim_{S\times H}} Ind^{N_{S\times S}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},S,J)})(C_G(Q)\times 1)}(e(M(\Delta(Q, \varphi, R))f \otimes_{FC_H(R)}$$

$fN(\Delta(R, \varphi^{-1}, Q)))e)$ *as $FN_{S\times S}(\Delta(Q))(e \otimes e^o)$-modules, where J is the inertial group of $FC_H(R)f$ in $N_H(R)$.*

Proof. The hypothesis of the lemma is the special case of the hypothesis of lemma 7.5 with $G = K, (Q, e) = (U, \epsilon), S = T$ and α the identity automorphism of Q. Thus (1) follows immediately from (1) of lemma 7.5, and hence it remains to show that (2) is a consequence of (2) of lemma 7.5.

As $C_G(Q) \le S \le N_G(Q), N_{S\times S}(\Delta(Q)) = \Delta(S)(C_G(Q) \times 1)$ by lemma 2.8.2. Thus as $C_G(Q) \times 1$ acts trivially on Γ and for $s \in S$ and $(R, f, \varphi) \in \Gamma$, ${}^{(s,s)}(R, f, \varphi) = {}^s(R, f, \varphi)$, the $N_{S\times S}(\Delta(Q))$-orbits of Γ are precisely the S-orbits of Γ. Let $(R, f, \varphi) \in \Gamma$, and let J be the inertial of $FC_H(R)f$ in $N_H(R)$. Then $N_{S\times H}((\Delta(Q, \varphi, R), e\otimes f^o)) = N_{S\times J}(\Delta(Q, \varphi, R))$, so $N_{S\times H}((\Delta(Q, \varphi, R), e \otimes f^o)) * N_{H\times S}((\Delta(R, \varphi^{-1}, Q), f \otimes e^o)) = \Delta(N_{(\varphi^{-1},S,J)})(C_G(Q) \times 1)$ by lemma 2.9 and lemma 2.10. Thus (2) follows from (2) of lemma 7.5. $\square$

Chapter 8

Adjointness of $\otimes$ and Hom

Throughout this section, let F be a field and let G and H be finite groups. Let $X \leq G \times H$, let U be an FX-module, let V be an FX^o-module, and let W be an $F(X * X^o)$-module.

Lemma 8.1. *(1) $Hom_{Fk_1(X)}(U, W)$ is an FX^o-module via*
$((h, g)\sigma)(u) = (g, g)\sigma((g, h)^{-1}u)$ for $(h, g) \in X^o, \sigma \in Hom_{Fk_1(X)}(U, W)$, and $u \in U$.
(2) $Hom_{Fk_1(X)}(V, W)$ is an FX-module via $((g, h)\sigma)(v) = (g, g)\sigma((h, g)^{-1}v)$ for $(g, h) \in X, \sigma \in Hom_{Fk_1(X)}(V, W)$, and $v \in V$.

Proof. This is a straightforward verification. □

By viewing $Fk_1(X)$ as an $F(X * X^o)$-module via left and right multiplication, $Hom_{Fk_1(X)}(U, Fk_1(X))$ acquires an FX^o-module structure and $Hom_{Fk_1(X)}(V, Fk_1(X))$ acquires an FX-module structure as in lemma 8.1.

Lemma 8.2. *(1) $U^o \cong Hom_{Fk_1(X)}(U, Fk_1(X))$ as FX^o-modules.*
(2) $V^o \cong Hom_{Fk_1(X)}(V, Fk_1(X))$ as FX-modules.

Proof. (1) The map $\alpha : U^o \to Hom_{Fk_1(X)}(U, Fk_1(X))$ defined by $\alpha(\sigma)(u) = \sum_{g \in k_1(X)} \sigma(g^{-1}u)g$ for $\sigma \in U^o$ and $u \in U$ is an FX^o-module isomorphism with inverse $\beta : Hom_{Fk_1(X)}(U, Fk_1(X)) \to U^o$ defined by the condition that $\beta(\tau)(u)$ is the 1-coefficient of $\tau(u)$ for $\tau \in Hom_{Fk_1(X)}(U, Fk_1(X))$ and $u \in U$.
(2) This is analogous to (1). □

Lemma 8.3. *(1)* $Hom_{F(X*X^o)}(U \otimes_{Fk_2(X)} V, W) \cong Hom_{FX^o}(V, Hom_{Fk_1(X)}(U, W))$ *as F-spaces.*
(2) $Hom_{F(X*X^o)}(U \otimes_{Fk_2(X)} V, W) \cong Hom_{FX}(U, Hom_{Fk_1(X)}(V, W))$ *as F-spaces.*

Proof. (1) The map $\alpha : Hom_{F(X*X^o)}(U \otimes_{Fk_2(X)} V, W) \to Hom_{FX^o}(V, Hom_{Fk_1(X)}(U, W))$ defined by $(\alpha(\sigma)(v))(u) = \sigma(u \otimes v)$ for $\sigma \in Hom_{F(X*X^o)}(U \otimes_{Fk_2(X)} V, W), v \in V$, and $u \in U$ is an F- isomorphism with inverse $\beta : Hom_{FX^o}(V, Hom_{Fk_1(X)}(U, W)) \to$ $Hom_{F(X*X^o)}(U \otimes_{Fk_2(X)} V, W)$ defined by $\beta(\tau)(u \otimes v) = \tau(v)(u)$ for $\tau \in Hom_{FX^o}(V, Hom_{Fk_1(X)}(U, W)), u \in U$, and $v \in V$.
(2) This is analogous to (1). □

Lemma 8.4. *(1) If U is a projective $Fk_1(X)$-module, then* $Hom_{Fk_1(X)}(U, W) \cong Hom_{Fk_1(X)}(U, Fk_1(X)) \otimes_{Fk_1(X)} W$ *as FX^o-modules.*
(2) If V is a projective $Fk_1(X)$-module, then $Hom_{Fk_1(X)}(V, W) \cong W \otimes_{Fk_1(X)} Hom_{Fk_1(X)}(V, Fk_1(X))$ *as FX-modules.*

Proof. (1) By [13](the beginning of section 2.2.2), the map $\alpha : Hom_{Fk_1(X)}(U, Fk_1(X)) \otimes_{Fk_1(X)} W \to Hom_{Fk_1(X)}(U, W)$ defined by $\alpha(\tau \otimes w)(u) = \tau(u)w$ for $\tau \in Hom_{Fk_1(X)}(U, Fk_1(X)), w \in W$, and $u \in U$, is an $(Fk_2(X), Fk_1(X))$-bimodule isomorphism, so it only remains to show that α is an FX^o-module homomorphism, which is a straightforward verification.
(2) This is analogous to (1). □

Lemma 8.5. *(1) If U is a projective $Fk_1(X)$-module, then* $Hom_{F(X*X^o)}(U \otimes_{Fk_2(X)} V, W) \cong Hom_{FX^o}(V, U^o \otimes_{Fk_1(X)} W)$ *as F-spaces.*
(2) If V is a projective $Fk_1(X)$-module, then $Hom_{F(X*X^o)}(U \otimes_{Fk_2(X)} V, W) \cong Hom_{FX}(U, W \otimes_{Fk_1(X)} V^o)$ *as F-spaces.*

Proof. This follows from lemma 8.2, lemma 8.3, and lemma 8.4. □

Lemma 8.6. *Let Q and R be isomorphic subgroups of G and H respectively and let $\varphi : R \to Q$ be an isomorphism. Let $C_G(Q) \leq I \leq N_G(Q)$ and let $C_H(R) \leq J \leq N_H(R)$. Let M and N be $FN_{I\times J}(\Delta(Q, \varphi, R))$-modules such that N is projective as an $FC_G(Q)$-module, and let V and W be $FN_{I\times I}(\Delta(Q))$-modules. Then* $Hom_{FN_{I\times I}(\Delta(Q))}(V \otimes_{FC_G(Q)} Ind^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(M \otimes_{FC_H(R)} N^o), W) \cong$ $Hom_{FN_{I\times J}(\Delta(Q,\varphi,R))}(V \otimes_{FC_G(Q)} M, W \otimes_{FC_G(Q)} N)$ *as F-spaces.*

Proof. By lemma 2.9, $k_1(N_{I\times J}(\Delta(Q,\varphi,R))) = C_G(Q)$, and $p_1(N_{I\times J}(\Delta(Q,\varphi,R))) = N_{(\varphi^{-1},I,J)}$, so by lemma 2.10, $N_{I\times J}(\Delta(Q,\varphi,R)) * (N_{I\times J}(\Delta(Q,\varphi,R)))^o = \Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)$, so as $k_2(N_{I\times J}(\Delta(Q,\varphi,R))) = C_H(R)$, we may view $M \otimes_{FC_H(R)} N^o$ as an $F\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)$-module. Note that $N_{I\times I}(\Delta(Q)) = (N_{I\times I}(\Delta(Q)))^o$, so by lemma 2.8.2 and lemma 2.10, $N_{I\times I}(\Delta(Q)) * N_{I\times I}(\Delta(Q)) = N_{I\times I}(\Delta(Q))$. Thus as $k_2(N_{I\times I}(\Delta(Q))) = C_G(Q)$ by lemma 2.8.2, we may view $V \otimes_{FC_G(Q)} Ind^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(M \otimes_{FC_H(R)} N^o)$ as an $FN_{I\times I}(\Delta(Q))$-module by lemma 7.1. This shows that the F-vector space on the left hand side of the isomorphism in the statement of the lemma makes sense.

By lemma 2.9.2, $k_1(N_{I\times J}(\Delta(Q,\varphi,R))) = C_G(Q)$, so as $N_{I\times I}(\Delta(Q)) = \Delta(I)(C_G(Q)\times 1)$ by lemma 2.8.2, it follows that $N_{I\times I}(\Delta(Q)) * N_{I\times J}(\Delta(Q,\varphi,R)) = N_{I\times J}(\Delta(Q,\varphi,R))$ by lemma 2.11. Thus it follows from lemma 7.1, that $V \otimes_{FC_G(Q)} M$ and $W \otimes_{FC_G(Q)} N$ may be viewed as $FN_{I\times J}(\Delta(Q,\varphi,R))$-modules, so the F-vector space on the right hand side of the isomorphism in the statement of the lemma makes sense.

By lemma 2.8.2 and lemma 2.11, $N_{I\times I}(\Delta(Q)) * \Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1) = \Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)$, so as $N_{I\times I}(\Delta(Q)) * N_{I\times I}(\Delta(Q)) = N_{I\times I}(\Delta(Q))$ as noted in the first paragraph, it follows from lemma 7.3 that $V \otimes_{FC_G(Q)} Ind^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(M \otimes_{FC_H(R)} N^o) \cong Ind^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(V \otimes_{FC_G(Q)} M \otimes_{FC_H(R)} N^o)$. Thus by Frobenius reciprocity, $Hom_{FN_{I\times I}(\Delta(Q))}(V \otimes_{FC_G(Q)} Ind^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(M \otimes_{FC_H(R)} N^o), W) \cong Hom_{F\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(V \otimes_{FC_G(Q)} M \otimes_{FC_H(R)} N^o, Res^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(W))$. As noted in the second paragraph, $V \otimes_{FC_G(Q)} M$ may be viewed as an $FN_{I\times J}(\Delta(Q,\varphi,R))$-module, and as N is projective as an $FC_G(Q)$-module, N^o is projective as a right $FC_G(Q)$-module. Thus as $N_{I\times J}(\Delta(Q,\varphi,R)) * (N_{I\times J}(\Delta(Q,\varphi,R)))^o = \Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)$ as was noted in the first paragraph, it follows from lemma 8.5.2 that $Hom_{F\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(V \otimes_{FC_G(Q)} M \otimes_{FC_H(R)} N^o, Res^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(W)) \cong Hom_{FN_{I\times J}(\Delta(Q,\varphi,R))}(V \otimes_{FC_G(Q)} M, Res^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(W) \otimes_{FC_G(Q)} N)$ as F-spaces. But now as $p_1(N_{I\times J}(\Delta(Q,\varphi,R))) = N_{(\varphi^{-1},I,J)}$ by lemma 2.9.1, $\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1) * N_{I\times J}(\Delta(Q,\varphi,R)) = N_{I\times J}(\Delta(Q,\varphi,R))$ by lemma 2.11. Thus as $N_{I\times I}(\Delta(Q)) * N_{I\times J}(\Delta(Q,\varphi,R)) = N_{I\times J}(\Delta(Q,\varphi,R))$ as was noted in the second paragraph, and $k_2(N_{I\times I}(\Delta(Q))) = k_2(\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)) = k_1(N_{I\times J}(\Delta(Q,\varphi,R))) = C_G(Q)$, it follows that $Res^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(W) \otimes_{FC_G(Q)} N \cong W \otimes_{FC_G(Q)} N$ as $FN_{I\times J}(\Delta(Q,\varphi,R))$-modules. Thus the lemma holds. □

Chapter 9

Perfect virtual characters and isometries

Throughout this section, let p be a prime and let $(K, \mathcal{O}, F)$ be a p-modular system with F algebraically closed and K large enough.

Definition 9.1. *For a finite group G, a normal subgroup N of G, and an idempotent e in $(FN)^G$, we denote by $\tilde{e}$ be the unique lift of e in $(\mathcal{O}N)^G$.*

Definition 9.2. *For a finite group G, we denote by $R(KG)$ the **Grothendieck group** of KG-modules with respect to the relation $[[M]] = [[L]] + [[N]]$ whenever there is a short exact sequence of $0 \to L \to M \to N \to 0$ of KG-module homomorphisms, where $[[M]]$ denotes the image of the KG-module M in $R(KG)$. We denote the K-vector space $K \otimes_{\mathbb{Z}} R(KG)$ by $KR(KG)$. We identify $KR(KG)$ with the group of K-valued K-linear functions on KG which are invariant under the conjugation action of G on KG, and we identify $R(KG)$ with the **character group** of G, which is the subgroup of $KR(KG)$ generated by the irreducible K-characters of KG. For a central idempotent e of KG, we denote by $R(KGe)$ the subgroup of $R(KG)$ generated by the images in $R(KG)$ of the irreducible KGe-modules, or equivalently, the subgroup generated by the irreducible K-characters of KGe. We denote by $KR(KGe)$ the K-subspace of $KR(KG)$ generated by $R(KGe)$. If in addition, H is a finite group and f is a central idempotent of KH, we denote $R(KGe \otimes KHf^o)$ by $R(KGe, KHf)$ and we denote $KR(KGe \otimes KHf^o)$ by $KR(KGe, KHf)$.*

For the remainder of the section, let G and H be finite groups, let $A = FGe_A$ be a block of FG, and let $B = FHe_B$ be a block of FH.

We recall the notion of a perfect virtual character, as defined in [5].

Definition 9.3. *We say that a virtual character $\mu \in R(KG, KH)$ is* ***perfect*** *if the following conditions hold:*
(1) For $g \in G$ and $h \in H, \mu(g,h) \in |C_G(g)|\mathcal{O} \cap |C_H(h)|\mathcal{O}$.
(2) For $g \in G$ and $h \in H$ such that $\mu(g,h) \neq 0, g$ is a p'-element if and only if h is a p'-element.

