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A SQUARE-WELL POTENTIAL

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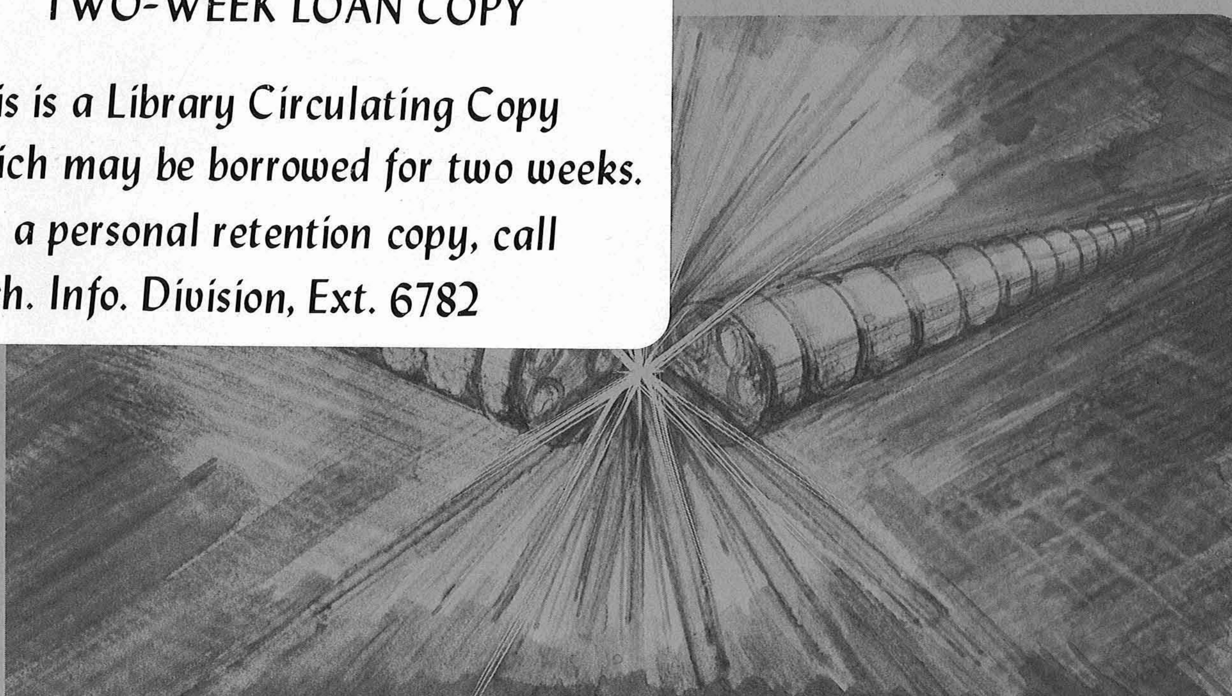

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A Study of Microwave Instabilities by means of a Square-Well Potential

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I. Introduction

The subject of microwave instabilities has attracted a lot of theoretical activity recently. A series of papers by Sacherer[1] has played the leading role in the field. Further development of his work is being actively pursued by several authors[2]. However, the mathematical complexity of the theory makes it very hard to grasp the essential physics underlying microwave instabilities. This is rather unfortunate since the qualitative features of microwave instabilities are easy to understand[3] by applying coasting beam theory[4].

In this paper, microwave instabilities are analyzed in a simple model, in which the usual synchrotron oscillation of a particle is replaced by particle motion in a square-well potential. The motivation for doing this was the following: In the usual synchrotron oscillation, a particle moves along an elliptic trajectory. The most natural coordinates for such a motion are the action and the angle variables. On the other hand, the distribution of the particles along the ring is most conveniently described by azimuthal variables. The complexity of the theory of microwave instabilities derives from the fact that the two sets of the variables are not simply related. The difficulty disappears if the synchrotron motion is approximated by the motion in a square-well potential.

