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Equilibrium and Media Exchange in a Convex Trading Post Economy with Transaction Costs

BY

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Equilibrium and Media of Exchange in a Convex Trading Post Economy with Transaction Costs

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PRELIMINARY: NOT FOR QUOTATION

"[An] important and difficult question...[is] not answered by the approach taken here: the integration of money in the theory of value..."
—— Gerard Debreu, *Theory of Value* (1959)

Abstract

General equilibrium is investigated with N commodities traded at $\frac{N(N-1)}{2}$ commodity-pairwise trading posts. Trade is a resource-using activity recovering transaction costs through the spread between bid (wholesale) and ask (retail) prices (quoted as commodity rates of exchange). Budget constraints are enforced at each trading post separately implying demand for a carrier of value between trading posts, commodity money. Existence of general equilibrium is established under conventional convexity and continuity conditions while structuring the price space to account for distinct bid and ask prices. Trade in media of exchange (commodity money) is the difference between gross and net inter-post trades.

JEL Classification: C62, E40

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1 Introduction

It is well-known that the Arrow-Debreu model of Walrasian general equilibrium cannot account for money. Professor Hahn (1982) writes

"The most serious challenge that the existence of money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. A first, and...difficult...task is to find an alternative construction without...sacrificing the clarity and logical coherence ... of Arrow-Debreu."

This paper pursues development of foundations for a theory of money based on elaborating the detail structure of an Arrow-Debreu model. The elementary first step is to create a general equilibrium where there is a well defined demand for a medium of exchange — a carrier of value between transactions. This is arranged by replacing the single budget constraint of the Arrow-Debreu model with the requirement that the typical household or firm pays for its purchases directly at each of many separate transactions. Transactions take place at commodity-pairwise trading posts. Then a well-defined demand for media of exchange (commodity monies, not necessarily unique) arises endogenously as an outcome of the market equilibrium. Media of exchange are characterized as the carrier of value between transactions (not fulfilling final demands or input requirements themselves), the difference between gross and net trades [1].

1.1 Structure of the Model

Trade takes place at commodity pairwise trading posts (Cournot (1838), Shapley and Shubik (1977), Walras (1874)) with budget constraints (you pay for what you get in commodity terms) enforced at each post. Prices — bid (wholesale) and ask (retail) — are quoted as commodity rates of exchange. Trade across trading posts is arranged by firms, typically buying at bid prices and selling at ask prices, incurring transaction costs (resources used up in the transaction process) and recouping them through the bid/ask spread. Market equilibrium occurs when bid and ask prices at each trading post have adjusted so that all trading posts clear.

[1] The present model proposes a foundation for a theory of (commodity) money as a medium of exchange, alternative to the fiat money models of overlapping generations, Wallace (1980), and of search, Kiyotaki and Wright (1989). There is a separate and independent family of issues regarding how to accommodate — in the Arrow-Debreu setting — an intrinsically worthless fiat money trading at a positive value and used as a common medium of exchange. The rationale is that taxes payable in fiat money provide for a positive value, and low transaction cost ensures use as medium of exchange, Goldberg (2005), Starr (2003A, 2003B). The literature on general equilibrium with transaction costs includes *inter alia* Foley (1970), Hahn (1971), and Starrett(1973).

1.2 Structure of the Proof

The structure of the proof of existence of general equilibrium follows the approach of Arrow and Debreu (1954), Debreu (1959), and Starr (1997). The usual assumptions of continuity, convexity (traditional but by no means innocuous in this context), and no free lunch/irreversibility are used. The price space at a trading post for exchange of one good at bid price for another at ask price is the unit 1-simplex, allowing any possible nonnegative relative price ratio. The price space for the economy as a whole then is a Cartesian product of unit 1-simplices. The attainable set of trading post transactions is compact. As in Arrow and Debreu (1954), the model considers transaction plans of firms and households artificially bounded in a compact set including the attainable set as a proper subset. Price adjustment to a fixed point with market clearing leads to equilibrium of the artificially bounded economy. But the artificial bounds are not a binding constraint in equilibrium. The equilibrium of the artificially bounded economy is as well an equilibrium of the original economy.

