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Information-theoretic significance of the Wigner distribution

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A coarse-grained Wigner distribution $p_W(x,\mu)$ obeying positivity derives out of information-theoretic considerations. Let $p(x,\mu)$ be the unknown joint probability density function (PDF) on position and momentum fluctuations x, μ for a particle in a pure state $\psi(x)$. Suppose that the phase part $\Psi(x,z)$ of its Fourier transform $T_F[p(x,\mu)] \equiv |G(x,z)| \exp[i\Psi(x,z)]$ is constructed as a hologram. (Such a hologram is often used in heterodyne interferometry.) Consider a particle randomly illuminating this phase hologram. Let its two position coordinates be measured. Require that the measurements contain an extreme amount of Fisher information about true position, through variation of the phase function $\Psi(x,z)$. The extremum solution gives an output PDF $p(x,\mu)$ that is the convolution of the Wigner $p_W(x,\mu)$ with an instrument function defining uncertainty in either position x or momentum μ. The convolution arises naturally out of the approach, and is one dimensional, in comparison with the *ad hoc* two-dimensional convolutions usually proposed for coarse graining purposes. The output obeys positivity, as required of a PDF, if the one-dimensional instrument function is sufficiently wide. The result holds for a large class of systems: those whose amplitudes $\psi(x)$ are the same at their boundaries [examples: states $\psi(x)$ with positive parity; with periodic boundary conditions; free particle trapped in a box].

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I. INTRODUCTION

Note: To avoid confusion, the word "phase" below is reserved to describe only the phase part of a complex amplitude. Phase is never used to describe "phase space" of statistical mechanics, i.e., joint position and momentum values (x,μ). These are always demarked as "position-momentum" space or (x,μ) space.

Consider a single, mass particle moving in one dimension and in a pure state $\psi(x)$, where the random variable x defines an *intrinsic fluctuation*, i.e., one that would exist even in the presence of a perfect (noise free) detector. The state $\psi(x)$ can be defined, e.g., by the nonrelativistic Schrödinger wave equation. By Fourier transform of $\psi(x)$, this also gives the particle's probability amplitude $\varphi(\mu)$ on intrinsic momentum μ. Let these two amplitude laws be known. (Note that all functions in this analysis depend as well upon the time; for brevity, this is suppressed from the notation.)

Note that $\psi(x)$ and $\varphi(\mu)$ are single-variable, *marginal* probability amplitudes, leaving open the question of the *joint* dependence of the joint fluctuations (x,μ). Quantum mechanics, regarded as a statistical theory, is not fully consistent probabilistically, since it does not make use of, or define, joint or conditional probabilities such as $p(x,\mu)$, $p(x|\mu)$, $p(\mu|x)$, etc. Here we consider the question of what the joint probability density function (PDF) $p(x,\mu)$ should be. How should it relate to $\psi(x)$? Is there a universal PDF $p(x,\mu)$, i.e., a unique function of $\psi(x)$, or should the function depend upon the particulars of the *given* measurement scenario?

Wigner [1] proposed the well-known joint PDF

$$p_W(x,\mu) = \frac{1}{2\pi}\int dz\, e^{-iz\mu}\psi^*(x-\hbar z/2)\psi(x+\hbar z/2) \quad (1)$$

for constructing a measure of the joint fluctuations in position-momentum space. Here $\hbar$ is Planck's constant$/2\pi$ and $i=\sqrt{-1}$. Conversely, given a $p_W(x,\mu)$ obeying (1) the wave function $\psi(x)$ may be reconstructed to within an irrelevant constant phase value. Hence (1) is often considered to be a generally complex quantum formulation that is equivalent to Schrödinger's. It is useful for visualizing the joint evolution of the position-momentum values. Result (1) has also been shown to follow from various operational viewpoints [2] (see as well the extensive bibliography and background for the problem given in [2]).

Unfortunately, for a general state function $\psi(x)$ Eq. (1) is known to incur negative values and, hence, cannot represent a well defined probability law. (The only case that does not incur negatives is the normal case, including squeezed or chirped versions.) This is also consistent with limitations set by the Heisenberg uncertainty principle, according to which precise joint values (X,M) of position and momentum do not exist on the quantum level [3,4].

These limitations are taken to imply that, given a marginal amplitude $\psi(x)$, there is *no single* joint PDF $p(x,\mu)$ that is generally well defined. Again, this is for *intrinsic* fluctuations (x,μ). On the other hand, a *real* measurement scenario, whether experimental or gedanken, is guaranteed to *obey* a well-defined PDF on its total fluctuations (including noise of detection). Therefore, from here on, by (x,μ) we mean *total* fluctuations in position and momentum.

In summary, our view is that a variety of physically meaningful PDFs $p(x,\mu)$ exist, where each is defined for the particular measurement scenario out of which it derives. In general, such a law depends upon the physics of the measurement scenario, both through the state $\psi(x)$ *and* through properties of the detector or other influences on the measurement. This general view was previously taken [5] as well (see note at Ref.). Consequently the PDF $p(x,\mu)$ that we obtain below is limited in validity to a particular gedanken measurement scenario, namely that of particle location in a hologram. An immediate benefit is that, in describing a real measurement, the fluctuations (x, μ) must now describe those in the total experimental measurement, including possible noise of detection. Thus, a theory of measurement emerges.