The square-well potential may seem extremely unphysical. However, it should be remarked that the form of the potential with addition of a Landau cavity looks more or less like a square-well. At any rate, the main motivation of introducing the square-well here is to simplify the mathematics of and thereby gaining some insight into microwave instabilities.

The model is exactly soluble. The results are in general agreement with the conclusions obtained from qualitative arguments[3] based on coasting beam theory. However, some of the detailed features of the solution, for example the behavior of ω^2 as a function of impedance, are surprising.

In section II, the model is defined precisely. In section III, the model is solved. The paper is concluded in section IV by discussing the properties of the solution.

II. The Model

The canonically conjugate variables are:

σ: The azimuthal distance from the reference particle.

ε: The energy difference $E - E_s$, where E and E_s are the energy of the particle under consideration and the reference particle, respectively.

Let $\Psi(\sigma,\varepsilon,t)$ be the distribution function in phase space. It satisfies the following Vlasov's equation:

$$\frac{\partial\Psi}{\partial t} + \Gamma\cdot\varepsilon\frac{\partial\Psi}{\partial\sigma} + \left(F_{ext} + \frac{ev\Omega_{rev}}{2\pi} U(\sigma,t)\right)\frac{\partial\Psi}{\partial\varepsilon} = 0 \quad . \tag{1}$$

Here, the constant Γ is defined in terms of R = machine radius, Ω_{rev} = revolution frequency, $\beta = v/c$ and $\eta = \gamma_t^{-2} - \gamma^{-2}$ as follows:

$$\Gamma = \frac{R\Omega_{rev}}{\beta^2 E_s} \quad . \tag{2}$$

$U(\sigma,t)$ is the collective potential given by

$$U(\sigma,t) = \int d\sigma' \; G(\sigma'-\sigma)\rho(\sigma',t) \quad . \tag{3}$$

Here $\rho(\sigma,t)$ is the line density,

$$\rho(\sigma,t) = \int d\varepsilon \; \Psi(\sigma,\varepsilon,t) \quad . \tag{4}$$

$G(\sigma)$ in eq.(5) is the Fourier transform of the impedance function $Z(k)$;

$$G(\sigma) = -\int Z(k)\; e^{-ik\sigma}\, dk, \quad Z(k) = \frac{-1}{2\pi}\int e^{ik\sigma}G(\sigma)\, d\sigma \quad . \tag{5}$$

So far, everything is quite general. The model enters in specifying the form of the external force F_{ext}. In a usual synchrotron oscillation, a particle moves in a harmonic potential as shown in Fig.(1.a.). Thus F_{ext} is proportional to σ and the motion in phase plane is elliptic as shown in Fig.(1.b.). This leads to the difficulties discussed in the Introduction. In this paper, I will replace the harmonic oscillator potential by the square-well potential shown in Fig.(2.a.). The corresponding trajectory in phase plane is shown in Fig.(2.b.). Here, the particle moves from the point A to the point B with a constant velocity, jumps to the point C, then moves again with a constant velocity to the point D, jumps to the point A, etc..

One of the simplicities of the potential being a square-well is that F_{ext} vanishes inside the well, $0<\sigma<L$.The sharp potential barrier at the edge $\sigma = 0$ and $\sigma = L$ could in principle be taken into account by introducing a certain δ-function type force. However, the use of such a singular function can become quite tricky. The difficulty is easily avoided; the reflection at the barrier can be expressed mathematically by means of suitable boundary conditions on Ψ. Consider a particle moving toward the barrier at $(0,-\varepsilon)$ in phase plane. As soon as it arrives at the point $(0,-\varepsilon)$, it jumps to the point $(0,\varepsilon)$ immediately. This means that the points $(0,\varepsilon)$ and $(0,-\varepsilon)$ should be identified. The same is true for the points (L,ε) and $(L,-\varepsilon)$. Therefore, the proper boundary conditions are

$$\Psi(0,\varepsilon,t) = \Psi(0,-\varepsilon,t)$$

$$\Psi(L,\varepsilon,t) = \Psi(L,-\varepsilon,t) \quad . \tag{6}$$

The model is therefore defined by the Vlasov's equation (1) with $F_{ext} = 0$ together with the boundary conditions (6).

