# Lawrence Berkeley National Laboratory 

Lawrence Berkeley National Laboratory
Title
EQUILIBRIUM AND STABILITY IN STRONGLY INHOMOGENEOUS PLASMAS
Permalink
https://escholarship.org/uc/item/49s2218z

## Author

Mynick, Harry E.

## Publication Date

1978-10-01

# ③ Lawrence Berkeley Laboratory UNIVERSITY OF CALIFORNIA 

## Accelerator \& Fusion Research Division

## EQUILIBRIUM AND STABILITY IN STRONGLY INHOMOGENEOUS PLASMAS

Harry E. Mynick (Ph. D. thesis)

October 1978

HEGEDVED
4 A Whemble
gerkgigi lamonarory
JUL 23999

MIBRARY AND DOCUMENTTS SECTHON

## TWO-WEEK LOAN COPY

This is a Library Círculating Copy which may be borrowed for two weeks. For a personal retention copy, call Tech. Info. Dívision, Ext. 6782


Prepared for the U. S. Department of Energy under Contract W-7405-ENG-48

## EQUILIBRIUA AND STABILITY IN STRONGLY INHOMOGENEOUS PLASMAS

## TABLE OF CONTENTS

ABSTRACT ..... vii
ACKNOWLEDOMENTS ..... x
INIRODUCTION ..... xi
PART 1. VLASOV SLAB EQUILIBRIA: INTERPRETATICN AND PRACTICAL APPLICATION OF THE CHANNEL FORMLISM ..... 1
I.A. Introduction to Part I ..... 1
I.B. Generalized Channell Formalism ..... 4
I.C. Interpretive Methods ..... 6
I.C.1. Connecting the Physical and Canonical Variables ..... 7
I.C.2. Obtaining $f_{s}$ from $f_{s}$ ..... 11
I.C.3. Explicit Example:
Loss Cone Distribution ..... 13
I.C.4. Obtaining $U$ from $f_{s}$ ..... 16
sc
I.C.5. Self-Consistency of $A$ and $A$ ..... 19
I.C.6. Nunerical Application ..... 20
APPENDIX A: SEMIQUANTITATIVE ESTDATE OF U ..... 23

[^0]PART II. PARTICIE MOTION IN REALISTIC GEOMEIRIES AND THE FINITE GYRORADIUS GJIDING CENTER HAMILTONIAN ..... 28
II.A. Introduction to Part II ..... 28
II.B. Overview of the Procedure for Obtaining K ..... 31
II.C. Coordinate System, Initial Hamıltonian ..... 36
II.D. Generation Function S;
Formal Description ..... 38
II.E. Lie Methods; Preparation for Explicit Calculation ..... 41
II.E.1. Dragt-Finn Lie Method ..... 41
II.E.2. Near-Slab Lie Method ..... 46
II.F. The "Frozen $\lambda$ " Problem ..... 50
II.G. Generating Function; Explicit Form. ..... 55
II.H. Calculation of $\mathrm{H}^{\circ}$ ..... 58
II.I. Calculation of $K$ from $H^{\circ}$ ..... 60
II.J. Generalized Adiabatic Invariant ..... 65
II.K. Application: Slab Geometry with Shear ..... 69
II.K.1. Specialization to Slab Geonetry ..... 70
II.K.2. Particle Motion ..... 73
II.K.3. Adiabatic Invariant ..... 76
II.L. Directions for Extension of the Formalism ..... 78
APPENDIX B. ORDERING SCHENE OF PART II ..... 82
part iil. stability in strongly infomogeneous plasmas ..... 84
III.A. Introduction to Part III ..... 84
III.B. Poisson's Equation; General Formalism ..... 87
III.B.1. Exict Formulation ..... 87
III.B.2. Truncation of the Exact Equations ..... 95
III.B.3. Variational Method ..... 98
III.C. Conductivity K; Explicit Form ..... 105
III.C.1. Explicit Expression for $K_{s}$. ..... 106
III.C.2. On the Structure of $K_{s}$; Discussion ..... 109
III.C.3. Specializing $K_{s} B$ : Choice of Distribution $f_{s}^{s}$ ..... 113
III.C.4. $\quad \mathrm{K}_{\mathrm{sB}}$ for the Drifting Maxwellian ..... 119
III.C.5. Some Estimates and Ordering ..... 123
III.D. Application: Ion Drift Cyclotron Instability ..... 130
III.D.1. Overview ..... 130
III.D.2. IDC Instability with Negligible Shear ..... 135
III.D.3. IDC Instabjility with Moderate sihear ..... 143
III.D.4. Instability with Strong Shear ..... 151
III.E. Further Comments on the Formalism ..... 152
APPENDIX C: DERIVATION OF THE CONDUCTIVITY KERNEL ..... 155
appendix d: EVALUATION OF $\mathrm{S}_{\mathrm{sB}}$ IN THE REGIME $\rho_{S} / L_{\phi}>1$ ..... 160
APPENDIX E: ONE TERM APPROXIMATION TO THE IDC GROWTH RATE ..... 169
SYMBOL TABLES FOR PARTS I, II, AND III ..... 171
REFERENCES ..... 184
FIGURE CAPTIONS ..... 186
FIGURES ..... 191

# EQUILIBRIUM AND STABILITY IN 

 STRONGLY INHDMOGENEOUS PLASMASHarry Elliot Mynick<br>Lawrence Berkeley Laboratory University of Califormia<br>Berkeley, California 94720

## ABSTRACT

In some confinement schemes of current interest, for example in conventional and field-reversed mirrors, and in the Tormac sheath, the parameter $n$, defined as the ratio of ion gyroradius to the plasma scale length $L_{\perp}$ perpendicular to the magnetic field, may take on appreciable values, as large as $1 / 3$. In this work, formalisms appropriate to the study of such strongly inhomogeneous plasmas are developed and applied, for three general aspects of plasma behavior. Each of these three topics is studied using Hamiltonian fomalism, which greatly aids in expressing results in a general and concise manner.

In Part I we study the equilibrium of strongly inhomogeneous, collisionless, slab plasmas, using a generalized version of a formalism previously developed, which permits the generation of self-consistent equilibria, for plasmas with arbitrary magnetic shear, and variation of magnetic field strength. The principal accomplishment of Part I is the development of an interpretive method, which connects the canonical variables, in terms of which the formalism is expressed, to the physical quantities needed to model a given physical system, and which is both
valid and practical, even for cases of strong magnetic inhomogeneity, and in particular for magnetic shear. The method provides a concrete physical content for the canonical formalism, permitting application to problems of interest. We apply the formalism to the particular problem of the Tormac sheath, modeling both the magnetic field and the loss-cone form of the ion distribution function in that region. The interpretive method of Part I is also of considerable use in the formalisms of Parts II and III.

Part II is concerned with the development of a systematic procedure for deriving the form of the guiding-center Hamiltonian $K$, for finite $n$, in an axisymmetric geometry. In the process of obtaining $K$, an expression for the first adiabatic invariant (the gyroaction) is obtained, which generalizes the usual expression $\frac{1}{2} m_{1}{ }^{2} / \Omega_{c}\left(\Omega_{c} \equiv e B / m c\right)$, to finite $n$ and magnetic shear.

In Part III a formalism is developed for the study of the stability of strongly-inhomogeneous, magnetized slab plasmas; it is then applied to the ion-drift-cyclotron instability. We restrict our study to longitudinal perturbations. Use of Hamiltonian formalism and operator techniques permits a very concise expression of the full nonlocal mode equation, in a form valid for arbitrary inhomogeneity, but which transparentiy reduces in the appropriate limit to the weakly inhomogeneous expressions usually used to study stability. Certain new effects of potential significance are found, arising naturally from the mathematics. In particular, we find a generalized expression for the diamagnetic drift
frequency is, of central importance in many plasma instabilities. For nonzers $v_{\|} /\left(\Omega_{c} L_{s}\right)$ (where $v_{\|}$is the parallel velocity of a particle at its guiding center, and $\dot{L}_{s}$ is the shear scale length), $\omega_{k}$ becomes a functicn of position in phase space, with a shift from its unsheared value $\omega_{*}^{0}$ given by $\omega_{k}=\omega_{k}^{0}\left[1+\left(v_{\|} / \Omega_{c} L_{s}\right)\right]^{-1}$.

The formalism is phrased in terms of a variational principle which, in addition to serving as a labor-saving device in parameter regimes where use of a finite-order differential equation is an adequate approximation to the full nonlocal mode equation, also permits the derivation of valid dispersion equations when the problem is fully nonlocal in character.

## ACKNOWLEDGMENTS

I would like to express my thanks to many of the members of the Plasma Physics groups both at Lewrence Berkeley Laboratory and in the Electrical Engineering Department of the University of California at Berkeley, for their technical and emotional support in the time I have been associated with them. Particular thanks are in order for my advisor, Allan Kaufman, whose enthusiasm for the subject, and ability as a physicist and teacher, have greatly enriched my time as a student of plasma physics.

This work was performed umder the auspices of the U. S. Department of Energy.

## INTRODUCTION

In plasmas of current interest, for example in the Tormak sheath region, in mirror machines, and in some Tokamak experiments, the ratio $\eta$ of the ion gyroradius to plasma scale lengths perpendicular to the magnetic field, may be an appreciably large parameter. The meaning of "appreciably large" will be clarified in each of the three parts of this thesis, but generally we mean by this that the inhomogeneity is strong enough to modify the actual form of the equations governing a given aspect of the plasma behavior, rather than entering the equations in only a paranetric fashion. In the latter situation, we shall say that the plasma is "weakly inhomogeneous."

Since in going from weak to strong inhomogeneity one removes a simplifying approximation from the theory, formalisms applicable to strong inhomogeneity will tend to be complicated and cumbersome. Therefore any formal means which makes the formalism more succinct and elegant is desireable. All three areas of study here are accordingly treated making maximal use of Haniltonian formalism, in terms of which the equations governing different plasma geometries tend to look the same, and those goveming the strongly inhomogeneous case have a transparent formal resemblance to their weakly inhomogeneous limits.

In the first part of this thesis we study the equilibrium of strongly inhomogeneous Vlasov slab plasmas, extending and clarifying the work of P. Channell. ${ }^{1}$ In Part II we develop a canonical formulation of guiding center motion valid for particles having appreciable n
('finite gyroradius'), moving in axisymetric geometries. Part Ill is concemed with the development and application of a fomalism appropriate to the systematic study of the effects on plasma stabiIity of strong inhomogeneity.

Each of these three parts has its own introdurtory section, in which the work of that section is described in greaier detail. The parts are presented in an order which allows for a natural progression in the development of ideas and formalism, and later pirts do make some reference to the preceding ones. Nevertheless the areas of study are distinct, and any of the three parts should be comprehensible without having to read the other two.

PART 1. VLASOV SLAB EQUILIBRIA: INTERPRETATION AND PRACTICAL APPLICATION OF THE CHANNEL FORMALISM

## I.A. Introduction To Part I

The work of P . Channell ${ }^{1}$ enables one to generate self-consistent, exact Vlasov equilibria in a slab geometry (i.e., a geometry with a single dirction of inhomogeneity, which we take to be the $x$ direction), allowing for arbitrary magnetic shear, as well as arbitrary variation in magnetic field strength $B \equiv|\underline{B}|$ and in plasma density, the scale lengths of these quantities ( $L_{S}, L_{B}$, and $L_{n}$, respectively) being limited only by the self-consistency of the equilibria generated. The possibility of generating such equilibria is particularly interesting for plasmas in which the ratio $n$ of the ion gyroradius $\rho_{i}$ to the magnetic scale length $L_{B} \equiv \min \left(L_{B}, L_{S}\right)$ is not a negligibly small parameter. For such plasmas, the ion orbits become complicated, distorted from the helical form valid in the small $n$ regime. Accordingly, calculations of plasma quantities such as density, pressure, and magnetic field variation, using the more elementary approaches adequate for small $n$, become of questionable validity.
in the Channell approach, from writing out the $x$-component of the pressure balance equation in terms of the constants of the motion ( $H, P_{y}, P_{z}$ ) (the single-particle energy, and the $y$ and $z$ components of canonical momentum), one obtains an equation for the development in $x$ of the vector potential $\underline{A}=\left(0, A_{y}, A_{z}\right)$, and the scalar potential $\Phi$, formally the same as the equation of motion of a particle in a three
dimensional space ( $(\wedge, \neq$ ), propagating in pseudo-time $x$, in an effective potential $U(\underline{A}, \Phi)$. (U tums out to be a constant times the $x x$-component $\Pi_{x x}$ of the pressure tensor $\Pi_{\text {. }}$ ) Imposing quasineutrality on this system enables one to eliminate one degree of freedom (say ©), and so have a two dimensional particle motion problem.

Channell's work did not allow for a scalar potential, and his formalism was derived only for a highly restricted class of distribution functions $f_{s}\left(H, P_{y}, P_{z}\right)$. H. Grad showed ${ }^{2}$ that, for the one dimensional pseudo-motion problem of an unsheared magnetic field, the same formalism holds, for arbitrary choice of $f_{s}$. Recently, A. N. Kaufman showed ${ }^{3}$ that one can choose $f_{s}$ arbitrarily for the sheared (2-D) problem Channell studied.

For the Channell formalism to be useful, one must be able to design the canonical form $f_{s}\left(H, P_{y}, P_{z}\right)$ of the distribution function to model systems of physical interest. Physically, one may be given a specification of the desired magnetic field $\underline{\mathbb{B}}(x)$, and so the reference trajectory $\underline{\tilde{A}}(x)$ which yields that $\underline{\widetilde{B}}$, and a general description of the physical form $f_{s}\left(X_{0}, v_{1}^{2}, v_{\|}\right)\left(X_{0} \equiv\right.$ guiding center position) of the distribution function (e.g., loss cone, with a prescribed density varıation with $X_{0}$ ) to be modeled. The present work establishes the connections between these physical quantities, and the carsnical quantities ( $\mathrm{H}_{2}, \mathrm{P}_{\mathrm{y}}, \mathrm{P}_{2}, \mathrm{f}_{\mathrm{s}}$,U) of the Channell formalism.

We begin in Sec. I.B by deriving the Channell formalism in its generalized form. In Sec. I.C, methods of interpreting the formalism are developed and applied. Section I.C.l discusses the connection
between the canonical variables $\left(H, P_{y}, P_{z}\right)$ and the physical ones ( $X_{0}, v_{1}$, $v_{\|}$). Graphical methods are developed which facilitate making this connection, and which enable one to easily and rapidly extract a large amount of information, even in situations where the corresponding algebraic expressions become analytically intractable. These methods are extended in Secs. I.C. 2 to I.C.4. In Sec. J.C. 2 we develop a prescription for determining $f_{s}$, given $f_{s}$ and $\tilde{A}$. Section I.C. 3 presents an explicit application of this procedure, modeling the Tormac sheath region, in which $\widetilde{f}_{s}$ has a loss-cone form, and where $\underline{\widetilde{A}}(x)$ gives a sheared magnetic field.

Having detemined $f_{s}$, in Sec. I.C. 4 we establish the connection between the topologies of $f_{s}$ and of $U$. The discussion is largely qualitative, again aided by graphical visualization of the algebraic statement5. This sort of analysis is continued in Sec. I.C.5, in which a remaining question of self-consistency is answered, tilereby showing the validity of the prescription for choosing $f_{s}$, laid down in Sec. I.C.2.

Quantitative support for the qualitatively-determined conclusions reached in Secs. I.C. 2 to I.C. 5 is given in Sec. I.C.6, and in Appendix A. Appendix A carries out in an analytic fashion an evaluation of $U$, given $f_{s}$, and arrives at a form having the same features as expected from the more qualitative analjsis of Sec. I.C.4. Section I.C. 6 displays the results of numerical adaptation by W. M. Sharp of the methods developed in the foregoing sections. The results bear out the conclusions and expectations developed in those sections.

## I.B. Generalized Channell Formalism

As described in the introduction to Part I, we consider a slab geometry, with $x$ the sole inhomogeneity direction. The magnetic field $\underline{B}(x)=\left(0, B_{y}(x), B_{z}(x)\right)$, obtained from the vector potential $\underline{A}(x) \equiv(0$, $\left.A_{y}(x), A_{z}(x)\right)$ through the relation

$$
\begin{equation*}
\underline{B}=\nabla \times \underline{A}=8 \times \mathrm{d} \underline{\mathrm{~A}} / \mathrm{dx}, \tag{1-1}
\end{equation*}
$$

has at this point arbitrary variation with $x$, as does the electrostatic potential $\Phi(x)$. (Their x dependence will be determined self-consistently from the Channell formalism.)

The pressure balance equation for species $s$ is

$$
\begin{equation*}
\underline{\nabla} \cdot \underline{\pi}^{s}=c^{-1} \dot{j}_{s} \times \underline{B}-\rho_{s} \underline{\nabla} \Phi . \tag{1-2}
\end{equation*}
$$

In slab geometry, its only nontrivial component is the x -component:

$$
\begin{equation*}
\partial \Pi_{x x}^{s} / \partial x=c^{-1} \hat{x} \cdot\left(j_{s} \times \underline{B}\right)-p_{s} \partial \phi / \partial x . \tag{I-3}
\end{equation*}
$$

Writing out $\pi_{x x}^{s}$ in terms of canonical variables, we will see shortly [cf. Eq. (I-8)] that it is a furctional of $A$ and $\Phi_{\text {, }}$ its $x$ dependence entering only through these functions of $x$. The canonical momentum $p \equiv\left(p_{x}, p_{y}, p_{z}\right)$ has $y$ and $z$ components which are constants of the motion, $\left(0, P_{y}, P_{z}\right) \equiv\left(0, p_{y}, P_{z}\right) \equiv \underline{P}=$ constant. A third constant of the motion is the single-particle Hamiltonian $H_{s}$, given by

$$
\begin{equation*}
H_{s}=p_{x}^{2} / 2 m_{s}+v_{s}(x \mid \underline{P}), \tag{I-4}
\end{equation*}
$$

where

$$
\begin{equation*}
V_{s}(x \mid \underline{P}) \equiv\left[\underline{P}-\left(e_{s} / c\right) \underline{A}(x)\right]^{2} / 2 m_{s}+e_{s} \Phi(x) \tag{I-5}
\end{equation*}
$$

is the effective $1-D$ potential in which a particle with momentum $\underline{P}$ oscillates.

With the distribution function $f_{s}$ written in terms of these three constarits, $\Pi_{X X}^{s}$ appears as

$$
\begin{equation*}
\pi_{x x}^{s} \equiv m_{s}^{-1} \int d^{3} p i_{x}^{2} f_{s}\left(H_{s}, P\right) \tag{I-6}
\end{equation*}
$$

We now define $h_{s}$ by

$$
\begin{equation*}
2 m_{s} h_{s} \equiv p_{x}^{2}=2 m_{s}\left(H_{s}-V_{s}\right)=2 m_{s}\left(\mu_{s}-e_{s} \Phi\right)-\left(\underline{P}-\left(e_{s} / c\right) A\right)^{2} \tag{I-7}
\end{equation*}
$$

and change the variables of integration in $工 q$. (I-6) from $p$ to $\left(h_{s}, \underline{P}\right.$ ), obtaining

$$
\begin{equation*}
\Pi_{x x}^{s}(\underline{A}, \phi)=2 \int d \underline{P} \int_{0}^{\infty} d h_{s}\left(2 m_{s} h_{s}\right)^{1 / 2} f_{s}\left\{H_{s}=h_{s}+e_{s} \phi+\left[\underline{P}-\left(e_{s} / c\right) \underline{A}\right]^{2 / 2 m} s, \underline{P}\right\} \tag{I-8}
\end{equation*}
$$

From this form of $\Pi_{x x}^{S}$, we see that we can write

$$
\begin{align*}
& \partial_{x} \Pi_{\mathrm{Xx}}^{\mathrm{S}}(\underline{A}, \Phi)=(\partial \underline{A} / \partial \mathrm{x}) \cdot\left(\partial \Pi_{\mathrm{xx}}^{\mathrm{S}} / \partial \underline{A}\right)+(\partial \Phi / \partial \mathrm{x})\left(\partial \Pi_{\mathrm{xx}}^{\mathrm{S}} / \partial \Phi\right)  \tag{I-9}\\
& \left.=\hat{x} \cdot\left[\left(\partial \Pi_{X \sim}^{S} / \partial A\right) \times \underline{B}\right]+(\partial \Phi / \partial x)\left(\partial \Pi_{X X}^{S} / \partial \Phi\right)\right] .
\end{align*}
$$

Putting this into ( $\mathrm{I}-3$ ) and identifying the functions accompanying the $\underline{B}$ and $\Phi$ terms on the right and left hand sides, we find

$$
\begin{equation*}
\partial \pi_{\mathrm{xx}}^{\mathrm{s}} / \partial \underline{A}=\mathrm{c}^{-1} \dot{j}_{\mathrm{s}} \tag{I-10}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial \Pi_{\mathrm{xx}}^{\mathrm{S}} / \partial \Phi=-\rho_{\mathrm{s}} \tag{I-11}
\end{equation*}
$$

Using Eqs. ( $\mathrm{I}-10$ ) and ( $\mathrm{I}-11$ ) in Ampere's and Poisson's equations, respectively, gives

$$
\begin{equation*}
d^{2} \underline{A} / d X^{2}=-\hat{A}_{n} c^{-1} \dot{L}=-(\partial / \partial \underline{A}) U(\underline{A}, \dot{W}) \tag{1-12}
\end{equation*}
$$

and

$$
\begin{equation*}
d^{2} \Phi / d x^{2}=-4 \pi \rho=-(\partial / \partial \phi) U(\underline{A}, \Phi) \tag{1-13}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{U}\left(\mathrm{~A}, \Phi^{J} \equiv A \pi \sum_{\mathrm{S}} \Pi_{\mathrm{XX}}^{\mathrm{S}}(\mathrm{~A}, \phi)\right. \tag{1-14}
\end{equation*}
$$

Equations (I-12) and (I-13) are a ciosed set of equations for exact, self-consistent equilibria, i.e., they embody the generalized Channell formalism. We may obtain a somewhat simpler version, if we replace (1-13) by the quasineutrality condition,

$$
\begin{equation*}
0=-4 \pi \rho=(\partial / \partial \Phi) \cup(A, \phi) \tag{I-15}
\end{equation*}
$$

One may solve this for $\Phi(A)$, and combine this with (I-12) to obtain

$$
\begin{equation*}
d^{2} \underline{A} / d x^{2}=-(d / d \underline{A}) \cup[\underline{A}, \Phi(\underline{A})] \tag{I-16}
\end{equation*}
$$

Equation (1-16) describes the motion of a pseudoparticle moving in a 2 -dimensional pseudospace $A$, evolving in pseudotine $x$. It is the form of the Channell formalism we shall work with here.

## I.C. Interpretive Methods

We want to develop an understanding of the connections between the canonical variables ( $e . g ., \underline{p}^{p}, H_{S}, f_{s}, U$ ) in terms of which the Channeil formalism is couched, and the physical variables (e.g., $X_{0}, v_{1}^{2}\left(X_{0}\right), v_{\|}\left(X_{0}\right)$, $\tilde{\mathbf{f}}_{s}$ ) in terms of which a system to be modeled is specified. Here we describe a simple graphical method to aid one in making these connections.

Using this approach, much of the relevant canonical and physical information can be visually ascertained at a glance. To each graphically obtained piece of information, there is a corresponding algebraic statement. The graphical results are in principle as precise as the algebraic ones, and enable one to make qualitative statements about the relations among the various quantities in the theory, even in situations where the algebraic counterparts become mathematically intractable. The technique will also be useful in Parts II and III.

## 1.C.1. Connecting the Physical and Canonical Variables

We begin by noting that $A$ makes its appearance in the theory only in the combination ( $\left.\underline{P}-\left(e_{s} / c\right) \underline{A}\right)$. Because of this, it will prove convenient to visualize all quantities over a single $\underline{P}$ or $A$ plane, where $A \equiv\left(e_{s} / c\right)$ A. The Channell formalism becones useful in the regime of appreciable gyroradius to scale length, where insuring self-consistency of an equilibrium becomes difficult. Therefore, in the following discussion, we shall be considering the ion species, $e_{s}=e_{i}=e$. Though less useful, the formalism is equally valid for the electrons, however, and everything that will be said here concerning the ions can be applied (with appropriate changes of signs and of mass) to the electrons. This is elaborated upon at the end of Sec. I.C.4.

Suppose we are given the physical information to be modeled, $\widetilde{f}_{s}\left(X_{0}, v_{1}^{2}, v_{\|}\right)$and the desired magnetic field $\underline{B}(x)$, through the reference trajectory $\underset{\mathbb{A}}{( }(x)$. We want to design a canonical distribution $f_{s}(H, \underline{P})$ which conforms to these physical requirements as closely as possible. We first draw in $\underset{A}{A}(x)$, as in Fig. 1. Using Eq. (I-1), we can then draw in the corresponding magnetic field at various points $x_{1}, x_{2}, x_{3} \ldots$
along the trajectory. We have chosen $\tilde{A}(x)$ to model Tormac, the elbow in $\tilde{A}$ yielding the shear in $\underline{\mathbb{B}}$ which occurs across the sheath region.

Now suppose we have found the self-consistent magnetic field $\underline{B}^{S C}(x)$ and its $\underline{A}^{S C}(x)$. (We are designing things so that $\underline{A}^{S C}$ lies as near $\tilde{A}$ as possible). We can make a plot analogous to Fig. 1 for $\underline{A}^{S C}$. Given $A^{S C}$, the canonical variables ( $H, \underline{p}$ ) acquire a definite physical significance. A natural definition of a particle's guiding center position $X_{0}$ is the bottom of the well $V(x \mid \underline{P})$ in which it is sloshing, i.e., we define $X_{0}$ by

$$
\begin{equation*}
0=\left.(\partial / \partial x) V(x \mid \underline{P})\right|_{x=x_{c}} \tag{I-17}
\end{equation*}
$$

One may in principle solve this for $X_{o}(P)$. In practice, this will in general be analytically intractable, so one may have to solve numerically to obtain explicit results. A graphical visualization of (1-17) is, however, mach simpler. We refer to Fig. 2, which neglects eФ in Eq. (I-5). Neglect of e $\Phi$ gives a $V$ which is simpler to analyse graphically (seiting $c=m_{i}=1$,

$$
\begin{equation*}
V(x \mid \underline{P})=\frac{1}{2}\left[\underline{P}-\underline{A}^{S C}(x)\right]^{2} . \tag{I-18}
\end{equation*}
$$

In Tomac or mirror machines, where $\Phi$ comes from the escape of electrons along field lines, $e \Phi \sim T_{e}$, and so for a hot-ion plasma, $e \Phi \sim T_{e} \ll T_{i}$ $\sim \frac{1}{2}(\mathrm{P}-\mathrm{A})^{2}$. Neglect of $e \Phi$ is thus a physically valid approxination in these cases, in addition to aiding pedagogic clarity.

From (I-18), we see that $V(x \mid \underline{P})$ is just half the square of the distance, in the $\underline{P}-\underline{A}$ plane, from the point $\underline{P}$ which characterize a
given particle, to the point $A^{\text {SC }}(x)$ on the trajectory. Exerema of this distance, i.e., points $\underset{A}{ }(x)$ at which Eq. (I-17) is satisfied, occur when the line passing from $\underline{P}$ to $A^{S C}(x)$ is nommal to the trajectory $A^{S C}$ at $A^{S C}(x)$. Therefore all points $\underline{P}$ lying along this line have the same guidirg center position. This is illustrated in Fig. 2. All points $\underline{P}$ on the lines labeled 1,2 , and 3 have as their guiding-center position $x_{1}, x_{2}$, and $x_{3}$, respectively.

We have drawn Fig. 2 so that a certain ambiguity becomes apparent; some values of P lie on more than one $\mathrm{X}_{\mathrm{o}}$ line, such as point a. This corresponds to the potential $V(x)$ having more than one extremum. For the trajectory $A^{\text {SC }}(\mathrm{x})$ considered here, a little thought shows that $\mathrm{V}(\mathrm{x} \mid \underline{\mathrm{P}})$ has either two minima and one maximum, as for point a, or a single miniman and no maxima, as for points $b$ and $c$. The former situation is illustrated in Fig. 3. Particles with energy $\mathrm{H}=\mathrm{H}_{2}$ may be trapped in either of the two wells, and so the canonical distribution $f_{s}(H, \underline{P})$ must in general be double-valued for such ( $\mathrm{H}, \mathrm{P}$ ).

Given $X_{0}(\underline{P})$, we are in a position to use Fig. 2 to read off $\mathrm{v}_{\text {II }}\left(\mathrm{X}_{\mathrm{o}}\right)$ for particles havins a given $\underline{P}$. We introduce the x -dependent coordinate system with right handed triad ( $\hat{b}_{1}, \hat{b}_{2}, \hat{b}_{3}$ ), where $\hat{b}_{1} \equiv \hat{x}$, $\hat{b}_{3} \equiv \hat{B}(x)$ points along the magnetic field at $x$, and $\hat{b}_{2} \equiv \hat{b}_{3} \times \hat{b}_{1}$ forms the third orthogonal wit vector. At a given point $x=X_{0}$, Hamilton's equations tell us that the parallel velocity is

$$
\begin{equation*}
v_{\| I}\left(X_{0}, \underline{P}\right)=b_{3}\left(X_{0}\right) \cdot\left[\underline{P}-A\left(X_{0} B \equiv\left(P_{3}-A_{3}\right)\left(X_{0}\right)\right.\right. \tag{I-19}
\end{equation*}
$$

Graphically, $v_{\|}$is just the distance along the constant $X_{o}$ line from the trajectory $\underline{A}^{S C}$ to the point $\underline{p}$ in question, as shown for point $c$ in Fig. 2 . Since $X_{0}=X_{o}(\underline{P})$, we may write $v_{\mathrm{li}}=v_{i l}(\underline{P})$.

The final piece of physical information, $v_{1}^{2}\left(X_{o}\right)$, may now be obtained using conservation of energy:

$$
H=\frac{1}{2}\left[v_{1}^{2}\left(x_{o}\right)+v_{\| l}^{2}\left(x_{0}\right)\right]+e \Phi\left(x_{0}\right),
$$

and thus

$$
\begin{equation*}
v_{1}^{2}\left(X_{0}\right) \equiv v_{1}^{2}\left(X_{0}\right)+v_{2}^{2}\left(X_{0}\right)=2\left[H-e \Phi\left(X_{0}\right)\right]-v_{1}^{2}\left(X_{0}\right) \tag{I-20}
\end{equation*}
$$

We see that $v_{\perp}^{2}$ is a function of $P$ and $H$.
To clarify these ideas, we consider a simple illustration, viz., a magnetic field $\underline{B}(x)$ with constant modulus $B_{o}$ and shear length $L_{S} \equiv \kappa_{S}{ }^{-1}$; $\underline{B}(x)=B_{0}\left(\hat{y} \sin k_{S} x+\hat{z} \cos \kappa_{s} x\right)$. This field is given by the circular nseudotraiectory with radius $A_{0} \equiv \rho R_{0} I_{s}$,

$$
\underline{A}(x)=A_{0}\left(\hat{y} \sin k_{s} x+\hat{z} \cos k_{s} x\right)=\operatorname{el}_{s} \underline{B}(x)
$$

as shown in Fig. 4. For any point $\underline{P}=P\left(\hat{y} \sin \theta_{p}+\hat{z} \cos \theta_{p}\right),(P \geqslant 0)$, the effective potential $V(x \mid \underline{P})=\frac{1}{2}[\underline{P}-\underline{A}(x)]^{2}$ is then given by

$$
V(x \mid P)=\frac{1}{2}\left(P^{2}+A_{o}^{2}\right)-P A_{o} \cos \left(\kappa_{s} x-\theta_{p}\right)
$$

This has an infinite number of evenly spaced minima $X_{o n}$, satisfying $\kappa_{\mathrm{s}} X_{o n}=\theta_{\mathrm{p}}+2 \pi n(\mathrm{n}=0, \pm 1, \pm 2, \ldots)$. The parallel velocity at any of these guiding center positions $X_{o n}$ is $v_{\|}\left(X_{0}\right)=P-A_{0}$. The $X_{o}=$ constant lines in the $\underline{P}$ plane project radially outward from the origin $\underline{P}=0$.

The point $\underline{P}=0$ has $P=0$, and so $V(x \mid P)=\frac{1}{2} A_{0}^{2}$, independent of $x$. All other points $\underline{P}$ have a sinusoidally varying $V(x)$.
1.C.2. Obtaining $f_{s}$ from $\tilde{f}_{s}$

Now that we know how to move between the canonical variables ( $H, \underline{P}$ ) and the physical ones $\mid x_{0}, r_{1}^{2}\left(x_{0}\right), r_{\|}\left(x_{0} \mid\right)$, we are in a position to consider what $f_{S}(H, P)$ to write down to give the desired physical distribution $\vec{f}_{\mathrm{s}}$. We visualize $\mathrm{f}_{\mathrm{S}}$ over the same $\underline{P}$ - plane as in Figs. 1 and 2, with the direction nomal to this plane serving as the H - axis. Given $A^{\text {SC }}(\mathrm{x})$, the physical distribution $\tilde{f}_{\mathrm{S}}$ can be determined from $f_{s}$, using the interpretive methods just described. However, one is not given $A^{S C}$ from the outset (this is determined self-consistently from Eq. (I-16), which requires a knowledge of $f_{S}$ to determine $U$ ), but rather the reference trajectory $\tilde{\mathbb{A}}$. We thus proceed as follows. Treating $\underset{A}{ }$ as though it were the real trajectory $\underline{A}^{\text {SC }}$, we determine the functions $\widetilde{X}_{o}(\underline{P})$, $\tilde{v}_{\|}(\mathrm{P})$, and $\tilde{v}_{\perp}^{2}(\mathrm{H}, \underline{\mathrm{P}})$ in the same manner as $\mathrm{X}_{0}, v_{\| \mid}$and $v_{\perp}$ are determined if $A^{\text {SC }}$ is a known trajectory in the $\underline{P}$ plane. The functions $\tilde{X}_{0}$ and $\tilde{x}_{\|}$provide a new coordinate system (in certain regions double-valued) of the $\underline{P}$ plane. The basic prescription for writing down $f_{s}$ from $\tilde{f}_{s}$ is then to replace the argunents $X_{0}, v_{1}^{2}, v_{\|}$in $\tilde{f}_{s}$ with these new functions $\tilde{X}_{0}, \tilde{v}_{1}^{2}, \tilde{v}_{\|}$of the canonical variables $\mathrm{H}, \underline{\mathrm{P}}$ :

$$
\begin{equation*}
f_{s}(H, \underline{P}) \equiv\left|\Omega T^{-1}\right| \tilde{f}_{s}\left[\tilde{x}_{0}(P), \tilde{Y}_{\perp}^{2}(H, \underline{P}), \tilde{v}_{\| \mid}(\underline{P})\right] . \tag{I-21}
\end{equation*}
$$

Two points in Eq. (1-21) require elaboration. The first is the presence of the factor $\mid \Omega I^{-1}$, which adjusts between the normalizations of $f_{s}$ and $\tilde{f}_{s}$. Here $\Omega \equiv \theta \equiv \partial H / \partial J$ is the gyrofrequency of a particle
with momenta $\underline{P}$ and gyroaction $J$, and $I$ is the Jacobian $I \equiv \partial\left(P_{y}, P_{z}\right) /$ $\partial\left(\tilde{X}_{0}, \tilde{v}_{\|}\right)$of our two parametrizations of the $\underline{P}$ plane. The distribution $\tilde{f}_{S}$ is normalized so that

$$
\begin{equation*}
n_{s}\left(X_{0}\right)=\iint d v_{\|} d^{2} \underline{v}_{1} \tilde{f}_{s}=\iint d v_{\|} d v_{1}^{2} \widetilde{\mathrm{r}}_{s}\left(X_{0}, v_{1}^{2}, v_{\|}\right) \tag{I-22}
\end{equation*}
$$

where $n_{s}$ is the density of guiding centers at $X_{0}$.
Distribution $f_{s}$ is nomalized over the canonical phase space $\underline{Z} \equiv\left(\theta, J ; Y, P_{y} ; Z, P_{z}\right)$. (Here J and $O$ are the canonical gyroaction and gyrophase, and $Y$ and $Z$, the variables conjugate to $P_{y}$ and $P_{z}$, are the guiding center $y$ and $z$ coordinates.)

Thus

$$
\mathrm{d} \underline{Z}=\mathrm{d} \theta \mathrm{~d} J \mathrm{~d} \underline{P} \mathrm{~d} Y \mathrm{~d} Z=\left(\Omega^{-1} I\right) \mathrm{d} \Theta \mathrm{~d} H \mathrm{dv}_{\|} \mathrm{d}^{3} \underline{X},
$$

where $d^{3} \underline{X} \equiv d \tilde{X}_{0} d Y d z$ is a volume element. Using ( $1-20$ ), we have

$$
\mathrm{dH} \tilde{\mathrm{~d}}_{\|}=\tilde{\mathrm{d}}_{1}^{2} \mathrm{~d} \tilde{v}_{\|} \partial\left(\mathrm{H}, \tilde{v}_{\|}\right) / \partial\left(\tilde{v}_{1}^{2}, \tilde{v}_{\|}\right)=\frac{1}{2} \mathrm{~d} \tilde{v}_{1}^{2} \mathrm{~d} \tilde{v}_{\|},
$$

and so the integral over velocity space analogous to (1-22) gives

$$
\begin{equation*}
n_{s}\left(\tilde{X}_{o}\right)=\pi \int \tilde{d_{i}} \tilde{v}_{v 1}\left|\Omega^{-1} I\right| f_{s}, \tag{I-2j}
\end{equation*}
$$

and Eq. (I-21) follows.
The second point concerning Eq. (I-21) which requires clarification is that, for a given $\mathrm{P}, \mathrm{H}$ is always greater than or equal to the value $H_{\min }(P) \equiv V\left(X_{o} \mid \underline{P}\right)$, and so it is only over this restricted, "physical region" of the $(H, P)$ - space that $f_{s}$ has any physical meaning. Inside the physical region, we must have

$$
\begin{equation*}
f_{s} \geqslant 0 \tag{I-24}
\end{equation*}
$$

and we take $f_{s}$ there to be given by Eq. (I-21), unless (I-21) violates condition (I-24). In this case we may take $f_{s}=0$.

Outside the physical region the value of $f_{s}$ is in principle irrelevant, and so one may assign $f_{s}$ any convenient value there. However, the physical region is determined from the self-consistent trajectory $A^{\prime}(A)$ or $A^{-2}(x)$, which is not known when writing down the distribution $f_{s}$. What is known is the reference trajectory $\tilde{A}(x)$, which hopefully lies close to $\underline{A}^{S C}(x)$, but in general not precisely on it. Therefore we require condition (I-24) to hold not only in the region which would be physical if $\underline{A}^{S C}=\tilde{A}$, but also in a sufficiently large neighborhood about this "quasi-physical" region that (I-24) will hold in the true physical region as well. A convenient choice which sutisfies these criteria is to tal.e $f_{s}$ from (I-21) in the quasi-physical region, to have it go rapidly to zero at this region's boundaries, and remain zero elsewhere in the $(H, \underline{P})$ space.

## I.C.3. Explicit Example: Loss Cone Distribution

We now illustrate these ideas more concretely. We denote by $\mathrm{H}^{\prime}$ the particle kinetic energy at $X_{o}, H^{\prime} \equiv H-e \Phi\left(X_{0}\right)=\frac{1}{2}\left[v_{1}^{2}\left(X_{0}\right)+v_{\|}^{2}\left(X_{0}\right)\right]$ $=h\left(X_{0}\right)+\frac{1}{2}\left[\underline{P}-\underline{A}\left(X_{0}\right)\right]^{2}$. Analogously, we denote by $\tilde{H}^{\prime}$ the kinetic energy the particle would have if $\underline{A}^{5 C}$ lay precisely on $\tilde{\tilde{A}}, \tilde{H}, \equiv H-e \Phi\left(X_{0}\right)$ $\equiv \frac{1}{2}\left[\tilde{v}_{1}^{2}\left(X_{0}\right)+\vec{v}_{\|}^{2}\left(X_{0}\right)\right]$. Since the reference potentials $\tilde{A}, \tilde{\Phi}$ are known functions, $\tilde{H}^{\prime}$ is a function of only the canonical variables ( $\mathrm{H}, \underline{\mathrm{P}}$ ).

We begin by visualizing the locus of the physical and quasiphysical domains in the ( $\mathrm{H}, \mathrm{P}$ ) space. As in Sec. I.C.1, we neglect e $\phi$ in comparison with $H$ in this visualization, for simplicity. Including the
effect of $\mathrm{e} \Phi$ would distort the sketches somenhat from those shown (for example, the line $X_{o}=$ constant in Fig. 4a would no longer be exactly straight, and the domain boundaries of Fig. 4 b would no longer be exactly parabolic), but the basic features would remain.

In Fig. 5a we draw the $\underline{P}$ plane of the ( $\mathrm{H}, \mathrm{P}$ ) space, showing the self-consistent and nearby refei snce trajectories $A_{-}^{S C}$ and $\underset{A}{\tilde{A}}$, a single $X_{0}=$ constant line, and a number of sample points $A_{-j}$ on and off the trajectories.

The pseudopotential $U(\underline{A})$ defined by Eqs. (I-8) and (I-14) is valid for all $A$ (or $A)$. The lower limit $\tilde{H}_{\min }^{\prime} \simeq H_{\min } \simeq \frac{1}{Z}(\underline{P}-\underline{A})^{2}$. In Fig. 5b we draw the integration domains for each of the $A_{j}$ of Fig. 5 a , in the slice of the ( $H, \underline{P}$ ) space having $X_{o}=$ constant. The boundiaries are 1 abeled by the corresponding $j$. The area above the 0 -boundary is the physical region, and above the 1 -boundary is the quasiphysical region.

Now we specify a distribution function. In the quasiphysical region, we shall take $f_{s}$ to be of the general form

$$
\begin{equation*}
f(H, \underline{P})=\sum_{j=0}^{N} g_{j}(\underline{P}) \exp \left(-\gamma_{j} H\right) . \tag{I-25}
\end{equation*}
$$

In his work ${ }^{1}$, Channell assumed a special case of the form (I-25) throughout the ( $H, \underline{p}$ ) space, namely $f=e^{-\gamma H} \mathrm{~g}_{(\underline{P})}$. Here we will be modeling a loss-cone type distribution, such as exists in the Tormac sheath. We assume a substracted Maxwellian form. We shall thus take $N=1$ in (I-25), with $g_{0}$ positive, and $g_{1}$ negative. As a result, $f$ will be positive in the quasiphysical region, going to zero at the quasiphysical boundary. Use of form (I-25) would make $f$ negative outside the quasiphysical domain,
in violation of (I-24). We accordingly take $f=0$ outside the quasiphysical region.

For the physical form, $\tilde{f}_{L C}$, of our distribution function, we write

$$
\begin{equation*}
f_{L C}\left(x_{0}, r_{1}^{2}, r_{\|}\right)=n e^{-\gamma_{n} \gamma_{i l}^{2} / 2}\left[\bar{g}_{0} e^{-\gamma_{1} \gamma_{1}^{2} / 2}-\bar{g}_{1} e^{-R \gamma_{1} v_{1}^{2} / 2}\right] \tag{1-26}
\end{equation*}
$$

Here $n=n\left(X_{0}\right)$ is the guiding center density, and $R=R\left(X_{0}\right) \geqslant 1$ gives the loss cone width at $X_{o}, R \rightarrow \infty$ corresponding to a vanishingly narrow loss conc. In Tormac, we expect $R$ to vary from $\infty$ in the interior, dropping as one moves across the sheath to some value $R_{0} \geq 1$, given by the sheath mirror ratio. The normalization condition (1-22) requires

$$
\begin{equation*}
1=\left(2 \pi / \gamma_{\|}\right)^{1 / 2}\left(2 \pi / \gamma_{\perp}\right)\left[\bar{g}_{0}-\bar{g}_{1} / R\right] \tag{I-27}
\end{equation*}
$$

For simplicity we shall take $\gamma_{\|}=\gamma_{1} \equiv \gamma$, and $\bar{g}_{\mathrm{O}}=\bar{g}_{1}$. In preparation for writing down $f_{L C}(H, \underline{P})$, we write (I-26) in the more nearly canonical form

$$
\begin{equation*}
\tilde{f}_{L C}=n \bar{g}_{o}\left(e^{-\gamma H^{\prime}}-e^{-R \gamma I^{\prime}} e^{(R-1) \gamma_{\|}^{2} / 2}\right) \tag{I-28}
\end{equation*}
$$

Finally, we apply Eq. (I-21) and the guidelines discussed subsequently, to define $f_{\mathrm{JC}}$ :

$$
f_{L C}(H, \underline{p})=\max \left\{\begin{array}{l}
\left|\Omega I^{-1}\right| n \bar{g}_{0}\left(e^{-\tilde{\gamma} H^{\prime}}-e^{-R \gamma \tilde{H}^{\prime}} e^{(R-1) \gamma \tilde{v}_{\|}^{2} / 2}\right) .  \tag{I-29}\\
0
\end{array}\right.
$$

Here $n, R$, and $\tilde{v}_{\|}^{2}$ are all evaluated at $\tilde{X}_{o}(P) . f_{L O}$ is of the form of (I-25) in the quasiphysical region, with $\gamma_{0}=\gamma, \gamma_{1}=R \gamma$, and $g_{0}$ and $g_{1}$ given by

$$
\begin{aligned}
& g_{0}(P)=\left|\Omega I^{-1}\right| n \vec{g}_{0} e^{\gamma \operatorname{ve\Phi }\left(\tilde{x}_{0}\right)} \\
& \left.g_{1}(\underline{P})=-\left|\Omega I^{-1}\right| n \bar{g}_{0} e^{\operatorname{Rre\phi }\left(\tilde{x}_{0}\right)} e^{(R-1) \cdot \tilde{v}_{\|}^{2} / 2}\right)
\end{aligned}
$$

in the region where $f_{L C}>0$, and $g_{0}=g_{1}=0$ elsewhere.
Using the relation $\tilde{H}^{\prime}=\frac{1}{2}\left(\tilde{v}_{\|}^{2}+\tilde{v}_{1}^{\prime}\right)$, from (I-26) or (I-29) one sees that $f_{L C}$ passes through zero on the curve $\tilde{H}^{\prime}=\tilde{i}_{\|}^{2} / 2$. The loss-cone boundary lies, roughly at $\tilde{v}_{1}^{2}=(R-1)^{-1} \tilde{v}_{\|}^{2}$ or $\tilde{H}^{\prime}=\left(1-R^{-1}\right)^{-1} \tilde{v}_{\|}^{2}$. In Fig. 6 we sketch the contours of $f_{L C}=$ constant, in an $\tilde{X}_{0}=$ constant slice of the $(\tilde{H}, \quad \underline{P})$ space.

## I.C.4. Obtaining $U$ from $f_{S}$

From Eqs. (I-8) and (I-14), we write $U_{S}(A)$, the contribution to $\mathrm{U}(\mathrm{A})$ from species s .

$$
\begin{align*}
& U_{s}(A)=\int d \underline{P} F_{s}(\underline{P}, A), \quad \text { where } \\
& F_{s}(\underline{P}, \underline{A}) \equiv 8 \pi \int_{0}^{\infty} \mathrm{dh}(2 h)^{1 / 2} f_{s}\left[H=h+\frac{1}{2}(\underline{P}-\underline{A})^{2}+e \Phi, \underline{P}\right] \tag{1-30}
\end{align*}
$$

Here we are again setting $m_{s}=c=1$, and suppressing the $\Phi$ dependence in $F_{s}(\underline{P}, A)$, it being understood that we will let $\Phi=\Phi(A)$ afterwards.) In the form written, $U_{s}$ appears as a simple integral transform of the $h$ integrated distribution function. Using (I-25) for $f_{s}$, we have

$$
\begin{equation*}
F_{s}(P, A)=C_{1} \sum_{j} \gamma_{j}^{-3 / 2} e^{-\gamma_{j} e \Phi} e^{-\gamma_{j}(P-A)^{2} / 2} g_{j}(P) \tag{1-31}
\end{equation*}
$$

where

$$
C_{1} \equiv 8 \pi \sqrt{2} \int_{0}^{\infty} d y \sqrt{y} e^{-y}=2(2 \pi)^{3 / 2}
$$

The factor $\exp -\gamma_{j}(P-A)^{2} / 2$ in $(I-31)$ is of the same form as the propagator for a diffusion equation, and accordingly doing the $\underline{p}$-integration on $F_{s}$ to get $U_{s}$ will smear out the form of $g_{j}$ over a region of radius $v_{j} \equiv \gamma_{j}^{-1 / 2}$. The topologies of $U_{s}$ and the $g_{j}$ will thus be related in this simple way; $U_{s}$ will tend to have peaks where the $g_{j}$ have peaks and troughs where the $g_{j}$ have troughs. (Of course, if the differing $g_{j}$ have their peaks and troughs in different places, one must look more closely at Eqs. (I-30) and (I-31) to develop valid qualitative expectations.) In particular, since the distribution $f_{\text {LC }}$ we have chosen in (I-29) is large only around $A=A^{S C}$, with a height varying as $n\left(\tilde{X}_{0}\right)$, we expect $U$ to have this same topology, i.e., a ridge centered about $A^{s C}$, and decreasing with $\tilde{X}_{o}$ as $n$ does.

We also observe that, since $F_{S}$ has integrated over the $h$-dependence of $f_{s}, U_{s}$ should be insensitive to the particular h-structure of $f_{s}$. Thus, whether $f_{s}$ is of the loss cone type (I-29) or just a Maxwellian with a similar $n\left(\bar{X}_{0}\right)$-variation, the qualitative features of $U_{s}$ should be the same. Physically, this is reasonable, since it is only very gross properties of the distribution function, e.g., its pressure and parallel currents, which determine the form $\underline{B}(x)$ of the magnetic field.

All the features discussed so far for the forms of $f_{s}, F_{s}$, and $U_{s}$ are in principle applicable to the electrons as well as the ions. Retaining the original definition for $\underline{A}, \underline{A} \equiv\left(e_{i} / i\right) \underline{A}=(e / c) \underline{A}$, we see that the electron quasiphysical region lies around the trajectory $\underline{A}=-\underline{\tilde{A}}$, and so the part of $f_{e}$ of principal physical relevance should be centered around this trajectory. The width of this region about $\tilde{A}$ is typically much narrower for electrons than for ions, however, the ratio of thermal spreads in this momentum space being $\left(p_{e} / p_{i}\right)=\left(m_{e} T / m_{i} T_{i}\right)^{1 / 2} \ll 1$.

In moving from $f_{e}$ to $U_{e}$, the fact that $e_{e}=-e_{i}$ introduces another reflection through the $A$ origin, causing $U_{e}$ to be centered along $\tilde{A}$, just as is $U_{i}$. The mass ratio causes $U_{e}$ to be narrower than $U_{i}$ by the same factor $\left(p_{e} / p_{i}\right)$, provided we choose $f_{e}$ according to the same principles as used for $\mathrm{f}_{\mathrm{i}}$ (namely, modeling the physical situation in the quasiphysical region, and falling rapidly to zero outside this region).

Having $f_{e}$ fall to zero outside the quasiphysical region is not necessary, however, nor is it always desirable. For comparable pressures, and so comparable heights of the $U_{s}$ ridges, since the electron ridge designed in this way is much narrower than the ion ridge, $\partial \mathrm{U}_{\mathrm{e}} / \partial A \gg \partial \mathrm{U}_{\mathrm{i}} / \partial A$, and so the electrons will dominate in causing any lateral accelerations in the trajectory $\mathbb{A}^{S C}(x)$, i.e., electron parallel current will be principally responsible for any magnetic shear. This may not be called for, by a given physical model. One may instead prescribe an $f_{e}$ which falls off much more slowly than $f_{i}$ as the two move away from their quasiphysical domains. This will cause the ion pseudopotential $\mathrm{U}_{\mathrm{i}}$ to be narrower than $U_{e}$, and so allow the ion parallel current to induce the shear. However, the electron physical region will still be much narrower than the ion physical region; this is independent of the choice of $f_{s}$ or the resultant $U_{S}$. In Sec. I.C. 6 we discuss a numerical application in which this latter method for choosing $f_{e}$ has been employed.

## I.C.5. Self-Consistency of $A^{S C}$ and $\tilde{A}$

In the following section and Appendix A, we will present more quantitative substantiation of the qualitative features of the pseudopotential U. However we are already in a position to answer a final important question concerning the Channell formalism's practical application. This is a question of self-consistency. We have chosen our distributions $f_{S}$ so that, if the self-consistent trajectory $A^{s C}$ lies close to the reference trajectory $\tilde{A}$, then the distribution function will have the physical features prescribed by $\mathrm{f}_{\mathrm{s}}$. The question is, then, given the $f_{S}$ we have chosen, is the form of $U$ capable of yielding an $A^{s C}$ lying close to $\tilde{A}$ ? The answer to this question is "yes", as we now explain.

From the preceding subsection, we see that $U_{S}$ for both species, and hence $U$, have the form of a ridge centered on $\tilde{A}\left(\tilde{X}_{0}\right)$, and with height varying as $n\left(\tilde{X}_{0}\right)$. On a pseudopotential of this form, we want to be able to shoot in our pseudoparticle, so that its trajectory $\underline{A}^{S C}(x)$ follows the ridge crest. Clearly, for the incoming pseudo-kinetic energy $\frac{1}{2}|\mathrm{dA} / \mathrm{dx}|^{2}=\frac{1}{2} \underline{\mathrm{~B}}^{2}$ not too large compared to the height of the pseudopotential ridge $U=4 \pi \Pi_{x x}$, one can do this, injecting the particle far enough on the lower downhill side of the ridge that it is accelerated around the elbow in $\tilde{A}$. The ratio (pseudopotential/pseudo kinetic energy) $=8 \pi \Pi_{x x} / B^{2}$ is just the plasma $\beta$, so the constraint just described says that the amount of shear a plasma can support becomes smaller with the plasma $B$.

Since the optimal trajectory $\underline{A}^{\text {SC }}$ must lie somewhat downhill from the ridge crest (which lies at $\tilde{A}$ ), we see that, for a nonzero shear,
$\tilde{A}$ cannot lie exactly on $\tilde{A}$, for the $f_{s}$ we have chosen. This disparity points to an inappropriate feature of the model (I-26) we chose for $f_{s}$; as written, $f_{\text {LC }}$ there has no parallel current, which is necessary for the shear stipulated by $\tilde{\mathbf{A}}$. From Fig. 6, we see that it is exactly the effect of $\underline{A}^{S C}$ lying somewhat downhill from $\tilde{A}$ to make $v_{\| \mid} \neq \tilde{v}_{\|}$, and thereby to give the true distribution the parallel current necessary to produce the shear in $A^{S C}$. If desired, one could now adjust the choice of $\tilde{f}_{S}$, for example introducing a parallel drift velocity $w_{\|}\left(\tilde{X}_{o}\right)$, and adjusting the $\tilde{X}_{0}$ dependence in order to make $\underline{A}^{\text {SC }}$ more nearly equal $\underline{A}$. he shall use this refined form for $f_{s}$ in chonsing our distribution in Part III.

Support for the validity of these qualitative statements is given in the next section, where we present results of numerical application of the interpretive methods thus far presented, and in Appendix A, where we make a more explicit estimate of the size and shape of the potential ridge U.

## I.C.6. Numerical application

In the foregoing sections we have seen how to apply and extract information from the Channell formalism, using graphical methods, from which qualitative conclusions may be relatively easily reached, even in situations where the corresponding mathematics becomes analytically intractable. W. M. Sharp ${ }^{5}$ has employed these methods on a computer, numerically generating self consistent Vlasov equilibria for an $\tilde{A}$ and $\tilde{\mathrm{f}}_{\mathrm{s}}$ modeling the Tormac sheath region. Typical results are shown in Fig. 7.

The model taken is one in which the ions have a loss cone form, and supply the parallel current which induces the shear. As discussed at
the end of Sec. I.C.4, this is achievable by taking $f_{e}$ to vary slowly with $\underline{P}$ as one moves away from its quasiphysical region. Sharp took an $f_{e}$ with no $\underline{P}$ variation at all, $f_{e}=$ (constant) $\times e^{-\gamma^{H}}$. This yields $U_{e}=$ (constant) $\times e^{\gamma_{e} e^{e \Phi}}$, independent of $A$. The electron distribution is thus a local Maxwellian, with zero parallel current, and density variation given by the Boltzmann factor $e^{Y_{e} e^{\uparrow}}$.

In Fig. $7 a$ are shown the reference trajectory $\tilde{A}$ (dotted line) and self-consistent trajectory $\underline{A}^{\mathrm{SC}}$. Sharp allowed for a parallel velocity $w_{11}\left(X_{0}\right)$ in the physical form of the ion distribution, as discussed at the end of Sec. I.C.5, setting up an iterative procedure which adjusted $w_{\|}$to get progressively better fits of $A^{S C}$ to $\tilde{A}$.

Figure 7 b graphs the magnetic field versus $x$, and Fig. 7c shows $\Phi(x)$. The nonphysical behavior at large $x$ is a deficiency of taking $f_{e}$ to be independent of $\underline{p}$. Quasi-neutrality requires that $n_{e}=n_{i}$. As $x$ becomes large, $f_{i}$ is modeled to go to zero, as occurs in Tormac. Since $n_{e} \sim \exp \gamma_{e} e \Phi,-\Phi$ must become large as $x$ does to maintain quasineutrality. In Figs. 7d-7f are plotted contours of the ion distribution, in the Tormac model interior $\left(x / \rho_{i}=-4.0\right)$, sheath center $\left(x / \rho_{i}=0\right)$, and sheath exterior $\left(x / \rho_{i}=4.0\right)$. The left-hand column shows $f_{i}$ in terms of the reference variables $\tilde{v}_{\|} \tilde{v}_{1}$, while the right-hand colum plots $f_{i}$ in terms of the true physical variables.

The equilibrium described by these figures is fully self-consistent, yet from Fig. 7 b we see that $\eta$ in the sheath region is quite substantial ( $n \simeq 1 / 3$ ), so that individual particle trajectories are appreciably distorted from their form in the limit of a uniform magnetic field.

The principal objective of this work was to establish a prescription for writing down a canonical form of the distribution function which yields self-consistent equilibria in this appreciable - $n$ regime, consistent with given physical specifications on the forms of the magnetic field and of the physical distribution function. From the proximity of the reference and self-consistent trajectories in Fig. 7a and the close resemblance of the reference distributions to the physical ones in Figs. 7d-7f, we see that this goal has been achieved.

## APPENDIX A: SEMIQUANTITATIVE ESTIMATE OF U

In this Appendix we shall proceed further with the explicit evaluation of $U_{s}$ from $f_{s}$, to check that the qualitative expectations of the previous Sec. I.C. 4 were accurate.

We use $f_{L C}$ from Eq. ( $1-29$ ) in (I-30), first obtaining $F_{S}=F_{L C}$. For simplicity, we neglect the $\Phi$ dependences. Then $\tilde{H}^{\prime}=H^{\prime}=H=h$ $+\frac{1}{2}(\underline{P}-\underline{A})^{2}$. For a typical $A$, e.g., $A_{2}$ of Fig. 4a, the region of integration in $h$ lies above the boundary 2 of Fig. 4b. In part of this region, namely below the quasiphysical boundary (labeled 1 in Fig. 4b), $f_{L C}=0$, and so the resultant region of integration is the intersection of these two regions.