1.3 Conclusion: The medium(a) of exchange

The general equilibrium specifies each household and firm's trading plan. At the conclusion of trade, each has achieved a net trade. Gross trades include trading activity that goes to paying for acquisitions and accepting payment for sales rather than directly implementing desired net trades. It's easy to calculate gross trades and net trades at equilibrium. For households, the difference — gross trades minus net trades — represents trading activity in carriers of value between trades, media of exchange (perhaps including some arbitrage). Since firms perform a market-making function within trading posts, identification of media of exchange used by firms is not so straightforward. After netting out intra-post trades, the remaining difference between inter-post gross and net trades represents the firms' trade flows of media of exchange. In some examples (see Starr (2003A, 2003B))the medium of exchange may be a single specialized commodity (the common medium of exchange). The approach of the present model is intended to provide a Walrasian general equilibrium theory of (commodity) money as a medium of exchange[2]. It is sufficiently general to include both a single common medium of exchange and many goods simultaneously acting as media of exchange.

[2]There is a variety of significant issues suitable for a trading post model beyond the scope of this paper. When is there a non-trivial level of commodity money trade? There are examples where there is no monetary trade in equilibrium: Pareto efficient endowment or prohibitively high transaction costs. Conversely Starr (2003A, 2003B) provides examples of active monetary trade in equilibrium. What are the properties of equilibrium media of exchange? What are the efficiency properties of trading post equilibria with media of exchange? Are equilibria Pareto ranked? Hahn (1971) and Starrett (1973) discuss efficient and inefficient equilibria in a sequence economy. Do those results generalize to a trading post economy?

2 Trading Posts

There are N tradeable goods denoted 1, 2, ..., N. They are traded for one another pairwise at trading posts. {i, j} (or equivalently {j,i}) denotes the trading post where goods i and j are traded for one another. There are $\frac{\mathrm{N(N-1)}}{2}$ distinct trading posts.

3 Prices

Goods are traded directly for one another without distinguishing any single good as 'money'. Prices are then quoted as rates of exchange between goods. We distinguish between bid (selling or wholesale) prices and ask (buying or retail) prices. Thus the ask price of a hamburger might be 5.0 chocolate bars and the bid price 3.0 chocolate bars. Note that the ask price of a chocolate bar then is the inverse of bid price of a hamburger. That is, the ask price of a chocolate bar is 0.333 hamburger and the bid price of a chocolate bar is 0.2 hamburger.

Let Δ represent the unit 1-simplex. At trading post {i, j}, the (relative) ask price of good i and (relative) bid price of good j are represented as $p^{\{i,j\}} \equiv (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \in \Delta$. In a (minor) abuse of notation, the ordering of i and j in the superscript on p will matter. $p^{\{j,i\}} \equiv (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \in \Delta$. Thus there are two operative price 1-simplices at each trading post. The full price space then is $\Delta^{N(N-1)}$, the $N(N-1)$-fold Cartesian product of Δ with itself; its typical element is $p \in \Delta^{N(N-1)}$.

4 Budget Constraints and Trading Opportunities

The budget constraint is simply that at each pairwise trading post, at prevailing prices, in each transaction, payment is given for goods received. That is, at trading post {i, j}, an ask/bid price pair is quoted $p^{\{i,j\}} \equiv (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \in \Delta$ expressing the ask price of i in terms of j and a bid price of j in terms of i. A firm or household's trading plan $(y, x) \in R^{2N(N-1)}$ specifies the following transactions at trading post {i,j}: $y_i^{\{i,j\}}$ (at ask prices — retail) in i, $y_j^{\{i,j\}}$ (at ask prices — retail) in j, $x_i^{\{i,j\}}$ (at bid prices — wholesale) in i, $x_j^{\{i,j\}}$ (at bid prices — wholesale) in j. Positive values of these transactions are purchases. Negative values are sales. At each trading post (of two goods) there are four quantities to specify in a trading plan. Then the budget constraint facing firms and households at each trading post is that value delivered must equal value received. That is

$$0 = (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \cdot (y_i^{\{i,j\}}, x_j^{\{i,j\}}) \ , \qquad 0 = (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \cdot (y_j^{\{i,j\}}, x_i^{\{i,j\}}) \qquad \textbf{(B)}$$

(B) says that purchases of i at the bid price are repaid by sales of j at the ask price, purchases of i at the ask price are repaid by sales of j at the bid price.