As is usual, one linearizes the Vlasov's equation. Write

$$\Psi(\sigma,\varepsilon,t) = \Psi_0(\varepsilon) + \Psi_1(\sigma,\varepsilon,t) \quad . \tag{7}$$

Here $\Psi_0(\varepsilon)$ is the static solution in the absence of the collective force, and Ψ_1 is the perturbation. The linearized equation is, for $0<\sigma<L$,

$$\frac{\partial\Psi_1}{\partial t} + \Gamma\varepsilon\frac{\partial\Psi_1}{\partial\sigma} + \frac{eV\Omega_{rev}}{2\pi} U_1(\sigma,t)\frac{\partial\Psi_0(\varepsilon)}{\partial\varepsilon} = 0 \quad . \tag{8}$$

In the above,

$$U_1(\sigma,t) = \int_0^L d\sigma' \; G(\sigma'-\sigma)\; \rho_1(\sigma')\; d\sigma' \quad , \tag{9}$$

where ρ_1 is the line density associated with Ψ_1. The limits of the integration in (9) arise from the obvious fact that there are no particles outside the potential well. For a general impedance function $G(\sigma)$, the appearance of the finite integration limits in (9) makes the solution of eq.(8) difficult. However, the difficulty disappears if the function $G(\sigma)$ is sharply peaked at $\sigma = 0$. In other words, the interaction is similar to the one induced by the space charge effect. Explicitly, $G(\sigma)$ will be taken to be of the following form:

$$G(\sigma) = G_1\delta'(\sigma) + G_2\delta(\sigma) \quad . \tag{10}$$

Eq.(10) represents the overall features of the longitudinal impedance correctly. The limits of the integration in eq.(9) can now be replaced by $-\infty$ and $+\infty$, enabling one to solve eq.(8) by a simple Fourier transformation.

As for $\Psi_0(\varepsilon)$, I take the simplest choice

$$\Psi_0(\varepsilon) = \frac{I}{2V\Delta}\,\theta(\Delta-|\varepsilon|) \quad , \tag{11}$$

Where I is the peak current in the ring. Notice that $\Psi_0(\varepsilon)$ is an even function of ε and therefore satisfies the boundary conditions (6). Eq.(11) applies only inside the potential well, i.e., when $0<\sigma<L$. It is understood that Ψ_0 vanishes outside the well. The same remark holds for the functions Ψ_1, A and B in the equations below. By differentiation, one obtains

$$\frac{\partial\Psi_0}{\partial\varepsilon} = \frac{I}{2v\Delta}\,[\delta(\varepsilon+\Delta) - \delta(\varepsilon-\Delta)] \quad . \tag{12}$$

From eqs.(8) and (12), one sees that Ψ_1 is of the following form:

$$\Psi_1(\sigma,\varepsilon,t) = A(\sigma,t)\delta(\varepsilon+\Delta) + B(\sigma,t)\delta(\varepsilon-\Delta) \quad . \tag{13}$$

The functions A and B satisfy the following equations:

$$\left(\frac{\partial}{\partial t} + \Gamma\Delta\frac{\partial}{\partial\sigma}\right) A(\sigma,t) + \frac{e\Omega_{rev} I}{2\pi 2\Delta}\int d\sigma'\ G(\sigma'-\sigma)\ [A(\sigma',t) + B(\sigma',t)] = 0$$

$$\left(\frac{\partial}{\partial t} - \Gamma\Delta\frac{\partial}{\partial\sigma}\right) B(\sigma,t) - \frac{e\Omega_{rev} I}{2\pi 2\Delta}\int d\sigma'\ G(\sigma'-\sigma)\ [A(\sigma',t) + B(\sigma',t)] = 0 \tag{14}$$