Neglecting $\Phi$ turns the guiding center condition (I-17) into

$$
\begin{equation*}
\left.0=\hat{b}_{2} \mid \underline{P}-\underline{A}\left(X_{0}\right)\right] \equiv\left(P_{2}-A_{2}\right)\left(X_{0}\right), \tag{A-1}
\end{equation*}
$$

and so Eq. (I-19) says

$$
\begin{equation*}
\left.v_{\|}^{2}\left(X_{0}, \underline{P}\right)=v\left(X_{0} \mid \underline{P}\right)=\frac{1}{2} I \underline{P}-\mathbb{A}\left(X_{0}\right)\right]^{2} . \tag{A-2}
\end{equation*}
$$

This says that we are neglecting the ExB drift velocity, $v_{E} \equiv v_{2}\left(X_{0}\right) \ll v_{x}\left(X_{o}\right)$. If we now generalize the meaning of $v_{\|}{ }^{2}$ from (A-2) to arbitrary $A$,

$$
\begin{equation*}
v_{\|}^{2}(\underline{P}, A) \equiv \frac{1}{2}(\underline{P}-\underline{A})^{2}, \tag{A-3}
\end{equation*}
$$

we can write (I-29) in the form

$$
f_{L C}=\max \left\{\begin{array}{l}
\left|\Omega I^{-1}\right| n \bar{g}_{0} e^{-\gamma v_{\|}^{2} / 2}\left(e^{-\gamma h}-e^{-R \gamma h} e^{(R-1) \gamma h_{1}}\right)  \tag{A-4}\\
0
\end{array}\right.
$$

where $\quad h_{1} \equiv\left(\tilde{v}_{\|}^{2}-v_{\|}^{2}\right) / 2$.

The region of integration of $h$ russ from $h=h_{2}$ to $\sigma$, where

$$
h_{2} \equiv \max \left(0, h_{1}\right) .
$$

The only term in (A-4) which has an $h$-dependence besides those which explicitly appear is $\Omega$, whose dependence on $h$ is quite weak, as we shall see in Parts II and III. (The $\mathrm{v}_{\|}$- dependence of $\Omega$ is more important.) Neglecting this weak $h$-dependence, we perform the $h$-integration, obtaining

$$
\begin{align*}
F_{L C}(\underline{P}, \underline{A}) & =\left|\Omega 1^{-1}\right| n \bar{g}_{O} e^{-\gamma v_{\|}^{2} / 2}\left(8 \pi \sqrt{2} \gamma^{-3 / 2}\right) \\
& \times\left[\Gamma\left(\frac{3}{2}, \gamma h_{2}\right)-R^{-3 / 2} e^{(R-1) \gamma h_{1}} \Gamma\left(\frac{3}{2}, R \gamma h_{2}\right)\right], \tag{A-5}
\end{align*}
$$

where $\Gamma(n, y) \equiv \int_{y}^{\infty} d x x^{n-1} e^{-x}$ is the incomplete $\Gamma$-function of index $n$. ${ }^{4}$ Since we have designed $f_{L C}$ in (A-4) to be nonnegative, we know that the quantity in square brackets in ( $A-5$ ) is also nonnegative, equaling zero only when $R=1$.

In Fig. 8a we sketch the behavior of $\mathrm{r}\left(\frac{3}{2}, y\right)$. Using this form, we show in Fig. 8b the appearance of the term $\left.-R^{-3 / 2} \operatorname{expl}(R-1) \gamma h_{1}\right]$ $\Gamma\left(\frac{3}{2}, R Y h_{2}\right)$ in Eq. (A-5). It is a lopsided downward spike, peaking at $\tilde{v}_{\|}=v_{\| \mid}$, having magnitude $R^{-3 / 2} \pi^{1 / 2} / 2$. Because of the factor $R^{-3 / 2}$, this term will be negligible for large $R$. Even for smaller $R$, since the basic dependence of $F_{L C}$ comes from the overall factor $\exp \left(-\gamma v_{\|}^{2} / 2\right)$, we drop this factor in our present effort to get a rough estimate of $U(A)$. We are thus evaluating $U$ for a Maxwellian distribution, i.e., we are
letting $R \rightarrow \infty$. Equation (I-27) then gives

$$
\begin{equation*}
\bar{g}_{0}=(\gamma / 2 \pi)^{3 / 2} \tag{A-6}
\end{equation*}
$$

Now we perform the $\underline{P}$ integration to get $U$ from F. From Eqs. (I-30), $(A-5)$ and $(A-6)$, we have

$$
\begin{equation*}
U(\underline{A})=4 \int \mathrm{~d}_{\underline{p}}^{n\left|s 1^{-1}\right| \pi^{-1 / 2}} \mathrm{e}^{-n_{1}^{2} / 2} \Gamma\left(3 / 2, \operatorname{Rrh}_{2}\right) . \tag{A-7}
\end{equation*}
$$

As discussed in Sec. I.C.4, the factor $\exp \left(-\gamma v_{11}^{2} / 2\right) \equiv \exp \left[-\gamma(\underline{P}-A)^{2} / 2\right]$ is a diffusive propagator-like term, smearing out the form of the rest of the integrand over a region of radius $v_{t h} \equiv \gamma^{-1 / 2}$. The $\Gamma$-factor gives the ridge with crest at $\underline{A}$, and the factor $n$ modulates the height of this ridge. To evaluate ( $\mathrm{A}-7$ ), we assume that the falloff radius $v_{\text {th }}$ is sufficiently small in comparison with the scales over which $n, \Omega$ and $I$ vary, that we may take them outside the integral. In particular, this means that the shear, which gives rise to the elbow in $\tilde{A}$, is sufficiently weak that, at least for $\underline{A}$ in the neighborhood of $\tilde{A}$ (where $\Gamma$, and so $U$, will be nonnegligible), we may set up a locally rectangular coordinate system $\left(P_{2}, \tilde{v}_{\|}=P_{3}\right)$, centered at that $\tilde{A}\left(\tilde{X}_{0}\right)$ suci that $A$ lies on the line $\tilde{X}_{0}=$ constant, and with coordinate axes in the increasing $\tilde{X}_{0}$ (unit vector $\hat{b}_{2}$ ) and $\widetilde{v}_{\|}$(umit vector $\hat{b}_{3}$ ) directions. This is illust ated in Fig. 9. In this coordinate system, vectors $\underline{P}$ and $A$ are described by $\left(P_{2}, P_{3}=\tilde{v}_{\|}\right)$ and $\left(O, A_{3}\right)$, respectively, and so $v_{\| I}^{2} \equiv(\underline{P}-\underline{A})^{2}=P_{2}^{2}+\left(P_{3}-A_{3}\right)^{2}$, and $h_{1} \equiv \hat{v}_{\mathrm{l}}{ }^{2}-v_{\|}^{2}=2 P_{3} A_{3}-A_{3}{ }^{2}-P_{2}^{2}$. The ridge from $\Gamma$ has a flat top out to the point $P_{31}$, given by $0=h_{1}\left(P_{31}\right)$. This yields $P_{31}=\left(P_{2}{ }^{2}+A_{3}{ }^{2}\right) / 2 A_{3}$.

As $P_{s}$ increases beyond this, the $\Gamma$ ridge falls off, over a falloff distance given by $\left.h_{2}=h_{1}\left(P_{32}\right)=R^{-1} V_{t h}^{2}\right) / 2 A_{3}$.

In this coordinate system, and within the approximations made,
Eq. (A-7) becomes

$$
\begin{equation*}
\left.U(\underline{A})=4 \pi^{-1 / 2} n \mid \Omega I^{-}\right\} \int d P_{2} e^{-\gamma P_{2}^{2} / 2} \int d P_{3} e^{-\gamma\left(P_{3}-A_{3}\right)^{2}} \Gamma\left(\frac{3}{2}, R \gamma h_{2}\right) \tag{A-8}
\end{equation*}
$$

The factor $h_{2}$ has both $P_{2}$ and $P_{3}$ dependence. The factor $\exp \left(-\frac{1}{2} \gamma P_{2}^{2} / 2\right)$ makes $P_{2} \leqslant v_{t h}$ in regions of significant contribution to $U$. Considered as a function of $A_{3}, P_{21}$ goes to infinity both as $A_{3}$ goes to zero and infinity, with a minimum value of $P_{2}$ at $A_{3}=P_{2} \leqslant v_{\text {th }}$. Thus for $A_{3} / v_{\text {th }} \rightarrow 0$ (i.e., $A$ near $\tilde{A}$ ), $P_{31} \rightarrow \infty$, and $h_{2}=0$, making $\Gamma=\pi^{1 / 2} / 2$ over the entire integration region, and so

$$
\begin{equation*}
\mathrm{U}\left(\mathrm{~A}_{3} / v_{\mathrm{th}} \ll 1\right) \simeq 4 \pi \mathrm{n}\left|\Omega I^{-1}\right| v_{\mathrm{th}}^{2} . \tag{A-9}
\end{equation*}
$$

For $A_{3} / v_{\text {th }} \gg 1,2 P_{31}=A_{3}$, and so $\Gamma$ has fallen off to zero in the region where $\exp \left[-\gamma\left(P_{3}-A_{3}\right)^{2} / 2\right.$ ] is substantial. Thus

$$
\begin{equation*}
U\left(A_{3} / v_{t h} \gg 1\right) \simeq 0 \tag{A-10}
\end{equation*}
$$

Finally, in the intermediate region $A_{3} / v_{t h} \sim \frac{1}{2}$, we eliminate the $P_{2}$ dependence of $h_{2}$ in (A-8) by letting $P_{2}$ take a typical value $P_{21} \leqslant v{ }_{\text {th }}$. Then $P_{31} \simeq\left(P_{21}^{2}+A_{3}^{2}\right) / 2 A_{3} \sim v_{t h}$, and so $\Gamma$ will introduce a cutoff in one of ine integration limits in the $P_{3}$ integral of the Gaussian:

$$
\begin{equation*}
\int d P_{3} e^{-\gamma\left(P_{3}-A_{3}\right)^{2} / 2} \Gamma \simeq \Gamma\left(\frac{3}{2}, 0\right) \int_{-\infty}^{\left(P_{31}-A_{3}\right)} d u e^{-\gamma u^{2} / 2} \equiv\left(\frac{\pi v}{\sqrt{2}}\right) E\left[\left(P_{31}-A_{3}\right) / v v_{t h}\right] \tag{A-11}
\end{equation*}
$$

( E is normalized to satisfy $\mathrm{E}(z+\infty)=1$. One may readily express it in terms of the error function, erí (z). Inserting (A-11) into Eq. (A-8). we obtain

$$
\begin{equation*}
\mathrm{U}\left(\mathrm{~A}_{3} / v_{t h} \sim 1\right)=4 \pi n\left|\Omega T^{-1}\right| v_{t h}^{2} E\left\{\left[\mathrm{P}_{31}\left(P_{21}\right)-A_{3} / v_{t h}\right\} .\right. \tag{A-12}
\end{equation*}
$$

From Eqs. ( $A-9$ ), ( $A-10$ ) and ( $A-12$ ) we see the qualitative expectations for $U(\underline{A})$ developed in Sec. I.C. 4 are born out; $U$ has the form of a ridge, with height proportional to $n\left(\bar{X}_{0}\right)$ and width on the order of $v_{\text {th }}$, and with ridge crest at $\underline{A}=\tilde{A}$.

Finally, we can put Eqs. ( $A-9$ ) and (A-12) into a more recognizable form, by evaluating the factor $\left|\Omega T^{-1}\right|$. This factor is to be evaluated at $\underline{P}$ on or near the reference trajectory $\tilde{A}$. For such points $\underline{P}$, we may again use the locally rectangular coordinate system ( $P_{2}, P_{3}$ ). Thus $I \equiv \partial\left(P_{y}, P_{2}\right) / \partial\left(\tilde{X}_{0}, \tilde{v}_{\|}\right)=\partial\left(P_{2}, P_{3}\right) / \partial\left(\tilde{x}_{0}, \tilde{v}_{\|}\right)=\left(\partial P_{2} / \partial \tilde{X}_{0}\right)=\left(\partial \tilde{I}_{2} / \partial \tilde{X}_{0}\right)$, where the last equality follows from the guiding center condition, Eq. (A-1) (with $A$ replaced by $\tilde{A}$, and $X_{0}$ by $\tilde{X}_{0}$ ). The potential $V$ is given approximately by $V \simeq \frac{1}{2}(\delta x)^{2}\left(\partial \tilde{A}_{2} / \partial \tilde{X}_{0}\right)^{2}$, where $\delta x \equiv x-\tilde{X}_{0}$, and so the associated gyrofrequency $\Omega$ satisfies $\Omega^{2}=\left(\partial \tilde{A}_{2} / \partial \tilde{X}_{0}\right)^{2}$. We thus see that (restoring factors of $m$ ) $\left|\Omega I^{-1}\right|=m$, and so the coefficient in Eqs. (A-9), (A-12) is given by

$$
\begin{equation*}
4 \pi n v_{t h}^{2}\left|\Omega I^{-1}\right|=4 \pi n T, \tag{A-13}
\end{equation*}
$$

$4 \pi$ times the usual scalar pressure.

PART II. PARTICIE MOIION IN REALISTIC GECMETRIES AND THE FINITE GYRORADIUS GUIDING CENTER HAMILTONIAN

## II.A. Introduction to Part II

In the present portion of this thesis, we will study particle motion in realistic geometries, with particular interest in cases in which the perpendicular inhomogeneity parameter $n$, defined (as in the Introduction to Part I) as the ratio of gyroradius $\rho$ to the magnetic scale length $L_{\perp}$ perpendicular to the magnetic field, is a nonnegligible parameter. Such strong inhomogeneity occurs, for example, in mirror machines and in the Tormac sheath region, where $n$ may take on values as large as $1 / 3$. In contrast to the slab geometry assumed in Part I, the geometries considered here are assumed to have a small but finite inhomogeneity along the field lines. We characterize this longitudinal inhomogeneity by a second parameter $\varepsilon$, the ratio of $\rho$ to the parallel magnetic scale length $L_{\|}$(or equivalently, the ratio of the longitudinal bounce frequency $\omega_{b}$ to the gyrofrequency $\Omega$ ). We shall consider systems which have a single symmetry direction, as does any axisymmetric device. The resultant particle motion problem then has a single exactly conserved canonical momentum (versus two for the slab geometry), and two nontrivial degrees of freedom.

The assumption that $\varepsilon \ll 1$ ("near-slab" geometry) implies a separation between the time scales on which the gyromotion and longitudinal motion occur. This time scale separation enables one to find a good adiabatic invariant, the gyroaction J. In the linit of $n+0$ (along
with certain additional restrictions to be discussed), J reduces to mc/e times the usual magnetic moment $\frac{1}{2} \mathrm{mv}{ }_{\perp}^{2} / \mathrm{B}$, but it is important to note that, as long as $\varepsilon \ll 1$, a good invariant $J$ can be found, even for $\eta$ comparable to one.

Use of $J$ in the equations of motion reduces the number of nontrivial degrees of freedom in the problem by one, giving a guiding center theory. Previous work ${ }^{6,7,8}$ on finite gyroradius ( $n \neq 0$ ) guiding center motion has made the assumption that the scale lengths $L_{\|}$and $L_{\perp}$ were of the same order, and an expansion in the single parameter $\varepsilon \sim n$ was performed. The assumption $\varepsilon \sim \eta$ is insufficiently flexible for the configurations to which the theory to be developed here is applicable.

Some inroads have been made in studying finite gyroradius guiding center motion. Gardner ${ }^{6}$ outlined a prescription by which one could in principle obtain a finite gyroradius guiding center Hamiltonian $K$, in which one performs a series of canonical transformations on the original Hamiltonian $H$ to remove the dependence of $H$ on the gyrophase $\theta$ to successively higher orders. Stem ${ }^{7}$ then carried out this scheme explicitly to first order. Even to this order, the algebra involved becomes cunbersome, and finding the appropriate canonical transformations is an unsystematic process. Yet the first order result is simple; $K$ has the same form as its zero order limit, but with the variables having a slightly different meaning.

More recently, Northrop and Rome ${ }^{8}$ studied guiding center motion to second order, using a non-canonical framework. In so doing they dispense with the need for transformations which are canonical, but sacrifice
the advantages of a Hamiltonian theory, where all of the equations of motion come from the single function $K$, and symmetries and conservation laws are most readily apparent. As a result, these authors find a breakdown at second order in conservation of canonical angular momentum $p_{\phi}$, which a canonical formulation automatically avoids.

The findings of Ref. 8 parallel those of Ref. 7; to first order, the equations of motion have the same form as the zero order equations, but with a generalized form of gyroaction. The second order corrections to the perpendicular drifts are also obtained in Ref. 8, but require, in the words of those authors, an anount of algebra which is "truly extensive and presents an almost infinite opportunity for mistakes."

Here we describe a treatment of the guiding center motion problem which attempts to minimize the difficulties and shortcomings encountered in the work just described. We adopt a canonical framework, to take advantage of the elegance and power of canonical methods, but employ a rather different method from that of Refs. 6 and 7 for canonically transforming H. This method (described in Sec. II.B) is systematic; and enables one to obtain explicit expressions for the guiding center Hamiltonian $K$ to arbitrarily high order in both $\varepsilon$ and $\eta$. The transformation process relies heavily on Lie perturbative methods, ${ }^{9}$ which greatly simplify the task of making canonical transformations.

The organization of Part II is as follows. In Sec. II.B, we describe in general terms the procedure we shall use to transform from H to K. Sections II.C through II. I supply the specifics of the procedure outlined in Sec. II.B, obtaining the guiding center Hamiltonian $K$,
valid to $O\left(\varepsilon^{2}, \varepsilon \eta, \eta^{2}\right)$, in Eq. (II-89) of Sec. II.I. This is the most important result of Part II.

In Sec. II.J we use the form of $K$ and transformation equations obtained in previous sections to derive another important result, a generalized expression for the gyroaction $J$, valid for finite $\varepsilon$ and $\eta$, and in the presence of magnetic inhomogeneity, including shear. Section II.K applies the formalism developed to that point. Specializing to the negligible e regime, we obtain more explicit expressions for J , as well as developing results describing particle motion in an inhomogeneous magnetic field, which will be of use in Part III. Finally, in Sec. II.L we discuss possible directions for extension and application of the formalism.

## II.B. Overview of the Procedure for Obtaining K

Ke now describe the general nature of the transformation process which takes us from $H$ to $K$. The more specific mechanics of each stage of the process will be presented in the following sections.

The theory is first phrased in a form which is perturbative in $\varepsilon$, but non-perturbative in $n$, reflecting the fact that it is only $\varepsilon \ll 1$ which is necessary for a valid guiding center theory. To obtain explicit expressions for $K$, one must then also do an expansion in $n$. Two variants of Lie methods are employed to facilitate the two expansions.

The procedure is carried out to second order in both $\varepsilon$ and $\eta$. The resultant guiding center Hamiltonian $K$ will then be expressed as a double expansion,

$$
\begin{equation*}
K=\sum_{\ell, m=0}^{\infty} E^{\ell} n^{m} K_{\ell m}, \tag{II-1}
\end{equation*}
$$

where we explicitly calculate $K_{00}, K_{01}, K_{10}, K_{11}, K_{02}$, and $K_{20} . K_{00}$ turns out to have the same form as the zero-gyroradius guiding center Hamiltonian usually calculated. In addition, we find that $K_{01}=K_{10}=$ $K_{11}=0$, so that $K$ equals $K_{00}$ plus tems second order in $\varepsilon$ and $\eta$. This is in accordance with the findings of Refs. 7 and 8 , whose first order guiding center equations were of the same form as the zero order ones, but with modified definitions of the variables.

The transformation process from H to K proceeds in two stages, designated $S$ and $T$. The original coordinates and momenta $\underline{z}$, in terms of which $H$ is expressed, include $\alpha$, which measures the poloidal radius, and its conjugate momentum $P_{\alpha}$. These variables correspond to $x$ and $p_{x}$ of the slab model Hamiltonian ( $\varepsilon=0$ ), Eq. ( $1-4$ ) of Part I; they are the variables of the gyromotion. Stage $S$ of the transformation process transforms from ( $\alpha, p_{\alpha}$ ) to action angle variables $\left(\theta^{\circ}, j^{\circ}\right)$. The resultant Hamiltonian $H^{0}$ has an $O(\varepsilon)$ dependence on $\theta^{\circ}$, so that $J^{\circ}$ is not quite the adiabatic invariant $J$ in terms of which $K$ is expressed. The remaining portion $T$ of the transformarion process is perturbative in character. The angle dependence of the Hamiltonian is removed to successively higher orders in $\varepsilon$, by the sequence of transformations $\left\{T_{1}, T_{2} \cdots\right\}$. These transformations are induced using Lie perturbative methods, which are computationally much more efficient than the mixed-variable generating function method traditionally employed. ${ }^{10}$

The transformation $S$ from ( $\alpha, p_{\alpha}$ ) to $\left(0^{\circ}, J^{\circ}\right)$ is a finite transformation, i.e., nonperturbative in character, and so is induced using a generating function $S(q, \underline{P})$ of the mixed variable type. Our "near-slab" Hamiltonian $H$ is formally the same as the slab Hamiltonian of Part I [ compare Eq. (II-5) to Eq. (I-4)]. The slab problem is one dimensional, and so can be solved exactly, for arbitrary $\eta$, in terms of quadratures involving the potential $V$, which is in general anharmonic. Explicit evaluation of these quadracures is difficult, and will invorve an expansion in $n$.

Analogously, exploiting the fomal similarity of the near-slab to the real-slab $H$, we choose the generating function $S$ for the near slab problem in the same manner which has been used to solve the slab problem, ${ }^{11}$ i.e., $S=\int^{\alpha} d \alpha^{\prime} p_{\alpha}\left(\alpha^{\prime}\right)$. As for the slab problem, this $S$ induces a transformation to action angle variables which is valid for arbitrary $n$. And as for the slab problem, obtaining explicit expressions from the formal ones involves an expansion in $r$, difficult to achieve using standard methods.

The standard approach to solving the one dimensional problem by action angle variables possesses two principal difficulties. One of these, alluded to just above, is the evaluation of integrals of the form $\int d x\left[1-a x^{2}-b x^{3}-c x^{4} \cdots\right]^{1 / 2}$, needed to calculate $S$, and the action $J$ as a function of the energy $E=H$. The second major difficulty is solving the mixed-variable coordinate transformations and performing the functional inversion of $J(H)$ to obtain $H$ in terms of the new variables.

Here again lie methods are of great use. The one dimensional problem, soluble as just described using generating function $S$, may be more easily solved using Lie techniques. One first makes a trivial finite canonical transformation $R_{o}$ to action angle variables ( $\theta^{0}, j^{0}$ ), in terms of which the Hamiltonian $h^{0}$ would be $\theta^{0}$ independent only for a harmonic potential V . The remainder of the problem is perturbative in character, and so may be readily solved using lie methods, inducing a sequence $R=\left\{R_{1}, R_{2}, \cdots\right\}$ of perturbative transformations, which remove the angle dependence of the Hamiltonian to successively higher orders in $\eta$, fully analogous to the sequence of transformations $\mathrm{T}=\left\{\mathrm{T}_{1}, \mathrm{~T}_{2}, \cdots\right\}$ used in stage two of the transformation process. The result is an angle independent Hamiltonian $h(j)$, expressed as an expansion in $n$, and the coordinate transformation, with the original and final variables already in unnixed form. The difficulties of the standard solution procedure just described are thereby completely circumvented.

In trying to generalize this approach in a straightforward way to the near-slab problem, one finds that the initial transformation $R_{o}$ to action-angle variable used for the slab problem is no longer canonical. The way around this difficulty is the following. We know that the mixed variable generating function $S=\int^{\alpha} d \alpha^{\prime} p_{\alpha}$ properly induces a canonical transformation from $\left(\alpha, p_{\alpha}\right)$ to $\left(\theta^{\circ}, \mathrm{J}^{\circ}\right)$; the problem was explicitly calculating $S$. We also note that $S$ has precisely the form as that for a one-dimensional problem, where the old coordinate $B$ and new momentum $P_{b}{ }^{\circ}$ associated with the second degree of freedom are treated simply as
constant parameters. He can solve this one-dimensional problem using Lie methods, as previously described, circumenting the usual difficulties encountered in solving by calculating $S$, using the coordinate transformation $R^{\prime}:\left(\alpha, p_{\alpha}\right) \rightarrow(\theta, j)\left(\right.$ so $\left.R^{\prime}=R \cdot R_{o}\right)$ in $S=\int^{\theta(\alpha)} d \theta^{\prime}\left(d \alpha / d \theta^{\prime}\right) \times$ $p_{a}\left\{\alpha\left(\theta^{\prime}\right)\right\}$, is a simple matter. For a truly one-dimensional problem, calculation of $S$ after transformation $R^{\prime}$ is obtained is superfluous. In the near-slab problem, however, we need $S$ to be able to go on to Stage two; it insures that the new variables $\theta^{\circ}=\theta, J^{0}=j$, and the others of the set $\underline{z}^{0}$, are canonically conjugate.

To sumarize the procedure then: from the original Hamiltonian $H$, developed in Sec. II.C, we set up a one degree of freedom problem, involving a simply related Hamiltonian $h$ defined on the restricted, one-degree-of-freedom phase space. This l-D problem is treated in Sec. II.F. The Lie methods needed for the treatment. are developed in Sec. II.E.

From the transformation $R^{\prime}$ obtained in solving the $1-D$ problem, in Sec. II.G we explicitly calculate the generating function $S$, which was given formal definition in Sec. II.D. S enables us to transform from $H(\underline{z})$ to $H^{\circ}\left(\underline{Z}^{0}\right)$; this is achieved in Sec. II.H. $H^{\circ}\left(\underline{Z}^{\circ}\right)$ emerges as an expansion in both $\eta$ (from the $R$ procedure) and $\varepsilon$ (from the fact that $\mathrm{H}^{\circ}$ is not completely independent of $0^{\circ}$ ), analogous to Eq. (II-I). Finally, given $H^{\circ}$, in Sec. II.I we apply lie-induced transformations $\left\{I_{n}\right\}(n=1,2, \cdots)$, obtaining the guiding center Hamiltonian $K$.

## II.C. Coordinate System, Initial Hamiltonian

We consider an axisymmetric magnetic geometry, independent of toroidal angle $\phi$. We choose orthogonal particle coordinates $q^{\mu}=$ $(\alpha, B, \phi)$, with conjugate momenta $p_{\mu}=\left(p_{\alpha}, p_{\beta}, p_{\phi}\right)$. Coordinates $\alpha$ and $B$ describe the particle variable, constant along poloidal field lines (specifically, we shall take $\alpha$ to be the poloidal flux enclosed by a poloidal field line), and B measuring the position along the poloidal field lines. The coordinate system is illustrated in Fig. 10.

The $\alpha$ and $B$ used here are not the Euler potentials used in Refs. 6 and 7. Use of the Euler potentials, allows one to pick an $A_{B}$ which depends only on $\alpha$. However this choice in general makes the $q^{H}$ nonorthogonal ( $\nabla \alpha \cdot \nabla \beta \neq 0$ ). We sacrifice the nice properties of the Euler potentials in favor of the orthogonality property. The metric tensor $g^{\mu \nu}(\alpha, \varepsilon \beta) \equiv \underline{\nabla} q^{\mu} \cdot \nabla q^{\nu}$ is then diagonal, with diagonal elements $g^{\alpha} \equiv|\nabla \alpha|^{2}, g^{B} \equiv|\nabla \beta|^{2}$, and $g^{\phi} \equiv|\nabla \phi|^{2}=R^{-2}(\alpha, \varepsilon \beta)$. The "near-slab" nature of the problem is expressed by having all quantities in the theory (e.g., $\mathrm{g}^{\mu \nu}$ and Hamiltonian $H$ ) depend on $\epsilon \beta$, instead of simply $B$.


$$
\begin{equation*}
\underline{B}=(\underline{\nabla} \times \underline{A})=\left(g^{1} g^{2} g^{3}\right)^{1 / 2}\left\{\underline{e}_{1}\left(\frac{\partial A_{3}}{\partial q^{2}}-\frac{\partial A_{2}}{\partial q^{3}}\right)+\text { cyclic permutations }\right\}, \tag{II-2}
\end{equation*}
$$

where $A_{\mu} \equiv{\underset{\mu}{\mu}}$ - .
Equation (II-2) is valid for a general orthogonal coordinate
j, Etem. Specializing to our present choice of coordinates, we know that $B^{\alpha} \equiv B^{1} \equiv{\underset{e}{ }}^{1} \cdot \underline{B}=0$. Our hypothesis that the $A_{\mu}$ are independent of $\Phi$,
and in particular our choice that $a$ be the poloidal flux enclosed by a poloidal flux line, allows the choice of gauge

$$
\begin{equation*}
A_{\mu}=\left[0, A_{B}(\alpha, \varepsilon \beta), \alpha\right], \tag{II-3}
\end{equation*}
$$

with magnetic field

$$
\begin{equation*}
B^{\mu} \equiv \underline{e}^{\mu} \cdot \underline{B}=\left(g^{1} g^{2} g^{3}\right)^{1 / 2} \quad\left(0,-\partial A_{\phi} / \partial \alpha=-1, \quad \partial A_{B} / \partial \alpha\right) . \tag{II-4}
\end{equation*}
$$

An important feature of Eq. (II-3) is that $A_{\alpha}=0$. This causes the Hamiltonian to be of the same form as the slab Hamiltonian used in part I [Eq. (I-4)]. Setting $m=c=1$, we have

$$
\begin{align*}
H\left(\alpha, p_{\alpha} ; \epsilon B, p_{B} ; p_{\phi}\right) & =\frac{1}{2}(p-e A)_{\mu} g^{\mu \nu}(p-e A)_{\nu}+e t  \tag{II-5}\\
& =\frac{1}{2} g^{\alpha}(\alpha, \varepsilon B) p_{\alpha}^{2}+V\left(\alpha \mid \varepsilon \beta, p_{B}, p_{\phi}\right)
\end{align*}
$$

where

$$
\begin{equation*}
V\left(\alpha \mid \varepsilon \beta, p_{\beta}, p_{\phi}\right) \equiv \frac{1}{2}\left\{g^{\beta}(\alpha, \varepsilon \beta)\left[p_{\beta}-e A_{B}(\alpha, \varepsilon \beta)\right]^{2}+g^{\phi}(\alpha, \varepsilon \beta)\left[p_{\phi}-\mathrm{e} \alpha{ }^{2}\right\}\right. \tag{II-6}
\end{equation*}
$$

$$
+e \Phi(\alpha, \varepsilon \beta)
$$

is the effective potential for the oscillation of the particle in $\alpha$. The first form given for $H$ in Eq. (II-5) is valid for a coordinate system $q^{U}$ of arbitrary metric; the second form is restricted to our specific choice.

The independence of $H$ from $\phi$ implies $p_{\phi}=$ constant. If we let $\varepsilon=0$, then $p_{B}$ would also be a constant, and our Hamiltonian would be precisely of the slab type used in Part I. In this case, the potential well $V(a)$ has a constant position, with well bottom $a_{0}$ parametrically dependent
on the constants $p_{\phi}$ and $p_{B}$. The particle gyromotion then corresponds to the high frequency oscillation in this constant well. For $\varepsilon \equiv 0$, $P_{B}$ is no longer exactly conserved. Driven by the $O(\varepsilon)$ dependence of $P_{B}=-\partial H / \partial B$ on $B, P_{B}$ has an $O(\varepsilon)$ jitter at the gyrofrequency. In addition, the slow variation of $H$ with $B$ as a particle moves along a field lune produces a corresponding variation of $p_{B}$ on the bounce period time scale. If the particle is trapped, oscillating in $B$ and $\phi, p_{B}$ will also experience a periodic variation on this time scale. This in turn produces an oscillation of the well $V(\alpha)$ and, in particular, of the well bottom $\alpha_{0}$, occurring once every bounce period. This is the explanation of the banana orbits of trapped particles in axisymmetric geometries, expressed in the canonical terminology we are using. Taking the $\alpha$ surface at which $\dot{\phi}=0$ as the "banana center" $\alpha_{b}$, we find from Hamilton's equation for $\dot{\phi}$ that $\alpha_{b}=p_{\phi} / e$. The guiding center position $\alpha_{0}=$ $\alpha_{0}\left(\varepsilon b, p_{B}, p_{\phi}\right)$ oscillates about $\alpha_{b}$, with the slow variation of $p_{B}$ and $\varepsilon \in$. The jitter of $p_{B}$ and $C B$ at the gyrofrequency induce a jitter of $V$ and the metric coefficient $g^{\alpha}$. This modifies the particle gyromotion, as well as giving rise to particle drifts.

In the following sections, we shall see the specific mathematics of these processes.

## I1.D. Generating Function S; Formal Description

In this section we describe stage one of the transformation process outlined in Sec. II.B, in the formal framework mentioned there, which is nonperturbative in $\eta$. That is, we shall write down the mixed variable
generating function $S$, which transforms to action angle variables ( $0^{\circ}, \mathrm{J}^{\circ}$ ), in terms of a certain integral. This integral is difficult to perform directly, if $V$ is anharmonic or $g^{\alpha}$ depends on $\alpha$.

The angular momentum $P_{\phi}=P_{\phi}$ is exactly conserved, and retains its value through all the transformations to be performed. We therefore drop it from explicit notation. We denote by $\lambda \equiv\left(\lambda_{1}, \lambda_{2}\right)$ the pair of variables describing motion along the field line. Before transformation $S$ is applied, these are the "old" variables $\lambda^{\circ} \equiv\left(\varepsilon \beta, p_{\beta}\right)$ and after $S$ they are the corresponding "new" pair $\lambda^{N} \equiv\left(\varepsilon b, P_{b}\right)$ (where both $b$ and $P_{b}$ may have a superscript $j$, denoting which of the sequence of tranformations $T_{j}$ was last applied). We shall also have need for the "mixed" pair $\lambda^{M} \equiv\left(\varepsilon \beta, P_{b}\right)$, in writing down $S$.
$S$ is then obtained in the following manner. The energy E of the particle is given by

$$
\begin{equation*}
E=H\left(\alpha, p_{\alpha} ; \lambda^{0}\right)=\frac{1}{2} g^{\alpha} p_{\alpha}^{2}+V\left(\alpha \mid \lambda^{0}\right) . \tag{II-7}
\end{equation*}
$$

We solve this for $p_{\alpha}$ :

$$
\begin{equation*}
p_{a}(\alpha, E, \lambda) \equiv\left\{2[E-V(\alpha \mid \lambda)] / g^{\alpha}(\alpha, \lambda)\right\}^{1 / 2}, \tag{II-8}
\end{equation*}
$$

and define the functional form of the action $J^{0}$ by

$$
\begin{equation*}
J^{o}(E, \lambda) \equiv(2 \pi)^{-1} \Phi \mathrm{~d} \alpha \mathrm{p}_{\alpha}(\alpha, E, \lambda) . \tag{II9}
\end{equation*}
$$

We then invert this, $J^{0}(E, \lambda) \rightarrow E\left(J^{0}, \lambda\right)$, and finally define $S$ by

$$
\begin{align*}
& S\left(\alpha, J^{0} ; \lambda^{M}\right) \equiv \beta P_{b}^{0}+S_{1},  \tag{II-10}\\
& S\left(\alpha, J^{0} ; \lambda^{M}\right) \equiv \beta \int^{\alpha} d \alpha^{\prime} p_{\alpha}\left[\alpha^{\prime}, E\left(J^{0}, \lambda^{M}\right), \lambda^{M}\right] . \tag{II-11}
\end{align*}
$$

Note that it is $E\left(J^{\circ}, \lambda^{\circ}\right)$ and not $E\left(J^{\circ}, \lambda^{M}\right)$ which is the particle's true energy. These two are different by a term of $O(E)$, and so in the slab 1imit $\varepsilon+0$ this choice for $S$ becomes the usual choice use $d^{31}$ to solve the one dimensional particle motion problem.

The transformation equations follow from $S(q, p)$ via the usual escription, 10

$$
\begin{equation*}
p=\partial S / \partial q, \quad Q=\partial S / \partial \underline{P} . \tag{II-12}
\end{equation*}
$$

Ot particular interest are
$I=\partial S / \partial \alpha=p_{\alpha}\left[\alpha, E\left(J^{0}, \lambda^{M}\right), \lambda^{M}\right]=\left\{2\left[E\left(J^{0}, \lambda^{M}\right)-V\left(\alpha \mid \lambda^{M}\right)\right] / g^{\alpha}(\alpha, \varepsilon \beta)\right\}^{1 / 2}$,
$p=\partial S / \partial \beta=P_{b}^{0}+\varepsilon \partial S_{1} / \partial(\varepsilon \hat{\beta}) \equiv p_{b}^{0}+\varepsilon \delta p_{\beta}\left(\alpha, J^{0}, \lambda^{M}\right)$,
ar. $]$

$$
\begin{equation*}
\beta=b^{0}-\partial S_{1} / \partial P_{b}^{O} \equiv b^{0}+\delta \beta\left(\alpha, \rho, \lambda^{M}\right) \tag{11-15}
\end{equation*}
$$

Using Eq. (II-13) in (II-5), we obtain a non-canonical form $\mathrm{H}^{\prime}$ for the Hamiltonian which we shall need in getting an explicit expression for $H^{\circ}$ (cf. Sec. II-H):
$H\left(\alpha, p_{\alpha} ; \lambda^{o}\right)=\frac{1}{2} g^{\alpha}(\alpha, E \beta) p_{\alpha}^{2}+V\left(\alpha \mid \lambda^{o}\right)=\left[E\left(J^{o}, \lambda^{M}\right)-V\left(\alpha \mid \lambda^{M}\right)\right]+V\left(\alpha \mid \lambda^{o}\right)$, an! thus
$H\left(, p_{\alpha} ; \lambda^{O}\right)=H^{\prime}\left(\alpha, J^{\circ}, \lambda^{\circ}, \lambda^{M}\right) \equiv E\left(J^{\circ}, \lambda^{M}\right)+V\left(\alpha \mid \lambda^{0}\right)-V\left(\alpha \mid \lambda^{M}\right)$.
One then uses Eqs. (II-14) and (II-15) in $\mathrm{H}^{\prime}$ to remove the variat es $\lambda^{\circ}$ in favor of the new onee $\lambda^{N}$. gettins in additivi cemrs in $\delta \hat{A}$, $\delta n_{;}$, which are higher order in $\varepsilon$. Since $\delta \beta$ and $\delta p_{\beta}$ are themselves functions of the mixed pair $\lambda^{M}$, this process must be applied iteratively, to
as high order in $\varepsilon$ as desired. The result is $H^{0}\left[Z^{0} \cong\left(\theta^{0}, J^{0} ; E b^{0}, P_{b}{ }^{0}\right)\right]$, expressed as an expansion in $\varepsilon$. In Sec. II. 4 we carry out this procedure in detail.

## II.E. Lie Methods; Preparation for Explicit Calculation

Now that stage one of the transformation process has been described in terms of the formal definition $c_{2} S$ [Eqs. (II-10) and (II-11)], we are ready to begin obtaining more explicit results. In order to do this, we will make use of two variants of lie methods, the type used being appropriate to the particular form of the Hamiltonian being trea*ed. In stage one, the appropriate variant is essentially the normal form type used by Dragt and Finn. ${ }^{9}$ Stage two requires a second variant, which we call the 'near-slab' type. The philosophy underlying the two is, however, just the same, and many of the important equations dictating their use are formally identical.

In this section we present our formulation of both types of Lie methods. We first describe the Dragt-Finn variety, and then more briefly describe the near-slab type, pointing out the most inportant differences between the two.

## II.E.1. Dragt-Finn Lie Method

We assume a Hamiltonian $h^{\prime}$, expressed as a power series expansion in its carcnical variables $z^{\prime}$. The phase point $\underline{z}^{\prime}=0$ is a fixed point of the system. The "distance" of phase point $\underline{z}$ ' away from this fixed point, expressed in terms of some suitable metric, is proportional
to the parameter $\eta$, and so the $m^{\text {th }}$ order, term in this multivariable expansion for $h$ ' is proportional to $\eta^{m}$.

Because $\underline{z}^{\prime}=0$ is a fixed point, $h_{1}^{\prime} \equiv \partial h^{\prime} /\left.\partial z^{\prime}\right|_{z^{\prime}=0}=0$. The zero-order term in $h^{\prime}, h_{0}{ }^{\prime} \equiv h^{\prime}\left(\underline{z}^{\prime}=0\right)$, will turn out to be of the order of the parallel kinetic energy at the particle gu:ding center, and so for a typical particle is of the same size as the perpendicular kinetic energy at the guiding center, which is on the order of the second order term in the expansion. To indicate that these two terms are of comparable size, $w$ express, this second order term without an accompanying $n$, $h_{2}^{\prime} \equiv \underline{z} \underline{z}^{\prime} \underline{z^{\prime}}:\left(\frac{1}{2} \partial^{2} h^{\prime} / \partial \underline{z^{\prime}} \underline{z}^{\prime}\right){ }_{1=0}$, even though evidently $h_{2}^{\prime} \sim\left(\underline{z^{\prime}}\right)^{2}$, while $h_{0}{ }^{\prime} \sim\left(\underline{z}^{\prime}\right)^{0}$. We thus write the expansion of $h^{\prime}$ as

$$
\begin{equation*}
h^{\prime}\left(\underline{z}^{\prime}\right)=h_{0}^{\prime}+\sum_{m=2}^{\infty} n^{m-2} h_{m}^{\prime}\left(\underline{z}^{\prime}\right), \tag{II-17}
\end{equation*}
$$

where it is uncerstood that $\eta$ is only present in the actual expression for $h$ ' implicity, i.e., it is a formal device which we use to concisely describe our ordering.

The term $h_{0}$ ' is independent of the dymanical variables in the restricted space, and so could normally be thrown away. In the present application, however, $h_{0}$ ' is dependent upon the variables $\lambda$, which are frozen in the one-dimensional problem of stage one, but which are dynamical variables for the full problem, and so we retain $h_{o}$ ', though it has no effect on the 1-D problem.

The second order term $h_{2}{ }^{\prime}$ describes a set of $N$ simple harmonic oscillators. $(\mathbb{N}$ is the number of degrees of freedom in the phase space.

In Sec. II.F, we shall be taking $N=1$. ) Discarding all $h_{m}^{\prime}$ for $m>2$ yields the unperturbed problem, exactly soluble, and expressible, if desired, in terms of action variables $\dot{L}^{\circ}$. The conjugate angle variables $\underline{\theta}^{\circ}$ enter only in the $m>2$ terms of the full problem. The Hamiltonian becomes

$$
\begin{equation*}
h^{\prime}\left(\underline{z}^{\prime}\right)=h^{0}\left[\underline{z}^{0} \equiv\left(\underline{\theta}^{0}, \underline{j}^{0} j\right] \equiv h_{0}+h_{2}^{0}\left(j^{0}\right)+\sum_{m=3}^{\infty} \eta^{m-2} h_{m}^{0}(\underline{z})\right. \tag{II-18}
\end{equation*}
$$

where $h_{0} \equiv h_{0}^{\prime}, h_{2}{ }^{0}=\underline{\omega}^{\circ} \cdot j^{0}$, and $\underline{\omega}^{0}$ is the set of frequencies of the $N$ unperturbed oscillators.

The objective of the perturbative technique is to apply canonical transformations $\left\{R_{1}, R_{2}, R_{3}, \cdots\right\}$ to transform away the dependence of the Hamiltonian on angles $\theta$ to successively higher orders. Transformaion $R_{n}$ transforms from $h^{n-1}\left(\underline{z}^{n-1}\right)$ to $h^{n}\left(\underline{z}^{n}\right)$, removing the dependence of Hamiltonian on $\underline{\theta}$ up to $O\left(n^{n}\right)$ :

$$
\begin{aligned}
& h^{n}(\underline{z})=k^{n}(\underline{j})+\sum_{m=n+1}^{\infty} n^{m} h_{m+2}^{n}(\underline{z}) \\
& k^{n}(j) \equiv h_{0}+\sum_{m=0}^{n} n^{m} h_{m+2}^{n}(j)
\end{aligned}
$$

For any phase function $G\left(z^{\prime}\right)$, we define the corresponding Lie opei tor $\stackrel{\text { G by }}{ }$

$$
\begin{equation*}
\dot{\mathrm{G}} \equiv\{\mathrm{G}, \mathrm{\}}, \tag{II-20}
\end{equation*}
$$

where , \} denotes Poisson brackets. The key to Lie perturbative nethods lies in the fact that the operator

$$
\begin{equation*}
R_{G} \equiv \exp \dot{G} \equiv 1+\dot{G}+\frac{1}{2} \dot{G}^{2}+\cdots \tag{II-21}
\end{equation*}
$$

induces a canonical transformation.
We therefore take the $R_{n}$ to be of the form

$$
\begin{equation*}
R_{n} \equiv \exp \left(\eta^{n} \stackrel{r}{G}_{n+2}\right) \tag{11-22}
\end{equation*}
$$

The generator $\eta^{n} G_{n+2}$ is a polynomial in $\underline{z}^{\prime}$ of order $(n+2), n^{n} G_{n+2}$ $\sim\left(z^{1}\right)^{n+2}$ (just as for $n^{n} h_{n+2}$ ), and so the operator $n^{n} G_{n+2}^{\sim} \sim\left(z^{\prime}\right)^{n}$. The specific form of the generators will be determined shortly.

One can show ${ }^{9}$, for any two fhase functions $f_{1}$ and $f_{2}$, that the operator $R_{G}$ satisfies $R_{G}\left(f_{1} f_{2}\right)=\left(R_{G} f_{1}\right)\left(R_{G} f_{2}\right)$, and $\left.R_{G}\left(f_{1}+f_{2}\right)=R_{G} f_{1}\right)+\left(R_{G} f_{2}\right)$. Therefore, writing any phase function $g$ as a power series expansion in the phase variables $z, R_{G}$ has the additional property $R_{G} g(z)=g \mid R_{G}=j$. We shall use this in Eq. (II-24).

The coordinate transformations are given by

$$
\begin{equation*}
\underline{z}^{n}=R_{n}^{-1} \underline{z}^{n-1} \equiv \exp \left(-n^{n} \quad \check{G}_{n+2}\right) \underline{z}^{n-1} \tag{I-23}
\end{equation*}
$$

If we have any phase function $f^{0}\left(\underline{z}^{0}\right)$ which we wish to express in terms of $\underline{z}^{n}$, i.e., $f^{n}\left(\underline{z}^{n}\right) \equiv f^{0}\left(\underline{z}^{0}\right)$, then

$$
\begin{equation*}
f^{n}\left(\underline{z}^{n}\right)=f^{0}\left(R_{n} R_{n-1} \cdots R_{1} z\right)=\left(R_{n} R_{n-1} \cdots R_{1}\right) f^{0}(\underline{z}) \tag{II-24}
\end{equation*}
$$

In particular, we want to take $f^{0}=h^{0}$, and choose the $G_{n}$ 's to eliminate the $\underline{\theta}$ - dependence of $h^{n}$.

We carry out this procedure explicitly to second order, i.e. we $=\operatorname{btain} R_{1}, R_{2}$, or $G_{3}, G_{4}$, and thereby $k^{1}, k^{2}$.

Expanding the exponential in (II-22), for $n=1$ we have

$$
\begin{align*}
h^{1}(\underline{z}) & \equiv R_{1} h^{0}(\underline{z})=\left(1+n \dot{C}_{3}+\frac{1}{2} n^{2} \dot{G}_{3}^{2}+\cdots\right)\left(h_{0}^{0}+h_{2}^{0}+n h_{3}^{0}+n^{2} h_{4}^{0}+\cdots\right) \\
& =k^{0}+n\left(h_{3}^{0}+\dot{G}_{3} h_{2}^{0}\right)+n^{2}\left(h_{4}^{0}+G_{3}^{0}+\frac{1}{2} \dot{G}_{3}^{2} h_{2}^{0}\right)+\cdots . \tag{II-25}
\end{align*}
$$

Henceforth we drop the superscript ' 0 " on the $h_{m}{ }^{0} . G_{3}$ is determined by requiring that the tem $h_{3}{ }^{1} \equiv h_{3}+\dot{G}_{3} h_{2}$ in Eq. (II-25) be independent of $\underline{\theta}$, and that the generator $G_{3}$ have no secular terms. From the definition (II-20 of a Lie operator, $\check{G}_{3} h_{2}=-\check{h}_{2} G_{3}$, and from the same definition, $-\dot{h}_{2}$ is just the time derivative ( $\left.\mathrm{d} / \mathrm{dt}\right)_{o}$ along the unperturbed trajectory in phase space. The above definition of $h_{3}{ }^{1}$ thus becomes a differential equation for $G_{3}$ :

$$
\begin{equation*}
(\mathrm{d} / \mathrm{dt})_{0} G_{3} \equiv-\mathrm{h}_{2} \mathrm{G}_{3}=\mathrm{h}_{3}^{1}-\mathrm{h}_{3}, \tag{II-26}
\end{equation*}
$$

with solution

$$
\begin{equation*}
G_{3}=-\tilde{h}_{2}{ }^{-1}\left(h_{3}{ }^{1}-h_{3}\right) \equiv \int^{t} d t\left(h_{3}{ }^{1}-h_{3}\right) . \tag{II-27}
\end{equation*}
$$

The nonsecularity condition on $G_{3}$ requires that $h_{3}{ }^{1}$ be chosen to be the average $h_{3}$ along the unperturbed trajectory of $h_{3}$ :

$$
\begin{equation*}
\mathrm{h}_{3}{ }^{1} \equiv \mathrm{~h}_{3} . \tag{II-28}
\end{equation*}
$$

This choice of $h_{3}{ }^{1}$ is independent of $\underline{\theta}$, as required.
One proceeds analogously to second order. We have $h^{1}=k^{1}+$ $\eta^{2} h_{4}{ }^{1}+\cdots$, where, from Eqs. (II-21), (II-25), (II-26) and (II-28)

$$
\begin{align*}
& k^{1} \equiv k^{0}+\eta h_{3}^{1}=\left(h_{0}+h_{2}\right)+\eta F_{3}, \quad \text { and }  \tag{II-29}\\
& h_{4}^{1} \equiv h_{4}+\dot{G}_{3} h_{3}+\frac{1}{2} \dot{G}_{3}^{2} h_{2}=\frac{1}{2} \dot{G}_{3}\left(h_{3}+\hbar_{3}\right) . \tag{array}
\end{align*}
$$

Analogous to Eq. (II-25), then, we have

$$
\begin{align*}
h^{2} & \equiv R_{2} h^{1}=\left(1+n^{2} \dot{G}_{4}+\cdots\right)\left(k^{1}+n^{2} h_{4}^{1}+\cdots\right) \\
& =k^{1}+n^{2}\left(h_{4}^{1}+\dot{G}_{4} h_{2}\right)+\cdots  \tag{II-31}\\
& \equiv k^{1}+n^{2} h_{4}{ }^{2}+\cdots \equiv k^{2}+\cdots
\end{align*}
$$

We determine $G_{4}$ in the same manner as $G_{3}$ was determined before, requiring $h_{4}^{2} \equiv h_{4}{ }^{1}-\check{h}_{2} G_{4}$ to be $\underline{\theta}$-independent, and $G_{4}$ to be nonsecular. Then we have

$$
\begin{align*}
& h_{4}^{2} \equiv \overline{h_{4}^{I}}=h_{4}+\frac{1}{2} \bar{G}_{3}\left(h_{3}+\bar{h}_{3}\right), \text { and }  \tag{II-32}\\
& -\dot{h}_{2} G_{4}=h_{4}^{2}-h_{4}^{1}=\overline{h_{4}^{1}}-h_{4}^{1} \tag{II-33}
\end{align*}
$$

with solution

$$
\begin{equation*}
\mathrm{G}_{4}=-\stackrel{h}{h}_{2}^{-1}\left(\mathrm{~h}_{4}^{1}-\mathrm{h}_{4}^{1}\right) \equiv \int^{\mathrm{t}} \mathrm{dt} t^{\prime}\left(\mathrm{h}_{4}^{1}-\mathrm{h}_{4}^{1}\right) \tag{1I-34}
\end{equation*}
$$

The generalization of these equations to arbitrary order is obvious. We note that at each stage $R_{n}$ leaves $k^{n-1}$ unchanged, and gives an additional term $h_{n+2}^{n} \equiv \overline{h_{n+2}^{n-1}}$ which, added to $k^{n-1}$, yields $k^{n}$. This and most other important features are present in the near-slab type of Lie method, which we now describe.

## II.E.2. Near-Slab Lie Nethod

At the beginning of stage two of the transformation process, we shall be faced with a Haniltonian $H^{0}$ the near-slab form

$$
\begin{equation*}
H^{0}\left(\underline{z}^{0}\right)=H_{o}\left(J^{0} ; \varepsilon b^{0}, P_{b}^{0}\right)+\sum_{\ell=1}^{\infty} \varepsilon^{\ell} H_{\ell}\left(0^{0}, J^{0} ; \varepsilon b^{0}, P_{b}^{0}\right) . \tag{II-35}
\end{equation*}
$$

This type of Hamiltonian was treated by McNamara and Whiteman, ${ }^{12}$ using Lie methods to find a good adiabatic invariant $J$. Here we want to do more; we shall develop a Lie perturbative technique to transform the Hamiltonian $H^{\circ}$ and all its variables $\underline{2}^{\circ}$, to eliminate its $\theta$-dependence and thereby obtain the guiding center Hamiltonian $K$, in the process funding $J$.

There are two important features which distinguish this nearslab form from that of $h^{0}\left(\underline{z}^{0}\right)$ of the preceding section. The first is that, whereas $\eta$ there was only present in $h^{\circ}$ implicitly, here $\varepsilon$ is an explicit small parameter, i.e., a particular small constant. In this respect the appropriate Lie method should resemble the sort employed by Deprit and Cary ${ }^{13}$ more than that of Dragt and Finn. The second inmortant feature, and the principal reason a different variant of Lie methods is needed to treat this problem, is tic appearance of $\varepsilon$ in two places in Eq. (II-35); accompanying the term $H_{l}$, as did $n$ in Eq. (II-17), and also multiplying the variable $b$ in the arguments of each $H_{\ell}$. This second $\varepsilon$ dependence, which gives $\mathrm{H}^{\circ}$ its near-slab nature, changes the ordering somewhat from that of Sec. II.E.1. Without this dependence, we would take $H_{o}$ to be our unperturbed Hamiltonian, giving the upperturbed trajectories one must integrate along in calculating the transformation generators. The trajectories of $H_{o}$ are not of the simple harmonic nature which those of $k^{0} \equiv h_{0}+h_{2}$ were, and so these orbit integrals would be difficult to perform explicitiy. We shall see shortly that the shuffling in the ordering which this near-slab $\varepsilon$ dependence introduces is just such as to make a simple integral over gyrophase the appropriate one.

The procedure parallels that of the Dragt-Finn method of the last section. We use a sequence $\left\{\mathrm{T}_{1}, \mathrm{~T}_{2}, \mathrm{~T}_{3} \cdots\right\}$ of transformations to remove the dependence of the Hamiltomian on $\theta$, the transformation $T_{i 1}$ transforming $H^{n-1}\left(\underline{Z}^{n-1}\right)$ into $H^{n}\left(\underline{Z}^{n}\right)$. Analogous to Eq. (II-19), $\mathrm{H}^{n}$ is of the form

$$
\begin{align*}
& H^{n}\left(\underline{2}^{n}\right)=K^{n}\left(J, \varepsilon b, P_{b}\right)+\sum_{\ell=n^{+1}}^{\infty} \varepsilon^{\ell} H_{\ell}^{n}\left(0, J, \varepsilon b, P_{b}\right)  \tag{iI-36}\\
& K^{n}\left(J, \varepsilon b, P_{b}\right)+\sum_{\ell=0}^{n} \varepsilon^{\ell}\left(J, \varepsilon b, P_{b}\right)
\end{align*}
$$

so that $H^{\infty}=K^{\infty}=K$, the guiding center Hamiltonian. The $T_{n}$ are given by

$$
\begin{equation*}
T_{n} \equiv \exp \varepsilon^{n} \dot{W}_{n} . \tag{II-37}
\end{equation*}
$$

As in the Dragt-Finn technique, the generators $W_{n}$ are taken to be of the same functional form as the terms in the Hamiltonian they are designed to eliminate, $W_{n}=W_{n}\left(0, J, \varepsilon b, P_{b}\right.$ ). Paralleling Eq. (II-25), we write

$$
\begin{align*}
H^{1} & \equiv T_{1} H^{0}=\left(1+\varepsilon \dot{W}_{1}+\frac{1}{2} \varepsilon^{2} \dot{W}_{2}^{2}+\cdots\right)\left(H_{0}+\varepsilon H_{1}+\varepsilon^{2} H_{2}+\cdots\right) \\
& =H_{0}+\varepsilon\left(H_{1}+\dot{W}_{1} H_{0}\right)+\varepsilon^{2}\left(H_{2}+\dot{W}_{1} H_{1}+\frac{1}{2} \dot{W}_{1}^{2} H_{0}\right)+\cdots . \tag{II-38}
\end{align*}
$$

It is here that the second, near-slab dependence of H on $\varepsilon$ enters, changing the ordering. We write

$$
\begin{equation*}
-\check{W}_{1} H_{o}=H_{o} \check{W}_{1}=H_{o \theta} \check{W}_{1}+\varepsilon H_{o b} \check{W}_{1}, \tag{II-39}
\end{equation*}
$$

where the operators $\dot{H}_{o \theta}, \dot{H}_{o b}$, both of $O(1)$, are defined by

$$
\begin{equation*}
\check{H}_{o \theta} \equiv \frac{\partial H_{o}}{\partial \theta} \frac{\partial}{\partial J}-\frac{\partial H_{o}}{\partial J} \frac{\partial}{\partial \theta}=-\frac{\partial H_{o}}{\partial T} \frac{\partial}{\partial \theta}, \dot{H}_{o b} \equiv \frac{\partial H_{o}}{\partial \varepsilon b} \frac{\partial}{\partial P_{b}}-\frac{\partial H_{0}}{\partial P_{b}} \frac{\partial}{\partial \varepsilon b} . \tag{II-40}
\end{equation*}
$$

Defining the zero-order gyrofrequency by $\Omega_{o} \equiv \partial H_{o} / \partial \mathrm{J}$, we see that - $\left.H_{o e}=\Omega_{0} \partial / \partial \theta \equiv d / d t\right)_{0}$, the time derivative along the simple trajectory in which 0 increases linearly in time, and the other variables ( $J, \in b, P_{b}$ ) are frozen. Since $\dot{H}_{o b}$ enters at higher order in $\varepsilon$ than $\dot{H}_{o \theta}$, it is these trajectories which are appropriate to integrate along, to calculate the $k_{n}$.