An *ad hoc* supplement to (1) that forces positivity is to mathematically convolve (1) with a chosen kernel function [2]. This could have a physical origin in coarse graining [2] the space (x, μ). The resulting measure is then taken to describe the joint probability law of the coarse grained space. Coarse graining has so far been proposed, by convolution of (1) with a suitably broad kernel function in x and μ, for example a Gaussian. This results from the well-known result that the convolution of two Wigner distribution functions obey positivity [2]. The minimum amount of coarse graining that suffices to give positive Wigner values has been established [6] as that obeying the Heisenberg uncertainty principle.

This convolution step is usually implemented as an *ad hoc* mathematical addon to $p_W(x, \mu)$. By comparison, our view is, as above, that a valid convolution step should be a consequence of the physics of the particular measurement problem. Indeed, by our approach it will be found to follow as a consequence of determining a particle position in a phase hologram. In its emphasis upon measurement, the approach is reminiscent of a previous analysis [5], which showed that a positive-constrained, Wigner-like PDF results from considering a scattering experiment in the Born approximation. Such Wigner-like PDFs have likewise followed as the outputs of optical heterodyne imaging experiments [7–11] wherein phase object profiles are estimated. See also [1].

Such a convolution is equivalent to blurring at the microlevel of points (x, μ). This blurring will have an important consequence to the point-level J of information. This is that the microlevel information level $J=0$ (see Sec. II F).

We next show that Wigner's function (1) follows from this overall viewpoint. In particular this will be out of the gedanken measurement of a particular scenario. This is of the position and momentum of a particle irradiating a holographic object. The scenario is suggested by past successful Wigner-like answers for optical heterodyning [7–11] methods, which likewise serve to determine phase objects. The analytical approach to be used is that of extreme physical information (EPI) [12–15]. (Note that EPI avoids the use of standard operator quantum mechanics.) The approach is handy in being basically statistical in nature, thereby enabling both quantum and classical statistical effects to be derived. Indeed, regarding the requisite convolution (above), [5] "The idea that in any realistic measurement a detector and a filtering device [as here] are required is not really quantum mechanical in nature."

The EPI approach is well suited to the problem, since (a) it has a strong track record of deriving probability amplitude laws [12–15]; and (b) has derived both quantum and classical PDFs. The result will be the convolution of the quantum Wigner law $p_W(x, \mu)$ with a classical noise distribution on either momentum or position. We emphasize that this is a *one*-dimensional convolution in place of the usual two-dimensional one mentioned above. Also, the statistical nature of the approach will allow the convolution kernel to be interpreted as straightforwardly a PDF on noise of detection.

II. EPI APPROACH

EPI is a general approach for calculating amplitude laws, PDFs, and input-output laws for the fluctuations of unknown systems. The approach centers on the flow of information that occurs during the measurement of a required parameter by an observer. EPI is briefly defined in the introductory paper [12], and fully developed in the books [13–15].

A. Extremum condition

The EPI approach requires solving an extremum problem

$$I - J = \text{extremum} \tag{2}$$

for the system amplitude or probability law. In general I is the Fisher information in the data and J is that in the source. Equation (2) states that the loss of information from source to data is an extreme value (usually a minimum). This condition is the central ansatz of EPI, and has been abundantly verified [13,14] by application. The ansatz is obeyed rigorously [12–14] in the presence of a unitary transformation. Accordingly, an obvious transformation of this type will be utilized. Also, it will be shown below that effectively $J=0$ in this problem, so that the EPI principle Eq. (2) simplifies here to $I=$extremum.

B. Rotation space

All quantum EPI calculations start with a rotation of either coordinates or amplitude functions. Such a rotation is demanded by the length-preserving nature of Fisher information under unitary transformation [13,14]. Note, e.g., that information quantity (11) is a sum (integral) of squares and, hence, invariant under such transformation. As in (11), which is an integral over the space of the amplitude Ψ, the invariance is specifically with respect to amplitude (not PDF) laws. Past examples of such rotations are from four-position space into four-momentum space in deriving the Klein-Gordon, Dirac, and Wheeler–DeWitt equations of quantum mechanics [12–14]; rotation by a complex angle in deriving the Lorentz transformation of special relativity [14]; and rotation by the Weinberg angle in Higgs mass theory [14].

In an unknown scenario, the user has to use physical intuition in choosing the appropriate rotation. However, as an aid, EPI is exhaustive under such rotations [14]: In practice every well-defined rotation leads to a new physical solution for the amplitude function ψ. What, then, should be rotated here?