The boundary condition becomes

$$A(0,t) = B(0,t), \quad A(L,t) = B(L,t) \quad . \tag{15}$$

Eq.(14) is applicable to coasting beam as well if σ is interpreted as the distance from a fixed point on the ring, say at $\sigma = 0$. However, the boundary conditions are modified as follows: Let the circumference of the ring be C. Since the points $\sigma = 0$ and $\sigma = C$ are identical, the boundary conditions become

$$A(0,t) = A(C,t), \quad B(0,t) = B(C,t) \quad . \tag{16}$$

III. The Solution

Let us forget about the boundary condition for the moment. Write

$$A(\sigma,t) = \Sigma e^{i(\omega t + k\sigma)} a_k$$
$$B(\sigma,t) = \Sigma e^{i(\omega t + k\sigma)} b_k \tag{17}$$

Eq.(14) becomes

$$(\omega - \gamma k + ig(k))a_k + ig(k)b_k = 0$$
$$-ig(k)a_k + (\omega + \gamma k - ig(k))b_k = 0 \quad , \tag{18}$$

where

$$\gamma = \Gamma\Delta \text{ and } g(k) = \frac{\Omega_{rev}}{2\pi}\,\frac{eZ(k)I}{2\Delta} \quad . \tag{19}$$

γ in the above is the velocity of the particle relative to the reference particle at the top or the bottom of the stack. In view of eq.(10), g(k) is of the following form:

$$g(k) = i\alpha k + \beta \ , \tag{20}$$

where α and β are real constants. The solubility of eq.(18) requires the following dispersion relation:

$$\omega = \pm\omega(k), \quad \omega(k) = \sqrt{(\gamma k)^2 - 2i\gamma k g(k)} \ . \tag{21}$$

In eq.(21), the square root is defined so that $\omega(k)$ has a positive real part.

To complete the solution, one should take into account the boundary condition. For coasting beam, eq.(16) requires that k be real and discrete as follows:

$$k = k_n \equiv \frac{2\pi n}{C} \qquad n = 0, \pm 1, \pm 2, \ldots \ . \tag{22}$$

The corresponding ω^2 is

$$\omega^2 = \omega_n{}^2 \equiv (\gamma k_n)^2 - 2i\gamma k_n g(k_n) = \gamma Y k_n{}^2 - 2i\beta\gamma k_n \ , \tag{23}$$

where

$$Y = \gamma + 2\alpha \ . \tag{24}$$

If $n>0$ and $\omega = +\omega_n$, it follows from eq.(18) that $b_k >> a_k$ when g is small. Thus the disturbance runs mainly along the top of the stack. See eq.(13). Analogously the case $\omega = -\omega_n$ corresponds to the bottom wave. If the impedance is purely resistive, $\alpha = 0$ and $\beta>0$. It is then easy to show that the bottom wave grows and the top wave damps. All of these features are well known from coasting beam theory[4].

For bunched beam, the relevant boundary condition is given by eq.(15). The top and the bottom waves couple with each other in an essential way. To proceed, notice first that the boundary condition applies at all times, so that the contributions from different frequencies can be analyzed separately. Therefore, it is necessary to find the k's which correspond to the same ω^2. Consider the equation

$$\omega^2 = (\gamma k)^2 - 2i\gamma k g(k) \quad , \tag{25}$$

which is equivalent to eq.(21). From eqs.(20) and (25), one obtains

$$k = k_\pm = \frac{i\beta}{Y} \pm \sqrt{\frac{\omega^2}{Y\gamma} - \left(\frac{\beta}{Y}\right)^2} \quad , \tag{26}$$

where Y is defined in eq.(24). The functions A and B that behave as $e^{i\omega t}$ are, in view of eqs.(17) and (18), as follows:

$$A(\sigma,t) = e^{i\omega t}\left(e^{ik_+\sigma}\, a_+ + e^{ik_-\sigma}\, a_-\right)$$

$$B(\sigma,t) = e^{i\omega t}\left(e^{ik_+\sigma}\, b_+ + e^{ik_-\sigma}\, b_-\right) \tag{27}$$

where

$$b_\pm = D_\pm a_\pm \ , \ D_\pm = \frac{i\, g(k_\pm)}{\omega + k_\pm - ig(k_\pm)} \quad . \tag{28}$$