Definition 9.4. *We say that a virtual character $\mu \in R(KG, KH)$ is* ***quasi-perfect*** *if condition (2) of definition 9.3 holds.*

Definition 9.5. *Let x be a p-element of G and let e be a central idempotent of $FC_G(x)$. We define the* ***generalized decomposition map*** $d_G^{(x,e)} : KR(KG) \to KR(KC_G(x)\tilde{e})$ *by $d_G^{(x,e)}(\chi)(x') = \chi(xx'\tilde{e})$ if $x' \in C_G(x)$ is a p'-element and $d_G^{(x,e)}(\chi)(x') = 0$ if $x' \in C_G(x)$ is not a p'-element. If $e = 1$, we denote $d_G^{(x,e)}$ by d_G^x. If $x = 1, d_G^x$ is called the* ***decomposition map*** *and is denoted by d_G.*

Definition 9.6. *For $\mu \in R(KG, KH)$, we denote by $I_\mu : KR(KH) \to KR(KG)$ the K-linear map defined by $\chi \mapsto \mu \otimes_{KH} \chi$ for $\chi \in KR(KH)$.*

Lemma 9.7. *Let $\mu \in R(KG, KH)$ and let $\chi \in KR(KH)$. For $g \in G, I_\mu(\chi)(g) = (1/|H|) \sum_{h \in H} \mu(g,h)\chi(h)$.*

Proof. See [5]. □

Lemma 9.8. *Let $\mu \in R(KG, KH)$. The following are equivalent:*
(1) The virtual character μ is quasi-perfect
(2) $d_G \circ I_\mu = I_\mu \circ d_H$ as maps from $KR(KH)$ to $KR(KG)$.
(3) $d_H \circ I_{\mu^\circ} = I_{\mu^\circ} \circ d_G$ as maps from $KR(KG)$ to $KR(KH)$.

Proof. We only show that (1) is equivalent to (2), as the proof that (1) is equivalent to (3) is analogous. Suppose that (1) holds. Let $\chi \in KR(KH)$ and let g be an element in G. First suppose that g is not a p'-element. By lemma 9.7, $I_\mu(d_H(\chi))(g) = (1/|H|) \sum_{h \in H} \mu(g,h) d_H(\chi)(h)$. As g is not a p'-element and μ is quasi-perfect, if h is a p'-element, then $\mu(g,h) = 0$. On the otherhand, if h is not a p'-element, then $d_H(\chi)(h) = 0$. Therefore, $I_\mu(d_H(\chi))(g) = 0 = d_G(I_\mu(\chi))(g)$ as g is not a p'-element. Thus we may assume that g is a p'-element. In this case, as μ is quasi-perfect, $d_G(I_\mu(\chi))(g) = I_\mu(\chi)(g) =$

$(1/|H|) \sum_{h \in H_{p'}} \mu(g,h)\chi(h) = (1/|H|) \sum_{h \in H} \mu(g,h) d_H(\chi)(h) = I_\mu(d_H(\chi))(g)$ by lemma 9.7. Thus we have shown that $d_G \circ I_\mu = I_\mu \circ d_H$, so (1) implies (2).

Now suppose that (2) holds. Let $g \in G$ be a p'-element and let $h \in H$ such that h is not a p'-element. Assume that $\mu(g,h) \neq 0$. Let $\chi \in KR(KH)$ be the class function such that for $y \in H, \chi(y) = 1$ if y is H-conjugate to h, and $\chi(y) = 0$ if y is not H-conjugate to h. As g is a p'-element and by lemma 9.7, $d_G(I_\mu(\chi))(g) = I_\mu(\chi)(g) = (1/|C_H(h)|)\mu(g,h) \neq 0$ as $\mu(g,h) \neq 0$. However, by lemma 9.7, $I_\mu(d_H(\chi))(g) = (1/|H|) \sum_{y \in H_{p'}} \mu(g,y)\chi(y) = 0$ by definition of χ and the fact that h is not a p'-element. Therefore, $d_G(I_\mu(\chi)) \neq I_\mu(d_H(\chi))$, which is a contradiction to (2). Thus we have shown that if g is a p'-element in G and h is not a p'-element in H, then $\mu(g,h) = 0$, so it suffices to show that if $g \in G$ is not a p'-element and $h \in H$ is a p'-element, then $\mu(g,h) = 0$.

Assume that $\mu(g,h) \neq 0$. Let $\chi \in KR(KH)$ be the class function such that for $y \in H, \chi(y) = 1$ if y is H-conjugate to h, and $\chi(y) = 0$ if y is not H-conjugate to h. By lemma 9.7 and the definition of $\chi, I_\mu(d_H(\chi))(g) = (1/|C_H(h)|)\mu(g,h) \neq 0$, while $d_G(I_\mu(\chi))(g) = 0$ as g is not a p'-element. This shows that $d_G(I_\mu(\chi)) \neq I_\mu(d_H(\chi))$, which is a contradiction to (2), so $\mu(g,h) = 0$ and hence (1) holds. Thus we have shown that (2) implies (1), so the lemma holds. □

Definition 9.9. *We say that $\mu \in R(KG\tilde{e_A}, KH\tilde{e_B})$ is an* ***isometry*** *if $\mu \otimes_{KH} \mu^o = [[KG\tilde{e_A}]]$ in $R(KG\tilde{e_A}, KG\tilde{e_A})$, and $\mu^o \otimes_{KG} \mu = [[KH\tilde{e_B}]]$ in $R(KH\tilde{e_B}, KH\tilde{e_B})$.*

Lemma 9.10. *Let $\mu \in R(KG\tilde{e_A}, KH\tilde{e_B})$ be a quasi-perfect virtual character such that there is a nonempty subset Ω of $Irr(A)$ such that $\mu \otimes_{KH} \mu^o = \sum_{\chi \in \Omega} \chi \otimes \chi^o$. The virtual character μ is an isometry.*

Proof. The hypothesis of the lemma implies that there is a subset Λ of $Irr(B)$ such that $\mu^o \otimes_{KG} \mu = \sum_{\zeta \in \Lambda} \zeta \otimes \zeta^o$. Thus as μ^o is quasi-perfect, by symmetry, it suffices to show that $\Omega = Irr(A)$. For $\chi, \chi' \in Irr(A)$, let $m_{\chi',\chi} \in \mathbb{Q}$ denote the Schur inner product of $d_G(\chi)$ with χ' and let M be the $Irr(A) \times Irr(A)$ matrix with (χ', χ)-entry $m_{\chi',\chi}$. As M is a symmetric matrix and $m_{\chi',\chi} \neq 0$ for all $\chi, \chi' \in Irr(A)$ such that χ has height zero by [9] (Lemma 6.34.ii), it follows that the rows and columns of M cannot be rearranged so as to make M a block diagonal matrix consisting of more than one block. Thus to complete the proof, it suffices to show that for $\chi \in \Omega$ and $\chi' \in Irr(A)$, if $m_{\chi',\chi} \neq 0$, then $\chi' \in \Omega$. In other words, it suffices to show that $d_G(\Omega) \subseteq < \Omega >_K$.

Let $\chi \in \Omega$. By the hypothesis of the lemma, there is a unique $\zeta \in \Lambda$ such that $I_\mu(\zeta) = \chi$. Thus as μ is quasi-perfect, by lemma 9.8, $d_G(\chi) = I_\mu(d_H(\zeta)) \in I_\mu(KR(KH\tilde{e_B}))$ $\subseteq < \Omega >_K$. This shows that $d_G(\Omega) \subseteq < \Omega >_K$, so the lemma holds. □

Chapter 10

Grothendieck groups and p-permutation equivalences

Throughout this section, let F be an algebraically closed field of characteristic $p > 0$.

Definition 10.1. *Let G be a finite group and let B be a direct sum of blocks of FG. We denote by $R(B)$ the **Grothendieck group** of B-modules with respect to the relation $[[M]] = [[L]] + [[N]]$ whenever there is a short exact sequence of $0 \to L \to M \to N \to 0$ of B-module homomorphisms, where $[[M]]$ denotes the image of the B-module M in $R(B)$. We identify $R(B)$ with the **Brauer character group** of B, which is the group of virtual Brauer characters of B.*

If G and H are finite groups, A is a direct sum of blocks of FG, and B is a direct sum of blocks of FH, we denote $R(A \otimes B^o)$ by $R(A, B)$.

Definition 10.2. *Let G be a finite group and let B be a direct sum of blocks of FG. The **trivial source group** of B, denoted by $T(B)$, is the Grothendieck group of trivial source FG-modules with respect to the relation $[M] = [L] + [N]$ whenever there is a split short exact sequence $0 \to L \to M \to N \to 0$ of B-module homomorphisms, where $[M]$ denotes the image of the B-module M in $T(B)$. For $\gamma \in T(B)$ and an indecomposable trivial source B-module M, we say that M **appears in** γ if the coefficient of $[M]$ in γ is nonzero with respect to the $\mathbb{Z}$-basis of $T(B)$ consisting of indecomposable trivial source B-modules. For $\gamma \in T(B)$, we denote by $[\gamma]$ the image of γ under the canonical homomorphism from $T(B)$ to $R(B)$.*

Lemma 10.3. *Let (P, e) be a Brauer pair for FG, let I be the inertial group of $FC_G(P)e$ in $N_G(P)$, and let $\omega \in T(FG)$ such that no indecomposable FG-module appearing in ω has Brauer pair properly containing (P, e). For an indecomposable trivial source FG-module M such that (P, e) is an M-Brauer pair, M has the same coefficient in ω as $eM(P)$ has in $e(\omega(P)) \in T(FIe)$.*

Proof. This follows from lemma 6.5. □

Definition 10.4. *For $\omega \in T(FG)$, an ω-**Brauer pair** is a Brauer pair (P, e) of FG such that $e\omega(P) \neq 0$ in $T(FIe)$, where I is the inertial group of $FC_G(P)e$ in $N_G(P)$.*

Definition 10.5. *Let X be a subgroup of $G \times H$, and let Λ be a direct sum of blocks of FX. We denote by $T^{\Delta}(\Lambda)$ the subgroup of $T(\Lambda)$ spanned by those indecomposable trivial source Λ modules that have a twisted diagonal vertex. If A is a direct sum of blocks of FG and B is a direct sum of blocks of FH, we denote $T^{\Delta}(A \otimes B^o)$ by $T^{\Delta}(A, B)$.*

Definition 10.6. *Let G and H be finite groups, let A be a direct sum of blocks of FG, and let B be a direct sum of blocks of FH. We denote by $T^{\Delta}_o(A, B)$ the set of all elements of γ of $T^{\Delta}(A, B)$ such that $\gamma \otimes_B \gamma^o = [A]$ in $T^{\Delta}(A, A)$ and $\gamma^o \otimes_A \gamma = [B]$ in $T^{\Delta}(B, B)$. An element of $T^{\Delta}_o(A, B)$ is called a p-**permutation equivalence**.*

Chapter 11

Brauer pairs for p-permutation equivalences

Throughout this section, let p be a prime and let $(K, \mathcal{O}, F)$ be a p-modular system with F algebraically closed and K large enough. Let G and H be finite groups, let $A = FGe_A$ be a direct sum of blocks of FG and let $B = FHe_B$ be a direct sum of blocks of FH. Suppose that A and B are p-permutation equivalent and let $\gamma \in T_o^{\Delta}(A, B)$ be a p-permutation equivalence.

Lemma 11.1. *Let (Q, e) be an A-Brauer pair and let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$.*
(1) For an irreducible $KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes\tilde{e}^o)$-module V, there is a unique $I\times H$-conjugacy class of triples (R, f, φ) such that (R, f) is a B-Brauer pair, $\varphi : R \to Q$ is an isomorphism, and $[[V]] \otimes_{KC_G(Q)} (e\gamma(\Delta(Q, \varphi, R))f)^K \neq 0$ in $R(KN_{I\times J}(\Delta(Q, \varphi, R))(\tilde{e}\otimes\tilde{f}^o))$, where J is the inertial group of $FC_H(R)f$ in $N_H(R)$. Moreover, $[V]] \otimes_{KC_G(Q)} (e\gamma(\Delta(Q, \varphi, R))f)^K$ is plus or minus an irreducible $KN_{I\times J}(\Delta(Q, \varphi, R))(\tilde{e}\otimes\tilde{f}^o)$-module.
(2) For an irreducible $(KC_G(Q)\tilde{e}, KC_G(Q)\tilde{e})$-bimodule V, there is a unique H-conjugacy class of triples (R, f, φ) such that (R, f) is a B-Brauer pair, $\varphi : R \to Q$ is an isomorphism, and $[[V]] \otimes_{KC_G(Q)} (e\gamma(\Delta(Q, \varphi, R))f)^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$. Moreover, $[[V]] \otimes_{KC_G(Q)} (e\gamma(\Delta(Q, \varphi, R))f)^K$ is plus or minus and irreducible $(KC_G(Q)\widetilde{e}, KC_H(R)\widetilde{f})$-bimodule.

Proof. (1) As $eA(\Delta(Q))e \cong FC_G(Q)e$ as $FN_{I\times I}(\Delta(Q))$-modules and $\gamma \in T_o^{\Delta}(A, B)$, $[FC_G(Q)e] =$

$\sum_{(R,f,\varphi)\in\Gamma/\sim_{I\times H}} Ind^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f \otimes_{FC_H(R)} f\gamma^o(\Delta(R,\varphi^{-1},Q))e)$ in $T^{\Delta}(FN_{I\times I}(\Delta(Q))(e\otimes e^o))$ by lemma 7.6.2. Therefore, it follows from lemma 4.20 and lemma 7.4.1 that $[[KC_G(Q)\tilde{e}]] =$
$\sum_{(R,f,\varphi)\in\Gamma/\sim_{I\times H}} Ind^{N_{I\times I}(\Delta(Q))}_{\Delta(N_{(\varphi^{-1},I,J)})(C_G(Q)\times 1)}((e\gamma(\Delta(Q,\varphi,R))f)^K \otimes_{KC_H(R)} (f\gamma^o(\Delta(R,\varphi^{-1},Q))e)^K)$.
As V is irreducible, $dim_K(Hom_{KN_{I\times I}(\Delta(Q))}(V\otimes_{KC_G(Q)} KC_G(Q)\tilde{e}, V)) =$ $dim_K(End_{KN_{I\times I}(\Delta(Q))}(V)) = 1$, so it follows from lemma 8.6 that $1 =$ $\sum_{(R,f,\varphi)\in\Gamma/\sim_{I\times H}} dim_K(End_{KN_{I\times J}(\Delta(Q,\varphi,R))}([[V]]\otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K))$, where J is the inertial group of $FC_H(R)f$ in $N_H(R)$, and hence (1) holds.
(2) As $eA(\Delta(Q))e \cong FC_G(Q)e$ as $(FC_G(Q), FC_G(Q))$-bimodules, and $\gamma \in T^{\Delta}_o(A,B)$, $[FC_G(Q)e] = \sum_{(R,f,\varphi)\in\Gamma/\sim_H} e\gamma(\Delta(Q,\varphi,R))f \otimes_{FC_H(R)} f\gamma^o(\Delta(R,\varphi^{-1},Q))e$ in $T^{\Delta}(FC_G(Q)e, FC_G(Q)e)$ by lemma 7.6.1. Therefore, $[[KC_G(Q)\tilde{e}]] =$ $\sum_{(R,f,\varphi)\in\Gamma/\sim_H} (e\gamma(\Delta(Q,\varphi,R))f)^K \otimes_{KC_H(R)} (f\gamma^o(\Delta(R,\varphi^{-1},Q)))e)^K$. As V is irreducible, $dim_K(Hom_{K(C_G(Q)\times C_G(Q))}(V\otimes_{KC_G(Q)} KC_G(Q)\tilde{e}, V)) = dim_K(End_{K(C_G(Q)\times C_G(Q))}(V)) =$ 1, so it follows from lemma 8.5.2 that
$1 = \sum_{(R,f,\varphi)\in\Gamma/\sim_H} dim_K(End_{K(C_G(Q)\times C_H(R))}([[V]]\otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K))$, and hence (2) holds. □

Lemma 11.2. *Let (Q,e) be an A-Brauer pair and let (R,f) be a B-Brauer pair such that R is isomorphic to Q and let $\varphi : R \to Q$ be an isomorphism. Let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and let J be the inertial group of $FC_H(R)f$ in $N_H(R)$. Then $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$ in $R(KN_{I\times J}(\Delta(Q,\varphi,R))(\tilde{e}\otimes\tilde{f}^o))$ if and only if $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$.*