Given a price vector $p \in \Delta^{N(N-1)}$ the array of trades fulfilling **(B)** is the set of trades fulfilling the $N(N-1)$ local budget constraints at the trading posts. Denote this set

$$\mathbf{M}(p) \equiv \{(y,x) \in R^{2N(N-1)} | (y,x) \text{ fulfills } \textbf{(B)} \text{ at } p \text{ for all } i,j = 1,...,N, i \neq j\}$$

5 Firms

The heavy lifting in this model is done by firms. They perform the market-making function, incurring transaction costs. The population of firms is a finite set denoted F, with typical element $f \in F$. Thus, firm f's technology set may specify that f's purchase of labor (retail) in exchange for i on the {i, labor} market and purchase of i and j wholesale on the {i,j} market allows f to sell i and j (retail) on the {i, j} market. That's how f can become a market maker. If there is a sufficient difference between bid and ask prices so that f can cover the cost of its inputs with a surplus left over, that surplus becomes f's profits, to be rebated to f's shareholders.

5.1 Transaction and Production Technology

Firm f's technology set is Y^f. We assume

P.0 $\quad Y^f \subset R^{2N(N-1)}$

The typical element of Y^f is (y^f, x^f), a pair of $N(N-1)$-dimensional vectors. The $N(N-1)$-dimensional vector y^f represents f's transactions at ask (retail) prices; the $N(N-1)$-dimensional vector x^f represents f's transactions at bid (wholesale) prices. The 2-dimensional vector $y^{f\{i,j\}}$ represents f's transactions at ask (retail) prices at trading post {i,j}; the 2-dimensional vector $x^{f\{i,j\}}$ represents f's transactions at bid (wholesale) prices at trading post {i,j}. The typical co-ordinates $y_i^{f\{i,j\}}, x_i^{f\{i,j\}}$ are f's action with respect to good i at the {i,j} trading post. Since f may act as a wholesaler/retailer/market maker, entries anywhere in $(y^{f\{i,j\}}, x^{f\{i,j\}})$ may be positive or negative — subject of course to constraints of technology Y^f and prices $\mathbf{M}$(p). This distinguishes the firm from the typical household. The typical household can only sell at bid prices and buy at ask prices.

The entry $y_i^{f\{i,j\}}$, represents f's actions at ask prices with regard to good i at trading post {i,j}. $y_i^{f\{i,j\}} > 0$ represents a purchase of i at the {i,j} trading post (at the ask price). $y_i^{f\{i,j\}} < 0$ represents a sale of i at the ask price.

The entry $x_i^{f\{i,j\}}$, represents f's actions at bid prices with regard to good i at trading post {i,j}. $x_i^{f\{i,j\}} > 0$ represents a purchase of i at the trading post (at the bid price). $x_i^{f\{i,j\}} < 0$ represents a sale of i at the bid price.

A firm that is an active market-maker at {i,j} will typically buy at the bid price and sell at the ask price. A firm that is not a market-maker may have to pay retail — like the rest of us — selling at the bid price and buying at the ask price.

In addition to indicating the transaction possibilities, Y^f includes the usual production possibilities. The usual assumptions on production technology apply. For each $f \in F$, assume

P.I Y^f is convex.

P.II $0 \in Y^f$, where 0 indicates the zero vector in $R^{2N(N-1)}$.

P.III Y^f is closed.

The aggregate technology set is the sum of individual firm technology sets. $Y \equiv \sum_{f\in F} Y^f$. It fulfills the familiar *no free lunch* and *irreversibility* conditions.

P.IV [(a)] if $(y, x) \in Y$ and $(y, x) \neq 0$, then $y_i^{\{i,j\}} + x_i^{\{i,j\}} > 0$ for some i, j.
[(b)] if $(y, x) \in Y$ and $(y, x) \neq 0$, then $-(y, x) \notin Y$.