The physical intuition here is that complex object distributions $Z(x,z)$ tend to be well approximated by their phase parts $\Psi(x,z)$. In fact, it was shown by Kermisch [16] for a class of holograms that the information about photon locations in $Z(x,z)$ is carried by about 78% of the photons that form $\Psi(x,z)$. This is one reason why phase-only holograms are practical as information storage devices. Also, phase distributions are noted for having high local gradients [17], and Fisher information—a local measure of information—is notably sensitive to such gradients. This further agrees with the need for using the channel capacity form of I, i.e., its maximized form (see above).

The preceding two paragraphs suggest that the rotation for the problem be a T_F (Fourier transform) operation on position-momentum space [2],

$$Z(x,z) \equiv \int d\mu\, p(x,\mu)e^{iz\mu} \equiv |G(x,z)|\exp[i\Psi(x,z)]. \quad (3)$$

(Note that $|G|=|Z|$ if Ψ is purely real.) Function $G(x,z)$ will be imposed below, with phase $\Psi(x,z)$ to be found. A two-dimensional phase hologram $\Psi(x,z)$ can in principle be formed optically [18], digitally [19] by generation of computer holograms, or by other means. *The EPI gedanken measurement for this problem will accordingly be that of the emergent position X,Z of a particle from the phase hologram* $\Psi(x,z)$. This will allow $\Psi(x,z)$ to be reconstructed from the principle. However, we emphasize that this is a gedanken measurement: Neither the hologram nor the measurement is actually implemented. Hence, our results express a contingency: *If* such a hologram were formed, and measured, then the EPI principle implies that the unknown joint PDF $p(x,\mu)$ would be the convolution mentioned above. In this way, *the PDF $p(x,\mu)$ is seen to represent an ideal state of information*, as occurs in deriving other laws of physics via EPI [12–14].

The problem of reconstructing a phase hologram has a long history, particularly by the use of heterodyne interferometry [7–11]. Here the latter is replaced by the use of an EPI gedanken measurement, and accompanying use of principle (2).

By the completeness of the Fourier description (3), an EPI problem of estimating $p(x,\mu)$ is thereby replaced with the problem of estimating the function $Z(x,z)$. Once the latter is known, (3) shows that $p(x,\mu)$ may be computed as follows:

$$p(x,\mu) = T_F(Z) \equiv \frac{1}{2\pi}\int dz Z(x,z)e^{-iz\mu}$$
$$\equiv \frac{1}{2\pi}\int dz|G(x,z)|\exp[i\Psi(x,z)]e^{-iz\mu}. \quad (4)$$

However, there is a drawback to this mathematical approach. As mentioned above, Fisher information is an invariant L^2 length in amplitude space and not PDF space. Hence, the rotation (3) in PDF space does not strictly comply with EPI. This is confirmed by the fact that, in depending upon the PDF $p(x,\mu)$, it ignores phase information in the amplitude function whose square is $p(x,\mu)$. Hence the answer we get must be approximate, lying somewhere between quantum and classical physics.

C. Holographic aspect of rotation

We next show that $Z(x,z)$ and its phase part $\Psi(x,z)$ can be represented as holograms in two-dimensional position space. This requires showing that coordinate z is proportional to a position. As usual, position-momentum space is subdivided into elemental cells of sides $\Delta l, \Delta\mu$, with areas $\Delta l\Delta\mu=\hbar/2$ by the Heisenberg principle. Hence, a position determination $l_z=n_z\Delta l$, for an appropriate integer n_z. Combining the last two relations gives $l_z=n_z\hbar/(2\Delta\mu)$. Also, in the exponent of (4), $z\mu$ must be unitless, so that we may express

$$z = n_z/\Delta\mu. \quad (5)$$

Combining the last two relations gives

$$l_z = (\hbar/2)z \quad \text{and} \quad dl_z = (\hbar/2)dz \quad (6)$$

as its differential. Hence $z \propto l_z$, or, the transform space coordinate z is effectively back in position space defining a corresponding position coordinate $(\hbar/2)z$. Since coordinate x is likewise a position, effectively functions $Z(x,z)$ and $\Psi(x,z)$ lie entirely in position space. This facilitates the required position measurement (X,Z), which will now be of the rectangular position in a two-dimensional hologram (see next).

D. Reconstruction step

In the reconstruction step, the phase hologram $\Psi(x,z)$ is illuminated with a uniform plane wave called a reference beam. Here either material particles or photons may be used. To be definite, we choose material particles.

The aim of the gedanken experiment is to measure the ideal joint position values (X,Z) of a randomly chosen particle as it passes through the phase hologram. In practice, the measurement is imperfect, with respective error fluctuations (x,z). The question that EPI will seek to answer is, What phase profile $\Psi(x,z)$ extremizes the acquired information I about position (X,Z) in the data? Once this is known, its use in (4) gives the required PDF $p(x,\mu)$ on position and momentum for the illuminating particle.

As mentioned above, such phase profiles $\Psi(x,z)$ have previously been found using heterodyne approaches [7–11], and these do tend to follow Wigner-type distributions, as required.

The phase law $\Psi(x,z)$ has a corresponding intensity profile

$$P(x,z) \equiv \Psi^*(x,z)\Psi(x,z), \quad (7)$$

where * denotes the complex conjugate. This is also the PDF on particle positions in the phase hologram. It is to be noted that Ψ is in general complex, allowing absorption as well as phase shift.