The boundary condition (15) becomes

$$(1 - D_+)\, a_+ + (1 - D_-)\, a_- = 0 \tag{29.a}$$

$$(1 - D_+)e^{ik_+L}\, a_+ + (1 - D_-)e^{ik_-L}\, a_- = 0 \quad . \tag{29.b}$$

One way to satisfy eq.(29) is to require

$$D_\pm = 1 \text{ and } a_\mp = 0 \tag{30}$$

After some algebra, one finds that eqs.(25) and (30) imply

$$\omega = 0,\ k = i\kappa = 2i\frac{\beta}{Y} \quad . \tag{31}$$

For this value of ω and k, A is identical to B and given by

$$A(\sigma,t) = B(\sigma,t) = e^{\kappa\sigma} \quad . \tag{32}$$

This solution is time independent and therefore stable.

Eq.(29) can also be satisfied if the two amplitudes a_+ and a_- are related by (29.a) and, furthermore, k_+ and k_- are related as follows:

$$k_+ - k_- = 2K_n = \frac{2\pi n}{L} \quad , \quad n = 0,1,2,\ldots \quad . \tag{33}$$

From (26), one obtains

$$\omega^2 = \omega_n{}^2 = \gamma\left(YK_n{}^2 + \frac{\beta}{Y}^2\right) \ . \tag{34}$$

and

$$k_{\pm} = k_{\pm n} = \pm K_n + i\frac{\beta}{Y} \quad . \tag{35}$$

The corresponding functions A and B are easily obtained. One gets

$$A(\sigma,t) = e^{i\omega_n t}\left(e^{ik_{+n}\cdot\sigma} a_{+n} + e^{ik_{-n}\cdot\sigma} a_{-n}\right) C$$

$$B(\sigma,t) = e^{i\omega_n t}\left(e^{ik_{+n}\cdot\sigma} b_{+n} + e^{ik_{-n}\cdot\sigma} b_{-n}\right) C \ , \tag{36}$$

In the above $(\quad)_{\pm}$ means that k should be replaced by $k_{\pm n}$, and C is an arbitrary normalization constant. This completes the solution of the problem.

IV. Discussion and Conclusions

Let us now discuss the properties of the solutions obtained in the previous section. First, if the impedance is small, eq.(34) can be approximated by

$$\omega_n \sim \gamma K_n = \gamma\frac{\pi n}{L} \quad . \tag{38}$$

To understand this formula, recall that γ is the particle velocity at the top or the bottom of the stack. Therefore the quantity $2L/\gamma$ can be interpreted as the period of one "synchrotron" oscillation. Thus, eq.(38) can be written in the following expected form:

$$\omega_n = n\Omega_s \tag{39}$$

where $\Omega_s = 2\pi\gamma/2L$ is the angular synchrotron frequency.

Next, it is interesting to compare the frequency spectrums for coasting beam and bunched beam given by eq.(23) and eq.(34), respectively. For coasting beam, the presence of a resistive part β in the impedance always implies an instability. The situation is quite different for the case of coasting beam; the resistive part enters as β^2 in eq.(34), and a bunched beam can be stable even if $\beta \neq 0$. Instabilities occur if the quantity $Y = \gamma + 2\alpha$ becomes negative. Therefore, a bunched beam is always stable if the impedance is small and hence $Y > 0$. This conclusion is in accord with the one reached by intuitive arguments[3] based on coasting beam theory.