Denote the initial resource endowment of the economy as $r \in R_+^N$. Then we define the attainable production plans of the economy as

$$\hat{Y} \equiv \{(y, x) \in Y | r_i \geq \textstyle\sum_j (y_i^{\{i,j\}} + x_i^{\{i,j\}}) \text{ all } i = 1, 2, ..., N\}$$

Attainable production plans for firm f can then be described as

$$\hat{Y}^f \equiv \{(y^f, x^f) \in Y^f | \text{ there is } (y^k, x^k) \in Y^k \text{ for each } k \in F, k \neq f \text{ , so that}$$
$$[\sum_{k\in F, k\neq f} (y^k, x^k) + (y^f, x^f)] \in \hat{Y}\}.$$

Lemma 5.1: Assume P.0 - P.IV. Then $\hat{Y}$ and $\hat{Y}^f$ are closed, convex, and bounded.
Proof: Starr (1997), Theorem 8.1, 8.2.

5.2 Firm Maximand and Transactions Function

The firm formulates a production plan and a trading plan. The firm's opportunity set for net yields after transactions fulfilling budget is $E^f(p) \equiv [\mathbf{M}(p) - Y^f] \cap R_+^{2N(N-1)}$. That is, consider the firm's production, purchase, and sale possibilities, net after paying for them, and what's left is the net yield. Using the sign conventions we've adopted — purchases are positive co-ordinates, sales are negative co-ordinates — the net yield is then the negative co-ordinates (supplies) in a trading plan that are not absorbed by payments due and the net purchases not required as inputs to the firm. The supplies are subtracted out, so the surpluses enter $E^f(p)$ as positive co-ordinates.

A typical element of these surplus supplies is denoted $(y', x') \in E^f(p)$. In this notation y' and x' are dummies, not actual marketed supplies and demands.

Now consider $(y', x') \in E^f(p)$. In each good i, the net surplus available in good i is $w_i^f \equiv \sum_{j=1}^{N}(y_i'^{\{i,j\}} + x_i'^{\{i,j\}})$ and firm f's surplus is the vector w^f of these co-ordinates. To give this notion a functional notation, let $W(y', x') \equiv w^f$ described here.

There are N-1 trading posts where each good i is traded, at N-1 rates of exchange. The notion of 'profit' is not well defined. In the absence of a single family of well-defined prices, it is difficult to characterize optimizing behavior for the firm. *Fautes de mieux* we'll give the firm a scalar maximand with argument p, y', x'. Firm f is assumed to have a real-valued, continuous maximand $v^f(p; y', x')$. We take v^f to be monotone and concave in (y', x').

The firm's market behavior (without any constraint requiring actions to stay in a bounded range) then is described by

$$S^f(p) \equiv \{(y, x; w) | (y, x) - (y^o, x^o) = (y', x'), \text{ where } (y', x') \text{ maximizes } v^f(p; y', x') \\ \text{subject to } (y', x') \in E^f(p) \text{ and } (y^o, x^o) \in Y^f \text{ and } (y, x) \in \mathbf{M}(p); w = W(y', x')\}$$

The logic of this definition is that $(y', x') \geq 0$ is the surplus left over after the firm f has performed according to its technology and subject to prevailing prices.

It is possible that $S^f(p)$ is not well defined, since the opportunity set may be unbounded. In the light of Lemma 5.1, there is a constant $c > 0$ sufficiently large so that for all $f \in F$, $\hat{Y}^f$ is strictly contained in a closed ball, denoted B_c of radius c centered at the origin of $R^{2N(N-1)}$. Following the technique of Arrow and Debreu (1954), constrained market behavior for the firm will consist of limiting its production choices to $Y^f \cap B_c$. This leads to the constrained surplus

$$\tilde{E}^f(p) \equiv [[\mathbf{M}(p) \cap B_c] - [Y^f \cap B_c]] \cap R_+^{2N(N-1)}$$

Lemma 5.2: Assume P.0 - P.IV. Then $\tilde{E}^f(p)$ is nonempty, upper and lower hemicontinuous.

Proof: Upper hemicontinuity and convexity follow from closedness and convexity of the underlying sets. $0 \in \tilde{E}^f(p)$ always, so nonemptiness is fulfilled. Lower hemicontinuity requires some work.

Let $p^\nu \to p^o$, $(y^o, x^o) \in \tilde{E}^f(p^o)$. We seek $(y^\nu, x^\nu) \in \tilde{E}^f(p^\nu)$ so that $(y^\nu, x^\nu) \to (y^o, x^o)$. If $(y^o, x^o) = 0$, lower hemicontinuity is trivially satisfied. Suppose instead $(y^o, x^o) \geq 0$ (the inequality applies co-ordinatewise). Then in an ϵ-neighborhood of (y^o, x^o), for ν sufficiently large, there is $(y^\nu, x^\nu) \in \tilde{E}(p^\nu)$. (y^ν, x^ν) is the required sequence.