E. Parameters to be measured

To review, the unknown parameters that are to be gedanken measured are the X position and effective Z position (as above) of a randomly selected particle that passes through the phase hologram $\Psi(x,z)$. These are measured with respective errors x and z, using an instrument that generally suffers from noise. The total measurement errors (x,z) are therefore inclusive of both this noise and the holographic object. We next seek the PDF on these total measurement errors using EPI.

F. Source information J

The general flow of information in an EPI measurement procedure is from the information source to the measurement space,

$$J \rightarrow I. \tag{8}$$

Here, by definition the information source is at the point level (x,z) of the joint position fluctuations. However, there is effectively no given information at that level since data averaging (coarse graining) will be taken in (x,μ) space, as discussed above, and this causes effective data averaging in (x,z) space as well, via the Fourier transform operation (4). That is, coarse graining for the original particle scenario translates into coarse graining in the hologram scenario as well. Hence, in this calculation

$$J = 0. \tag{9}$$

There is effectively no information on the microscale of this gedanken experiment. Note that this coarse graining will be a result of the calculation, not an assumption. That is, the EPI solution to the problem will be self-consistently smeared out sufficiently in momentum or in position *to have* this property.

In general, use of $J=0$ indicates an EPI calculation of lowest precision, level (c) [13,14]. Results must be regarded as approximate or contingent [as with the Wigner answer (1), which is contingent upon coarse graining or a Gaussian $\psi(x)$]. This is the second approximation that is made in the overall approach. The other was the rotation in PDF or energy space rather than in amplitude space (see above).

A check on the assumption $J=0$ will be the calculated I at solution. Since $I=\kappa J$ according to EPI theory, then the solution should obey $I=0$ as well. This is verified in the Appendix A for a wide class of state functions $\psi(x)$.

G. Data information I

With $J=0$, the entire calculation Eq. (2) hinges on the information functional I. The positional errors of the problem are the positions x and $(\hbar/2)z$ [see (6)]. We choose to Wick rotate the latter into an imaginary coordinate $i(\hbar/2)z$ (see the above remarks about rotation and EPI). The Fisher coordinates of the problem are accordingly

$$(x_1,x_2) \equiv (x,i(\hbar/2)z). \tag{10}$$

Note that this rotation is arbitrary, and represents prior knowledge on the part of the observer. As usual in EPI problems, it will be justified on the grounds that the solution is reasonable and gives new insight into the problem. The same Wick rotation is of course commonly used to represent the time coordinate in relativistic effects [20].

With these as Fisher coordinates, quantity I has the significance of being the Fisher information in an attempt at measuring the ideal coordinates $(X,i(\hbar/2)Z)$ of a randomly selected particle in the phase hologram.

The differentials of the Fisher coordinates are $dx_1=dx$, $dx_2\equiv dl_z=i(\hbar/2)dz$ by (6), so that $dx_1|dx_2|=(\hbar/2)dxdz$. Then the Fisher channel capacity information is [12–14]

$$I = \frac{8}{\hbar}\int\int dxdz\left[\left(\frac{\partial\Psi^*}{\partial x}\right)\left(\frac{\partial\Psi}{\partial x}\right)-\left(\frac{2}{\hbar}\right)^2\left(\frac{\partial\Psi^*}{\partial z}\right)\left(\frac{\partial\Psi}{\partial z}\right)\right],$$

$$P = \Psi^*\Psi. \tag{11}$$

The minus sign arises out of squaring the imaginary i in dx_2 ([14], Appendix C). The unknown phase profile $\Psi(x,z)$ that attains extreme information in the data therefore has this level of information I.

III. EPI IMPLEMENTATION

The EPI extremization principle is $I-J=$extremum. Then by (9) and (11), the principle is

$$\int\int dxdz\left[\left(\frac{\partial\Psi^*}{\partial x}\right)\left(\frac{\partial\Psi}{\partial x}\right)-\left(\frac{2}{\hbar}\right)^2\left(\frac{\partial\Psi^*}{\partial z}\right)\left(\frac{\partial\Psi}{\partial z}\right)\right] = \text{extremum}, \quad \Psi = \Psi(x,z). \tag{12}$$

We ignored an irrelevent multiplicative constant. The rest is algebra.