For coasting beam, $|\omega^2|$ generally increases as the impedance (or current) increases. The situation is again quite different for bunched beam. Fig(3). shows the behavior of ω^2 as a function of Y for a fixed n. If $\beta = 0$, the curve is similar to the case of coasting beam. However, the curve for the case $\beta \neq 0$ is qualitatively different; It is singular at $Y = 0$ and has minima at $Y = \pm\beta/k_n$. The presence of a resistive term therefore has an important bearing on the behavior of ω^2 in bunched beam, although it does not directly influence the stability criteria. Whether this feature is due to the specific model discussed in this paper remains to be seen. However, I suspect that it is a general phenomena arising from the strong interference of the top and bottom waves inevitable in bunched beams.

Finally, it is also interesting to compute the line density $\rho(\sigma,t) = A(\sigma,t) + B(\sigma,t)$. The result is complicated, but when $\beta = 0$ it becomes

$$\rho(\sigma,t) = e^{i\omega_n t} \cos(K_n\sigma), \qquad \omega_n = \sqrt{\gamma Y}\, K_n \quad . \tag{40}$$

That is, $\rho(\sigma,t)$ is a standing wave, as would be reasonable for a purely reactive impedance. However, the distributions $A(\sigma,t)$ and $B(\sigma,t)$ are <u>not</u> separately standing waves.

In this paper, the physics of microwave instabilities have been studied using a simple model for the synchrotron oscillation. The results verify to a certain extent the validity of intuitive arguments. However, a certain aspect of the results is unexpected. One hopes that the insight gained in this analysis will be helpful in attacking a more realistic theory of microwave instabilities.

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V. Addded Notes

If the resistive part β is due to the skin effect, it is of the form

$$\beta = (1 + i)\beta_0 \quad ,$$

where β_0 is a real constant. From this and eq.(23), one sees that β_0 is the growth rate for coasting beam in the case of $\gamma >> \alpha$. The growth rate for bunched beam is found by inserting the above formula into eq.(34), and one finds

$$\text{Im}\ \omega \sim \frac{\beta_0}{n}\left(\frac{\beta_0}{\Omega_s}\right) \quad .$$

In other words, the growth rate for bunched beam is reduced by a factor β_0/Ω_s compared to the growth rate of coasting beam, β_0.

Dr. L. Smith has obtained the dispersion relation (34) by using the technique of integrating over unperturbed orbit. The same technique can be used to justify the boundary condition (6) on a more rigorous basis.

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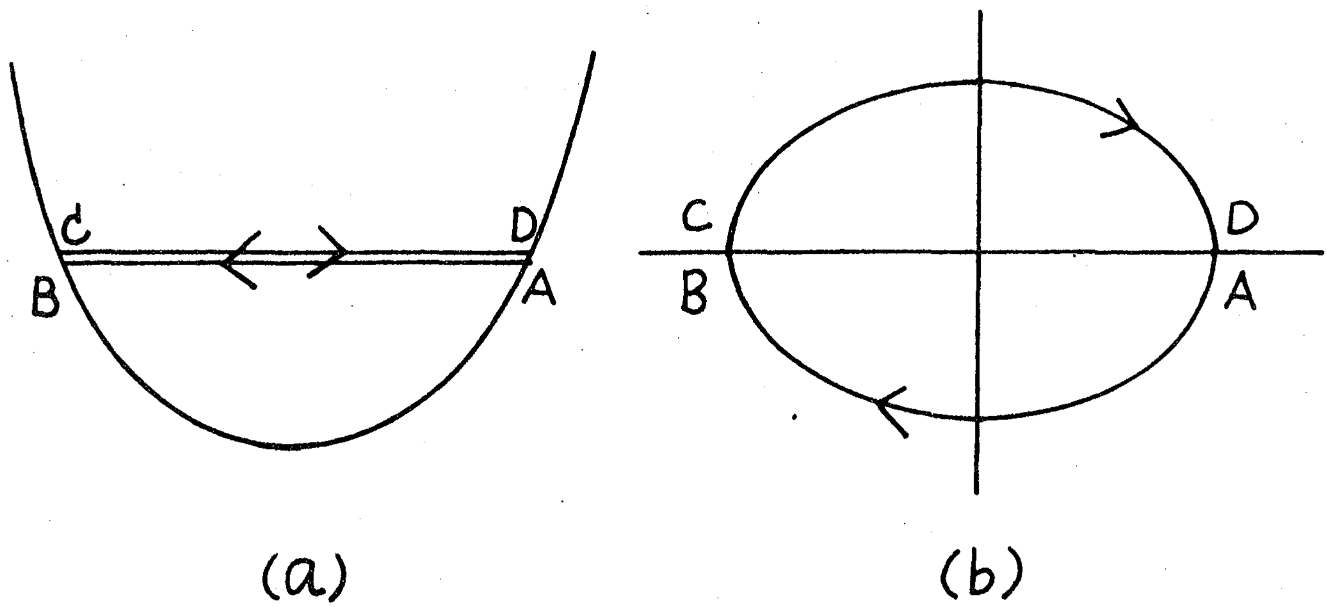