Proof. The "if" part of the statement clearly holds, so it suffices to prove the "only if" part. Suppose $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$ in $R(KN_{I\times J}(\Delta(Q,\varphi,R))(\tilde{e}\otimes\tilde{f}^o))$. As $[[KC_G(Q)e]]\otimes_{KC_G(Q)}$ $(e\gamma(\Delta(Q,\varphi,R))f)^K = (e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$, there is an irreducible $KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes\tilde{e^o})$-module V such that $[[V]]\otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$, and hence by lemma 11.1.1, $[[V]]\otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K$ is plus or minus an irreducible $KN_{I\times J}(\Delta(Q,\varphi,R))(\tilde{e}\otimes\tilde{f^o}))$-module. Therefore,
$[[Res^{N_{I\times I}(\Delta(Q))}_{C_G(Q)\times C_G(Q)}(V)]]\otimes_{KC_G(Q)} Res^{N_{I\times J}(\Delta(Q,\varphi,R))}_{C_G(Q)\times C_H(R)}((e\gamma(\Delta(Q,\varphi,R))f)^K) =$
$Res^{N_{I\times J}(\Delta(Q,\varphi,R))}_{C_G(Q)\times C_H(R)}([[V]]\otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K) \neq 0$, and hence $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq$

0 in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$, so the lemma holds. □

Lemma 11.3. *(1) If (Q, e) is an A-Brauer pair, (R, f) is a B-Brauer pair, and $\varphi : R \to Q$ is an isomorphism such that $(e\gamma(\Delta(Q, \varphi, R))f)^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$, then $(e\gamma(\Delta(Q, \varphi, R))f)^K$ is an isometry and*
$[e\gamma(\Delta(Q, \varphi, R))f] \neq 0$ in $R(FC_G(Q)e, FC_H(R)f)$.
(2) For an A-Brauer pair (Q, e), there is a unique H-conjugacy class of triples (R, f, φ) such that (R, f) is a B-Brauer pair, $\varphi : R \to Q$ is an isomorphism, and $(e\gamma(\Delta(Q, \varphi, R))f)^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$.
(3) For a B-Brauer pair (R, f), there is a unique G-conjugacy class of triples (Q, e, φ) such that (Q, e) is an A-Brauer pair, $\varphi : R \to Q$ is an isomorphism, and $(e\gamma(\Delta(Q, \varphi, R))f)^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$.

Proof. As (2) and (3) follow from (1) and lemma 11.1.2, it suffices to prove (1). By lemma 11.1.2, $[[V]] \otimes_{KC_G(Q)} (e\gamma(\Delta(Q, \varphi, R))f)^K$ is either zero or plus or minus and irreducible $(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$-bimodule for any irreducible $(KC_G(Q)\tilde{e}, KC_G(Q)\tilde{e})$-bimodule V, and $[[W]] \otimes_{KC_H(R)} ((e\gamma(\Delta(Q, \varphi, R))f)^o)^K$ is either zero or plus or minus an irreducible $(KC_H(R)\tilde{f}, KC_G(Q)\tilde{e})$-bimodule for any irreducible $(KC_H(R)\tilde{f}, KC_H(R)\tilde{f})$-bimodule W. Thus as $(e\gamma(\Delta(Q, \varphi, R))f)^K$ is a perfect virtual character by [5](Theorem 1.5(2)), it follows that $(e\gamma(\Delta(Q, \varphi, R))f)^K$ is an isometry by lemma 9.10. Let $\mu = (e\gamma(\Delta(Q, \varphi, R))f)^K$. As μ is a perfect, $I_{d_{C_G(Q)\times C_H(R)}(\mu)} \circ d_{C_H(R)} = I_\mu \circ d_{C_H(R)} = d_{C_G(Q)} \circ I_\mu$ as maps from $KR(KC_H(R)\tilde{f})$ to $KR(KC_G(Q)\tilde{e})$ by lemma 9.8.2. As μ is an isometry, $d_{C_G(Q)} \circ I_\mu \neq 0$, so $[e\gamma(\Delta(Q, \varphi, R))f] = d_{C_G(Q)\times C_H(R)}(\mu) \neq 0$, and hence (1) holds. □

Lemma 11.4. *Let (D, e) be an A-Brauer, and let (E, f) be a B-Brauer such that there is an isomorphism $\varphi : E \to D$. Let $R \leq E$, let $Q = \varphi(R)$, let $\sigma \in bli(FC_G(Q))$ such that $(Q, \sigma) \leq (D, e)$ and let $\tau \in bli(FC_H(R))$ such that $(R, \tau) \leq (E, f)$. If $(e\gamma(\Delta(D, \varphi, E))f)^K \neq 0$ in $R(KC_G(D)\tilde{e}, KC_H(E)\tilde{f})$, then $(\sigma\gamma(\Delta(Q, \varphi, R))\tau)^K \neq 0$ in $R(KC_G(Q)\tilde{\sigma}, KC_H(R)\tilde{\tau})$.*

Proof. Arguing by induction on $[D : Q]$, we may assume that $Q \trianglelefteq D$ and D/Q is a cyclic group. Thus there exists $x \in D$ such that $D = Q < x >$. Let $y = \varphi^{-1}(x)$, and note that $E = R < y >$. Assume that $(\sigma\gamma(\Delta(Q, \varphi, R))\tau)^K = 0$ in $R(KC_G(Q)\tilde{\sigma}, KC_H(R)\tilde{\tau})$. By lemma 11.2, $(\sigma\gamma(\Delta(Q, \varphi, R))\tau)^K = 0$ in $R(KN_{I\times J}(\Delta(Q, \varphi, R)))(\tilde{\sigma} \otimes \tilde{\tau^o}))$, so by lemma 4.17 and lemma 4.18,

$[Br_D(\sigma)\gamma(\Delta(D,\varphi,E))Br_E(\tau)] = [(\sigma\gamma(\Delta(Q,\varphi,R))\tau)(< (x,y) >)] =$ $d^{(x,y)}_{N_{I\times J}(\Delta(Q,\varphi,R))}((\sigma\gamma(\Delta(Q,\varphi,R))\tau)^K) = 0$. Thus as $eBr_D(\sigma) = e$ and $fBr_E(\tau) = f$, $[e\gamma(\Delta(D,\varphi,E))f] = 0$, which is a contradiction to lemma 11.3 as $(e\gamma(\Delta(D,\varphi,E))f)^K \neq 0$, so the lemma holds. □

Lemma 11.5. *Let (Q,e) be an A-Brauer pair and let (R,f) be a B-Brauer pair such that there is an isomorphism $\varphi : R \to Q$. If there is an indecomposable $A \otimes B^o$-module M that appears in γ such that $(\Delta(Q,\varphi,R), e\otimes f^o)$ is an M-Brauer pair, then $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$.*

Proof. Let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and let J be the inertial group of $FC_H(R)f$ in $N_H(R)$. By lemma 11.4, we may assume that $(\Delta(Q,\varphi,R), e\otimes f^o)$ is a maximal $A\otimes B^o$-Brauer pair subject to the condition that $(\Delta(Q,\varphi,R), e\otimes f^o)$ is an M-Brauer pair for some indecomposable $A\otimes B^o$-module appearing in γ. It follows that for any indecomposable $A\otimes B^o$-module M that appears in γ such that $(\Delta(Q,\varphi,R), e\otimes f^o)$ is an M-Brauer pair, $(\Delta(Q,\varphi,R), e\otimes f^o)$ is a maximal M-Brauer pair, and hence $eM(\Delta(Q,\varphi,R))f$ is a projective indecomposable $FN_{I\times J}(\Delta(Q,\varphi,R))/\Delta(Q,\varphi,R)$-module by lemma 4.16 and lemma 6.4. Therefore, $e\gamma(\Delta(Q,\varphi,R))f$ is a virtual projective $N_{I\times J}(\Delta(Q,\varphi,R))/\Delta(Q,\varphi,R)$-module. By our hypothesis, there is an indecomposable $A\otimes B^o$-module M that appears in γ such that $(\Delta(Q,\varphi,R), e\otimes f^o)$ is an M-Brauer pair, so by the maximality of $(\Delta(Q,\varphi,R), e\otimes f^o)$ and lemma 10.3, the indecomposable $FN_{I\times J}(\Delta(Q,\varphi,R))$-module $eM(\Delta(Q,\varphi,R))f$ has nonzero coefficient in $e\gamma(\Delta(Q,\varphi,R))f$, so $e\gamma(\Delta(Q,\varphi,R))f \neq 0$ in $T^\Delta(FN_{I\times J}(\Delta(Q,\varphi,R)))$. Thus as $e\gamma(\Delta(Q,\varphi,R))f$ is a virtual projective $FN_{I\times J}(\Delta(Q,\varphi,R))/\Delta(Q,\varphi,R)$-module, $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$ in $R(KN_{I\times J}(\Delta(Q,\varphi,R)))$ by the injectivity of the adjoint to the decomposition map. Therefore, $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$ by lemma 11.2, so the lemma holds. □

Lemma 11.6. *Let (Q,e) be an A-Brauer pair and let (R,f) be a B-Brauer pair such that there is an isomorphism $\varphi : R \to Q$. Let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and let J be the inertial group of $FC_H(R)f$ in $N_H(R)$. The following are equivalent:*

(1) The $A\otimes B^o$-Brauer pair $(\Delta(Q,\varphi,R), e\otimes f^o)$ is a γ-Brauer pair.

(2) $e\gamma(\Delta(Q,\varphi,R))f \neq 0$ in $T^\Delta(FC_G(Q)e, FC_H(R)f)$.

(3) $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$ in $R(KN_{I\times J}(\Delta(Q,\varphi,R))(\tilde{e}\otimes\tilde{f}^o))$.

(4) $(e\gamma(\Delta(Q,\varphi,R))f)^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$.

(5) $[e\gamma(\Delta(Q,\varphi,R))f] \neq 0$ *in* $R(FN_{I\times J}(\Delta(Q,\varphi,R))(e\otimes f^o))$.
(6) $[e\gamma(\Delta(Q,\varphi,R))f] \neq 0$ *in* $R(FC_G(Q)e, FC_H(R)f)$.

Proof. Clearly each of conditions (2)-(6) implies (1) and (6) implies each of conditions (2)-(5), so it suffices to show that (1) implies (6). Now (1) implies (4) by lemma 11.5, and (4) implies (6) by lemma 11.3, so the lemma holds. □

Lemma 11.7. *(1) For every A-Brauer pair (Q,e), there is a unique H-conjugacy class of triples (R,f,φ) such that (R,f) is a B-Brauer pair and $\varphi : R \to Q$ is an isomorphism such that $(\Delta(Q,\varphi,R), e\otimes f^o)$ is a γ-Brauer pair.*
(2) For every B-Brauer pair (R,f), there is a unique G-conjugacy class of triples (Q,e,φ) such that (Q,e) is an A-Brauer pair and $\varphi : R \to Q$ is an isomorphism such that $(\Delta(Q,\varphi,R), e\otimes f^o)$ is a γ-Brauer pair.

Proof. This follows from lemma 11.3 and lemma 11.6 □

Lemma 11.8. *For each block direct summand $A' = FGe_{A'}$ of A, there is a unique block direct summand $B' = FHe_{B'}$ of B such that $e_{A'}\gamma e_{B'} \neq 0$, and for each block direct summand $B' = FHe_{B'}$ of B, there is a unique block direct summand $A' = FGe_{A'}$ of A such that $e_{A'}\gamma e_{B'} \neq 0$. Furthermore, if $A' = FGe_{A'}$ is a block direct summand of A and $B' = FHe_{B'}$ is a block direct summand of B such that $e_{A'}\gamma e_{B'} \neq 0$, then $e_{A'}\gamma e_{B'}$ is a p-permutation equivalence.*

Proof. The first statement follows from lemma 11.7 with Q the trivial subgroup of G and R the trivial subgroup of H. The last statement follows from the first and the fact that γ is a p-permutation equivalence. □

Now suppose that A is a block of FG and B is a block of FH.

Theorem 11.9. *The set of γ-Brauer pairs is closed under inclusion and $G\times H$-conjugation. The maximal γ-Brauer pairs are precisely the γ-Brauer pairs of the form $(\Delta(D,\varphi,E), e\otimes f^o)$, where (D,e) is a maximal A-Brauer pair, (E,f) is a maximal B-Brauer pair, and $\varphi : E \to D$ is an isomorphism. Furthermore, the maximal γ-Brauer pairs are all $G\times H$-conjugate.*

Proof. The set of γ-Brauer pairs is easily seen to be closed under $G\times H$-conjugation, and it is closed under inclusion by lemma 11.4 and lemma 11.6. The second statement follows from lemma 11.3, lemma 11.4 and lemma 11.6. The final statement holds by lemma 11.3 and the fact that G acts transitively by conjugation on the maximal A-Brauer pairs. □

Chapter 12

Invariants preserved by p-permutation equivalences

Throughout this section, let p be a prime and let $(K, \mathcal{O}, F)$ be a p-modular system with F algebraically closed and K large enough. Let G and H be finite groups, let $A = FGe_A$ and $B = FHe_B$ be p-permutation equivalent blocks of FG and FH respectively, and let $\gamma \in T_o^{\Delta}(A, B)$ be a p-permutation equivalence.

Theorem 12.1. *Let $(\Delta(D, \varphi, E), e \otimes f^o)$ be a maximal γ-Brauer pair, where (D, e) is a maximal A-Brauer pair, (E, f) is a maximal B-Brauer pair, and $\varphi : E \to D$ is an isomorphism. Let $\mathcal{A}$ be the fusion system associated with (D, e) and let $\mathcal{B}$ be the fusion system associated with (E, f). The isomorphism $\varphi : E \to D$ is an isomorphism between $\mathcal{B}$ and $\mathcal{A}$.*

Proof. We need to show that if R and R' are subgroups of $E, Q = \varphi(R)$, and $Q' = \varphi(R')$, then $\varphi^{-1} \circ Hom_{\mathcal{A}}(Q', Q) \circ \varphi = Hom_{\mathcal{B}}(R', R)$. By symmetry, it suffices to show that $\varphi^{-1} \circ Hom_{\mathcal{A}}(Q', Q) \circ \varphi \subseteq Hom_{\mathcal{B}}(R', R)$. Thus by Alperin's fusion theorem we may assume that $Q = Q'$. Let σ be the unique block idempotent of $FC_G(Q)$ such that $(Q, \sigma) \leq (D, e)$ and let τ be the unique block idempotent of $FC_H(R)$ such that $(R, \tau) \leq (E, f)$. Let I be the inertial group of $FC_G(Q)\sigma$ in $N_G(Q)$ and let J be the inertial group of $FC_H(R)\tau$ in $N_H(R)$. Let $g \in I$. As $(\Delta(Q, \varphi, R), \sigma \otimes \tau^o)$ is a γ-Brauer pair by lemma 11.9, $(\Delta(Q, c_g\varphi, R), \sigma \otimes \tau^o) = {}^{(g,1)}(\Delta(Q, \varphi, R), \sigma \otimes \tau^o)$ is a γ-Brauer pair, so by lemma 11.7, there exists $h \in H$ such that $(R, \tau, c_g\varphi) = {}^{h}(R, \tau, \varphi) = ({}^{h}R, {}^{h}\tau, \varphi c_h^{-1})$, and hence $h \in J$ and $\varphi^{-1} \circ c_g \circ \varphi = c_h^{-1}$. This shows that $\varphi^{-1} \circ Aut_I(Q) \circ \varphi \subseteq Aut_J(R)$, so as $Aut_{\mathcal{A}}(Q) = Aut_I(Q)$ and $Aut_{\mathcal{B}}(R) = Aut_J(R)$, the lemma holds. ∎

Theorem 12.2. *Let $(\Delta(Q, \varphi, R), e \otimes f^o)$ be a γ-Brauer pair, where (Q, e) is an A-Brauer, (R, f) is a B-Brauer pair and $\varphi : R \to Q$ is an isomorphism. Let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$, and let J be the inertial group of $FC_H(R)f$ in $N_H(R)$.*
(1) $\varphi \circ J/C_H(R) \circ \varphi^{-1} = I/C_G(Q)$.
(2) Let $C_G(Q) \leq S \leq I$ and let $C_H(R) \leq T \leq J$ such that $\varphi \circ T/C_H(R) \circ \varphi^{-1} = S/C_G(Q)$. The element $Ind^{S\times T}_{N_{S\times T}(\Delta(Q,\varphi,R))}(e\gamma(\Delta(Q, \varphi, R))f) \in T^{\Delta}(FSe, FTf)$ is a p-permutation equivalence.