Using Eq. (II-39) in (II-38), we find that the first order part of $H^{1}$ is only part of the factor $\left(H_{1}+\dot{W}_{1} H_{o}\right)$ there, $H_{1}{ }^{1}=\left(H_{1}-\dot{H}_{o \theta} W_{1}\right)$, the remainder $-\varepsilon \stackrel{H}{\mathrm{H}}_{\mathrm{ob}} \mathrm{W}_{1}$ going into $\mathrm{H}_{2}{ }^{1}$. Imposing the same conditions as in Sec. II.E. 1 gives the defining equations for $W_{1} ; H_{1}{ }^{1}=\bar{H}_{1}$, and

$$
\begin{equation*}
\mathrm{d} /\left.\mathrm{dt}\right|_{o} \mathrm{w}_{1} \equiv-\stackrel{\check{H}}{o \theta} \mathrm{w}_{1}=\bar{H}_{1}-\mathrm{H}_{1}, \tag{II-41}
\end{equation*}
$$

with solution

$$
\begin{equation*}
\mathrm{w}_{1}=-\dot{H}_{\mathrm{o} \theta}\left(\overline{\mathrm{H}}_{1}-\mathrm{H}_{1}\right) \equiv \int^{\mathrm{t}} \mathrm{dt} \cdot\left(\overline{\mathrm{H}}_{1}-\mathrm{H}_{1}\right), \tag{II-42}
\end{equation*}
$$

analogous to Eqs. (II-26) and (II-27). This gives a $W_{1}$ with the same near-slab dependences as the $H_{\ell}$, and so we write $\dot{W}_{1}=\dot{W}_{1 \theta}+\varepsilon \dot{W}_{1 b}$, ana1ogous to (II-39) and (II-40), to maintain the proper ordering in the terms of order two and higher. Using this and Eqs. (II-39), (II-42) in Ea. (II-38), we find

$$
\begin{equation*}
\mathrm{H}_{2}^{1}=\mathrm{H}_{2}+\frac{1}{2} \stackrel{W}{\mathrm{~W}}_{1 \theta}\left(\mathrm{H}_{1}+\bar{H}_{1}\right)-\check{H}_{\mathrm{ob}} \mathrm{~W}_{1}, \tag{II-43}
\end{equation*}
$$

slightly different from Eq. (II-30) for $h_{4}{ }^{1}$.
The procedure to second order is just as in Sec. II.E.1. One obtains

$$
\begin{align*}
\mathrm{H}_{2}^{2} & =\overline{\mathrm{H}}_{2}^{1}, \quad \text { and }  \tag{II-44}\\
-\mathrm{H}_{\mathrm{o} \mathrm{\theta}} \mathrm{~W}_{2} & =\overline{\mathrm{H}}_{2}^{1}-\mathrm{H}_{2}^{1}, \tag{II-45}
\end{align*}
$$

with solution

$$
\begin{equation*}
W_{2}=-\dot{H}_{o \theta}^{-1}\left(\bar{H}_{2}^{1}-H_{2}^{1}\right) \equiv \int^{t} d t^{\prime}\left(\overline{\mathrm{H}}_{2}^{1}-\mathrm{H}_{2}^{1}\right) \tag{II-46}
\end{equation*}
$$

II.F. The "Frozen $\lambda$ " Problem

With the methods of Sec. II.E.I in hand, we are prepared to set up and solve the $1-D$, "frozen $\lambda$ " problem sketched in Sec. II.B. We begin with the initial Hamiltonian $H\left(\alpha, p_{\alpha} ; \lambda\right)$, and treat the longitudinal variables $\lambda$ as constant parameters. This leaves us with a one degree of freedom phase space, with Hamiltonian $H$.

We define the particle's $\alpha$-center $\alpha_{0}$ as in Eq. (I-17):

$$
\begin{equation*}
0=\partial / \partial \alpha V\left(\alpha|\lambda|_{\alpha_{0}}\right. \tag{II-47}
\end{equation*}
$$

which may be solved to give $\alpha_{0}=\alpha_{0}(\lambda)$. We then make the simple transformation $\left(\alpha, p_{\alpha}\right) \rightarrow \underline{z}^{\prime} \equiv\left(\delta \alpha \equiv \alpha-\alpha_{0}, p_{\alpha}\right)$, which is canonical in the reduced phase space. The new Hamiltonian $h^{\prime}\left(z^{\prime}\right)$ is given by

$$
\begin{align*}
h^{\prime}(\underline{z} ; \lambda) & \left.=H\left(\alpha, p_{\alpha} ; \lambda\right)=\frac{1}{2} g^{\alpha} p_{\alpha}^{2}+V\left|\alpha_{0}(\lambda)+\delta \alpha\right| \lambda\right]  \tag{II-48a}\\
& =V_{0}+\sum_{m=2}^{\infty}\left[\frac{1}{2} g_{m-2} p_{\alpha}^{2}(\delta \alpha)^{m-2}+V_{m}(\delta \alpha)^{m}\right],
\end{align*}
$$

where

$$
\begin{align*}
& g^{\alpha}\left(\alpha_{0}+\delta \alpha, \lambda\right) \equiv \sum_{m=0}^{\infty} g_{m}\left(\alpha_{0}, \lambda\right)(\delta a)^{m}, \quad \text { and }  \tag{II-48b}\\
& V\left(x_{0}+\delta \alpha, \lambda\right) \equiv \sum_{m=0}^{\infty} v_{m}\left(\alpha_{0}, \lambda\right)(\delta \alpha)^{m} . \tag{II-48c}
\end{align*}
$$

Corparing Eq. (II-48a) with (II-17), we make the identifications

$$
\begin{aligned}
h_{o}^{\prime} & =V_{o}, h_{1}^{\prime}=v_{1}=0, \text { and } \\
n^{m-2} h_{m}^{\prime} & =\frac{1}{2} g_{m-2} p_{\alpha}^{2}(\delta \alpha)^{m-2}+V_{m}(\delta \alpha)^{m}, m>1 .
\end{aligned}
$$

Following the discussion in Sec. I.E. 1 , we trivially make the finite transformation $R_{0}: \underline{z}^{\prime}+\underline{z}^{0} \equiv\left(\theta^{0}, j^{0}\right)$ to action-angle variables, canonical in the reduced space, given by

$$
\begin{equation*}
\delta \alpha=a\left(\theta^{\circ}, j^{\mathrm{o}} \mid \lambda\right), p_{\alpha}=p\left(\theta^{\mathrm{o}}, \mathrm{j}^{\mathrm{o}} \mid \lambda\right), \tag{II-49}
\end{equation*}
$$

where the phase functions $a$ and $p$ are defined as

$$
\begin{aligned}
& a(\theta, j \mid \lambda) \equiv\left(\Omega_{0} / V_{2}\right)^{1 / 2} j^{1 / 2} \sin \theta \equiv \bar{a}(\lambda) j^{1 / 2} \sin \theta, \\
& p(\theta, j \mid \lambda) \equiv\left(2 \Omega_{0} / g_{o}\right)^{1 / 2} j^{1 / 2} \cos \theta \equiv \bar{p}(\lambda) j^{1 / 2} \cos \theta .
\end{aligned}
$$

Here $\Omega_{0}^{2} \equiv 2 \mathrm{~g}_{\mathrm{o}} \mathrm{V}_{2}$. The frequency $\Omega_{0}$ is the $\mathrm{n} \rightarrow$ o gyrofrequency, including effects due to finite ( $v_{\|} / \Omega_{c} L_{s}$ ), which exist even in the zero gyroradius 1 imit. (Here $\Omega_{c} \equiv e \mathrm{e} /$ пп , the unsheared gyrofrequency, and we recall that $\mathrm{L}_{\mathrm{s}}$ is the shear scale length.) Using (II-49) in (II-48), we find the Hamiltomian $h^{\circ}\left(\underline{z}^{\circ}\right)$ :
$h^{0}(\underline{z})=\left(V_{0}+\Omega_{0} j\right)+\sum_{m=3}^{\infty}\left(\frac{1}{2} \bar{g}_{m-2} \bar{p}^{-2-m-2} \cos ^{2} \theta \sin ^{m-2} \theta+V_{m} \bar{a}^{m} \sin ^{m} \theta\right] j^{m / 2}$.
Comparing this with Eq. (II-18), we find (dropping the superscript 0 on
the $h_{m}{ }^{0}$, as before)

$$
\begin{align*}
h_{0}(\lambda) & \equiv V_{0}, h_{2}(\underline{z} ; \lambda) \equiv \Omega_{0} j  \tag{II-51}\\
n h_{3}(z ; \lambda) & \equiv \Omega_{0} \bar{a} j^{3 / 2}\left[\left(g_{1} / g_{0}\right) \cos ^{2} \theta \sin \theta+\left(V_{3} / V_{2}\right) \sin ^{3} \theta\right]  \tag{II-52}\\
& =\Omega_{0} \bar{i} j^{3 / 2}\left[c_{3} \sin 3 \theta+c_{1} \sin \theta \mid / 4\right. \\
\eta^{2} h_{4}(z ; \lambda) & \equiv \Omega_{0} \bar{a}^{2} j^{2}\left[\left(g_{2} / g_{0}\right) \cos ^{2} \theta \sin ^{2} \theta+\left(V_{4} / V_{2}\right) \sin ^{4} \theta\right]  \tag{II-53}\\
& =-\Omega_{0} \bar{a}^{2} j^{2}\left[d_{4} \cos 4 \theta+d_{2} \cos 2 \theta-d_{0}\right] / 8
\end{align*}
$$

Here $c_{m}$ (in $h_{3}$ ) and $d_{m}$ (in $h_{4}$ ) are the coefficients accompanying sinm $\theta$ or cosme and in particular

$$
\begin{aligned}
& c_{3} \equiv\left(g_{1} / g_{0}-v_{3} / V_{2}\right), c_{1} \equiv\left(g_{1} / g_{0}+3 V_{3} / V_{2}\right) \\
& d_{4} \equiv\left(g_{2} / g_{0}-v_{4} / v_{2}\right), \dot{d}_{2} \equiv 4 v_{4} / v_{2}, d_{0} \equiv\left(g_{2} / g_{0}+3 v_{4} / V_{2}\right)
\end{aligned}
$$

The parameters $c_{m}, d_{m}, \bar{a}, \Omega_{0}, g_{m}$, and $V_{m}$ are all functions of $\lambda$, and the dependence of the $h_{m}$ on $\underline{z} \equiv(\theta, j)$ is explicitly displayed.

We now apply the general formulae obtained in Sec. II.E.1. From Eqs. (II-28) and (II-52), we see that

$$
\begin{equation*}
\mathrm{h}_{3}^{1}=\overleftarrow{h}_{3}=0 \tag{II-54}
\end{equation*}
$$

Using this in (II-27) yields

$$
\begin{align*}
\eta G_{3} & =-\Omega_{o}^{-1} f^{\theta} d \theta^{\prime} \eta h_{3} \\
& =\bar{a} j^{3 / 2}\left[c_{3}(\cos 3 \theta) / 12+c_{1}(\cos \theta) / 4\right] \tag{II-55}
\end{align*}
$$

From Eqs. (II-55), (II-53) and (II-30), we then find

$$
\begin{align*}
\eta^{2} h_{4}^{1} & =\eta^{2}\left(h_{4}+\frac{1}{2} G_{3} h_{3}\right) \\
& =-\Omega_{0} \bar{i}^{-2} j^{2}\left\{\left(d_{4}+c_{3} c_{1} / 4\right) \cos 4 \theta+\left(d_{2}+c_{3} c_{1}\right) \cos 2 \theta-\left(d_{0}-\left(\omega_{1}\right)\left(c_{3}^{2}+c_{1}^{2}\right)\right]\right\} / 8 \tag{1I-56}
\end{align*}
$$

From this and (II-32), we read off $\eta^{2} h_{4}{ }^{2}$ :

$$
\begin{equation*}
\eta^{2} h_{4}^{2}=\eta^{2} h_{4}^{1}=\Omega_{0} \bar{a}^{2} j^{2}\left[d_{0}-(3 / 8)\left(c_{3}^{2}+c_{1}^{2}\right)\right] / 8 \tag{1I-57}
\end{equation*}
$$

Equations (1I-56), (II-57) and (II-34) give us $G_{4}$ :

$$
\begin{align*}
\eta^{2} G_{4} & =\Omega_{0}^{-1} \int^{\theta} d \theta^{\prime} \eta^{2}\left(\overline{h_{4}^{1}}-h_{4}^{1}\right) \\
& =a^{-2} j^{2}\left[\left(d_{4}+c_{3} c_{1} / 4\right)(\sin 4 \theta) / 32+\left(d_{2}+c_{3} c_{1}\right)(\sin 2 \theta) / 16\right] \tag{II-58}
\end{align*}
$$

This is to as high order as we shall explicitly carry the calculation, though clearly the extension to arbitrarily high order is straightforward. To this order $h^{\infty}(j ; \lambda) \approx k^{2}(j ; \lambda)=h_{0}+h_{2}+\eta h_{3}{ }^{1}+\eta^{2} h_{4}{ }^{2}$, which we can read off from Eqs. (II-51), (II-54) and (II-57). In addition, because $h^{\infty}$ is just the energy for the $1-D$ particle motion problem specified by parameter $\lambda$, and $j \equiv j^{\infty}$ is the action, these are to be identified with $E\left(J^{0}, \lambda\right)$ and $J^{0}$ respectively, which resulted from solving this I-D problem using the traditional mixed-variable generating function prescription, as discussed in Sec. II.D. Therefore

$$
\begin{align*}
E^{\prime}(j, \lambda) & =h^{\infty}(j, \lambda) \simeq k^{2}(j, \lambda)=\left(h_{0}+h_{2}\right)+\eta^{2} h_{4}^{2}  \tag{II-59}\\
& =\left(V_{0}+\Omega_{o} j\right)+\Omega_{0} \bar{a}^{2} j^{2}\left[d_{0}-(3 / 8)\left(c_{3}^{2}+c_{1}^{2}\right)\right] / 8
\end{align*}
$$

and

$$
\begin{equation*}
j \equiv j^{\infty}=j^{0} . \tag{II-60}
\end{equation*}
$$

We observe that the Lie method of solution has completely circumented the difficulties of cumbersome integrals and functional inversions encompered in the traditional solution procedure, as discussed in Sec. II.B.

Note that knowledge of $\mathrm{G}_{4}$ was not necessary for writing out the form of $k^{2}$; the stipulation that $h_{4}^{2}=\overline{h_{4}^{1}}$ in Eq. (II-32) completed our knowledge of $k^{2}$, and only then is $G_{4}$ determined. However if we desirc the interpretation of the variables $\underline{z}^{2}$ appearing in $\mathrm{k}^{2}$, we need the transformation $R_{2}$, and so $G_{4}$.

In the next section we shall calculate the generating function $S$, which requires use of the coordinate transformations just obtained. To the order we are working, we will only need the transformation equations for $\delta \alpha$ and $p_{\alpha}$ to first order, i.e., we will need only $G_{3}$. From Eqs. (II-23) and (II-49),
$\left.\delta \alpha \equiv \alpha-\alpha_{0}(\lambda)=a\left(\underline{z}^{0} \mid \lambda\right)=R_{1} \mathrm{a}\left(\underline{z}^{1} \mid \lambda\right) \approx R_{1} a(\underline{z} \mid \lambda) \simeq\left(1+\eta \dot{G}_{3}\right) a(\underline{z}) \mid \lambda\right)$,
and similarly

$$
\begin{equation*}
p_{\alpha}=p\left(\underline{z}^{o} \mid \lambda\right) \simeq\left(1+n \dot{G}_{3}\right) p(\underline{z} \mid \lambda) . \tag{11-62}
\end{equation*}
$$

Denoting by $y$ any of the variables $\delta \alpha, p_{\alpha}, \delta \beta, \delta p_{\beta}$, we may write $y$ as an expansion in $n$,

$$
\begin{equation*}
y=\sum_{m=0}^{\infty} n^{m} y_{m} . \tag{II-63}
\end{equation*}
$$

Using Eq. (II-55) in (II-61) and (II-62), we explicitly write out the first two terms in this $n$-expansion of $\delta a, p_{\alpha}$ :

$$
\begin{align*}
& \left.\delta \alpha_{0}=a(\underline{z} \mid \lambda), \quad p_{\alpha 0}=p(\underline{z}) \mid \lambda\right),  \tag{II-64}\\
& n \delta \alpha_{1} \equiv n \dot{G}_{3} a(\underline{z} \mid \lambda)=-\bar{a}^{2} j\left(c_{1}+c_{\alpha} \cos 2 \theta\right) / 4  \tag{II-65}\\
& n p_{\alpha \underline{1}} \equiv n \dot{G}_{3} p(\underline{z} \mid \lambda)=\bar{p} \bar{a} j c_{p} \sin 2 \theta / 2=j c_{p} \sin 2 \theta \tag{II-66}
\end{align*}
$$

where

$$
c_{\alpha} \equiv\left(c_{1}+c_{3}\right) / 2=\left(g_{1} / g_{0}+v_{3} / v_{2}\right), \text { and } c_{p} \equiv\left(c_{1}-c_{3}\right) / 4=v_{3} / v_{2} .
$$

## II.G. Generating Function; Explicit Form

In the present section, we calculate the generating function $S$, and the associated functions $\delta \beta$, $\delta p_{\beta}$, defined in Eqs. (II-14) and (II-1r). These will be necessary to obtain Hamiltonian $H^{\circ}$ from $H^{\prime}$, as described in Sec. II.D. We want to obtain $H^{\circ}$ explicitly to $O\left(\varepsilon^{2}, \varepsilon \eta, \eta^{2}\right)$ which will necessitate finding $\delta B, \delta p_{\beta}$ up to one order further than the $n \rightarrow 0$ simple harmonic limit, i.e., we will need up to ny ${ }_{1}$, in the terminology of Eq.
(II.63). For this, we need $S$ to one order higher than its simple harmonic value, and this in tum requires $\delta \alpha$ and $p_{a}$ up to first order as well.

We calculate $S_{1}$ using the coordinate transformations obtained in the last section:

$$
\begin{equation*}
S_{1} \equiv \int^{\alpha} d \alpha^{\prime} p_{\alpha}\left(\alpha^{\prime}\right)=\int^{\theta} d \theta^{\prime}\left(d \delta \alpha / d \theta^{\prime}\right) p_{\alpha}\left(\theta^{\prime}, j \mid \lambda^{M}\right) \tag{II-67}
\end{equation*}
$$

This integral over $\theta$ has integrand

$$
\begin{equation*}
(d \delta \alpha / d \theta) p_{\alpha}\left(\theta, j \mid \lambda^{M}\right)=\left(\bar{a} j^{1 / 2} \cos \theta+\frac{1}{2} \bar{a}^{2} j c_{\alpha} \sin 2 \theta\right)\left(\bar{p} j^{1 / 2} \cos \theta+\frac{1}{2} \bar{p} \bar{a} j c_{p} \sin 2 \theta\right)(I \tag{I-68}
\end{equation*}
$$

+ higher order,
and so we compute $S_{1}$ by a trivial integration:

$$
\begin{equation*}
S_{1}\left(\theta, j \mid \lambda^{M}\right)=j\left(\theta+\frac{1}{2} \sin \theta\right)-\frac{1}{2}-j^{3 / 2} c_{s}(\cos \theta+\cos 3 \theta / 3)+\text { higher order } \tag{1I-69}
\end{equation*}
$$

where

$$
c_{s} \equiv c_{\alpha}+c_{p}=\left(g_{1} / g_{o}+2 V_{3} / V_{2}\right) .
$$

From Eqs. (11-14), (11-15), we have $\delta p_{B} \equiv \partial S_{1} / \partial \lambda_{1}^{M}, \delta B \equiv \partial S_{1} / \partial \lambda_{2}^{M}$, where $\lambda^{M} \equiv\left(\lambda_{1}{ }^{M}, \lambda_{2}{ }^{M}\right) \equiv\left(\varepsilon \beta, P_{b}{ }^{0}\right)$, and so we are interested in the derivative $\partial S_{1} / \partial \lambda_{1,2}$, where $\alpha, j$ and the other $\lambda_{i}$ are to be held constant. We want to express these derivatives in terms of the form $S_{1}(\theta, j \mid \lambda)$ we now have for $S_{1}$ [cf. Eq. (II-70)]. We have

$$
\left.\partial S_{1} / \partial \lambda_{1}\right)_{\alpha, j, \lambda_{2}}=\left(\partial S_{1} / \partial \theta\right)_{j, \lambda}\left(\partial \theta / \partial \lambda_{1}\right)_{\alpha, j, \lambda_{2}}+\left(\partial S_{1} / \partial \lambda_{1}\right)_{\theta, j, \lambda_{2}} .
$$

From (II-67)

$$
\left(\partial S_{1} / \partial \theta\right)_{j, \lambda}=(d \delta \alpha / d \theta) p_{\alpha}(\theta, j \mid \lambda)
$$

We have $\delta \alpha(\theta, j \mid \lambda)$, which we can in principal invert to obtain $\theta\left(\delta \alpha \equiv \alpha-\alpha_{0}, j \mid \lambda\right)$, whence

$$
\begin{aligned}
\left(\partial \theta / \partial \lambda_{1}\right)_{\alpha, \mathfrak{j}, \lambda_{2}} & =-\left(\partial \alpha_{0} / \partial \lambda_{1}\right)_{\lambda_{2}}(\partial \theta / \partial \delta \alpha)_{j, \lambda}+\left(\partial \theta / \partial \lambda_{1}\right)_{\delta \alpha, j} \\
& =-\left[\left(\partial \alpha_{0} / \partial \lambda_{1}\right)_{\lambda_{2}}+\left(\partial \delta \alpha / \partial \lambda_{1}\right)_{\theta, j}, \lambda_{2}\right](\partial \delta \alpha / \partial \theta)_{j, \lambda}^{-1} .
\end{aligned}
$$

Putting these pieces together, we find, finally,

$$
\begin{align*}
\left(\partial S_{1} / \partial \lambda_{1}\right)_{\alpha, \mathfrak{i}, \lambda_{2}} & =-\rho_{\alpha}(\theta, j \mid \lambda)\left[\left(\partial \alpha_{0} / \partial \lambda_{1}\right)_{\lambda_{2}}+\left(\partial \delta \alpha / \partial \lambda_{1}\right)_{\theta, j, \lambda_{2}}\right]  \tag{II-70}\\
& +\left(\partial S_{1} / \partial \lambda_{1}\right)_{\theta, j, \lambda_{2}}
\end{align*}
$$

There are three terms in (II-70), the two in the square brackets, and a third term $\left(\partial S_{1} / \partial \lambda_{1}\right)_{\theta, j, \lambda_{2}}$. From Eq. (II-69), this Iast term is given by

$$
\left(\partial S_{1} / \partial \lambda_{1}\right)_{e, j, \lambda_{2}}=-\frac{1}{2} \partial\left(\bar{a} c_{S}\right) / \partial \lambda_{1} j^{3 / 2}(\cos \theta+\cos 3 \theta / 3)
$$

From the ordering scheme outlined in Appendix B, this term is down from the first by $\eta^{2}$, and so we neglect it completely, to the order we are working. Similarly the second term is down by a factor $\eta$ from the first, and so contributes to $\delta p_{B 1}$. Analogous equations to those above hold for $\partial S_{1} / \partial \lambda_{2}$. Using the $n$ expansion of $p_{\alpha}$ in (II-70) gives

$$
\begin{align*}
& \delta \beta_{0}=\bar{p} j^{1 / 2} \cos \theta\left(\partial \alpha_{0} / \partial \lambda_{2}\right), \quad \delta p_{\beta O}=-\bar{p} j^{1 / 2} \cos \theta\left(\partial \alpha_{0} / \partial \lambda_{1}\right)  \tag{II-71}\\
& n \delta \beta_{1}=d_{p} j \sin 2 \theta, \quad n \delta p_{\beta 1}=-d_{b} j \sin 2 \theta \tag{II-72}
\end{align*}
$$

where

$$
\begin{aligned}
& d_{p} \equiv c_{p}\left(\partial \alpha_{0} / \partial \lambda_{2}\right)+\partial(\ln \bar{a}) / \partial \lambda_{2}, \quad \text { and } \\
& d_{b} \equiv c_{p}\left(\partial \alpha_{0} / \partial \lambda_{1}\right)+\partial(\ln \bar{a}) / \partial \lambda_{2}
\end{aligned}
$$

Here $\lambda$ is understood to be evaluated at $\lambda^{M}$. The dependence of the $\delta \beta_{m}$, $\delta p_{\beta n}$ on $\theta$ and $j$ appears explicitly, as will be necessary in finding the explicit dependence of $\mathrm{H}^{\mathrm{O}}$ on these variables. In the next section, we proceed to do this.

## II.H. Calculation of $\mathrm{H}^{\circ}$

We have now completed the work of stage one of the transformation procedure, and are ready to proceed to stage two. Once we have obtained an explicit expression for $H^{\circ}$ (the task of this section), stage two is a simple matter, using the lie method of Sec. H.E.2.

As described following Eq. (II-16), $H^{\circ}$ is obtained by using CE and $\delta p_{\beta}$ in the mixed-variable form $H^{\prime}$ of the Hamiltonian, to eliminate the old variables in favor of the new. This iterative procedure is in principle executable to arbitrary order in both $\varepsilon$ and $\eta$, yielding $H^{\circ}$ as a double expansion analogous to (II-1). Forhigher order, howeter, the expansion rapidly becomes complicated. This is the price to be paid for using a mixed-variable generating function; the transformation equations must subsequently be unmixed. Because the transformation is finite, Lie methods are not useful, and we have been forced to pay this price. We carry out the expansion to $O\left(\varepsilon^{2}, \varepsilon \eta, \eta^{2}\right)$. The first iteration of the expansion of Eq. (II-16) yields

$$
\begin{aligned}
H^{\prime} & =E\left(\lambda^{M}\right)+V\left(\alpha^{M} \mid \lambda^{0}\right)-V\left(\alpha^{M} \mid \lambda^{M}\right) \\
& =E\left(\lambda^{N}\right) \\
& \left.+\varepsilon[\delta \beta / \partial \varepsilon b) E\left(\lambda^{N}\right)+\delta p_{\beta}\left(\partial / \partial P_{b}\right) V\left(\alpha^{M} \mid \lambda^{M}\right)\right] \\
& +\frac{1}{2} \varepsilon^{2}\left[(\delta \beta)^{2}(\partial / \partial \varepsilon b)^{2} E\left(\lambda^{N}\right)+\left(\delta p_{\beta}\right)^{2}\left(\partial / \partial P_{b}\right)^{2} V\left(\alpha^{M} \mid \lambda^{M}\right)\right]+\text { higher order. }
\end{aligned}
$$

We have suppressed some of the arguments here for notational simplicity, and the notation $\alpha^{M}$ denotes that the transformation $S$ gives us $\alpha$ in terms of the mixed variables, $\alpha=\alpha_{0}\left(\lambda^{M}\right)+\delta \alpha\left(\theta, j \mid \lambda^{M}\right) \equiv \alpha^{M}$, so that we must expand this function about $\lambda^{N}$ in the next iteration.

The $O\left(\varepsilon^{0}\right)$ term in (II-73), $E\left(\lambda^{N}\right)=k^{0}+n^{2} h_{4}{ }^{2}+O\left(n^{3}\right)$, is already in terms of the new variables and so we identify

$$
\begin{equation*}
H_{\infty}=k^{0}, \quad n H_{01}=0, n^{2} H_{02}=n^{2} h_{4}{ }^{2} . \tag{11-74}
\end{equation*}
$$

From Eq. (II-6), we have

$$
\begin{gather*}
\left.V^{1}\left(a \mid \lambda^{M}\right) \equiv\left(\partial / \partial P_{b}\right) V\left(a \mid \lambda^{M}\right)=g^{B}\left(a, M^{M}\right) \mid P_{b}-e A_{\beta}\left(a \mid \lambda^{M}\right)\right], \\
\left(a^{2} / \partial P_{b}^{2}\right) V\left(a \mid \lambda^{M}\right)=g^{B}\left(a, \lambda^{M}\right) . \tag{II-75}
\end{gather*}
$$

In the $O\left(\varepsilon^{2}\right)$ term, we may replace $\lambda^{M}$ by $\lambda^{N}$, $\delta B$ by $\delta \beta_{0}, \delta p_{B}$ by $\mathrm{in}_{50}$, and $a$ by $\alpha_{0}{ }^{N} \equiv \alpha_{0}\left(\lambda^{N}\right)$ in $g^{B}$. The $\mathcal{O}(\varepsilon)$ term in (II-73) gives rise to a number of terms, contributing to $\mathrm{H}_{10}, \mathrm{H}_{11}$, and $\mathrm{H}_{20}$. One must expand $\delta \varepsilon, \delta p_{E}$, and $V^{1}$ in both $\varepsilon$ and $n$ :

$$
\begin{aligned}
& \left.\delta E\left(\lambda^{M}\right)=\delta \beta_{0}\left(\lambda^{N}\right)+\varepsilon \delta \beta_{0}(\partial / \partial \varepsilon b) \delta \beta_{0}\left(\lambda^{N}\right)+n \delta \beta_{1}\left(\lambda^{N}\right)+\text { higher order (h.o. }\right) \\
& \delta p_{E O}\left(\lambda^{M}\right)=\delta p_{B O}\left(\lambda^{N}\right)+\varepsilon \delta \beta_{0}(\partial / \partial \varepsilon b) \delta p_{B o}\left(\lambda^{N}\right)+\eta \delta p_{B 1}\left(\lambda^{N}\right)+\text { h.o. }, \\
& V^{\prime}\left(a^{M} \mid \lambda^{M}\right)=V^{\prime}+\varepsilon \delta \beta_{0}\left[\partial / \partial \varepsilon b+\left(\partial \alpha_{0} / \partial \varepsilon b\right) \partial / \partial \alpha_{0}\right] V^{\prime}+\delta \alpha_{0} \partial / \partial \alpha_{0} V^{\prime}+\text { h.o., }
\end{aligned}
$$

where $V^{\prime}$ on the right hand side is evaluated at $\alpha_{0}^{N}, \lambda^{N}$. Combining the various contributions yields

$$
\begin{align*}
H_{10} & =\delta \beta\left(\partial k^{O} / \partial \varepsilon b\right)+\delta p_{B O} V^{\prime}\left(\alpha_{0}^{N} \mid \lambda^{N}\right)  \tag{II-76}\\
\eta H_{11} & =\eta \delta \beta_{1}\left(\partial k^{0} / \partial \varepsilon b\right)+\eta \delta p_{B 1} V^{\prime}\left(\alpha_{0}^{N} \mid \lambda^{N}\right)+\delta p_{B O} \delta \alpha_{0}\left(\partial / \partial \alpha_{0}\right) V^{\prime}  \tag{II-77}\\
H_{20} & =\frac{1}{2}\left[\left(\delta B_{0}\right)^{2}(\partial / \partial \varepsilon b)^{2} k^{O}+\left(\delta p_{B O}\right)^{2} g^{\beta}\left(\alpha_{0}, \lambda^{N}\right)\right] \\
& +\delta \beta_{0}\left[\left(\partial k^{O} / \partial \varepsilon b\right)(\partial / \partial \varepsilon b) \delta \beta_{0}+V^{\prime}(\partial / \partial \varepsilon b) \partial p_{B O}\right]  \tag{II-78}\\
& +\delta p_{B O} \delta \beta_{0}\left[(\partial / \partial \varepsilon b)+\left(\partial \alpha_{0} / \partial \varepsilon b\right)\left(\partial / \partial \alpha_{0}\right)\right] V^{\prime}
\end{align*}
$$

All terms here are evaluated at the new variables $\underline{Z}^{0}$. The explicit dependence of all quantities here on $(G, j)=\left(\theta^{\circ}, J^{\circ}\right)$ are known from results already obtained. The dependences on $\lambda^{N}=\left(c b^{\circ}, \mathrm{P}_{\mathrm{b}}{ }^{\mathrm{o}}\right)$ are determined by the magnetic geometry in the particle's vicinity.

## H.I. Calculation of K from $\mathrm{H}^{\mathrm{O}}$

In this section we complete the transformation process, performing the transformations $T_{n}$ of stage two, using the formalism developed in Sec. II.E.2, to obtain the finite $\varepsilon$, finite $n$, guiding center Hamiltonian K.

We proceed explicitly to $O\left(\varepsilon^{2}\right)$, i.e., we shall calculate $K_{20}$. As noted in Sec. II.F in the context of the frozen $-\lambda$ problem, to obtain the second-order term, $\varepsilon \varepsilon_{2}$, of $K$, we only need to calculate the generator $\varepsilon W_{1}$, and not $\varepsilon^{2} W_{2}$. In addition, just as each $K_{\ell}$ is an expansion in $n, K_{\ell}=\sum_{m=0}^{\infty} \eta^{m} K_{\ell m}$, the generators $W_{\ell}$ are also expressed as $\eta$ expansions, $W_{\ell}=\sum_{m=0} n^{m} W_{\ell m}$. Because we are seeking $K_{20}$ but neglecting $n K_{2 i}$, it will only be necessary to calculate $W_{10}$.

To begin with, fron Eqs. (II-35), (II-36), (II-74), and (II-60), we have the $O\left(\varepsilon^{\circ}\right)$ tern of $H^{\circ}$,

$$
\begin{align*}
\mathrm{K}^{\mathrm{O}}\left(\mathrm{~J} ; \lambda^{\mathrm{N}}\right) & \equiv \mathrm{H}_{0}\left(\mathrm{~J} ; \lambda^{N}\right)=E\left(J ; \lambda^{N}\right) \cong k^{0}+\eta^{2} h_{4}^{2} \\
& =\left[V_{0}\left(\lambda^{\mathrm{N}}\right)+\Omega_{0} \mathrm{~J}\right]+\Omega_{0} \bar{a}^{2} \mathrm{~J}^{2}\left[\mathrm{~d}_{0}-(3 / 8)\left(c_{3}{ }^{2}+c_{1}^{2}\right)\right] / 8 . \tag{II-79}
\end{align*}
$$

For $\varepsilon \rightarrow 0$, we have $K=H^{\circ}=K^{\circ}=E$, and (II-79) is the full guiding center Hamiltonian, to $O\left(n^{2}\right)$.

At $O(\varepsilon)$, we begin by using the expressions obtained for $\delta \alpha_{0}, \delta \alpha_{1}$, $\delta \beta_{0}, \delta \beta_{1}, \delta p_{\beta o}, \delta p_{\beta 1}$ [EqS. (II-64), (II-65), (II-71), (II-72] in expres-
sions (II-76) and (II-77), and rewrite $H_{10}$ and $H_{11}$, explicitly displaying their dependences on $\odot$ and $J$ :

$$
\begin{align*}
H_{10} & =\left(C_{13} J^{3 / 2}+C_{11} J^{1 / 2}\right) \cos  \tag{II-80}\\
n H_{11} & =\left(C_{24} J^{2}+C_{22} J\right) \sin 20 . \tag{II-8I}
\end{align*}
$$

Here $C_{m n}$ designates the coefficient of the term in $H_{1}$ in $\cos m o r \sin m=$ and $\mathrm{J}^{n^{\prime 2}-}$. In particular

$$
\begin{aligned}
& C_{13} \equiv \bar{p}\left(\partial \alpha_{o} / \partial P_{b}\right)\left(\partial \Omega_{o} / \partial \varepsilon b\right), \\
& C_{11} \equiv \bar{p}\left[\left(\partial \alpha_{o} / P_{b}\right)\left(\partial V_{o} / \partial \varepsilon b\right)-\left(\partial \alpha_{o} / \partial \varepsilon b\right) V^{\prime}\right], \\
& C_{24} \equiv d_{p}\left(\partial \Omega_{o} / \partial \varepsilon b\right), \quad \text { and } \\
& C_{22} \equiv d_{p}\left(\partial V_{o} / \partial \varepsilon b\right)-d_{b} V^{\prime}-\left(\partial \alpha_{o} / \partial P_{b}\right)\left(\partial V^{\prime} / \partial \alpha_{o}\right) .
\end{aligned}
$$

Following Sec. II.E.2, we can now read off $K_{10}, K_{11}$ :

$$
\begin{equation*}
K_{10}=\bar{H}_{10}=0, \quad n K_{11}=n H_{11}=0 \tag{II-82}
\end{equation*}
$$

In addition, from Eq. (II-74) or (II-79), we have

$$
\begin{equation*}
K_{00}=k^{0}, \eta K_{01}=\eta H_{01}=0, \eta^{2} K_{02}=\eta^{2} H_{02}=\eta^{2} h_{4}^{2} . \tag{II-83}
\end{equation*}
$$

The only term remaining to be found, to the order we are working, is $\mathrm{K}_{20}$. To $O(\varepsilon, \varepsilon \eta, \Pi)$, we see that $K=K_{00}=V_{0}+\Omega_{C} J$, formally the same as the $\varepsilon, \eta \rightarrow 0$ guiding center usually used, but with slightly different definitions of the variables, in agreement with the findings of Refs. 7 and 8. This difference in meaning is dealt with in Sec. II.J.

We now proceed to calculate $K_{20^{\circ}}$. Using Eq. (II-80) in (II-42), we find the $O\left(\eta^{0}\right)$ part, $W_{10}$, of $W_{1}$ to be

$$
\begin{equation*}
W_{10}=-\Omega_{0}^{-1} \int d O^{\prime} H_{10}=-\Omega_{0}^{-1}\left(C_{13^{3}} 3^{3 / 2}+C_{11} J^{1 / 2}\right) \sin \theta, \tag{11-84}
\end{equation*}
$$

where we have used that $\partial H_{0} / \partial J=\partial H_{o o} / \partial J+O\left(n_{i}^{2}\right)=\Omega_{0}+O\left(n_{i}^{2}\right)$. From (II-43) we see that we will need $\mathrm{H}_{20}$ and $1 / 2 \dot{\mathrm{H}}_{10 \theta} \mathrm{H}_{10}$ for $\mathrm{H}_{20}^{1}$. From Eqs. (II-84) and (1I-80) we calculate the latter of these, finding, after a bit of algebra,

$$
\begin{align*}
\frac{1}{2} \dot{W}_{10 \theta} H_{10} & \equiv \frac{1}{2}\left[\left(\partial{k_{10}}_{10} \partial \Theta\right)\left(\partial H_{10} / \partial \mathrm{J}\right)-\left(\partial W_{10} / \partial \mathrm{J}\right)\left(\partial H_{10} / \partial 0\right)\right]  \tag{I-85}\\
& =-\left(4 \Omega_{0}\right)^{-1}\left(C_{11}+C_{13} J\right)\left(C_{11}+3 C_{13} J\right)
\end{align*}
$$

From Eq. (II-78) we rewrite $\mathrm{H}_{20}$, explicitly displaying its $0, \mathrm{~J}$ dependences:

$$
\begin{align*}
H_{20} & =2\left(D_{2} J+D_{4} J^{2}\right) \cos ^{2} \theta,  \tag{II-86}\\
D_{2} & \equiv \frac{1}{2} \bar{p}^{2}\left\{\frac{1}{2}\left(\partial \alpha_{0} / \partial P_{b}\right)^{2}(\partial / \partial \varepsilon b)^{2} v_{0}+\left(\partial \alpha_{0} / \partial \varepsilon b\right)^{2} g^{B}\left(\alpha_{o}, \varepsilon b\right)\right] \\
& +\left(\partial \alpha_{0} / \partial P_{b}\right)^{2}\left(\partial V_{o} / \partial \varepsilon b\right)(\partial / \partial \varepsilon b) \ln \left\{\bar{p}\left(\partial \alpha_{0} / \partial P_{b}\right)\right] \\
& \left.-\left(\partial \alpha_{0} / \partial P_{b}\right)\left(\partial \alpha_{o} / \partial \varepsilon b\right)\left[V^{\prime}(\partial / \partial \varepsilon b) \ln \left[\bar{p}\left(\partial \alpha_{0} / \partial \varepsilon b\right)\right]+(d / d \varepsilon b) V^{\prime}\right]\right\}, \\
D_{4} & \equiv \frac{1}{2} \bar{p}^{2}\left(\partial \alpha_{0} / \partial P_{b}\right)^{2}\left\{\frac{1}{2}(\partial / \partial \varepsilon b)^{2} \Omega_{0}+\left(\partial \Omega_{\mathrm{o}} / \partial \varepsilon b\right)(\partial / \partial \varepsilon b) \ln \left[\bar{p}\left(\partial \alpha_{0} / \partial P_{b}\right)\right]\right\},
\end{align*}
$$

where

$$
(d / d \varepsilon b) \equiv(\partial / \partial \varepsilon b)+\left(\partial \alpha_{0} / \partial \varepsilon b\right)\left(\partial \alpha_{0}\right)
$$

From Eq. (II-86) we see that

$$
\begin{equation*}
\mathrm{H}_{20}=\left(\mathrm{D}_{2} \mathrm{~J}+\mathrm{D}_{4} \mathrm{~J}^{2}\right) \tag{II-87}
\end{equation*}
$$

The final term in Eq. (II-43), $-\dot{H}_{o b} W_{1}=\dot{W}_{1 b} H_{0}$, gives a contribution to $\mathrm{H}_{20}{ }^{1}$ of

$$
\dot{\vec{x}}_{10 \mathrm{~b}} \mathrm{H}_{00} \equiv\left(\partial \mathrm{w}_{20} / \partial \mathrm{cb}\right)\left(\partial \mathrm{H}_{00} / \partial \mathrm{P}_{\mathrm{b}}\right)-\left(\partial \mathrm{W}_{10} / \partial \mathrm{P}_{\mathrm{b}}\right)\left(\partial \mathrm{H}_{00} / \partial c \mathrm{~b}\right) .
$$

From Eqs. (II-74) and (11-84), one sees that this term is proportional to sino, and so its contribution $\overline{\dot{b}_{10 \mathrm{~b}} \mathrm{H}_{00}}$ to $\mathrm{K}_{20}=\mathrm{H}_{20}{ }^{2}$, is zero. From Eqs. (II-43), (II-44), (II-85) and (II-87), therefore, we obtain $\mathrm{K}_{20}$ :

$$
\begin{align*}
\mathrm{K}_{20} & ={H_{20}}^{2} \equiv \overline{{H_{20}}^{1}} \\
& =\left(D_{2} J+D_{4} J^{2}\right)-\Omega_{0}^{-1}\left(C_{11}+C_{13} J\right)\left(C_{11}+3 C_{13} J\right) / 4 . \tag{I-88}
\end{align*}
$$

Assembling the various contributions from Eqs. (II-82), (II-83), and (II-88), then, we write out the guiding center Hamiltonian K, valid to $O\left(\varepsilon^{2}, \varepsilon \sqcap, \eta^{2}\right)$ :

$$
\begin{align*}
K & =K_{00}+\eta^{2} K_{02}+K_{20}+h .0 . \\
& =\left(V_{0}+\Omega_{0} J\right)+\Omega_{0} \bar{a}^{2} J^{2}\left[d_{0}-(3 / 8)\left(c_{3}^{2}+c_{1}^{2}\right)\right] / 8  \tag{II-89}\\
& +\varepsilon^{2}\left[-\Omega_{0}^{-1} C_{11} / 4+\left(D_{2}-\Omega_{0}^{-1} C_{11} C_{13}\right) J+\left(D_{4}-3 \Omega_{0}^{-1} C_{13}^{2} / 4 J^{2}\right] .\right.
\end{align*}
$$

The term $K_{00}$ is the guiding center Hamiltonian, valid to $O\left(\varepsilon, \varepsilon n, r_{1}\right)$. It is formally the same as the lowest order Hamiltonian usually used. As already mentioned in Sec. II.F, however, the gyrofrequency $\Omega_{0}$ is more general than its $L_{s}{ }^{-1} \rightarrow 0$ limit. We shall evaluate this more general form of $\Omega_{0}$ more explicitly in Sec. II.K.

This lowest order term in $K$ yields the expressions commonly used for the particle drifts, including the drift in $\alpha_{0}$ which produces orbits with finite banana width, as described in Sec. II.C. The higher order terms $\eta^{2} K_{o 2}$ and $\varepsilon^{2} K_{20}$ give corrections to these drifts. (We note that the part of this slow shift of $\alpha_{0}(\lambda)$ which is due to a change (with $\beta$ ) of the
contribution $e$ to $V$ corresponds physically to a particle drift in the direction of $E$, due to the particle seeing a time-varying E-field, i.e., to a polarization drift.)

The term $n^{2} K_{02}$ gives the lowest order nonzero corrections to K from the anharmonicity of the potential well $V$, or from the a dependence of $g^{t}$. It is thus a simple finite gyroradius (finite $n$ ) correction, present even in the slab limit $\varepsilon \rightarrow 0$, in which the parameter $\lambda$ is truly frozen.

The term $\varepsilon^{2} K_{20}$ comes from the fact that $\lambda$ is not frozen, but has a jitter at the gyrofrequency. The contribution $\varepsilon^{2} \partial K_{20} / \partial J$ of this term to the gyrofrequency $\Omega$ is the mathematical expression of the modification of $\Omega$ due to the jitter in the potential well $V(\alpha)$, described qualitatively at the end of Sec. II.C. The contribution of the term $\varepsilon^{2} K_{20}$ to the particle drifts in $b, p_{b}\left[\right.$ and so in $\left.a_{0}(\lambda)\right]$, is analogous to the pondemotive force $F_{p}$ for a particle jittering in an electric field $\varepsilon E(x, t)(E \sim 1)$. $F_{p}$ arises from a nonzero time-averaged acceleration at second order in $\varepsilon$. Solving the force equation $\ddot{x}=\varepsilon E(x, t)$ iteratively, one finds a first order jitter in $x(t)$ by freezing $x$ at $x^{(0)}$ on the right hand side: $\ddot{x}^{(1)}=\varepsilon E\left(x^{(0)}, t\right)$. Integrating this to obtain $x^{(1)}(t)=x^{(0)}+\varepsilon \delta x^{(1)}$, the time average of the next iteration gives $F_{p}$ :

$$
\overline{\ddot{x}^{(2)}}=\varepsilon \overline{E\left(x^{(0)}+\varepsilon \delta x^{(1)}, t\right)} \cong \varepsilon \overline{E\left(x^{(0)}, t\right)}+\varepsilon^{2} \overline{\partial x^{(1)} \partial E\left(x^{(0)}, t\right) / \partial x}
$$

Analogously, one may calculate the frozen $-\lambda$ gyroorbit $\alpha^{(0)}(t)$ (for which $n$ is arbitrary, but $\varepsilon$ is effectively set equal to zero), and from this calculate the first order jitter $\varepsilon \delta \beta^{(1)}, \varepsilon \delta p_{B}{ }^{(1)}$ in $\beta$ and $p_{B}$ from

Hamilton's equations for $\dot{B}$ and $\dot{p}_{\beta}$, e.g., $\dot{B}^{(1)}=\left(\partial / \partial p_{\beta}\right) H\left(\alpha^{(0)}, p_{\alpha}{ }^{(0)}\right.$; $\left.{ }_{\varepsilon E}{ }^{(0)}, P_{E}{ }^{(0)}\right)$. Solving for $\varepsilon \delta B^{(1)}(t)$ and taking the time average of the next iteration again gives a nonzero contribution of order $\varepsilon^{2}$.

The present canonical form for K could be readily employed monerically to find guiding center orbits. Specifying the drift surface $a_{b}$ near which the guiding center trajectory is to move determines $P_{\phi}$, as explained at the end of Sec. II.C. Purther specifying the initial guiding center position $\left[\alpha_{0}\left(t_{0}\right), b\left(t_{o}\right)\right]$ then fixes the value of $g_{b}\left(t_{o}\right)$, through solving $\alpha_{0}\left(t_{o}\right)=\alpha_{0}\left[\varepsilon b\left(t_{0}\right), P_{b}\left(t_{0}\right), P_{\phi}\right]$. Finally, specifying the value of the action, Hamilton's equations may be integrated in a straightforward manner. The trajectory in terms of the physical variables ( $\left.\alpha_{0}, b\right)$ may then be easily computed from the canonical variables ( $\mathrm{b}, \mathrm{P}_{\mathrm{b}}, \mathrm{P}_{\phi}$ ) using $\alpha_{o}\left(c b, P_{b}, P_{\phi}\right)$. This is the only additional step required by the canonical formulation beyond those needed in numerically applying the non-canonical formulation employed by Northrop and Rome; ${ }^{8}$ while their heuristically determined expression for $P_{\phi}$ in terms of the physical guiding center variables is foum not to be conserved, the time invariance of $P_{\phi}$ in this canonical formulation is automatic.

## II.J. Generalized Adiabatic Invariant

In the previous section we obtained the guiding center Hamiltonian $K$, to $O\left(\varepsilon^{2}, \varepsilon \eta, \eta^{2}\right)$, expressed in terms of the gyroaction J. Since it is the constancy of $J$ which makes $K$ useful, reducing the number of nontrivial degrees of freedom by one, it is important to be able to express J in terms of more directly physical variables. This expression is the
generalization of the expression $J=\left(m v_{1}^{2} / 2 \Omega_{c}\right) \equiv(m c / e) \mu$ usually used, to the finite $\varepsilon$, finite $n$ regime, and allowing for shear in the magnetic field.

We shall find $J$ valid to $O\left(\varepsilon, \eta^{2}\right)$, ignoring tems of order $\varepsilon n$, $\varepsilon^{2}$, and higher. As for the calculation of $K$, the procedure to higher order is the same, but grows rapidly more complicated.

We assume knowledge of the coordinate system $g^{\mu}(\underline{x}) \equiv\left|\partial q^{\mu} / \partial \underline{x}\right|^{2}$, of the potentials $\underline{A}$ and $\Phi$ which give the magnetic and electric fields, and of the particle position $\underline{x}$ and velocity $\underline{v} \equiv \underline{x}$ at any instant $t$ that we want to compute J. Given the physical specification $\underline{x}, \underline{v}$ of the particle, the values of the variables $\underline{z} \equiv\left(q^{\mu}, p_{\nu}\right)$ in terms of which our initial Hamiltonian $H$ are directly determined, through $q^{\mu}(\underline{x})$, and $p_{\nu}=\left(g^{\nu}\right)^{-1} \dot{q}_{\nu}+e A_{\nu}$. We therefore want to express $J$ in terms of $\underline{z}$.

We begin by writing out the particle energy $E=H(\underline{z})$ in terms of J . From Eq. (II-89), to the order we are working, $E$ is given by

$$
\begin{align*}
E & =K=K_{o o}+\eta^{2} K_{o 2}+\text { h.o. } \\
& =V_{o}+\Omega_{o} J+\Omega_{2} J^{2}+\text { h.o. } \tag{II-90}
\end{align*}
$$

where

$$
V_{0}, \Omega_{0}, \text { and } \Omega_{2} \equiv \Omega_{0} \bar{a}^{2}\left[d_{0}-(3 / 8)\left(c_{3}^{2}+c_{1}^{2}\right)\right] / 8
$$

are functions of $\lambda^{N} \equiv\left(\varepsilon b, P_{b}\right)$. To the order we are working, $\underline{z}=\underline{z}^{1}$, and sc in particular we take $\varepsilon b=\varepsilon b^{1}, P_{b}=P_{b}^{1}$, and we want to express $J$ in terms of $\lambda^{0} \equiv\left(\varepsilon \beta, p_{\beta}\right)$. We thus need to compute the differences between these parameters:

$$
\begin{aligned}
& P_{b}-p_{\beta}=\left(P_{b}-P_{b}^{o}\right)+\left(P_{b}^{o}-p_{\beta}\right)=-\varepsilon\left[W_{10} P_{b}^{o}+\delta p_{R o}\right]+\text { h.o. } \\
& \quad=-\varepsilon\left[W_{10} / \partial \varepsilon \beta-\left(\partial a_{o} / \partial \varepsilon \beta\right) p_{\alpha}\right]+o\left(\varepsilon^{2}, \varepsilon n\right)
\end{aligned}
$$

Using Eq. (II-84), and the relation $\mathrm{J}^{1 / 2} \sin \theta=\delta \alpha / \overline{\mathrm{a}}+O(\varepsilon, \mathrm{n})$ (where $\delta \alpha \equiv \alpha-\alpha_{0}\left(\lambda^{0}\right) 1$, this gives

$$
\begin{align*}
P_{b}-P_{\beta} & =\varepsilon\left\{\left[J_{0}(\partial / \partial \varepsilon \beta)\left(C_{13} / \Omega_{0}\right)+(\partial / \partial \varepsilon \beta)\left(C_{11} / \Omega_{o}\right)\right](\delta \alpha / \bar{a})\right.  \tag{II-91}\\
& \left.+\left(\partial \alpha_{0} / \partial \varepsilon \beta\right) p_{\alpha}\right\}+O\left(\varepsilon^{2}, \varepsilon \eta\right) .
\end{align*}
$$

Here $J_{0}(E, \lambda) \equiv E_{1} / \Omega_{0} \equiv\left[E-V_{c}(\lambda) / \Omega_{0}(\lambda)\right.$ is the $O\left(\varepsilon^{0}, \eta^{0}\right.$, $v$.ue of the adiabatic invariant, somewhat generalized over the expression $m_{l}^{2} / 2 \Omega_{c}$ to allow for finite $\left(v_{\|} / \Omega_{c} L_{s}\right)$ or ( $\left.v_{E} / \Omega_{c} L_{B}\right)$ effects which enter, ev 1 in the zero $n$ limit (c.f. Sec. II.K). $E_{\perp}$ is the kinetic energy in the a direction which the particle would have at its guiding center positic. $\alpha_{0}(\lambda)$, if $\lambda$ were frozen at its instantaneous value.

Similarly, for the B coordinate we get

$$
\begin{equation*}
\varepsilon(b-\beta)=-\varepsilon \delta \beta_{o}+O\left(\varepsilon^{2}, \varepsilon n\right)=-\varepsilon\left(\partial \alpha_{o} / \partial p \quad p_{\alpha}+O\left(\varepsilon^{2}, \varepsilon n\right) .\right. \tag{II-92}
\end{equation*}
$$

Now we invert Eq. (II-90):

$$
\begin{align*}
J=J\left(E, \lambda^{N}\right) & \equiv E_{1} / \Omega_{0}-\Omega_{0} E_{1}^{2} / \Omega_{0}^{3}+\text { h.o. } \\
& =J_{0}\left(E, \lambda^{N}\right)-\left(\Omega_{2} / \tau J J_{0}^{2}+\right.\text { h.o. } \tag{II-93}
\end{align*}
$$

The term in $\Omega_{2}$ is already $O\left(n^{2}\right)$, and so can be evaluated at $\lambda=\lambda^{\circ}$. The term. J $\left(E, \lambda^{N}\right.$ ) we expand to $O(\varepsilon)$ in $\lambda^{N} \cdot \lambda^{\mathrm{O}}$, using Eqs. (II-91) and (II-92). This gives, finally,

$$
\begin{align*}
J & =J\left(E, \lambda^{0}\right)+\varepsilon(b-\beta)\left(\partial J^{0} / \partial \varepsilon \beta\right)+\left(P_{b}-p_{\beta}\right)\left(\partial J_{0} / \partial p_{\beta}\right)+\text { h.o. } \\
& =J_{0}\left(E, \lambda^{0}\right)-\left(\Omega_{2} / \Omega_{0}\right) J_{0}^{2} \\
& +\varepsilon\left\{p_{\alpha}\left[(\partial \alpha / \partial \varepsilon \beta)\left(\partial J_{0} / \partial p_{\beta}\right)-\left(\partial \alpha_{0} / \partial p_{\beta}\right)\left(\partial J_{0} / \partial \varepsilon \beta\right)\right]\right.  \tag{II-94}\\
& \left.+(\delta \alpha / \bar{a})\left[J_{0}(\partial / \partial \varepsilon \beta)\left(C_{13} / \Omega_{0}\right)+(\partial / \partial \varepsilon \beta)\left(C_{11} / \Omega_{0}\right)\right]\right\} \\
& +O\left(\varepsilon^{2}, \varepsilon n, n^{3}\right) .
\end{align*}
$$

All parameters here are evaluated at $\lambda=\lambda^{0}$ and $\alpha_{0}\left(\lambda^{0}\right)$, i.e., at the true particle position. The term in $\Omega_{2}$ is the lowest order finite-n correction which is nonzero. It is of order $\eta^{2}$. The term in $\varepsilon$ enters to compensate for the jitter in $J_{0}\left(E, \lambda^{0}\right)$ coming from its dependence on $\lambda^{0}$, which has a corresponding $O(\varepsilon)$ jitter.

The $\lambda$ dependence of $J$ here enters both explicitly, and through the function $\alpha_{0}(\lambda)$. Until now we have been assuming $\alpha_{0}$ is a known function. We can more explicitly express $\alpha_{0}$, defining

$$
\delta \alpha \equiv \alpha-\alpha_{0}(\lambda) \equiv \delta \alpha_{0}+n \delta \alpha_{1}+n^{2} \delta \alpha_{2}+\cdots,
$$

and solving for the successive $\delta \alpha_{m}$ by expanding the $\alpha$-center condition (II-47) about $\alpha$ :

$$
\begin{align*}
0=V_{-}(\alpha, \lambda) & =V_{1}(\alpha, \lambda)-\delta \alpha(\partial / \partial \alpha) V_{1}(\alpha, \lambda)+\frac{1}{2}(\delta \alpha)^{2}(\partial / \partial \alpha)^{2} V_{1}+\cdots  \tag{11-95}\\
& =V_{1}(\alpha, \lambda)-2 \delta \alpha V_{2}(\alpha, \lambda)+3(\delta \alpha)^{2} V_{3}(\alpha, \lambda)-\cdots
\end{align*}
$$

Substituting the $\eta$ expansion of $\delta \alpha$ into (II-95) and collecting like powers in $\eta$ yields

$$
\begin{align*}
\delta \alpha_{0} & =\left(v_{1} / 2 V_{2}\right), \eta \delta \alpha_{1}=\left(3 V_{3} / 2 V_{2}\right)\left(\delta \alpha_{0}\right)^{2}  \tag{II-96}\\
\eta^{2} \delta \alpha_{2} & =\left[2\left(3 V_{3} / \alpha V_{2}\right)^{2}-\left(4 V_{4} / 2 V_{2}\right)\right]\left(\delta \alpha_{0}\right)^{3}
\end{align*}
$$

and so forth. The $V_{m}$ here are evaulated at $\alpha, \lambda$. Inserting (II-96) into $\alpha_{0}=\alpha-\delta \alpha\left(\alpha, \lambda^{0}\right)$, expression (II-94) for $J$ is expressed purely in terms of quantities defined at the particle position. To more explicitly evaluate $J$, one must assume specific forms for the metric coefficients $g^{\mu}$ and the potentials $\underline{A}$ and $\$$. In the following section, we assume specific simple forms for these, namely forms appropriate to the slab geometry of

Parts I and III, and apply the results of Part II, to prepare for Part III.

## II.K. Application: Slab Geometry with Shear

The stability work of Part III begins with the conductivity kernel of a slab plasma, with arbitrary inhomogeneity in the inhomogeneity direction $x$. This kernel requires a knowledge of the particle orbits and drift frequencies in such a mignetic geometry. In preparation for Part IIl, in this sectiion we apply the results thus far obtained in the present portion of this work, to a slab geometry, allowing for shear and variation in the magnetic field strength $\left(\rho / \underline{L}_{\underline{B}} \neq 0\right)$. As part of this application, we shall be able to specialize the results of Sec. II.J, obtaining a more explicit expression for the gyroaction in the presence of strong perpendicular magnetic inhomogeneity, and, in particular, in the presence of magnetic shear.

The appropriate Hamiltonian is that of Part I [Eq. (I-4)], which is in terms of Cartesian coordinates, for which $g^{\mu \nu}$ is the identity matrix $g^{\mu \nu}=\delta^{\mu \nu}$. This is not the same as the $\varepsilon \rightarrow 0$ limit of the Hamiltonian of Part II, for which the results of this part were developed. (In the limit $\varepsilon \rightarrow 0, \mathrm{H}$ in Eq. (II-5) is the Hamiltonian of a particle in a system with cylindrical symmetry.) However it is one of the advantages of the canonical formulation we are using that formulae in different geometries tend to look the same. Everything developed in the present part of this work is also valid for the Cartesian slab geometries of Parts I and III.

The adoption of a slab geometry, i.e., taking only the $\varepsilon^{0}$ terms in the expressions obtained thus far, is an extreme limit of the theory, but a limit with relevance to the Tormac sheatin, where one may well have $n^{2} \gg \varepsilon$.
II.K.l Specialization to Slab Geometry

Going back to the notation of Part I, we have the sole inhomogeneity direction $x$, corresponding to the coordinate a of Part II, the $y$ direction corresponding to $\beta$, and the zdirection c: responding to $\phi$. The expansion cocfficients $g_{\text {on }}$ are now all zero, except for $g_{0}=1$. We consider a given particle, with guiding center $X_{o}(\underline{P})$, and go into the Cartesian coordinate system natural to the magnetic ficld at $X_{0}$, evaluating the $x$-dependent system $\left[\hat{b}_{1}, \hat{b}_{2}(x), \hat{b}_{3}(x)\right]$ of Part I at $X_{0}$. We recall the definitions of this right handed triad: $\hat{b}_{1} \equiv \hat{x}, \hat{b}_{2} \equiv$ $\hat{b}_{2}\left(X_{0}\right) \equiv \hat{b}_{3} \times \hat{b}_{1}, \hat{b}_{3} \equiv \hat{b}_{3}\left(X_{0}\right) \equiv \hat{B}\left(X_{0}\right)$, the magnetic field direction at $X_{0}$. The corresponding coordiates we denote by ( $x_{1}, x_{2}, x_{3}$ ).