Figure (1). Harmonic oscillator potential and the motion in phase plane.

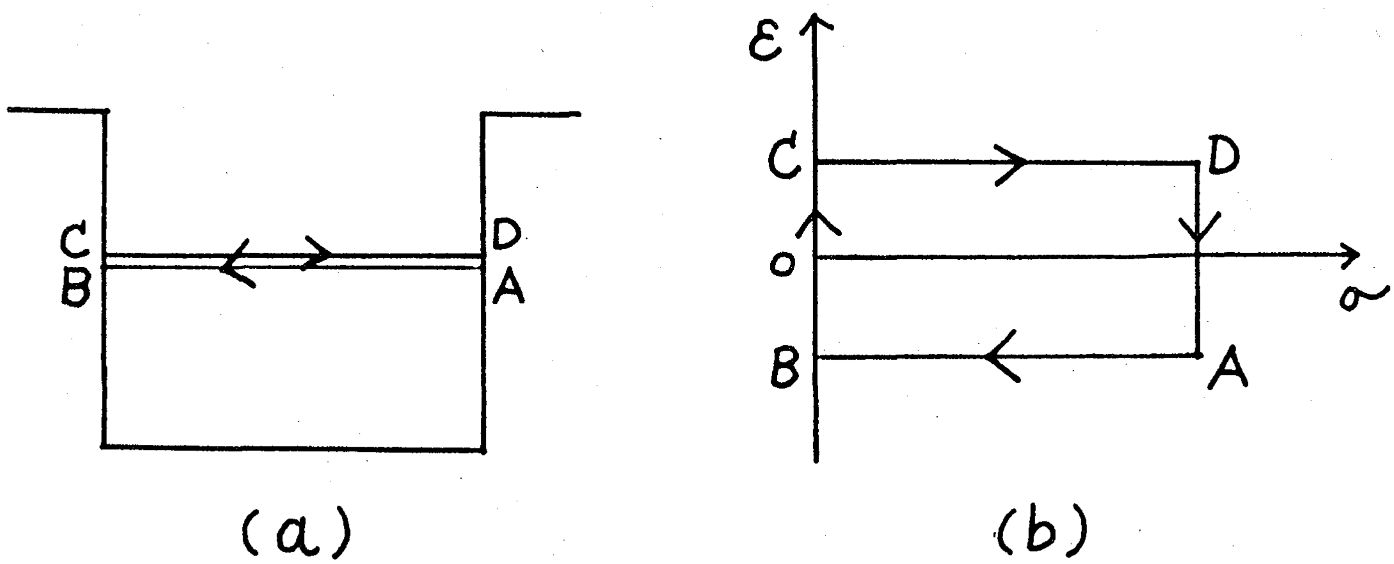

Figure (2). Square-well potential and the motion in phase plane.

