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UNIVERSITY OF CALIFORNIA, SAN DIEGO

SAN DIEGO STATE UNIVERSITY

Negotiating Meaning for the Symbolic Expressions for Vectors and Vector Equations
in a Classroom Community of Practice

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics and Science Education

by

George Franklin Sweeney

Committee in Charge:

University of California, San Diego

      Professor Mark Appelbaum
      Professor Jeff Rabin

San Diego State University

      Professor Chris Rasmussen, Chair
      Professor Ricardo Nemirovsky
      Professor Jessica Pierson
      Professor Michelle Zandieh

The Dissertation of George Franklin Sweeney is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

San Diego State University

2012

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Acknowledgements

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Education

PhD, Mathematics and Science Education   Fall 2007-2012
San Diego State University   San Diego, California, USA
University of California, San Diego   San Diego, California, USA
Advisor: Dr. Chris Rasmussen

M.S. Mathematics   Fall 2005-Spring 2007
Cal Poly, Pomona   Pomona, California, USA
Specialization: Stochastic Analysis and Real Analysis

Post-Baccalaureate- Mathematics   Summer 2004-Spring 2005
University of California, Davis   Davis, California, USA
Specialization: Pure Mathematics

Post-Baccalaureate- Mathematics Spring 2004
Sacramento City College   Sacramento, California, USA

B.A. American Literature and Culture   Fall 1995-Spring 1999
University of California, Los Angeles   Los Angeles, California, USA

Teaching and Instructional Activities

San Diego State University   San Diego, California, USA
Instructor
Fall 2007
1. Intructor of record for two sections of Number Systems in Elementary Mathematics- a course covering base number systems to fractions for pre-service elementary school teachers
2. Taught using an inquiry-oriented teaching methods including group work
3. Utilized smart classroom technology

Cal Poly Pomona   Pomona, California, USA
Instructor   September, 2006-August, 2007
1. Instructor of record for 1st, 2nd and 3rd semester intermediate algebra- traditional and computer-aided courses
2. Educational Opportunity Program's summer program in college algebra (Instructor of Record)
3. Summer experience for high school students in geometry and college algebra (Instructor of Record)

4. Treisman group discussion sections for college algebra students

Elite Educational Institute   La Habra, California, USA
Instructor   June, 2005- Present

1. Taught classroom courses in college statistics for advanced high school students, Algebra II, Junior high school algebra and geometry, Math IIC preparatory course, SAT math preparatory, SAT reading and writing preparatory
2. Tutoring for individual students in mathematics, essay writing, English, US history and European history.

Sylvan Learning Centers   Davis and Sacramento, California, USA
Instructor and Math Coordinator   January 2004-June 2005
                                                September 2001-December 2002

1. Taught K-12 mathematics specializing in middle and high school math, utilizing both proprietary materials and tutoring of individual programs
2. Conducted mathematics assessments of students for formative and summative purposes
3. Designed course curriculum for students based upon assessments and student performance on given curriculum
4. Also taught reading K-12 reading, writing grades 4-12, and study skills

Sylvan Learning Centers   Sacramento, California, USA
Director of Education   December 2002-January 2004

1. Performed assessments for students in reading, writing, study skills and mathematics
2. Designed programs based upon assessments and individual student needs
3. Counseled parents about courses of action and needs of particular children
4. Interacted with school teachers and administrators in order to tailor programs to students in-school needs
5. Conducted community outreach as a part of expanding our role within the Florin Road community

Professional Experience

San Diego State University   San Diego, California, USA
Research Assistant   Spring 2008-Present
Advisor: Dr. Chris Rasmussen
    1. Projects

a. Linear algebra instructional design
b. Inquiry-oriented teaching in differential equations
c. Discursive moves in inquiry-oriented teaching
d. Large scale study of calculus programs across the US, focusing on increasing student persistence in STEM programs

1. Responsibilities and activities
   a. Responsible for all data collection, storage and coordination
   b. Designing classroom instruction, interviews and other data collection activities
   c. Analysis of collected data
   d. Presentation of findings from analysis as talks, journal articles and book chapters
   e. Grant writing

San Diego State University   San Diego, California, USA
Collaborator on multi-institution projects   Spring 2008-Present