Proof. Note that (1) holds by lemma 11.9 and lemma 12.1, so it suffices to prove (2). By lemma 11.7, if (R', f', φ') is a triple such that (R', f') is a B-Brauer pair and $\varphi' : R' \to Q$ is an isomorphism and (R', f', φ') is not H-conjugate to (R, f, φ), then $e\gamma(\Delta(Q, \varphi', R'))f' = 0$ in $T^{\Delta}(N_{I\times J'}(\Delta(Q, \varphi', R'))(e\otimes(f')^o))$, where J' is the inertial group of $FC_H(R)f'$ in $N_H(R')$. Thus as $eA(\Delta(Q))e \cong FC_G(Q)e$ as $FN_{S\times S}(\Delta(Q))$-modules, it follows from lemma 7.6.2 that $[FC_G(Q)e] = e\gamma(\Delta(Q, \varphi, R))f \otimes_{FC_H(R)} f\gamma^o(\Delta(R, \varphi^{-1}, Q))e$ in $T^{\Delta}(FN_{S\times S}(\Delta(Q))(e \otimes e^o))$. Thus it follows from lemma 7.2 and lemma 7.3 that $[Ind^{S\times S}_{N_{S\times S}(\Delta(Q))}(FC_G(Q)e)] = Ind^{S\times T}_{N_{S\times T}(\Delta(Q,\varphi,R))}(e\gamma(\Delta(Q, \varphi, R))f) \otimes_{FT} Ind^{T\times S}_{N_{T\times S}(\Delta(R,\varphi^{-1},Q))}(f\gamma^o(\Delta(R, \varphi^{-1}, Q))e)$. Now as $C_G(Q)$ is a transitive $N_{S\times S}(\Delta(Q))$-set and $\Delta(S)$ is the point stabilizer of the identity element, $FC_G(Q) \cong Ind^{N_{S\times S}(\Delta(Q))}_{\Delta(S)}(F_{\Delta(S)})$, so by lemma 4.1, $Ind^{S\times S}_{N_{S\times S}(\Delta(Q))}(FC_G(Q)e) \cong eInd^{S\times S}_{\Delta(S)}(F_{\Delta(S)})e \cong FSe$ as (FSe, FSe)-bimodules. Therefore, $[FSe] = Ind^{S\times T}_{N_{S\times T}(\Delta(Q,\varphi,R))}(e\gamma(\Delta(Q, \varphi, R))f) \otimes_{FT} (Ind^{S\times T}_{N_{S\times T}(\Delta(Q,\varphi,R))}(e\gamma(\Delta(Q, \varphi, R))f))^o$, and by symmetry, $[FTf] = (Ind^{S\times T}_{N_{S\times T}(\Delta(Q,\varphi,R))}(e\gamma(\Delta(Q, \varphi, R))f))^o \otimes_{FS} Ind^{S\times T}_{N_{S\times T}(\Delta(Q,\varphi,R))}(e\gamma(\Delta(Q, \varphi, R))f)$, so (2) holds. ∎

Lemma 12.3. *Let $(\Delta(D, \varphi, E), e\otimes f^o)$ be a maximal γ-Brauer pair, where (D, e) is a maximal A-Brauer, (E, f) is a maximal B-Brauer pair and $\varphi : E \to D$ is an isomorphism. Let $\mathcal{A}$ and $\mathcal{B}$ denote the fusion systems associated with (D, e) and (E, f) respectively. For $Q \leq D$ and $R \leq E$, let e_Q denote the unique block idempotent of $FC_G(Q)$ such that $(Q, e_Q) \leq (D, e)$ and let f_R denote the unique block idempotent of $FC_H(R)$ such that $(R, f_R) \leq (E, f)$. Let $R \leq E$ and let $Q = \varphi(R)$. Let I be the inertial group of $FC_G(Q)e_Q$ in $N_G(Q)$, and let J be the inertial group of $FC_H(R)f_R$ in $N_H(R)$.*
(1) The $FC_G(Q)e_Q \otimes FC_H(R)f^o_R$-Brauer pair $(\Delta(C_D(Q), \varphi, C_E(R)), e_{QC_D(Q)} \otimes f^o_{RC_E(R)})$ is

a maximal $e_Q\gamma(\Delta(Q,\varphi,R))f_R$*-Brauer pair if and only if* Q *is fully* $\mathcal{A}$*-centralized. In particular,* $(\Delta(Z(Q),\varphi,Z(R)),e_Q\otimes f_R^o)$ *is a maximal* $e_Q\gamma(\Delta(Q,\varphi,R))f_R$*-Brauer pair if and only if* Q *is* $\mathcal{A}$*-centric.*

(2) The $FIe_Q\otimes FJf_R^o$*-Brauer pair* $(\Delta(N_D(Q),\varphi,N_E(R)),e_{N_D(Q)}\otimes f_{N_E(R)}^o)$ *is a maximal* $Ind_{N_{I\times J}(\Delta(Q,\varphi,R))}^{I\times J}(e_Q\gamma(\Delta(Q,\varphi,R))f_R)$*-Brauer pair if and only if* Q *is fully* $\mathcal{A}$*-normalized.*

Proof. (1) If $(\Delta(C_D(Q),\varphi,C_E(R)),e_{QC_D(Q)}\otimes f_{RC_E(R)}^o)$ is a maximal $e_Q\gamma(\Delta(Q,\varphi,R))f_R$-Brauer pair, then as $e_Q\gamma(\Delta(Q,\varphi,R))f_R$ is a p-permutation equivalence by lemma 12.2, $C_D(Q)$ is a defect group of $FC_G(Q)e_Q$ by lemma 11.9 and hence Q is fully $\mathcal{A}$-centralized by [6] (Theorem 3.11). Conversely, suppose that Q is fully $\mathcal{A}$-centralized. As $e_{QC_D(Q)}(e_Q\gamma(\Delta(Q,\varphi,R))f_R)(\Delta(C_D(Q),\varphi,C_E(R)))f_{RC_E(R)} = e_{QC_D(Q)}\gamma(\Delta(QC_D(Q),\varphi,RC_E(R)))f_{RC_E(R)}$ in $T^\Delta(FC_G(QC_D(Q))e_{QC_D(Q)},FC_H(RC_E(R))f_{RC_E(R)})$ and $(\Delta(QC_D(Q),\varphi,RC_E(R)),e_{QC_D(Q)}\otimes f_{RC_E(R)}^o)$ is a γ-Brauer pair by lemma 11.9, it follows that $(\Delta(C_D(Q),\varphi,C_E(R)),e_{QC_D(Q)}\otimes f_{RC_E(R)}^o)$ is an $e_Q\gamma(\Delta(Q,\varphi,R))f_R$-Brauer pair. As Q is fully $\mathcal{A}$-centralized, $C_D(Q)$ is a defect group of $FC_G(Q)e_Q$ by [6] (Theorem 3.11), and hence $(\Delta(C_D(Q),\varphi,C_E(R)),e_{QC_D(Q)}\otimes f_{RC_E(R)}^o)$ is a maximal $e_Q\gamma(\Delta(Q,\varphi,R))f_R$-Brauer pair by lemma 11.9, so (1) holds.

(2) If $(\Delta(N_D(Q),\varphi,N_E(R)),e_{N_D(Q)}\otimes f_{N_E(R)}^o)$ is a maximal $Ind_{N_{I\times J}(\Delta(Q,\varphi,R))}^{I\times J}(e_Q\gamma(\Delta(Q,\varphi,R))f_R)$-Brauer pair, then $N_D(Q)$ is a defect group of FIe_Q by lemma 11.9, and hence Q is fully $\mathcal{A}$-normalized by [6] (Theorem 3.11). Conversely, suppose that Q is fully $\mathcal{A}$-normalized. By lemma 11.9 and [6] (Theorem 3.11), it suffices to show that $(\Delta(N_D(Q),\varphi,N_E(R)),e_{N_D(Q)}\otimes f_{N_E(R)}^o)$ is an $Ind_{N_{I\times J}(\Delta(Q,\varphi,R))}^{I\times J}(e_Q\gamma(\Delta(Q,\varphi,R))f_R)$-Brauer pair. By lemma 4.19,
$e_{N_D(Q)}(Ind_{N_{I\times J}(\Delta(Q,\varphi,R))}^{I\times J}(e_Q\gamma(\Delta(Q,\varphi,R))f_R))(\Delta(N_D(Q),\varphi,N_E(R)))f_{N_E(R)}$
$= e_{N_D(Q)}\gamma(\Delta(N_D(Q),\varphi,N_E(R))f_{N_E(R)}$, so (2) follows as $(\Delta(N_D(Q),\varphi,N_E(R)),e_{N_D(Q)}\otimes f_{N_E(R)}^o)$ is a γ-Brauer pair by lemma 11.9. □

Lemma 12.4. *Let* $(\Delta(Q,\varphi,R),e\otimes f^o)$ *be a* γ*-Brauer pair, where* (Q,e) *is an* A*-Brauer pair,* (R,f) *is a* B*-Brauer pair, and* $\varphi: R\to Q$ *is an isomorphism.*

(1)The block $FC_G(Q)e$ *has defect group* $Z(Q)$ *if and only if the block* $FC_H(R)f$ *has defect group* $Z(R)$*.*

(2) Suppose that $FC_G(Q)e$ *has defect group* $Z(Q)$*.*

(a) The element $e\gamma(\Delta(Q,\varphi,R))f\in T^\Delta(FC_G(Q)e,FC_H(R)f)$ *is plus or minus the unique*

indecomposable trivial source $(FC_G(Q)e, FC_H(R)f)$*-bimodule with vertex* $\Delta(Z(Q), \varphi, Z(R))$.
(b) $Def^{C_G(Q)\times C_H(R)}_{C_G(Q)\times C_H(R)/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f)$ *is plus or minus the unique irreducible* $(FC_G(Q)e, FC_H(R)f)$*-bimodule.*
(c) The element $(Def^{N_{I\times J}(\Delta(Q,\varphi,R))}_{N_{I\times J}(\Delta(Q,\varphi,R))/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f))^K \in$
$R(KN_{I\times J}(\Delta(Q,\varphi,R)))$ *is plus or minus an irreducible* $KN_{I\times J}(\Delta(Q,\varphi,R))(\tilde{e}\otimes\tilde{f}^o)$*-module.*
(d) The element $[Def^{N_{I\times J}(\Delta(Q,\varphi,R))}_{N_{I\times J}(\Delta(Q,\varphi,R))/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f)] \in$
$R(FN_{I\times J}(\Delta(Q,\varphi,R)))$ *is plus or minus an irreducible* $FN_{I\times J}(\Delta(Q,\varphi,R))(e\otimes f^o)$*-module.*

Proof. (1) By lemma 11.9, there is a maximal A-Brauer pair (D,σ) and a maximal B-Brauer pair (E,τ) such that $\varphi : R \to Q$ extends to an isomorphism $\varphi : E \to D$ and $(\Delta(D,\varphi,E), \sigma\otimes\tau^o)$ is a maximal γ-Brauer pair. Now by lemma 12.1, Q is centric in the fusion system associated with (D,σ) if and only of R is centric in the fusion system associated with (E,τ), so (1) holds by [6] (Theorem 3.11).
(2) As $Z(Q)$ is the defect group of $FC_G(Q)e$, it follows from [9] (Theorem 5.8.10 and Theorem 5.8.11) that $F(C_G(Q)/Z(Q))\overline{e}$ is a block of $F(C_G(Q)/Z(Q))$ of defect zero, so it follows that $K(C_G(Q)/Z(Q))\tilde{\overline{e}}$ is an irreducible $(KC_G(Q)\tilde{e}, KC_G(Q)\tilde{e})$-bimodule. Thus by lemma 11.3 and 11.6, $Def^{C_G(Q)\times C_H(R)}_{C_G(Q)\times C_H(R)/(Z(Q)\times 1)}((e\gamma(\Delta(Q,\varphi,R))f)^K) =$
$[[K(C_G(Q)/Z(Q))\tilde{\overline{e}}]] \otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K$ is plus or minus an irreducible $(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$-bimodule. Let M be an $(FC_G(Q), FC_H(R))$-bimodule appearing in $e\gamma(\Delta(Q,\varphi,R))f$. As $\Delta(Q,\varphi,R)$ acts trivially on every module appearing in $e\gamma(\Delta(Q,\varphi,R))f$ viewed as an element of $T^\Delta(F(N_{I\times J}(\Delta(Q,\varphi,R)))(e\otimes f^o))$, $\Delta(Z(Q),\varphi,Z(R))$ acts trivially on M and hence is contained in any vertex U of M. On the otherhand, U is a twisted diagonal subgroup of $C_G(Q)\times C_H(R)$ with $p_1(U) \le Z(Q)$ as $Z(Q)$ is the defect group of $FC_G(Q)e$. Therefore, $U = \Delta(Z(Q),\varphi,Z(R))$ is the vertex of M. Thus we may view $e\gamma(\Delta(Q,\varphi,R))f$ as virtual projective $F(C_G(Q)\times C_H(R)/\Delta(Z(Q),\varphi,Z(R)))$-module, and $Def^{C_G(Q)\times C_H(R)}_{C_G(Q)\times C_H(R)/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f)$ as a virtual $(FC_G(Q)/Z(Q), FC_H(R)/Z(R))$-bimodule belonging to the block $F(C_G(Q)/Z(Q))\overline{e}\otimes F(C_H(R)/Z(R))\overline{f^o}$ of defect zero. Thus as $(Def^{C_G(Q)\times C_H(R)}_{C_G(Q)\times C_H(R)/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f))^K$ is plus or minus an irreducible $(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$-bimodule, $hd(e\gamma(\Delta(Q,\varphi,R))f) =$
$Def^{C_G(Q)\times C_H(R)}_{C_G(Q)\times C_H(R)/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f)$ is plus or minus the unique irreducible $(FC_G(Q)e, FC_H(R)f)$-bimodule, and hence $e\gamma(\Delta(Q,\varphi,R))f$ is plus or minus the unique projective indecomposable $F(C_G(Q)\times C_H(R)/\Delta(Z(Q),\varphi,Z(R)))$-module, so (a) and (b) hold.

As $(Def^{C_G(Q)\times C_H(R)}_{C_G(Q)\times C_H(R)/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f))^K \neq 0$ in $R(KC_G(Q)\tilde{e}, KC_H(R)\tilde{f})$, $[[K(C_G(Q)/Z(Q))\tilde{\tilde{e}}]] \otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K =$ $(Def^{N_{I\times J}(\Delta(Q,\varphi,R))}_{N_{I\times J}(\Delta(Q,\varphi,R))/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f))^K \neq 0$ in $R(KN_{I\times J}(\Delta(Q,\varphi,R)))$, so as $K(C_G(Q)/Z(Q))\tilde{\tilde{e}}$ is irredicible as an $(KC_G(Q)\tilde{e}, KC_G(Q)\tilde{e})$-bimodule and hence as a $KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes \tilde{e^o})$-module, (c) holds by lemma 11.1.1.