The firm's constrained (to B_c) market behavior then is defined as

$\tilde{S}^f(p) \equiv \{(y, x; w)|(y, x) - (y^o, x^o) = (y', x')$, where (y', x') maximizes $v^f(p; y', x')$
subject to $(y', x') \in \tilde{E}^f(p)$; $(y^o, x^o) \in Y^f \cap B_c$ and $(y, x) \in \mathbf{M}(p); w = W(y', x')\}$

Lemma 5.3: Assume P.0 - P.IV. Then $\tilde{S}^f(p)$ is well defined, non-empty, upper hemicontinuous, and convex-valued for all $p \in \Delta^{N(N-1)}$.

Proof: Theorem of the Maximum, continuity and concavity of v^f.

$(y, x; w) \in \tilde{S}^f(p)$ implies $(y, x) \in B_c$, a bounded set. $w \in R^N_+$ is f's profits. By construction there is $K > 0$ so that w is contained in the nonnegative quadrant of a ball of radius K centered at the origin, denoted $B_K \subset R^N_+$.

5.3 Inclusion of constrained supply in unconstrained supply

Lemma 5.4: Let $p \in \Delta^{N(N-1)}$ such that $\tilde{S}^f(p) \in \hat{Y}^f \times B_K$. Then $S^f(p)$ is well defined and nonempty. Further $S^f(p) = \tilde{S}^f(p)$.

Proof: Recall that B_c strictly includes $\hat{Y}^f$. Then the result follows from convexity of Y^f and $\hat{Y}^f$ and concavity of $v^f(p; y', x')$. The proof follows the model of Starr (1997) Theorem 8.3.

6 Households

There is a finite set of households, H, with typical element h.

6.1 Endowment and Consumption Set

$h \in H$ has a possible consumption set, taken for simplicity to be the nonnegative quadrant of R^N, R^N_+. $h \in H$ is endowed with $r^h >> 0$ assumed to be strictly positive to avoid boundary problems. $h \in H$ has a share $\alpha^{hf} \geq 0$ of firm f, so that $\sum_{h \in H} \alpha^{hf} = 1$.

6.2 Trades and Payment Constraint

$h \in H$ chooses $(y^h, x^h) \in R^{2N(N-1)}$ subject to the following restrictions. A household always balances its budget, sells wholesale and buys retail:

(i) $0 \geq x_i^{h\{i,j\}}$ for all i, j.

(ii) $y_i^{h\{i,j\}} \geq 0$ for all i, j.

(iii) $(y^h, x^h) \in \mathbf{M}(p)$

6.3 Maximand and Demand

Household h's share of profits from firm f is part of h's endowment and enters directly into consumption. When the profits of all firms $f \in F$, w^f in $(y^f, x^f; w^f)$, are well defined, f distributes to shareholders w^f, and h's consumption of good i is

(iv) $c_i^h \equiv r_i^h + [\sum_{f \in F} \alpha^{hf} w^f]_i + \sum_{j=1}^N x_i^{h\{i,j\}} + \sum_{j=1}^N y_i^{h\{i,j\}}$

However, prices p may be such that $S^f(p)$ is not well defined for some f. Then we may wish to discuss the constrained version of (iv), where $\tilde{w}^f$ comes from $(y^f, x^f; \tilde{w}^f) \in \tilde{S}^f(p)$. (iv′) $c_i^h \equiv r_i^h + [\sum_{f \in F} \alpha^{hf} \tilde{w}^f]_i + \sum_{j=1}^N x_i^{h\{i,j\}} + \sum_{j=1}^N y_i^{h\{i,j\}}$

In addition, h's consumption must be nonnegative.

(v) $c^h \geq 0$. The inequality applies co-ordinatewise.

C.I For all $h \in H$, h's maximand is the continuous, concave, real-valued, strictly monotone, utility function $u^h(c^h)$.

h's planned transactions function is defined as $D^h{:}\Delta^{N(N-1)} \times R^{N\#F} \to R^{2N(N-1)}$. Let w denote $(w^1, w^2, w^3, ..., w^f, ..., w^{\#F})$.