1. International collaboration with Mexican group on instructional design in linear algebra
   a. International meeting in Mexico City
   b. Working group on modeling in linear algebra at PME-NA 2010
1. Cross-disciplinary collaboration with physical chemistry group on argumentation in Physical Chemistry
   o Multiple intergroup collaborative meetings at Purdue University and San Diego State
   o Multiple presentations at math and science education conferences
   o REESE Phase 1 grant to study argumentation, mathematical symbolization and teacher discourse in physical chemistry
   · Cross research group on embodied cognition and complex numbers

San Diego State University   San Diego, California, USA
Jackie Robinson YMCA LabZone afterschool program   January 2009-Present
Committee member and designer
- Collaborating with the Jackie Robinson YMCA on an afterschool program in informal mathematics and science
- Participated in the design of a classroom course in informal learning in math and science for SDSU students (Class was successfully included in the College of Letters and Science)
- Grant writing for initial funding for the project

Recruitment of students for the SDSU class

University of Michigan: Ann Arbor   Ann Arbor, Michigan, USA
San Diego State University   San Diego, California, USA
Teaching and Research Practicum   June 2008-May 2009

- Designing instruction and tasks for use with ThEMaT animations in geometry and middle school and high school algebra
- Advised on online environments utilizing ThEMT animations
- Presented on the uses and applications of ThEMaT animations at SDSU
- Recruited professors for piloting of animations in their classrooms
- Designed instructional sequences and piloted use of the animations in three SDSU classes
- Reported on results for ThEMaT group

Publications

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (in press). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences.*

Wawro, M., Sweeney, G., & Rabin, J. (in press). Subspace in linear algebra: Investigating students' concept images and interactions with formal definition. *Educational Studies in Mathematics.*

Cole, R., Towns, M., Rasmussen, C., Becker, N., Sweeney, G., & Wawro, M. (in press). Adapting a methodology from mathematics education research to chemistry education research: Documenting collective activity to an undergraduate chemistry class. *International Journal of Science and Mathematics education.*

In Process Publications

Sweeney, G. & Rasmussen, C. Modeling Reconceived. Book chapter for book on gestures and embodiment in mathematics education. (In Press)

Awards

Chancellor's Doctoral Incentive Program Grant Recipient	Fall 2008-Present
TRUSE Grant Recipient	Spring 2010
CDIP Travel Grant Recipient	Spring 2010
Sacramento Sylvan Learning Center- Teacher of the Year	2002
Davis Sylvan Learning Center- Teacher of the Year	2004
Golden Key Honor Society	June 1999

UCLA Collegiate Honors	June 1999
English department- Highest Honors	June 1999

Professional Activities

Reviewer for Journal for Research in Undergraduate Mathematics

Member of the Mathematics Association of America

Member of SIG-MAA on Research in Undergraduate Mathematics

Member of the National Council of Teachers of Mathematics

Member of the Association of Mathematics Teacher Educators

Member of the American Educational Research Association

ABSTRACT OF THE DISSERTATION

Negotiating Meaning for the Symbolic Expressions for Vectors and Vector Equations
in a Classroom Community of Practice

by

George Franklin Sweeney

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2012
San Diego State University, 2012

Professor Chris Rasmussen, Chair

In this study, I analyze the development of meaning for the symbolic expressions for vectors and vector equations in an introductory, inquiry-oriented linear algebra course. Linear algebra is one of the first in-depth experiences that students have with vectors and vector equations, and as such my study examines these meanings in their formative and advanced states. The analysis in this study seeks to answer two fundamental questions:

1. What are the different meanings that this classroom
   community develops for vectors and vector equations?

2. In what ways do individual students contribute to and take

responsibility for the different meanings that this classroom

community develops for vectors and vector equations?

Both of these questions deal with meaning making from two different, but mutually

informing perspectives: what is created and how the individual feels about what is

created.

Answering the first question entailed examining the collective meanings that

the community develops for symbolic expressions for vectors and vector equations.

For this part of the study, I used the Toulmin argumentation scheme to analyze

classroom argumentation from the first six days of class. In this analysis, video and

transcripts from whole class discussion were used. To answer the second question, I

analyzed a series of six focus group interviews made at the beginning and end of the

semester. I used a grounded theory and thematic analysis of the transcripts and videos

of these focus groups in order to ascertain the kinds of responsibility and contribution

that these individual students demonstrated and expressed as members of the

classroom community.

There are several contributions that this study makes to the field of

mathematics education. First, the examination of the collective creation, use and

interpretation of vectors and vector equations will add insight into the process of

meaning making for these specific expressions and mathematical expressions in

general. Second, my analysis examines the often overlooked affective aspects of

meaning making and the notion that this process is tied to the roles that the individual

feels that they occupy within the community. Third, I examine the possible and

observed connections between the identity of the individual and knowledge

construction during classroom discourse.