A. Forming Lagrangian

The Lagrangian of the problem is directly the integrand of (12),

$$\mathcal{L} = \left(\frac{\partial\Psi^*}{\partial x}\right)\left(\frac{\partial\Psi}{\partial x}\right)-\left(\frac{2}{\hbar}\right)^2\left(\frac{\partial\Psi^*}{\partial z}\right)\left(\frac{\partial\Psi}{\partial z}\right). \tag{13}$$

B. Semiclassical solution

The general Euler-Lagrange solution obeys

$$\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial\Psi^*/\partial x)}+\frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{\partial(\partial\Psi^*/\partial z)} = \frac{\partial\mathcal{L}}{\partial\Psi^*}.$$

Hence by (13) the Euler-Lagrange solution to this problem obeys

$$\frac{\partial^2\Psi}{\partial x^2}-\left(\frac{2}{\hbar}\right)^2\frac{\partial^2\Psi}{\partial z^2} = 0. \tag{14}$$

This is a wave equation for two traveling waves of "velocities" $\pm\hbar/2$ with z standing in for the "time." Thus, as first shown by Tatarskii [21], the solution is the sum of these waves

$$\Psi = F_1\left(x+\frac{\hbar}{2}z\right)+F_2\left(x-\frac{\hbar}{2}z\right), \quad F_1,\ F_2 \text{ arbitrary}. \tag{15}$$

Then by (3),

$$Z = |G|\exp\left[iF_1\left(x+\frac{\hbar}{2}z\right)\right]\exp\left[iF_2\left(x-\frac{\hbar}{2}z\right)\right] \equiv |G|f_1\left(x+\frac{\hbar}{2}z\right)f_2\left(x-\frac{\hbar}{2}z\right) \tag{16}$$

after defining

$$f_1(x) = e^{iF_1(x)}, \quad f_2(x) = e^{iF_2(x)}. \tag{17}$$

Then by (4) and (16),

$$p(x,\mu) \equiv T_F(Z) = T_F\left[|G|\, f_1\left(x + \frac{\hbar}{2}z\right) f_2\left(x - \frac{\hbar}{2}z\right)\right]. \tag{18}$$

Then by the theorem for the T_F of a product,

$$p(x,\mu) = g(\mu|x) \otimes p_W(x,\mu) \equiv \int d\mu' g(\mu - \mu'|x) p_W(x,\mu'), \tag{19}$$

where $\otimes$ denotes a one-dimensional convolution, the kernel function $g(\mu|x)$ obeys

$$g(\mu|x) \equiv T_F(|G|) \equiv \frac{1}{2\pi}\int dz\, e^{-iz\mu}|G(x,z)|, \tag{20}$$

and $p_W(x,\mu)$ obeys

$$p_W(x,\mu) = T_F\left[f_1\left(x + \frac{\hbar}{2}z\right) f_2\left(x - \frac{\hbar}{2}z\right)\right]$$
$$= \frac{1}{2\pi}\int dz e^{-iz\mu} f_1\left(x + \frac{\hbar}{2}z\right) f_2\left(x - \frac{\hbar}{2}z\right). \tag{21}$$

The probabilistic notation $|$ in $g(\mu|x)$ denotes the conditional "if," that is, $g(\mu|x)$ is the probability density on a random value of momentum μ in the presence of (if) a fixed value of x.

Finally, as noted [5], if we regard $f_1(x)$ and $f_2(x)$ as square integrable, and $p_W(x,\mu)$ as real and normalized, then Eq. (21) takes the particular Wigner form Eq. (1)

$$p_W(x,\mu) = \frac{1}{2\pi}\int dz\, e^{-iz\mu}\psi^*(x - \hbar z/2)\psi(x + \hbar z/2) \tag{22}$$

for the appropriate choice of functions

$$f_1(x) = \psi(x), \quad f_2(x) = \psi^*(x). \tag{23}$$

Equations (19) and (22) are the main results of the paper. These show that the EPI solution for the PDF of the measurement problem is the Wigner function convolved with a kernel function $g(\mu|x)$ along μ. One can regard the Wigner function as the quantum part of the answer, with the convolving kernel function a classical part. The convolution is only along the momentum coordinate μ. Statistically, such a convolution denotes the presence of added classical noise of detection [22] of momentum. Hence, the solution pictures a measurement scenario where the detection is generally imperfect, suffering from noise in the momentum reading. The noise is classically characterized by a PDF $g(\mu|x)$. As above, this means the probability of a random variable value μ, in the presence of a general but fixed value of x. Hence the noise in μ is signal dependent, if we regard fluctuation value x as being the signal. This noise effectively allows coarse graining of momentum space in this problem. The kernel function $g(\mu|x)$, as a measure of detection noise, is often called the instrument function of the measurement experiment. This is, then, the physical significance of g and, by Eq. (20), $|G|$).

This result distinguishes the solution from most past approaches [2] to a practical Wigner function in two ways: Past approaches postulate (not derive, as here) convolution with a kernel function, and in two dimensions rather than the one dimension here. Two-dimensional convolution is well-known to achieve positivity if the kernel function is chosen to be another Wigner distribution (in particular, a Gaussian of sufficient width).

Although of secondary importance, the estimated amplitude law $\Psi(x,z)$ obeyed by the phase hologram is, by (15), (17), and (23),

$$\Psi(x,z) = -i\ln[\psi(x + \hbar z/2)\psi^*(x - \hbar z/2)]. \tag{24}$$

It will be shown next that, alternatively, a one-dimensional convolution *along* x can arise in the representation for $p(x,\mu)$.