To prove (d), we may assume without loss of generality that $(Def^{N_{I\times J}(\Delta(Q,\varphi,R))}_{N_{I\times J}(\Delta(Q,\varphi,R))/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f))^K$ is an irreducible $KN_{I\times J}(\Delta(Q,\varphi,R))(\tilde{e}\otimes \tilde{f^o})$-module by (c). Therefore, every module appearing in $[Def^{N_{I\times J}(\Delta(Q,\varphi,R))}_{N_{I\times J}(\Delta(Q,\varphi,R))/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f)]$ has positive coefficient, so as $[Res^{N_{I\times J}(\Delta(Q,\varphi,R))/(Z(Q)\times 1)}_{C_G(Q)\times C_H(R)/(Z(Q)\times 1)} Def^{N_{I\times J}(\Delta(Q,\varphi,R))}_{N_{I\times J}(\Delta(Q,\varphi,R))/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f)]$ is the irreducible $(FC_G(Q)e, FC_H(R)f)$-bimodule by (b), it follows that $[Def^{N_{I\times J}(\Delta(Q,\varphi,R))}_{N_{I\times J}(\Delta(Q,\varphi,R))/(Z(Q)\times 1)}(e\gamma(\Delta(Q,\varphi,R))f)]$ is an irreducible $FN_{I\times J}(\Delta(Q,\varphi,R))$-module, so (d) holds. □

Theorem 12.5. *Let $(\Delta(Q,\varphi,R), e\otimes f^o)$ be a γ-Brauer pair, where (Q,e) is an A-Brauer pair, (R,f) is a B-Brauer pair, and $\varphi : R \to Q$ is an isomorphism. Let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and let J be the inertial group of $FC_H(R)f$ in $N_H(R)$. Suppose that the defect group of $FC_G(Q)e$ is $Z(Q)$ so that the defect group of $FC_H(R)f$ is $Z(R)$ by lemma 12.4.1. Let $[\overline{\alpha}] \in H^2(I/QC_G(Q), F^\times)$ be the Külshammer-Puig class associated with (Q,e), and let $[\overline{\beta}] \in H^2(J/RC_H(R), F^\times)$ be the Külshammer-Puig class associated with (R,f). Let $\psi : Aut(R) \to Aut(Q)$ be the isomorphism defined by $\sigma \mapsto \varphi\circ\sigma\circ\varphi^{-1}$ for $\sigma \in Aut(R)$. As $\varphi(R) = Q, \psi(RC_H(R)/C_H(R)) = QC_G(Q)/C_G(Q)$, and by lemma 12.1, $\psi(J/C_H(R)) = I/C_G(Q)$, so ψ induces an isomorphism $\overline{\psi} : J/RC_H(R) \to I/QC_G(Q)$. Furthermore, we have that $[\overline{\alpha}\circ(\overline{\psi}\times\overline{\psi})] = [\overline{\beta}]$ in $H^2(J/RC_H(R), F^\times)$.*

Proof. We only need to prove the last statement. Let M be the unique irreducible $FQC_G(Q)e$-module and let N be the unique irreducible $FRC_H(R)f^o$-module. By definition 3.5, there is an $F_\alpha I$-module V and an $F_{\beta^{-1}}J$-module W such that $Res^I_{QC_G(Q)}(V) \cong M$ and $Res^J_{RC_H(R)}(W) \cong N$. By lemma 12.4.2(b) and lemma 12.4.2(d), there is an irreducible $FN_{I\times J}(\Delta(Q,\varphi,R))$-module L such that $Res^{N_{I\times J}(\Delta(Q,\varphi,R))}_{C_G(Q)\times C_H(R)}(L)$ is the unique irreducible $(FC_G(Q)e, FC_H(R)f)$-bimodule. Therefore, $Res^{N_{I\times J}(\Delta(Q,\varphi,R))}_{\Delta(Q,\varphi,R)(C_G(Q)\times C_H(R))}(L)$ is the unique irreducible $F\Delta(Q,\varphi,R)(C_G(Q)\times C_H(R))(e\otimes f^o)$-module, and hence $Res^{N_{I\times J}(\Delta(Q,\varphi,R))}_{\Delta(Q,\varphi,R)(C_G(Q)\times C_H(R))}(L) \cong Res^{QC_G(Q)\times RC_H(R)}_{\Delta(Q,\varphi,R)(C_G(Q)\times C_H(R))}(M\otimes N)$. Thus as $V\otimes W$ is an $F_{\alpha\beta^{-1}}(I\times J)$-module and $Res^{I\times J}_{\Delta(Q,\varphi,R)(C_G(Q)\times C_H(R))}(V\otimes W) \cong Res^{QC_G(Q)\times RC_H(R)}_{\Delta(Q,\varphi,R)(C_G(Q)\times C_H(R))}(M\otimes$

$N), \overline{\alpha\beta^{-1}}$ is cohomologous to the trivial $N_{I\times J}(\Delta(Q,\varphi,R))/\Delta(Q,\varphi,R)(C_G(Q)\times C_H(R))$ 2-cocycle by [9](Theorem 3.5.7.(iii)). Thus by viewing $N_{I\times J}(\Delta(Q,\varphi,R))/\Delta(Q,\varphi,R)(C_G(Q)\times C_H(R)) = \Delta(I/QC_G(Q),\overline{\psi},J/RC_H(R))$ as a subgroup of $I/QC_G(Q)\times J/RC_H(R)$, we find that $\overline{\alpha}\overline{\beta^{-1}}$ is cohomologous to the trivial $\Delta(I/QC_G(Q),\overline{\psi},J/RC_H(R))$ 2-cocyle, and hence the lemma holds. □

Lemma 12.6. *Let $(\Delta(D,\varphi,E), e\otimes f^o)$ be a maximal γ-Brauer pair, where (D,e) is a maximal A-Brauer pair, (E,f) is a maximal B-Brauer pair, and $\varphi: E\to D$ is an isomorphism. Let I be the inertial group of $FC_G(D)e$ in $N_G(D)$ and let J be the inertial group of $FC_H(E)f$ in $N_H(E)$. The virtual module $e\gamma(\Delta(D,\varphi,E))f$ is plus or minus an indecomposable trivial source $FN_{I\times J}(\Delta(D,\varphi,E))(e\otimes f^o)$-module with vertex $\Delta(D,\varphi,E)$, and $Def^{N_{I\times J}(\Delta(D,\varphi,E))}_{N_{I\times J}(\Delta(D,\varphi,E))/(Z(D)\times 1)}(e\gamma(\Delta(D,\varphi,E))f)$ is plus or minus an irreducible $FN_{I\times J}(\Delta(D,\varphi,E))(e\otimes f^o)$-module.*

Proof. By Theorem 3.11.ii in [6], $(\Delta(D,\varphi,E))(Z(D)\times 1)$ is the defect group of $FN_{I\times J}(\Delta(D,\varphi,E))(e\otimes f^o)$, and hence $FN_{I\times J}(\Delta(D,\varphi,E))/\Delta(D,\varphi,E)(Z(D)\times 1)(\overline{e}\otimes\overline{f^o})$ is a semisimple F-algebra. Thus as $[Def^{N_{I\times J}(\Delta(D,\varphi,E))}_{N_{I\times J}(\Delta(D,\varphi,E))/(Z(D)\times 1)}(e\gamma(\Delta(D,\varphi,E))f)]$ is plus or minus an irreducible $FN_{I\times J}(\Delta(D,\varphi,E))/\Delta(D,\varphi,E)(Z(D)\times 1)(\overline{e}\otimes\overline{f^o})$-module by lemma 12.4.2(d), $Def^{N_{I\times J}(\Delta(D,\varphi,E))}_{N_{I\times J}(\Delta(D,\varphi,E))/(Z(D)\times 1)}(e\gamma(\Delta(D,\varphi,E))f)$ is plus or minus an irreducible $FN_{I\times J}(\Delta(D,\varphi,E))/\Delta(D,\varphi,E)(Z(D)\times 1)(\overline{e}\otimes\overline{f^o})$-module. Thus as $e\gamma(\Delta(D,\varphi,E))f$ may be viewed as a virtual projective $FN_{I\times J}(\Delta(D,\varphi,E))/\Delta(D,\varphi,E)$-module and $Def^{N_{I\times J}(\Delta(D,\varphi,E))}_{N_{I\times J}(\Delta(D,\varphi,E))/(Z(D)\times 1)}(e\gamma(\Delta(D,\varphi,E))f) = hd(e\gamma(\Delta(D,\varphi,E))f), e\gamma(\Delta(D,\varphi,E))f$ is plus or minus an indecomposable trivial source $FN_{I\times J}(\Delta(D,\varphi,E))(e\otimes f^o)$-module with vertex $\Delta(D,\varphi,E)$, so the lemma holds. □

Lemma 12.7. *There is a unique indecomposable (A,B)-bimodule M appearing in γ such that M has a vertex of the form $\Delta(D,\varphi,E)$, where D is a defect group of A, E is a defect group of B, and $\varphi: E\to D$ is an isomorphism. Moreover, $[M]$ has coefficient 1 or -1 in γ.*

Proof. By lemma 11.9, there is a maximal A-Brauer pair (D,e), a maximal B-Brauer pair (E,f), and an isomorphism $\varphi: E\to D$ such that $e\gamma(\Delta(D,\varphi,E))f\neq 0$ in $T^{\Delta}(FN_{I\times J}(\Delta(D,\varphi,E))(e\otimes f^o))$. Thus there is an indecomposable (A,B)-bimodule M that appears in γ such that $(\Delta(D,\varphi,E), e\otimes f^o)$ is an M-Brauer pair. Therefore, $\Delta(D,\varphi,E)$ is contained in a vertex U of M. On the otherhand, U is twisted diagonal and as M is an (A,B)-bimodule, $p_1(U)$ is contained in a defect group of A, so it follows that $\Delta(D,\varphi,E)=U$, and hence $\Delta(D,\varphi,E)$ is a vertex of M.

Let N be an indecomposable (A,B)-bimodule that appears in γ with vertex $\Delta(D',\varphi',E')$, where D' is a defect group of A, E' is a defect group of B, and $\varphi' : E' \to D'$ is an isomorphism. Let e' and f' be block idempotents of $FC_G(D')$ and $FC_H(E')$ respectively such that $(\Delta(D',\varphi',E'), e'\otimes(f')^o)$ is a maximal N-Brauer pair. Then as G is transitive on the maximal A-Brauer pairs by lemma 3.4.2, we may assume that $(D,e)=(D',e')$. Thus by lemma 10.3, $eN(\Delta(D,\varphi',E'))f'$ appears in $e\gamma(\Delta(D,\varphi',E'))f'$, so $e\gamma(\Delta(D,\varphi',E'))f'\neq 0$. Therefore, by lemma 11.7, (E',f',φ') is H-conjugate to (E,f,φ), and hence we may assume that $(E',f',\varphi')=(E,f,\varphi)$. Thus as $e\gamma(\Delta(D,\varphi,E))f$ is plus or minus an indecomposable $FN_{I\times J}(\Delta(D,\varphi,E))(e\otimes f^o)$-module by lemma 12.6 and $eM(\Delta(D,\varphi,E))f$ and $eN(\Delta(D,\varphi,E))f$ appear in $e\gamma(\Delta(D,\varphi,E))f$,
$eM(\Delta(D,\varphi,E))f\cong eN(\Delta(D,\varphi,E))f$, and hence $M\cong N$ by lemma 6.5.

This shows that M is the unique indecomposable (A,B)-bimodule that appears in γ with vertex of the form given in the statement of the lemma. Furthermore, by lemma 12.6, the coefficient of $eM(\Delta(D,\varphi,E))f$ in $e\gamma(\Delta(D,\varphi,E))f$ is plus or minus 1, so the coefficient of M in γ is plus or minus 1 by lemma 10.3. Thus the lemma holds. □

Definition 12.8. *The indecomposable (A,B)-bimodule M of lemma 12.7 is called the* ***maximal module*** *of γ.*

Theorem 12.9. *(1) For every indecomposable (A,B)-bimodule M that appears in γ, every M Brauer pair is a γ-Brauer pair.*
(2) Let M be the maximal module of γ.
(a) The set of M-Brauer pairs is equal to the set of γ-Brauer pairs.
(b) The (A,B)-bimodule M is the unique indecomposable (A,B)-bimodule that appears in γ such that every γ-Brauer pair is an M-Brauer pair.

Proof. Note that (1) holds by lemma 11.5 and lemma 11.6, so it suffices to prove (2). By lemma 6.4 and lemma 11.9, to prove (a) it suffices to show that the set of maximal M-Brauer pairs is equal to the set of maximal γ-Brauer pairs, which is clearly the case from (1), the definition of M, and lemma 11.9, so (a) holds. Now note that (b) holds by (a), lemma 11.9, and lemma 12.7. □

Lemma 12.10. *Let* $(\Delta(D,\varphi,E), e\otimes f^o)$ *be a maximal* γ*-Brauer pair, where* (D,e) *is a maximal* A*-Brauer pair,* (E,f) *is a maximal* B*-Brauer pair, and* $\varphi : E \to D$ *is an isomorphism. Let* I *be the inertial group of* $FC_G(D)e$ *in* $N_G(D)$ *and let* J *be the inertial group of* $FC_H(E)f$ *in* $N_H(E)$*. Let* $C_G(Q) \le S \le I$ *and let* $C_H(E) \le T \le J$ *such that* $\varphi^{-1}\circ Aut_S(D)\circ\varphi = Aut_T(E)$*, and let* M *be the maximal module of* γ*. The* (FSe, FTf)*-bimodule*
$Ind_{N_{S\times T}(\Delta(D,\varphi,E))}^{S\times T}(eM(\Delta(D,\varphi,E))f)$ *induces a Morita equivalence between* FSe *and* FTf*.*

Proof. This follows from lemma 12.2.2 and lemma 12.7. □

Theorem 12.11. *Let* D *be a defect group of* A*, let* E *be a defect group of* B*, and let* $\varphi : E \to D$ *be an isomorphism such that* $\gamma(\Delta(D,\varphi,E)) \ne 0$*. Let* $a \in Bl(FN_G(D))$ *and* $b \in Bl(FN_H(E))$ *be the Brauer correspondents of* A *and* B *respectively, and let* M *be the maximal module of* γ*. The* (a,b)*-bimodule* $Ind_{N_{G\times H}(\Delta(D,\varphi,E))}^{N_G(D)\times N_H(E)}(M(\Delta(D,\varphi,E)))$ *induces a Morita equivalence between* a *and* b*.*

Proof. As $\gamma(\Delta(D,\varphi,E)) \ne 0$, there exists block idempotents e of $FC_G(D)$ and f of $FC_H(E)$ such that $e\gamma(\Delta(D,\varphi,E))f \ne 0$. Let I be the inertial group of $FC_G(D)e$ in $N_G(D)$ and let J be the inertial group of $FC_H(E)$ in $N_H(E)$. By lemma 12.10, the (FIe, FJf)-bimodule $Ind_{N_{I\times J}(\Delta(D,\varphi,E))}^{I\times J}(eM(\Delta(D,\varphi,E))f)$ induces a Morita equivalence between FIe and FJf. Thus as the (a, FIe)-bimodule $FN_G(D)e$ induces a Morita equivalence between a and FIe, and the (FJf, b)-bimodule $fFN_H(E)$ induces a Morita equivalence between FJf and b, it follows that the (a,b)-bimodule $FN_G(D)e \otimes_{FI} Ind_{N_{I\times J}(\Delta(D,\varphi,E))}^{I\times J}(eM(\Delta(D,\varphi,E))f) \otimes_{FJ} fFN_H(E)$ induces a Morita equivalence between a and b. Thus as
$FN_G(D)e \otimes_{FI} Ind_{N_{I\times J}(\Delta(D,\varphi,E))}^{I\times J}(eM(\Delta(D,\varphi,E))f) \otimes_{FJ} fFN_H(E) \cong$
$Ind_{N_{I\times J}(\Delta(D,\varphi,E))}^{N_G(D)\times N_H(E)}(eM(\Delta(D,\varphi,E))f) \cong$
$Ind_{N_{G\times H}(\Delta(D,\varphi,E))}^{N_G(D)\times N_H(E)} Ind_{N_{I\times J}(\Delta(D,\varphi,E))}^{N_{G\times H}(\Delta(D,\varphi,E))}(eM(\Delta(D,\varphi,E))f)$
$\cong Ind_{N_{G\times H}(\Delta(D,\varphi,E))}^{N_G(D)\times N_H(E)}(M(\Delta(D,\varphi,E)))$, the lemma holds. □