Chapter 1 : Research Questions and Significance

The focus of this dissertation is how a classroom community created meaning for vectors and vector equations in linear algebra. In particular, I examined the meanings that students negotiated for vectors and vector equations and the extent to which identity played a role in their shaping and negotiating those meanings. In this chapter I first present the research questions. Then I consider how the research questions relate to the theoretical constructs that I will be using, particularly with regard to meaning-making and identity. In this section, I define the key terms that are necessary to understand the questions and their analysis. I conclude the chapter by specifying how answering the particular question will contribute to the study of student learning and teaching of linear algebra and to the field more broadly by adding to our understanding of symbolizing and identity.

Research questions

My dissertation focuses on important symbolic forms in linear algebra, vectors and vector equations. Linear algebra is rich with opportunities for students to create and use various symbolic systems for use in computation and reasoning about algebraic concepts. The first question in my dissertation deals directly with this issue, with the classroom community as the unit of analysis:

1. What are the different meanings that this classroom
   community develops for vectors and vector equations?

Conceptual and operative meanings for symbols are important. What it means for individuals to do mathematics and be a member of a classroom community also plays a role in the development of meaning for symbolic expressions. The second question for my dissertation deals with identity and its role in the meaning-making process for individuals. The second question that I will be addressing in my dissertation is:

>   2. In what ways do individual students contribute to and take responsibility for the different meanings that this classroom community develops for vectors and vector equations?

How students contribute to and take responsibility for the meanings of the classroom reveals the roles that the students play in the classroom community's knowledge building process. By answering this question I address how the collective process of creating mathematical meaning affects individual student identities and their roles and responsibilities within the classroom community.

Communities of practice: A theoretical perspective

The particular theoretical perspective that my dissertation draws upon is a socio-cultural theory called communities of practice (Wenger, 1998; Lave & Wenger, 1991). In broad terms, the communities of practice perspective deals with organizational dynamics and how those dynamics shape the meanings and production of an organization. Hence, the theoretical perspective can be seen as dealing primarily with collections of individuals who are engaged in a common purpose, in this case studying linear algebra. In fact, my first research question deals exclusively with the meanings that the collective classroom community developed. Although Wenger does

not see the role of the community as being necessarily deterministic in the shaping of

meaning for individuals, he does posit many relationships between individual

members of the community, their role within the community, and the activities of the

community. In this section, I discuss one of those relationships, identity, and how

identity relates to collective production of meaning.

Negotiation of meaning: Understanding the meaning making process as a collective activity

Meanings for symbolic expressions accrue over time and these meanings are

always bound up in the contexts of their use. Students and teachers have goals in

developing symbolic expressions, and the way that they develop meaning is via

activity and use. Roth (2008) illustrates this position:

> First, pointing (indicating) is based in the tool nature of a
> sign, which orients it to the what-for and in-order-to
> already indicates that a sign cannot be considered in and
> for itself because it is always already bound up with
> intentions, purposes, motives, and the objects of actions.
> (p. 84)

Roth states that a sign (which includes symbols) is a tool for use and hence the

purposes that a symbol will be used for helps dictate its meanings. Different meaning

can represent use in different situations, under different constraints, when there is a

difference in intentions for those using them. How students determine the what-for of

a particular symbolic expression arises from their participation in activities that

concern those expressions and in communicating with others who are also pursuing

meaning-making.

Although the previous quote from Roth was in reference to individuals, the sentiment he expressed is equally applicable to the classroom community. In the case of mathematics and symbolizing, the what-for and in-order-to of the symbols is intimately tied to activity in the classroom, participation with the larger mathematics community, and subsequent activity with those objects. It is in this regard that my analysis focuses on the meanings that function-as-if-shared in the classroom (Rasmussen & Stephan, 2008), examining the what-for and in-order-to that are tacitly and explicitly agreed upon by the collective as a whole.

The overarching theoretical lens through which I examine the creation of collective meaning making is through Wenger's (1998) construct of the community of practice. A community of practice is a group of individuals, institutions, and objects that share a common set of goals and mission. Individuals within a community of practice may have differing reasons for being members of the community, but as members of the community they share the mission of the group. Similar to the what-for and the in-order-to mentioned by Roth (2008), a community of practice is often defined by what it is for. For example, a group of claims processors processes claims, that is the in-order-to of the community. As a result of this common mission or activity, the community develops tools, activities, language, and identities to accomplish these goals. In many ways, an individual can become identified with a particular community of practice by their engagement in the activities, tradition, language and tool-use of the community. Furthermore, the community has certain traditions and activities that allow it to deal with new problems and sustain its

existence. A very important aspect to the community of practice is its history, culture, and modes of enculturation, as these are the modes by which it continues its existence.