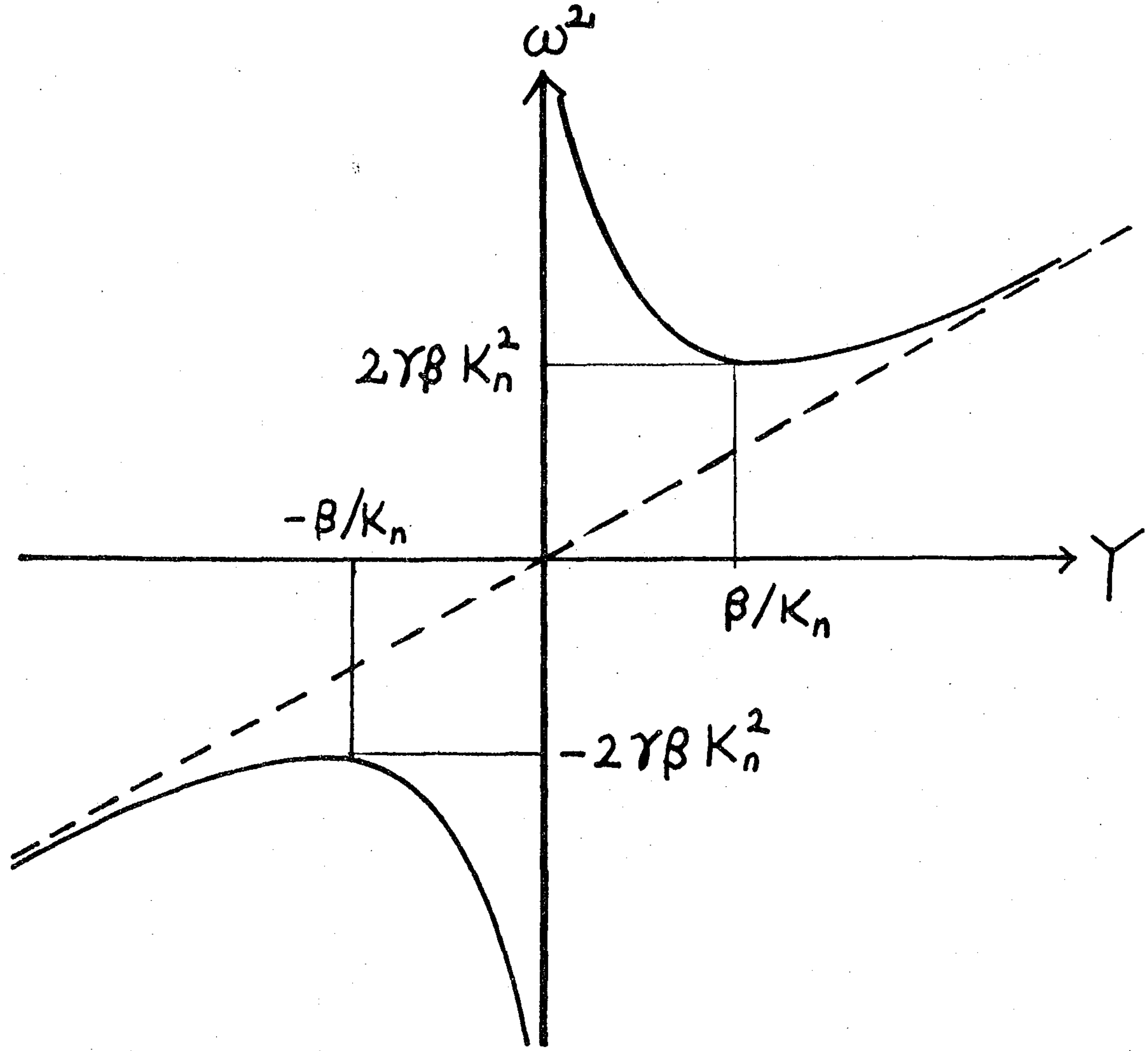

Figure (3). ω^2 versus $Y = \gamma + 2\alpha$ for a fixed K_n. The solid curve is for the case $\beta \neq 0$, while the dashed one represents the case $\beta = 0$.

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