We expand the potentials $\underline{A}=\left(0, A_{y}, A_{z}\right)$ and $\phi$ about $x \approx X_{0}$ :
$\delta \underline{A}\left(\delta x, X_{0}\right) \equiv \underline{A}(x)-\underline{A}\left(X_{0}\right)=\sum_{n=1}^{\infty}(\delta x)^{n}\left(\partial / \partial X_{0}\right)^{n} A_{0} / n!$

$$
\begin{equation*}
=\left[\sum_{n=0}^{\infty}(\delta x)^{n+1}\left(\partial / \partial x_{0}\right)^{n} \underline{B}_{0} /(n+1)!\right] \times \hat{x} \tag{II-99}
\end{equation*}
$$

$$
\begin{equation*}
=-\sum_{n=0}^{\infty}(\delta x)^{n+1}\left(\partial / \partial x_{0}\right)^{n} E_{0} /(n+1)!, \tag{II-100}
\end{equation*}
$$


we express the effective potential $V$ of Eq. (I-5) as

$$
\begin{align*}
V(x \mid \underline{P}) & =V\left(X_{0} \mid \underline{P}\right)+\frac{1}{2} e^{2}|\delta \underline{A}|^{2}-e \delta \underline{A} \cdot v_{0}+c \delta \\
& \equiv \sum_{m=0}^{\infty} V_{m}\left(X_{0}, \underline{P}\right)(\delta x)^{m} \tag{II-101}
\end{align*}
$$

Using (II-99) and (II-100), we can read off the $V_{m}$ s. Procesding up to $m=4$, we find

$$
V_{0} \equiv V\left(X_{0} \mid P ;=\frac{1}{2} V_{0}^{2}+e \Phi_{0}\right.
$$

$$
v_{1} \equiv-e\left(\underline{v}_{0} \times \underline{B}_{0}\right) \cdot \hat{x}-e E_{0}=-e\left(v_{02} B_{0}+E_{0}\right)=0
$$

$$
V_{2} \equiv \frac{1}{2}\left[e^{2} \mathrm{~B}_{0}^{2}-\mathrm{e}{\underset{-0}{ }}_{v_{0}} \times\left(\partial \underline{B}_{0} / \partial X_{0}\right) \cdot \hat{x}-e\left(\partial E_{0} / \partial X_{0}\right)\right]
$$

$$
\begin{equation*}
v_{3} \equiv \frac{1}{2} e^{2} \underline{B}_{0} \cdot\left(\partial \underline{-}_{0} / \partial X_{0}\right)-(e / 3!)\left[\underline{v}_{0} \times\left(\partial^{2} \underline{B}_{0} / \partial x_{0}^{2}\right) \cdot \hat{x}+\partial^{2} E_{0} / \partial x_{0}^{2}\right] \tag{II-102}
\end{equation*}
$$

$$
V_{4} \equiv e\left[\underline{B}_{0} \cdot\left(\partial^{2} \underline{B}_{0} / \partial X_{0}^{2}\right) / 3!+\left(\partial \underline{B}_{0} / \partial X_{0}\right)^{2} / 8\right]
$$

$$
-\left(e^{2} A!\right)\left[\underline{v}_{0} \times\left(\partial^{3} \underline{B}_{0} / \partial X_{0}^{3}\right) \cdot \hat{x}+\partial^{3} E_{0} / \partial X_{0}^{3}\right]
$$

Solving the second of Eqs. (II-102) for $v_{o 2}$ gives

$$
v_{o 2}=-c E_{0} / B_{0} \equiv v_{E}
$$

where we have reinstated the $c$. To $O(\eta)$ this velocity, $v_{o 2} \equiv \dot{x}_{2}\left(X_{0}\right)$, is the same as the guiding center drift $\overline{\dot{x}}_{2}$ in the $\hat{\mathrm{b}}_{2}$ direction, which is therefore the usual $E \times B$ drift $v_{E}$, in this small $\eta$ limit.

For convenience we define

$$
\begin{equation*}
\hat{b}_{ \pm}(x) \equiv \frac{1}{2}\left[ \pm i \hat{b}_{2}(x)+\hat{b}_{3}(x)\right] \equiv \frac{1}{2}\left[ \pm i \hat{y} \quad-\because x p-i \theta_{s}(x)\right. \tag{II-104}
\end{equation*}
$$

and

$$
B(x) \equiv|B(x)| \equiv \bar{B} \exp \Theta_{B}(x)
$$

Here $\bar{B}$ is a constant, and $\kappa_{B, S}(x) \equiv L_{B, S}^{-1}(x) \equiv d 0_{B, S} / d x$. Using these definitions and $\kappa \equiv \kappa_{B}-i \kappa_{S}$, we have

$$
\begin{align*}
& \underline{B}(x)=B(x)\left(\hat{b}_{+}+\hat{b}_{-}\right)=B(x)\left(\hat{b}_{+}+c \cdot c \cdot\right)  \tag{11-105}\\
& \left.\partial B / \partial x=B(x)\left(r \hat{b}_{+}+c \cdot c \cdot\right)=B(x) \hat{b}_{3^{*} B}+\hat{b}_{2}^{r} S\right) \tag{II-106}
\end{align*}
$$

If we in addition neglect all derivatives of $k_{B, S}(x)$, we have in general

$$
\begin{equation*}
\left.(\partial / \partial x)^{n} \underline{B}=B(x) \mid k^{n} \hat{b}_{+}+c \cdot c \cdot\right\} \tag{II-107}
\end{equation*}
$$

Using Eqs. (II-106), (II-103) in the third of expressions (II-102) gives

$$
\begin{align*}
v_{2} & =\frac{1}{2}\left[e^{2} B_{0}^{2}-e\left(v_{0} 2^{k} B-v_{0} 3^{k} S B_{0}-e\left(\partial E_{0} / \partial X_{0}\right)\right]\right.  \tag{11-108}\\
& =\frac{1}{2}\left[e^{2} B_{0}^{2}+e^{k}{ }_{B} E_{0}+e^{k} S{ }_{0}{ }_{0} B_{0}-e\left(\partial E_{0} / \partial X_{0}\right)\right]
\end{align*}
$$

and therefore (restoring the $m$ and $c$ ),

$$
\begin{equation*}
\Omega_{0}^{2} \equiv 2 g_{0} v_{2}=\Omega_{c}^{2}+k_{S} v_{0} \Omega_{c}+\left(\kappa_{E}-\kappa_{B}\right) v_{E}^{\prime \Omega_{c}} \tag{1I-109}
\end{equation*}
$$

where $\quad k_{E} \equiv d \ln E_{o} / d x$.
We see that the gyrofrequency $\Omega$ is different from its homogeneous $\underline{B}$ value $\Omega_{c}$, even in the zero gyroradius $(\eta \rightarrow 0)$ limit, for finite $\left(\kappa_{S} v_{03} / \Omega_{c}\right)$ or ( $\left.\kappa_{B}{ }^{-\kappa_{E}}\right) v_{E} / \Omega_{c}$, as mentioned previously. For substantial ${ }^{\kappa_{S}} v_{03}$, the particle experiences a restoring force $\sim-v_{03} B_{2}(x)$ in the $x$ direction, in addition to the $v_{2} B_{3}$ force which gives the usual $\Omega_{c}$ part of $\Omega_{0}$. Since $B_{2} \sim B_{0}\left(\delta x / L_{S}\right) \sim \delta x$, this additional force senses finite shear, even in the $\eta \rightarrow 0$ limit. Similarly, for substantial $K_{B} y_{E}$, the particle senses a restoring force $\sim v_{2} \delta B_{3}(x)=v_{E} B_{0}\left(\delta x / L_{B}\right)$ in addition to the force giving $\Omega_{c}$. Again, this force is linear in $\delta x$, and so rerormalizes $\Omega_{0}$.

The shifts in $\Omega_{0}$ due to finite $\left(\kappa_{S} v_{03} / \Omega_{C}\right)^{14}$ and $\left(\kappa_{E} v_{E} / \Omega_{C}\right)^{15}$ have been noted by other workers. We can rewrite the last term in Eq. (I1-109) as $\left(\kappa_{E}{ }^{-K_{B}}\right) u_{E} \Omega_{C}=\Omega_{C}\left(\partial v_{E} / \partial x\right)$. This form arose in Ref. 15 for the special case ${ }^{K_{B}}=0$.

## II.K.2. Particle Motion

In preparation for Part III, we solve the particle motion problem to $O\left(r_{1}\right)$. Since $E=0$, the pseudanotion problem of Sec. II.F is the full problem, i.e., $K(J, \lambda)=E(J, \lambda)=h^{\infty}(J, \lambda)$, in the terminology of Eq. (11-59). To $O\left(\varepsilon^{0}, \eta^{1}\right)$, then, we have

$$
\begin{equation*}
K(J, \lambda)=K_{00}(J, \lambda)=V_{0}+\Omega_{0} J \tag{II-110}
\end{equation*}
$$

The $x$ motion has already been solved in Sec. II.F. Replacing da by $5 x$, Eqs. (II-64), (II-65) give us

$$
\begin{align*}
\delta x_{0} & =\bar{x}_{0} \sin \theta  \tag{II-111}\\
n \delta x_{1} & =-\bar{x}_{0}^{2}\left(v_{3} / 4 v_{2}\right)(3+\cos 2 \theta) \tag{II-112}
\end{align*}
$$

where $\bar{x}_{0} \equiv \bar{a} \mathrm{~J}^{1 / 2}$ is the $O\left(n^{0}\right)$ approximation to the particle gyroradius, and $\bar{a} \equiv\left(\Omega_{0} N_{2}\right)^{1 / 2}$. Explicit expressions for $V_{2}$ and $\Omega_{0}$ are given in Eqs. (II-108), (II-109). Using (II-107) in Eqs. (II-102), we obtain a simpler expression for $V_{3}$ :

$$
\begin{align*}
v_{3} & =\frac{1}{2} \kappa_{B}\left(e B_{0}\right)^{2}-\left(e B_{0} 3!\right)\left[\left(\kappa_{B}^{2}-\kappa_{S}^{2}\right) v_{o 2}-2 \kappa_{B} \kappa_{S} v_{o 3}\right]  \tag{II-113}\\
& =m\left\{\kappa_{B} \Omega_{C}^{2} / 2+\left[\left(\kappa_{S}^{2}-\kappa_{B}^{2}\right) v_{E}+2 \kappa_{B} \kappa_{S} v_{o 3}\right] \Omega_{C} / 3!\right\}
\end{align*}
$$

Here, and henceforth, we neglect all derivatives of $E(x)$. In the second line of (II-113), we have restored the $m$ and $c$. Similarly, we may write
$v_{4}$ as

$$
\begin{align*}
& V_{A}=\left(7{k_{B}}^{2}{ }^{-k_{S}}{ }^{2}\right)\left(e B_{0} r^{2} / 24-\left(e B_{0} / 4!\right)\left[\left(x_{B}^{3}-3 r_{B} r_{S}{ }^{2}\right) v_{02}-\left(3 r_{B}{ }^{2} k_{S}{ }^{-k_{S}}{ }^{3}\right) v_{03}\right]\right. \tag{11-114}
\end{align*}
$$

Higher order $\mathrm{V}_{\mathrm{m}}$ 's can be readıly obtained, if desıred.
Equations (II-111) and (II-112), along with the fact that $\Omega=\partial K / \partial J=\Omega_{0}+O\left(r_{1}{ }^{2}\right)$, give a complete description of the $x$ motion. The $x_{2}$ and $x_{3}$ motion then comes from a simple time integration of Hamilton's equations,

$$
\begin{equation*}
\dot{x}_{2}=\{p-e \mathrm{eA}[x(t)]\} \cdot \hat{b}_{2}, \quad \dot{x}_{3}=\{p-\mathrm{eA}[x(t)]\} \cdot \hat{b}_{3} . \tag{11-115}
\end{equation*}
$$

Using $x=x_{0}+\delta x_{0}+n \delta x_{1}$, and the expansion (II-99) for $\dot{\delta} \underline{A}$, we find

$$
\begin{equation*}
x_{2}(0)=x_{20}+\delta x_{20}+\eta \delta x_{21}, x_{3}(\theta)=x_{30}+\eta \delta x_{31} \tag{II-116}
\end{equation*}
$$

where the guiding center coordinates $X_{20}, X_{30}$ evolve 1inearly in time (or 0 ), at rates

$$
\begin{align*}
& \dot{x}_{20}=v_{E}+\left(\bar{x}_{0}^{2} \Omega_{c} / 4\right)\left(3 v_{3} / v_{2}-\kappa_{B}\right) \equiv v_{E}+v_{B},  \tag{II-117}\\
& \dot{x}_{30}=v_{03}+\kappa_{S}\left(\bar{x}_{0}^{2} \Omega_{c} / 4\right), \tag{II-118}
\end{align*}
$$

and the gyromotion $\delta x_{1,2,3}(0, \mathrm{~J})$ about this guiding center, to $O(n)$, is given by Eqs. (II-1i1) (II-112), and:

$$
\begin{align*}
\delta x_{20} & =\bar{x}_{20} \cos \theta, \quad \bar{x}_{20} \equiv\left(\Omega_{c} / \Omega_{o}\right) \bar{x}_{0}  \tag{II-119}\\
n \delta x_{21} & =\left(\bar{x}_{0}^{2} \Omega_{c} / 8 \Omega_{0}\right)\left(3 v_{3} / N_{2}+\kappa_{B}\right) \sin 2 \theta  \tag{II-120}\\
n \delta x_{31} & \left.=-k_{s} \bar{x}_{0}^{2} \Omega_{c} / 8 \Omega_{0}\right) \sin 2 \theta . \tag{II-121}
\end{align*}
$$

Thas completes ti:e mathematical description of the particle motion, to $O(n)$. Several aspects of the motion should be noted. First, already mentioned, is the dependence of the grofrequency on $v_{03} \equiv v_{1}\left(x_{0}\right)$, for finite shear. As we shall see in Part JIJ, this dispersion in sthas a significant effect on the ion Bemstein modes, and the instabilities associated with them. The other two 'frequencies" (i.e., time deritatues $\therefore \equiv \dot{E}$ of the canonical coordinates $\underline{\underline{G}}$ ) in the problem, $\dot{X}_{20}$ and $\dot{X}_{30}$, are also modified by shear. In the limits of $x \rightarrow 0$ and neglagible " $\mathrm{B}^{\mathrm{r}} \mathrm{F} / \mathrm{C}$, , one has $r_{3} / N_{2}=\kappa_{B}$, and $\bar{x}_{0}^{2}=\left(v_{1}^{2} / 2 r_{c}\right)$, and so Eq. (II-117) reduces to
 see that shear also modifies the particle parallel volucity, a fact noted previously in the literature. ${ }^{16}$

We also note modifications in the gyromotion about the guiding center, induced by shear. Comparing Eqs. (II-111) and (II-119), one sees that, even at $O\left(\eta^{\circ}\right)$, finite shear distorts the gyrooroit from a c rcular to an elliptical shape. At $O(n)$, the corrections nox $\mathrm{x}_{\mathrm{il}}$ ( $\mathrm{i}=1,2,3$ ) further distort the orbit. In particular, $n \delta x_{31}$ causes the orbit to twist in the direction of magnetic shear, cuming the elliptical orbit into a "'potato-chip" form.

The canonical framework and graphical techniques of Part I enaole one to understand qualitatively many of these shear-induced modifications. For example, the correction to $\dot{X}_{30}$ in Eq. (II-118) may be understood from referring to Fig. 2. We recall that $v_{2} \hat{b}_{2}+v_{3} \hat{\mathrm{~b}}_{3}=\underline{P}-\mathrm{e} A(x)$. As $x$ oscillates about $X_{0}, \underline{A}(x)$ oscillates on the $A$ trajectory about A( $X_{0}$ ). For a particle with positive parallel velocity, such as at point c in Fig. 2, the shear causes the erajectory $A(x)$ to bend away from point
$P$ as $x$ moves in either direction from $X_{0}$, thereby causing the parallel velocity $v_{3}$ to have a positive addition to its value $v_{03}$ at $X_{0}$. Conversely, the curvature of A toward point b causes $\left|\dot{x}_{30}\right|$ to be smaller, for particles with negative parallel velocity. In a similar way, one sees that the curvature $v_{2}$ of the potential well $v=\frac{1}{2}(P-e A)^{2}$ is enhatced by shear for a particle with momenti $P$ at point $c$, and lessened for P at point b . This explains the finite ${ }^{x}$ correction io $?_{0}$ in Eq. (1I-109).

## H.K. 3 Adrabatic Invariant

We now specialize the results of Sec. II.J to the present geometry, calculating $J$ to $O\left(r_{1}\right)$. Defining $v_{o l} \cong \mathrm{p}_{x}^{2}\left(X_{0}\right)(m=1)$, we have $\mathrm{E}=\frac{1}{2}\left(\mathrm{v}_{\mathrm{o} 1}{ }^{2}+\mathrm{v}_{\mathrm{o} 2}{ }^{2}+\mathrm{v}_{03}{ }^{2}\right)+e \downarrow\left(\mathrm{x}_{\mathrm{o}}\right)$, and so $\mathrm{E}_{1} \equiv \mathrm{E}-\mathrm{V}_{\mathrm{o}}=\frac{1}{2} \mathrm{v}_{\mathrm{o} 1}{ }^{2}$, the kinetic energy in the $x$ direction at the guiding center. Equation (II-94) the tells us that ( $\mathrm{x} \equiv \alpha, \mathrm{x}_{\mathrm{o}} \equiv \alpha_{0}$ )

$$
\begin{align*}
J & =J_{0}\left(E, x_{0}, \lambda\right)+v\left(n^{2}\right)  \tag{II-122}\\
& =m v_{o 1}^{2} / 2 \Omega_{0}\left(x_{0}\right)+o\left(n^{2}\right)
\end{align*}
$$

To $O\left(n^{0}\right)$ we may replace $\Omega_{0}\left(X_{0}\right)$ by $\Omega_{0}(x)$ and (II-122) evidently reduces to the usual expression $\mathrm{mv}_{1}{ }^{2} / 2 \Omega_{c}$, valid in the limits of negligible $\left(\kappa_{S} v_{o 3} / \Omega_{c}\right)$ and ( $\kappa_{B} v_{o 2} / \Omega_{c}$ ). Even to $O\left(n^{0}\right)$, however, we see that these parameters modify $J$.

We note that, expressed in terms of $J_{0}\left(E, X_{0}, \lambda\right)$, the form of $J$ at $O(n)$ is the same as that for $O\left(\eta^{0}\right)$. If we express $X_{0}$ as $x-\delta x(x, \lambda)$, however, the first order form is different from the one valid at zero order. Consistent with the ordering described in Appendix B, we have

$$
\begin{align*}
v_{0}\left(x_{0}\right) & =v_{0}(x)-\delta x v_{1}(x)+(\delta x)^{2} v_{2}(x)-(\delta x)^{3} v_{3}(x)+0\left(n^{2}\right)  \tag{II-123}\\
& =\left[v_{0}-\left(\delta x_{0}\right)^{2} v_{2}\right]-\left(\delta x_{0}\right)^{3} 3 v_{3}+0\left(n^{2}\right)
\end{align*}
$$

where the $V_{m}$ on the second line are evaluated at $x$, and $\delta x_{0} \equiv V_{1} / 2 V_{2}$, from the first of EqS. (II-96).

In addition, for $\Omega_{0}$ we have

$$
\begin{align*}
\Omega_{0}\left(x_{0}\right) & =\Omega_{0}(x)-\delta x(\partial / \partial x) \Omega_{0}(x)+o\left(n^{2}\right)  \tag{II-124}\\
& =\Omega_{0}(x)\left[1-\left(\delta x_{0}\right)\left(3 V_{3} / 2 V_{2}\right]+o\left(n^{2}\right)\right.
\end{align*}
$$

Putting (II-123) and (II-124) into (II-122), we obtain an expression for $J$ entirely in terms of the quantities $E, V_{0}, V_{1}, V_{2}, V_{3}$, all evaluated at $x$ :

$$
\begin{align*}
J & =\Omega_{0}^{-1}\left(x_{0}\right)\left[E-v_{0}\left(X_{0}\right)\right]  \tag{II-125}\\
& =\Omega_{0}^{-1}(x)\left[E-v_{0}(x)+\left(\delta x_{0}\right)^{2} v_{2}+3\left(\delta x_{0}\right)^{3} v_{3}\right][1 \\
& \text { At } O\left(n^{0}\right), \text { this reduces to } \\
J & =\Omega_{0}^{-1}(x)\left[E-v_{o}(x)+\left(\delta x_{0}\right)^{2} v_{2}\right]+O(n)  \tag{II-126}\\
= & \frac{1}{2}\left[p_{x}^{2}+\left(\dot{p}_{x} / \Omega_{0}\right)^{2}\right] / \Omega_{0}(x)+O(n),
\end{align*}
$$

$$
=\Omega_{0}^{-1}(x)\left[E-V_{0}(x)+\left(\delta x_{0}\right)^{2} V_{2}+3\left(\delta x_{0}\right)^{3} V_{3}\right]\left[1-\left(\delta x_{0}\right)\left(3 V_{3} / 2 V_{2}\right)\right]^{-1}+0\left(\pi^{2}\right)
$$

where we have used $V_{0}(x) \equiv V(x \mid \lambda)=E-\frac{1}{2} p_{x}^{2}$,
$V_{1}(x) \equiv(\partial / \partial x) V(x \mid \lambda)=-\dot{p}_{x}$, and $V_{2}(x)=\frac{1}{2} \Omega_{0}^{2}(x)$. In the limit of negligible $v_{E} / v_{1}, \dot{p}_{x}=v_{2} \Omega_{c}(x)$ (where $v_{i} \equiv b_{i}(x) \cdot \underline{v}$ is the particle velocity in the $i$ direction of the coordinate system natural to position $x$ ), and, using $P_{x} \equiv v_{1}$, Eq. (II-126) may be written

$$
\begin{equation*}
J=\frac{1}{2}\left[v_{1}^{2}+v_{2}^{2}\left(\Omega_{c} / \Omega_{0}\right)^{2}\right] / \Omega_{0}+O(n) \tag{II-127}
\end{equation*}
$$

where again, all quantities are evaluated at $x$. We note the "elliptical distortion" of this form from the unsheared circular limit in which $v_{1}^{2}=v_{1}^{2}+{v_{2}}^{2}$ appears, parallel to the distortion in Eqs. (II-111), (11-119) of the $O\left(n^{0}\right)$ gyroorbit.

The higher onder expression for $J$, Eq. (II-125), involves $V_{3}(x)$, which is given in Eqs. (II-102) or (II-113). The calculation of $J$ to still higher orders may be analogously carried out, bringing in higher$m V_{m}$, and correspondingly higher order derivatives of $B(x)$.

## II.L. Directions for Extension of the Formalism

In the development of the formalism of the present part of this thesis, we have made a number of sinplifying assumptions. These were useful in developing the fomal procedures, but now that the procedures have been established, the assumptions may be weakened, and the range of applicability of the theory may thereby be enlarged.

One assumption inplicity made throughout is that resonance effects between the gyromotion and bounce motion are negligible. Thus, one removes the dependence of H on the gyrophase $\theta$ to all orders in $\varepsilon$ ending finally with a 0 -independent Hamiltonian $K$, for which no resonance effects are possible.

To study possible resonance effects in the framework developed here, we begin with the Hamiltonian $\mathrm{H}^{\mathrm{O}}\left(\underline{2}^{\mathrm{O}}\right)$ of Sec. II.H, the starting point for stage two of the transformation process. Instead of neglecting resonance effects as done in stage two, one may instead first solve for the umperturbed bounce motion, taking $H_{0}^{0}\left(J^{0}, \varepsilon b^{0}, P_{b}^{0}\right)$ as the unperturbed Hamiltonian. $H_{0}^{0}$ has one nontrivial degree of freedom, and so the bounce
motion may be solved for exactly. Analogous to the way in which the one-dimensional, frozen-x problem for the gyromotion was solved, we transform fron $\mathrm{f}^{\circ}$, $\mathrm{P}_{\mathrm{b}}{ }^{\circ}$ ) to $\left(\mathrm{C}_{\mathrm{b}}{ }^{\circ}, \mathrm{J}_{\mathrm{b}}^{\mathrm{o}}\right.$ ), the action-angle variables of the bounce motion. The Hamiltonian $f^{\circ}$ ini terms of these variables then has a lowest order term $H_{0}^{\circ}=H_{0}^{\circ}$ which is independent of both angle variables, $\pi_{0}^{0}=H_{0}^{\circ}\left(J^{\circ}, J_{h}^{0}\right)$. The higher order tems $\mathcal{H}_{\hat{X}}^{0}$ are in general dependent on both angle variables, and this coupling allows for the study of resonance effects. From this, one obtains the resonance condition $0=n_{1}-n_{2} \psi_{n}$ (where $\overline{-}$ is the bounce averaged grofrequency), the condition for violation of the perturbative nature of the transformations made in stage two.

Another assumption we have made is that of perfect axisymetry. One would like to remove this assumption to study geometries for which the toroidal symmetry is weakly broken, either due to unintentional field imperfections, as in Tokamaks, or due to an intentional design for confinement, as in Elmo Bumpy Torus.

The central difficulty overcome by the formalism we have developed here is inducing the finite canonical transformation $S$ to action angle variables $\left(\Theta^{\circ}, J^{o}\right)$, which have the property of lying sufficiently close to the desired variables ( $\theta, \mathrm{J}$ ) (in terms of which K is expressed) that the remaining transformation $\mathrm{T}:\left(\theta^{\circ}, J^{\circ}\right) \rightarrow(\theta, J)$ may be achieved by purely perturbative means. Inducing the perturbative transformations comprising T is a relatively simple and straightforward matter, especially when aided by Lie techniques. In view of this, the procedure developed here may be expected to be applicable to particle motion in any geonetry in which the procedure we have used for inducing $S$ again gives actionangle variables which are only perturbatively far away from the true gyroaction $J$ (the adiabatic invariant) and its conjugate angle $\theta$.

In the present concext, one may consider breaking the toroidal symmetry in either of two ways. One may allow any or all of the metric coefficients $g^{\mu}$ (as well as $\&$ and the components $A_{\mu}$ of the vector potential), formerly independent of $\phi$, to have either a small additional $¢$ dependent part; $g^{\mu}=g_{0}^{\mu}(\alpha, \varepsilon \varepsilon)+\delta g_{1}^{d}(\alpha, \mathcal{E}, \ddagger)$, or a slow dependence on $\varphi$, as well as on $\varepsilon ; g^{\mu}=g^{\mu}(\alpha, \varepsilon \in, \delta \&)$. (The quantity $\delta$ here is the parameter of smallness.)

In the former case, a nonzero perturbation $A_{\alpha 1}$ of $A_{\alpha}$ is permussible; in the latter case it is not, since there would then be no acceptable wiperturbed Hamiltonian having the same formal similarity to a onedimensional particle motion problem as does H in Eq . (II-5). In both cases, one may follow a procedure fully analogous to that described in Sec. II.D to induce the transformation $S$. The resultant Hamiltonian $H^{\circ}$ will then have an $O\left(\varepsilon^{\circ}, \delta^{\circ}\right)$ part independent of $0^{\circ}$, and $0^{\circ}$-dependent tems higher order in both $E$ and $\delta$. The remainder of the transformation process is thus again perturbative in character, and so should be executable without great difficulty. The generators of these transformations will now be $\phi$ dependent. Resonance effects may also be studied as outlined above for the $\phi$-dependent case, with a more flexible resonance condition, $0=n_{1} \bar{\Omega}+n_{2} u_{b}+n_{3} \omega_{\phi}$, arising from the new characteristic frequency $\omega_{\phi}$, describing the time required for the particle to traverse one period in the toroidal perturbation.

Finally, one may relax the assumption of time-translational invariance, i.e., one may allow the $g^{\mu}$ and potentials $\phi$ and $A$ to have a weak time dependence, in either of the two senses just discussed in connection vith breaking of rotational symmetry. In this way one may study, for example, the effects of an electrostatic perturbation $\delta \Phi_{1} \sim e^{-i \omega t}$ on
the particle motion. The transformations would then be time dependent, and the resonance condition would take the form

$$
c=n_{1} \bar{\lambda}+n_{2} u_{b}+n_{3} u_{d}+n_{4} w
$$

## APPENDIX E , ORDERING SCHEME OF PART II

In this Appendix we clarify certain aspects of the ordering scheme we use in comparing terms we find in the expansions performed in Part 11.

Wc define the scale length $\bar{\alpha}_{0}$ of perpendicular inhomogeneity by

$$
\begin{equation*}
\left.\bar{\alpha}_{\mathrm{o}} \equiv \min \left(V_{2} / V_{3}, \xi_{n} / g_{1}\right)=\min \left[\partial \ln V_{2} / \partial \alpha_{0}\right]^{-1},\left(\partial \ln g_{0} / \partial \alpha_{0}\right)^{-1}\right] . \tag{B-1}
\end{equation*}
$$

The perpendicular expansion parameter $\eta$ is then given by $n \equiv \delta \alpha / \bar{s}_{0}$, and we assume that

$$
\begin{equation*}
v_{m} \delta \alpha / V_{m-1} \leqslant n<1(m \geqslant 3), \quad y_{m} \delta \alpha / g_{m-1} \leqslant n<1(m \geqslant 1) . \tag{B-2}
\end{equation*}
$$

In contrast to the ordering in ( $B-2$ ), since $V_{0}$ is on the order of the particle's parallel energy at the guiding center, and $V_{2}(\delta \alpha)^{2}$ is on the order of its perpendicular energy, one has $v_{0} \sim v_{2}(\delta \alpha)^{2}$, for a typical particle. From this orderang, and from the definitions [cf. Secs. II.F, II.G] of the $c_{m}$ (including $c_{\alpha}, c_{p}$, and $c_{s}$ ), all of these parameters are on the order of $\bar{\alpha}_{0}{ }^{-1}$, and all the $d_{m}(m=0,2,4)$ are on the order of $\bar{\alpha}_{0}{ }^{-2}$.

While the expansion coefficients $V_{m}\left(\alpha_{0}, \lambda\right)$ have this slow dependence on $\alpha_{0}, \partial V_{m} / \partial \alpha_{0} \sim V_{m} / \bar{\alpha}_{0}$, we note that the potential $V(\alpha \mid \lambda)$ itself has a stronger $\alpha$ dependence: $\quad \partial V / \partial \alpha \sim V_{2} \delta \alpha \sim V / \delta \alpha=\eta^{-1} \mathrm{~V} / \ddot{\alpha}$.

In addition to an $\alpha_{0}$ dependence, many of the parameters of the theory (including $\alpha_{o}$ itself) have a dependence on $\lambda_{1} \equiv \varepsilon \beta$ (or $\varepsilon b$ ), $\lambda_{2} \equiv p_{\bar{\beta}}$ (or $P_{b}$ ). We define characteristic scale lengths $\bar{\lambda}_{i}(i=1,2)$ in the variables $\lambda_{i}$ by

$$
\begin{equation*}
\left.\lambda_{i} \equiv \min \left[\partial \ln v_{2} / \partial \lambda_{i}\right)^{-1},\left(\partial \ln g_{0} / \partial \lambda_{i}\right)^{-1}\right] \tag{B-3}
\end{equation*}
$$

in analogy with (B1). We assume that the various parameters $f$ of the theory (e.g., $f=v_{m}, g_{m}, \delta a, h_{m}$ ) all have comparable dependences on the $\lambda_{i}$. In ordering the various terms arising in calculations (as in, e.g., Sec. II.G), we shall thus assume

$$
\begin{equation*}
\partial f / \partial \alpha_{0} \sim f / \bar{a}_{0}, \quad d f / \partial \lambda_{i} \sim f / \bar{\lambda}_{i} . \tag{B-4}
\end{equation*}
$$

We now apply this ordering scheme to a particular example, ordering the three terms in Eq. (II-70). We have
$\left(\right.$ term $2 /$ term 1) $=\left(\partial \delta \alpha / \partial \lambda_{i}\right) /\left(\partial \alpha_{0} / \partial \lambda_{i}\right) \sim\left(\delta \alpha / \lambda_{1}\right) /\left(\alpha_{0}{ }_{0} \lambda_{1}\right)$

$$
\begin{equation*}
=\delta \alpha / a_{0}=\eta, \tag{B-5}
\end{equation*}
$$

and
$($ term $3 /$ term 1$) \sim\left(\bar{a} j^{1 / 2} c_{s} / \bar{\lambda}_{1}\right) j /\left(p_{\alpha 0} \bar{\alpha}_{0} / \bar{\lambda}_{1}\right)$

$$
\begin{equation*}
\sim n j /\left(p_{\alpha 0} \bar{\alpha}_{o}\right) \sim n^{2} \tag{B-6}
\end{equation*}
$$

where we have used $j \sim p_{d o} \delta \alpha$ and $\delta \alpha \sim \bar{a} j^{1 / 2}$. The, are the orderings used in Sec. II.G in calculating $\delta B$ and $\delta p_{\beta}$.

## PART III. STABILITY IN STRONGLY INHOMOGENEOUS PLASMAS

## III.A. Introduction to Part III

The final part of this work will be concerned with the development of a systematic framework for obtaining disperson equations for the study of normal modes in a strongly inhomogeneous plasma. The formalism to be developed begins with Poisson's equation, and is thus restricted in validity to electrostatic pertubations. In addition, we assume a slab geometry, as in Part I. Generalization of the formalism to the study of arbitrary electromagnetic perturbations should not be difficult, and generalization to allow for more complex geometries may also be possible. These matters are discussed further in Sec. III.E. The formalism to be developed is aimed at characterizing the problem of normal modes in strongly inhomogeneous plasmas in a manner which resembles as closely as possible the more familiar formulations of the simpler problem of stability in weakly inhomogeneous plasmas, while still maintaining an exact formulatic $\Omega$, so that when information is finally discarded in order to achieve a mathematically tractable problem, it can be done in a systematic and consistently ordered fashion.

We are aided in this pursuit by two mathematical tools in particular. The first is a canonical framework, as in Parts I and II. The second is a variational formulation of the problem, which is useful both in insuring that approximation schemes are applied in a consistent manner, ${ }^{17}$ and also in lessening the work involved in obtaining dispersion equations. $18,19,20$

Use of the variational formulation is introduced in Sec. III.B, in which the formalism is developed in a very general but abscract fashion, using only a single requisite property [Eq. (III-4)] of the conductivity kernel $\bar{K}_{s}$. From the variational formulation, one may follow two routes. One can develop differential equations describing the spatial variation of the eigennodes $\phi(x)$, which are approximations to the infinite order differential equation (d.e.) which describes the exact problem. Alternatively, if one can make a guess at the form of the solutions of these d.e. $s$, one can derive dispersion equaticns directly from the variational principle, even in parameter regimes where truncation of the d.e. at finite order is invalid. We shall follow both of these routes. The two approaches, solving d.e.'s or use of a variational principle, complement each other. The d.e. approach has more direct connection to the physics, and to the original equation we want to solve, Eq. (III.3). The truncated d.e. we will be solving will be formally very similar to the Schrödinger equation, and this formal resemblance will supply additional insight into what sorts of solutions $\phi(x)$ to expect. Given a good guess at how $\phi(x)$ looks, determined from insight derived from the d.e. approach, the variational approach supplies a computationally more efficient means for obtaining dispersion equations.

In Sec. III.C the abstract formalism of Sec. III.B is given physical content, by introducing the explicit form of the conductivity kernel $\bar{K}_{s}$. It is here that use of a canonical framework becomes particularly helpful. The resultant formalism is applicable to slab geometries with arbitrary inhomogeneity in the density and magnetic field (including

Shear), and yet emerges in a form which transparently reduces in the appropriate ilmits to the weakly inhomogeneous expressions comnonly used. Two particularly useful aspects of this should be pointed out. The first of these is the conceptually unifying aspect of such a formalism. Dispersion equations applicable to different particular geometries now arise as different limits of the same dispersion equation, eliminating the need for going through an analogous but new derivation of the dispersion equation for each particular geonetry. The second useful aspect is that such a formalism permits a more careful derivation of the various effects present in a given situation. Much previous work has involved ad hoc derivations or statements of the appropriate equation to be studied, obtained by intuitive genexalizations of the equations applicable to simpler situations. Most notable among these are the equations given for the study of stability of sheared plasmas. In certain cases the present formalism serves as a more rigorous justification of previous work, or finds limits on the validity of that work. In still Other cases, we have found effects (to our knowledge previously undiscovered), which can be significant in certain situations (in particular in sheared plasmas). These emerge from the canonical formulation in an automatic and natural fashion, and the physical origin is discernable. One example of these effects is the fact that, in a sheared magnetic field, the diamagnetic drift frequency $\omega_{*}$, which plays a central role in dispersion equations for drift modes, is modified, having a dependence on the same paraneter ( $v_{\|} / L_{s} \Omega_{c}$ ) which, as we saw in Part II, also modifies the gyrofrequency.

In Sec, III.D we apply the formalism to a study of the ion drift cyclotron instability, in both unsheared and sheared magnetic fields. Study of this instability allows some of the useful and novel aspects of the formalism to be illustrated. Discussion of possible useful directions for extension of the formalism is given in Sec. III.E.

For the present part of the thesis, we need to distinguish between the various inhomogeneity parameters designated by $n$ in Parts I and II. Thus we denote by $n_{n}, n_{s}, n_{B}$ and $\eta_{B}$ the ratio of ion gyroradius $\rho_{i}$ to the scale lengths of density ( $L_{n}$ ), shear ( $L_{S}$ ), magnetic field modulus ( $L_{B}$ ), and magnetic field $\left[L_{B} \equiv \min \left(L_{S}, L_{B}\right)\right]$, respectively, and define $n \equiv \max \left(\eta_{n}, \eta_{B}\right)$.

## III.B. Poisson's Equation; General Formalism

## III.B.1. Exact formulation

Restricting ourselves to electrostatic perturbations, we take for our wave equation Poisson's equation,

$$
\begin{equation*}
\nabla^{2} \phi(\underline{x})=-4 \pi \rho(\underline{x}) . \tag{III-1a}
\end{equation*}
$$

Here $\phi(\underline{x})$ is the perturbed electrostatic potential, and $\rho(\underline{x})$ is the perturbed charge density at point $x$.

We assume that $\rho(\underline{x})$ is given by the sum over contributions $\rho_{s}(\underline{x})$ of the linear response of each species s to $\phi$. Taplace transforming in time, we write this response in the frequency domain as

$$
\begin{equation*}
-4 \pi \rho_{s}(\underline{x} ; \omega)=\int d \underline{x}^{\prime} \bar{K}_{s}\left(\underline{x}, \underline{x}^{\prime} ; \omega\right) \phi\left(\underline{x}^{\prime} ; \omega\right) \tag{III-1b}
\end{equation*}
$$

where $w$ is the component of frequency. Using Eq. (III-lb) in (III-1a),
we obtain the wave equation for $\phi$, exact, within the confines of linear theory, regardless of geometry:

$$
\begin{equation*}
\nabla^{2} \phi(\underline{x} ; \omega)=\sum_{s} \int d \underline{x}^{\prime} \bar{K}_{s}\left(\underline{x}, \underline{x}^{\prime} ; \omega\right) \phi\left(\underline{x}^{\prime} ; \omega\right) . \tag{III-1C}
\end{equation*}
$$

Specializing to a slab geometry makes the dependence of $\phi$ on the homogeneity directions $y$ and 2 trivial,

$$
\begin{equation*}
\phi(\underline{x})=\phi(x) \exp i\left(k_{y} y+k_{z} z\right) . \tag{111-2}
\end{equation*}
$$

This leaves us with a one dimensional wave equation to solve, parametrically dependent on the constants $k_{y}, k_{z}$, just as the assumption of slab geometry in Parts I and II in the study of particle motion reduced that problem to a one dimensional one, parametrically dependent on the momenta $\underline{P} \equiv\left(P_{y}, P_{z}\right)$. Inserting (III-2) into (III-1) yields

$$
\begin{equation*}
0=\left(-\partial_{x}^{2}+k_{y}^{2}+k_{z}^{2}\right) \phi(x)+\sum_{s} \int d x^{\prime} K_{s}\left(x, x^{\prime} ; \lambda\right) \phi\left(x^{\prime}\right) \tag{III-3}
\end{equation*}
$$

where $\lambda \equiv\left(\omega, k_{y}, k_{z}\right)$, and $\partial_{x}$ denotes derivative with respect to $x_{\text {. }}$ In Sec. III. C we will consider the explicit form of the conductivity a kernel $K_{S}$, derived from first principles in Appendix C. Here we need only use a certain property of $\bar{K}_{s}$ to be shown there, namely that $K_{s}$ can be written as an ineegral over contributions $K_{S}$ to $\bar{K}_{s}$ from particles having guiding center $X_{0}$ :

$$
\begin{equation*}
\overline{\mathrm{R}}_{\mathrm{s}}\left(x, x^{\prime} ; \lambda\right)=\int d X_{0} \mathrm{~K}_{\mathrm{s}}\left(\sigma, X_{0}, \sigma^{\prime} ; \lambda\right) \tag{III-4}
\end{equation*}
$$

where $\sigma \equiv x-X_{0}, \sigma^{\prime} \equiv x^{\prime}-X_{0} \cdot K_{s}$ bears a more direct relation to the expressions for susceptibilities usually used in weakly inhomogeneous stability calculations, as we shall see.

As noted in the Introduction to Part III, it will be computationally useful to phrase the stability problem in terms of a variational principle. For any two functions $\phi(x), \psi(x)$, normalized so that $\int d x|\phi(x)|^{2}=1=\int d x|\psi(x)|^{2}$, we define the functional $S(\psi, \phi)$ by $S(\psi, \phi) \equiv S_{V}(\psi, \phi)+\sum_{S} S_{S}(\psi, \phi)$

$$
\begin{equation*}
\equiv \int d x \psi^{*}(x)\left(\partial_{x^{3}} \vec{x}^{+} k_{y}^{2}+\mathrm{K}_{z}^{2}\right) \phi(x)+\sum \int \mathrm{d} x \int \mathrm{~d} x^{\prime} \psi^{*}(x) \overline{\mathrm{K}}_{\mathrm{s}}\left(\mathrm{x}, \mathrm{x}^{\prime}\right) \phi\left(\mathrm{x}^{\prime}\right), \tag{I1I-5}
\end{equation*}
$$

where $\vec{\partial}_{x} \equiv \partial_{x}$, and $\dot{\partial}_{x}$ denotes an $x$ derivative acting to the left. (The subscript ' $V$ ' on $S_{v}$ denotes 'vacuum.") $S(\phi, \phi)$ is closely related to the additional energy present in the plasma in the presence of perturbation $\phi$, as noted in the variational formulation of Berk and Dominguez, ${ }^{17}$ Setting the variation of $S$ with respect to $\psi^{*}$ equal to zero gives back the basic wave equation (II1-3):

$$
\begin{equation*}
0=\left[\delta / \delta \psi^{*}(x)\right] S(\psi, \phi) \tag{III-6}
\end{equation*}
$$

Here $\delta / \delta f(x)$ denotes functional differentiation with respect to any function $f(x)$. Though we will not be making use of it subsequently, we note that a related equation for $\psi^{*}(x)$ is obtained by taking $\delta S / \delta \phi(x)$ :

$$
\begin{equation*}
0=-\left(\partial_{x}^{2}+k_{y}^{2}+k_{z}^{2}\right) \psi^{*}(x)+\sum_{s} \int d x^{\prime} \psi^{*}\left(x^{\prime}\right) \bar{K}_{s}\left(x^{\prime}, x ; \lambda\right) . \tag{III-7}
\end{equation*}
$$

From the explicit form [Eq. (III-48)] for $\bar{K}_{s}(x, x ; w)$, introduced in Sec. III.C.1, we see that

$$
\begin{equation*}
\left[\bar{K}_{s}\left(x, \underline{x} \underline{x}^{\prime} ; w\right)\right]^{*}=\bar{K}_{s}\left(\underline{x}^{\prime}, x ; w^{*}\right), \tag{III-8}
\end{equation*}
$$

where "*' denotes complex conjugation. Actually, this form for $\bar{K}_{s}$ is valid only for $\omega$ in the upper-half plane; for $\omega$ in the lower-half plane, an analytic continuation of this form must be made.

However, for the moment we denote by $\overline{\mathrm{K}}_{\mathrm{s}}$ only the form given in (III-48), so that Eq. (III-8) holds, as does the analogous relation for the $1-\mathrm{L})$ kernel, $\quad\left[\bar{K}_{s}\left(x, x^{\prime} ; \omega, k_{y}, k_{z}\right)\right]^{*}=K_{s}\left(x^{\prime}, x\right.$; $\omega^{*}, k_{y}, k_{z}$ ). Using this in (III-7) and complex conjugating, one obtains Eq. (III-3), with $\psi(x)$ replacing $\phi(x)$. Thus, $o=\delta S / \delta \phi(x)$ yields Poisson's equation for $\psi(x)$, taking this interpretation of $\bar{K}_{s}$.

Now we make use of the form (III-4). Putting this form into the expression in Eq. (III-5) for the contribution $S_{S}$ to $S$ and replacing $X_{0}$ by $x$, by the following manipulations we obtain a compact expression for $S_{s}$ :

$$
\begin{align*}
S_{S}(\psi, \phi) & =\int d x \int d \sigma \int d \sigma^{\prime} \psi^{*}(x+\sigma) K_{S}\left(\sigma, x \sigma^{\prime}\right) \phi\left(x+\sigma^{\prime}\right) \\
& =\int d x \int d \sigma \int d \sigma^{\prime} \psi^{*}(x) \exp \left(\sigma^{\star}{ }_{x}\right) K_{S}\left(\sigma, x, \sigma^{\prime}\right) \exp \left(\sigma^{\prime} \vec{\partial}_{x}\right) \phi(x) \\
& =\int d x \psi^{*}(x) K_{S}\left(i \mho_{x}, x, i \vec{J}_{x}\right) \phi(x) . \tag{III-9}
\end{align*}
$$

In the third line, the infinite order differential operator $K_{s}\left(i \delta_{x}, x, i \vec{J}_{x}\right)$ is understood to mean the kernel $\mathrm{K}_{\mathrm{s}}$, Fourier transformed in arguments $\sigma$ and $\sigma^{\prime} ;$

$$
\begin{equation*}
K_{s}\left(k, x, k^{\prime}\right) \equiv \int d \sigma \int d \sigma^{\prime} e^{-i k \sigma} K_{s}\left(\sigma, x, \sigma^{\prime}\right) e^{-i k^{\prime} \sigma^{\prime}}, \tag{III-10}
\end{equation*}
$$

and with the replacements $k+i J_{x}, k^{\prime}+i J_{x}$.
(Using the same symbol $K_{s}$ for both $K_{s}\left(\sigma, x, \sigma^{\prime}\right)$ and its Fourier transform presents no ambiguity, if one simply notes whether the units in the arguments are $\sigma \sim \mathrm{x}$. or $\mathrm{k} \sim \mathrm{x}^{-1}$. The formal manipulations using the displacement operator $\exp \left(\sigma \partial_{x}\right)$ in (III-9) may be verified, using more standard, but far lengthier procedures, involving Fourier transforming with only numbers (instead of operators) as arguments. Since both the operators $\stackrel{\delta}{x}$ and $\vec{\partial}_{x}$ in (III-9) point outward, neither of them acts on the
$x$-dependence of $K_{s}$ itself.
Given the form (III-9) for $S_{s}$, expression (III-5) may be more neatly expressed as

$$
\begin{equation*}
S(\psi, \phi)=\int d x \psi^{*}(x) K\left(i{ }_{x}^{\star}, x, i{\underset{\partial}{x}}^{*}\right) \phi(x), \tag{III-11}
\end{equation*}
$$

where

$$
K\left(k, x, k^{\prime}\right) \equiv K_{v}\left(k, k^{\prime}\right)+\sum_{s} K_{s}\left(k, x, k^{\prime}\right) \text {, and } K_{v}\left(k, k^{\prime}\right) \equiv k k^{\prime}+k_{y}^{2}+k_{2}^{2}
$$

We now use this form of $S$ in the variational equation, Eq. (III-6).
To take the variation $\delta / \delta \psi^{*}(x)$ of $S$, we must integrate all derivatives $\bar{\delta}_{x}$ (which act on $\psi^{*}$ ) by parts, turning them into derivatives $-\Xi_{x}$ acting on $\phi$ and on $K^{\prime}$ 's own $x$ dependence, yielding $S(\psi, \phi)=\int d x \psi^{*}(x) K\left(-i J_{x}\right.$, $\left.x, i \vec{J}_{x}\right) \phi(x)$. (We note the sign change in the left-hand derivative in
K.) Poisson's equation, Eq. (III-6), then reads

$$
\begin{equation*}
0=K\left(-i \vec{\jmath}_{x}, x, i \vec{\partial}_{x}\right) \phi(x) \tag{III-12}
\end{equation*}
$$

Without loss of generality, we could at first have allowed $\phi$ and $\psi$ to have some dominant $x$ behevior, $\psi(x)=\psi_{1}(x)$ exp ikx, $\phi(x)=\phi_{1}(x) \exp$ ikx. Equation (III-11) then gives

$$
\begin{align*}
S(\psi, \phi) & =\int \mathrm{d} x \psi_{1}{ }^{*}(x) K\left(k+i ڭ_{x}, x,-k+i J_{x}\right) \phi_{1}(x) \\
& =\int d x \psi_{1}{ }^{*}(x) K\left(k-i J_{x}, x,-k+i J_{x}\right) \phi_{1}(x), \tag{III-13}
\end{align*}
$$

and Poisson's equation appears as

$$
\begin{equation*}
0=k\left(k-i \exists_{x}, x,-k+i \exists_{x}\right) \phi_{1}(x) \tag{III-14}
\end{equation*}
$$

Equations (III-12) and (III-14) are exact formulations of the full nonlocal Poisson's equation, Eq. (III-13). The integral operator in (III-3) has been converted into an infinite order differential operator,
which may be truncated at some finite order to enable analytic progress to be made. These differential formalations could have been obtained directly from the integral equation (III-3), without first going through the variational formulation. We have taken the variational route for two reasons. First, in performing the truncation at finite order of (II-12) or (II-14), one must be careful to perform the truncation in a consistent manner, so as not to introduce terms leading to spurious growth rates and frequency shifts, as pointed out by Berk and Dominguez. ${ }^{17}$ This consistr icy is achieved through the variational principle, performing the truncation on $S$ in a manner which preserves the symmetry property (III-8) of $S$ (i.e., we truncate at the same order in $\bar{\delta}_{x}$ and $\vec{\partial}_{\mathrm{x}}$ ), then integrating by parts and taking the variation of S , as done above, to obtain a differential equation.

The second reason for the variational formulation is that it is in its own right a powerful means for obtaining dispersion relations, without actually having to solve a differential equation first. Denoting by $L_{\phi}$ the mode localization width, if $\rho_{s} / L_{\phi}>1$, the Fourier representation of $\phi(x)$ will have a width $\Delta k \sim L_{\phi}^{-1}>\rho_{s}^{-1}$, and so one expects that the operator $K_{s}\left(i \stackrel{\rightharpoonup}{x}_{x}, x, i J_{x}\right)$ in (III-9) cannot validly be truncated at some finite order in $\partial_{x}$ about some average value of $k$. Proper application of $K_{s}$ thus falls naturally into two regimes. When $\rho_{s} / L_{\phi}<1$ is satisfied, the variational principle can be used to obtain results that could also be gotten from solving a differential equation (d.e.), with less labor than is required for the d.e. approach. In the regime $\rho_{s} / L_{\phi}>1$, truncation at finite order (in $\partial_{x}$ ) of $K_{s}$ becomes invalid, and the contribution if $K_{s}$ to Poisson's equation is truly integral in nature.

Here, the variational formulation is especially useful in obtaining valid approximate dispersion equations. We discuss these matters further in Secs. III.B. 2 and II.B.3.

Fommlations of Poisson's equation similar to Eq. (III-12) or
(III-14) have been derived by other authors. Gerver, Birdsall, Langdon, and Fuss ${ }^{21}$ derive a form

$$
\begin{equation*}
0=D_{G}\left(x, k-i J_{x} ; \lambda\right) \phi_{1}(x) \tag{III-15}
\end{equation*}
$$

Whereas Eq. (III-12) is exact, however, certain approximations were made in obtaining (III-15), which break down in the regime of appreciable $n$. These are manifested in the fact that the operator in (III-15) has no $x$ derivatives acting on the $x$ dependence of $D_{G}$ itself. Essentially, $D_{G}$ is what the operator $K$ would be if the left hand $\partial_{x}$ there acted only on $\phi_{1}: \quad D_{G}\left(x, k-i J_{x}\right) \phi_{1}(x)=\left.K\left(k-i J_{x}, \quad \xi,-k+i J_{x}\right) \phi_{1}(x)\right|_{x=\xi}$. Berk and Book ${ }^{22}$ derive an exact formanlation, equivalent to (III-14), but without using property (III-4). Rewriting $\mathbb{R}\left(x, x^{\prime}\right)$ as $D_{B}\left[s \equiv x^{\prime}-x, \bar{x} \equiv \frac{1}{2}\left(x+x^{\prime}\right)\right]$, they derive a formulation which is exact and which, like Eq. (III-15), has only a two argment form. Operator methods like those used above greatly simplify the derivation and expression of their result. In our temmology, their fommlation may be expressed as

$$
\begin{equation*}
0=\left.D_{B}\left(i \partial_{x}+i \partial_{\xi} / 2, \xi\right) \phi(x)\right|_{x=\xi}, \tag{III-16}
\end{equation*}
$$

where $D_{B}$ here is the kernel $D_{B}(s, \bar{x})$ just above, Fourier transformed in the first argment, and with the replacements $k \rightarrow i \partial_{x}+i \partial_{\xi} / 2, \bar{x}+\xi$. Equation (III-16) is equivalent to (III-12), with $\partial_{x}$ acting on that of $D_{B}$. However, the use of variables $s, \bar{x}$ in $D_{B}$ masks the symmetry in $x, x$ '
which $\bar{K}$ possesses. As a result, when the explicit form for the kernel is put into the formalism, use of the three-argument kernel $\mathrm{K}\left(\sigma, \mathrm{X}_{\mathrm{o}}, \sigma^{\prime}\right)$ yields expressions for the weakly inhomogeneous limit and its higher order corrections in a more direct and transparent fashion (cf. Sec. III.C).

The formalism developed thus far is quite close to that of Berk and Dominguez. ${ }^{17}$ At the present stage of development, the advantage of the present formalism lies in the use of operator techniques, which allows for a much quicker derivation and more compact expression of results than previous work. In the development in the following sections, we shall find two principal additional advantages.

The first of these comes from using the variational formulation to obtain dispersion equations directly, similar in spirit to the methods of Refs. 18-20. As already mentioned, the variational method is a labor-saving device in the regime $p_{i} / L_{\phi}<1$ where a differential equation approach like that of Ref. 17 is valid, and more importantly, it enables one to obtain dispersion equations valid in the fully nonlocal regime $\rho_{i} / L_{\phi}>1$, where the validity of the differential equation approach fails.

A second chief advantage lies in our use of a canonical formulation. As we shall see in Sec. III.C, the conciseness which the canonical formulation and the operator techniques already employed lend to the formalism, pernits various effects of potential significance, some to our knowledge previously unknown, to arise automatically from the mathematics.

## III.B.2. Truncation of the exact equations

To obtain an analytically tractable problem, we now begin to apply approximaions to the exart equation [Eq. (III-12) or (III-14) to be solved. We shall expand to second order in $\partial_{x}$ that portion $K_{<}$of the operator $K\left(i \delta_{x}+k, x,-k+i J_{x}\right)$ which is "nearly local," so that the finite-order truncation is valid, obtaining the contribution ${ }^{\circ}$ < from $K_{<}$ to $S$. (The subscript " $<$ " refers to the "nearly local" criterion $\rho_{s} / L_{\phi}<1$.) These nearly local concributions to $S$ are all but the nonlocal portion of the ion contribution in the regime $\rho_{i} / L_{\phi}>1$. This term, discussed in more detail in Secs. III.B. 3 and III.C., is evaluated in Appendix D.

Dropping the subscript on $S_{<}, K_{<}$(or assuming $\rho_{i} / L_{\phi}<1$, so that $S_{<}=S, K_{<}=K$ ), we expand $K$ to second order, analogous to the procedure of Ref. 17:

$$
\begin{aligned}
& S(\psi, \phi)=\int d x \psi_{1}{ }^{*} K\left(k, x,-k^{\prime}\right) \phi_{1}
\end{aligned}
$$

$$
\begin{align*}
& +\int d x \psi_{1}{ }^{*}\left[-\frac{1}{2} \delta_{x}{ }^{2} \partial_{k}{ }^{2} F^{\prime}\left(k, x,-k^{\prime}\right)+\delta_{x^{\prime}} \partial_{k^{\prime}} \partial_{k^{\prime}} K\left(k, x, k^{\prime}\right) \partial_{x}\right.  \tag{III-17}\\
& \left.-\frac{1}{2} \partial_{k^{\prime}}{ }^{2} K\left(k, x, k^{\prime}\right){ }^{\prime}{ }_{x}{ }^{2}\right]\left.\right|_{k=k}{ }^{\prime} \phi_{1} \\
& +O\left(\partial_{x}^{3} \partial_{k}^{3}\right) \text {. }
\end{align*}
$$

We discuss the explicit meaning of the formal expansion parameter $\partial_{x}{ }^{\circ}{ }_{k}$, in Sec. III.C.5.)