Fundamental to the communities of practice perspective is that the problems, the situations, contexts, and the methods for solving those problems that individuals encounter in the course of developing mathematical knowledge are shaped by group membership and activity, whether in the classroom or as a member of the larger mathematics community. A conventional interpretation of meaning often privileges the individual and ascribes particular meanings to individuals. Thus one might say that an individual has acquired or appropriated a particular meaning (Sfard, 1998). However, from the perspective of the classroom community, saying that a classroom community has acquired or appropriated a particular meaning presents a framing challenge. Wenger deals with this problem via the construct of the negotiation of meaning:

> I will use the concept of the negotiation of meaning very
> generally to characterize the process by which we experience the
> world and our engagement in it as meaningful. (p. 53)

The negotiation of meaning is the process by which human beings make sense of their activities. Wenger characterizes the negotiation of meaning as the process by which communities of practice come to have common and working meanings for particular objects and practices. The construct of negotiability of meaning underscores that meanings are not fixed in practice, but rather are continuously shifting and changing as members of the community encounter new problems and situations. Thus, meaning is flexible and is developed in social interaction through use. Negotiation of meaning is the process by which a community creates meaning when engaging in activities or

when dealing with certain objects. New meanings are both tacitly and explicitly agreed upon by the members of the community as new problems and situations arise.

Communities of practice and their component members negotiate meaning through the dual mechanisms of reification and participation. Reification is very generally, "the process of giving form to our experience by producing objects that congeals this experience into 'thingness' (p. 58)." Reification, thought of this way, establishes an object around which activity can be done. It becomes the center of activity. For example, the symbolic expression for a vector equation can be thought of as a reified object in the context of a particular community.   A particular vector expression, for example:

$$2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Could mean to multiply the first vector by 2 and the second vector by 3 and then add the two of them together. In this first case, I might consider this particular symbolic representation as resulting in computation and a resulting vector. Or it could mean that I want to consider the addition of two geometric vectors that have been stretched by a given amount. In this case that could entail imagining a parallelogram made up of the component vectors, each of which is stretched by the given amount. Hence, the result is considered to be a diagram. Or the equation could be a statement of what happens when an individual flies a magic carpet for two turns and a hoverboard for three turns (more on this later in the methods section). As students participate in using and creating these symbolic expressions, thereby developing these various meanings, what the symbolic expression means changes. The symbolic expressions have been reified

because the students relate to them as the collection of meanings that they have created. They have developed a meaning around which the members of the classroom can further participate.

According to Wenger (1998), reification is only part of the process of the negotiation of meaning in a community of practice. Reification occurs when members of a community engage in activities using those particular concepts or objects. Use may mean that the objects are tools in the activity or that individuals are engaged in activities attempting to make sense of the objects. This use (Wenger refers to this as participation) has a symbiotic relationship with reification. One cannot exist without the other, and both contribute equally to the creation of meaning. In a community of practice, participation is engagement in purposeful activity that centers on the reified objects of the community. The example of working with the magic carpet ride clearly exemplifies the role that participation plays in the construction of meaning. Students may come to see the equation as being movement on a magic carpet and a hoverboard only if they have experience in relating those modes of transportation as being represented by vectors and their movement as being represented by the vector equation that gets created to describe its movement. My first research question focuses on the meanings that students create through their participation, considering those patterns of participation and use as integral to the meanings that are created. Reification/Participation is the fundamental duality in Wenger's formulation of meaning. Via participation, reified objects are created and imbued with meaning. And via reified objects the community preserves a mutually shared meaning for various activities.

Negotiability: The power to shape one's own mathematical activity

Understanding what meanings get negotiated in the classroom is an important question, but it is also important that students feel that the meanings that get constructed in the classroom are their own and a product of their work and processes of understanding. For over 40 years, researchers have examined student beliefs about mathematics and their beliefs about themselves as nascent mathematicians (Pajares & Miller, 1994; Malmivuori, 2001; Muis, 2004). According to Wenger (1998) a major part of an individual's identity is their role within a community of practice. A student's ability to contribute to and take responsibility for meaning-making within the classroom plays a large role in whether or not students continue in mathematics (Solomon, 2007; Boaler & Greeno, 2000). It may also potentially shape their use in various contexts and creation as expressions for mathematical reasoning in interesting ways. My second question deals specifically with student's ability to contribute to the meaning-making of the classroom.