Theorem 12.12. *Let* $(\Delta(D,\varphi,E), e\otimes f^o)$ *be a maximal* γ*-Brauer pair, where* (D,e) *is a maximal* A*-Brauer pair,* (E,f) *is a maximal* B*-Brauer pair, and* $\varphi : E \to D$ *is an isomorphism. Let* I *be the inertial group of* $FC_G(D)e$ *in* $N_G(D)$ *and let* J *be the inertial group of* $FC_H(E)f$ *in* $N_H(E)$*. Let* $a \in Bl(FN_G(D))$ *be the Brauer correspondent of* A *and let* $b \in Bl(FN_H(E))$ *be the Brauer correspondent of* B*.*
(1) Let $DC_G(D) \le S \le I$ *and* $EC_H(E) \le T \le J$ *such that* $\varphi^{-1}\circ Aut_S(D)\circ\varphi = Aut_T(E)$*,*

let i be a source idempotent of FSe associated with (D, e) and let j be a source idempotent of FTf associated with (E, f). The source algebras $iFSi$ and $jFTj$ are isomorphic as interior E-algebras, where we view $iFSi$ as an interior E-algebra via the isomorphism $\varphi : E \to D$.
(2) Let $C_G(D) \leq S \leq I$ and let $C_H(E) \leq T \leq J$ such that $\varphi^{-1} \circ Aut_S(D) \circ \varphi = Aut_T(E)$, let $i \in pi(FC_G(D)e)$, and let $j \in pi(FC_H(E)f)$. The algebras $iFSi$ and $jFTj$ are isomorphic as interior $E \cap T$-algebras, where we view $iFSi$ as an interior $E \cap T$-algebra via the isomorphism $\varphi : E \cap T \to S \cap D$.
(3) Let i be a source idempotent of a associated with (D, e) and j is a source idempotent of b associated with (E, f). The source algebras iai of a and jbj of b are isomorphic as interior E-algebras.

Proof. By lemma 5.1 and lemma 5.2.3, it suffices to prove (2). Let M be the maximal module of γ, and let $V = Ind^{I\times J}_{N_{I\times J}(\Delta(D,\varphi,E))}(eM(\Delta(D, \varphi, E))f)$. As V induces a Morita equivalence between FIe and FJf by lemma 12.10 and $V(\Delta(D, \varphi, E)) \cong eM(\Delta(D, \varphi, E))f$ by lemma 4.19, V is indecomposable and $\Delta(D, \varphi, E)$ is a vertex of V. Thus as V is a trivial source module, $V|FIe \otimes FJf^o \otimes_{F\Delta(D,\varphi,E)} F_{\Delta(D,\varphi,E)}$. Now by viewing FIe as an (FIe, FE)-bimodule via the isomorphism $\varphi : E \to D$, we may view $FIe \otimes_{FE} FJf$ as (FIe, FJf)-bimodule. Furthermore, $FIe \otimes_{FE} FJf \cong FIe \otimes FJf^o \otimes_{F\Delta(D,\varphi,E)} F_{\Delta(D,\varphi,E)}$ as (FIe, FJf)-bimdules, and hence $V|FIe \otimes_{FE} FJf$. Thus as V is indecomposable, there exists primitive idempotents $i' \in FC_G(D)e$ and $j' \in FC_H(E)f$ such that $V|FIi' \otimes_{FE} j'FJ$. As the blocks $FC_G(D)e$ and $FC_H(E)f$ each have a unique projective indecomposable module, i is conjugate to i' in $FC_G(D)$ and j is conjugate to j' in $FC_H(E)$. Thus it follows that $FIi \cong FIi'$ as (FI, FE)-bimodules and $jFJ \cong j'FJ$ as (FE, FJ)-bimodules. Therefore, $V|FIi \otimes_{FE} jFJ$, and hence $Vj \cong FIi$ as (FI, FE)-bimodules by [8](Theorem 4.1). As $V(\Delta(D, \varphi, E)) \cong eM(\Delta(D, \varphi, E))f, (eM(\Delta(D, \varphi, E))f)j \cong V(\Delta(D, \varphi, E))j \cong Vj(\Delta(D, \varphi, E)) \cong FIi(\Delta(D, \varphi, E)) \cong FC_G(D)i$ as $FN_{S\times E}(\Delta(D, \varphi, E))$-modules. Let $W = Ind^{S\times T}_{N_{S\times T}(\Delta(D,\varphi,E))}(eM(\Delta(D, \varphi, E))f)$. As $(eM(\Delta(D, \varphi, E))f)j \cong FC_G(D)i$ as $FN_{S\times E}(\Delta(D, \varphi, E))$-modules, it follows that $Wj \cong Ind^{S\times(E\cap T)}_{N_{S\times E}(\Delta(D,\varphi,E))}((eM(\Delta(D, \varphi, E))f)j)$ $\cong Ind^{S\times(E\cap T)}_{N_{S\times E}(\Delta(D,\varphi,E))}(FC_G(D)i) \cong FSi$, and hence $Wj \cong FSi$ as $(FS, F(E\cap T))$-bimodules. Thus as W induces a Morita equivalence between FSe and FTf by lemma 12.10, it follows that $jFTj \cong (End_{FS\times 1}(Wj))^o \cong (End_{FS\times 1}(FSi))^o \cong iFSi$ as interior $E \cap T$-algebras, so the lemma holds. □

Lemma 12.13. *Let $(\Delta(D, \varphi, E), e\otimes f^o)$ be a maximal γ-Brauer pair, where (D, e) is a max-*

imal A-Brauer, (E, f) is a maximal B-Brauer pair and $\varphi : E \to D$ is an isomorphism. Let $\mathcal{A}$ and $\mathcal{B}$ denote the fusion systems associated with (D, e) and (E, f) respectively. For $Q \leq D$ and $R \leq E$, let e_Q denote the unique block idempotent of $FC_G(Q)$ such that $(Q, e_Q) \leq (D, e)$ and let f_R denote the unique block idempotent of $FC_H(R)$ such that $(R, f_R) \leq (E, f)$. Let $R \leq E$ and let $Q = \varphi(R)$. Let I be the inertial group of $FC_G(Q)e_Q$ in $N_G(Q)$, and let J be the inertial group of $FC_H(R)f_R$ in $N_H(R)$.

(1) Suppose that Q is fully $\mathcal{A}$-centralized, let $C_G(QC_D(Q)) \leq S \leq N_{C_G(Q)}((C_D(Q), e_{QC_D(Q)}))$ and let $C_H(RC_E(R)) \leq T \leq N_{C_H(R)}((C_E(R), f_{RC_E(R)}))$ such that $\varphi^{-1} \circ Aut_S(C_D(Q)) \circ \varphi = Aut_T(C_E(R))$. The element $Ind^{S\times T}_{N_{S\times T}(\Delta(C_D(Q),\varphi,C_E(R)))}(e_{QC_D(Q)}\gamma(\Delta(QC_D(Q), \varphi, RC_E(R))f_{RC_E(R)}) \in T^{\Delta}(FSe_{QC_D(Q)}, FTf_{RC_E(R)})$ is plus or minus an indecomposable $(FSe_{QC_D(Q)}, FTf_{RC_E(R)})$-bimodule that induces a Morita equivalence between $FSe_{QC_D(Q)}$ and $FTf_{RC_E(R)}$.

(2) Suppose that Q is fully $\mathcal{A}$-normalized. Let $C_G(N_D(Q)) \leq S \leq N_I((N_D(Q), e_{N_D(Q)}))$ and let $C_H(N_E(R)) \leq T \leq N_J((N_E(R), f_{N_E(R)}))$ such that $\varphi^{-1} \circ Aut_S(N_D(Q)) \circ \varphi = Aut_T(N_E(R))$. The element $Ind^{S\times T}_{N_{S\times T}(\Delta(N_D(Q),\varphi,N_E(R)))}(e_{N_D(Q)}\gamma(\Delta(N_D(Q), \varphi, N_E(R)))f_{N_E(R)}) \in T^{\Delta}(FSe_{N_D(Q)}, FTf_{N_E(R)})$ is plus or minus an indecomposable $(FSe_{N_D(Q)}, FTf_{N_E(R)})$-bimodule which induces a Morita equivalence between $FSe_{N_D(Q)}$ and $FTf_{N_E(R)}$.

Proof. Note that (1) holds by lemma 12.3.1, lemma 12.10, and lemma 4.17, and (2) holds by lemma 12.3.2, lemma 12.10, and lemma 4.19. ☐

Chapter 13

Finiteness of the set $T_o^{\Delta}(A,B)$

Throughout this section, let p be a prime and let $(K, \mathcal{O}, F)$ be a p-modular system with F algebraically closed and K large enough. Let G and H be finite groups and let $A = FGe_A$ and $B = FHe_B$ be blocks of FG and FH respectively. Let $\mathcal{S}^{\Delta}(A,B)$ denote the poset of $A \otimes B^o$-Brauer pairs of the form $(\Delta(Q,\varphi,R), e\otimes f^o)$, where (Q,e) is an A-Brauer pair, (R,f) is a B-Brauer pair, and $\varphi : R \to Q$ is an isomorphism, and let $\tilde{\mathcal{S}}^{\Delta}(A,B)$ denote the set of $G \times H$-conjugacy classes of $\mathcal{S}^{\Delta}(A,B)$.

Let $(\Delta(Q,\varphi,R), e\otimes f^o) \in \mathcal{S}^{\Delta}(A,B)$, let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and let J be the inertial group of $FC_H(R)f$ in $N_H(R)$. Define the group homomorphism

$\Psi_{(\Delta(Q,\varphi,R),e\otimes f^o)} : T^{\Delta}(A,B) \to Hom(R(KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes \tilde{e^o})), R(KN_{I\times J}(\Delta(Q,\varphi,R))(\tilde{e}\otimes \tilde{f^o})))$ by $\Psi_{(\Delta(Q,\varphi,R),e\otimes f^o)}(\gamma)([[M]]) = [[M]] \otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K$ for $\gamma \in T^{\Delta}(A,B)$ and a M a $KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes\tilde{e^o})$-module.

Let

$$\Psi = \bigoplus_{(\Delta(Q,\varphi,R),e\otimes f^o)\in\tilde{\mathcal{S}}^{\Delta}(A,B)} \Psi_{(\Delta(Q,\varphi,R),e\otimes f^o)} : T^{\Delta}(A,B) \to \bigoplus_{(\Delta(Q,\varphi,R),e\otimes f^o)\in\tilde{\mathcal{S}}^{\Delta}(A,B)} Hom(R(KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes\tilde{e^o})), R(KN_{I\times J}(\Delta(Q,\varphi,R))(\tilde{e}\otimes\tilde{f^o}))).$$

Lemma 13.1. *The group homomorphism Ψ is injective.*

Proof. If $(X,\sigma),(Y,\tau) \in \mathcal{S}^{\Delta}(A,B)$ such that $(X,\sigma) \not\leq_{G\times H} (Y,\tau)$, and M is an indecomposable trivial source (A,B)-bimodule such that (Y,τ) is a maximal M-Brauer pair, then $\Psi_{(X,\sigma)}([M]) = 0$ by lemma 6.4. Thus by an upper triangular matrix argument, it suffices to show that for $(\Delta(Q,\varphi,R), e\otimes f^o) \in \mathcal{S}^{\Delta}(A,B)$ and $\gamma \in T^{\Delta}(A,B)$ such that every indecomposable (A,B)-bimodule that appears in γ has maximal Brauer pair $(\Delta(Q,\varphi,R), e\otimes f^o)$, if $\Psi_{(\Delta(R,\varphi,Q),e\otimes f^o)}(\gamma)=0$, then $\gamma=0$. As $\Psi_{(\Delta(R,\varphi,Q),e\otimes f^o)}(\gamma)=0, (e\gamma(\Delta(Q,\varphi,R))f)^K = [[KC_G(Q)]]\otimes_{KC_G(Q)}(e\gamma(\Delta(Q,\varphi,R))f)^K = 0$. Let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and let J be the inertial group of $FC_H(R)f$ in $N_H(R)$. As $e\gamma(\Delta(Q,\varphi,R))f$ is a virtual projective $FN_{I\times J}(\Delta(Q,\varphi,R))/\Delta(Q,\varphi,R)$-module and $(e\gamma(\Delta(Q,\varphi,R))f)^K = 0, e\gamma(\Delta(Q,\varphi,R))f=0$ in $T^\Delta(FN_{I\times J}(\Delta(Q,\varphi,R))(e\otimes f^o))$ by the injectivity of the decomposition map, and hence $\gamma=0$ by lemma 10.3. Thus the lemma holds. □

Theorem 13.2. *$T_o^\Delta(A,B)$ is a finite set and $T_o^\Delta(A,A)$ is a finite group.*

Proof. It follows from lemma 11.1.1 that $\Psi(T_o^\Delta(A,B))$ is finite, so as Ψ is injective by lemma 13.1, $T_o^\Delta(A,B)$ is finite. □

Chapter 14

Isotypies

Throughout this section, let p be a prime and let $(K, \mathcal{O}, F)$ be a p-modular system with F algebraically closed and K large enough. Let G and H be finite groups, let $A = FGe_A$ be a block of FG, and let $B = FHe_B$ be a block of FH.

We recall the notion of an isotypy, which was defined in [5](Définition 4.6 and Remarque 2 following it.)

Definition 14.1. *An* ***isotypy*** *is a tuple* $I = (D, e, \varphi, E, f, (\mu_R)_{R \leq E})$ *satisfying the following:*

(1) (D, e) *is a maximal A-Brauer pair,* (E, f) *is a maximal B-Brauer pair and* $\varphi : E \to D$ *is an isomorphism. Let $\mathcal{A}$ be the fusion system associated with (D, e) and let $\mathcal{B}$ be the fusion system associated with (E, f). For $Q \leq D$, let e_Q denote the unique block idempotent of $FC_G(Q)$ such that $(Q, e_Q) \leq (D, e)$ and for $R \leq E$, let f_R denote the unique block idempotent of $FC_H(R)$ such that $(R, f_R) \leq (E, f)$.*

(2) The isomorphism $\varphi : E \to D$ is an isomorphism between $\mathcal{B}$ and $\mathcal{A}$.

(3) For $R \leq E, \mu_R$ is a perfect isometry between $FC_H(R)f_R$ and $FC_G(Q)e_Q$, where $Q = \varphi(R)$. We denote by $I_R : KR(KC_H(R)\tilde{f_R}) \to KR(KC_G(Q)\tilde{e_Q})$ the K-linear map defined by $\chi \mapsto \mu \otimes_{KC_H(R)} \chi$ for $\chi \in KR(KC_H(R)\tilde{f_R})$.