Integrating certain terms in (III-15) by parts and rearranoing cerms, we cast it in a more convenient form:

$$
\begin{align*}
& S=\int d x \psi_{1}^{*} K(k, x,-k) \phi_{1} \\
& +\int \mathrm{dx} \psi_{1}^{*}\left(\delta_{x} \Gamma+\Gamma^{*}{ }_{\mathrm{x}} \mathrm{~J}_{\mathrm{x}}\right) \phi_{1}  \tag{III-18}\\
& +\int d x \psi_{1}^{*}\left(\partial_{x} \frac{1}{2} d_{k}^{2} K \partial_{x}\right) \phi_{1} \quad+O\left(\partial_{x}{ }^{3} \partial_{k}{ }^{3}\right) \\
& \equiv S_{0}+S_{1}+S_{2}+O\left(\partial_{x}^{3} \partial_{k}{ }^{3}\right) \text {. }
\end{align*}
$$

Here, $\Gamma \equiv r_{1}+\Gamma_{2}, r^{*} \equiv r_{1}{ }^{*}+\Gamma_{2}{ }^{*}$, where

$$
\begin{align*}
& \left.r_{1} \equiv \Gamma_{1}(k, x) \equiv i \partial_{k} K\left(k, x,-k^{\prime}\right)\right|_{k=k^{\prime}},\left.\Gamma_{1}{ }^{*} \equiv i \partial_{k}{ }^{\prime} K\left(k, x,-k^{\prime}\right)\right|_{k=k^{\prime}}, \\
& \left.\Gamma_{2} \equiv \Gamma_{2}(k, x) \equiv \partial_{x^{\prime} \partial_{k}}{ }^{2} K\left(k, x,-k^{\prime}\right)\right|_{k=k^{\prime}}, \Gamma_{2}^{*} \equiv \partial_{\left.x^{\prime} \partial^{\prime}{ }^{\prime}{ }^{2} K\left(k, x,-k^{\prime}\right)\right|_{k-k^{\prime}},} .
\end{align*}
$$

and the symivi $\mathrm{d}_{\mathrm{k}}$ denotes total derivative with respect to k , so $d_{k}^{2} K \equiv(d / d k)^{2} K(k, x,-k)$. Henceforth, unless explicitly denoted otherwise, $K$ will mean $K(k, x,-k)$ (i.e., evaluated at $k=k^{\prime}$ ). The terms $\Gamma_{2}, \Gamma_{2}{ }^{*}$ are down from $\Gamma_{1}, \Gamma_{1}{ }^{*}$ by $O\left(\partial_{x} \partial_{k}\right)$, and will eventually be neglected, but we may carry them along for the moment at no extra expense. The terms $r$ and $r$ " are "formal complex conjugates" of each other, in the sense to be described in Sec. III.C.1, and so we write $\Gamma$ as the sum of its formally real and imaginary parts:

$$
\Gamma \equiv \Gamma_{R}+i \Gamma_{\mathrm{I}}, \Gamma^{*} \equiv \Gamma_{\mathrm{R}}-i \Gamma_{\mathrm{I}},
$$

and similarly for $\Gamma_{1} \equiv \Gamma_{1 R}+i \Gamma_{1 I}$ and $\Gamma_{2} \equiv \Gamma_{2 R}+i \Gamma_{2 I}$.
We now obtain the consistently-truncated form of Poisson's equation (assuming $\rho_{i} / L_{\phi}<1$, so that $S_{<}=S$ ), taking $\delta S / \delta \psi_{1}^{*}(x)$ as described previously. Using (IIT-18) for S, one finds

$$
\begin{equation*}
o=\left(A-B \vec{\partial}_{x}-C \vec{\partial}_{x}^{2}\right) \phi_{1}, \tag{III-20}
\end{equation*}
$$

where $A, B$, and $C$ are functions (not operators) of $x$ and $k$, defuned by $A \equiv k-\partial_{x} \Gamma, B \equiv \frac{1}{2} \partial_{x} d_{k}{ }^{2} K+2 i \Gamma_{j}$, and $C \equiv \frac{1}{2} d_{k}{ }^{2} K$. The term B will be negligible, to the order we will work, and so (III-20) takes the form

$$
\begin{equation*}
o=\left(Q-\partial_{x}^{2}\right) \phi_{1} \tag{111-21}
\end{equation*}
$$

where $Q(k, x, \lambda) \equiv A / C$. This has the form of the Helmholtz equation for heat flow in one dimension, or of the time-independent Schrödingei equation, $o=\left\{[V(x)-E]+\frac{1}{2} p^{2}\right\} \psi$, with (V-E) replaced by $Q$. We note in addition that, even if $B$ were not negligible, (Ill-20) could still be put into the form of Eq. (III-21), inserting ${ }_{1} \equiv \mathrm{f} \Phi_{2}$ into (III-20), and choosing the function $f(x)$ so that the coefficient $B^{\prime}$ oi $\partial_{x} \phi_{2}$ vanishes. This yields

$$
\begin{align*}
f(x) & =f_{0} \exp \left(-\frac{1}{2} \int^{x} d x^{\prime} B / C\right), \quad \text { and } \\
0 & =\left(Q^{\prime}-\partial_{x}^{2}\right) \phi_{2} \tag{III-22}
\end{align*}
$$

where $\quad Q^{\prime} \equiv A^{\prime} / C^{\prime}, \quad A^{\prime} \equiv A-B\left(\partial_{x} f / f\right)-C\left(\partial_{x}{ }^{2} f / f\right)$, and $C^{\prime} \equiv C$.
We have thus succeeded in putting the truncated problem into the familiar form of the Schrödinger equation, with no assumptions yet made on the shape of the "well" $Q(x)$, or on the localization of $\phi$. One must remember that $Q(x)$ is in general complex, more general than the usual Schrödinger equation studied in quantum mechanics.

## III.B.3. Variational method

He appeal to the fact that $S$ is variational with respect to both $\psi$ and $\phi$, i.e., that it is stationary when $\psi$ and $\psi$ are solutions to their respective equations. When they are solutions to their equations, we also have $S=0$. The variational method is thus the following. One Thakes a guess at the form of the solutions $\phi$ and $\psi$, parametrically dependent upon the set of variables $\underline{\mu} \equiv\left\{\mu_{1}, \mu_{2}, \cdots \mu_{N}\right\}$. Evaluating $S(\psi, \phi ; \lambda)$ using these forms gives $S(\mu ; \lambda)$. The stationarity condition on $S$ then corresponds to the N conditions

$$
\begin{equation*}
o=\partial S / \partial \mu_{i}, \quad(i=1,2 \cdots N) . \tag{III-23}
\end{equation*}
$$

As is the case when $\phi, \psi$ are exact solutions, one then sets

$$
\begin{equation*}
0=S(\mu ; i) \tag{1II-24}
\end{equation*}
$$

which yields a dispersion equation for $\omega\left(k_{y}, k_{z}\right)$. If the guess made for $\phi$ and $\psi$ is a good one, with difference between the exact solutions and optimized trial solutions on the order of some small parameter $\varepsilon$, the error in the resultant dispersion relation (III-24) is $O\left(\varepsilon^{2}\right)$.

We have already adopted a first parameter, $\mu_{1} \doteq \mathrm{k}$. We now take $\psi_{1}^{*}=\phi_{1}$. This gives a variational formulation paralleling that used by Ross and Mahajan, ${ }^{18,19}$ and Hazeltine and Ross, ${ }^{20}$ in their variational approach to studying drift waves. Defining ${K_{\phi}}(x) \equiv \partial_{x} \ln \phi_{1}$, and (f) $\equiv \int \mathrm{d} x \phi_{1}^{2}(x) f(x)$, for any function $f$, Eq. (III-18) may be written (restoring the subscript <)

$$
\begin{equation*}
S_{<}=\left(K_{<}\right)-\left\langle\partial_{x} \Gamma_{R}\right)+\left(\frac{1}{2} K_{\phi}^{2} d_{k}^{2} K_{<}\right\rangle+O\left(\partial_{x} \partial_{k}^{3}\right) \tag{III-25}
\end{equation*}
$$

We recall that for $p_{i} / L_{\phi}<1$, one has $S_{<}=S$. In Sec. III.C, we shall see that the contribution $\mathbb{R}_{s}\left(\underline{x}, \underline{x}^{\prime}\right)$ to $\mathbb{K}\left(\underline{x}, \underline{x}^{\prime}\right)$ consists of two parts, a purely local part, the "adiabatic term" $\bar{K}_{S A}=\delta\left(\underline{x}, \underline{x}{ }^{\prime}\right) \lambda_{i}{ }^{-2}(x)$, and a nonlocal part $\bar{R}_{S B^{\prime}}$ In the regime $\rho_{i} / L \phi>1, \bar{K}_{i A}$ may still be combined with the 'nearly local" contributions to $S$, but $\bar{K}_{i B}$ must be treated differently. In Appendix $D$, we evaluate the contribution $S_{i B}$ of $\bar{K}_{i B}$ to $S$, in the regime $\rho_{i} / L_{\phi}>1$, using a straight line orbit approxination for the ion orbits in the region of mode localization. In this regime, we refer to $S_{i B}$ as $S_{>}$, and so we have

$$
\begin{equation*}
S=S_{<}+S_{i B} \equiv S_{<}+S_{>}, \quad\left(\rho_{i} / L_{\phi}>1\right) \tag{III-26}
\end{equation*}
$$

and from before,

$$
S=S_{<}, \quad S_{\rangle}=0 \quad\left(\rho_{i} / L_{\phi}<1 j\right.
$$

In both regimes, $S_{<}$has the form of (III-25).
Before further specifying the parameters $\mu_{i}$, we can apply Eq. (III-23) for $\mu_{i}=\mu_{1} \equiv k$. In Sec. III.C. 2 we will see that, to the order we are working both $K_{<}$and $\Gamma_{R}$ in $S_{<}$are even functions of $k$, and in Appendix D, we will find that $S_{>}$is independent of $k$, assuming that $k<L_{\phi}^{-1}$. Therefore $S$ is an even function of $k$, in both regimes, and so the variational condition for $k$ is

$$
\begin{equation*}
0=\partial S / \partial k=2 k \partial S\left(k^{2}\right) / \partial k^{2} \tag{III-27}
\end{equation*}
$$

satisfied by $k=0$. We shall choose this solution henceforth.
Each of the variational equations (III-23) we will use has a counterpart in the differential equation approach as used, for example, by Gerver et al. ${ }^{21}$ These authors expand Eq. (III-15) to second order
in both $\delta x \equiv x-x_{0}$ and $\delta k \equiv i \partial_{x}$, and to first order in $\delta \omega \equiv \omega-\omega_{0}$, where the values of $x_{0}, k$ are determined from the conditions

$$
\begin{equation*}
0=\partial_{k} D_{G}\left(x_{0}, k ; \lambda\right), \quad 0=\partial_{x_{0}} D_{G}\left(x_{0}, k ; \lambda\right), \tag{11I-28}
\end{equation*}
$$

and $\omega_{0}$, is determined from the local dispersion relation

$$
\begin{equation*}
0=U_{0}\left[x_{0}, k ; \lambda_{0} \equiv\left(\omega_{0}, k_{y}, k_{z}\right)\right] . \tag{11I-29}
\end{equation*}
$$

The first of Eqs. (III-28) corresponds to our Eq. (III-27), and gives the same result, $k=0$. Equation (III-29) corresponds to taking a lowestorder approximation to our Eq. (III-24). The higher order contributions to the dispersion relation $S=0$ held in $S_{<}$in Eq. (III-25) correspond to the shift $\delta \omega$ in frequency found in solving the Weber equation which arises in the differential equation approach of Ref. 21. We refer to these contributions to the dispersion relation as $K_{d . e}$. (the subscript "d.e." denoting "differential equation").

Now we further specify the form of the trial solutions. We expect solutions localized at some (real) point $x_{0} \equiv \mu_{2}$, with half width $L_{\phi} \equiv|\alpha|^{-1 / 2}$, given by the parameter $\alpha \equiv \mu_{3}$, in general complex. Defining $\delta x \equiv x-x_{0}, \xi \equiv \alpha^{-1 / 2} \delta x$, we choose $\phi_{1}$ to be given by the eigenstates of the Weber equation

$$
\begin{equation*}
\left(-\partial_{\xi}^{2}+\xi^{2}\right) \phi_{1}=f \phi_{1} \tag{III-30}
\end{equation*}
$$

This has solutions $\phi_{1 n}(n=0,1,2, \cdots)$ given by

$$
\begin{equation*}
\phi_{1 n}(\xi) \equiv N_{x n}(\alpha) H_{n}(\xi) e^{-\xi^{2} / 2} \tag{III-31}
\end{equation*}
$$

where the $H_{n}$ are the Hernite polynomials, and $N_{x n}$ is chosen so that $I=\int_{-\infty}^{\infty} d x\left|\phi_{1 n}(\xi)\right|^{2}$. The $\phi_{1 n}$ are solutions of (III-30) with eigen-
value $f=f_{n} \equiv(2 n+1)$. The solutions may be localized either by a potential well in the real part $Q_{R}$ of $Q(x)$ in (III-21), or by the effect of the imaginary part $Q_{I}$, physically corresponding to a Landau resonance effect, in the manner first elucidated by Pearlstein and Berk ${ }^{23}$ in studying the so-called universal instability.

We define the moments $M_{m}$ of the Gaussian by

$$
\begin{equation*}
M_{m}(\alpha) \equiv \int_{-\infty}^{\infty} d y y^{m} e^{-\alpha y^{2}} \tag{III-32}
\end{equation*}
$$

so that

$$
\begin{equation*}
M_{0}(\alpha) \equiv(\pi / \alpha)^{1 / 2}, \quad M_{1}(\alpha) \equiv 0, \quad M_{m}(\alpha)=(m-1) M_{m-2}(\alpha) / 2 \alpha \tag{III-33}
\end{equation*}
$$

The Hermite polynomials may be defined by

$$
\begin{equation*}
H_{n}(\xi)=e^{\xi^{2}}\left(-\partial_{\xi}\right)^{n} e^{-\xi^{2}} \tag{III-34}
\end{equation*}
$$

In particular, $H_{0}(\xi)=1$, so that normalizing $\phi_{10}=N_{x O} e^{-\xi^{2} / 2}$ yields

$$
\begin{equation*}
N_{x 0}(\alpha)=M_{0}^{-1 / 2}(\alpha)=(\alpha / \pi)^{1 / 4} \tag{III-35a}
\end{equation*}
$$

Using (III-34), one can show that ${ }^{24}$ in general

$$
\begin{equation*}
N_{x n}(\alpha)\left(2^{n} n!\right)^{-1 / 2} \tag{III-35b}
\end{equation*}
$$

Expanding $k_{<}(x)$ about $x=x_{0}$

$$
\begin{equation*}
K_{<}(x)=\sum_{m=0}^{\infty}(\delta x)^{m} \partial_{x_{0}} K_{<}\left(x_{0}\right) / m!, \tag{III-36}
\end{equation*}
$$

we evaluate $\left(K_{<}\right)$in (III-25), for $\phi_{1}=\phi_{10}$ :

$$
\begin{align*}
\left\langle K_{<}\right\rangle & =M_{0}^{-1}\left\{M_{0} K_{<}\left(x_{0}\right)+M_{2} \partial_{x_{0}}^{2} K_{s} / 2!+\cdots\right\}  \tag{III-37}\\
& =K_{<}\left(x_{0}\right)+(4 \alpha)^{-1} \partial_{x_{0}}{ }^{2} K_{<}+O\left(L_{\phi}^{4} / L_{Q}^{4}\right)
\end{align*}
$$

!ere $L_{Q} \equiv\left|\partial_{x} \ell n Q\right|^{-1}$ is the scale length of the potential $Q$ localizing
the mode. An expansion analogous to (III-37) may be done for the other terms in (III-25) but, to the order we are working (namely, keeping only up to second derivatives on $K_{<}$, we may keep only the zero order parts of these. In addition, it will be convenient (and consistent with our ordering) to approximate $K_{<}$in these higher order terms by $K_{0}$, the difference $K_{<}-K_{o}$ being of higher order. We shall choose $K_{o}$ to be all contributions to $K$ except that from $K_{i B}$ which, though inportant for determining the growth rate of the instability to be studied, is nevertheless small:

$$
\begin{equation*}
K_{o}=K-K_{i B}=K_{v}+K_{e}+K_{i A} \tag{III-38}
\end{equation*}
$$

A more explicit form for $K_{0}$ will be given in Sec. III.D.1. In the regime $\rho_{i} / L_{\phi}<1$, where $K_{<}=K=K_{c}+K_{i B}$, one therefore has

$$
\begin{equation*}
\left\langle K_{<}\right\rangle \cong\left\langle K_{o}\right\rangle+K_{i B} \cong K_{o}+K_{i B}+(4 \alpha)^{-1} \partial_{x_{o}}^{2} K_{o} \tag{III-39a}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle-\partial_{x} \Gamma_{R}\right\rangle \simeq-\partial_{x_{0}} \Gamma_{R} \simeq \sum_{s}\left(-i \partial_{x_{0}} \partial_{k} K_{s B}+c . c .\right) \tag{III-39b}
\end{equation*}
$$

where we have used (III-17), and the facts that $K_{v}$ is independent of $x$ and that $K_{S A} \sim \lambda_{s}{ }^{-2}(x)$ is independent of $k$ (as already mentioned), so that only the terms $K_{s B}$ in $K$ contribute to $-\partial_{X} \Gamma_{R}$. For $o_{i} / L_{\phi}>1, K_{<}=K$ $-K_{i B}=K_{o}$, and therefore

$$
\begin{equation*}
\left\langle K_{<}\right\rangle=\left\langle K_{0}\right\rangle \simeq K_{0}+(4 \alpha)^{-1} \partial_{x_{0}}^{2} K_{0} \tag{III-40a}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle-\partial_{x_{R}} \Gamma_{R} \simeq-\partial_{x_{0}} \Gamma_{R} \simeq\left(-i \partial_{x_{0}} \partial_{k} K_{e B}+c . c .\right)\right. \tag{III-40b}
\end{equation*}
$$

In Sec. III.C. 4 we will see that the electron contribution to $\Gamma_{R}$ is down from that of the ions by a factor $\sim\left(\Omega_{i} / \Omega_{\mathrm{e}}\right)$. Therefore, we may
retain only the ion term in (III-39b), and totally neglect the term in (III-40b) .

With these approximations, using (III-26) in (III-24), the dispersion relation assumes the following forms in the small and large $-o_{i} / L_{\neq}$regimes:

$$
\begin{equation*}
0=S=K_{o}+S_{i B}+K_{\text {n.l. }}+K_{\text {d.e. }}, \rho_{i} / L_{\phi}<1 \tag{III-41a}
\end{equation*}
$$

and

$$
\begin{equation*}
0=S=K_{0}+S_{i B}+{ }^{\prime} \text { i.e., } \quad \rho_{i} / L_{\phi}>1 \tag{III-41b}
\end{equation*}
$$

For $\rho_{i} / L_{\phi}<1, S_{i B}=\left\langle K_{i B}\right\rangle=K_{i B}$. We evaluate $S_{i B}$ for $\left.\rho_{i} / L_{\phi}\right\rangle 1$ in Appendix $D$, and find that it has a form quite similar to its form in the regime $\rho_{i} / L_{\phi}<1$. This is discussed further in Sec. III.C. The term $K_{n, \ell .} \equiv\left\{-\partial_{x} \Gamma_{R}\right\}=-i \partial_{x_{0}} \partial_{k} K_{i B}+c . c$. is the lowest order correction due to the nonlocal (n. र.) nature of the conductivity kernel. We see that $S_{i B}\left(\rho_{i} / L_{\phi}<1\right)+K_{n . \ell .}$ in Eq. (III-41a) is replaced by $S_{i B}\left(\rho_{i} / L_{\phi}>1\right)$ in (III-41b). Finally, $\mathrm{K}_{\mathrm{d}}$.e., whose explicit form will be given shortly, is the contribution arising from solving a differential equation in $x$, instead of just a local dispersion equation [viz. $K(x)=0$ ]. All quantities in Eqs. (III-41) are evaluated at $x_{0}$. For the Gaussian form of $\phi_{i 0}(x)$, and the quadratic approximation of $K_{0}(x)$ we have made expanding about $x=x_{0}, K_{d . e}$. corresponds to the ground state energy coming from solving the Schrödinger equation for a simple harmonic oscillator (SHO); ${ }^{K}$ d.e. is then given by $K_{d . e . ~}=\frac{1}{4}\left(\alpha^{-1} \partial_{X_{o}}{ }^{2} K_{o}+\alpha d_{k}{ }^{2} K_{o}\right)$. Since the differential equation to be solved here is of the same form as the Schrödinger equation, we may use cur knowledge of the nature of the solutions of that enuation to generalize Eq . (III-41) to be valid both for the higher $n$
"excited state" solutions $\phi_{1 n}$, and for anharmonic corrections to the "potential" $K_{0}(x)$. For the quadratic potential, we generalize the expression for $K_{d . e}$ to

$$
\begin{equation*}
K_{\text {d.e. }}=K_{d . e .}^{(n)} \equiv\left(f_{n} / 4\right)\left(\alpha^{-1} \partial_{x_{0}} 2_{0}+\alpha d_{k}^{2} K_{o}\right) \tag{III-42}
\end{equation*}
$$

where we recall $f_{n} \equiv 2 n+1$. Letting anharmonic corrections enter will shift the eigenvalues from their harmonic well values, which we may account for by allowing the coefficients $f_{n}$ to shift from their harmonic values $2 n+1$, as the anharmonicity is turned on.

We now write out the variational equations for $\mu_{2} \equiv x_{0}$ and $\mu_{3} \equiv a$. The term $S_{i B}$ in both $\rho_{i} / L_{\phi}$ regimes will turn out to be equal to a sum over harmonic number $\ell$ of terms proportional to $\left(\omega-\omega_{*_{i}}\right)(\omega-\ell \Omega)^{-1}$. This term is essential for obtaining the growth rates for the modes we shall be studying, but it is nevertheless small in comparison to the other tems in $S$. We can therefore neglect it in the $x_{0}$ and $\alpha$ variational equations. For the $x_{0}$ equation, $K_{d . e}$. and $K_{n . l}$. are also higher order contributions, leaving only $K_{0}$ in expressions (III-41) for $S$ :

$$
\begin{equation*}
0=\partial S / \partial x_{0}=\partial_{x_{0}} K_{0}+h .0 \tag{III-43}
\end{equation*}
$$

Equation (III-43) is the variational counterpart of the second of conditions (III-28). It localizes the mode at an extremem of $K_{0}$, just as solving the differential equation (III-21) localizes the mode at an extremm of $Q(x)$. The difference between these localization criteria of the variatiional and differential equation approaches is negligible, to zne order we are working.

Our final variational equation gives us $a$ :

$$
\begin{equation*}
0=2 S / 3 \alpha=\left(f_{n} / 4\right)\left(-\alpha^{-2} \partial_{x_{0}}^{2} K_{0}+\dot{\alpha}_{k}^{2} K_{0}\right) \tag{III-44}
\end{equation*}
$$

and

$$
\begin{equation*}
a=\left(\frac{1}{2} \dot{o}_{x_{0}}^{2} K_{0} / \frac{1}{2} d_{k}^{2} K_{0}\right)^{1 / 2} \tag{III-45}
\end{equation*}
$$

This gives

$$
\begin{equation*}
\left.k_{d . e .}=f_{n}\left[\frac{1}{2} d_{k}^{2} K_{o}\right)\left(\frac{1}{2} \partial_{x_{0}}^{2} K_{o}\right)\right]^{1 / 2} \tag{III-46}
\end{equation*}
$$

In Sec. III.d.l we will find that $\mathrm{d}_{\mathrm{k}}{ }^{2} \mathrm{~K}_{\mathrm{o}}>0$. Whether $\alpha$, and $\mathrm{K}_{\mathrm{d} . e .}$, are real or imaginary thus depends on whether sine mode is localized at a minimam or maximum of $\mathrm{K}_{\mathrm{o}}$ (or, approximately, of Q ).

In order for the localized mode assumption to be reasonable, the potential $Q$ must be of sufficient width and magnitude that $\left(L_{\phi} / L_{Q}\right)<1$. Using $L_{\phi}{ }^{2} \simeq f_{n} \alpha^{-1}$, and for the purposes of estimation assuming a well in $Q_{R}$ which is quadratic from a maximum depth $Q_{d} \equiv\left|Q\left(x_{0}\right)\right|$ up to $Q=0$, so that $L_{Q}{ }^{2} \approx Q_{d} /\left(\frac{1}{2} \partial_{x_{0}}^{2} Q\right)$, we obtain the criterion for localization

$$
\begin{equation*}
1>\left(L_{\phi} / L_{Q}\right)^{2} \simeq\left|K_{d . e .} / K_{d}\right| \tag{III-47}
\end{equation*}
$$

where $K_{d} \sim K_{0}$ is the corresponding depth of the well in $K_{0}$.
To apply this abstract formalism, one mast look at the specific form of the function $K$. We proceed to do this in the following section.

## III.C. Conductivity X ; Explicit Form

In this section we insert the physics into the abstract formalism of Sec. III.B, by using the explicit form for the conductivity kernel in a plasma, expressed in a canonical framework. The work of Parts I and II will prove useful to us here, in providing a description of the particle orbits, in choosing an appropriate distribution function, and
in understanding the physical origin of certain effects, which automatically arise from the mathematics of the canonical formulation. Most notable among these are a generalized evpression for the diamagnetic drift frequency in the presence of shear, and ion resonance effects which may occur for appreciable shear, which previous ad hoe derivations have missed.

## III.C.1. Explicit expression for $\mathrm{K}_{\mathrm{S}}$

In Appendix C we derive general expressions for the conductivity kernel $\overline{\mathrm{K}}_{\mathrm{s}}$ in a three dimensional plasma, in a canonical formalation. As there, we denote by I the three canonical momenta whose invariance makes the time rate of change $\underline{\Omega}(\underline{I}) \equiv \dot{\theta}(\underline{I}) \equiv \partial H / \partial \underline{I}$ of the corresponding coordinates $\underline{\theta}$ constant in time. For the slab geometry, we take $\underline{I} \equiv T, P_{y}, P_{z}$ ) $\equiv(J, \underline{P})$, where $J$ is the generalized gyroaction discussed in $P$.t II. The conjugate coordinates are $\theta \equiv(\theta, \mathrm{Y}, \mathrm{Z})$, with time derivatives $\Omega \equiv(\Omega$, $\dot{Y}, \dot{Z}$ ). In Appendix $C$, we find that

$$
\begin{equation*}
\bar{K}_{s}\left(\underline{x}, \underline{x}^{\prime} ; w\right)=\bar{K}_{s A}\left(\underline{x}, \underline{x}^{\prime}\right)+\overline{\mathrm{K}}_{s B}\left(\underline{x}, \underline{x}^{\prime}, w\right), \tag{III-48a}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{S A}\left(\underline{x}, \underline{x}^{\prime}\right) \equiv \lambda_{s}^{-2}(x) \delta\left(\underline{x}-\underline{x}^{\prime}\right), \tag{III-48b}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{K}_{\mathrm{SB}}\left(\underline{x}, \underline{x}^{\prime} ; \omega\right) \equiv(4 \pi)(2 \pi)^{3} \sum_{\ell} \int \underline{d I}_{\underline{l}} \underline{(x \mid \underline{I})} \frac{\omega \partial \mathrm{f}_{s} / \partial H-\ell \cdot \partial \mathrm{f}_{s} / \partial \underline{I}}{\omega+\underline{\ell} \cdot \underline{\Omega}} \rho_{\ell}{ }^{*}\left(x^{\prime} \mid \underline{I}\right) \tag{III-48c}
\end{equation*}
$$

Here, $f_{s}$ is the equilibrium distribution function (denoted $f_{o s}$ in Appen$\operatorname{dix} C), \ell \equiv\left(\ell, \ell_{y}, \ell_{z}\right)$, and the other symbols are defined in Appendix $C$.

First we deal with the nonlocal part, $\mathbb{K}_{s B}$. Specializing to a slab geometry, we Fourier transform out the $y, z, y^{\prime}$, and $z^{\prime}$ dependences in $R_{S B}$. This involves the corresponding transforms of $\rho_{\underline{\ell}}(\underline{x})$ and $F_{\underline{\ell}}^{*}\left(\underline{X^{\prime}}\right)$. Writing $\underline{r}(\underline{Z}) \equiv \underline{R}+\delta \underline{r}$, where $\underline{R} \equiv\left(X_{0}, Y, Z\right)$ is the guiding center position and $\delta \underline{I} \equiv(\delta x, \delta y, \delta z)$ describes the gyromotion about $\underline{R}$, one finds

$$
\begin{equation*}
\rho_{\ell}\left(x, k_{y}, k_{z}\right)=e \delta\left(\ell_{y}+k_{y}\right) \delta\left(l_{z}+k_{z}\right) \tilde{J}_{\ell}\left(0, k_{y}, k_{z} \mid \underline{I}\right), \tag{III-49}
\end{equation*}
$$

where $0 \because x-X_{0}$, and
$\tilde{J}_{\ell}\left(0, k_{y}, k_{z} \mid \underline{I}\right) \equiv(2 \pi)^{-1} \int_{0}^{2 \pi} d 0 \delta\left(x-X_{0}-\delta x\right) e^{-l \theta} \exp \left[-i\left(k_{y} \delta y(\theta, I)+k_{z} \delta z(\theta, \underline{I})\right]\right.$. Using Eqs. (III-48C), (III-49), and (III-ib), we find the one dimensional analog of (III-1b), for the contribution $\rho_{S B}(\underline{x})$ to $\rho_{S}(\underline{x}):-4 \pi \rho_{S B}(x ; \lambda)$ $=\int d x^{\prime} \mathrm{R}_{\mathrm{SB}}\left(x, x^{\prime} ; \lambda\right)$, where
$\bar{K}_{s B}\left(x, x^{\prime} ; \lambda\right) \equiv(4 \pi)(2 \pi) \sum_{\ell=-\infty}^{\infty} \int \frac{\omega I}{} \frac{\omega \partial f_{s} / \partial H-\ell \cdot \partial f_{s} / \partial \underline{I}}{\omega+\ell \cdot \underline{\Omega}} \tilde{J}_{\ell}\left(\sigma, k_{y}, k_{z} \mid \underline{I}\right) \tilde{J}_{\ell}^{*}\left(\sigma^{\prime}, k_{y}, k_{z} \mid \underline{I}\right)$.

We assume that we have chosen our distribution function $f_{s}$ in terms of the canonical variables well, so that the self consistent equilibrium potentials, determined by the formalism of Part I, are precisely the desired reference potentials, and sc $X_{0}=\tilde{X}_{0}(\underline{P}), v_{3} \equiv v_{03} \equiv v_{1}\left(X_{0}\right)=\tilde{v}_{1}(\underline{P})$. As in Part $I$, we shall take $f_{s}=f_{s}(H, P)$, dependent on $J$ only through H. Therefore $-\ell \cdot \partial f_{s} / \partial \underline{I}=-\underline{\ell} \cdot \partial f_{s} / \partial \underline{P}=\hat{\underline{k}} \cdot \partial f_{s} / \partial \underline{P}$, where $\hat{\hat{k}} \equiv \hat{y} \hat{k}_{y}+\hat{z} k_{z}$. Changing from integration variables $I$ to ( $\mathrm{H}, \underline{\mathrm{P}}$ ), then, we have

$$
\begin{equation*}
d^{3} \underline{I} \equiv d^{2} \underline{p} d J=d X_{c} d v_{3} d H \Omega^{-1} I \tag{III-51}
\end{equation*}
$$

where $1 \equiv \partial\left(P_{y}, P_{z}\right) / \partial\left(X_{0}, v_{3}\right)$, as introduced in Sec. I.C.C. Putting (III-51) into (III-50), comparison with Eq. (III-4) allows us to read
off $K_{S 1}$ :
$K_{s B}\left(\sigma, X_{0}, \sigma^{0}, \lambda\right)=$

$$
\begin{equation*}
8 \pi^{2} \mathrm{e}^{2} \sum_{\ell=-\infty}^{\infty} \iint \mathrm{dv}_{3} \mathrm{dH} \ell^{-1} I-\frac{\omega \partial \mathrm{f}_{s} / \partial H+\hat{\hat{k}^{\prime}} \cdot \partial \hat{f}_{s} / \partial \underline{P}}{\mu+\underline{\ell} \cdot \underline{\Omega}} \tilde{J}_{\ell}\left(\sigma, \mathrm{k}_{y} k_{z}\right) \tilde{J}_{k}^{*}\left(\sigma^{\prime}, k_{y}, k_{z}\right) \tag{III-52}
\end{equation*}
$$

A central quantity of the abstract formalism of Sec. III.B is $K_{<}\left(k, X_{0},-k\right)$. We therefore Fourier transform ( $I \perp I-52$ ) with respect to both $\sigma$ and $\sigma^{\prime}$. We use

$$
\begin{align*}
\tilde{J}_{\ell}\left(k_{,} k_{y}, k_{z} \mid \underline{I}\right) & \equiv \int d \sigma e^{-k \sigma} \tilde{J}_{\ell}\left(\sigma, k_{y}, k_{z} \mid \underline{I}\right) \\
& =(2 \pi)^{-1} \int_{0}^{2 \pi} d \theta e^{-i \ell \theta} \int d \sigma \delta(\sigma-\delta x) \exp \left[-i\left(k \sigma+k_{y} \delta v+k_{z} \delta z\right)\right] \\
& =(2 \pi)^{-1} \int_{0}^{2 \pi} d \theta e^{-i \ell \theta} e^{-k} \cdot \delta \underline{r}(\theta),\left(\underline{k} \equiv \hat{x}+k_{y} \hat{y}+k_{z} \hat{z}\right) \tag{III-53}
\end{align*}
$$

in Eq. (III-52), obtaining the important formula

$$
\begin{align*}
& K_{s B}\left(k, x_{0},-k^{\prime} ; \lambda\right)= \\
& 8 \pi^{2} e^{2} \sum_{\ell=-}^{\infty} \iint d v_{3} d H \Omega^{-1} I \frac{\omega \partial f_{s} / \partial H+\hat{k} \cdot \partial f_{s} / \partial \underline{P}}{\omega+\underline{\ell} \cdot \underline{\Omega}} \tilde{J}_{l}(\underline{k} \mid I)^{*}\left(\underline{k^{\prime}} \mid \underline{I}\right) . \tag{III-54}
\end{align*}
$$

Because it is purely local, the contribution $S_{S A}$ of the "adiabatic term" $\bar{K}_{s A}$ to $S$ is easier to calculate. Using (III-48b) in the first line of Eq. (III-9), one finds

$$
\begin{equation*}
S_{S A}(\psi, \phi)=\int d x \psi^{*}(x) \lambda_{S}^{-2}(x) \phi(x) \tag{III-55}
\end{equation*}
$$

Comparing, this with the third line of (III-9), we read off $\mathrm{K}_{\mathrm{sA}}$ :

$$
\begin{equation*}
K_{s A}(k, x,-k)=K_{s A}(x)=\lambda_{s}^{-2}(x) \tag{III-56}
\end{equation*}
$$

As already noted, these forms for $\mathrm{K}_{\mathrm{SA}}$ and $\mathrm{S}_{\mathrm{SA}}$ are valid and useful for any value of $\rho_{s} / L_{\phi}$, while $K_{s B}$ in (III-45) is only of use in the "nearly local" regime $\rho_{s} / L_{\phi}<1$. In Appendix $D$ we calculate a form for $S_{S B}$ which is valid for the regime $\rho_{s} / L_{\phi}>1$, starting from $K_{s B}\left(\sigma, X_{o}\right.$, $\sigma^{\prime}$ ) as given in Eq. (III-52). We thus obtain expressions for $S$ and dispersion equations valid in both regimes, and can then smoothly connect the behavior of $S$ in these two regions to obtain a picture of mode stability valid for all $\rho_{S} / L_{\phi}$.

## III.C.2. On the Structure of $\mathrm{K}_{\mathrm{s}}$; Discussion

In the present section we discuss certain important features of the Kernel $\mathrm{K}_{\mathrm{s}}$ [given by Eqs. (III-54), (III-56)] which is a central object in the abstract formalism of Sec. III.B. We note that $k_{s}\left(k, x_{0},-k\right.$ ) has the same structure as the usual expression for the susceptibility $x_{s}$ of a weakly inhomogeneous plasma in a uniform magnetic field; indeed, in that limint $K_{s}$ reduces to just $\underline{k}^{2} \chi_{S}$. It is much more general, however, in two basic respects. The first is that all the quantities (e.g. $\underline{I}, \underline{\Omega}(\underline{I}), \tilde{\mathcal{J}}_{\ell}$ ) appearing in $K_{s}$ have a meaning generalized from their unifom $\underline{B}$ limits to arbitrary magnetic inhomogeneity. The second is that $k \neq k^{\prime}$, so that $K_{s}$ contains more information than $X_{s}$. This is reasonable, since $K_{s}$ describes the full nonlocal conductivity properties of the plasma, while $X_{s}$ is a quantity derived under a local ( $n+0$ ) approximation. It arises from the present formalism Icf. Eq. (III-14)] by letting the operators $\vec{\partial}_{x}$ in $K$ act only on the $x$ dependence of $\phi_{1}$ and not of K itself. This corresponds to setting $k=k^{\prime}$.

The factor $J_{\ell}^{2}\left(k_{1} \rho_{s}\right)\left(J_{2}\right.$ is the Bessel function of the first kind), appearing in the uniform $\underline{B}$ expressions for the conductivity, has been repiaced in Eq. (III-54) by the factor $\tilde{J}_{\ell}(\underline{k} \mid \underline{I}) \tilde{\mathcal{J}}_{\ell}^{*}(\underline{k} \mid \underline{I})$. This is more general than $J_{\ell}^{2}$ in both of the ways just mentioned. We recover $J_{\ell}$ by letting $\delta \underline{r}(\theta)$ go to its simple harmonic, small gyroradius ( $\eta_{\underline{B}}+0$ ) limit. W'e use the Bessel identity

$$
\begin{equation*}
J_{\ell}(\mu)=(2 \pi)^{-1} \int_{0}^{2 \pi} d \theta e^{-i \ell \theta} e^{i \mu \sin \theta}=i^{\ell}(2 \pi)^{-1} \int_{c}^{2 \pi} d \theta e^{-i \ell \theta} e^{-i \mu \cos 0} \tag{III-57}
\end{equation*}
$$

and Eqs. (II-111), (II-119) of Part II for $\delta \underline{r}$, to obtain the $n_{\underline{B}} \rightarrow o$ limit of $\tilde{J}_{\hat{\ell}}$. These equations tell us that, in the coordinate system $\left[\hat{b}_{1} \equiv \hat{x}, \hat{b}_{2} \equiv \hat{b}_{3} \times \hat{b}_{1}, \hat{b}_{3} \equiv \hat{B}\left(x_{0}\right)\right]$ natural to the magnetic geometry at $x=X_{0}$, the $\eta_{\underline{B}}+0$ limit of the gyromotion is given by

$$
\delta \underline{r}(\theta)=\left[\delta x_{0}(\theta), \delta x_{20}(\theta), 0\right],
$$

where $\delta x_{0}(\theta)=\bar{x}_{0} \sin \theta$ is the excursion in the $x \equiv \underline{x} \cdot \hat{b}_{1}$ direction, $\delta x_{20}(\theta)=\bar{x}_{20} \cos \theta$ is the excursion in the $x_{2} \equiv \underline{x} \cdot \hat{b}_{2}$ direction, and the excursion in the $x_{3} \equiv \underline{x} \cdot \hat{b}_{3}$ direction is zero. The oscillation amplitudes $\bar{x}_{0}$ and $\bar{x}_{20}$ are related by $\bar{x}_{20}=\bar{x}_{o}\left(\Omega_{c} / \Omega_{0}\right)$, where $\Omega_{c} \equiv e B_{o} / m c$ is the usual cyclotron frequency at $X_{0}$, and $\Omega_{0}$ is the $\eta_{\underline{B}} \rightarrow 0$ limit of the gyrofrequency which, we recall from Sec. II.K.1, is not in general equal to $\Omega_{c}$.

From these results of Part II, we write

$$
\begin{equation*}
\underline{k} \cdot \delta \underline{r}(\theta) \equiv k \delta x_{0}(\theta)+k_{2} \delta x_{20}(\theta) \equiv \mu_{0} \cos \left(\theta-\theta_{0}\right), \tag{III-58a}
\end{equation*}
$$

where $\mathrm{k} \equiv \underline{\mathrm{k}} \cdot \hat{\mathrm{b}}_{1}$ and $\mathrm{k}_{2} \equiv \mathrm{k} \cdot \hat{\mathrm{b}}_{2}$. (Similarly, we define $\mathrm{k}_{3} \equiv \underline{\mathrm{k}} \cdot \hat{\mathrm{b}}_{3}$ ). From this follow the definitions of $\mu_{0}, \theta_{0}$ :

$$
\begin{equation*}
\mu_{0}^{2} \equiv k^{2} \bar{x}_{0}^{2}+k_{2}^{2} \bar{x}_{20}^{2}, k \bar{x}_{0}=\mu_{0} \sin \theta_{0}, k_{2} \bar{x}_{20}=\mu_{0} \cos \theta_{0} \tag{III-58b}
\end{equation*}
$$

Using (III-58a) and (III-57) in (III-53), we obtain, finally,

$$
\begin{equation*}
\lim _{\underline{\eta_{\underline{B}}} \rightarrow 0} \tilde{J}_{\ell}\left(\underline{k}(\underline{1})=(-i)^{\ell} e^{-i l \theta_{0}} J_{l}\left(\mu_{0}\right)\right. \tag{III-59}
\end{equation*}
$$

For $\delta \underline{r}(0)$ describing anharmonic gyromotion, we can still obtain explicit expressions for the $\bar{J}_{\ell}$. Proceeding only up to $O\left(n_{\underline{B}}\right)$ as in Sec. II.K.2, we can krite
$\underline{k} \cdot \delta \underline{r}(\theta)=\mu_{0} \cos \left(\theta-\theta_{0}\right)+\mu_{1} \cos 2\left(\theta-\theta_{1}\right)$, where $\mu_{1} / \mu_{0} \sim 0\left(\eta_{\underline{B}}\right)$.
Using the Bessel identity
$\exp (i \mu \sin 0)=\sum_{\ell} e^{i \ell \theta_{l}} J_{\ell}(\mu)$, or $\exp (-i \mu \cos \theta)=\sum_{\ell}(-i)^{\ell} e^{i \ell \theta^{\prime}} J_{\ell}(\mu), \quad(1 I I-60)$ one finds $\bar{J}_{\hat{l}}$ is given $b_{j}$

$$
\begin{equation*}
\tilde{J}_{\ell}(\underline{h} \dot{\underline{1}})=(-i)^{\ell} e^{-i \ell \theta_{0}} \sum_{m} e^{2 i m\left(\theta_{0}-\theta_{1}\right)} J_{\ell-2 m}\left(\mu_{0}\right) J_{m}\left(\mu_{1}\right) \tag{111-61}
\end{equation*}
$$

One could, if desired, proceed similarly to arbitrary order in the $n_{B}$ expansion of $\delta \underline{r}(0)$.

In general, one may define $J_{\ell}$ by

$$
\begin{equation*}
\tilde{J}_{\ell}(\underline{k} \mid \underline{I}) \equiv J_{\ell}(\underline{k} \mid \underline{I}) e^{-i \ell \theta_{k}} \tag{III-62}
\end{equation*}
$$

where $\theta_{k}$ is chosen to make $J_{\ell}(\underline{k} \rightarrow 0)$ real and positive. Then $\theta_{k} \rightarrow \theta_{0}$, and $J_{\ell} \rightarrow J_{\ell}$ in the $\eta_{\underline{B}} \rightarrow$ limit. The $J_{\ell}$ retain certain of the useful properties of their simple harmonic limits, the $J_{\ell}$. For example, using Parseval's theorem and the definition (III-53) of $\tilde{J}_{\ell}$, one finds the relation

For $k \neq k^{\prime}$, there is a phase factor $\exp i \ell\left(\theta_{k},-\theta_{k}\right)$ present in expression (III-54), absent in the weakly inhomogeneous limit ( $k=k^{\prime}$ )
usually used. For substantial $\ell$, this factor can give a significant correction to the dispersion relation, contributing to the term $-3 x^{\Gamma} R$ in Eq. (III-25) or (III-41) (cf. Sec. Ill.D.2). This correction has been found previously by other workers. ${ }^{17,27}$

For $k=k$, one sees from Eq. (III-54) that $k_{s B}$ is a 'formally real" quantity, i.e. if we replaced the resonance denominator ( $\omega+\ell \cdot \Omega$ ) (which gives $K_{S B}$ a real and imaginary part) by a real function of the momenta $\underline{I}, K_{s B}$ itself would be real. Similarly, one may speak of functions which are "formally imaginary" or "formally complex conjugates," as in the discussion of $\Gamma$ and $\Gamma^{*}$ in Sec. III.B.2. For example, from (III-54) one sees that $K_{S B}\left(k, X_{0},-k^{\prime}\right)=K_{s B}^{*}\left(k^{\prime}, X_{0},-k\right)$, where the " ${ }^{\prime}$ " denotes the formal complex conjugate.

The entire $k$ dependence of $\mathrm{K}_{\mathrm{sB}}\left(\mathrm{k}, \mathrm{X}_{0},-\mathrm{k}\right)$ is contained in the factor $J_{\ell}^{2}(\underline{k})$. For anharmonic orbits, this factor, and so $K_{s}$, are not in general even in $k$, as is the $\eta_{\underline{B}}$ o limit $J_{\ell}^{2}\left(\mu_{0}\right)$. However, these anharmonic corrections, of order $\eta_{\underline{B}}$, are negligible in our ordering scheme, as will be discussed in Sec. III.C. 5.

We shall break $K$ into a lowest order part $K_{h}$, plus a term $K_{o . m}$. which is of higher order, and may be neglected, consistent with our ordering (cf. Sec. III.C.5). The subscript " $h$ " in $K_{h}$ indicates that in evaluating $K_{h}$ we use the small gyroradius, harmonic limits of the particle motion, and therefore the $J_{\ell}$ limit of the $J_{\ell}$. The subscript "o.m." in $K_{o . m}$. refers to all the various anharmonic orbit modification effects which enter expression (III-54) for $K_{s B}$, when $s=i \equiv$ ion species. Thus $K_{o . m .} / K_{h} \sim \eta_{\underline{B}}$. The term kept, $K_{h}\left(k, X_{o},-k^{\prime}\right)$, is an integral over
 $k_{h}\left(k, X_{0},-k\right)$ is even in $k$, as is $d_{k}^{2} k_{h}$. From Eqs. (III-19), we see that the remaining term, $-\partial_{x} \Gamma_{R} \simeq-\partial_{x} \Gamma_{1 R}$ in Eq. (III-25), has a $k$ dependence given by

$$
\begin{align*}
\Gamma_{1 R} & \left.\equiv \frac{1}{2}\left(\Gamma_{1}+\Gamma_{l}^{\star}\right) \equiv \frac{1}{2}\left(i \partial_{k}-i \partial_{k^{\prime}}\right) k^{\prime}\left(k, x,-k^{\prime}\right)\right|_{k=k^{\prime}} \\
& -\left.(i / 2)\left(\partial_{k^{\prime}}-\partial_{k^{\prime}}\right) e^{-i \ell\left(\theta_{k}-\theta_{k^{\prime}}\right)} J_{\ell}\left(\mu_{0}\right) J_{\ell}\left(\mu_{0}^{\prime}\right)\right|_{k=k^{\prime}} \\
& =\ell\left(\partial_{k} \theta_{k} j_{N_{l}}^{2}\left(\mu_{0}\right) .\right. \tag{III-64}
\end{align*}
$$

From Eqs. (III-59), the factor

$$
\begin{align*}
\partial_{k} \theta_{k} & =\partial_{k} \Theta_{o}=\partial_{k} \operatorname{arc} \tan \left(k \bar{x}_{0} / \nu_{o}\right) \\
& =k_{2}^{2} \bar{x}_{20}\left(\bar{x}_{0} / \mu_{0}\right)\left[\mu_{0}^{2}+\left(k \bar{x}_{o}\right)^{2}\right]^{-1} \tag{III-65}
\end{align*}
$$

is even in $k$, as is $J_{\ell}^{2}\left(\mu_{0}\right)$. Thus each of the terms in expression (III-25) for $S_{<}$is even in $k$, to the order we are working, as assumed in Eq. (III-27) in obtaining the condition $k=0$. For $k=0$, Eq. (III-65) reduces to

$$
\begin{equation*}
\tilde{\partial}_{k} \Theta_{k}=\bar{x}_{o} / \mu_{0}=\left(\Omega_{0} / \Omega_{c}\right) k_{2}^{-1} \tag{III-66}
\end{equation*}
$$

## III.C.3. Specializing $K_{S B}$ : Choice of Distribution $f_{S}$

We proceed further with expression (III-54) for $\mathrm{K}_{\mathrm{sB}}$, by choosing a particular form for the distribution fucntion $f_{s}$, so that the integrals over momenta appearing in $K_{s B}$ can be perfomed. We are guided in this choice by the work of Part I. As in Sec. III.C.1, we assume we have adjusted the variable parameters in $f_{s}\left\{\operatorname{such}\right.$ as $w_{\| s}\left(X_{0}\right)$, the average
parallel velocity at $X_{0}$ of particles of species $s$ having guiding center $X_{0}$, so that the self-consistent and reference variables are the same. Then the physical and quasiphysical regions of the ( $H, P$ ) space are also the same, and so the value of $f_{s}$ outside this region is irrelevant.

We begin by specifying $f_{s}$ quite generally, and then progressively specialize it to get more spcific results. We modify the form of $f_{s}$ from that of Part I to account for the parallel drift velocity $w_{l \mid s}\left(X_{0}\right)$ (just defined in more precise terms) of the distribution, and for the perpendicular drift $v_{02}=v_{E}$. As in Part $I$, we want to express $f_{s}$ in terms of functions of the invariants which have a readily discernable physical meaning. One of these is $X_{0}$, already incorporated into the formalism. The second is the parallel velocity $u$ of a particle at $x_{0}$, relative to the average parallel velocity $w_{\| s}$ of the particles of that species with guiding center $X_{o}, u(\underline{P}) \equiv v_{3}-w_{\| l}$. Finally, we need a function of the invariants describing the perpendicular kinetic energy (again, at the guiding center). From Sec. II.K.3, we recall that the perpendicular kinetic energy at $X_{0}$ is $\frac{1}{2}\left(v_{01}^{2}+v_{02}^{2}\right)=\frac{1}{2}\left(v_{01}^{2}+v_{E}^{2}\right)$. Going into the frame drifting with perpendicular velocity $v_{02}=v_{E}$ leaves only the kinetic energy in the $\hat{\mathrm{b}}_{1} \equiv \hat{\mathrm{x}}$ direction, $\mathrm{h}^{\prime} \equiv \frac{1}{2} v_{01}^{2}$ (which is what one means by perpendicular kinetic energy when writing, for example, $\mu=m v_{1}^{2} / 2 B$, as we saw in Secs. II.J and II.K.3). We shall thus express $f_{s}$ in terms of $X_{o}, u$, and $h^{\prime}$ or $H^{\prime} \equiv h^{\prime}+\frac{1}{2} u^{2}=H-e_{s} \Phi\left(X_{0}\right)-\frac{1}{2} v_{E}^{2}-w_{\| \mid} v_{3}+\frac{1}{2} w_{\|}^{2}$.

We now perform manipulations on the factor $\omega \partial f_{S} / \partial H+\hat{k} \cdot \partial \underline{P}$ appearing in (III-45), and will then work on the denominator $\omega+\ell \cdot \Omega$. Expressing $f_{s}$ in terms of $H^{\prime}$ instead of $H$, we have

$$
\begin{align*}
& \left.\left.\partial f_{s} / \partial \underline{P}\right)_{H}=\partial f_{s} / \partial \underline{P}\right)_{H} \\
& \left.\quad+\partial f_{s} / \partial H^{\prime}\left\{\partial X_{0^{\prime}}^{\prime} \partial \underline{P}\right)\left[e_{s} E_{0}-u\left(\partial w_{H} / \partial X_{0}\right)-v_{E}\left(\partial v_{E} / \partial X_{0}\right)\right]-w_{\|}\left(\partial v_{3} / \partial \underline{P}\right)\right\} \tag{III-67}
\end{align*}
$$

where $E_{0} \equiv E\left(X_{c}\right) \equiv-\partial \psi / \partial X_{0}$, as in Part II.
We shall take $f_{s}$ as a sum over terms $f_{s j}$, as in Sec. I.C.3.
Eq. (III-67) then also holds for each $o^{c}$ the $f_{s j}$. Using (III-67), we find that
$\left.\left.\omega \partial f_{s j} / \partial H\right)_{\underline{P}}+\underline{\hat{k}} \cdot \partial f_{s j} / \partial \underline{P}\right)_{H}=-\bar{Y}_{s j}\left(\omega^{\prime}-\omega_{*} s j\right) f_{s j}+\left[k_{3}-k_{2}\left(\partial w_{\|} / \partial X_{o}\right)\right] \partial f_{s j} / \partial u$,
where $k_{j} \equiv \hat{b}_{j} \cdot \underline{k} \quad(j=2,3)$, and where we have the generalized forms of familiar symbols:
$\omega^{\prime} \equiv \omega-k_{2} u_{E}-k_{3} w_{\|}$,
$u_{E} \equiv\left(\partial X_{o} / \partial P_{2}\right)\left[-e_{s} E_{0}+u\left(\partial \omega_{| |} / \partial X_{o}\right)+v_{E}\left(\partial r_{E} / \partial X_{o}\right)\right]$
$\omega_{*_{5 j}} \equiv k_{2}{ }_{f j}\left(\partial X_{0} / \partial P_{2}\right) \bar{\gamma}_{s j}^{-1}, \quad\left(\partial X_{0} / \partial P_{2}\right) \equiv \hat{b}_{2} \cdot\left(\partial X_{0} / \partial P\right)$,
$k_{f j} \equiv\left(\partial / \partial X_{o}\right) \ln f_{s j}, \quad \bar{\gamma}_{s j} \equiv-\left(\partial / \partial H^{\prime}\right) \ln f_{s j}$,
and used $f_{s}=f_{s}\left(H^{\prime}, u, X_{c}\right)$ in computing $\left.\partial f_{s} / \partial \underline{P}\right)_{H^{\prime}}$. We note that $\omega_{*} j^{\prime}$, $\kappa_{f j}$, and $\bar{\gamma}_{s j}$ are not functions of $X_{o}, u$, and $h$. Taking the $H^{\prime}$ dependence
 uniform temperature), we have $\bar{\gamma}_{s j}=\gamma_{j}=$ constant. The variation of $f_{s}(H, \underline{P})$ with respect to $P_{2}$, i.e., $\hat{b}_{2} \cdot\left(\partial f_{s} / \partial P\right)$, is daminated by the variation of the density $n\left(X_{o}\right)$ [cf. Eq. (III-83)], and thus $\kappa_{f j} \neq \kappa_{n} \equiv$ $\left(\partial / \partial X_{o}\right) \ln n\left(X_{0}\right)$, In the limits of no shear and $\partial v_{E} / \partial X_{0} \rightarrow 0, \partial X_{0} / \partial P_{2} \rightarrow$ $\left(m \Omega_{c}\right)^{-1}$, and so

$$
u_{E}+v_{E}=-c E_{0} / B_{o}, \quad \text { and } \quad \omega_{*_{s j}}+k_{2} \kappa_{n} v_{s j}^{2} / \Omega_{c}
$$

$\left(v_{s j}^{2} \equiv r_{s j}^{-1}\right)$, the usual dianagnetic drift frequency. With shear, the factor ( $\partial X_{0} / \partial P_{2}$ ) becomes strongly $v_{3}$ - dependent [ $c f$ Eq. (111-85)], in fact becoming singular as ( $1+\kappa_{s} v_{3} / \Omega_{c}$ ) goes to zero. Accordingly, the expressions for $u_{E}$ and $\omega_{\omega_{s j}}$ are modified by finite $\kappa_{s} v_{3} / \Omega_{c}$, just as found in Sec. II.K for the gyrofrequency $\Omega$. In particular, for values of $\underline{p}$ where the function $X_{0}(\underline{P})$ approaches the point of double-valuedness (as for point a in Fig. 2), $\partial X_{0} / \partial P_{2}$ becomes infinite, as do $U_{E}, \omega_{*_{s j}}$, and $\Omega_{0}-1$. The aprroach of $\Omega$ to zero is a real one, arising as the septum creating the two wells in Fig. 3 appears, as $\left|v_{3}\right|$ is increased from zero, so that even a particle with $\eta_{\underline{B}} \equiv \rho_{\mathrm{i}} / \mathrm{L}_{\underline{B}} \simeq 0$ can make very large excursions in $x$, in this very flat-bottoned well. In this unstable situation, even a very slight perturbation $\delta \mathrm{E}_{2} \sim \mathrm{k}_{2} \phi$ can cuase large changes in $X_{0}$; hence the large contributions to the response $k_{2} u_{E}, \omega_{*}{ }_{j} \sim k_{2}$ in (III-68) and (III-69). The response cannot really become infinite, of course; as these terms go to infinity, the linearization approximation upon which the conductivity kernel was derived breaks down.

We turn now to expressing the resonant denominator $\omega+\underline{\ell} \cdot \underline{\Omega}$ in terms of the new variables ( $u, h^{\prime}$ ). We rewrite Eqs. (II-117) and (II-i18) as

$$
\begin{equation*}
\dot{x}_{20}=v_{E}+v_{B}\left(u, h^{\prime}\right), \quad \dot{x}_{30}=w_{1}+u \tag{III-73}
\end{equation*}
$$

where $v_{B}$, the generalization of the $B \times \nabla B$ drift, may be read off from Eq. (II-117). We also expand the gyro-frequency $\Omega$ (which, we recall, is $v_{3}$-dependent) about its value at the drift frame velocity $\hat{b}_{2} v_{E}+\hat{b}_{3} w_{11}$.

From Eq. (II-109), we have:

$$
\begin{align*}
& \Omega=\Omega_{c}\left[1+\left(\partial v_{E} / \partial X_{o}\right) / \Omega_{c}+\kappa_{s} v_{3} / \Omega_{c}\right]^{1 / 2}+O\left(n_{B}^{2}\right) \\
\simeq & \Omega_{0}+\frac{1}{2}\left(\Omega_{c} / \Omega_{o}\right)\left(\kappa_{s} u\right) \equiv \Omega_{0}+\delta \Omega \tag{III-74}
\end{align*}
$$

where we redefine $\Omega_{0}$ slightly from its meaning in Part II, by $\Omega_{0} \equiv \Omega(u=0) \equiv \Omega_{c}\left[1+\left(\partial v_{E} / \partial X_{0}\right) / \Omega_{c}+k_{s} w_{\|} / \Omega_{c}\right]^{1 / 2}$. Using (III-73), (III-74), we then have

$$
\begin{align*}
\omega+\ell \cdot \underline{\Omega} & =\omega^{\prime \prime}-\ell \Omega_{0}-\left(k_{2} v_{B}+k_{3} u+\ell \delta \Omega\right),  \tag{III-75}\\
& =\omega^{\prime \prime}-\ell \Omega_{0}-\left(k_{2} v_{B}+k_{3}^{\prime} u\right),
\end{align*}
$$

where $w^{\prime \prime} \equiv \omega-k_{2} v_{E}-k_{3} W_{\|}$is the wave frequency as seen from the drift frame, and $k_{3}{ }^{\prime} \equiv k_{3}+\ell\left(\Omega_{c} / \Omega_{0}\right) k_{s} / 2$. We see that the dispersion in $\Omega$ i.e., its dependence upon $u$ ) has introduced a term which makes $\delta k_{3} \equiv k_{3}{ }^{\prime}-k_{3} \neq 0$. Since this term is proportional to $\ell$, even for weak shear, $\delta \mathrm{k}_{3}$ may be substantial, for sizeable $\ell$. This can have significant effects on growth rates of instabilities such as the ion drift cyclotron instability (to be studied in Sec. III.D), where the assumption $k_{3} \simeq 0$ is usually made.