2. In what ways do individual students contribute to and take responsibility for the different meanings that this classroom community develops for vectors and vector equations?

In Wenger's (1998) formulation, reified objects such as vectors and vector equations have more than one meaning as they are encountered in one or more

communities of practice. In this dissertation, the classroom community is the community that is explicitly analyzed. However, the community of professional mathematicians implicitly plays a role in this analysis because it is the intent of instruction to allow students to participate with the concepts, symbolizations, and practices of the mathematics community, albeit on a limited scale.

The meanings that professional mathematicians and the members of this classroom community develop for vectors and vector equations can vary. One of the reasons that Wenger posits for this difference in the ways that novice mathematics students and professional mathematicians relate to mathematics is because they belong to two very different communities. As mathematicians, members of that community engage in activities that are fundamentally different than those of mathematics students. One of the primary differences is that professional mathematicians are responsible for the creation of new mathematics, whereas mathematics students are frequently asked to do and understand mathematics that has been established over time. Consequently, when mathematicians encounter mathematics, they do so with the knowledge that there is a potential for the math to change and that they are capable of making those changes. Students in introductory mathematics courses may not relate to mathematics in the same way. They may view the ability to shape and develop meaning for mathematics as being outside of their control. They are capable of understanding and doing mathematics, but they are not necessarily responsible for the mathematics that they create. The ability to create new mathematics and shape existing mathematics is, according to Wenger, part of the professional mathematicians membership in their community. This is one of the expectations of being in their

community and so it brings a kind of agency and power that may be different from those who are just beginning to learn advanced mathematics.

Belonging within a particular community, whether a mathematics classroom or the community of professional mathematicians is a part of what Wenger calls identity. Identity thought of in this way is the collection of expectations, roles, and responsibilities that an individual exhibits as a result of their membership in their various communities. This formulation ties identity to the community because these roles, expectations and responsibilities extend from the individual's membership and place within a community. In the example of professional mathematics, one of the goals of the mathematics community is to create new mathematics. Hence, each member of that community is responsible for that creation. They are empowered to create new mathematics because that is a part of being in the community.

Because the development of new mathematical knowledge is such an integral job of the community and a chief responsibility of the members of the mathematical community, this aspect of the individual's identity creates ways of acting and behaving that are consistent with accomplishing that goal. From this perspective, concentrating on the development of identity for students can be particularly productive for the teaching and learning of mathematics. One aspect of identity that Wenger presents in his formulation of identity is the ability to shape and create new meanings. He refers to this quality as negotiability:

> Negotiability refers to the ability, facility, and legitimacy
> to contribute to, take responsibility for, and shape the
> meanings that matter within a social configuration.
> Negotiability allows us to make meanings applicable to
> new circumstances, to enlist the collaboration of others, to

> make sense of events, or to assert our membership. (p. 197)

For example, in one classroom, students may feel that the ways that vector equations are symbolized is rigid and that they are required to memorize the meanings that experts have already developed. On the other hand, students in a different class may feel that the meanings for vectors and vector equations are dependent upon their particular uses and hence may show flexibility in their use. In a classroom where students are empowered to a high degree to negotiate the meanings of mathematical objects, students would display collaboration with their peers and responsibility for making sense of the problems and the mathematical objects under consideration.

Understanding the kinds of negotiability that students develop as a member of a classroom community can lead to better understanding the identities that they develop. By analyzing the kinds of responsibilities and contributions that they are expected to make as members, I can gain insight into what it means for a person to be a member of this classroom community. The responsibilities and contributions that I examine in my dissertation are intimately tied to membership in the community.. Thus their characterization is a characterization of the individuals and also a characterization of the community as well. However, this characterization is not a characterization of the individual's identity itself. The identity that an individual exhibits is actually the product of the intersection of all the communities to which they belong. In the case of my dissertation, I am only looking at their membership within one of these communities, their introductory linear algebra course.

My second research question deals directly with the different ways that individual members of this classroom community demonstrate negotiability and how that relates to their roles in the classroom community. Examining negotiability allows me to infer who individual students are in relation to the classroom community and identify productive ways of being in the classroom and productive ways of interacting with mathematical symbols beyond their understanding of the concepts and their correct or incorrect use.