IV. ALTERNATIVE CASE OF SPREAD IN x

The preceding analysis indicated that a conditional PDF $g(\mu|x)$ convolved with the Wigner distribution is the answer for the net $p(x,\mu)$. That derivation started with the definition (4) of $p(x,\mu)$, whereby variable z of its spectrum $Z(x,z)$ is integrated over. We can instead choose to work with a representation where the other variable x of $Z(x,z)$ is effectively integrated over, as follows:

$$p(x,\mu) \equiv \frac{1}{2\pi}\int dk Z(k,\mu)e^{-2ikx/\hbar} \tag{25}$$

[cf. Eq. (4)]. Going through the analogous EPI derivation in Secs. II G–III B now gives

$$p(x,\mu) = g(x|\mu) \otimes p_W(x,\mu) \equiv \int dk g(x - k|\mu) p_W(k,\mu) \tag{26}$$

[cf. Eq. (19)], where now

$$p_W(k,\mu) = \frac{1}{\pi\hbar}\int dk e^{-2ixk/\hbar}\varphi^*(\mu + k)\varphi(\mu - k) \tag{27}$$

[cf. Eq. (1)] in terms of the momentum eigenfunction $\varphi(\mu)$ of the system.

The convolution in (26) is again one-dimensional, but now along x. Effectively, this means the presence of noise of detection of position, rather than of momentum as in the preceding. The notation $g(x|\mu)$ signifies a conditional PDF on values of x in the presence of each fixed value of momentum μ. As before, this is generally signal-dependent noise, and also equates to coarse graining of coordinate position space. The kernel $g(x|\mu)$, as a specifier of detection noise, is often called the instrument function of the measurement experiment.

We need to show next that either of the one-dimensional convolution answers (19) or (26) suffices to give a *positive* $p(x,\mu)$. This question seems to have not been addressed before in the voluminous literature on the Wigner distribution.

A. Positivity property of coarse graining in μ

We first treat convolution along momentum coordinate μ, then alternatively along x.

Eq. (19) in Fourier space is

$$Z(x,z) = \hat{p}_W(x,z)|G(x,z)|, \tag{28}$$

where a caret indicates the Fourier transform of the function beneath it. Equations (3) and (20) were also used. By Eq. (22),

$$\hat{p}_W(x,z) = \psi^*(x-\hbar z/2)\psi(x+\hbar z/2). \tag{29}$$

So far function $|G(x,z)|$ is arbitrary. Temporarily let

$$|G(x,z)| \equiv f_X(x)\exp(-\sigma^2 z^2/2), \quad f_X(x) \geq 0, \quad \sigma \text{ large}, \tag{30}$$

with $f_X(x)$ a positive but otherwise arbitrary function. Using Eqs. (29) and (30) in (28) gives

$$Z(x,z) \approx f_X(x)\lim_{\sigma\to\infty}\psi^*(x-\hbar z/2)\psi(x+\hbar z/2)\exp(-\sigma^2 z^2/2). \tag{31}$$

Then by (4),

$$p(x,\mu) = \frac{f_X(x)}{2\pi}\int dz \lim_{\sigma\to\infty}\exp(-\sigma^2 z^2/2)\psi^*(x-\hbar z/2) \times\psi(x+\hbar z/2)\exp(-iz\mu). \tag{32}$$

With σ sufficiently large only integrand values $z\approx 0$ contribute to the integral, so that

$$p(x,\mu) \approx \frac{f_X(x)}{2\pi}dz\psi^*(x)\psi(x). \tag{33}$$

This obeys positivity since $f_X(x)\geq 0$ by (30), since $dz>0$ and because $\psi^*(x)\psi(x)=p(x)$ is a well-defined probability law on position fluctuation x. Expanding out factors $\psi^*(x-\hbar z/2)$ and $\psi(x+\hbar z/2)$ in (32) about the point x would allow one to extend the property of positivity for finite values of σ as well. A minimum necessary grain size σ for achieving positivity for a given state $\psi(x)$ could be found in this way. This has yet to be done.

B. Positivity property of coarse graining in x

Results for convolution along x, instead, are completely analogous. In place of (32), one gets

$$p(x,\mu) = \frac{f_M(\mu)}{2\pi}\int dk \lim_{\sigma\to\infty}\exp(-\sigma^2 k^2/2)\varphi^*(\mu+k) \times\varphi(\mu-k)\exp(-2ikx/\hbar), \tag{34}$$

where $f_M(\mu)\geq 0$. Again taking the $\lim\sigma\to\infty$, only values of $k\approx 0$ contribute, so that

$$p(x,\mu) \approx \frac{f_M(\mu)}{2\pi}dk\varphi^*(\mu)\varphi(\mu). \tag{35}$$

This is again positive.

V. AMOUNT OF RECEIVED INFORMATION

The first tenet of EPI is the variational principle (2), $I-J=$extremum. This was used to form the above solution. The second tenet [12–15] is that $I=\kappa J$ at solution, with κ a constant on the interval $(0,1)$. Here, by (9), $J=0$. Therefore the data information (11) at solution should likewise be zero. This is an important check on the consistency of the theory, and we show in the Appendix that it is satisfied, for a wide class of wave functions $\psi(x)$. These either have the same values at their boundaries or, if the common boundary value is zero, approach zero at the same rate. The simplest example is a free particle in a box of length $2b$, b finite. More general classes include wave functions having positive parity; or wave functions with periodic boundary conditions.