(4) Let $R \leq E$ and let $Q = \varphi(R)$. For $g \in G$ and $h \in H$ such that $c_g \in Hom_{\mathcal{A}}(Q, D)$ and $c_h \in Hom_{\mathcal{B}}(R, E)$ such that $c_h = \varphi^{-1} \circ c_g \circ \varphi$ in $Hom_{\mathcal{B}}(R, E)$,
$I_{{}^hR} = {}^{(g,h)}I_R$, where ${}^{(g,h)}I_R$ denotes the K-linear map $c_g \circ I_R \circ c_{h^{-1}}$.

(5) Let $R \leq E$, let $Q = \varphi(R)$, let $y \in C_E(R)$ and let $x = \varphi(y) \in C_D(Q)$. The equality $d^{(x, e_{Q<x>})}_{C_G(Q)} \circ I_R = I_{R<y>} \circ d^{(y, f_{R<y>})}_{C_H(R)}$ holds.

Lemma 14.2. *For a tuple $I = (D, e, \varphi, E, f, (\mu_R)_{R \leq E})$ such that I satisfies conditions (1)-(4) of definition 14.1, I is an isotypy if and only if the equation in condition (5) of definition 14.1 is satisfied for all $R \leq E$ such that R is fully $\mathcal{B}$-centralized and for all $y \in C_E(R)$.*

Proof. We adopt the notation of definition 14.1. Suppose that I satisfies conditions (1)-(4) of definition 14.1 and that the equation in condition (5) of definition 14.1 is satisfied for all $R \leq E$ such that R is fully $\mathcal{B}$-centralized and for all $y \in C_E(R)$. Let $R \leq E$, let $Q = \varphi(R)$, let $y \in C_E(R)$ and let $x = \varphi(y) \in C_D(Q)$. We need to show that $d^{(x,e_{Q<x>})}_{C_G(Q)} \circ I_R = I_{R<y>} \circ d^{(y,f_{R<y>})}_{C_H(R)}$. By the extension axiom for fusion systems and the fact that every fully $\mathcal{B}$-normalized subgroup of E is fully $\mathcal{B}$-centralized by [7](Proposition 2.5), there is an $h \in H$ such that ${}^hR$ is a fully $\mathcal{B}$-centralized subgroup of E and $c_h \in Hom_{\mathcal{B}}(R < y >, E)$. Thus as φ is an isomorphism between $\mathcal{B}$ and $\mathcal{A}$, there exists $g \in G$ such that $c_g \in Hom_{\mathcal{A}}(Q < x >, D)$ and $c_h = \varphi^{-1} \circ c_g \circ \varphi$ in $Hom_{\mathcal{B}}(R < y >, E)$. Therefore, $\varphi({}^hy) = {}^gx$, so as ${}^hR$ is fully $\mathcal{B}$-centralized, $d^{({}^gx, e_{{}^gQ<{}^gx>})}_{C_G({}^gQ)} \circ I_{{}^hR} = I_{{}^hR<{}^hy>} \circ d^{({}^hy, f_{{}^hR<{}^hy>})}_{C_H({}^hR)}$ by our hypothesis. As I satisfies condition (4) of definition 14.1, $I_{{}^hR} = {}^{(g,h)}I_R$, and $I_{{}^hR<{}^hy>} = {}^{(g,h)}I_{R<y>}$. As $c_g \in Hom_{\mathcal{A}}(Q < x >, D)$ and $c_h \in Hom_{\mathcal{B}}(R < y >, E)$, $e_{{}^gQ<{}^gx>} = {}^ge_{Q<x>}$ and $f_{{}^hR<{}^hy>} = {}^hf_{R<y>}$, so $d^{({}^gx, e_{{}^gQ<{}^gx>})}_{C_G({}^gQ)} = {}^gd^{(x,e_{Q<x>})}_{C_G(Q)}$ and $d^{({}^hy, f_{{}^hR<{}^hy>})}_{C_H({}^hR)} = {}^hd^{(y,f_{R<y>})}_{C_H(R)}$. Therefore, $d^{(x,e_{Q<x>})}_{C_G(Q)} \circ I_R = {}^{g^{-1}}d^{({}^gx, e_{{}^gQ<{}^gx>})}_{C_G({}^gQ)} \circ {}^{(g,h)^{-1}}I_{{}^hR} = {}^{(g,h)^{-1}}(d^{({}^gx, e_{{}^gQ<{}^gx>})}_{C_G({}^gQ)} \circ I_{{}^hR}) = {}^{(g,h)^{-1}}(I_{{}^hR<{}^hy>} \circ d^{({}^hy, f_{{}^hR<{}^hy>})}_{C_H({}^hR)}) = {}^{(g,h)^{-1}}({}^{(g,h)}I_{R<y>} \circ {}^hd^{(y,f_{R<y>})}_{C_H(R)}) = I_{R<y>} \circ d^{(y,f_{R<y>})}_{C_H(R)}$, so the lemma holds. □

Lemma 14.3. *Let $I = (D, e, \varphi, E, f, (\mu_R)_{R \leq E})$ be a tuple such that I satisfies conditions (1)-(3) of definition 14.1. Adopt the notation of definition 14.1. The tuple I is an isotypy if and only if I satisfies the following conditions:*

(1) Let $R \leq E$, let $Q = \varphi(R)$, let $h \in H$ such that $c_h \in Hom_{\mathcal{B}}(R, E)$, and let $g \in G$ such that $c_g \in Hom_{\mathcal{A}}(Q, D)$ and $c_h = \varphi^{-1} \circ c_g \circ \varphi$ in $Hom_{\mathcal{B}}(R, E)$. The equality $\mu_{{}^hR} = {}^{(g,h)}\mu_R$ holds.

(2) Let $R \leq E$ such that R is fully $\mathcal{B}$-centralized, and let $Q = \varphi(R)$.

(a) Let $y \in C_E(R)$ and let $x = \varphi(y) \in C_D(Q)$. For x' a p'-element in $C_G(Q < x >)$ and y' a p'-element in $C_H(R < y >)$, $\mu_R(xx'\widetilde{e_{Q<x>}} \otimes yy'\widetilde{f^o_{R<y>}}) = \mu_{R<y>}((x', y'))$.

(b) Let $(< x >, \sigma)$ be an $FC_G(Q)e_Q$-Brauer pair and let $(< y >, \tau)$ be an $FC_H(R)f_R$-Brauer pair, where x is a p-element of $C_G(Q)$ and y is a p-element of $C_H(R)$. If there exist

p'-elements $x' \in C_G(Q < x >)$ and $y' \in C_H(R < y >)$ such that $\mu_R(xx'\tilde{\sigma} \otimes yy'\tilde{\tau^o}) \neq 0$, then there exists $z \in C_E(R)$ such that (y, τ) is $C_H(R)$-conjugate to $(z, f_{R<z>})$ and (x, σ) is $C_G(Q)$-conjugate to $(\varphi(z), e_{Q<\varphi(z)>})$.

Proof. Clearly condition (4) of definition 14.1 is equivalent to condition (1), so by lemma 14.2, it suffices to show that if I satisfies condition (4) of definition 14.1, then I satisfies condition (2) if and only if for $R \leq E$ such that R is fully $\mathcal{B}$-centralized and for $y \in C_E(R)$ the equality in condition (5) of definition 14.1 is satisfied. Let $R \leq E$ such that R is fully $\mathcal{B}$-centralized. As φ is an isomorphism between $\mathcal{B}$ and $\mathcal{A}$, we also have that Q is fully $\mathcal{A}$-centralized. Thus by [6](Theorem 3.11), $(C_D(Q), e_{QC_D(Q)})$ is a maximal $FC_G(Q)e_Q$-Brauer pair and $(C_E(R), f_{RC_E(R)})$ is a maximal $FC_H(R)f_R$-Brauer pair, so the lemma holds by [5](Proposition 4.7). □

Lemma 14.4. *Let $I = (D, e, \varphi, E, f, (\mu_R)_{R \leq E})$ be a tuple such that I satisfies conditions (1)-(3) of definition 14.1. Adopt the notation of definition 14.1. The tuple I is an isotypy if and only if I satisfies the following conditions:*

(1) Let $R \leq E$, let $Q = \varphi(R)$, let $h \in H$ such that $c_h \in Hom_{\mathcal{B}}(R, E)$, and let $g \in G$ such that $c_g \in Hom_{\mathcal{A}}(Q, D)$ and $c_h = \varphi^{-1} \circ c_g \circ \varphi$ in $Hom_{\mathcal{B}}(R, E)$. The equality $\mu_{{}^hR} = {}^{(g,h)}\mu_R$ holds.

(2) Let $R \leq E$ and let $Q = \varphi(R)$.

(a) Let $y \in C_E(R)$ and let $x = \varphi(y) \in C_D(Q)$. For x' a p'-element in $C_G(Q < x >)$ and y' a p'-element in $C_H(R < y >)$, $\mu_R(xx'\widetilde{e_{Q<x>}} \otimes yy'\widetilde{f^o_{R<y>}}) = \mu_{R<y>}((x', y'))$.

(b) Let $(< x >, \sigma)$ be an $FC_G(Q)e_Q$-Brauer pair and let $(< y >, \tau)$ be an $FC_H(R)f_R$-Brauer pair, where x is a p-element of $C_G(Q)$ and y is a p-element of $C_H(R)$. If there exist p'-elements $x' \in C_G(Q < x >)$ and $y' \in C_H(R < y >)$ such that $\mu_R(xx'\tilde{\sigma} \otimes yy'\tilde{\tau^o}) \neq 0$, then there exist elements $g \in G$ and $h \in H$ such that $c_g \in Hom_{\mathcal{A}}(Q, D), c_h \in Hom_{\mathcal{B}}(R, E), c_h = \varphi^{-1} \circ c_g \circ \varphi$ in $Hom_{\mathcal{B}}(R, E)$, and there exists $z \in C_E({}^hR)$ such that ${}^h(y, \tau) = (z, f_{{}^hR<z>})$ and ${}^g(x, \sigma) = (\varphi(z), e_{{}^gQ<\varphi(z)>})$.

Proof. By lemma 14.3, it suffices to show that if I satisfies condition (1), then I satisfies condition (2) if and only if I satisfies condition (2) of lemma 14.3. Suppose that I satisfies condition (1).

Suppose that I satisfies condition (2) of the lemma. Clearly, I satisfies condition (2)(a) of lemma 14.3, so it suffices to show that I satisfies condition (2)(b) of lemma 14.3. Let $R \leq E$ such that R is fully $\mathcal{B}$-centralized and let $Q = \varphi(R)$. Let $(< x >, \sigma)$ be an $FC_G(Q)e_Q$-Brauer pair and let $(< y >, \tau)$ be an $FC_H(R)f_R$-Brauer pair, where x is a p-element of $C_G(Q)$ and y is a p-element of $C_H(R)$. Suppose there exist p'-elements $x' \in C_G(Q < x >)$ and $y' \in C_H(R < y >)$ such that $\mu_R(xx'\widetilde{\sigma} \otimes yy'\widetilde{\tau^o}) \neq 0$. As R is fully $\mathcal{B}$-centralized, $(C_E(R), f_{RC_E(R)})$ is a maximal $FC_H(R)f_R$-Brauer pair by [6](Theorem 3.11). Therefore, by lemma 3.4.2, $(< y >, \tau)$ is $C_H(R)$-conjugate to a subpair of $(C_E(R), f_{RC_E(R)})$. Thus we may assume that $y \in C_E(R)$ and $\tau = f_{R<y>}$. As condition (2) of the lemma holds, there exist elements $g \in G$ and $h \in H$ such that $c_g \in Hom_{\mathcal{A}}(Q, D), c_h \in Hom_{\mathcal{B}}(R, E), c_h = \varphi^{-1} \circ c_g \circ \varphi$ in $Hom_{\mathcal{B}}(R, E)$, and there exists $z \in C_E({}^hR)$ such that ${}^h(y, f_{R<y>}) = (z, f_{{}^hR<z>})$ and ${}^g(x, \sigma) = (\varphi(z), e_{{}^gQ<\varphi(z)>})$. As ${}^h(R < y >, f_{R<y>}) = ({}^hR < z >, f_{{}^hR<z>}), c_h \in Hom_{\mathcal{B}}(R < y >, E)$, so as φ is an isomorphism between $\mathcal{B}$ and $\mathcal{A}$, there exists $s \in G$ such that $c_s \in Hom_{\mathcal{A}}(Q < \varphi(y) >, D)$ and $c_h = \varphi^{-1} \circ c_s \circ \varphi$ in $Hom_{\mathcal{B}}(R < y >, E)$. Therefore, ${}^s\varphi(y) = \varphi({}^hy) = \varphi(z) = {}^gx$, so ${}^{g^{-1}s}\varphi(y) = x$. As $c_s \in Hom_{\mathcal{A}}(Q < \varphi(y) >, D)$, ${}^se_{Q<\varphi(y)>} = e_{{}^gQ<\varphi(z)>}$, so ${}^{g^{-1}s}e_{Q<\varphi(y)>} = {}^{g^{-1}}e_{{}^gQ<\varphi(z)>} = \sigma$. Therefore, ${}^{g^{-1}s}(\varphi(y), e_{Q<\varphi(y)>}) = (x, \sigma)$ and as $\varphi^{-1} \circ c_g \circ \varphi = c_h = \varphi^{-1} \circ c_s \circ \varphi$ in $Hom_{\mathcal{B}}(R, E)$, it follows that $g^{-1}s \in C_G(Q)$, so condition (2)(b) of lemma 14.3 holds.

Now suppose that I satisfies condition (2) of lemma 14.3. First we show that I satisfies condition (2)(a). Let $R \leq E$ and let $Q = \varphi(R)$. Let $y \in C_E(R)$, let $x = \varphi(y) \in C_D(Q)$, let x' be a p'-element in $C_G(Q < x >)$ and let y' be a p'-element in $C_H(R < y >)$. Let $h \in H$ such that ${}^hR$ is fully $\mathcal{B}$-centralized and $c_h \in Hom_{\mathcal{B}}(R < y >, E)$. As φ is an isomorphism between $\mathcal{B}$ and $\mathcal{A}$, there exists $g \in G$ such that $c_g \in Hom_{\mathcal{A}}(Q < x >, D)$ and $c_h = \varphi^{-1} \circ c_g \circ \varphi$ in $Hom_{\mathcal{B}}(R < y >, E)$. Thus as I satisfies condition (1) and (2) of lemma 14.3, $\mu_R(xx'\widetilde{e_{Q<x>}} \otimes yy'\widetilde{f^o_{R<y>}}) = \mu_{{}^hR}({}^gx\,{}^gx'\widetilde{e_{{}^gQ<{}^gx>}} \otimes {}^hy\,{}^hy'\widetilde{f^o_{{}^hR<{}^hy>}}) = \mu_{{}^hR<{}^hy>}({}^gx', {}^hy') = \mu_{R<y>}(x', y')$, so I satisfies condition (2)(a). Thus it remains to show that I satisfies condition (2)(b).