Contribution to the study of student thinking and teaching

The importance of examining the development of meaning for the symbols for vectors and vector equations

As is detailed in greater depth in chapter 2, students have had difficulties in developing meaning for vectors and vector equations (Dorier, Robert, Robinet, & Rogalski, 2000; Harel, 1989a, 1989b; Hillel, 2000; Sierpinska, 2000) . These difficulties stem from a variety of sources, including but not limited to connecting geometric expressions for vectors and vector equations to formal notions, relating the symbolic expressions to definitions and theorems, and working in dimensions higher than two and 3. However, none of the literature has addressed the collective meaning making for vectors and vector equations for classroom communities. By examining collective meaning-making activity, my dissertation will expand our understanding of how groups of people make sense of symbolic expressions in linear algebra and the practice of symbolizing in general.

In addition, my dissertation will contribute to the teaching of linear algebra by identifying a set of functioning-as-if-shared concepts and classroom mathematics practices that are developed in the course of inquiry-oriented learning. Teachers can use these ideas and practices in order to guide classroom activity, by indicating what concepts could be developed in the course of classroom discussion and then leveraging them to create a more robust learning environment. As well, instructional designers will have a set of collective meanings that are a possible result of actual classroom activity. They can then use these meanings to develop more robust and informed activities that allow for high levels of student involvement.

The importance of examining identity and student meaning-making

The actual mathematical symbols that classroom communities produce and use on a regular basis are crucial in understanding how individuals and communities learn mathematics. But the conceptual knowledge that that students develop for a set of symbols is not the only meaning that an individual ascribes to those symbols. In addition to their understanding of symbolic expressions and the concepts that work with those expressions, who the individuals are in relation to the classroom community also plays a role in how the student is able to use them for familiar situations and create new ones for different problems and situations.

A basic premise of my analysis is that an individual's experience in a classroom plays a particularly powerful role in shaping how one participates in the classroom and the extent to which one feels empowered in his or her use of vectors and vector equations. Who the individual is in relation to the classroom community

addresses directly the issue of identity. Several researchers have studied how identity and belonging play a role in mathematics success. Boaler (1998), for example, examined how the practices at two institutions impacted how students viewed mathematics and how they viewed themselves as doers of mathematics. Since that study, Boaler has continued to examine identity and its role in student's sense of belonging as members of different mathematics communities (Boaler, Wiliam, & Brown, 2000; Boaler, Wiliam, & Zevenbergen, 2000). Her research has focused on the sense of belonging that students adopt and how the institutional forces that students interact with shape their beliefs about themselves and how they understand what it means to do mathematics.

Boaler's research demonstrates that student identity affects persistence in mathematics and mathematics-related fields. She has demonstrated that developing meaning for mathematics and mathematical objects is intimately tied to the communities that one is a part of and the practices of those communities. This work has demonstrated that success in mathematics is not only a result of conceptual and procedural skill, but also having a productive attitude towards the mathematics and classroom community. Boaler & Greeno (2000) have found that students who feel that they are constructing knowledge for themselves feel a greater connection to the field. This leads to greater ability to create and take responsibility for mathematical meaning because the student feels that they the knowledge that they are creating is their own.

Exploring students' negotiability with respect to vectors and vector equations will add to our understanding of how students become enculturated into the practice of

mathematics. As Boaler & Greeno (2000) have pointed out, an increase in student"s

feelings of purpose and the extent to which the student finds the activity personally

meaningful can lead to a greater desire to stay in the field and continue pursuing

mathematics. Conversely, higher levels of alienation can lead to students feeling that

they do not want to be members of the professional mathematics community or

communities that require comfort with mathematics. Recently, Solomon (2007)

examined "the ownership of meaning" that students have for mathematics in the

undergraduate mathematics community versus the professional mathematics

community. Ownership of meaning is an aspect of Wenger's (1998) form of identity

that is closely tied to negotiability. Solomon found that the rote nature of

undergraduate mathematics and the lack of meaningful activity in the classroom led to

students becoming alienated because these students had no say over the meanings that

they were asked to learn. Consequently, she found that this was a key reason why

students, particularly women, left the mathematics field.

Conclusion

   This study examines two specific questions with regard to the symbolic system

for vectors and vector equations. They are:

1. What are the different meanings that this classroom community develops for
    vectors and vector equations?

2. In what ways do individual students contribute to and take responsibility for
    the different meanings that this classroom community develops for vectors and
    vector equations?