Using the results obtained thus far in this section, we may write Eq. (III-54), valid for arbitrary functions $f_{s j}\left(H^{\prime}, u, X_{o}\right)$, as $K_{s B}=\sum_{j} K_{s j B}, \quad K_{s j B}=K_{s j B 1}+K_{s j B 2}$
$K_{s j(B 1, B 2)}\left(k, X_{0},-k^{\prime}\right) \equiv 8 \pi^{2} e^{2} \sum_{\ell} \int d u \int d h^{\prime}\left|\Omega^{-1} I\right| \tilde{J}_{\ell}(\underline{k}) \tilde{J}_{\ell}{ }^{*}\left(\underline{k}^{\prime}\right) F_{B 1 . B 2}$,
where the factors $F_{B 1, B 2}$ are given by

$$
\begin{align*}
& F_{B 1} \equiv-\bar{\gamma}_{s j}\left(\omega^{\prime}-\omega_{*_{s j}}\right) f_{s j}(\omega+\underline{\ell} \cdot \underline{\Omega})^{-1},  \tag{III-77}\\
& F_{B 2} \equiv\left[k_{3}-k_{2}\left(\partial w_{\|} / \partial X_{o}\right)\right]\left(\partial f_{s j} / \partial u\right)(\omega+\underline{\ell} \cdot \underline{\Omega})^{-1},
\end{align*}
$$

and $(\omega+\underline{\ell} \cdot \underline{\Omega})$ is given by (III-75).
Now we further specialize the form of $f_{s}$. Following (I-21) and (I-26), we take

$$
\begin{equation*}
f_{s}\left(H^{\prime}=h^{\prime}+\frac{1}{2} u^{2}, u, X_{0}\right)=n_{s}\left|\Omega I^{-1}\right| e^{-\gamma_{s} u^{2} / 2}\left(\bar{g}_{0} e^{-\gamma_{s} h^{\prime}}-\bar{g}_{1} e^{-R \gamma_{s} h^{\prime}}\right) \tag{III-80}
\end{equation*}
$$

The normalization, analogous to (I-23), is taken as

$$
\begin{equation*}
\mathrm{n}_{\mathrm{s}}\left(\mathrm{X}_{0}\right)=2 \pi \int_{0}^{\infty} \mathrm{dh} h^{\prime} \int_{-\infty}^{\infty} \mathrm{du}\left|\Omega^{-1} I\right| \mathrm{f}_{\mathrm{s}} . \tag{III-81}
\end{equation*}
$$

Putting (III-70) jnto (III-71) gives

$$
\begin{equation*}
1=\left(2 \pi / \gamma_{s}\right)^{3 / 2}\left(\bar{g}_{0}-\bar{g}_{1} / R\right) \tag{III-82}
\end{equation*}
$$

as in (I-27).
By taking $0<\overline{\mathrm{g}}_{1} \leqslant \overline{\mathrm{~g}}_{\mathrm{O}}$, one may consider a loss cone type distribution, with loss cone partially filled $\left(\bar{g}_{1}<\bar{g}_{0}\right)$, if desired. In the application to be made in Sec. III.D, we shall take $\overline{\mathrm{g}}_{1}=0$, a drifting Maxwellian distribution. We assume $\bar{g}_{1}=0$ henceforth, and so may drop the $j$ subscript on $f_{s j=0}=f_{s}$. Equation (III~80) then reduces to

$$
\begin{equation*}
f_{s}\left(H^{\prime}, X_{0}\right)=n_{s}\left|\Omega I^{-1}\right| \bar{g}_{0} e^{-\gamma_{s} H^{\prime}} \tag{III-83}
\end{equation*}
$$

where $\bar{g}_{0}=\left(\gamma_{S} / 2 \pi\right)^{3 / 2}$. Since $f_{s}\left(H^{\prime}, X_{0}\right)$ is not explicitly u-dependent, $\mathrm{F}_{\mathrm{B} 2}$ in (III-77) and $\mathrm{K}_{\mathrm{SB} 2}$ in (III-76) vanish, so that $\mathrm{K}_{\mathrm{SB}}=\mathrm{K}_{\mathrm{SB} 1}$. In the following section, we use our choice (III-83) for $f_{s}$ in expressions (III-76) and (III-77) to explicitly evaluate the integrals prescribed there.

## III.C.4. $K_{S B}$ for the Drifting Maxwellian

In this section we use the specialized form (III-83) for $f_{s}$ to obtain a more explicit expression for the contribution $K_{s B}\left(k, x,-k^{\prime}\right)$ to the conductivity. From (III-76) and (III-77),

$$
\begin{align*}
K_{s B}\left(k, X_{0},-k^{\prime}\right) & =-\lambda_{s}^{-2} \sum_{\ell}(2 \pi) \int d u \int d h^{\prime} \bar{g}_{0} e^{-\gamma_{s} u^{2} / 2} e^{-\gamma_{s} h^{\prime}}  \tag{III-84}\\
& \times e^{-i \ell\left(\Theta_{k}-\Theta_{k^{\prime}}\right)} J_{\ell}(\underline{k}) J_{\ell}\left(\underline{k}^{\prime}\right)\left(\omega^{\prime}-\omega_{\star_{s}}\right)\left(\omega^{\prime \prime}-\ell \Omega_{0}-k_{2} v_{B}-k_{3}^{\prime} u\right)^{-1} .
\end{align*}
$$

Orbit modification effects enter here in several places, all of which we lump into the term $K_{o . m}$. There is the $O\left(n_{\underline{B}}\right)$ correction to $\tilde{J}_{\ell} \equiv J_{\ell} e^{-i \ell \epsilon_{k}}$ from its harmonic limit, as already discussed. There is also the $u$ dependence of $u_{E}$ and $\omega_{*_{S}}$, in addition to that for $\Omega$, already expressed in Eq. (III-74). From Eqs. (III-70), (III-71), we see that both these are proportional to $\left(\partial X_{0} / \partial P_{2}\right) \equiv\left(\partial X_{0} / \partial \underline{P}\right) \cdot \hat{b}_{2}$. From Part $I$, we know that along the pseudotrajectory $A_{i}^{S C}(X),\left(\partial X_{0} / \partial P_{2}\right)=\left(m \varepsilon_{c}\right)^{-1}$. Off this trajectory $\left(v_{3} \neq 0\right),\left(\partial X_{0} / \partial P_{2}\right)$ is shifted from this value by the local radius of curvature $P=e B / K_{S}$ of $\underline{A}^{S C}$ in the $\underline{P}$ plane:

$$
\begin{equation*}
\left(\partial X_{0} / \partial P_{2}\right)=\left(m \Omega_{c}\right)^{-1}\left(1+v_{3} / P\right)^{-1}=\left(m \Omega_{c}\right)^{-1}\left(1+k_{s} v_{3} / \Omega_{c}\right)^{-1} \tag{III-85}
\end{equation*}
$$

As in (III-74), we may thus expand $u_{E}, \omega_{*_{s}}$ about $v_{3}=w_{\|}$, the first order corrections being $O\left(k_{s} u / \Omega_{C}\right) \sim O\left(n_{\underline{B}}\right)$ down from the zero order terms. In the case of $\Omega$, this $O\left(\eta_{\underline{B}}\right)$ correction appears in $K_{S B}$ multiplied by harmonic number $\ell$, and so we retain it in $K_{h}$. The corrections to $K_{s B}$ due to this dispersion in $u_{E}, w_{*_{S}}$ are only $O\left(\eta_{\underline{B}}\right)$, and so are put into $K_{o . m \text {. }}$. Each of these $O\left(n_{\underline{B}}\right)$ ierms comprising $K_{O . m}$. can be explicitly evaluated, if desired. These terms are of higher order than we require, as already
mentioned, however, and so shall not be explicitly displayed here. Also lumped into the neglected $K_{0 . m}$. contribution is the tern: from $k_{2} v_{B}$ in (III-84). Finally, we neglect the terms in $\partial w_{1 \mid} / \partial X_{0}$ and $\partial v_{E} / \partial X_{0}$ in (III-70), so that $u_{E}$ may be written

$$
\begin{equation*}
u_{E}\left(v_{3}\right)=v_{E}\left(1+\kappa_{S} v_{3} / \Omega_{c}\right)^{-1} \tag{III-86}
\end{equation*}
$$

We denote by $U_{E}, \omega_{*_{S}}$ the zero-order $\left(v_{3}=w_{\|}\right)$values of these quantities. From EqS. (III-58b) we note chat $\rho_{0} \equiv \arcsin \left(\bar{x}_{0} / \mu_{0}\right)$ is independent of the action $J$, and hence of $h^{\prime}$, and for $k / k_{2} \ll 1$, of $u$. We may thus approximate (III-84) by its contribution to $K_{h}$ :

$$
\begin{align*}
& K_{s B}\left(k, X_{c},-k^{\prime}\right)=-\lambda_{s}^{-2} \sum_{\ell} \mathrm{e}^{-i \ell\left(\theta_{k}-\theta_{k}{ }^{\prime}\right)} 2 \pi \int d u \int d h^{\prime} \bar{g}_{0} e^{-\gamma_{s} u^{2} / 2} e^{-\gamma_{s} h^{\prime}} \\
& \times J_{\ell}\left(\mu_{0}\right) J_{\ell}\left(\mu_{0}^{\prime}\right)\left(\omega^{\prime}-\omega_{*_{s}}\right)\left(\omega^{\prime \prime}-\ell \Omega_{0}-k_{3}^{\prime} u\right)^{-1}  \tag{III-87}\\
&=-\lambda_{s}^{-2} \sum_{\ell} e^{-i \ell\left(\theta_{k}-\theta_{k^{\prime}}\right)}\left[1-\hbar^{\prime}\left(z_{\ell s}\right)\right]\left(\Omega_{1} / \Omega_{2}\right) I_{\ell}\left(\mu_{s} \mu_{s}^{\prime}\right) \exp \left[\frac{1}{2}\left(\mu_{s}^{2}+\mu_{s}^{\prime}\right)\right] .
\end{align*}
$$

Here $\Omega_{1} \equiv \omega^{\prime}-\omega_{*_{s}}, \Omega_{2} \equiv \omega^{\prime \prime}-\ell \Omega_{0}, z_{\ell s} \equiv \Omega_{2} /\left|k_{3}^{\prime} v_{s}\right|, v_{s} \equiv \gamma_{s}^{-1 / 2}$, $\mu_{s}^{2} \equiv \mu_{s}^{2}(k) \equiv\left(v_{s} / \Omega_{0}\right)^{2}\left[k^{2}+k_{2}^{2}\left(\Omega_{c} / \Omega_{0}\right)^{2}\right], \mu_{s}^{\prime} \equiv \mu_{s}\left(k^{\prime}\right), I_{\ell}$ is the modified Bessel function of index $\ell$, and

$$
\begin{equation*}
W(z) \equiv(2 \pi)^{-1 / 2} \int_{-\infty}^{\infty} d x x e^{-x^{2} / 2}(x-z)^{-1}=1+z Z(z), \tag{III-88}
\end{equation*}
$$

for $z \equiv z_{R}+i z_{I}$ in the upper half plane, where

$$
\begin{equation*}
z(z) \equiv(2 \pi)^{-1 / 2} \int_{-\infty}^{\infty} d x e^{-x^{2} / 2}(x-z)^{-1} \tag{III-89}
\end{equation*}
$$

is the plasma dispersion function. In obtaining (III-87) we have made use of the Bessel identity ${ }^{28}$

$$
\begin{equation*}
\int_{0}^{\infty} d x x e^{-x^{2} / 2} J_{\ell}(q x) J_{\ell}\left(q^{\prime} x\right)=I_{\ell}\left(q q^{\prime}\right) \exp \left[-\frac{1}{2}\left(q^{2}+q^{\prime}\right)\right] \tag{III-90}
\end{equation*}
$$

and of tiee fact that $\bar{x}_{0}^{2}=v_{01}^{2} / 2 v_{2}=\left(v_{o l} / x_{0}\right)^{2}$, so that

$$
\begin{equation*}
\mu_{0}^{2}=\bar{x}_{0}^{2}\left[k^{2}+k_{2}^{2}\left(\Omega_{c} / \Omega_{0}\right)^{2}\right]=\left(v_{o l} / \Omega_{0}\right)^{2}\left[k^{2}+k_{2}^{2}\left(\Omega_{c} / \Omega_{0}\right)^{2}\right] . \tag{III-91}
\end{equation*}
$$

From (III-87), the contribution of $K_{s B}$ to $K=K\left(k, X_{0}-k\right)$ is

$$
\begin{equation*}
K_{s B}\left(k, x_{0},-k\right)=-\lambda_{s}^{-2} \sum_{l}\left(1-k_{l S}\right)\left(\Omega_{1} / r_{2}\right) \Lambda_{\ell S} \tag{III-92}
\end{equation*}
$$

where $\Lambda_{\ell s} \equiv \Lambda_{\ell}\left(b_{s}\right), b_{s} \equiv \mu_{s}{ }^{2}, \Lambda_{\ell}(b) \equiv I_{\ell}(b) e^{-b}$, and $W_{\ell s} \equiv W\left(z_{\ell S}\right)$.
For $K_{n . \ell}$, we also need the contributior. of species s to $\Gamma_{1} \equiv i \partial_{k} K\left(k, X_{o}\right.$, $\left.-k^{\prime}\right)\left.\right|_{k=k}{ }^{\prime}$. Since $K_{S A}$ in Eq. (III-56) is $k$ - independent, this contribution is given by

$$
\begin{equation*}
\left.\left.i \partial_{k}^{K}{ }_{s B}\left(k, X_{0},-k^{\prime}\right)\right|_{k=k^{\prime}}=-\lambda_{s}^{-2} \sum_{\ell}\left(1-W_{l S}\right)\left(\Omega_{1} / \Omega_{2}\right)\left[\ell \partial_{k} \theta_{k}\right) \Lambda_{\ell s}+(i / 2)\left(\partial_{k} b_{s}\right) \Lambda_{\ell s}^{\prime}\right], \tag{III-93}
\end{equation*}
$$

where $\Lambda_{\ell S}^{\prime} \equiv \partial \Lambda_{\ell}\left(b_{S}\right) / \partial b_{S}$. The term in $\left(\ell \partial_{k} \theta_{k}\right)$ yields $\Gamma_{1 R}$, and the one in ( $\partial_{k} b_{s}$ ) gives $\Gamma_{1 I}$. In the frequency regime we shall be consjdering, we have $\omega<\ell \Omega_{i} \ll \Omega_{e}$, and so $\Omega_{2} \leqslant \ell \Omega_{i}$ for the ions, and $\Omega_{2} \sim \ell \Omega_{e}$ for the electrons. Therefore the electron contribution to $\Gamma_{1}$ is down from the ion contribution by a factor $\leqslant\left(\Omega_{i} / \ell \Omega_{\mathrm{e}}\right)$. The electron contribution is thus completely negligible, reflecting the fact that the small electron gyroradius causes the electron contribution to the conductivity to be almost truly local. Since in addition $K_{s A}$ does not contribute to $K_{\text {n. .., }}$ nor does the vacuam term $K_{V}$ (being $x$ independent), one has the form

$$
K_{n, \ell .}=-\partial_{x}\left[\left.i \partial_{k} K_{i B}\left(k, x,-k^{\prime}\right)\right|_{k=k^{\prime}}+c . c .\right]
$$

already stated in Sec. III.B.3. Using

$$
\begin{equation*}
\ell \Omega_{1} / \Omega_{2}=-\Omega_{1} / \Omega_{0}+\left(\omega^{\prime \prime} / \Omega_{0}\right)\left(\Omega_{1} / \Omega_{2}\right) \tag{III-94}
\end{equation*}
$$

and the Bessel identity

$$
\begin{equation*}
1=\sum_{\ell=-\infty}^{\infty} \Lambda_{\ell}(b), \tag{III-95}
\end{equation*}
$$

we find

$$
\begin{align*}
K_{n, \ell .} & =\partial_{x}\left[\lambda_{i}^{-2}\left(\partial_{k} \theta_{k}\right) \sum_{\ell}\left(1-W_{l i}\right)\left(\ell \Omega_{1} / \Omega_{2}\right) \Lambda_{\ell i}\right]  \tag{III-96}\\
& =\partial_{x}\left[-\lambda_{i}^{-2}\left(\partial_{k} \theta_{k}\right) I_{1}\left(\Omega_{1} / \Omega_{0}\right)+\lambda_{i}^{-2}\left(\partial_{k} \theta_{k}\right)\left(\omega^{\prime \prime} / \Omega_{0}\right) \sum_{\ell}\left(1-W_{\ell i}\right)\left(\Omega_{1} / \Omega_{2}\right) \Lambda_{\ell i}\right]
\end{align*}
$$

where the species label $s=i$ on $\Omega_{1}, \Omega_{2}, \omega^{\prime \prime}$, and $\Omega_{0}$ is implicit, and $I_{1} \equiv 1-\sum_{l} W_{\ell i} A_{\ell i}$. In the limit of no shear $\left(\eta_{S} \rightarrow 0\right)$, we will consider modes in which $k_{3}^{\prime}=k_{3}=0, W_{\hat{k} i}=W(2=\infty)=0$, for all $\ell$, and thus $I_{1}=1$. In Fig. 11 we sketch the behavior of the real and imaginary parts, $K_{R}$ and $W_{I}$, of $W(z)$. These functions are substantial in the region $|z| \leqslant 1$, going rapidly to zero for larger $|z|$. We write $\omega^{\prime \prime}=(\ell+\Delta \ell) \Omega_{o i}$, with $\Delta \ell$ some substantial fraction of one. Thus $\left.\left.z_{\ell^{\prime} i} \equiv\left(\omega^{\prime \prime}-\ell^{\prime} \Omega_{o i}\right) / / k_{3}+\frac{1}{2} \ell^{\prime} \kappa_{s}\right) v_{i}\right]$ $\simeq(\ell+\Delta \ell-\ell ') /\left[\left(k_{3}+\frac{1}{2} \ell ' r_{s}\right) \rho_{i}\right]$. Again assuming $k_{3} \rho_{i} \ll 1$, we sketch the $\ell '$ dependence of this in Fig. 12. We see that it is only for $\ell^{\prime}$ in the $v i$ cinity of $\ell$ that $\left|z_{\ell \prime}\right| \sim^{\prime}$, and so it is only a few tems of the sum in $I_{1}$ that contribute. Thus, even in the presence of shear, we expect $\left|I_{1}\right|$ to be sane number $\leq 1$. We also note that $K_{n . l .}$, and hence $\Gamma_{1 R}$, have both real and imaginary parts, reflecting the distinction between truly real and formally real.

We have now evaluated the terms called for by the dispersion reI.tion, Eq. (III-41). Before going on to an application, we can now use the explicit expressions just obtained to make more precise the various assumptions or: ordering we have made up to this point. We proceed to do this is in the following section.

## III.C.S. Some Estimates and Ordering

In the development thus far, we have made a number of assumptions on the relative sizes of the various terms under consideration. In the application of the fomalism in Sec. III.D, we shall make a number of additional ordering assumptions. Now that we have derived explicit expressions for the quantities called for by the abstract formalism of Sec. III. $B$, we are in a position to consider more precisely what these approximations imply.

There are a number of small parameters which enter into the theory. To begin with, there are the parameter: $\eta_{S}, \eta_{B}$, $\eta_{n}$, etc., defined in the Introduction to Part III, and used frequently throughout each part of this thesis. As has been noted, in Sec. III.D we shall be considering mode frequencies $\omega \sim \ell \Omega_{i}$, with $\ell$ a moderately large number, perhaps in the range $5<\ell \leq 20$, for which an analysis using separate Bernstein harmonics is preferable to one involving the straight line orbit approxination. Thus $\ell^{-1}$ is another small parameter.

In the trumation procedure of Sec. III.B.2, we neglected terms with derivatives of $K_{<}$of higher than second order, first in $k$, and later in $x$. From the structure of $\mathrm{K}_{\mathrm{sB}}$ in Eqs. (III-54), (III-84), or (III-87), we see that $\partial_{k}$ acts on two places in $K_{s B}$; on the factor $e^{-i l \theta_{k}}$, and on the factor $J_{\ell}\left(\mu_{0}\right) J_{\ell}\left(\mu_{0}^{\prime}\right)$ or $\Lambda_{\ell}\left(\mu_{s} \mu_{s}^{\prime}\right)$. (Here, $\ell$ satisfies $\ell \Omega_{s}=\omega$, hence $5 \leq \ell \leqslant 20$ for the ions, as noted above, and $\ell \simeq 0$ for the electrons.) Using EqS. (III-6S) or (III-66), one has that

$$
\begin{equation*}
\partial_{k}^{n} e^{-i l \theta_{k}} \sim\left(l \partial_{k} \theta_{k}\right)^{n} e^{-i l \theta_{k}} \sim\left(l / k_{2}\right)^{n} e^{-i l \theta_{k}}, \tag{III-97}
\end{equation*}
$$

and recalling that $b_{s} \equiv \mu_{s}^{2}=\left(k_{1} \rho_{s}\right)^{2}$, one finds

$$
\begin{equation*}
\partial_{k}^{2 n} \Lambda_{\ell S} \leq k_{2}^{-2 n} \Lambda_{\ell S}, \partial_{k}^{2 n+1} \Lambda_{\ell, S} \leq\left(k / k_{2}^{2 n}\right) A_{\ell, S} \tag{III-98}
\end{equation*}
$$

In view of these scalings, the action of $\partial_{k}$ on the former factor dominates the latter, for the ions, by the harmonic number \&. Also, by Eq. (III-27), ( $k / k_{2}$ ) is equal to (or very nearly equal to) zero. Thus, we have

$$
\begin{equation*}
\Gamma_{1 I} / \Gamma_{1 R} \sim\left(\partial_{k} \ln b_{s}\right) /\left(\ell \partial_{k} \theta_{k}\right) \sim\left(k / k_{2} \ell\right) \ll 1 \tag{III-99}
\end{equation*}
$$

For $K(k, x,-k)$, the factor in $\theta_{k}$ is not present, and so $\partial_{k}$ acts purely on the $\mathrm{b}_{\mathrm{s}}$ dependence. Thus

$$
\begin{equation*}
\frac{1}{2} \mathrm{~d}_{\mathrm{k}}^{2} \mathrm{~K} \sim \mathrm{k}_{2}^{-2} \mathrm{k} \tag{III-100}
\end{equation*}
$$

Now we can discuss the condition for convergence of the infinite order operator $K$ of Eq. (III-13) or (III-14), and the validity of truncation at finite order. In Sec. III.B, these conditions were abstractly symbolized by the condition $\partial_{x} \partial_{k}<1$. Making the straightforward replacements $\partial_{k} \rightarrow \ell / k_{2}, \partial_{\chi} \rightarrow L_{\phi}$, we arrive at the convergence condition

$$
\begin{equation*}
1>\partial_{x} \partial_{k} \sim\left(\ell / k_{2} L_{\phi}\right) \equiv \varepsilon_{\phi \ell} . \tag{III-101}
\end{equation*}
$$

For the electrons, where $\ell \approx 0$, this condition is easily satisfied, even for very small $L_{\phi}$. The truncation at second order for the electrons is thus a very good approximation, reflecting the highly local nature (small $\mathrm{k} \rho_{\mathrm{e}}$ ) of the electron response. We therefore henceforth consider (III-101) only in connection with the ions. For the ions, this condition can be rather stringent. The ratio of the parameter $\varepsilon_{\phi \ell}$ to $\rho_{i} / L_{\phi}$
(which wis the parameter for detemining whether the nearly local or fully integral form of $S_{i B}$ should be employed) is given by $\varepsilon_{\phi l} /\left(\rho_{i} / L_{\phi}\right)=\left(\ell / k_{2} \rho_{i}\right)$. In Eq. (III-104), we shall find that $\left(l / k_{2} \rho_{i}\right)$ - $\eta_{n}<1$. Thus Eq. (III-101) is a samewhat more stringent condition than $F_{i} / L_{\phi}<1$ for loss of validity of the nearly-local form of $S_{i B}$. At the end of this section [see Eq. III-116)], we discuss why condition (III-101) is probably more stringent than necessary, and may be replaced by $\varepsilon_{\phi 1}<1$. This criterion is of stringency comparable to $\rho_{i} / L_{\phi}<1$. Thus, when the validity of the nearly-local approximation for $S_{i B}$ breaks down, the regime where $\rho_{i} / L_{\phi}>1$ is entered. In this regime, dispersion equation (III-41b) brecomes valid.

Applying the same sort of scaling argument as used to obtain (III-101), and using (III-45) for $L_{\phi}^{-2}=|\alpha|$, we reexpress condition (III-47) for mode localization:

$$
\begin{equation*}
1>\left(L_{\phi} / L_{Q}\right)^{2} \simeq\left(k_{2} L_{Q}\right)^{-1} \tag{III-102}
\end{equation*}
$$

This condition is met whenever (III-101) is.
We can estimate the size of $k_{2}$, for the application of Sec. III-D, using the fact that we shall be considering modes in a frequency regime where

$$
\begin{equation*}
\omega \sim \ell \Omega_{i} \simeq \omega_{t_{i}} \simeq k_{2} \kappa_{n} \rho_{i} v_{i} \tag{III-103}
\end{equation*}
$$

This gives

$$
\begin{equation*}
k_{2} \rho_{i}=\ell L_{n} / \rho_{i} \equiv \ell \eta_{n}^{-1}>\ell \tag{III-104}
\end{equation*}
$$

and so (III-93) gives a condition on $\mathrm{I}_{\mathrm{Q}}$ :

$$
\eta_{Q} \equiv \rho_{i} / L_{Q}<\ell \eta_{n}
$$

Before we can compare the relative sizes of the various terms in dispersion equations (III-41), we must estimate the size of $L_{Q}$. Without shear, $L_{Q}$ is given by the scale length $L_{p} \sim L_{n}$ of the plasma parameters. As $\kappa_{s} \equiv L_{s}^{-1}$ is increased from zero, however, a new scale length Loe, defined by

$$
\begin{equation*}
L_{o e} \equiv\left[\hat{o}_{x}\left|\ell n z_{o e}\right|\right]^{-1}=L_{s}\left(\omega / k_{2} v_{e}\right) \ll L_{s} \tag{III-106}
\end{equation*}
$$

moves rapidly in from infinity, becoming the dominant (shorter) scale length of the potential $Q$. Thus,

$$
L_{Q} \equiv \min \left(L_{n}, L_{o e}\right)
$$

The changeover from $L_{Q}=L_{n}$ to $L_{Q}=L_{\text {oe }}$ occurs when

$$
\begin{equation*}
\kappa_{s} / \kappa_{n} \equiv L_{n} / L_{s} \simeq \omega / k_{2} v_{e} \simeq \eta_{n}\left(v_{i} / v_{e}\right) \ll 1 \tag{III-108}
\end{equation*}
$$

where we have used Egs. (III-103) and (III-104) in the second approximation. For $T_{e} \simeq T_{i}$, and $\eta_{n} \simeq \frac{1}{5}$, this gives $k_{s} / k_{n} \simeq(1 / 200)$ at changeover. Thus, essentially because $v_{i} / v_{e} \ll 1$, shear effects become strong long before the naive guess $k_{s} / k_{n} \sim 1$ one might make.

We now conpare the relative sizes of the tems appearing in the dispersion relation, using the scalings discussed in this section. We find

$$
\begin{align*}
& K_{\text {d.e. }} / K \sim f_{n}\left(k_{2} L_{Q}\right)^{-1}, K_{n . \ell .} / K \sim\left(\ell / k_{2} L_{p}\right),  \tag{III-109}\\
& K_{o . m .} / K \sim \eta_{B}, K_{o} \sim K_{n} \sim K .
\end{align*}
$$

We note that $K_{n . l}$. scales as the reciprocal of $L_{p} \equiv \min \left(L_{n}, L_{\underline{B}}\right)$ instead of as $L_{Q}^{-1}$, as does $K_{d . e . ~ T h i s ~ r e f l e c t s ~ t h e ~ f a c t, ~ n o t e d ~ i n ~ S e c . ~ I I I . C . ~}^{\text {. }}$, tfiat the electron contribution to $\Gamma_{1 R}$ is down from the ion contribution by ( $m_{e} / m_{i} \ell$ ). The action of $\partial_{x}$ in $K_{n, \ell .}=-\partial_{x} \Gamma_{1 R}$ brings in (through $L_{o e}$ )
a factor $\left(m_{i} / m_{e}\right)^{1 / 2}$ which partially compensates, but still leaves the ion contribution dominant. The dependence of $K_{n . \ell}$. on $L_{S}^{-1}$ is thus much weaker than that of $\mathrm{K}_{\mathrm{d} . e}$.

From Eqs. (III-109) and (III-104), we have that $K_{o . m .} / K_{d . e} \leq$ $\eta_{\underline{B}} k_{2} L_{Q}-\left(L_{Q} / L_{\underline{B}}\right) l \eta_{n}^{-1}$. For both the regime $L_{Q}=L_{n}$ and $L_{Q}=L_{o e}$, using this condition and expression (III-108) gives

$$
\begin{equation*}
K_{o . m .} / K_{\text {d.e. }} \leq\left(v_{i} / v_{e}\right)_{-} \ll 1 \tag{III-110}
\end{equation*}
$$

 placing $K$ there by $K_{h}$.

In Fig. 13 we display the scaling of these various ratios $\left(K_{x} / K_{h}\right)\left(x=h\right.$, d.e., n.l., o.m.) with $\kappa_{s} / \kappa_{n}$. We enploy the parameters $\varepsilon_{p \hat{\imath}} \equiv \ell / k_{2} L_{p} \quad \varepsilon_{p 1} \equiv 1 / k_{2} L_{p}$, and $\eta_{n}$ as calibrations on the ordinate axis (not drawn to scale). We note the elbow in ( $\mathrm{K}_{\mathrm{d} . e}{ }^{/} \mathrm{K}_{\mathrm{h}}$ ) due to the transition from $L_{Q}=L_{n}$ to $L_{Q}=L_{o e}$, occurring at $\kappa_{s} / \kappa_{n} \sim\left(\omega / k_{2} v_{e}\right)$, and see that $K_{o . m .} / K_{d . e .} \ll 1$, as shown just above algebraically. Using $L_{\phi} \simeq \alpha^{-1 / 2}$ $\sim L_{Q}\left(k_{2} L_{Q}\right)^{-1 / 2}$ and (III-104), we express (III-101) as

$$
\begin{equation*}
1>\ell\left(k_{2} L_{Q}\right)^{-1 / 2} \simeq\left(\ell \eta_{n} \eta_{Q}\right)^{1 / 2} \tag{III-111}
\end{equation*}
$$

This convergence condition is violated at $\kappa_{s} / \kappa_{n} \simeq\left(v_{i} / v_{e}\right)\left(\ell n_{n}\right)^{-1} \simeq$ ( $L_{o e^{\prime}} / L_{s}$ ) $\left(\ell \eta_{n}{ }^{2}\right)^{-1}$, indicated by line a in Fig. 13. To the right of line $b$, the less stringent condition (III-116) or (III-117) breaks down, and the criterion $\rho_{i} / L_{\phi}<1$ fails at some point in the vicinity of this line. To the right of this line, use of the form $S_{i B}$ valid for $\rho_{i} / L_{\phi}>1$ becomes appropriate.

## * From the "point of view of the ions, the appearance of the local-

 ized mode in the nearly local $\left(\rho_{i} / L_{\phi}<1\right.$ and fully integral ( $\rho_{i} / L_{\phi}>1$ )regimes is quite different. However, we now show that the dispersion equations for these two different regines are, nevertheless, quite similar. Using the large argument limit of $\Lambda_{\ell}$,

$$
\begin{equation*}
\Lambda_{l}(b) \underset{b>l^{2}}{ }(2 \pi b)^{-1 / 2}, \tag{III-112}
\end{equation*}
$$

and putting this into Eq. (D29), one has

$$
\begin{equation*}
\Lambda_{\ell i}\left(\rho_{s} / L_{\phi}<1\right) \simeq\left(2 \pi b_{i}\right)^{-1 / 2} \simeq\left(2 \pi k_{1}^{2} \rho_{i}^{2}\right)^{-1 / 2} \tag{III-113}
\end{equation*}
$$

On the other hand, taking the $n=0$ value of (i)30),

$$
\begin{equation*}
\Lambda_{\ell i}\left(\rho_{s} / L_{\phi}>1\right)=\left(2 \pi^{2}\right)^{-1 / 2}\left[\alpha \rho_{i}^{2}+k_{2}^{2} \rho_{i}^{2}\right]^{-1 / 2} \tag{III-114}
\end{equation*}
$$

If we write $a=\left(k_{x}{ }^{2}\right)$, some average value of the wavevector in the $\hat{x}=\hat{b}_{1}$ direction, we see that (III-113) is the same as (III-114), up to a factor of $\pi^{1 / 2}$. Thus, up to this factor, the contribution of $S_{i B}$ to dispersion equations (III-41a) and (III-41b) is the same. Moreover, from Fig. 13 we see that $K_{n . \ell . ~ h a s ~ b e c o m e ~ s m a l l ~ c o m p a r e d ~ t o ~} K_{d . e}$. in the regime where $p_{s} / L_{\dot{\phi}}>1$ becomes valid, and so may be neglected in (III-41a). Equations (III-41a) and (III-41b) are then almost identical, though applying to quite different regimes. This similarity in form is corroborated by the numerical findings of Ref. 21. These authors perform a stability analysis for a plasma with strong density inhomogeneity using two methods, Oie (which they call the "local method") enploying truncation of the exact kernel at second order, just as we do here, and the other (termed the 'nonlocal method') solving poisson's equation in $k$ space, taking account of the coupling of wavenumbers caused by Vn. . Applying these mathods nu-* merically, these authors find the results of their local method to be in good agreement with the nonlocal method into the regime $\left(k_{2} L_{\phi}>1\right)$ where
the latter method is strictly valid, but the fomer is not.
The authors of Ref. 21 postulate a much more easily satisi.ed convergence condition than (III-101), namely $1>\varepsilon_{Q 1}$, where $\varepsilon_{Q Q} \equiv \ell / k_{2} L_{Q}$. (Hamonic number $\ell$ does not enter into their considerations because, as noted in Sec. Ill.B.1, they make the approximation $k=k^{\prime}$ from the beginning, and so set $e^{-i \ell\left(\theta_{k}-O_{k}\right)}=1 . j$ Estimating the sizes of the $n=0,1,2$ tems in Eq. (III-18), we find

$$
\begin{align*}
& (\mathrm{n}=0 \text { term }) \sim K, \quad(\mathrm{n}=1 \text { term }) \sim \mathrm{i} \partial_{x} \partial_{k} K \sim \varepsilon_{Q \ell} K,  \tag{III-115}\\
& (\mathrm{n}=2 \text { terms }) \sim L_{Q}^{-2} d_{k}^{2} K \sim \varepsilon_{Q 1}^{2} K, \text { and } \partial_{x}^{2} \partial_{k}^{2} K \sim \varepsilon_{Q \ell}^{2} K .
\end{align*}
$$

We thus see that, by manipulating terms as done in obtaining (III-18) from (III-17), we can change the parameter $\varepsilon_{\phi \ell}$ of (III-101) to either $\varepsilon_{\phi 1}$ (by collecting terms in $\partial_{k}^{m} \partial_{k^{\prime}}^{n-m}$ to yield a term in $d_{k}^{n}$ ), or to $\varepsilon_{Q Q}$ (by integrating derivatives $\partial_{x}$ on $\Phi_{1}$ by parts so that they act only on $K$ ). If we thus postulate that the nth order terms of $S$ scale as $E_{\phi 1}^{m} \varepsilon_{Q Q}^{n-m}$ $(m=0,1, \cdots n)$, where $\varepsilon_{Q \ell} \equiv \ell / k_{2} L_{Q}$, condition (III-101) should be replaced by

$$
\begin{equation*}
1>\varepsilon_{\phi 1}, \varepsilon_{Q \ell} . \tag{ITI-116}
\end{equation*}
$$

Using this weaker condition, we obtain the convergence boundary

$$
\begin{equation*}
1 \simeq F_{Q Q} \simeq \eta_{n} \eta_{Q} \text {, or } \kappa_{s} / \kappa_{n} \simeq\left(L_{0 e} / L_{s}\right) \eta_{n}^{-2}, \tag{III-117}
\end{equation*}
$$

instead of that given by Eq. (III-111).

## III.D. Application: Ion Drift Cyclotron Instability

In this Section we apply the formalism thus far developed, studying the ion drift cyclotron (IDC) instability, ${ }^{28}$ in both unsheared and sheared magnetic geometries. This instability is closely related to the drift cyclotron loss cone (DCLC) jastability, ${ }^{21,27}$ but finds its free energy source only in the density gradient, as opposed to the DCLC, where a loss cone ion distribution is the prinary free energy source. The application displays some (but not all) of the useful features of the formalism. In Sec. III.D.1, we characterize the different shear regimes, into which the scaling of the various terms entering into dispersion equations (III-41) divides the problem. The subsequent sections consider each of these regimes in turn.

## III.D.1. Overview

In the application of this section, we use dispersion equation (III-41a) in the regime where $\rho_{i} / L_{\phi}>1$. However, in light of the discussion in Sec. III.C. 5, the form of the dispersion equation in these two regimes is almost the same, as are the expressions for mode growth rates. The analysis divides itself naturally into three regimes, as a function of shear strength:
a) 'Weak Shear" $\left(\kappa_{s} / \kappa_{n}<L_{o e} / L_{s}\right.$, i.e. $\left.L_{Q} \simeq L_{n}\right)$ : In this regime, the shear scale length $L_{o e}$ is too long to significantly affect the form of. . ....... the potential $Q(x)$. Modes are then localized in a potential well, of width on the order of $L_{n}$. Then $K_{d . e}$, and $\alpha$ are real quantities. From Fig. 13 or Eq. (III-109), we see that in this range $K_{d . e}(n=0) \ll K_{\text {n.l. }}$ and
so $k_{\text {d.e. }}$. becomes appreciable only for the higher-n modes ( $2 n \sim \ell$ ). Equation (III-4la) is the appropriate dispersion equation.
b) "Intermediate Shear" $\left[L_{\mathrm{Oc}} / L_{\mathrm{s}}<\kappa_{\mathrm{s}} / \kappa_{\mathrm{n}}<\left(\mathrm{L}_{\mathrm{OC}} / L_{\mathrm{s}}\right)\left(\Omega / \eta_{\mathrm{n}}{ }^{2}\right)\right]$ : In this range, the density and shear scale leugths $L_{n}$ and $L_{o c}$ determining $L_{Q}$ are comparable. Shear causes the real part $Q_{R}$ of the potential $Q$ to have a hump at the point where $k_{3}=0$, surrounded by two wells at $\left|z_{o e}\right| \sim 1$ (cf. Fig. 17b). Away from this hump, the mode is strongly Landau damped by electrons, and so the mode is localized in the Berk-Pearlstein fashion. ${ }^{23}$ (This will be described in more detail at the end of this section.) Correspondingly, $K_{\text {d.e. }}$ and $\alpha$ are imaginary. $K_{d . c}(n=0)$ is now comparable to $K_{n . \ell}$, and so both should be retained in dispersion equation (III-41a), for $\rho_{i} / L_{\phi}<1$. As the shear is increased still further in this regime, $L_{Q} \simeq L_{\text {oe }}$ becomes small enough that the range $\rho_{i} / L_{\phi}>1$ is entered. Then Eq. (III-41b) is the appropriate dispersion equation, the nonlocal corrections entering through $S_{i B}$.
c) 'Strong Shear"' $\left[\kappa_{s} / \kappa_{n}>\left(L_{o e} / L_{s}\right)\left(\ell / \eta_{n}^{2}\right)\right]$ : Here $L_{o e}$ has become so small that $Q(x)$ cannot localize modes, even in the Berk-Pearlstein manner. (The tems 'weak", "intermediate", and "strong" here refer to the effect on the problem of $L_{o e} \ll L_{s}$. Thus $L_{s} / L_{n}$ is said to be "small" when the effect of shear on $Q(x)$ dominates that of $L_{n} \cdot$ ) In the following sections, we shall discuss each of these regimes in turn.

Before proceeding to consider the IDC instability in these specific parametom regimes, we develop some more general expressions which are of use for more than one regime of $\kappa_{s} / \kappa_{n}$. The IDC instability arises from the interaction of a drift mode, the "ion-shielded electron convection" (ISEC) wave, and the ion Bernstein harmonics. We are thus in a
frequency regime $\omega \sim l \Omega_{i} \ll \Omega_{e}$, and so keep only the $\ell=0$ term in the electron contribution $K_{\mathrm{eB}}$. From Eqs. (III-11), (III-56), (III-92), and (III-96), we write

$$
\begin{align*}
K & +K_{n . \ell .} \simeq K_{h}+K_{n . \ell .} \\
& =\underline{k}^{2}+\lambda_{e} e^{-2}\left\{1-\Lambda_{o e}\left(\frac{\omega_{e}^{\prime}-\omega_{*_{e}}}{\omega_{e}^{\prime \prime}}\right)\left(1-W_{o e}\right)\right\}  \tag{III-118}\\
& +\lambda_{i}^{-2}\left\{I_{2}-I_{3}\left[\Lambda_{\ell l}\left(\frac{\omega_{i}-\omega_{\star_{j}}}{\omega_{i}^{\prime \prime}-\ell \Omega_{\mathrm{oi}}}\right)\left(1-W_{\ell i}\right)\right\}\right.
\end{align*}
$$

Here, $\underline{k}^{2} \equiv \mathrm{k}^{2}+\mathrm{k}_{2}^{2}+\mathrm{k}_{3}{ }^{2}$, and we have summed $\mathrm{K}_{\mathrm{iB}}$ and $\mathrm{K}_{\mathrm{n} . \ell .}$, yielding the factors $I_{2} \equiv 1-\left(\kappa_{p} \partial_{k} \theta_{k}\right)\left(\Omega_{1} / \Omega_{o i}\right) I_{1}$, and $I_{3} \equiv 1-\left(\kappa_{p} \partial_{k} \theta_{k}\right)\left(\omega_{i} " / \Omega_{o i}\right)$. We have made the replacement $\partial_{X} \rightarrow \kappa_{p}$ here, where $\kappa_{p}$ is the inverse plasma scale length (for the ions). In the weak shear regime, and well into the intermediate shear regime, $k_{p}=k_{n}$. Even into the strong shear regime, where $\kappa_{s}$ is only becoming comparable to $\kappa_{n}$, one still has $\kappa_{p} \sim \kappa_{n}$. Noting that $\omega_{i}^{\prime} \sim \omega_{i}^{\prime \prime} \sim \ell \Omega_{i}, \quad \Omega_{i} \equiv \omega_{i}^{\prime}-\omega_{*_{i}} \sim \Omega_{i}$, and $\partial_{k}{\theta_{k}}^{\sim} k_{2}{ }^{-1}$, we see that $I_{2} \sim 1-\varepsilon_{p 1}, I_{3} \sim 1-\varepsilon_{p \ell}$.

For $\omega_{i}$ " not near $\ell \Omega_{o i}$, one may neglect the $K_{i B}$ contribution (the term $\Sigma_{\ell}$ ) in (III-118), which is the origin of the ion Bernstein harmonics ( $\omega \sim l \Omega_{\mathrm{oi}}$ ). The remaining expression is our choice for $K_{o}$ in the regime $\rho_{i} / L_{\phi}<1$, whose form is dominantly responsible for determining the mode localization:

$$
\begin{equation*}
K_{o} \equiv \mathrm{k}^{2}+\mathrm{I}_{2} \lambda_{\mathrm{i}}^{-2}+\lambda_{\mathrm{e}}{ }^{-2}\left[1-\Lambda_{\mathrm{oe}}\left(1-\omega_{\mathrm{te}_{\mathrm{e}}} / \omega_{\mathrm{e}}{ }^{\prime}\right)\left(1-W_{\mathrm{oe}}\right)\right] \tag{III-119}
\end{equation*}
$$

For the regime $\rho_{i} / L_{\phi}>1$ where we have no contribution $K_{n, \ell .}, K_{o}$ is given by (III-119), replacing $I_{2}$ there by unity. In (III-119) we have used (III-86), noting that $\left|\omega_{e}^{\prime}-\omega_{e}^{\prime \prime}\right|=\left|k_{2}\left(u_{E}-v_{E}\right)\right| \sim\left|\left(k_{2} v_{E}\right)\left(\kappa_{s} \rho_{e}\right)\right| \ll\left|k_{2} v_{E}\right|$.

Setting the (formally) real part of this to zero gives the dispersion relation for the ISEC mode:

$$
\begin{equation*}
\omega_{\text {ISEC }}^{\prime} \equiv \omega_{\text {ISEC }}-k_{2} v_{E}-k_{3} w_{\| e}=-\omega_{\pi_{e}} \Lambda_{\mathrm{oe}} g_{\mathrm{Re}}\left[\left(\kappa \lambda_{\mathrm{e}}\right)^{2}+\tau I_{2}+\left(1-\Lambda_{\mathrm{oe}} g_{\mathrm{Re}}\right)\right]^{-1}, \tag{III-120}
\end{equation*}
$$

where $\tau \equiv \lambda_{e}^{2} / \lambda_{i}^{2} \equiv T_{e} / T_{i}$, and $g_{R e} \equiv R_{e} g_{e} \equiv R_{e}\left(1-W_{o e}\right)$. In the usual limit in which this dispersion relation is derived, $\varepsilon_{\mathrm{p} 1} \rightarrow 0, \mathrm{k}_{3} \rightarrow 0$, and $\omega_{\text {ISEC }} \rightarrow{ }^{\omega}$ ISEC , so that $g_{e} \rightarrow 1, I_{2} \rightarrow 1$, and Eq. (III-120) reads

$$
\begin{align*}
\omega_{\text {ISEC }} & =-\omega_{k} \Lambda_{o e}\left[\left(k \lambda_{e}\right)^{2}+\tau+b_{e}\right]^{-1}  \tag{III-121}\\
& \simeq-\omega_{*_{e}} \Lambda_{o e}\left[k_{2}^{2}\left(\lambda_{e}^{2}+\rho_{e}^{2}\right)+\tau\right]^{-1}
\end{align*}
$$

where we have used $\left(1-\Lambda_{o e}\right) \simeq b_{e} \simeq k_{2}{ }^{2} \rho_{e}{ }^{2}$.
We consider $\omega_{\text {ISEC }}$ as a function of $k_{2} \simeq|\underline{k}|$. From Eqs. (III-71) and (III-121), we have that ${ }_{\text {WSEC }} \propto k_{2}\left[k_{2}^{2}\left(\lambda_{e}{ }^{2}+\rho_{e}^{2}\right)+\tau\right]^{-1}$, and thus ${ }^{u}{ }_{I S E C}$ has a maximum value $\max _{\text {ISEC }}$, occurring at $k_{2}^{2}\left(\lambda_{e}^{2}+\rho_{e}^{2}\right) \simeq \tau$. This gives a maximum harmonic number, $\ell_{\text {max }}$, for which the IDC instability can occur:

$$
\begin{equation*}
\ell_{\max } \simeq \omega_{\text {ISEC }}^{\max } / \Omega_{i} \simeq \frac{1}{2}\left(\kappa_{n} \rho_{i}\right)\left(\rho_{i} / \lambda_{e}\right) \tau^{1 / 2}\left(1+\omega_{e}^{2 / \Omega_{e}}\right)^{-1 / 2} \tag{III-122}
\end{equation*}
$$

where $\omega_{e}(x) \equiv\left[4 \pi n_{e}(x) e^{2} / m_{e}\right]^{1 / 2}$ is the electron plasma frequency for a plasma with density given by the guiding center density $n_{e}(x)$. As the density gradient $\kappa_{n}$ weakens, $\ell_{\max }$ decreases, until finally $\ell_{\max }$ drops below unity, and the IDC instability disappears altogether.

For calculating $\mathrm{K}_{\mathrm{d} . \mathrm{e}}$, we need to compute $\frac{1}{2} \mathrm{~d}_{\mathrm{k}}{ }^{2} \mathrm{~K}_{\mathrm{o}}$. The derivative $d_{k}$ acts only on $\underline{k}^{2}$ and $\Lambda_{o e} \simeq 1-b_{e}$ in (III-119), giving

$$
\begin{equation*}
\frac{1}{2} d_{k}^{2} K_{o}=1+\left(\omega_{e}^{2} / \Omega_{o e}^{2}\right)\left(1+\tau \omega_{\star}^{\prime} / \omega_{e}^{\prime}\right) g_{e} \tag{III-123}
\end{equation*}
$$

where $\omega_{*}^{\prime} \equiv k_{2} k_{n} v_{i}^{2} / \Omega_{c i}$, so $\omega_{*} / \omega_{*}^{\prime}=-\tau$. Since from (III-120), $\omega_{e}^{\prime} \sim \omega_{\text {ISEC }}^{\prime} \sim \omega_{*}$, we see from (III-123) that $R_{e}\left(\frac{1}{2} d_{k}{ }^{2} K_{0}\right)$ is a positive quantity, a fact quoted in Sec. III.B.3.

Now using (III-123) and (III-119) in (III-46), we obtain an explicit expression for $K_{\mathrm{d}} . \mathrm{e}$ :

$$
\begin{align*}
K_{d . e} & =f_{n}\left[\frac{1}{2}\left(d_{k}^{2} K_{0}\right) \frac{1}{2} \partial_{x_{0}}^{2} K_{0}\right]^{1 / 2} \\
& =f_{n}\left\{\left[1+\left(\omega_{e}^{2} / \Omega_{o e^{2}}^{2}\right)\left(1+\tau \omega_{*}^{\prime} / \omega^{\prime}\right) \xi_{e}\right]^{\frac{1}{2}} \partial_{x_{0}}^{2}\left[I_{2} \lambda_{1}^{-2}+\lambda_{e}^{-2}\left(1-\Lambda_{o e^{\prime}} g^{\left(1+\tau \omega_{*}^{\prime} / \omega^{\prime}\right)}\right)\right]\right\}^{1 / 2}, \tag{II-124}
\end{align*}
$$

where $\omega^{\prime} \equiv \omega_{e}^{\prime \prime}$. (In the present shear regime, $\omega^{\prime}=\omega_{e}^{\prime}=\omega_{i}^{\prime}=\omega-k_{2} v_{e}$.)
Taking $g_{e}$ and $\omega^{\prime}$ real, we have just seen that $\frac{1}{2} d_{k}{ }^{2} K_{o}$ is real and positive. The term $\frac{1}{2} \partial_{x_{0}}{ }_{0}{ }_{0}$ will also be positive if the mode is localized at a minimum of $K_{o}(x)$, and negative if localized at a maximum. Thus the product of these terms, $K_{\text {d.e. }}{ }^{2}$, and their ratio, $\alpha^{2}$, will be positive or negative according as the mode is localized at a well bottom or a hill top of $K_{o}$, respectively. Moreover from Eq. (III-41), $K_{d . e}=\left(\frac{1}{2} d_{k}{ }^{2} K_{o}\right) a$, so that ( $\mathrm{K}_{\mathrm{d} . e}$. ${ }^{\text {a }}$ will be positive. In the case of weak shear, we will find that the mode is localized at a well bottom. For the assumed form of the eigennodes, $\phi^{\sim} \exp \left(-\alpha x^{2} / 2\right)$, we see that $\alpha$ must be real and positive to behave suitably as $|x|+\infty$, and accordingly so must $K_{d . e .}$. For stronger shear, the modes will be localized at a hill top. The appropriate boundary conditions in this case are outgoing wave solutions, ${ }^{23}$ for which $\alpha$, and so $K_{\text {d.e. }}$, have negative inaginary parts $\left(\alpha \simeq i \alpha_{I}, \alpha_{I}<0\right)$.

## III.D.2. IDC Instability with Negligible Shear

In this section we begin our study of the IDC instability in the simply case in which it was originally studied, ${ }^{30}$ a uniform magnetic field $\left(n_{B}-0\right)$, with density gradient only. We shall retain the correction $k_{n . \ell .}$, however, making the results valid for larger $\eta_{n}$ than in the work of Ref. 30. Study of this simple configuration will serve to clarify the nature of the instability, in preparation for the more complicated problem of a sheared field, in addition to showing the effect of $\mathrm{K}_{\mathrm{n} . \ell .}$ on the instability.

For $\mathrm{k}_{3} \neq 0$, electron Landau damping enters (III-118) [through $\left.W_{\text {Ioe }} \equiv I_{m}\left(W_{o e}\right)\right]$ to stabilize the mode. To maximize the growth rate, one therefore sets $k_{3}=0$ in the dispersion equation. Thus $W_{\ell S} \rightarrow W(\infty)=0$ (cf. Fig. 11), and $g_{e} \equiv 1-W_{\ell i}+1$. For future use, we retain $g_{e}$, and $g_{\ell}$ in the disprsion equation, remembering that in this unsheared case these factors equal unity.

Using Eq. (iII-118) in this unsheared limit, Eq. (III-41a)
becones

$$
\begin{align*}
0 & =K+K_{n . \ell .}+K_{d . e} . \\
& =\underline{k}^{2}+\lambda_{e}^{-2}\left[1-\Lambda_{o e} g_{e}\left(1+\tau \omega_{*} \prime / \omega^{\prime}\right)\right]+\lambda_{i}^{-2}\left[I_{2}-I_{3} \sum_{\ell} \Lambda_{\ell i} g_{\ell} \frac{\omega^{\prime}-\omega_{* i}}{\omega^{\prime}-\ell \Omega_{c i}}\right]+K_{d . e} . \tag{III-125}
\end{align*}
$$

where $\omega_{e}^{\prime}=\omega_{i}^{\prime}=\omega^{\prime} \equiv \omega-k_{2} v_{E}$ For $I_{2}, I_{3}+1, K_{d . e .} \rightarrow 0$, this is the stan: . dard dispersion"equadion of Ref. 30 for the IDC instability. It is to be evaluated at $x=x_{0}$, where $x_{0}$ is found from application of (III-43).

We solve Eq. (III-125) for the IDC growth rate $\gamma$. We first write it in a slightly different form:

$$
\begin{gather*}
\left.0=\left[\underline{k} \lambda_{e}\right)^{2}+\tau I_{2}+\left(1-\Lambda_{o e} g_{e}\right]+\lambda_{e}^{2} K_{d e}\right]-\tau \Lambda_{o e} \omega_{*} / \omega^{\prime}  \tag{111-126}\\
-\tau I_{3} \sum_{\ell} \Lambda_{\ell i} g_{\ell} \frac{\omega^{\prime}-\omega_{*}}{\omega^{\prime}-\rho \Omega_{c i}},
\end{gather*}
$$

where we define $\omega_{*} \equiv \omega_{*_{i}}$, and note that in this regime $\omega_{*}=\omega_{*}$. Designating the thre terms in (III-126) by $A^{\prime} \equiv A+\lambda_{e}{ }^{2} K_{d . e .}, B$, and $C$, in Fig. 14 we graphically visualize the dispersion equation to be solved. In each frequency incerval $\ell^{\prime} \in \omega^{\prime} / \Omega_{\mathrm{ci}}<\ell^{\prime}+1$ except for the one containing $\omega_{*}\left(\ell x_{a} \omega_{*} / \Gamma i_{i}<\ell+1\right)$, term $C$ runs from $-\infty$ to $\infty$, and so an intersection of $B+C$ with $-A$ ' in Fig. 14, hence a real root of 4 . (III126), is guaranteed $j$, each of these intervals $\ell$ '. The IDC instability can thus arise only in the $\ell^{\text {th }}$ interval, if $B+C$ does not extend far enough down (for $\omega^{\prime}$ eal) to reach $-A^{\prime}$.

From this griphical representation of the origin of the instability, we see that $r$ taining only the two terms $\ell$ and $\ell+1$ in factor $C$ accurately represents the dispersion relation in the frequency interval where the instability arises. This two term approximation is graphically displayed in Fig. 15. Previous work on the IDC instability has solved the IDC dispersion equation both numerically ${ }^{27}$ and analytically. $30,31,32$ In contrast to the two-term approximation we shall use here, the previous analytic solutions $r$ tained only a single term of facior $C$, illustrated in Fig. 15b. The radically different behavior of this one-term approx-- imation $0 \leqslant C$ around ihe $s^{\text {th }}$ frequency interval yields an expression for $\gamma$ which, as we shall see, scales quite differently with $I_{2}, I_{3}, g_{e}, g_{\ell}$ and $K_{\text {d.e. }}$ than the $\gamma$ from the two term approximation. In the unsheared, small $\eta_{14}$ limit of Re, 30 the first four of these hecome unity: $K_{d . e . ~ i s ~}^{\text {. }}$
neglected, and this improper scaling is irrelevant. In the present work, however, we are interested in the corrections arising from these factors, and so the ro-term approximation becomes essential.

Using the two-tern approximation to $C$, we see that $C$ has two zeroes, one at $\omega^{\prime}=\omega_{*}$, and the other at

$$
\begin{equation*}
\bar{\omega} \equiv\left(\Lambda_{\ell} g_{\ell} \omega_{\ell+1}+\Lambda_{\ell+1} g_{\ell+1} \omega_{\ell}\right) /\left(\Lambda_{\ell} g_{\ell}+\Lambda_{\ell+1} g_{\ell+1}\right)=\omega_{\ell+1 / 2}, \tag{III-127}
\end{equation*}
$$

where $\omega_{\ell} \equiv \ell s_{c i}$. The final near equality follows from (III-104), and the large argument limit (III-112) on th? $\Lambda_{l}$. We define $\bar{\Omega}$ as the value of $\omega^{\prime}$ at which $C$ is a minimum; this occurs roughly midway between $\omega_{*}$ and $\overline{\text { w. }}$

$$
\begin{equation*}
\bar{\Omega}=\frac{1}{2}\left(\omega_{*}+\bar{\omega}\right) . \tag{III-128}
\end{equation*}
$$

We need the derivatives

$$
\begin{align*}
& \partial_{\omega} C=-\tau I_{3}\left[\Lambda_{\ell} g_{\ell}\left(\omega_{*}-\omega_{\ell}\right)\left(\omega^{\prime}-\omega_{\ell}\right)^{-2}+\Lambda_{\ell+1} g_{\ell+1}\left(\omega_{*}-\omega_{\ell+1}\right)\left(\omega^{\prime}-\omega_{\ell+1}\right)^{-2}\right],  \tag{III-129}\\
& \partial_{\omega}^{2} C=2 \tau I_{3}\left[\Lambda_{\ell} g_{\ell}\left(\omega_{*}-\omega_{\ell}\right)\left(\omega^{\prime}-\omega_{\ell}\right)^{-3}+\Lambda_{\ell+1} g_{\ell+1}\left(\omega_{\ell+1}-\omega_{*}\right)\left(\omega_{\ell+1}-\omega^{\prime}\right)^{-3}\right], \tag{III-130}
\end{align*}
$$

$$
\begin{equation*}
\partial_{\omega} B=\tau \Lambda_{o e} g_{e} \omega_{\star} \omega^{\prime} \omega^{2}, \tag{III-131}
\end{equation*}
$$

$$
\begin{equation*}
\partial_{\omega}{ }^{2} \mathrm{~B}=-2 \tau \Lambda_{\mathrm{oe}} \mathrm{~g}_{\mathrm{e}} \omega_{\star} / \omega^{\prime}{ }^{3} . \tag{III-132}
\end{equation*}
$$

Now we solve (III-126), expanding in $\delta \omega \equiv \omega^{\prime}-\bar{\Omega}$. At $\omega^{\prime}=\bar{\Omega}$, we have $\partial_{\omega} \mathrm{C}=0$, and
so that we may neglect $\partial_{\omega}{ }^{2}$ B. The ISEC mode satisfies

$$
\begin{equation*}
0=\left.(A+B)\right|_{\omega_{\text {ISEC }}} \tag{III-134}
\end{equation*}
$$

so the full dispersion equation (III-126) may be written

$$
\begin{equation*}
0=[\delta \omega+(\bar{\Omega}-\omega \operatorname{ISEC})] \partial_{\omega} B+\frac{1}{2}(\delta \omega)^{2} a_{\omega}^{2} C+C+\lambda_{e}^{2} K_{d . e} . \tag{ILI-135}
\end{equation*}
$$

where all coefficients are evaluated at $\bar{\Omega}$. This is a quadratic equation in $\delta \omega$ with solution
$\delta \omega \equiv \omega^{\prime}-\bar{\Omega}=-\left(\partial_{\omega} B / \partial_{\omega}{ }^{2} \mathrm{C}\right)$

$$
\begin{equation*}
\pm\left\{\left(\partial_{\omega}{ }^{B / \partial}{ }_{\omega}^{2} \mathrm{C}\right)^{2}-2\left[\left(\bar{\Omega}-\omega_{I S E C}\right) \partial_{\omega} B+C+\lambda_{e}^{2} K_{d . e .}\right] / \partial_{\omega}^{2} C\right\}^{1 / 2} . \tag{III-136}
\end{equation*}
$$

Instability arises when the expression under the radical is negative. The destabilizing terms are $-2 \lambda_{\mathrm{e}}{ }^{2} \mathrm{~K}_{\mathrm{de}} / \partial_{\omega}{ }^{2} \mathrm{C}$ and $-2\left(\bar{\Omega}-\omega_{\text {ISEC }}\right) \partial_{\omega} \mathrm{B} / \partial_{\omega}{ }^{2} \mathrm{C}$. For instability, we must have

$$
\begin{equation*}
\left(\bar{\Omega}-\omega_{\text {ISEC }}\right)+\lambda_{\mathrm{e}}{ }^{2} \mathrm{~K}_{\mathrm{d} . \mathrm{e} .}{ }^{1 \partial}{ }_{\omega} \mathrm{B}>\frac{1}{2} \partial_{\omega}{ }^{B / \partial}{ }_{\omega}{ }^{2} \mathrm{C}-\mathrm{C} / \partial{ }_{\omega} \mathrm{B} . \tag{III-137}
\end{equation*}
$$

We maximize the growth rate $\gamma$ by letting $\omega_{*} \bar{\omega} \bar{\omega}$, so that $C(\bar{\Omega})=0$. From Eqs. (III-130), (III-31), we estimate the size of the other stabilizing term, $\left(\partial_{\omega} B / \partial_{\omega}^{2} C\right)^{2}$, from

$$
\begin{align*}
\frac{1}{2} \partial_{\omega} B / \partial_{\omega}{ }_{\omega}^{2} C & \simeq\left(\omega_{\ell} \Lambda_{o e^{g}} g_{e} \omega_{\ell}^{-2}\right) /\left(8 I_{3} \Lambda_{\ell} g_{\ell} \omega_{1 / 2}^{-2}\right)  \tag{III-138}\\
& \simeq\left(32 \ell^{2}\right)^{-1}\left(\Lambda_{o e^{g}} g_{e} / I_{z} \Lambda_{\ell} g_{\ell}\right) \omega_{*} \ll \Omega_{i}
\end{align*}
$$

Neglecting this term as well gives us a maximized growth rate:

$$
\begin{align*}
\gamma_{m x}^{2} & \simeq 2 \bar{\Omega}\left(1-\omega_{\text {ISEC }} / \bar{\Omega}\right) \partial_{\omega} B / \partial_{\omega}^{2} \mathrm{C}+2 \lambda_{e}^{2}{K_{d . e}}^{/ \partial_{\omega}}{ }^{2} \mathrm{C} \\
& \left.\simeq\left(\bar{\Omega} / 8 I_{3} A_{\ell} g_{\ell} \ell^{2}\right)\left[1-\omega_{\text {ISEC }} / \bar{\Omega}\right) A_{o e} g_{e}+\lambda_{i}^{2} K_{d . e}\right] \tag{III-139}
\end{align*}
$$

In Appendix if we use an analogous procedure for the one term approximation to the dispersiion equation, obtaining Eq. (E5) for $\gamma_{m x}{ }^{2}$, which (neglecting $\mathrm{K}_{\mathrm{d} . \mathrm{e}}$.) has a form quite similar to (III-139), but with a factor ( $I_{3} \Lambda_{\ell} g_{\ell} / \Lambda_{o e} g_{e}$ ), instead of the factor ( $\Lambda_{o e} g_{e} / 8 I_{3} \Lambda_{\ell} g_{\ell} \ell^{2}$ ) of (III-139). Using (III-112), and writing $b_{i}=\ell^{2}$, the ratio of these two factors is $\left(8{ }_{3}{ }^{2} \Lambda_{\ell} g_{\ell}{ }^{2} l^{2} / g_{Q}{ }^{2} \Lambda_{o c}{ }^{2}\right)=8 / 2 \pi=1$, where we have taken the limits $I_{3}, g_{\ell}, g_{e} \rightarrow 1$. As mentioned previously, therefore, though the scaling of $\gamma_{m x}$ with these factors is very different for the two approximations, in the limit that they are usually taken, the two expressions for the growth rate are approximately the same.