VI. DISCUSSION

This approach has shown that the Wigner law results from the following gedanken experiment: The T_F taken in (3) is of the unknown joint law $p(x,\mu)$. Its amplitude function G is the T_F of a sufficiently coarse graining function g. Its (generally complex) phase Ψ is to be found. This phase hologram is illuminated by a uniform reference beam of particles. One of these is randomly selected and measured for its position as it emerges. For the measured position to contain an extreme level of Fisher information about its ideal position, the joint law $p(x,\mu)$ must obey the convolution of the Wigner law (1) with a one-dimensional PDF on either x or μ. That is, $p(x,\mu)$ must obey results (19), (22), and (26). These show an alternative significance for the Wigner law: *The Wigner distribution represents an optimal joint distribution for purposes of conveying information about two-dimensional particle position in a phase hologram* $\Psi(x,z)$ *constructed out of* $p(x,\mu)$.

Equations (19) and (26) are also interesting in not expressing purely the Wigner law, or even the Wigner law convolved with a two-dimensional kernel function. Rather, they express the Wigner law convolved with a suitable *one-dimensional* kernel. This was shown [Eqs. (33), (35)] to obey the required property of positivity for a PDF under sufficiently coarse graining in one dimension. As discussed (Sec. I), the answer is specific to the given measurement problem, and does not represent all measurement problems. Also, it holds for a class of system states $\psi(x)$ (see the Appendix) and not all. The convolution is due to noise of detection of either momentum or position (but not necessarily both). This noise is the origin of coarse graining in this problem.

It was found that, under coarse graining in either position or momentum space, the law achieves positivity and therefore becomes a legitimate PDF. Hence, the EPI result agrees with the conventional quantum view that, on the point level (x,μ), a joint probability law $p(x,\mu)$ on intrinsic fluctuations does not generally exist, but that when localized averaging can be taken, it does exist. However, the result differs from the conventional view in showing that *the averaging arises naturally out of the measurement process.* It does not arise as merely an *ad hoc* appendage to the Wigner answer. Also, the averaging need not be done over *both* coordinates x and μ. It is done over either *the momentum* coordinate or *the position* coordinate.

Finally, the results (19) and (22) obey self-consistency in obeying the assumption $J=0$ [Eq. (9)] made in their derivation. Information J generally represents the source information for a given problem. Here this is the information that exists at the point level of (x,μ) space. With the ansatz $J=0$, we assumed that such information was not available. Either convolution (19) or (26) bears this out, stating that no such information is present because of the convolution (smearing) operation along either coordinate μ or x.

APPENDIX: SYSTEMS OBEYING $I=0$ AT SOLUTION

By Eq. (24),

$$\Psi(x,z) = -i[\ln \psi(x+\alpha z) + \ln \psi^*(x-\alpha z)], \quad \alpha \equiv \hbar/2. \tag{A1}$$

Differentiating gives

$$\Psi_X = -i[(\ln \psi)'_{x+\alpha z} + (\ln \psi^*)'_{x-\alpha z}], \tag{A2}$$

$$\Psi_Z = -i\alpha[(\ln \psi)'_{x+\alpha z} - (\ln \psi^*)'_{x-\alpha z}]. \tag{A3}$$

The subscript X means $\partial/\partial x$, subscript Z means $\partial/\partial z$, and the notation $(\ln \psi)'_{x\pm\alpha z}$ means $[\partial \ln \psi(w)/\partial w]$ evaluated, respectively, at $w=x\pm\alpha z$. At this point the calculation simplifies if we assume that $\psi(x)$ is purely real. Full complexity is retrieved at the end. Substituting Eqs. (A2) and (A3) into (12) gives just the cross term

$$I = \int\int dxdz (\ln \psi)'_{x+\alpha z}(\ln \psi)'_{x-\alpha z}. \tag{A4}$$

An irrelevant multiplier was again ignored.

It is convenient to change variables,

$$w \equiv x+\alpha z, \quad v \equiv x-\alpha z. \tag{A5}$$

Then $x=2^{-1}(w+v)$, $z=(2\alpha)^{-1}(w-v)$. The Jacobian of the transformation is then $|J(x,z/w,v)|=(2\alpha)^{-1}$, so that (A4) becomes

$$I = \int\int dwdv|J(x,z/w,v)|[\ln \psi(w)]'[\ln \psi(v)]'$$

$$= \int\int dwdv[\ln \psi(w)]'[\ln \psi(v)]' \tag{A6}$$

after use of the particular Jacobian, and as usual ignoring a multiplicative constant. The integral is actually a simple square,

$$I = \left[\int dw[\ln \psi(w)]'\right]^2. \tag{A7}$$

This is easily evaluated as follows:

$$I = \left[\int dw \frac{d}{dw} \ln \psi(w)\right]^2 = [\ln \psi(b) - \ln \psi(a)]^2 = \ln^2\left[\frac{\psi(b)}{\psi(a)}\right], \tag{A8}$$

where a,b are the boundary values of w. With the original boundaries at $x=\pm x_0$, $z=\pm z_0$, Eqs. (A5) show that $b=x_0+\alpha z_0$, $a=-b$, so that

$$I = \lim_{x\to b} \ln^2\left[\frac{\psi(x)}{\psi(-x)}\right], \quad b = x_0+\alpha z_0. \tag{A9}$$