Let $(< x >, \sigma)$ be an $FC_G(Q)e_Q$-Brauer pair and let $(< y >, \tau)$ be an $FC_H(R)f_R$-Brauer pair, where x is a p-element of $C_G(Q)$ and y is a p-element of $C_H(R)$. Suppose there exist p'-elements $x' \in C_G(Q < x >)$ and $y' \in C_H(R < y >)$ such that $\mu_R(xx'\widetilde{\sigma} \otimes yy'\widetilde{\tau^o}) \neq 0$. As I satisfies condition (1), we may assume that R is fully $\mathcal{B}$-centralized. Thus as I satisfies condition (2)(b) of lemma 14.3, it follows that I satisfies condition (2)(b), so the lemma holds. □

Theorem 14.5. *Suppose that A and B are p-permutation equivalent and let $\gamma \in T^{\Delta}_o(A, B)$*

be a p-permutation equivalence. Let $(\Delta(D,\varphi,E), e\otimes f^o)$ *be a maximal* γ*-Brauer pair where* (D,e) *is a maximal A-Brauer pair,* (E,f) *is a maximal B-Brauer pair, and* $\varphi : E \to D$ *is an isomorphism. Adopt the notation of definition 14.1. The tuple* $I = (D, e, \varphi, E, f, ((e_{\varphi(R)}\gamma(\Delta(\varphi(R),\varphi,R))f_R)^K)_{R\leq E})$ *is an isotypy.*

Proof. The tuple I satisfies condition (1) of definition 14.1 by construction, and condition (2) is satisfied by lemma 12.1. Next note that for $R \leq E$, every module that appears in $e_{\varphi(R)}\gamma(\Delta(\varphi(R),\varphi,R))f_R$ has twisted diagonal vertex and hence is projective as a left $FC_G(\varphi(R))$-module and as a right $FC_H(R)$-module, so I satisfies condition (3) of definition 14.1 by lemma 12.2.2 and [5](Theorem 1.5(2)). Thus by lemma 14.4, it suffices to show that I satisfies conditions (1) and (2) of lemma 14.4.

Let $R \leq E$, let $Q = \varphi(R)$, let $h \in H$ such that $c_h \in Hom_{\mathcal{B}}(R,E)$, and let $g \in G$ such that $c_g \in Hom_{\mathcal{A}}(Q,D)$ and $c_h = \varphi^{-1}\circ c_g \circ \varphi$ in $Hom_{\mathcal{B}}(R,E)$. As $\mu_{{}^hR} = (e_{{}^gQ}\gamma(\Delta({}^gQ,\varphi,{}^hR))f_{{}^hR})^K = ({}^ge_Q\gamma(\Delta({}^gQ, c_g\varphi c_{h^{-1}}, {}^hR))\,{}^hf_R)^K =$ ${}^{(g,h)}((e_Q\gamma(\Delta(Q,\varphi,R))f_R)^K) = {}^{(g,h)}\mu_R$, I satisfies condition (1) of lemma 14.4.

Let $R \leq E$, let $Q = \varphi(R)$, let $y \in C_E(R)$, let $x = \varphi(y) \in C_D(Q)$, let x' be a p'-element in $C_G(Q<x>)$ and let y' be a p'-element in $C_H(R<y>)$. As $\mu_R(xx'\widetilde{e_{Q<x>}} \otimes yy'\widetilde{f^o_{R<y>}}) = (e_Q\gamma(\Delta(Q,\varphi,R))f_R)^K(xx'\widetilde{e_{Q<x>}}\otimes yy'\widetilde{f^o_{R<y>}}) = ((e_Q\gamma(\Delta(Q,\varphi,R))f_R)(<(x,y)>))^K(x'\widetilde{e_{Q<x>}} \otimes y'\widetilde{f^o_{R<y>}}) =$ $(e_{Q<x>}\gamma(\Delta(Q<x>,\varphi,R<y>))f_{R<y>})^K((x',y')) = \mu_{R<y>}((x',y'))$, so I satisfies condition (2)(a) of lemma 14.4.

Let $R \leq E$, let $Q = \varphi(R)$, let $(<x>,\sigma)$ be an $FC_G(Q)e_Q$-Brauer pair and let $(<y>,\tau)$ be an $FC_H(R)f_R$-Brauer pair, where x is a p-element of $C_G(Q)$ and y is a p-element of $C_H(R)$. Suppose there exist p'-elements $x' \in C_G(Q<x>)$ and $y' \in C_H(R<y>)$ such that $\mu_R(xx'\widetilde{\sigma}\otimes yy'\widetilde{\tau^o}) \neq 0$. As $\mu_R(xx'\widetilde{\sigma}\otimes yy'\widetilde{\tau^o}) = (e_Q\gamma(\Delta(Q,\varphi,R))f_R)^K(xx'\widetilde{\sigma}\otimes yy'\widetilde{\tau^o}) = ((e_Q\gamma(\Delta(Q,\varphi,R))f_R)(<(x,y)>))^K(x'\widetilde{\sigma}\otimes y'\widetilde{\tau^o}) = (\sigma\gamma(\Delta(Q,\varphi,R)<(x,y)>)\tau)^K((x',y'))$, it follows that $\sigma\gamma(\Delta(Q,\varphi,R)<(x,y)>)\tau \neq 0$, and hence $(\Delta(Q,\varphi,R)<(x,y)>, \sigma\otimes\tau^o)$ is a γ-Brauer pair. Thus by lemma 11.9, there exists $g \in G$ and $h \in H$ such that $(\Delta(Q,\varphi,R)<(x,y)>,\sigma\otimes\tau^o) \leq (\Delta({}^gD, c_g\varphi c_{h^{-1}}, {}^hE), {}^ge\otimes {}^hf^o)$. Therefore, ${}^{g^{-1}}\sigma = e_{{}^{g^{-1}}Q<{}^{g^{-1}}x>}$ and ${}^{h^{-1}}\tau = f_{{}^{h^{-1}}R<{}^{h^{-1}}y>}$. As $(<x>,\sigma)$ is an $FC_G(Q)e_Q$-Brauer pair and $(<y>,\tau)$ is an $FC_H(R)f_R$-Brauer pair, $(Q,e_Q) \leq (Q<x>,\sigma)$ and $(R,f_R) \leq (R<y>,\tau)$, so ${}^{g^{-1}}(Q,e_Q) \leq {}^{g^{-1}}(Q<x>,\sigma) = ({}^{g^{-1}}Q < {}^{g^{-1}}x>, e_{{}^{g^{-1}}Q<{}^{g^{-1}}x>})$, and ${}^{h^{-1}}(R,f_R) \leq {}^{h^{-1}}(R<y>,\tau) = ({}^{h^{-1}}R < {}^{h^{-1}}y>, f_{{}^{h^{-1}}R<{}^{h^{-1}}y>})$. Therefore, $c_{g^{-1}} \in Hom_{\mathcal{A}}(Q,D)$ and

$c_{h^{-1}} \in Hom_{\mathcal{B}}(R, E)$. Furthermore, as $\Delta(Q, \varphi, R) < (x, y) > \leq \Delta({}^{g}D, c_g \varphi c_{h^{-1}}, {}^{h}E)$, it follows that $c_{h^{-1}} = \varphi^{-1} \circ c_{g^{-1}} \circ \varphi$ in $Hom_{\mathcal{B}}(R, E)$ and $\varphi({}^{h^{-1}}y) = {}^{g^{-1}}x$. Thus it follows that I satisfies condition (2)(b) of lemma 14.4, so the lemma holds. □

Chapter 15

A character-theoretic criterion for p-permutation equivalences

Throughout this section, let p be a prime and let $(K, \mathcal{O}, F)$ be a p-modular system with F algebraically closed and K large enough. Let G and H be finite groups, let $A = FGe_A$ be a block of FG and let $B = FHe_B$ be a block of FH.

Lemma 15.1. *Let (D, e) be a maximal A-Brauer pair and let (E, f) be a maximal B-Brauer pair such that there is an isomorphism $\varphi : E \to D$ such that φ induces an isomorphism between the fusion system $\mathcal{B}$ associated with (E, f) and the fusion system $\mathcal{A}$ associated with (D, e). For $Q \leq D$, let e_Q denote the unique block idempotent of $FC_G(Q)$ such that $(Q, e_Q) \leq (D, e)$, and for $R \leq E$, let f_R denote the unique block idempotent of $FC_H(R)$ such that $(R, f_R) \leq (E, f)$. Let $Q \leq D$ and let $R = \varphi^{-1}(Q)$. The triple (R, f_R, φ) lies in the unique H-conjugacy class of triples (R', f', φ') such that (R', f') is a B-Brauer pair, $\varphi' : R' \to Q$ is an isomorphism, and $(\Delta(Q, \varphi', R'), e_Q \otimes (f')^o) \leq_{G\times H} (\Delta(D, \varphi, E), e \otimes f^o)$.*

Proof. We need to show that if (R', f', φ') is a triple such that $(\Delta(Q, \varphi', R'), e_Q \otimes (f')^o) \leq_{G\times H} (\Delta(D, \varphi, E), e \otimes f^o)$, then (R, f_R, φ) and (R', f', φ') are H-conjugate. By hypothesis there exists $(g, h) \in G \times H$ such that $(\Delta({}^gQ, c_g\varphi' c_h^{-1}, {}^hR'), {}^ge_Q \otimes {}^h(f')^o) \leq (\Delta(D, \varphi, E), e \otimes f^o)$. As $(Q, e_Q), ({}^gQ, {}^ge_Q) \leq (D, e), c_g^{-1} : {}^gQ \to Q \in \mathcal{A}$, so as $\varphi : E \to D$ is an isomorphism between $\mathcal{B}$ and $\mathcal{A}, \varphi^{-1}c_g^{-1}\varphi : {}^hR' \to R \in \mathcal{B}$. Thus there exists $y \in H$ such that ${}^{yh}(R', f') = (R, f_R)$ and $c_g^{-1}\varphi = \varphi c_y$ as homomorphisms from ${}^hR'$ to Q. As $\Delta({}^gQ, c_g\varphi' c_h^{-1}, {}^hR') \leq \Delta(D, \varphi, E), \Delta(Q, \varphi', R') = \Delta(Q, c_g^{-1}\varphi c_h, R')$. Therefore, $(R', f', \varphi') =$
$(R', f', c_g^{-1}\varphi c_h) = (R', f', \varphi c_{yh}) = {}^{(yh)^{-1}}(R, f_R, \varphi)$, so (R, f_R, φ) and (R', f', φ') are H-

conjugate and hence the lemma holds. $\square$

Lemma 15.2. *Let $\gamma \in T^{\Delta}(A,B)$ such that the maximal γ-Brauer pairs form a $G \times H$-conjugacy class consisting of $A \otimes B^o$-Brauer pairs of the form $(\Delta(D,\varphi,E), e \otimes f^o)$, where (D,e) is a maximal A-Brauer, (E,f) is a maximal B-Brauer pair, and $\varphi : E \to D$ is an isomorphism of groups that induces an isomorphism between the fusion systems associated with (E,f) and (D,e) respectively. The element γ is a p-permutation equivalence between A and B if and only if $(e\gamma(\Delta(Q,\varphi,R))f)^K \otimes_{KC_H(R)} ((e\gamma(\Delta(Q,\varphi,R))f)^o)^K = [[KC_G(Q)\tilde{e}]]$ in $T^{\Delta}(KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes \tilde{e^o}))$ and $((e\gamma(\Delta(Q,\varphi,R))f)^o)^K \otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K = [[KC_H(R)\tilde{f}]]$ in $T^{\Delta}(KN_{J\times J}(\Delta(R))(\tilde{f}\otimes \tilde{f^o}))$ for any $A\otimes B^o$-Brauer pair $(\Delta(Q,\varphi,R), e\otimes f^o)$ contained in a maximal γ-Brauer pair, where I is the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and J is the inertial group of $FC_H(R)f$ in $N_H(R)$.*

Proof. We have seen in the proof of lemma 12.2.2 that if γ is a p-permutation equivalence, then $(e\gamma(\Delta(Q,\varphi,R))f)^K \otimes_{KC_H(R)} ((e\gamma(\Delta(Q,\varphi,R))f)^o)^K = [[KC_G(Q)\tilde{e}]]$ in $T^{\Delta}(KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes \tilde{e^o}))$ and $((e\gamma(\Delta(Q,\varphi,R))f)^o)^K \otimes_{KC_G(Q)} (e\gamma(\Delta(Q,\varphi,R))f)^K = [[KC_H(R)\tilde{f}]]$ in $T^{\Delta}(KN_{J\times J}(\Delta(R))(\tilde{f}\otimes \tilde{f^o}))$ for any $A\otimes B^o$-Brauer pair $(\Delta(Q,\varphi,R), e\otimes f^o)$ contained in a maximal γ-Brauer pair, where I is the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and J is the inertial group of $FC_H(R)f$ in $N_H(R)$, so it remains to prove the converse. By symmetry, it suffices to show that $\gamma \otimes_{FH} \gamma^o = [A]$ in $T^{\Delta}(A,A)$.

As the group homomorphism Ψ of lemma 13.1 is injective, it suffices to show that for any A-Brauer pair (Q,e), $(e(\gamma\otimes_{FH}\gamma^o)(\Delta(Q))e)^K = [[KC_G(Q)e]]$ in $T^{\Delta}(KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes \tilde{e^o}))$, and that for any $A\otimes A^o$-Brauer pair of the form $(\Delta(Q,\alpha,U), e\otimes \epsilon^o)$, where (Q,e) and (U,ϵ) are A-Brauer pairs and $\alpha : U \to Q$ is an isomorphism, if $(e(\gamma\otimes_{FH}\gamma^o)(\Delta(Q,\alpha,U))\epsilon)^K \neq 0$ in $T^{\Delta}(KN_{I\times L}(\Delta(Q,\alpha,U))(\tilde{e}\otimes \tilde{\epsilon^o}))$, then $(\Delta(Q,\alpha,U), e\otimes \epsilon^o)$ is $G\times G$-conjugate to $(\Delta(Q), e\otimes e^o)$, where I is the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and L is the inertial group of $FC_G(U)\epsilon$ in $N_G(Q)$.

Let (Q,e) be an A-Brauer pair and let I be the inertial group of $FC_G(Q)e$ in $N_G(Q)$. It follows from our hypothesis and from lemma 15.1 that there is a unique H-conjugacy class of triples (R,f,φ) such that (R,f) is a B-Brauer pair, $\varphi : R \to Q$ is an isomorphism, and $(\Delta(Q,\varphi,R), e\otimes f^o)$ is a γ-Brauer pair. Thus by lemma 7.6.2 and our hypothesis, $(e(\gamma\otimes_{FH}\gamma^o)(\Delta(Q))e)^K = (e\gamma(\Delta(Q,\varphi,R))f)^K \otimes_{KC_H(R)} ((e\gamma(\Delta(Q,\varphi,R))f)^o)^K = [[KC_G(Q)\tilde{e}]]$ in $T^{\Delta}(KN_{I\times I}(\Delta(Q))(\tilde{e}\otimes \tilde{e^o}))$.

Now suppose that $(\Delta(Q,\alpha,U), e\otimes\epsilon^o)$ is an $A\otimes A^o$-Brauer pair, where (Q,e) and (U,ϵ) are A-Brauer pairs and $\alpha : U \to Q$ is an isomorphism and that $(e(\gamma \otimes_{FH} \gamma^o)(\Delta(Q,\alpha,U))\epsilon)^K \neq 0$ in $T^{\Delta}(KN_{I\times L}(\Delta(Q,\alpha,U))(\tilde{e}\otimes\tilde{\epsilon^o}))$, where I is the inertial group of $FC_G(Q)e$ in $N_G(Q)$ and L is the inertial group of $FC_G(U)\epsilon$ in $N_G(U)$. By lemma 7.5.2, there exists a quadruple (R,f,φ,ψ), where (R,f) is a B-Brauer pair and $\varphi : R \to Q$ and $\psi : U \to R$ is are isomorphisms such that $\alpha = \varphi\circ\psi$ and $(\Delta(R,\varphi^{-1},Q), f\otimes e^o)$ and $(\Delta(R,\psi,U), f\otimes\epsilon^o)$ are γ^o-Brauer pairs. Thus by our hypothesis and lemma 15.1, there exists $g\in G$ such that ${}^g(U,\epsilon,\psi) = (Q,e,\varphi^{-1})$. Therefore, $\alpha = {}^g\psi^{-1}\circ\psi = c_g$, so ${}^{(1,g)}(\Delta(Q,\alpha,U), e\otimes\epsilon^o) = (\Delta(Q), e\otimes e^o)$, and hence the lemma holds. □

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