For the negligible shear regine, $\mathrm{g}_{\mathrm{c}}=\mathrm{g}_{\mathrm{\ell}}=1$. From (III-139), we see that $\gamma^{2}$ is affected by $I_{3}$, and also implicity by $I_{2}$, which changes ${ }^{\omega}$ ISEC as in (III-120). These effects are readily seen graphically. Increasing $I_{2}$ increases term $A^{\prime}$ in Eq. (III-126), thereby depressing the line $-A^{\prime}$ in Figs. 14 or $15 a$. This in turn increases the gap between $\left(-A^{\prime}\right)$ and the minimun in $(B+C)$, enhancing $\gamma^{2}$, as well as decreasing ${ }^{\omega}{ }_{\text {ISEC }}$ [and so increasing ( $1-\omega_{\text {ISEC }} / \bar{\Omega}$ )]. Increasing $I_{3}$ increases the curvature $\left.\partial_{\omega}^{2} C\right|_{\bar{\Omega}}$, and thereby is stabilizing, allowing the same gap in ( $B+C$ )-$(-A)$ to be bridged by $\gamma^{2} \partial_{\omega}{ }^{2} \mathrm{C}$, for smaller $\gamma^{2}$.

We must no consider the dependence of the quantities $\mathrm{I}_{2}, \mathrm{I}_{3}$ on the physical parameters. Without loss of generality, we may restrict ourselves to positive $k_{2}$, relying on the very general symmetry property ${ }^{25,26}$ of $\bar{K}_{s}$ in $E q$ (III-48), that if[ $\phi(x) \omega$ satisfies Eq. (III-1), so does $\left[\phi^{*}(x),-\omega^{*}\right]$, i.e. under the mapping $(\underline{k}, \omega) \rightarrow\left(-\underline{k},-\omega^{*}\right)$, the physics is unchanged. In addition, we may wi thout loss of generality take $\dddot{K}_{n}>0$. Taking $\dddot{k}_{n}<0$ will merely cause the drift mode $\left(\omega \sim \omega_{*}\right)$ to interact with a Bernstein harmonic
$w \sim-l \Omega_{i}$, instead of $\omega \sim \ell \Omega_{i}$. This change leaves $\varepsilon_{p l}=\ell \kappa_{n} / k_{2}$ unchanged. From the definitions following (III-118) of $I_{2}, I_{3}$, we see that $1-I_{2} \sim \Omega_{1}=\omega^{\prime}-\omega_{\star}$. At $\gamma=\gamma_{\operatorname{mx}}$ we have $\omega^{\prime} \simeq \bar{\Omega}=\omega_{\star}$, and so $\Omega=0$. The effect of $I_{2}$ on $\gamma_{m x}$ is therefore negligible. Also, we have $I_{3}=1-\ell k_{n} / k_{2}=1-\varepsilon_{p \ell}$. Since $\ell, k_{n}$, and $k_{2}$ are all positive, increasing $k_{n}$ decreases $I_{3}$ which, from the foregoing discussion, enhances $\gamma_{m x}{ }^{2}$ over the value one would expect if the nonlocal correction $K_{n . \ell}$. were neglected, especially for larger $\ell$. The validity of expression (III-139) and the concomitant scaling, $\gamma_{\mathrm{mxx}}^{2}-I_{3}^{-1} \simeq\left(1-\varepsilon_{p \ell}\right)^{-1}$, are limited, however, by the convergence condition (III-116), and by the validity of the truncation at second order in $\delta w$ used to obtain Eq. (III-135).

Just as for $I_{2}$, the effect of increasing $K_{d}$.e. in Figs. 14 or $15 a$ is to further depress the line ( $-A^{\prime}$ ) thereby enhancing $\gamma^{2}$, as is algebraically corroborated by Eq. (III-139). We shall see shortly that the eigennodes are localized at a minimum of $K_{0}$. Accordingly, as explained in Sec. III.D.1, $\mathrm{K}_{\mathrm{d}}$.e. is real and positive for this shear regime. Thus, for fixed $\ell$, the higher $-n$ modes (which have larger $K_{d . e .}^{(n)}=f_{n} K_{d . e .}^{(o)}$ ) should be more unstable. This enhancement may be understood in the following manner. ${ }^{33}$ The factor $K_{\text {d.e. arises }}$ in the dispersion equation as an average of $k_{x}{ }^{2}(x)$ over the extent of the mode, and so many be grouped with the vacum term $k_{2}{ }^{2}$ which also appears there. Neglecting the explicit factor $K_{\text {d.e. }}$ in (III-139), we have that $\gamma_{m x}^{2} \sim \Omega^{-2} \sim \omega_{\star}^{2} \sim k_{1}^{2} \equiv k_{2}^{2}+\left\langle k_{x}\right\}^{2}$. Thus higher - $n$ modes, having larger $\left(k_{X}{ }^{2}\right\rangle$, should have larger $\gamma^{2}$. From Fig. 15b, we note that use of the one term approximation would indicate
that these higher $n$ modes are more stable than those with lower $n$, just the opposite behavior from that just described.

This increasing instability with mode number $n$ has limits, however. The first of these is the mode localization condition, Eq. (III-47), which gives a maximum value $n_{\text {max }}$ to the mode index $n$. As will be shown shortly (cf. Fig. 16b), $K_{o}(x)$ has a well of depth $K_{d}=K_{d}=r \lambda_{j}^{-2}$, where $r$ is some fraction $0 \leqslant r<1$, whose exact value depends upon the particular density profile. As $n$ increases from zero, $K_{d . e}^{(n)}=f_{n} k_{d . e .}^{(o)}$ increases, and so does the mode growth rate. Finally, as $n \rightarrow n_{\max }, K_{d . e}^{(n)}$ takes on the maximum value for which (III-47) is not violated, i.e. $\left.k_{d . e}^{(n)}\right|_{n=n_{\max }} \simeq \kappa_{d}=r \lambda_{i}^{-2}$. This determines $n_{\max }$. From Eq. (III-139), the contribution of this to $\left(\gamma_{m x} / \Omega_{i}\right)^{2}$ is about $\left(r / 8 I_{3} \Lambda_{\ell} g_{\ell}\right) \sim(\ell r / 8)$.

The second restriction on this analysis is that, if $\left(\gamma_{m x} / \Omega_{i}\right)$ becomes greater than unity, the approximation of separate harmonics we have been using becomes invalid, ${ }^{34}$ and one should perform the sum over $\ell$ to obtain a valid dispersion equation, i.e., one should enploy a straight line orbit $a_{k} p$ poximation for the ions. This approximation is considered in connection with the IDC instability by Zelenyi. ${ }^{32}$ Here we shall employ only the separate harmonic approximation, our principal interest being to illustrate new effects brought out by the present formalism, rather than exhaustively studying this instability. Applying this separate-harmonic limitation

$$
\begin{equation*}
1>\left(\gamma_{\mathrm{mx}} / \Omega_{\mathrm{i}}\right)^{2} \tag{III-140}
\end{equation*}
$$

to the effect on $\left(\gamma_{m x} / \Omega_{i}\right)^{2}$ from $K_{d . e . ~}$ alone, we find the restriction

$$
\begin{equation*}
1>\ell r / 8 \tag{III-141}
\end{equation*}
$$

for the validity of the present analysis.

If (III-141) is satisfied, we may imagine decreasing $k_{n}$ from some appreciable value, and observing the instability of the modes for some fixed value of $\ell$. As the well width $\left(\sim L_{n}\right)$ is increased, keeping its depth about constant, $\gamma_{m x}$ for a mode with a given $n$ decreases, but higher $n$ modes arise, with the highest $n$ mode roughly satisfying $k_{\mathrm{d} . \mathrm{e} .}^{(\mathrm{n})} \sim \mathrm{r}_{\mathrm{i}}^{-2}{ }^{-2}$. Accordingly, we expect the maximum growth rate fron all modes of a given $\ell$ to decrease only slowly, due solely to the effect on Eq. (III-139) of increasing $\mathrm{I}_{3}$. This slow stabilization of the 2 -modes continues until $\kappa_{n}$ becomes so small that $\ell_{\max }$ of Eq. (III-122) drops below $\ell$. At this point, all modes with harmonic number $\ell$ become stable.

We now proceed to apply the variational condition (III-43), to determine the position $x_{0}$ at which the mode is locilized, and accordingly where (III-139) is to be evaluated. Taking $g_{e}=1$, Eq. (III-119) becomes

$$
\begin{equation*}
K_{o}=\underline{k}^{2}+I_{2} \lambda_{i}^{-2}+\lambda_{e}^{-2}\left[1-\Lambda_{o e}\left(1+\tau \omega_{\star} / \omega^{\prime}\right)\right] . \tag{III-142}
\end{equation*}
$$

Noting that $\lambda_{e}^{-2} \omega_{*} \sim n(x) \kappa_{n}(x) \equiv n^{\prime}(x) \equiv \partial_{x} n$, we see that the factor $\lambda_{e}^{-2} \tau \omega_{\star} / \omega^{\prime}$ reaches a maximm where $n^{\prime}(x)$ does. At this point $\omega_{\star} / \omega^{\prime} \simeq 1$. From this information, in Fig. 16a we sketch the factor $\lambda_{e}^{-2} \omega_{*} / \omega^{\prime}$, as well as $\lambda_{i}^{-2}=\lambda_{e^{-2}}$, versus $x$. Using this with Eq. (III-142), in Fig. 16b we sketch $K_{0}(x)$. We see that $K_{0}$ has a single minimum $x=x_{0}$, occurring a bit on the low density side of the point $x_{1}$ where $n$ ' is maximun, in agreement with the numerical findings of Ref. 21. The maximum in $K_{0}(x)$ to the low density side of $x_{0}$ is unsuitable for Berk-Pearlstein type mode localization, since in this shear regime $g_{e}=g_{\ell}=1$, for all $x$.

## III.D.3. IDC Instability with Moderate Shear

Now we are ready to consider the problem of the IDC instability in the regime of intermediate shear, where the scale lengths $L_{n}, L_{o e}$, which enter into $K_{0}(x)$ and $Q(x)$, may be comparable. For $\eta_{n}<1$ (which is a valid assumption even for strongly inhomogeneous plasmas such as the Tomac sheath), it is in this regime that the transition from $\rho_{i} / L_{\phi}<1$ to $\rho_{i} / L_{\phi}>1$ occurs. However, as discussed in Sec. III.C.5, the change in the actual form of the dispersion equation in making this transition is quite minor, so that the analysis in either range is essentially the same. For definiteness, we shall consider the regime $\rho_{i} / L_{\phi}>1$, so that Eq. (1II-41b) is the appropriate dispersion equation.

We can put the dispersion equation into a form very similar to Eqs. (III-125) or (III-126) (which applied to the negligible shear regine), by some generalizid definitions of the symbols used there. We define $\omega_{*}^{\prime}$ as $k_{2} k_{n} v_{i}^{2} / \Omega_{i}$, as before, and generalize the shearless definition of $\omega^{\prime}$ as follows: $\omega^{\prime} \equiv \omega_{\mathrm{e}}^{\prime \prime} \equiv \omega-\mathrm{k}_{2} \mathrm{v}_{\mathrm{E}}-\mathrm{k}_{3^{i *} \| \mathrm{e}}$. We also write $\omega_{\star} \equiv \omega_{*_{i}}$ $+k_{2}\left(u_{E i}-v_{E}\right)+k_{3} \Delta w_{\|}$, and $w_{\ell} \equiv \ell \Omega_{o i}+k_{3} \Delta w_{\|}$, where $\Delta k_{\|} \equiv w_{\| i}-w_{\| e}$, the difference in parallel drifts, which produces the shear. The dispersion equation (III-41b) then appears as

$$
\begin{align*}
0 & =K_{o}+S_{i B}+K_{\text {d.e. }} \\
& =\underline{k}^{2}+\lambda_{e}^{-2}\left\{1-\Lambda_{o e^{\prime}} g_{e}\left(1+\tau \omega_{*}^{\prime} / \omega^{\prime}\right)\right] \\
& +\lambda_{i}^{-2}\left[1-\sum_{\ell} \Lambda_{0, i} g_{\ell}\left(\frac{\omega^{\prime}-\omega_{*}}{\omega^{\prime}-\omega_{\ell}}\right)\right]+K_{d . e .} \tag{III-143}
\end{align*}
$$

where $\Lambda_{\ell i}$ is given by Eq . (D30) for t . e regime $\rho_{i} / L_{\phi}>1$.

We consider the size of $\Delta w_{\|}$, and the effects on the dispersion equation caused by it. Using Ampere's law, we estimate the size of $\mathrm{k}_{\mathrm{s}}$ or $\eta_{s} \equiv \kappa_{s} \rho_{i}:$

$$
\begin{equation*}
k_{s} \simeq(4 \pi n e / B)\left(\Delta w_{\|} / c\right), \text { or } \eta_{s} \simeq B\left(\Delta v_{\|} / v_{i}\right), \tag{III-144}
\end{equation*}
$$

where $B \equiv 8 \pi \mathrm{p} / \mathrm{B}^{2}$ is the usual plasma 2 . For the stronger-shear portion of this intermediate-shear range, [e.g., from the point $\left(\kappa_{s} / k_{n}\right)=\left(L_{o e} / L_{s} \ell n_{n}^{2}\right)$ where the strong convergence condition (III-111) breaks down, to the point $\left(\kappa_{s} / K_{n}\right)=\left(\ell L_{o e} / L_{s} \eta_{n}^{2}\right)$ where localization condition (III-102) is violated], using the values $\eta_{n} \sim 2, \ell \sim 10, \tau \sim 1$ as in Sec. HII.C.5, $\left(\kappa_{s} / r_{n}\right)$ iies in the range . 01 to 1 . Therefore $\pi_{s}=\left(\kappa_{s} / \kappa_{n}\right) n_{n}$ may take on values from about . 002 to .2. Strictly speaking, because it is based on Poisson's equation instead of the full Maxwell equations, the present formalisn is valid only for $10 \mathrm{w} B$ plasmas. Taking $\beta^{-} .01$, using (III-144) tells us that $\left(\Delta u_{\|} / v_{i}\right)$ falls in the range .2 to 20 .

In addition to the term $k_{3} \Delta w_{\|}$in the definition of $\omega_{*}$. finite $w_{\| i}$ enters $\omega_{*}$ through the terms $\omega_{*_{i}}$ and $k_{2}\left(u_{E i}-v_{E}\right)$. Using Eq. (III-86) for $u_{E i}$ and the analogous equation for $\omega_{*_{i}}$, i.e.,

$$
\begin{align*}
\omega_{*_{i}} \equiv \omega_{*_{i}}\left(v_{3}\right. & \left.={w_{\| i}^{\prime}}\right)=\omega_{*_{i}}\left(v_{3}=0\right)\left(1+\kappa_{s_{\| \|}} \mathbf{k}^{\prime} / \Omega_{\mathrm{ci}}\right)^{-1}  \tag{III-145}\\
& \equiv \omega_{*}^{\prime}\left(1+\kappa_{s^{w_{\| i}}} / \Omega_{\mathrm{ci}}\right)^{-1},
\end{align*}
$$

one finds

$$
\begin{equation*}
\omega_{*} \equiv \omega_{*_{i}}+k_{2}\left(u_{E i}-v_{E}\right)+k_{3} \Delta w_{\|} \simeq \omega_{*}^{\prime}-\left(\omega_{*}^{\prime}+k_{2} v_{E}\right)\left(k_{s^{\prime} \|_{i}} / \Omega_{c i}\right)+k_{3} \Delta_{w_{\|}} . \tag{III-146}
\end{equation*}
$$

Assuming ${\omega_{\| i}}^{\sim} \Delta w_{\| \mid}$, we see that $\omega_{*}$ may experinece a fractional shift from its shearless limit $\omega_{*}$ ) of $\left(k_{s} w_{\| i} / \eta_{c i}\right) \sim\left(\Delta{w_{| |}} / v_{i}\right) \eta_{s}=\beta\left(\Delta w_{| |} / v_{i}\right)^{2}$ which, with the parameter values just estimated, falls in the range $4 \times 10^{-4}$ to 4 . The shift in $\omega_{*}$ away from the value $\omega_{*}$ ' used in previous analyses of instabilities involving this diamagnetic drift frequency may therefore be quite significant.

As noted in Sec. III.D.2, for finite $\mathrm{k}_{5}(\mathrm{x})$, electron Landay damping will set in, localizing the mode. We therefore expect the mode amplitule (and source $0^{*}$ free energy) to be greatest at that point $x_{2}$ where $k_{3}\left(x_{2}\right)=0$. In contrast to the unsheared case, we are thus at liberty change the point of mode localization, by varying $k_{y}$ or $k_{z}$.

The localization of the mode is determined by $K_{0}$, which, from (III-119), is given by (recalling $I_{2} \rightarrow 1$ for $\rho_{i} / L_{\phi}>1$ )

$$
\begin{equation*}
K_{o}=\underline{k}^{2}+\lambda_{i}^{-2}+\lambda_{e}^{-2}\left[1-\Lambda_{o e} g_{e}\left(1+\tau \omega_{\star}^{\prime} / \omega^{\prime}\right)\right], \tag{III-147}
\end{equation*}
$$

the same as Eq. (III-142), but with the factor $g_{e} \equiv 1-W_{o e}$ now $x$ - dependent. Taking $x_{2}$ as our origin ( $x_{2}=0$ ), in the neighborhood of $x=x_{2}$ we have $z_{o e}=\omega /\left|k_{3}(x) v_{e}\right| \simeq L_{o e} /|x|$. Using Fig. 11, therefore, in Fig. 17a we draw $-g_{e}(x)$. Since $W_{I}(z) \sim e^{-z^{2} / 2}$, at $x=x_{2}=0\left(-g_{I e}\right)$ and all its derivatives are zero, giving it a flat bottomed appearance.

For $x / L_{o e} \ll 1, g_{e} \rightarrow 1$, and $K_{o}$ goes to its unsheared value, sketched in Fig. 16b. For $x / L_{o e} \gg 1, g_{e} \rightarrow 0$, and $K_{o}+\underline{\mathrm{k}}^{2}+\lambda_{e}^{-2}\left(I_{2}+\tau\right)$. In the region $x / L_{o e} \sim 1$, the variation of $g_{\text {Re }}$ gives the real part $K_{\text {Ro }}$ of $K_{o}$ a double-well form. This is sketched in Fig. 17 b , for $\mathrm{L}_{\mathrm{oe}}$ a bit shorter than $L_{n}$, and for $x=x_{2}$ taken to the higher-density side of the point $x_{1}$
where $n^{\prime}=0$. Also drawn (dashed curve) is the form of $K_{0}$ for the unsheared case, from Fig. 16b.

For weaker shear, the position of the double wells about $x=x_{2}$ caused by $g_{e}$ will move out toward $|x|=\infty$, the resultant shape of $K_{\text {Ro }}$ moving toward its shearless value in rhe region of the well induced by $n^{\prime}$, where a free energy source is available. Thus, for $L_{o e}$ greater than $L_{n}$, but comparable to it, we may still have a potential well $\left(\partial_{x}^{2} K_{R o}>0\right.$ at $x=x_{2}$ instead of the hump $\left(\partial_{x}{ }^{2} K_{R o}<U\right)$ shown in Fig. 17b. In the following, we shall assume $L_{o e}<L_{n}$, so that the situation is as in Fig. 17b. The analysis for the weaker shear case is, however, similar, connecting the shearless regime of Sec. III.D. 2 to the shear-dominated one.

The variational condition (III-43) for $x_{0}$ allows $x_{0}$ to be at any extremum of $K_{0}$. However modes centered at any $x_{0}$ besides $x_{0}=x_{2}$ will be strongly damped, and so we consider only this local maximum. In addition, in seeking the most unstable case, we choose $k_{y}, k_{z}$ so that $x=x_{2}$ falls at the bottom of the well we found in the shearless regime.

As discussed at the end of Sec. III.D.1, modes localized at a maximm of $K_{0}$ will have Berk-Pearlstein type localization, with $\alpha$ and $K_{d . e}$ being (formally) imaginary and negative ( $\alpha \simeq-i|\alpha|$ ). Using the asymptotic value $W(z) \underset{z \rightarrow \infty}{\rightarrow}-z^{2}$, and our assumption that the curvature of $K_{0}$ at $x_{o}$ is doninated by the factor $g_{e}$, we find

$$
\begin{equation*}
\frac{1}{2} \partial_{x_{0}}^{2} K_{0} \simeq-\mathrm{L}_{\mathrm{oe}}^{-2} \lambda_{2}^{-2} \Lambda_{\mathrm{e}}^{-2}\left(1+\tau \omega_{\star}^{\prime} / \omega^{\prime}\right) \tag{III-148}
\end{equation*}
$$

and (III-124) becomes
$K_{\text {d.e. }}=-i L_{o e}^{-1} \lambda_{e}^{-1}\left\{\left[1+\left(\omega_{e}^{2} / \Omega_{\mathrm{oe}}^{2}\right)\left(1+\tau \omega_{\star}^{\prime} / \omega^{\prime}\right)\right] \Lambda_{\mathrm{oe}}\left(1+\tau \omega_{*}^{\prime} / \omega^{\prime}\right)\right\}^{1 / 2}$.

Evaluating the dispersion equation (III-143) at $x_{0}=x_{2}$, we again have $g_{e}=1$. Analogous to (III-126), we rewrite (III-143) as

$$
\begin{align*}
0 & =\left[\left(k \lambda e^{2}+\tau+\left(1-\Lambda_{o e} g_{e}\right)+\dot{\lambda}_{e}{ }^{2} K_{d . e}\right]-\tau \Lambda_{o e^{\prime}} g_{e} \omega_{\star}^{\prime} / \omega^{\prime}\right. \\
& -\tau \sum_{\ell} A_{\ell i} g_{\ell}\left(\frac{\omega^{\prime}-\omega_{\star}}{\omega^{\prime}-\omega_{\ell}}\right)  \tag{11I-150}\\
& \equiv A^{\prime}+B+C, \quad A^{\prime} \equiv A+\lambda_{e}^{2} K_{d . e .}
\end{align*}
$$

This has essentially the same form as Eq. (III-126), and the same method of solution may be enployed. There are certain differences worthy of note, however.

To begin with, while $K_{d . e}$. was real for the shearless regime, here it is imaginary, and the corresponding effect of $\mathrm{K}_{\mathrm{d} . e}$. on the growth rate will be quite different.

Secondly, while $g_{e}=1$ for both the moderate shear and the unsheared cases, $g_{\ell}$ is not equal to one, due to the dependence of $\Omega_{i}$ on $v_{3}$ or $u$. We have $g_{\ell} \equiv 1-W\left(z_{\ell i}\right)$, where $z_{\ell i} \equiv \Omega_{2} /\left|k_{3}^{\prime} v_{i}\right|, \Omega_{2}=\omega^{\prime}-\omega_{\ell}$, and $\mathrm{k}_{3}{ }^{\prime} \equiv \mathrm{k}_{3}+\ell \mathrm{k}_{\mathrm{s}} / 2=\ell \mathrm{k}_{\mathrm{s}} / 2$. As described in the discussion following Eq. (III-96) and illustrated in Fig. 12, for $\omega^{\prime}$ near the $\ell^{\text {th }}$ Bernstein harmonic $\left(\omega^{\prime}-\omega_{\ell} \simeq \Delta \ell \Omega_{o i}, 0<\ell<1\right)$, the $\ell^{\text {th }}$ harmonic and nearby harmonics $\ell^{\prime}$ may have $\left|z_{\ell}\right| \lesssim 1$, and so $g_{\ell^{\prime}}$ is appreciably different from unity, with both real and imaginary parts. For the $\ell$ and $\ell+1$ terms kept in the twoterm approximation, one has $z_{\ell i} \simeq 2 \Delta \ell / \ell \eta_{s}$ and $z_{\ell+1} \simeq 2(\Delta \ell-1) /(\ell+1) n_{s}$. If we let $\Delta \ell \simeq \frac{1}{2}$, this gives $z_{\ell i} \simeq-z_{\ell+1, i} \simeq\left(\ell n_{S}\right)^{-1}$. Therefore we shall take $g_{\ell+1}=g_{\ell}$ in the following analysis.

In addition, although expressions (D29) and (D30) for $\Lambda_{\ell i}$ are quite similar, they are not entirely identical. Even if one writes $\alpha=\left\langle k_{x}{ }^{2}\right\rangle$ and sets $n=0$, as done in Sec. III.C. 5 to achieve a formal resemblance, $\Lambda_{\ell i}\left(\rho_{i} / L_{\phi}>1\right)$ is still down from $\Lambda_{\ell i}\left(\rho_{i} / L_{\phi}<1\right)$ by a factor of $\pi^{1 / 2}$. Also, $\alpha$ is now an imaginary number, and so will affect the mode growth rate differently, in the stronger-shear end of the moderate-shear regime, where $\alpha \sim k_{2}{ }^{2}$. And lastly, the higher $-n$ values of $\Lambda_{\ell i}$ are equal to $(-1)^{n} c_{n}$ times the $n=0$ value, $\Lambda_{\ell i}(n=0)$. For $n$ odd, $c_{n}=0$, and therefore $S_{i B}=0$, and the IDC instability vanishes. We shall therefore henceforth consider only even- $n$ perturbations, for which $\Lambda_{\ell i}(n)=c_{n} \Lambda_{\ell i}(0)$.

Finally, the effects of the finite parallel currents "ns present with finite shear make themselves known in (III-150), through the generalized meanings of $\omega_{\star}, \omega_{\ell}$, and $\Omega_{i o}$. Each of these effects may be seen either graphically from Fig. 14, or algebraically, in Eq. (IlI-136) or (III-139), which, with suitable reinterpretation, are valid for the moderate shear regime, as well as for the negligible-shear case for which they were derived.

For fixed $\omega_{*}$ ', Eq. (III-146) tells us that as $w_{\| j}$ is increased, $\omega_{*}$ decreases, while $\Omega_{o i} \simeq \Omega_{C}\left(1+\frac{1}{2} \kappa_{s} w_{\| i} / \Omega_{c}\right)$ increases. Thus the point $\omega^{\prime}=\omega_{*}$ at which term $C$ in (III-150) or Fig. 14 passes through zero shifts toward lower values, while the spacing $\Omega_{\text {oi }}$ between consecutive poles of $C$ increases. Terms $A$ and $B$ are unaffected. Thus the values of $\bar{\Omega}$ and $k_{e}$ at which maximm growth occurs will also be shifted.

The situation is different if instead the shear is induced by increasing $-w_{\| e}$, taking $w_{\| i}$ equal zero. In this case $\omega_{\star} \simeq \omega_{*}{ }^{\prime}, \Omega_{o i}=\Omega_{\text {ci }}$ and, aside from the difference induced by $K_{d . e .}$, the dispersion equation
is the same as in the unsheared case.
Now we solve (III-150) for the growth rate, using the two-term approximation, as in Sec. III.D.2. We again define $\bar{\Omega}$ as the point at which $\partial_{\omega} C=0$. Using the facts that $g_{\ell} \simeq g_{\ell+1}$, and that $\Lambda_{\ell} \simeq \Lambda_{\ell+1}$ [cf. Eqs. (III-114) or (III-113)], we find from (III-129) the relation

$$
\begin{align*}
\left(\bar{\Omega}-\omega_{\ell+1}\right)^{2} /\left(\bar{\Omega}-\omega_{\ell}\right)^{2} & =\left(\Lambda_{\ell+1} g_{\ell+1} / \Lambda_{\ell} g_{\ell}\right)\left(\omega_{\ell+1}-\omega_{\star}\right) /\left(\omega_{\star}-\omega_{\ell}\right)  \tag{III-151}\\
& =\left(\omega_{\ell+1}-\omega_{\star}\right) /\left(\omega_{\star}-\omega_{\ell}\right) .
\end{align*}
$$

Thus $\bar{\Omega}$ again lies on the real $\omega^{\prime}$ axis, between $\omega_{\ell}$ and $\omega_{\ell+1} \cdot \Gamma .$. thing now follows as in Sec. III.D.2. In particular, Eqs. (III-134) through (III-136) are again valid. The growth rate is again maximized by th? expression on the right hand side of (III-139), but nai this expression is no longer real, due to $K_{\text {d.e., }} \alpha$ in $\Lambda_{\ell}$, and $g_{\ell}$. Instead, one must write

$$
\begin{equation*}
\gamma_{\pi x}=\operatorname{Re}\left\{\left(\bar{\Omega}^{2} / 8 \mathrm{I}_{3}{ }^{\prime} g_{\ell} \ell^{2}\right)\left[\left(1-\omega_{\text {ISEC }} / \bar{\Omega}\right) \Lambda_{o \mathrm{e}} \mathrm{~g}_{\mathrm{e}}+\lambda_{\mathrm{i}}{ }^{2} \mathrm{~K}_{\mathrm{d} . \mathrm{e}}\right]\right\}^{1 / 2} . \tag{III-152}
\end{equation*}
$$

From Fig. 11 , one sees that as $z_{\ell i} \simeq\left(\ell \eta_{S}\right)^{-1}$ moves in from infinity, $\mathrm{g}_{\mathrm{R} \mathrm{\ell}} \equiv \operatorname{Re} \mathrm{~g}_{\ell}$ first increases from one, and then decreases to values below one in the (strong shear) regime where $z_{\ell i}<1$, while $g_{I \ell} \equiv \operatorname{Img}_{\ell}$ is negative, becoming largest around $z_{\ell i}=1$. Combining this with the facts that $K_{\text {d.e. }} \simeq-i\left|K_{d / e}\right|, \alpha \simeq-i \mid \alpha$, and the dependence of $\Lambda_{\ell i}$ in (D30) on $n$ and $\alpha$, we schematically represent (III-152) as

$$
\begin{equation*}
\gamma_{\operatorname{mx}}^{(n)}=c_{n}^{-1 / 2} \underset{\operatorname{mx}}{(n=0)}=(n!)^{1 / 2} \gamma_{m x}^{(n=0)}, \tag{III-153}
\end{equation*}
$$

$$
\gamma_{\operatorname{mx}}^{(n=0)}=\operatorname{ke}\left\{\left(k_{2}^{2}-i|\alpha|\right)^{1 / 2}\left(g_{R}-i g_{I}\right)(R-i I)\right\}^{1 / 2},
$$

where $g_{R}-i g_{I} \equiv g_{\ell}=g_{R \ell}-i\left|g_{I \ell}\right|$, R corresponds to the term with ( $1-\omega_{1 S E C} / \bar{\Omega}$ ), and I corresponds to the term in $K_{d . e . ~ F r o m ~ t h i s ~ w e ~}$ see that the effect of nonzern $a, K_{d . e}$, and $g_{I}$ is principaliy to cause a shift in the real part $\omega_{R}^{\prime}$ of $\omega^{\prime}$. The effect of increase (decrease) in $g_{R}$ from unity is stabilizing (destabilizing), and the higher $n$ ( $n$ even) modes again have larger growth rates than the $\mathrm{n}=0$ mode. The limits on this increase of $\gamma_{m x}$ with $n$, due to an upper limit $n_{\max }$ of $n$, and due to the breakdown of the validity of the present twu-term analysis at $\gamma_{m x} / \Omega_{i} \sim 1$, are the same as discussed in Sec. III.D. 2 .

The effect of shear on the IDC instability in the intermediate shear regine is thus the following. The distance $L_{\phi}$ over which a given mode is localized narrows, as $L_{\text {oe }}^{-1}$ dominates $L_{n}$. However, there is now a spread in the positions $x_{0}$ at which a given eigennode may be localized. An eigenmode is localized at the point $x_{0}$ where $k_{3}=0$, and its growth rate, being proportional to the free energy source $\left[\sim n^{\prime}(x) \equiv d n / d x\right]$, is greatest near the position where $n$ ' is greatest. Taking a superposition of the unstable modes localized near the most unstable position, one obtains a perturbation which is broader in $x$, which twists with the magnetic field, but which has a spread in frequency and growth rate. This is the "quasimode" picture, introduced by Roberts and Taylor ${ }^{35}$ in connection with MDD resistive instabilities.

For a given $\ell$, the larger $n$ modes ( $n$ even) have larger growth rates than the $n=0$ mode, as found in the negligible shear regime. The growth rate of the $n=0$ mode is not radically reduced by shear in this regime. It maybe somewhat enhanced, in fact, for $z_{\text {li }}$ becoming small
enough that $g_{R}<1$ (which, we recall, can occur due to the dependence of $\Omega_{o i}$ on $u \equiv v_{3}-w_{\| i}$ ). Finally, the values of $\bar{\Omega}$ and $k_{2}$ at which the maximm growth appears is modified by the fact that $\Omega_{o i} \neq \Omega_{c i}$, for appreciable $w_{\| i}$.

To obtain a criterion for shear stabilization, therefore, we must proceed to stronger shear. This is done in the following section.

## III.D.4. Instability with Strong Shear

In this section we consider what happens when shear becomes so strong that localization condition (III-47) is violated, even for the $n=0$ mode. In this regime, we find from Eq. (IIl-147) the well depth $K_{d} \sim\left(\lambda_{e}^{-2}+\lambda_{i}^{-2}\right)=\lambda_{e}^{-2}(1+\tau)$, and from (III-149), $\left\{K_{d . e} \mid \sim L_{o e^{-1} \lambda_{e}^{-1} \times}\right.$ $\left[1+\omega_{e}^{e} / \delta_{\mathrm{oe}}\right]^{1 / 2}=\mathrm{L}_{\mathrm{oe}}^{-1}\left[\lambda_{\mathrm{e}}^{-2}+\rho_{\mathrm{e}}^{-2}\right]^{1 / 2} \sim \mathrm{~L}_{\mathrm{oe}}^{-1}\left[\min \left(\lambda_{\mathrm{e}}, \rho_{\mathrm{e}}\right)\right]^{-1}$. Condition (III-47) then gives the localization criterion

$$
\begin{equation*}
1>\left(\lambda_{\mathrm{e}} / L_{\mathrm{oe}}\right) \max \left(1, \lambda_{\mathrm{e}} / \rho_{\mathrm{e}}\right) \tag{III-154}
\end{equation*}
$$

When (III-154) is violated, the final ( $n=0$ ) mode can no longer be localized by $K_{0}$, so that no modes satisfying appropriate boundary conditions exist. The IDC mode is then stabilized. For typical paraneters of confinement devices, $\lambda_{e} / \rho_{e} \sim 1$, and so (III-154) gives the stabilization criterion

$$
\begin{equation*}
1 \sim \lambda_{\mathrm{e}} / L_{\mathrm{oe}} \tag{III-155}
\end{equation*}
$$

Combining the results of this and the preceding section, for a given $\ell$, as shear is increased from zero, the maximum IDC growth rate is relatively insensitive to the shear, until the regime satisfying Eq.
(III-155) is approached, at which point the instability disappears.

This behavior, as well as stabilization criterion (III-155), is in qualitative agreemenc with the results of zelen:i. ${ }^{32}$

## III.E. Further Comments on the Formalism

The application of Sec. III.D made use of :ome, but not all, of the novel features of the stability formalism. One principal restriction on the formalism as developed here is that it is based on Poisson's equation, and therefore electromagnetic correcticns which one expects for perturbations in higher $\beta$ plasmas are not taken into account. Many of the interesting features of the formalism (e.g., the $v_{3}$ dependences of $\Omega_{o i}, u_{E}$, and $\omega_{*}$ ) become significant only for substantial shear, and for larger shear one needs larger $\beta$, for realistic plasmas.

The formalism should be readily generalized to allow for arbitrary electromagnetic perturbations, replacing Poisson's equation (III-1) with the full Maxwell equations

$$
\begin{equation*}
\nabla \times[\nabla \times E(\underline{x})]-(\omega / c)^{2} E(\underline{x})=\left(4 \pi i \omega / c^{2}\right) \dot{j}(\underline{x})=\sum_{S} \int d x^{\prime} \bar{K}_{S}\left(\underline{x}, \underline{x}^{\prime}, \omega\right) \cdot \underline{E}\left(x^{\prime}\right) . \tag{11I-156}
\end{equation*}
$$

The structure of the tensor conductivity $\mathbb{R}_{s}$ is analogous to that of the scalar conductivity $\bar{K}_{s}$ we have used, with the $\rho_{\ell}$ 's which appear in expressions (III-48) for $\overline{\mathrm{K}}_{s}$ replaced by $\underline{\underline{j}}_{\underline{\ell}}$ 's, where $\dot{j}(\underline{x} \mid \underline{Z}) \equiv \mathrm{e} \underline{v}(Z) \times$ $\delta[\underline{x}-\underline{r}(\underline{Z})]$, analogous to the definition of phase function $\rho(\underline{x} \mid \underline{Z}) .{ }^{26}$ This analogous structure insures that ${\underset{\sim}{k}}_{s}$ can be written in a form parallel to Eq. (III-4), and given this crucial property, the formalism developed here follows.

Thus, though the specific results which the present formalism yields cannot be taken too seriously for higher- $\beta$ plasmas (such as Tomac), the novel effects of appreciable shear indicated by the formalism will also arise in the electromagnetic formalism, and should be taken into account in any careful treatment of the stability of sheared plasmas.

In the present work, the variational formulation was used to obtain dispersion equations in the regime $\rho_{i} / L_{+}<1$, and in the regime $\rho_{i} /_{\phi}>1$, where the mode equation is intrinsically integral in nature $H$. Berk has suggested ${ }^{33}$ that in fact perhaps $S_{i B}$ can be evaluated in closed form for arbitrary $\rho_{i} / L_{\phi}$, using the Gaussian form $\phi_{10} \sim \exp \left[-\frac{1}{2} \alpha(E x)^{2}\right]$ for the trial function. If such an evaluation proved possible, the smooth interpolation done here between the small and large $-\rho_{i} / L_{\phi}$ regines could be checked, and additional insights might be gained on the treatment of such fully nonlocal problems.

Another possibly useful direction for generalization of the formalism is to relax the assumption of a slab geometry, considering, for example, the near slab geometry of Part II, or perhaps even breaking the toroidal symmetry as well. Provided one can find a complete set of particle invariants $I$, expressions (III-48) for $\bar{K}_{s}$, upon which mach of the formalism is founded, remain valid. For example, for the near slab case, the slab invariants $I=\left(J, P_{y}, P_{z}\right)$ used here are replaced by $I=\left(J, J_{b}, P_{\Phi}\right)$, where the canonical angulaı momentum $P_{\Phi}$ specifies the $\alpha$ surface $\alpha_{b}$ lying at the banana center, and the bounce action $J_{b}$ characterizes the area enclosed by the banana. In addition, expressions
(III-13) and (III-14), which are the abstract expression of the formalism for the one degree of freedom (slab) case, are readily generalized formally to the two or three degree of freedom case, merely by replacing the scalars $k, x$, and $\partial_{x}$ appearing there by vectors. To give physical content to these formal expressions, however, one must be able to write Eqs. (III-48) for these geometries in a form analogous to Eq. (III-4). This entails, basically, being able to associate some average position $X_{0}$ with any given value of the momenta $I$. The at tainment of a physically meaningful generalization of this sort, and applications to modes in which inhomogeneity parallel to the magnetic field plays an essential role, are problems for future study.

## APPENDIX C: DERIVATION OF THE CONDUCTIVITY KERNEL

In this Appendix we derive an expression for the conductivity Kernel $\bar{K}_{s}$ defined by Eq. (III-1.b), in a canonical framework. The expression derived is quite similar to those obtained in Refs. 25 and 26, but has a form which is more useful for our purposes. We perform the derivation in a bra-ket notation, similar to that of Lewis and Symon, ${ }^{36}$ which allows for a very concise expression of the results.

As in Sec. III.C.l, we consider a three-dimensional plasma, with unperturbed Hamiltonian $\mathrm{H}_{\mathrm{o}}(\underline{I})$ [denoted by $\mathrm{H}(\mathrm{I})$ in the main text), where I are three canonical momenta, constant in time, with conjugate coordinates $\underline{\theta}$. For any phase functions $F(\underline{Z}), G(\underline{Z})[\underline{Z}=(\theta, \underline{I})]$, we denote by $\dot{F}$, $\dot{G}$ the corresponding Lie operators as in Part II,

$$
\begin{equation*}
\dot{\mathrm{G} F} \equiv\{\mathrm{G}, \mathrm{~F}\}=-\{\mathrm{F}, \mathrm{G}\} \equiv-\dot{\mathrm{F}} \mathrm{G} . \tag{C1}
\end{equation*}
$$

Then the Vlasov equation reads (species label suppressed)

$$
\begin{equation*}
\left(\partial_{t}-\check{H}\right) f=0 \tag{C2}
\end{equation*}
$$

and writing $H=H_{0}+H_{1}, f=f_{0}+f_{1}$, the linearized Vlasov equation is

$$
\begin{equation*}
\left(\partial_{t}-\check{H}_{0}\right) f_{1}=\dot{H}_{1} f_{0}=-\dot{f}_{0} H_{1} . \tag{C3}
\end{equation*}
$$

Henceforth we shall for notational convenience employ bra-ket notation, with states defined over the phase space, spanned by the orthonormal basis $|\underline{Z}\rangle=|\underline{\theta}, \underline{\underline{1}}\rangle$. Thus, for example, $\mathrm{f}(\underline{Z}) \equiv\langle\underline{z} \mid f\rangle$. Then Eq. (C3) becomes

$$
\begin{equation*}
\left(\partial_{t}-\dot{H}_{0}\right)\left|f_{1}\right\rangle=-\dot{f}_{0}\left|H_{1}\right\rangle . \tag{C4}
\end{equation*}
$$

Solving this for $\mid f_{1}$,

$$
\begin{equation*}
\left|f_{1}\right\rangle=-\left(\dot{o}_{t}-\dot{H}_{0}\right)^{-1} \stackrel{\rightharpoonup}{f}_{0}\left|H_{1}\right\rangle \tag{C5}
\end{equation*}
$$

and therefore the perturbed charge density $\rho_{s}(\underline{x} ; t)$ due to species $s$ is given by

$$
\begin{equation*}
\rho_{S}(\underline{x} ; t)=-\langle\rho(x)|\left(\partial_{t}-H_{o}\right)^{-1} \dot{f}_{O S}\left|H_{1}\right\rangle \tag{C6}
\end{equation*}
$$

or in the frequency domain

$$
\begin{equation*}
\rho_{S}(\underline{x} ; w)=\langle\rho(\underline{x})|\left(i \omega+\check{H}_{o S}\right)^{-1} \stackrel{r}{f}_{O S}\left|H_{1}\right\rangle \tag{C,7}
\end{equation*}
$$

Here the state $\mid \rho(\underline{x})$ ) is the contribution to charge density of a particle at a given phase point $\underline{z}$ :

$$
\begin{align*}
\langle\underline{Z} \mid \rho(\underline{x})\rangle \equiv \mathrm{e} \delta[\underline{x}-\underline{r}(\underline{Z})] & =e \int d p^{\prime} \delta\left[p^{\prime}-\underline{p}(\underline{z})\right] \delta[\underline{x}-\underline{r}(z)]  \tag{C8a}\\
& =\langle\underline{Z}| e \int d p^{\prime}\left|\underline{x}, p^{\prime}\right\rangle, \quad \text { i.e. } \tag{C8~b}
\end{align*}
$$

We consider electrostatic perturbations,

$$
\begin{equation*}
\left\langle\underline{z} \mid H_{1}\right\rangle=e \phi[\underline{r}(\underline{Z})]=\int d \underline{x}^{\prime}\langle\underline{z} \mid \rho(\underline{x})\rangle \phi\left(x^{\prime}\right) . \tag{C9}
\end{equation*}
$$

Putting this into (C7) and comparing with Eq. (III-1b) yields

$$
\begin{equation*}
\bar{K}_{s}\left(\underline{x}, \underline{x}^{\prime} ; \omega\right)=-4 \pi\langle\rho(x)|\left(i \omega+\dot{H}_{o S}\right)^{-1} \dot{f}_{o S}\left|\rho\left(x^{\prime}\right)\right\rangle . \tag{C10}
\end{equation*}
$$

To put (C10) into a more explicit form, we introduce another orthonormal basis |ㅡ, $\underline{I}$ ):

$$
\begin{equation*}
\left\langle\underline{Z}_{0} \mid \underline{\ell}, \underline{I}\right\rangle \equiv N e^{-i \ell \cdot \theta_{0}} \delta\left(\underline{I}-\underline{I}_{0}\right) . \tag{C11}
\end{equation*}
$$

The nomalization constant N is decemined by the requirement of ortho. normality, $\delta\left(\underline{\underline{z}}_{1}-\underline{\underline{z}}_{2}\right)=\left(\underline{\underline{z}}_{1} \mid \underline{\underline{z}}_{2}\right)$, to be

$$
\begin{equation*}
N=(2 \pi)^{-3 / 2} \tag{C12}
\end{equation*}
$$

The basis $\mid \underline{\ell}, \underline{1}$ ) is chosen because these states are the eigenstates of the Liouville operator $\dot{H}_{o}$. One has

$$
\begin{equation*}
\tilde{H}_{0}(\underline{\ell}, \underline{I})=i \underline{\ell} \cdot \partial H_{0} / \partial \underline{I}|\underline{\ell}, \underline{I}\rangle \equiv i \underline{\ell} \cdot \underline{\Omega}|\underline{\ell}, \underline{I}\rangle . \tag{Cl4}
\end{equation*}
$$

Ke shall assume that $f_{0}$ is a function of both $H_{0}$ and I, in general. Then since $H_{0}=H_{0}(I)$, one has

$$
\begin{equation*}
\dot{f}_{0}=\left(\partial f_{0} / \partial H_{0}\right) \check{H}_{0}+\dot{f}_{o I}, \quad \ddot{f}_{o I} \equiv-\left(\partial f_{0} / \partial I\right) \cdot \partial / \partial \theta \tag{C15}
\end{equation*}
$$

The $|\underline{\ell}, \underline{I}\rangle$ are also eigenstates of $\dot{f}_{0}$ and $\check{f}_{\text {OI }}$;

$$
\begin{equation*}
\left.\hat{f}_{O I} \mid \ell, I\right)=i \underline{\ell} \cdot \partial f_{0} / \partial \underline{I}|\underline{\ell}, I\rangle \tag{C16}
\end{equation*}
$$

Also, one has that

$$
\begin{align*}
\left(i \omega+\dot{H}_{0}\right)^{-1} \dot{f}_{0} & =\left(i \omega+\dot{H}_{0}\right)^{-1}\left[\partial f_{0} / \partial H_{0}\left(i \omega+\dot{H}_{0}\right)-i \omega \partial f_{0} / \partial H_{0}+\dot{f}_{o I}\right] \\
& =\partial f_{0} / \partial H_{0}+\left(i \omega+\dot{H}_{0}\right)^{-1}\left(-i \omega \partial / \partial \dot{H}_{0}+\dot{f}_{o I}\right) \tag{C17}
\end{align*}
$$

In less formal terms, in ( C 17 ) we have written the factor ( $\partial \mathrm{f}_{0} / \partial \mathrm{H}_{0}$ ) $\dot{H}_{0}$ in $\dot{f}_{0}$ as the sum of a total time derivative $\left(\partial f_{0} / \partial H_{0}\right)\left(i \omega+\dot{H}_{0}\right)$ and a remainder, so that the integration along umperturbed trajectories induced by $\left(i \omega+\dot{H}_{0}\right)^{-1}$ becomes trivial. This technique has been used previously by other authors, 32,35 in a less formal, non-canonical framework.

Inserting the identity dyadic $1=[\mid \underline{\ell}, \mathrm{I})(\underline{\ell}, \underline{I} \mid$ between
 Eqs. (C14), (C16) and (C17), one obtains the decomposition:

$$
\begin{equation*}
\bar{K}_{S}\left(x, \underline{x}^{\prime} ; w\right)=\bar{K}_{S A}\left(\underline{x}, \underline{x}^{\prime}\right)+\bar{K}_{S B}\left(\underline{x}, \underline{x}^{\prime} ; \omega\right) \tag{C18a}
\end{equation*}
$$

The 'adiabatic term" $\bar{K}_{s A}$ is purely local, and w-independent, coming from the part $\partial f_{0} / \partial H_{o}$ of $\left(i \omega+\stackrel{\rightharpoonup}{H}_{o}\right)^{-1} \stackrel{r}{o}_{0}$ :
$\bar{K}_{S A}\left(\underline{x}, \underline{x}^{\prime}\right) \equiv 4 \pi ; \rho(\underline{x})\left|\partial f_{o_{S}} / \partial H_{o_{S}} \operatorname{l\rho }\left(\underline{x}^{\prime}\right)\right\rangle=\lambda_{S}^{-2}(\underline{x}) \varepsilon\left(x-x^{\prime}\right)$,
where $\lambda_{s}^{2}(\underline{x}) \equiv T_{s}(\underline{x}) / 4 \pi n_{s}(\underline{x}) e^{2}, n_{s}(\underline{x}) \equiv \int d \underline{Z} \delta[\underline{x} \underline{\underline{r}}(\underline{2})] f_{o s}$, and $T_{s}^{-1}(\underline{x}) \equiv-n_{s}^{-1}(\underline{x}) \int d \underline{E} \delta[\underline{x}-\underline{r}(\underline{z})] \partial f_{o_{s}} / \partial H_{o_{s}}$. The remainder, $\bar{K}_{s B}$, is given by

$$
\begin{align*}
& \bar{K}_{s B}\left(\underline{x}, \underline{x}^{\prime} ; \omega\right) \equiv-4 \pi\left(\rho(\underline{x})\left|\left(i \omega+\check{H}_{o_{s}}\right)^{-1}\left(-i \omega \partial f_{o_{s}} / \partial H_{o_{s}}+\check{f}_{o I}\right)\right| \rho\left(\underline{x}^{\prime}\right)\right\rangle \\
& =4 \pi \sum_{\underline{l}} \int \mathrm{dI}\left\{\rho(\underline{x})|\underline{\ell}, \underline{I}\rangle \frac{\omega \partial \mathrm{f}_{\mathrm{o}_{S}} / \partial \mathrm{H}_{\mathrm{o}_{S}}-\ell \cdot \partial \mathrm{f}_{\mathrm{o}_{S}} / \partial \underline{I}}{\omega+\underline{\ell} \cdot \underline{\Omega}}-\left\{\underline{\ell}, \underline{I}\left|\rho\left(x^{\mathrm{r}}\right)\right\rangle .\right.\right. \tag{C18c}
\end{align*}
$$

Defining $\rho_{\ell}(\underline{x} \mid \underline{I})$ by

$$
\begin{align*}
\rho_{\underline{\ell}}(\underline{x} \mid \underline{I}) & \equiv(2 \pi)^{-3 / 2}\langle\rho(\underline{x}) \mid \underline{\ell}, \underline{I}\rangle=(2 \pi)^{-3} \int \mathrm{~d} \underline{\theta} \mathrm{e}^{-\mathrm{i} \underline{\ell} \cdot \underline{\theta}} \rho(\mathrm{x} \mid \underline{Z})  \tag{C19}\\
& =(2 \pi)^{-3} \int \mathrm{~d} \underline{\theta} \mathrm{e}^{-i \underline{\ell} \cdot \underline{\theta}} \mathrm{e} \delta \underline{\mathrm{x}}-\underline{r}(\underline{Z})
\end{align*}
$$

we write ( C 18 c ) in the more standard form

$$
\begin{equation*}
\bar{K}_{\mathrm{SB}}\left(\underline{x}, \underline{x}^{\prime} ; \omega\right)=(4 \pi)(2 \pi)^{3} \sum_{\underline{l}} \int \mathrm{~d} \underline{I} \rho_{\underline{\ell}}(\underline{x} \mid \underline{I}) \frac{\omega \partial \mathrm{f}_{\mathrm{o}_{s}} / \partial \mathrm{H}_{\mathrm{o}_{s}}-\underline{\ell} \cdot \partial \mathrm{f}_{\mathrm{o}_{s}} / \partial \underline{I}}{\omega+\ell \cdot \Omega} \rho_{\ell} \underline{\underline{x}}^{\prime}(\underline{\mathrm{x}}) \tag{C20}
\end{equation*}
$$

Dropping the explicit dependence of $f_{0}$ on $H_{0}$, the adiabatic tem $\bar{K}_{S A}$ and the term in $\partial f_{0} / \partial H_{o}$ of (C20) vanish, and one ubtains a form paralleling that of Refs. 25 and 26. In our work, it is more convenient to instead drop the explicit dependence on one of the canonical momenta $I_{1}$ (the gyroaction $J$ ). The term $\partial f_{o} / \partial I$ in (C20) then has only two conmpnents.

APPENDIX D: EVALUATION OF $S_{S B}$ IN TFIE REGIME: $c_{S} / L_{\phi}>1$

We here ohtain an approximate expression for the contribution $\mathrm{S}_{\mathrm{SB}}$ of $\overline{\mathrm{K}}_{\mathrm{sB}}$ or $\mathrm{K}_{\mathrm{sB}}$ to S , valid in the regime where the mode localization length $L_{\phi}$ is small compared to the gyroradius "s of a typical particle of species $s$. We take for our distribution function the drifting Maxweliian form (III-83) used in the applacations of Sec. III.D, though this particular form is inessential to the evaluation method we use.

From the discussion in Sec. III.B, $\mathrm{S}_{\mathrm{SB}}$ is given by

$$
\begin{equation*}
\mathrm{S}_{\mathrm{SB}}(\psi, \phi) \equiv \int \mathrm{d} x \int \mathrm{~d} \sigma \int \mathrm{~d} \sigma^{\prime} \psi^{\star}(x+\sigma) \mathrm{K}_{\mathrm{SB}}\left(\sigma, x, \sigma^{\prime}\right) \psi\left(x+\sigma^{\prime}\right), \tag{Dl}
\end{equation*}
$$

analogous to Eq. (III-9). We use expression (III-83) for $f_{s}$ in Eq. (III-52) for $K_{s B}$. Performing the same manipulat ions as done in Sec. III.C. 3 and III.C. 4 to evaluate $K_{s B}\left(k, x,-k^{\prime}\right)$, and seîting $\phi(x)=\phi_{1}(x) e^{i k x}, \psi(x)=\psi_{1}(x) e^{i k x}$, with $\psi_{1}^{*}(x)=\phi_{1}(x)$ as in Secs.
III.B. 1 and III.B.3, we have

$$
\begin{equation*}
S_{s B}=\int d x K_{s B}(\psi, x, \phi), \tag{D2}
\end{equation*}
$$

where

$$
\begin{align*}
K_{s B}(\psi, x, \phi) & \equiv-\lambda_{s}^{-2}(x) \sum_{\ell} 2 \pi \int_{-\infty}^{\rho} d u \int_{0}^{\infty} d h^{\prime} \bar{g}_{0} e^{-\gamma_{s} u^{2} / 2} e^{-\gamma_{s} h^{\prime}}  \tag{D3}\\
& \times \tilde{J}_{\ell}(\psi) \tilde{J}_{\ell}^{*}(\phi)\left(\omega^{\prime}-\omega_{*_{s}}\right)\left(\omega^{\prime \prime}-\ell \Omega_{0}-k_{2} v_{B}-k_{3}^{\prime} u\right)^{-1}
\end{align*}
$$

The only difference between the quantity $K_{s B}(\psi, x, \phi)$ here and the expression for $k_{s B}\left(k, x,-k^{\prime}\right)$ in Eq. (III-84), is in the replacement of $\tilde{J}_{\ell}(\underline{k})$ defined by

$$
\begin{align*}
\bar{y}_{\hat{x}}(f \mid \underline{1}) & \equiv \int \mathrm{d} c \hat{j}_{\hat{l}}\left(\sim, k_{y}, k_{z} \mid 1\right) \phi^{*}\left(x=x_{0}+0\right) \\
& =(2 \pi)^{-1} \int_{0}^{2 \pi} d \because e^{-i n c} e^{-\hat{k} \cdot\{\underline{r}(0)} \phi^{*}\left[r_{x}(0)\right] \tag{D4}
\end{align*}
$$

where we recall that $\underline{\hat{k}} \equiv \hat{\mathbf{y}} \mathrm{k}_{y}+\dot{z}_{k_{z}}$. The difference between $\tilde{J}_{\ell}(\underline{k})$ and $\tilde{I}_{\mathrm{f}}(*)$, then, comes just in replacing a plane wave form $\mathrm{e}^{i k x}$ by the form
 as specified in Eq. (III-31). Putting thas form for $\phi$ into (D4) and using $r_{x}(\varepsilon)=x_{0}+\delta x(0)$, we have

$$
\begin{equation*}
\tilde{I}_{\Omega}(\phi)=e^{-i k \lambda_{0}}(2 \pi)^{-1} \int_{0}^{2 \pi} d \odot e^{-i \ell \theta} e^{-\underline{k} \cdot \delta \underline{r}(0)} \phi_{1}{ }^{*}\left[\alpha^{-1 / 2}\left(\delta x(\theta)-\sigma_{0}\right)\right] \tag{D5}
\end{equation*}
$$

where $c_{0} \equiv x_{0}-X_{0}$, the distance between the point $x_{0}$ of mode localization and the guiding center $x_{o}$ of any given particle. The factor $e^{-i k X_{o}}$ in (D5) is unimportant, cancelling a corresponding factor $e^{i k X_{0}}$ in $J_{\ell}^{*}(\phi)$ in ( D 3 ).