$D^h(p, w) \equiv \{(y^h, x^h) \in R^{2N(N-1)} | (y^h, x^h)$ maximizes $u^h(c^h)$, subject to (i), (ii), (iii), (iv) and (v) $\}$. However, $D^h(p, w)$ may not be well defined when opportunity sets are unbounded (when ask prices of some goods are zero) and w may not be well defined for p such that $S^f(p)$ is not well defined for some f. To treat this issue, let $B_K^{\#F}$ be the $\#F$-fold Cartesian product of B_K, and define $\tilde{D}^h{:}\Delta^{N(N-1)} \times B_K^{\#F} \to B_c$.

$\tilde{D}^h(p, w) \equiv \{(y^h, x^h) | (y^h, x^h)$ maximizes $u^h(c^h)$, subject to (i), (ii), (iii), (iv′), (v), and $(y^h, x^h) \in B_c\}$. The restriction to B_c in this definition assures that $\tilde{D}^h(p)$ represents the result of optimization on a bounded set, and is well-defined.

Lemma 6.1: Assume P.0 - P.IV, C.I. Then $\tilde{D}^h(p, w)$ is nonempty, upper hemi-continuous and convex-valued, for all $p \in \Delta^{N(N-1)}$, $w \in B_K^{\#F}$. The range of $\tilde{D}^h(p, w)$ is compact. For (p, w) such that $|(y^h, x^h)| < c$ for all $(y^h, x^h) \in \tilde{D}^h(p, w)$, it follows that $\tilde{D}^h(p, w) = D^h(p, w)$.

Proof: Theorem of the Maximum. Concavity of u^h, convexity of constraints sets defined by (i)-(v) or by (i),(ii),(iii), (iv′), (v). Equivalence of $\tilde{D}^h(p, w)$ and $D^h(p, w)$ follows the pattern of Starr (1997) Theorem 9.1(b).

7 Excess Demand

Let $(p, w') \in \Delta^{N(N-1)} \times B_K^{\#F}$. Constrained excess demand and dividends at (p, w') is defined as

$\tilde{Z} : \Delta^{N(N-1)} \times B_K^{\#F} \to R^{2N(N-1)} \times B_K^{\#F}$.

$$\tilde{Z}(p, w') \equiv \{(\sum_{f \in F} (y^f, x^f) + \sum_{h \in H} \tilde{D}^h(p, w'),\ w^1, w^2, ..., w^f, ..., w^{\#F}) | (y^f, x^f, w^f) \in \tilde{S}^f(p)\}$$

.

Lemma 7.1: Assume P.0 - P.IV, and C.I. The range of $\tilde{Z}$ is bounded. $\tilde{Z}$ is upper hemi-continuous and convex-valued for all $(p, w') \in \Delta^{N(N-1)} \times B_K^{\#F}$.

Lemma 7.2 (Walras' Law): Let $(p, w') \in \Delta^{N(N-1)} \times B_K^{\#F}$. Let $(y, x, w) \in \tilde{Z}(p, w')$. The for each $i, j = 1, ..., N, i \neq j$, we have

$$0 = (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \cdot (y_i^{\{i,j\}}, x_j^{\{i,j\}}) \ , \qquad 0 = (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \cdot (y_j^{\{i,j\}}, x_i^{\{i,j\}}) \qquad \textbf{(W)}.$$

Proof : The element (y, x) of $(y, x, w) \in \tilde{Z}(p, w')$ is the sum of elements (y^f, x^f) of $\tilde{S}^f(p)$ and (y^h, x^h) of $\tilde{D}^h(p, w')$ each of which is subject to **(B)**.

8 Equilibrium

Let Ξ denote a compact convex subset of $R^{2N(N-1)}$ so that $\Xi \times B_K^{\#F}$ includes the range of $\tilde{Z}$. Let $z \in \Xi$, $z \equiv ((y_1^{\{1,2\}}, x_2^{\{1,2\}}), ..., (y_i^{\{i,j\}}, x_j^{\{i,j\}}), ..., (y_{N-1}^{\{N-1,N\}}, x_N^{\{N-1,N\}}))$. Define $\rho : \Xi \to \Delta^{N(N-1)}$

$\rho(z) \equiv \{p^o \in \Delta^{N(N-1)} |$ For each $i, j = 1, 2, ..., N$, $i \neq j$, $p^{o\{i,j\}} \in \Delta$ maximizes $p^{\{i,j\}} \cdot (y_i^{\{i,j\}}, x_j^{\{i,j\}})$ subject to $p^{\{i,j\}} \in \Delta\}$.