My intention in answering these questions is to better understand the process of meaning making for vectors and vector equations. Furthermore, I want to understand in what ways students feel that they can contribute to and shape the meanings that develop in their classroom community.

Studying student symbolizing and meaning-making activities is an important contribution to the field of mathematics education in and of itself. This research contributes to the understanding of the symbolic systems of linear algebra. It also further facilitates the development of instructional materials and instructional theory in linear algebra. More generally, the findings from this study of symbolic systems will aid in understanding how students make sense of algebraic symbols and use them to make meaning in their mathematics classes.

In addition, my research explores the ways and extent to which individual students perceive themselves as equal members in the classroom community when it comes to learning. This is important because students who identify with the activity that is taking place in the mathematics classroom will tend to want to remain as members of the mathematics community. By offering students an opportunity to shape their own learning and engage in doing mathematics, there is a greater likelihood that students will then contribute fruitfully to the mathematics community

Chapter 2 : Literature Review

My dissertation is primarily about how meaning is negotiated for the symbolic system of vectors and vector equations in a linear algebra classroom.   There has been a sizable amount of research on how students learn to interpret graphs and the difficulties that they may encounter as they work with the symbolic systems for mathematics (for an excellent overview of the research on this subject before 1989, see Leinhardt, Zaslavsky, & Stein, 1990). Furthermore, the National Council of Teachers of Mathematics (2000) has made the understanding of the symbolic systems and the communication of mathematics through symbolic systems a priority for math education as well. While much work has been done in the field of symbolizing, there continues to be a sizable need for continuing work on how classroom communities develop symbolic systems.

In this chapter, I examine the literature from perspectives that have been informative to the formulation of my research questions and methods and to my analysis. I first examine the literature on classroom mathematics practices and the development of normative discourse in mathematics classrooms. This literature is particularly important to my dissertation as Chapter 4 utilizes similar methodology to these works and one of the key contributions of my research is to understanding how the classroom community develops meaning for symbolic expressions. Second, I look

at the literature regarding linear algebra and place my questions within that context. Third, I detail why examining symbolic systems is important to the study of mathematics education. Fourth, I detail what it means to learn in a community of practice. Fifth, I discuss research on the role of discourse and its impact on the learning of mathematics. In this section I also examine some important views on the development and analysis of normative discourse in classrooms. Finally, I detail literature on the development of practices that contribute to meaning making in mathematics classrooms, focusing particularly on literature within collegiate mathematics.

Collective meaning-making in Mathematics Education

In this section, I review the development of classroom mathematical practices (CMP's) and discipline-specific practices such as algorithmatizing, symbolizing and defining. I also detail literature on how students develop models and then use those models to further their mathematical activity. Finally, I discuss literature that explores how understanding collective meaning-making can expand beyond what is said to other elements of the discourse especially gestures. As was stated in chapter 1, reification takes place via participation in the activities of the community. One of the ways that I examine the process of reification is via what meanings function-as-if-shared in the classroom. Rasmussen & Stephan (2008) explain that for a meaning to function-as-if-shared in the classroom, all students do not need to have identical meanings. Instead, a functioning-as-if-shared meaning is identified by student's ability to communicate effectively with the meaning and use those meanings for later

mathematical work. From this perspective, I posit that the process of reification for a symbolic notation occurs as students continuously develop functioning-as-if-shared meanings for that notation. The development of these collective, functioning-as-if-shared meanings can then be brought to bear in later mathematical work demonstrating the ways that the symbolic expression has become reified.

In my dissertation, the meanings that the classroom community established are classified as classroom symbolizing practices. Classroom symbolizing practices are a subset of classroom mathematics practices that deal explicitly with the collective meanings generated for symbolic expressions. A classroom mathematics practice is defined as a collection of functioning-as-if-shared ways of reasoning organized around a particular concept, activity or as is frequently my case symbolic expression or collection of symbolic expressions (Rasmussen & Stephan, 2008). More generally, a practice is an action that is used by members of a community in order to accomplish a given task or goal (Wenger, 1998). However, what it means to engage in that practice is a product of the negotiation of meaning in the classroom community. As students participate in the community, the practice becomes imbued with meaning in terms of what it means to engage in that practice and for what reasons one would engage in that practices. In Stephan and Rasmussen (2002), the researchers analyzed the collective meaning-making that occurred over the first 22 days of instruction in a semester long in a differential equations class. As a result of their analysis, the researchers identified a series of classroom mathematics practices that emerged during the course. In answering the first research question, I first identify ideas and ways of reasoning that function-as-if-shared in the classroom community's symbolic expressions and then