This shows that $I=0$ if $\psi(b)\to\psi(-b)$, i.e., the same value ψ is approached as the particle approaches its boundaries. An example is a system whose ψ has positive parity. Another is where the system is periodic, with period $2b$. A third is not necessarily periodic in all its values, but has boundary values that repeat. A possible complication is where the boundary values are zero, as in bound systems, since $0/0$ is indeterminate. Here what is required is that ψ approach zero symmetrically at the two boundaries of this system. This is obeyed by a wide class of bound systems. Examples are states with positive parity; or, as the simplest example, a free particle in a box of length $2b$, b finite, where $\lim_{x\to\pm a}\psi(x) = \lim_{x\to\pm b}\cos(n\pi x/2b)=+0$, $n=1,3,\ldots$. (Here both boundary values approach zero symmetrically from above.)

We assumed a real $\psi(x)$, for simplicity in deriving (A9). With $\psi(x)$ instead complex, the net I turns out to be the quantity (A9) plus the same expression in ψ^*. The latter expression is zero under the condition $\psi^*(b)=\psi^*(-b)$, which is the same condition as before.

In summary the condition $I=0$ holds, not for all systems, but for a wide class of systems. This is a natural consequence of the overall approach which, as discussed (Sec. I), is specific to the given measurement problem.

[1] E. Wigner, Phys. Rev. **40**, 749 (1932). Some optical examples are in A. W. Lohmann and B. H. Soffer, J. Opt. Soc. Am. A **11**, 1798 (1994) and X.-G. Xia, Y. Owechko, B. H. Soffer, and R. M. Matic, IEEE Trans. Signal Process. **44**(11), 2882 (1996).

[2] N. C. Dias and J. N. Prata, Ann. Phys. **313**, 110 (2004); also e-print hep-th/0402008.

[3] E. B. Davies, *Quantum Theory of Open Systems* (Academic, New York, 1976).

[4] J. Uffinck, Int. J. Theor. Phys. **33**, 199 (1994).

[5] K. Wodkiewicz, Phys. Rev. Lett. **52**, 1064 (1984). We quote: "Is it possible to define a realistic phase-space function that can be *recorded* [our italics] in the laboratory"?

[6] N. D. Cartwright, Physica A **83A**, 210 (1976); L. Diosi, e-print quant-ph/0212103.

[7] A. Wax and J. K. Thomas, Opt. Lett. **21**, 1427 (1996).

[8] L. McMackin, D. G. Voeltz, and M. P. Fetrow, Opt. Express **1**, 332 (1997).

[9] R. Thalmann and R. Dandliker, Proc. SPIE **599**, 141 (1985).

[10] P. Hariharan, Opt. Eng. **24**, 632 (1985).
[11] *Holographic Interferometry*, edited by P.K. Rastogi, Springer Ser. Opt. Sci. Vol. 68 (Springer-Verlag, Berlin, 1994).
[12] B. R. Frieden and B. H. Soffer, Phys. Rev. E **52**, 2274 (1995).
[13] B. R. Frieden, *Physics from Fisher Information* (Cambridge University Press, Cambridge, 1998).
[14] B. R. Frieden, *Science from Fisher Information* (Cambridge University Press, Cambridge, 2004).
[15] *Exploratory Data Analysis Using Fisher Information*, edited by B. R. Frieden and R. A. Gatenby (Springer-Verlag, London, 2006).
[16] D. Kermisch, J. Opt. Soc. Am. **60**, 15 (1970). We quote: "The phase information thoroughly dominates the process" [of image reconstruction]. "Up to 78% of the total image radiance reconstructs exactly the original image. The other 22% consists of [self] convolutions of the original image, and degrades the reconstructed image."
[17] S. Furhapter, A. Jesacher, S. Bernet, and M. Ritsch-Marte, Opt. Express **13**, 689 (2005).
[18] S. A. Benton, Opt. Eng. **19**, 686 (1980).
[19] A. Marquez, J. Campos, and M. J. Yzuel, Opt. Eng. **39**, 1612 (2000).
[20] Use of an imaginary coordinate is by so-called *Wick rotation*. The imaginary time coordinate was invented by H. Minkowski for defining a relativistic four space. A recent use of the Wick rotation is H. Gies and K. Klingmuller, Phys. Rev. D **72**, 065001 (2005).
[21] V. I. Tatarskii, Sov. Phys. Usp. **26**, 311 (1983).
[22] B. R. Frieden, *Probability, Statistical Optics and Data Testing*, 3rd ed. (Springer-Verlag, Berlin, 2001).