Lemma 8.1: ρ is upper hemi-continuous and convex-valued for all $z \in \Xi$.

Define $\Gamma : \Delta^{N(N-1)} \times \Xi \times B_K^{\#F} \to \Delta^{N(N-1)} \times \Xi \times B_K^{\#F}$.

$\Gamma(p, z, w') \equiv \rho(z) \times \tilde{Z}(p, w')$.

Lemma 8.2: Assume P.0 - P.IV, and C.I. Then Γ is upper hemi-continuous and convex-valued on $\Delta^{N(N-1)} \times \Xi \times B_K^{\#F}$. Γ has a fixed point (p^*, z^*, w^*) and $0 = z^*$.

Proof: Upper hemicontinuity and convexity are established in lemmas 7.1 and 8.1. Existence of the fixed point (p^*, z^*) then follows from the Kakutani fixed point theorem. To demonstrate that $z^* = 0$, note lemma 7.2 and strict monotonicity of u^h.

Definition: $(p^*, w^*) \in \Delta^{N(N-1)} \times B_K^{\#F}$ is said to be an equilibrium if

$$(0, w^*) \in \{(\sum_{f \in F} (y^f, x^f) + \sum_{h \in H} D^h(p^*, w^*),\ w^1, w^2, ..., w^f, ..., w^{\#F}) | (y^f, x^f, w^f) \in S^f(p^*)\}$$

where 0 is the origin in $R^{2N(N-1)}$.

Theorem 8.1: Assume P.0 - P.IV, C.I . Then there is an equilibrium $(p^*, w^*) \in \Delta^{N(N-1)} \times B_K^{\#F}$.

Proof: Lemmas 5.4, 6.1, 8.2.

9 Media of Exchange

Let $(y^h, x^h) \in D^h(p, w')$ be household h's $2N(N-1)$-dimensional transaction vector. The x co-ordinates are typically sales (negative sign) at bid prices; the y co-ordinates are typically purchases (positive sign) at ask prices. Then we can characterize h's gross transactions in good i as

$\sum_j y_i^{h\{i,j\}} - \sum_j x_i^{h\{i,j\}} \equiv \gamma_i^h$.

Further, the absolute value of h's net transactions in good i, is

$|\sum_j y_i^{h\{i,j\}} + \sum_j x_i^{h\{i,j\}}| \equiv \nu_i^h$.

The N-dimensional vector γ^h with typical element γ_i^h is h's gross trade. The N-dimensional vector ν^h with typical element ν_i^h is h's net trade vector (in absolute value). $\mu^h \equiv \gamma^h - \nu^h$ is h's flow of goods as media of exchange, gross trades minus net trades.

Since firms perform a market-making function, buying and selling the same good at a single trading post, a more complex view of their transactions is needed to sort out trading flows used as media of exchange. In particular, for firms, we should net out offsetting transactions within a single trading post. Thus for $f \in F$, f's gross transactions in i, netting out intra-post transactions is

$\sum_j |[y_i^{f\{i,j\}} + x_i^{f\{i,j\}}]| \equiv \gamma_i^f$.

The corresponding net transaction is

$|\sum_j [y_i^{f\{i,j\}} + x_i^{f\{i,j\}}]| \equiv \nu_i^f$.

The N-dimensional vector γ^f with typical element γ_i^f is f's gross inter-post trade. The N-dimensional vector ν^f with typical element ν_i^f is h's net inter-post trade vector (in absolute value). $\mu^f \equiv \gamma^f - \nu^f$ is f's flow of goods as media of exchange, gross (inter-post) trades minus net trades.

The total (N-dimensional vector) flow of media of exchange among households and firms is then $\sum_{h\in H} \mu^h + \sum_{f\in F} \mu^f$.

Thus the trading post equilibrium establishes a well-defined demand for media of exchange as an outcome of the market equilibrium. Media of exchange (commodity monies) are characterized as goods flows acting as the carrier of value between transactions (not fulfilling final demands or input requirements themselves), the difference between gross and net trades.

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