The integration $\int d u$ over parallel velocity in (D3) is unchanged from that done in Sec. III.C. 4 to evaluate $K_{s B}\left(k, x,-k^{\prime}\right)$. The integration $\int \mathrm{dh}^{\prime}$ over perpendicular velocity done there used a factor $\gamma_{s} / 2 \pi$ from $\bar{g}_{\mathrm{O}} \equiv\left(\gamma_{\mathrm{S}} / 2 \pi\right)^{3 / 2}$ to yield the integral

$$
\begin{equation*}
2 \pi\left(\gamma_{s} / 2 \pi\right) \int d h^{\prime} e^{-\gamma_{s} h^{\prime}} J_{\ell}^{2}\left(\mu_{0}\right) \equiv \Lambda_{\ell}\left(b_{s}\right) \tag{D6}
\end{equation*}
$$

in expression (III-92) for $K_{s B}(k, x,-k)$, in the approximiation of harmonic gyromotion. Our result for $K_{s B}$ here wiill thus be the same as that expression, but with the $J_{\ell}$ in (D6) replaced by $\tilde{J}_{\ell}(\Phi)$. It is therefore convenient to extend the definition of $\Lambda_{\ell S}$ from its value $I_{\ell S}$ (b) $e^{-b_{S}}$ for the regime $\rho_{s} / L_{\phi}>1$, namely

$$
\begin{equation*}
\Lambda_{\ell s}\left(\sigma_{0} ; \rho_{s} / L_{\phi}>1\right) \equiv \gamma_{s} \int d h^{\prime} e^{-\gamma_{s} h^{\prime}} \tilde{J}_{\ell}(\psi) \tilde{J}_{\ell}^{*}(\phi) . \tag{D7}
\end{equation*}
$$

(We note the dependence of this quantity on $0_{0} \equiv x_{0}-X_{0}$.)
To evaluate $\bar{J}_{\ell}(\phi)$, we refer to Fig. 18. Assuming hamonic gyromotion in this regime also, with
$\delta \underline{r}(0)=\hat{b}_{1} \delta x_{0}(c)+\hat{b}_{2} \delta x_{20}(0)$
$\delta x_{0}(\theta)=\bar{x}_{0} \sin \theta, \quad \delta x_{20}(\theta)=\bar{x}_{20} \cos \theta, \quad \bar{x}_{20}=\bar{x}_{0}\left(\Omega_{c} / \Omega_{0}\right)$,
as in Sec. III.C.2, we see that the particle trajectory passes through the region of wave localization at two places ( $0=\theta_{ \pm}$) for $\sigma_{0} / \bar{x}_{0}<1$, and not at all, for $\sigma_{0} / \bar{x}_{0}>1$. For particles having $\sigma_{0} / \bar{x}_{0}<1$, we approxinate the true trajectory in the region of wave localization by straight line orbits. From Fig. 18, one has that

$$
\begin{equation*}
\sin \theta_{ \pm}=\sigma_{0} / \bar{x}_{0} \tag{D9}
\end{equation*}
$$

and so, calling the particle velocities at $\theta_{ \pm}$in the $\hat{b}_{1}$ and $\hat{b}_{2}$ directions $v_{1 \pm}, v_{2 \pm}$, using (D8) gives $v_{1+}=-v_{1-}, v_{2+}=v_{2-} \equiv v_{2}$, with

$$
\begin{equation*}
v_{2}=-\Omega_{0} \bar{x}_{20} \sin \theta_{ \pm}=-\sigma_{0} \Omega_{c}, v_{1+}=\Omega_{0} \bar{x}_{0} \cos \theta_{ \pm}=\Omega_{0}\left(\bar{x}_{0}^{2}-\sigma_{0}^{2}\right)^{1 / 2} \tag{D10}
\end{equation*}
$$

The integration over $\theta$ in (D5) breaks into two parts, which we denote by $\left.\tilde{J}_{\ell \pm}: \tilde{J}_{\ell}=\sum_{ \pm}\right]_{\ell \pm}$. For each of these, we replace $\theta-\dot{\theta}_{ \pm}$by $\Omega_{0} t$, and extend the limits of integration to $\pm \infty$ for both, permissible since the Gaussian factor in $\phi_{1 n}$ introduces a cutoff in the integration. Using (D5), we then obtain

$$
\begin{equation*}
\tilde{J}_{\ell \pm}^{ \pm}=e^{i k X_{0}} e^{i \ell \Theta_{ \pm}} e^{-i k_{2} \delta x_{2 r}}\left(\Omega_{0} / 2 \pi\right) N_{x n} \int_{-\infty}^{\infty} d t e^{-\kappa t} e^{-A t^{2} / 2} H_{n}\left(A^{1 / 2} t\right), \tag{D11}
\end{equation*}
$$

where $k \equiv l \Omega_{0}+k_{2} v_{2}+k v_{1+}, A \equiv \alpha v_{1+}$ and $\delta x_{2 \pm} \equiv \delta x_{20}\left(\theta_{ \pm}\right)$. The exponential factors outside the integral, which for sherthand we refer to $n=$, will cancel against the factor in $J_{\ell}$. The integral itself is simply the Fourier transform from $t$-space to $k$-space of the function $e^{-A t^{2} / 2} H_{n}(-/ 2 t)$. For $n=0$ we have just the Fourier transform of a Gaussian,

$$
\begin{equation*}
\int_{-\infty}^{\infty} d t e^{-i \kappa t} e^{-A t^{2} / 2}=(2 \pi / A)^{1 / 2} e^{-\kappa^{2} / 2 A} \tag{212}
\end{equation*}
$$

Given the $(n-1)^{\text {th }}$ solution $\phi_{1(n-1)}$ of the Weber equation (III-? , it is easily verified that $\phi_{1 n}$ is given by

$$
\begin{equation*}
\phi_{1 n}(\xi)=(2 n)^{-1 / 2}\left(-\partial_{\xi}+\xi\right) \phi_{1(n-1)}(\xi) \tag{D13}
\end{equation*}
$$

(In fact, this is where expressions (III-34) and (III-35b) come from.)
Letting the "raising operator" $(2 n)^{-1 / 2}\left(-\partial_{\xi}+\xi\right)$ act on $\left.t\right\}$ Fourier representation of $\phi_{1 n}(\xi)=(2 \pi)^{-1} \int d q e^{i q \xi} \phi_{1 n}(q)$, one $f$ nds that

$$
\begin{equation*}
\phi_{1 n}(q)=i(2 n)^{-1 / 2}\left(-\partial_{q}+q\right) \phi_{1(n-1)}(q) \tag{D14}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\phi_{1 n}(q)=i^{n}\left(2^{n} n!\right)^{-1 / 2} H_{n}(q) \phi_{10}(q) . \tag{D15}
\end{equation*}
$$

i.e., the Fourier transform of the eigenfunctions of the Weber equation are again eigenfunctions of the Weber equation, a result to be expected, because of the symmetry of that equation with respect to $\xi$ and $-\partial_{\xi}$. Applying this to Eq. (D1I), one has

$$
\begin{equation*}
\vec{J}_{i \pm}^{\star}=e_{ \pm}\left(\Omega_{0} / 2 \pi\right) N_{k n} H_{n}(q) e^{-q^{2} / L} \tag{D16}
\end{equation*}
$$

where $q \equiv \kappa A^{-1 / 2}$, and $N_{k n}=i^{n}(2 \pi / A)^{1 / 2} N_{x n}$.
To evaiuate (D7), we need $\tilde{J}_{\ell}(\psi) \tilde{J}_{\ell}^{*}(\phi)$. Each $\tilde{J}_{\ell}$ here is a sum $\sum_{ \pm} \tilde{j}_{\ell \pm}$ over contributions from the crossings at $\theta_{ \pm}$. We are here interested in the ion contribution $S_{i B}$ for the ion drift cyclotron modes. For these modes, one has $k_{2} \rho_{i} \gg 1$. Therefore, in moving from $\theta_{+}$to $0_{-}$, the particle phase factor $e_{ \pm}$will be highly sensitive to any sorts of perturbations present, e.g., from other modes, from orbit modification due to magnetic inhomogeneities, etc. We thus assume that, in averaging over all the particles in the distribution, that contributions from crossterms $\tilde{J}_{\ell \pm} \tilde{J}_{\ell \mp}^{*}$ are negligible, i.e., we assume that

$$
\begin{equation*}
\tilde{J}_{\ell}(\psi) \tilde{J}_{\ell}^{*}(\phi) \simeq \sum_{ \pm} \tilde{J}_{\ell \pm}(\psi) \tilde{J}_{\ell \pm}^{*}(\phi) . \tag{D17}
\end{equation*}
$$

Using (D16), we then find

$$
\begin{align*}
\widetilde{J}_{\ell}(\psi) \widetilde{J}_{l}^{*}(\phi) & \left.\simeq 2 \widetilde{J}_{\ell+}(\psi) \tilde{J}_{l+}(\phi)=2\left[\Omega_{0} / 2 \pi\right) N_{l n} H_{\Omega}(q) e^{-q^{2} / 2}\right]^{2} \\
& =(-1)^{n} \Omega_{0}^{2}\left(\pi \alpha v_{1+}^{2}\right)^{-1} N_{x \Omega}^{2} H_{n}^{2}(q) e^{-q^{2}} \tag{D18}
\end{align*}
$$

where $q^{2} \equiv \kappa^{2} / A \equiv\left(l \Omega_{0}+k_{2} v_{2}+k v_{1+}\right)^{2}\left(\alpha v_{1+}\right)^{2}$.

We assume the orderings

$$
\begin{equation*}
\rho_{\mathrm{s}}^{-1} \mathrm{k} \ll L_{\phi}^{-1} \equiv \alpha^{1 / 2} \leqslant k_{2}, \tag{D19}
\end{equation*}
$$

of which only the assumption $\dot{k} \ll \alpha^{1 / 2}$ has not yet already been discussed. With this assur-tion, we neglect the term $\mathrm{kv}_{1+}$ in $q^{2}$, making $\mathrm{K}_{\mathrm{sB}}$ and $\mathrm{S}_{\mathrm{sB}}$ independent of k . This assumption was used in the variational equation (III-27), which lead to the result that $k=0$, consistent with $\mathrm{k} \ll \alpha^{1 / 2}$.

From (D10), the particle velocity $v_{o l}$ in the $\hat{b}_{1}$ direction at the guiding center is given by $v_{o l}=\Omega_{0} \bar{x}_{0}$, and the perpendicular energy of gyration $h^{\prime}$ is given by $h^{\prime}=\frac{1}{2} v_{o l}{ }^{2}$. Thus from (D10), we have

$$
\begin{equation*}
h \equiv \frac{1}{2} v_{1+}^{2}=h-\frac{1}{2}\left(\Omega_{0} \sigma_{0}\right)^{2} \tag{D20}
\end{equation*}
$$

Using this and (D18) in (D7) yields
$\Lambda_{l s}\left(\sigma_{0}\right)=(-1)^{n} e^{-\gamma_{s}\left(\Omega_{0} \sigma_{0}\right)^{2} / 2} \int_{0}^{\infty} d n\left(\mathrm{e}^{-\eta / \eta)\left(N_{x n}^{2} / 2 \pi\right)\left(\Omega_{0}^{2} \gamma_{s} / \alpha\right) H_{n}^{2}(q) e^{-q^{2}}, ~}\right.$
where $n \equiv \gamma_{s} h$, and using Eqs. (D10) and the above approximations,

$$
\begin{equation*}
q^{2} \simeq\left(\Omega_{0}^{2} \gamma_{s} / 2 a\right)\left[\ell-k_{2} \sigma_{0}\left(\Omega_{c} / \Omega_{0}\right)\right]^{2} n^{-1} \equiv n_{0} / n . \tag{D22}
\end{equation*}
$$

If we let $n$ (and so ${ }_{1+}$ ) go to zero, corresponding to the particle trajectory which has $\sigma_{0} / \bar{x}_{0}=1$ (so that the particle moves straight along the crest of the Gaussian), the factor $\mathrm{e}^{-\eta / \eta}$ in (D21) becomes infinite, but this factor is more than compensated for by the term $e^{-q^{2}}=e^{-\eta_{0} / \eta}$, which arises from the rapid phase oscillations a particle traveling
directly along the Gaussian crest sees, due to the factor ( $9 a_{0}+\mathrm{k}_{2} \mathrm{v}_{2}$ ). The term $\mathrm{e}^{-q^{2}}$ thus provides a cutoff at $q=1$ in the $n$-integration in (U21), and so we replace the factor $e^{-n_{0} / \eta}$ there with a step function, equal to unity for $n>n_{0}$, and zero for $n<n_{0}$.

If we consider for the moment the $n=0$ case, the $n$ - integral in (D21) is $\int_{\eta_{0}}^{\infty} \mathrm{d}_{\mathrm{n}} \epsilon^{-\eta} / \eta \equiv \mathrm{Ei}\left(-\eta_{0}\right)$, where Ei is the exponential integral function. ${ }^{29}$ For $n_{0}>0$, this function falls off monotonically. For $n_{0}>1$, the falloff is dcminated by the exponential factor $e^{-\eta}$ in the integrand, Ei $\left(-\eta_{0}\right) \sim e^{-\eta_{0}}$. As $\eta_{0} \rightarrow 0$, Ei has a logarithmic singularity. This limit corresponds to the situation $\left(l_{2}+k_{2} v_{2}\right) \rightarrow 0$, for which the straight line orbit approximation we have made yields an artificial resonance effect. In this limit, we expect that the curvature of the particle orbits, as well as the other orbit modification effects, will act to make the $n$-integration be well behaved as $n_{0} \rightarrow 0$. Combining these expectations for the large and small $n_{0}$ behaviors of the $n$-integration, we approximate that integral by $e^{-\eta_{0}}$, for all $n_{0}$. For $n=0$, then, Eq. (D21) becomes

$$
\begin{equation*}
\left.\Lambda_{l s}\left(\sigma_{0} ; n=0\right) \simeq e^{-\gamma_{s}\left(\Omega_{0} \sigma_{0}\right)^{2} / 2} e^{-\eta_{0}} N_{x 0}^{2} / 2 \pi\right)\left(\Omega_{0}^{2} \gamma_{s} / \alpha\right) \tag{D23}
\end{equation*}
$$

For larger $n$, the factor $H_{n}{ }^{2}(q)$ displaces a larger fraction of the integrand excluding $\mathrm{e}^{-\mathrm{q}^{2}}$ in (D21) toward larger $q$ values, or smaller $\eta$ values, so that a smaller and smaller proportion is left in the region where the cutoff factor $\mathrm{e}^{-\mathrm{q}^{2}}$ is substantial. We thus approximate $\mathrm{H}_{\mathrm{n}}{ }^{2}(\mathrm{q})$ in the region of substantial $e^{-q^{2}}$ (small $q$ ) by its $q=0$ value,

$$
H_{n}^{2}(q=0)= \begin{cases}0, & n \text { odd }  \tag{D24}\\ (n!), & n \text { even } .\end{cases}
$$

We thus see that for larger $n,\left|\Lambda_{\ell s}(n)\right|$ falls off rather rapidly from the value $\Lambda_{l s}(0)$. Defining $c_{n} \equiv\left|\Lambda_{l s}(n) / \Lambda_{l s}(0)\right|$, then, we have $c_{0}=1$, $c_{n}=0$ for $n$ odd, and $c_{n}=(n!)^{-1}$ for $n$ even, and so from (D21), (D23), and (111-30a)

$$
\begin{align*}
& \Lambda_{l S}\left(\sigma_{0} ; 1\right)=(-1)^{n} c_{n} \Lambda_{\ell S}\left(\sigma_{0} ; n=0\right)  \tag{D25}\\
& \quad=(-1)^{n} c_{n} e^{-\gamma_{S}\left(\Omega_{0} \sigma_{0}\right)^{2} / 2} e^{-\eta_{0}}\left(\gamma_{s} \Omega_{0}^{2 / 2 \pi^{3 / 2}}{ }_{a}^{1 / 2}\right) .
\end{align*}
$$

To evaluate $S_{S B}$, it remains to put expression (D25) for $\Lambda_{\ell S}$ into Eq. (III-92) for $K_{S B}$, and then perform the $x$ (or $o_{0}$ ) integration indicated by (D2). Assuming for simplicity that since $\eta_{p} \equiv \rho_{i} / L_{p}<1$, we may neglect the spatial variation of all terms in $K_{i B}$ but that of $\Lambda_{\ell i}$, we may readily evaluate the integration in $S_{S B}$ for the ion species. Again only for simplicity, we first neglect $\ell$ in comparison with $\mathrm{k}_{2} \mathrm{O}_{\mathrm{o}}$ in (D22), in view of Eq. (III-104). This gives

$$
\begin{equation*}
\eta_{0}=\frac{1}{2} \gamma_{s}\left(\Omega_{0} \sigma_{0}\right)^{2}\left(k_{2} \Omega_{c} / \Omega_{0}\right)^{2} \alpha^{-1}, \tag{D26}
\end{equation*}
$$

and so from (D25), defining $\rho_{s}^{2} \equiv\left(v_{s} / \Omega_{o}\right)^{2} \equiv \gamma_{s}^{-1} \Omega_{o}^{-2}$,
$\Lambda_{l s}(n) \equiv \int_{-\infty}^{\infty} d \sigma_{0} \Lambda_{l s}\left(\sigma_{0} ; n\right)=(-1)^{n} c_{n}\left(\gamma_{s} \Omega_{0}^{2} / 2 \pi \alpha_{\alpha}^{3 / 2} 1 / 2 \int_{-\infty}^{\infty} d \sigma_{0} e^{-\frac{1}{2} \gamma_{s} \Omega_{0}^{2}\left[1+\left(k_{2} \Omega_{c} / \Omega_{0}\right)^{2}\right] \sigma_{0}^{2}}\right.$

$$
\begin{equation*}
=(-1)^{n} c_{n} 2^{-1 / 2}{ }^{-1}\left[\alpha_{s}^{2}+\left(k_{2} \rho_{s}\right)^{2}\left(\Omega_{c} / \Omega_{0}\right)^{2}\right]^{-1 / 2} . \tag{D27}
\end{equation*}
$$

This expression simply replaces the factor $A_{8,1} \equiv I_{2}\left(b_{i}\right) e^{-b_{1}}$ appearing in the form for $S_{i B}$ valid in the small - $\mathcal{F}_{i} / L_{\phi}$ regine, i.e.,

$$
\begin{equation*}
S_{i B}=-\lambda_{i}^{-2} \sum_{\ell}\left(1-k_{\ell, 1}\right)\left(i_{1} / i_{2}\right) l_{\ell 3}, \tag{D28}
\end{equation*}
$$

where
$\dot{n}_{l i}\left(\rho_{1} / L_{\phi}<1\right) \equiv I_{\ell} \rho_{i} / e^{-b_{i}}, \quad$ and
$\Lambda_{\ell i}\left(\rho_{i} / L_{\phi}>1\right) \equiv(-1)^{n} c_{n} 2^{-1 / 2} \pi^{-1}\left[\alpha \rho_{s}{ }^{2}+\left(k_{2} \rho_{s}\right)^{2}\left(\rho_{c} / \lambda_{o}\right)^{2}\right]^{-1 / 2}$.

APPENDIX E: ONE TERM APPROXDMTIO: TO THE IDC GROWTH RATE

Here we derive an expression for the IDC growth rate, using the usual approximation made in analytic analyses of the IDC instability, in which we keep only the $\ell^{\text {th }}$ term in factor $C$ in Eq. (III-126). As shown in Figs. 15a, 15b, we nere redefine $\overline{/}$ from the definition used in the two term analysis of Sec. III.D. 2 by

$$
\begin{equation*}
0=\left.\partial_{\omega}(B+C)\right|_{\bar{\Omega}} \tag{E1}
\end{equation*}
$$

This gives

$$
\begin{equation*}
\left(\bar{\Omega}-\omega_{\ell}\right)^{2}=\bar{\Omega}^{2}\left(\mathrm{I}_{2} \Lambda_{\ell} g_{\ell} / \Lambda_{\mathrm{oe}} \mathrm{~g}_{\mathrm{e}}\right)\left(\omega_{*}-\omega_{\ell}\right) \omega_{*}^{-1} \tag{E2}
\end{equation*}
$$

As in Sec. III.D.2, we expand the dispersion equation (III-126) in $\delta \omega \equiv \omega^{\prime}-\Omega$, again neglecting $\partial_{\omega}^{2} B$ in view of (III-133). Using (III-134) as in Sec. III.D.2, and neglecting the term in $K_{d}$.e. for simplicity, we obtain the quadratic equation

$$
\begin{equation*}
0 \simeq\left(\bar{\Omega}-\omega_{\text {ISEC }} \partial_{\omega} B+\frac{1}{2}(\delta \omega)^{2} \partial_{\omega}^{2} C+C\right. \tag{E3}
\end{equation*}
$$

with solution

$$
\begin{equation*}
(\delta \omega)^{2}=-2\left[\left(\bar{\Omega}-\omega_{I S E C}\right) \partial_{\omega} B+C\right] / \partial_{\omega}^{2} C \tag{E4}
\end{equation*}
$$

In contrast to the two term approximation, a glance at Fig. 15b shows that here $\partial_{\omega}^{2} \mathrm{C}<0$, and so here the term $\mathrm{C} / \partial_{\omega}^{2} \mathrm{C}$ is destabilizing. Assuming that $\omega_{\text {ISEC }} \simeq \bar{\Omega}$, we neglect the term in ( $\bar{\Omega}-\omega_{\text {ISEC }}$ ) in (E4) and use (E2) to obtain the desired result:

$$
\begin{aligned}
r_{m x}^{2} & =2 C / \partial_{\omega}^{2} C=\left.\left(\omega^{\prime}-\omega_{\ell}\right)^{2}\left(\omega^{\prime}-\omega_{\theta}\right)\left(\omega_{\ell}-\omega_{\ell}\right)^{-1}\right|_{\omega^{\prime}=\bar{\Omega}} \\
& =\bar{\Omega}^{2}\left(I_{3} \Lambda_{\ell} g_{\ell} / \Lambda_{o e} g_{e}\right)\left(1-\overline{\sigma_{l} / \omega_{\ell}}\right)
\end{aligned}
$$

Using the fact that here $\bar{\Omega}=\omega_{\text {ISEC }}$, and that $u_{\text {. }}$ is what we call $\overline{\bar{\theta}}$ (at maximm growth) in the two term approximation, we see that the factor


SIMBOL TABLES FOR PARTS 1, 11, AND I11

The following tables provide an explanation of the symbols used, ci indicate where in the text the symbol is defined or introduced. A separate section is given for each of Parts $I, I I$, and IJl of the text. Symols whose use is restricted to the inmediate vicinity of the place tiey are defined are not included.

In Table A of each part is given a list of general notational conventions used throughout that part. Table $k$ is a listing, in alphabetical order (Latin alphabet farst, then Greek alphabet), of the symbols used in that part, followed by the sumbol's meaning or where to find it. The symbol is given, followed by a hyphen, followed by information giving the symbol's meaning. Here, if the defining equation referenced occurs in the same part as that to which the symbols table refers, the part number is not given. For example, in the table for Part IIl,

$$
\begin{equation*}
\Lambda_{\ell} \tag{83}
\end{equation*}
$$

means symbol $\Lambda_{2}$ is defined in Eq. (111-83). The notations "b(83)" and " $f(85)$ " indicate that the definition is made in the text in the immediate vicinity before (b) or following (f) Eq. (III-83).

When a symbol has more than one important equation indicating its meaning, a multiple notation may be given, e.g. for the symbol $k_{s}$ of part IlI,

$$
K_{s}-[(8),(45)]
$$

both Eqs. (III-8) and (III-45) are useful in toderstanding its meaning. In certain cases, where context should make the meaning clear, the sane symbol has been used for different quantities. In these few cases, the symbol itself is repeated in Table B.

Finally, some of the symbols with related meanings fall into "notational groups" of similar symbols, for example the various contributions to the Hamiltonians in leri ll. To aid the reader in remembering the significance of and differences among the se related quantities, in Table $C$ of Parts Il and IIl, a description is given of the more important notational groups of those parts.

## PART I

Table I.A) Conventions

| Notation | Meaning | Examples |
| :---: | :---: | :---: |
| tildc | physical specification of the quantity in question | $\tilde{f}_{s}, \widetilde{X}_{0}, \underline{A}$ |
| superscript 'sc' | "self consistent" | $\underline{A}^{\mathrm{SC}}, \underline{B}^{\mathrm{SC}}$ |
| componerits $\mathrm{i}=1,2,3$ of a vector $V$ | $\hat{b}_{1} \cdot \underline{V}$ | $A_{2}, P_{3}, V_{1}$ |

Table I.B) Symbols for Part I
A - [vector potential, b(1)], $\widetilde{A}-\S I . A ., A-\S I . C .1$, beginning, A-§I.C.l, beginning
$A^{S C}$ - §I.C.1, beginning, $A_{3}-(19)$
$B \equiv|\underline{B}|, \underline{B}-\left[\right.$ magnetic field, (1)], $\widehat{B}-£ I . C .1$, beginning, $\underline{B}^{\text {SC }}-\xi I . C .1$, beginning
$\hat{b}_{1} . \quad(i=1,2,3)-,b(19)$
$e_{s}$ - charge of particle of species $s$
$F_{s}-(30)$

H- \{(A), (species label suppressed)], H’-छ].C.3, beginning, $\mathrm{H}^{\prime}-9$ I.C. 3 , beginning, $\mathrm{H}_{\mathrm{min}}-\mathrm{f}(23)$
$h-\{(7),($ species label suppressed $)\}, h_{1,2}-f(4)$
J • f!2]!
J - groaction
$L_{B, R, S, n}$ - gl.A, beginring
$n\left(X_{0}\right)$ - Iguiding center density (species label suppressed), (23)]
I: $\left(P_{y}, P_{2}\right)-b(4), P_{2}-(A-1), P_{3}-(19)$
$\mathrm{Y}=\left(\mathrm{p}_{\mathrm{X}}, \mathrm{p}_{\mathrm{y}}, \mathrm{p}_{z}\right)-b(4)$
R - (26)
I - temperature (subscripts suppressed)
U-[EI.A, (14)]
V - (5) (species label suppressed)
$v_{i \|}-[\S 1 . A,(19)], v_{1}-[\xi 1 . A,(20)], \tilde{v}_{\| \mid}-\S 1 . C .2$, beginning
$v_{1}$ - 5I.C.2, beginning, $v_{1,2}-(20)$
$h_{i l}$ - [parallel drift velocity, 5 I.C. 5 , end ]
$X_{0}-[\xi I . A,(17)], \tilde{X}_{0}-\xi 1 . C .2$, beginning
$x-$-I.A, beginning
Y - guiding center y coordinate
$y-\xi I . A$
2 - guiding center $z$ coordinate, $\underline{Z}-\mathbf{f}(22)$

2-6I.A
Y - inverse temperature (subscripts suppressed)
n - §l.A, beginning
O - gyrophase
$\mathbb{I}_{\sim}$ - (2), $\Pi_{x x}$ (3) (species label suppressed)
$\Phi$ - slectrostatic potential, $\bar{\Phi}-5 I . C .3$, beginning
$\Omega$ - grofrequency, f(21)

PART I1
Table II.A) Conveitions
Sub- and superscripts
Meaning
Examples
$\ell$
order in e expansion $K=\sum_{\ell} E^{\ell} K_{\ell}$
m
order in $\eta$ expansion $K_{\ell}=\sum_{m} \eta^{m} K_{\ell m}$
n
number of perturbative trans fonnations applied
$\mathrm{H}^{\mathrm{n}}$
overbar
time average over umperturbed trajectory
$\bar{h}_{3}, \bar{H}_{1}$
inverted carot
Lie operator
$\stackrel{\rightharpoonup}{G}, \dot{H}_{1}$

Table II.B) Symbols for Part II
A - [vector potential, (2),(3)], $A_{\mu}$ - [vector potential,(2),(3)], $A_{0}-f(100)$
a-f(49), $\mathbf{a}-f(49)$
$\underline{B}$ - $\left\{\right.$ magnetic field, (41)\}, $B^{\mu}$ - [magnetic field,(4)\}, $\underline{B}_{0}-f(100)$
$\hat{b}_{j}-\hat{b}(99), b_{ \pm}-(104)$
$C_{\text {mn }}-f(81)$
$c_{1,3}-f(53), c_{a, p}-f(66), c_{s}-f(69)$
$\mathrm{D}_{2,4}-(86)$
$\mathrm{d}_{0,2,4}-\mathrm{f}(53)$
E - [particle energy, (71)], $E(x)-[$ electric field $\left.], E_{1}-f(9]\right), E_{0}-f(100)$
$e_{w}-b(2), e^{\mu}-b(2)$
$G_{n}-(22), \dot{G}_{n}-(22), \dot{G}-(20)$
$g^{L i}$ - Imetric tensor, 5 II.CJ, $g_{m}-(48 b), g^{\mu}(\mu=a, B, \phi)-b(2)$
H - (see also Table II.C for all symbols $H, h$ ):
H - [particle Hamiltonian, (5) ], $\mathrm{H}^{\prime}-(16), \mathrm{H}^{\circ}+5$ §II.B, §II. $\mathrm{H},(35) \mathrm{J}, \mathrm{H}^{\mathrm{n}}-(36)$, $\dot{H}_{o b}-(39), \dot{H}_{o C^{-(39)}}$
$h$ - EII.B, $h^{\prime}-\left[\right.$ §II.E.1, beginaing, (48a)], $h^{0}$ [§II.B,(50)], $h^{n}-(19)$
$J$ - groaction, $J^{0}-f(1), J_{0}-f(91)$
$j$ - $511 . \mathrm{B}, \mathrm{j}^{0}-51 \mathrm{I} . \mathrm{B}$
k - (see also Table II.C for all symbols $k, k$ ):
$k$ - I guiding center Haniltonian, (89)], $\mathrm{K}_{\ell m}-(1), \mathrm{K}^{n}-(36)$
$\mathrm{k}^{\mathrm{n}}$ - (19)
$L_{B}$ - magnetic modulus shear length, $L_{B} \equiv \min \left(L_{s}, L_{B}\right), L_{s}$-shear scale length, $L_{1, \|}-$ §II.A, beginning
$P_{b}$ - Table II.C, $P_{\phi}-\left\lceil\S I I . D\right.$, beginning, Taole II.C], $P_{b}^{\circ}-[\S I I . B$, Table II.C J
$p_{x}-f(1), p_{\mu}(\mu=\alpha, \beta, \phi)-$ SII. $C$
$q^{\mu}(\mu=\alpha, \beta, \phi)-5$ II.C

$$
\begin{aligned}
& R-[5 I I . B,(22)], R_{n}-[\xi I I . B,(22)], R_{0}-\xi I I . B, R^{\prime}-5 I I . B \\
& \text { S - [5II.B,10] } \\
& T-f(1), T_{n}-\S I I . B \\
& V^{-(6)}, V^{\prime}-(75), V_{m}-(48 c) \\
& v_{B}-(117), v_{E}-(103), v_{i}-f(126), v_{0 i}(i=2,3)-f(100), v_{01}-b(122) \\
& y_{-0}-f(100)
\end{aligned}
$$

$$
\begin{aligned}
& x_{0}-b(99), \dot{x}_{20^{-}}-(117), \dot{x}_{30^{-}}-(118) \\
& x-f(1), x_{j}-b(99), \bar{x}_{o}-f(112), \bar{x}_{20}(119) \\
& y_{m}-(63) \\
& \underline{2} \text { - (see table II.C for all symbols } \underline{z}, \underline{z} \text { ) } \\
& \alpha-[f(1), \S I I . C], \alpha_{b}-\left[' \text { banana center," §II.C,end], } \alpha_{0}\right. \text {-[5II.C, end(47], } \\
& \bar{\alpha}_{0}-(B 1), \alpha_{0}^{n}-f(75) \\
& \text { B-[5II.B,§II.C] } \\
& \delta x_{0}-(111), \delta x_{1}-(112), \delta x_{20^{-(119)}, \delta x_{21}-(120), \delta x_{31}-(121), \delta p_{\beta}-(14), ~}^{\text {, }} \\
& \delta p_{\beta 1}-(72), \delta \alpha-[f(47),(49)], \delta \alpha_{0}-(64), \delta \alpha_{1}-(65), \delta \beta-(15), \delta \beta_{0}-(71) \\
& \delta \beta_{1}-(72) \\
& \text { ع - SII.A, beginning } \\
& \text { n-5II.A, beginning } \\
& \theta \text { - gyrophase, } \theta^{0}-f(1) \\
& \theta \text { - 5II. } \mathrm{B}_{8} \theta^{\circ} \text { - }- \text { II. } \mathrm{B}
\end{aligned}
$$

$K-f(104), K_{B, S}-f(104), \kappa_{E}-(109)$
$\lambda \equiv\left(\lambda_{1}, \lambda_{2}\right)-5 I J . D$, beginning, $\bar{\lambda}_{j} \cdot(B 3), \lambda^{0, M, \lambda_{-}}$§II.D, beginning
$\downarrow$ - magnetic moment
p - [particle gyroradius, §li.A]
© - |electrostatic potential,(6)]
$\Phi$ - [toroidal angle, §II.C]
$\Omega$ - girofrequency, $\left.\Omega_{0}-\mid f(40), f(49)\right], \Omega_{2}-(90)$
cio - bounce frequency

Table II.C) Notat jonal Groups

1) Hamiltonians: In general, symbols H denote Hamiltonians (or parts of Hamiltonians) for the full (2-D) particle motion problem; symbols $h$ demote those for the frozen $-\lambda$ ( $1-D$ ) problem. Superscript $n$ denotes the Hamiltonian after $n$ perturbative transformations. $K^{n}$ and $k^{n}$ denote the angle-independent part of $H^{n}$ and $h^{n}$, respectively, and so the guiding center Hamiltonian $K$, being fully angle-independent, satisfies $K=H^{\infty}=K^{\infty}$.

Each $h^{n}$ is expressible as an expansion in $\Pi$ [cf Eq. (19)], with $m$ denoting the power of $n$ :

$$
h^{n}=h_{0}^{n}+\sum_{m=0}^{\infty} n^{m} h_{m+2}^{n}
$$

Each $H^{n}$ is expressible as an expansion in both $n$ and $\varepsilon$, with indices m and $\ell$, respectively:

$$
H^{n}=\sum_{\ell=0}^{\infty} \varepsilon^{\ell} H_{\ell}^{n} \sum_{\ell, m=0}^{\infty} \varepsilon^{\ell} n^{m} H_{\ell m}^{n}
$$

The original particle Hamiltoman, expressed in terms of the curwilinear coordinates ( $a, E, \Phi$ ) and their conjugate moment a, is denoted b) $H$. The symbol $\mathrm{H}^{\prime}$ denotes a useful non-canonical expression [EG. (16, ) for the Hamiltonian, and $h$ denotes the Hamiltonian for the frozen-: prohlem, before transformation to action-angle variables [Eq. (17)].
2) Lic Generators: The Lie generators for the frozen- $\lambda$ transfomations $k_{n}$ are denoted $G_{n}$; those for the transformations $T_{n}$ of stage two are denoted $H_{n}$. The corresponding Lie operators are denoted $\dot{G}_{n}$ and $\dot{k}_{n}$.
3) Phase variables: The variables for the restricted (1-D) phase space of the frozen- $\lambda$ problem are denoted by the general symbol $\underline{z}$, with a superscript matching that of the corresponding Haniltonian, i.e. $h^{\prime}(\underline{\prime})$, $h^{n}\left(E^{n}\right)$. Analogously, the variables for the full phase space are denoted by $\underset{Z}{z}$. When occurring without superscript, $\underline{z}$ denotes either the variables of the full problem in general (so that any $H^{n}$ may be considered as a function of 2 ), or the specific phase variables of the guiding center Hamiltonian $K \equiv H^{\infty}$. These variables are

$$
\underline{Z}=\left(\theta, J ; b, P_{b}\right),
$$

where we have suppressed the variables of the trivial toroidal degree of freedom, the (constant) canonical angular momentum $P_{\phi}=P_{\phi}$, and its conjugate toroidal coordinate, as done through most of the text. Khen occurring with superscript, i.e. $\underline{z}^{n}, \underline{Z}$ denotes the phase space variables of $\mathrm{H}^{\mathrm{n}}$.

By the symbol 2 , we refer either to the phase variables of the frozen- $\lambda$ problem, described above, or to the variables ( $\alpha, \beta, \phi, p_{\alpha}, p_{\beta}, p_{\phi}$ ) in terms of which the original Hamiltonian $H$ is expressed.

## PRT IIJ

Tahle III-A) Conventions

| Subscripts | Mcaning | Exargles |
| :---: | :---: | :---: |
| 5 | species s | $\mathrm{f}_{\mathrm{s}}$, " ${ }_{\mathrm{s}}$ |
| 5 | $\begin{aligned} & \text { contribution from part } f_{s j} \\ & \text { of distribution } f_{s} \end{aligned}$ | $*_{s j}, \bar{\gamma}_{s j}$ |
| ; | harmonic number | $H_{\& S}, M_{\ell}$ |
| $i_{2}, 1$ | (formally) real and maginary parts | $k_{\text {oe }}=k_{\text {Roe }}+i^{\text {roe }}$, |
|  |  | $\Gamma_{1}=\Gamma_{1 R}+i \Gamma_{1 I}$ |

Tatile III.B) Simbols for Part III
A (20), A-f(126), A'f(126)
E-(20), B-f(126)
C - (20), C-f(126)
$U_{G}-(15), D_{B}-b(16)$
$d_{k}$ - [total derivative w.r.t. $\left.k,(16)\right]$
$E_{0}-(67), e-f(48)$
$F_{B 1, P_{2}}$ (7?)
$f_{n}-(42), f_{s}-(48), f_{s j}-b(68)$
$\bar{g}_{0,1}-(82), g_{e, \ell}-b(125)=g_{R_{e}}-(120)$
$H-b(48), H^{\prime}-b(68), h^{\prime}-b(68), H_{n}(\xi)-(34)$
$\underline{I}-b(48), I-(51), I_{1}-(96), I_{2,3}-(118), I_{\ell}-(87)$
$\mathrm{J}-\mathrm{b}(48), \ell_{\ell}^{-5 I I I . C .2}, J_{\ell}-(62), \tilde{J}_{\ell}-(49)$

$$
\begin{aligned}
& K-(11), K_{d}^{-(47)}, K_{d . e}{ }^{-[f(29),(42),(46)], K_{d . e .}^{(n)}-(42), K_{h}-b(64), ~} \\
& K_{\text {n. . . }}-(41), K_{o . m .}-b(64), K_{s}-[(10),(54),(56)], \bar{K}_{s}-(48 a), \bar{K}_{s A, B}-[f(25), \\
& \text { (48) }], K_{S A^{-}}(56), K_{s B^{-}}[(52),(54)], K_{s j}-(76), K_{0}-[f(37),(119)] \\
& k-(10), k^{\prime}-(10), k-(53), \hat{k}-f(50), k_{y, z^{-}}(2), k_{2,3}-[(58 a),(68)], k_{3}^{\prime}-(75) \\
& L_{B, B, n, S} \quad\left[\mathrm{end} \text { of SIII.A], } L_{p}-f(109), L_{Q}-(37), L_{\phi}-b(30), L_{o e}-(106)\right. \\
& \underline{\ell}=\left(\ell, \ell_{y}, \ell_{z}\right)-(48), \ell_{\max }-(122) \\
& \mathrm{M}_{\mathrm{m}}-(32) \\
& m_{s} \text { - mass of particle of species } s \\
& N(\alpha)-(31) \\
& n-[\text { mode index, } f(41)], n\left(X_{0}\right)-f(72), n_{s}\left(X_{0}\right)-(81), n^{\prime}-f(142) \\
& \underline{P}=\left(P_{y}, P_{z}\right)-b(2), P_{2,3}-f(69) \\
& Q-(21), Q_{d}-b(47), Q_{R, 1^{-f}}(31), Q^{\prime}-(22) \\
& \text { R - (82), } \underline{R}-b(49) \\
& r-b(140), \delta r-b(49) \\
& S \cdot(5), S_{i B}-(26), S_{v, s^{-}}(5), S_{<}-\{b(17),(26)], S_{>}-(26) \\
& \text { s - species label } \\
& \mathrm{T}_{\mathrm{s}} \text { - temperature of species } \mathrm{s} \\
& u-b(67), u_{E}-[(70),(86)] \\
& v_{B}-[(73),(I I-117)], v_{E}-f(72), v_{s} \text {-thernal velocity of species } s \text {, } \\
& \left.v_{\mathrm{cj}} \text { - [thermal velocity of component } j \text { of species } s, f(72)\right] \text {, } \\
& v_{01}-\left[\mathrm{b}(67), \text { §II.K. 3] }, v_{02}-\lceil\mathrm{b}(67), 5 I I . K 3], v_{3}-[\mathrm{b}(51), 5 I I . K .3]\right.
\end{aligned}
$$

$$
\begin{aligned}
& W(z)-(88), W_{l 5}-(92), W_{R, I^{\prime}}-f(96) \\
& w_{\|}-[\mathrm{b}(67), 5 \mathrm{I} . \mathrm{C} .6,(\text { species label implicit)] } \\
& x_{0}-(4), \dot{x}_{20}-(73),{\stackrel{\circ}{x_{30}}}^{-(73)} \\
& \underline{x}-(1), \underline{x}^{\prime}-(1), x-(2), x^{\prime}-(3), x_{0}-(28), \bar{x}_{0}-[(58), f(I I-112)], x_{1}-f(142), \\
& x_{2}-b(147), \bar{x}_{20}[(58),(I I-119)] \\
& \text { r - b(48) } \\
& Z-b(48), Z(z)-(89), Z-f(48) \\
& z_{\ell S}-(87) \\
& \alpha-[\mathrm{b}(31),(45)], \alpha_{\mathrm{I}} \equiv \operatorname{Im} \alpha \\
& \text { B - (144) } \\
& \Gamma-(18), \Gamma_{R, I^{-}} f(19), \Gamma_{1,2^{-(19)}, \Gamma_{1 R^{-}}[f(19),(64)], \Gamma_{1 I^{-}}-f(19)} \\
& \gamma \text { - growth rate, } \gamma_{j}-f(72), \gamma_{m x}-(139), \gamma_{s}-(80), \bar{\gamma}_{s j}-(72) \\
& \Delta l-f(96), \Delta w_{\|}-b(143) \\
& \partial_{x, k} \text { - partial derivative w.r.t } x \text { or } k, \vec{\partial}_{x} \equiv \partial_{x}, \stackrel{\partial}{x}^{x} \text {-partial derivative } \\
& \text { w.r.t } x \text { acting to the left, } \delta / \delta f(x) \text {-functional derivative w.r.t. } \\
& \text { function } f(x), \delta x_{0}-[f(57),(I I-111)], \delta x_{20}-[f(57),(I I-130)] \text {, } \\
& \delta y-b(49), \delta z-b(49) \\
& \varepsilon_{Q l}-b(116), \varepsilon_{p l}-f(110), \varepsilon_{\phi l^{-}}-(101)
\end{aligned}
$$

$\theta$ - [gyrophase, $\mathrm{b}(48)], \theta_{\mathrm{k}}-(62), \theta_{\mathrm{o}}-(58 \mathrm{~b})$
$\underline{\theta}-b(48)$
$\kappa_{B, B, n, S}-5 I I I . A, k_{f j}-(72), \kappa_{p}-f(118), \kappa_{\phi}-b(25)$
$\Lambda_{\ell}-(92), \Lambda_{\ell i}-[(D 29),(D 30)], \Lambda_{\ell S}[(92), A p p . D]$
$\lambda \equiv\left(\omega, k_{y}, k_{z}\right), \lambda_{s}-[$ Debye length of species $s,(C 18)]$
$\underline{\mu}=\left\{\mu_{1}, \ldots \mu_{N}\right\}-b(23), \mu_{0}-(58 b), \mu_{o}^{\prime}-b(64), \mu_{s}-(87)$
$\rho(\underline{x})-(1), \rho_{i}$-[ion gyroradius, §III.A], $\rho_{\underline{\ell}}(\underline{x} \mid \underline{I})-(48)$
$\sigma-(4), \sigma^{\prime-}(4)$
$\tau$ - (120)
$\left.\phi(\underline{x})-\int(1), \S I I I . A\right], \phi(x)-(2), \phi_{1}-(13)$
$x_{s}$ - susceptibility of species s
$\psi(x)-(5), \psi_{1}-(13)$
$\Omega$ - [gyrofrequency,b(48)], $\underline{\Omega}^{-b(48)}, \Omega_{c}-(59), \Omega_{0}-(59), \Omega_{1,2}-(87)$, $\bar{\Omega}-[\mathrm{b}(128),(151)]$
$\omega$ - (1), $\omega^{\prime}-(69), \omega^{\prime} \equiv \omega_{e}^{\prime \prime}-[124, b(143)], \omega^{\prime \prime}-(75), \bar{\omega}-(127), \omega_{\star} \equiv \omega_{*_{i}}-(126)$, $\omega_{*_{*}}^{\prime}-(123), \omega_{*_{s}}-(84), \omega_{*_{s j}}-(69), \omega_{e}-[$ electron plasma frequency, (122], $\omega_{\text {ISEC }^{-}}(120), \omega_{\text {ISEC }}{ }^{-(120)}, \omega_{\ell}-\mathrm{b}(143)$
[Note: Species label $s$ has been suppressed in the frequencies $\Omega$, $\Omega_{c}$, $\left.\Omega_{0}, \Omega_{1,2}, \omega^{\prime}, \omega^{\prime \prime}.\right]$

Table III.C) Notational Groups

1) Scale lengths $L$, reciprocal scale lengths $k$, and inhonogeneity parameters of expansion $\varepsilon$ and $n$ :
$L_{y} \equiv$ scale length of any quantity $y(x)$, e.g. $L_{p} \equiv$ plasma scale length, $L_{\underline{B}} \equiv$ scale length for variation of magnetic field.

$$
\begin{aligned}
& \kappa_{y} \equiv L_{y}^{-1} \\
& \varepsilon_{y l} \equiv \ell / k_{2} L_{y}, n_{y} \equiv \rho_{i} / L_{y}
\end{aligned}
$$

2) Contributions $K$ to the dispersion equation:

Subscripts denote origin or function of term:
V - vacurm, s - contribution of species s , sj - contribution from portion $f_{s j}$ of distribution $f_{s}, d$ - well depth of $K(x)$, for localizing modes, d.e. - contribution due to solving a differential equation, $h$ - contribution from assuming small gyroradius (harmonic) particle orbits, n.l nonlocal contribution, o.m. - corrections due to anharmonic orbit modification effects, 0 - zero-order approximation to $K$, for determining mode localization, <- nearly local portion of K .

## REFERENCES

1. P. J. Channell, Phys. Fluids 19, 1541 (1976).
2. H. Grad, Phys. Fluids 4, 1366 (1961).
3. A. N. Kaufman, (private discussions).
4. M. Abromowitz and I. A. Stegun, Eds., Handbook of Mathematical Functions (National Bureau of Standards, Wash., D. C., 1967), p. 260.
5. H. E. Mynick, W. M. Sharp, and A. N. Kaufman, "Realistic Vlasov Slab Equilibria with Magnetic Shear," Phys. Fluids, (to be published).
6. C. S. Gardner, Phys. Rev. 115, 791 (1959).
7. David P. Stern, Goddard Space Flight Center preprin: X-641-71-56, (2/71).
8. T. G. Northrop and J. A. Rome, Phys. Fluids 21, 384 (1978).
9. A. J. Dragt and J. M. Finn, J. Math. Phys. 17, 2215 (1976).
10. H. Goldstein, Classical Mechanics, (Addison-Wesley, Reading, Mass., 1950), Sec. 8.1.
11. L. D. Landau and E. M. Lifshitz, Mechanics, Third Edition, (Pergamon Press, Oxford, 1976), Sec. 50.
12. B. McNamara and K. J. Whiteman, J. Math. Phys. 8, 2029 (1967).
13. J. R. Cary, 'Hamiltonian Perturbation Theory Using Lie Transforms," LBL Report \#6350, (1/78).
14. W. M. Nevins, (private discussions).
15. A. N. Kaufman, Phys. Fluids 3, 610 (1960).
16. See, for example, F. L. Hinton and R. D. Hazeltine, "Theory of Plasma Trnasport in Toroidal Confinement Systems," Reviews of Modern Physics 48, No. 2, Part 1, (4/76).
17. H. L. Berk, R. R. Dominguez, J. Plas. Phys. 18, Part 1, 31 (8/77).
18. D. W. Ross, S. M. Mahajan, Phys. Rev. Lett. 40, 324 (1978).
19. S. M. Mahajan, D. W. Ross, "Finite Beta Drift Waves in a Sheared Magnetic Field," (to be published).
20. R. D. Hazeltine, D. W. Ross, !hiversity of Texas Report No. FRCR 154, (10/77).
21. M. J. Gerver, C. U. Birdsall, A. B. Langdon, D. Fuss, Phys. Fluids 20, 291 (1977).
22. H. L. Berk, D. L. Book, Phys. Fluids 12, 649 (1969).
23. L. D. Pearlstein, H. L. Berk, Phys. Rev. Lett 23, 220 (1969).
24. See for example E. Merzbacher, Quantum Mechanics, (John Wiley and Sons, Inc., New York), Ch. 5.
25. A. N. Kaufman, T. Nakayama, Phys. Fluids 13, 956 (1970).
26. A. N. Kaufman, Phys. Fluids 15. 1063 (1972).
27. J. G. Cordey, L. G. Kuo-Petravic, M. Petravic, Nuc. Fus. 8, 153 (1968).
28. 29. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, New York, 1965) p. 710.
1. Ibid, p. 925.
2. A. B. Mikhailovskii, A. V. Timofeev, Zh. Eksp. Teor. Fiz. 44, 919 (1963) [Sov. Phys. - JETP 17, 626 (1963)].
3. C. S. Liu, D. K. Bhadra, Phys. Fluids 15, 2288 (1972).
4. L. M. Zelenyi, Sov. J. Plas. Phys. 1, 319 (1975).
5. H. L. Berk (private discussions).
6. H. L. Berk, L. D. Pearlstein, Phys. Fluids 14, 1810 (1971).
7. K. V. Roberts, J. B. Taylor, Phys. Fluids 8, 315 (1965).
8. H. R. Lewis, K. R. Symon, LASL Report LA-UR-77-1967, corrected 9/5/77 (submitted to Phys. Fluids).
9. A. B. Mikhailovskii, Theory of Plasma Instabilities, (Plenum Publishing Corp., New York, 1974), Sec. 1.7.

## FIGURE CAPTIONS

Fig. 1. Reference pseudotrajectory $\tilde{\mathbb{A}}(x)$ modeling the Tormac sheath region, and the corresponding magnetic field $\widetilde{\mathbb{B}}(x)$, for four values of x .

Fig. L. Illustration of the graphical method introduced in Sec. I.C.1. For three particular values of $x$ on the self-consistent pseudotrajectory $\underline{A}^{\text {SC }}(x)$ (labeled 1, 2, and 3) are drawn lines of constant guiding-center position $X_{0}$ in the $\underline{p}$-plane, which lie in the direction of the magnetic field $B$ at those points. The parallel velocity $v_{\|}$at $X_{o}$ is the distance from the pseudotrajectory to any point $\underline{P}$ considered, as shown for the point labeled " c " on line 2.

Fig. 3. Sketch of the form of the effective potential $V(x / P)$ in which a particle with canonical momentum $\underline{\mathrm{P}}$ moves, for points $\underline{\mathrm{P}}$ like point "a" of Fig. 2, which lies on two lines of constant guidingcenter, correponding to the two minima in $\mathrm{V}(\mathrm{x})$. Depending or, the value of the energy $H$, the particle may be confined to one of the two wells, or it may pass through both wells, in the course of an oscillation.

Fig. 4. Pseudotrajectory $A(x)$ for the simple example of magnetic field $\underline{B}(x)$ with constant modulus and shear scale length. For this example $\underline{B}$ is parallel to $\underline{A}$, as are the lines of constant guidingcenter $X_{0}(\underline{P})$, here drawn for the particular value $X_{0}=x_{2}$.

Fig. 5a. The $\underline{P}$ or $\underline{A}$ plane of the ( $\mathrm{H}, \underline{\mathrm{P}})$ space over which distribution $f_{S}$ is defined, showing the reference and self-consistent pseudotrajectories $\tilde{A}$ and $A^{s C}$, a single constant guiding-center line, and a number of sample points ${\underset{-}{A}}$, corresponding to the integration boundaries in Fig. 5b:

Fig. 5b. Slice of the ( $\mathrm{H}, \mathrm{F}$ ) space corresponding to the $X_{0}=$ constant line in Fig. 5a, showing the H-integration boundaries for points $A_{j}$ of Fig. 5a. The curve labeled ' 0 ', corresponding to point $A_{0}=A^{S C}\left(X_{0}\right)$ on the self-consistent pseudotrajectory, marks the boundary of the 'physical region." The curve labeled " 1 ", corresponding to ${\underset{A}{1}}=\tilde{A}\left(X_{0}\right)$, marks the boundary of the "quasiphysical region" (cf. Sec. I.C.2).

Fig. 6. Sketch of contours of constant $f(H, P)$, along an $X_{0}=$ constant slice of the ( $\mathrm{H}, \mathrm{P}$ ) plane, for the loss-cone distribution function of Sec. I.C.3. Outside the quasiphysical region, $f$ is taken equal to zero.

Fig. 7. Numerical results of W. M. Sharp, applying the methods of Part I: (a) Reference and self-consistent pseudotrajectories $\widetilde{A}$ and $\underline{A}^{S C}$ modeling the Tommac sheath. The proximity of $\underline{A}^{S C}$ to $\tilde{A}$ indicates that the goal of resemblance of the self-consistent and the reference magnetic fields has been achieved.
(b) Magnetic field $\underline{B}=\hat{y B} \hat{B}_{y}+\hat{z}_{z}$ and its modulus $|\underline{B}|$ as functions of $x$ (in units of the ion gyroradius $\rho_{i}$ ).
(c) Electrostatic potential $\Phi(x)$. The large values for large
$x / \rho_{i}$ reflect a deficiency of the particular model used for $f_{e}$ (cf. Sec. I.C.6).
(d-f) Contours of constant ion distribution $f_{i}$, at points in the Tormac interior (d), sheath center (e), and sheath exterior (f), for the model of Sec. I.C.6. The resemblance of the physically prescribed reference distributions (left side) and the self-consistent distributions (right side) reflects the achievement of the goal of Part I.

Figs. 8. Sketches of functions arising in the evaluation of the pseudopotential $U(A)$ in Appendix $A$.
Fig. 9. Coordinate system $\left(\hat{b}_{2}, \hat{b}_{3}\right)$ natural to a particular point $\underline{\tilde{A}}\left(\tilde{X}_{0}\right)$ on the reference pseudotrajectory, used in Appendix $A$ to evaluate pseudopotential $U(A)$.

Fig. 10. Axisymmetric coordinate system ( $\alpha, \beta, \phi$ ) adopted in Part II (cf. Sec. II.C).

Fig. 11. Real and imaginary parts of the plasma dispersion function $W(z) \equiv(2 \pi)^{-1 / 2} \int d x x^{-x^{2} / 2}(x-z)^{-1}$, for $z$ on the real axis.
Fig. 12. Sketch of the argunent $z_{\ell^{\prime} i}$ of the dispersion function $W\left(z_{\ell}{ }^{\prime}\right)$, as a function of harmonic number $\ell^{\prime}$, showing that for nonzero shear, a few values of $\ell$ ' may have $z_{\ell ' i}$ sufficiently small that $W\left(z_{\ell ' i}\right)$ is appreciable (cf. Secs. III.C.4, III.D.3).

Fig. 13. Sketch of the scaling with shear $\left(\kappa_{s} / \kappa_{n}\right)$ of the various contributions to the dispersion equation (cf. Sec. III.C.5). To the right of line a the stringent convergence condition
(III-101) is violated. To the right of line $b$ the less stringent convergence condition (IIT-116) is violated. When the convergence condition is satisfied (violated), the former (latter) of Eqs. (III-41) is the appropriate dispersion equation. To the right of line $c$, mode localization condition (III-102) is violated.

Fig. 14. Graphical depiction of the dispersion equation for the IDC instability, studied in Sec. III.D. Instability arises when the sum of terms $B$ and $C$ does not extend down to the $\omega^{\prime}$-independent line -A , in the interval containing $\omega_{*}$.

Figs. 15. Sketches of the two-term (a) and one-term (b) approximations to the full IDC dispersion equation, which is illustrated in Fig. 14. The two-term approximation accurately represents the full equation in the interval containing $\omega_{*}$, in which instability can occur, while the one-term approximation, used in previous analytic studies of the IDC instability, does not. Fig. 16a. Sketch of the factors $\lambda_{e}^{-2} \omega_{\star} / \omega^{\prime}$ and $\lambda_{i}^{-2}$ versus $x$, important in determining the form of the function $K_{0}(x)$ in Fig. $16 b$.
Fig. 16b. Sketch of the function $K_{0}(x)$ (for the negligible-shear regime of Sec. III.D.2), which acts as a potential well, at whose bottom unstable eigenmodes are localized.

Fig. 17a. Sketch of the real and imaginary parts of the factor $-\mathrm{g}_{\mathrm{e}} \equiv W\left(2_{\mathrm{oe}}\right)-1$, important in determining the position and width of unstable eigennodes in the appreciable-shear regimes (Secs. III.D.3, III.D.4).

Fig. 17b. Sketch of the real part $K_{R O}(x)$ of the function $K_{0}(x)$, for shear length $L_{s}$ large enough that $L_{o e} \simeq L_{s}\left(\omega / k_{2} v_{e}\right)$ is somewhat shorter than density scale-length $L_{n}$ (Sec. III.D.3). Unstable eigenmodes are localized at the local maximum near point $x_{2}$ where $k_{3}\left(x_{2}\right)=0$, in Berk-Pearlstein fashion. Away from this point, the mode is strongiy damped through $K_{I O}$ by electron Landau danping.

Fig. 18. Depiction of the gyroorbit of a particle passing through a region of mode localization, in the regime in which the particle gyroradius $\rho_{s} \approx \bar{x}_{0}$ is much greater than the mode localization widtn $L_{\phi} \approx \alpha^{-1 / 2}$. In this regime, one may approxinate the particle trajectory in the mode localization region by a straight line (cf. Appendix D).


XBL 78\|-12796
Figure 1


Figure 2


XBL 7811-13300

Figure 3


Figure 4


X8L 7811-12793
Figure 5a


Figure 5b


XBL 7811-12791
Figure 6


Figure 7a



XBL 78II-12799

Figure 7 b (top), 7 c (bottom)


XBL 7811-13302
Figure 7d(top), 7e(middle) and 7 f (bottom)



Figure 8a(top), 8b (botton)


Figure 9


XBL 7811-12797

Figure 10

Figure 11


XBL 7812-13364
Figure 12


XBL 7812-13365
Figure 13


Figure 14
-208-



XBL 7812-13368
Figure 15a(top), 15b (bottom)




XBL 7812-13369
Figure 17a(top), 17 b (bottom)


XB1-7812-13366
Figure 18


[^0]:    This report was psepared as an accounl of work aponoted by the Uniled Slates Govemment. Neither the United Sisies rion the United States Depatment of Energy, nor any of their employees, nor any of their contractors, subcontractions, or their employes, mand
    any waranty, express or implied، or asumes any ligal any warrinty, express or implied, or asom, completeness
    liability or responsibilty for the akentacy, or usfuleses of any information, apparalus, produet or prosess disclaved, or feprecents that its use would not Infruge privately owned rights.