group them around a set of common themes to detail the classroom mathematics practices that emerged in the classroom for vectors and vector equations. Rasmussen and Stephan (2008) expanded upon the findings from Cobb, Stephan, McClain and Gravemeijer (2001), who characterized classroom mathematics practices in a second grade math classroom. Stephan and Rasmussen (2002) found that a specific idea or way of reasoning can be part of more than one classroom mathematics practice. They also found that classroom mathematics practices do not necessarily proceed in chronological order, but instead they may progress non-linearly. Both of these findings are particularly important for the study of undergraduate mathematics because of the complexity and difficulty of advanced mathematics; students may not develop meaning in a linear fashion and more complex concepts require multiple interconnected ideas. This is particularly evident in my analysis.

A second kind of practice, discipline specific practices, are less organized by normative ways of reasoning and more by the normative activities that generate the normative ways of reasoning. In this case, the discipline in question is the discipline of mathematics. Discipline specific practices are not specific to a particular content area, but instead may arise in various different content areas. What they have in common is that they are utilized by members of the mathematics community to produce knowledge and solve problems. Discipline specific practices are sanctioned ways of creating knowledge within the community of mathematics practitioners. Rasmussen, Zandieh, King, and Teppo (2005) examined the symbolizing, algorithmatizing and defining activities of the members of undergraduate mathematics classrooms. The authors analyzed how three different mathematics classrooms engaged in discipline

specific practices. Developing more sophisticated ways of symbolizing, algorithmatizing, defining or proving was a hallmark of becoming a more central participant in the mathematical community. The process by which students develop and refine discipline specific practice is called mathematizing. This dual process consists of horizontal and vertical mathematizing. By horizontal mathematizing, the authors mean "formulating a problem situation that makes it amenable to further mathematical analysis (p. 54)." The process and products of the process is fundamentally situated in the context of the problem scenario. By this I mean that students activity at this level refers to the problem scenario or features of the problem scenario. Vertical mathematizing is understood in relation to the student's current mathematical activity. This activity works with already mathematized activity and develops further mathematics from that activity. This activity is often more formal than horizontal mathematizing as it is working with previously mathematized activity. One way to consider the processes of vertical and horizontal mathematizing is via the process of reification. As students consider mathematical problems and scenarios, they develop tools for solving these problems. These tools become imbued with meaning through participation and are hence utilized to solve other problems and develop more mathematics. As they are used for more and different kinds of mathematics, their reified meanings transform into new and expanded meanings.

Rasmussen & Blumenfeld (2006) expand on this research by analyzing how students solved systems of differential equations in an inquiry-oriented classroom. The authors' analysis of the student activity serves to illustrate the process of emergent models (Gravemeijer, 1999) that can occur in mathematics classrooms. Models, as

defined by the authors, are student-generated ways of dealing with mathematical activity. These models are characterized as emergent because the tools and symbols that are used in student reasoning arise with that reasoning. The student's use of analytic expressions arose from their use of vectors and vector fields. In this way, analytic expressions functioned as a model-of the student's activity. This allowed for the model to become in and of itself an object of focus and hence reified. But the modeling process does not stop once the model has become an object of focus, but instead the students utilized analytic expressions as a model-for more generalized and formal mathematical reasoning. In the author's case students, finding analytic solutions to differential equations led to general solutions to systems of linear equations. This gives an example of a process of reification that could take place in a given classroom. The student's activity becomes reified in the inscription and hence that inscription takes on meaning for members of the class. Through further participation, the meaning of the object transformed as it was used in different ways to accomplish new mathematical goals.

The "emergent model" heuristic has not just been used to understand students modeling activities. Zandieh and Rasmussen (2010) expanded on the heuristic to include student's activity with definitions. The authors analyzed how a group of students took their pre-existing definitions about triangles and expanded and reinvented them to discuss the properties of triangles on a sphere. Students in a math class for prospective high school teachers began with their familiar definitions of a triangle. Using these familiar definitions, the authors illustrate how the students proceeded through the definition-of a triangle on a sphere to using the definition-for

further mathematical activity. The continuum that the author's used to demonstrate the classroom community's defining utilizes four different kinds of activity, situational, referential, general and formal. Situational activity refers to students mathematizing in an experientially real mathematical setting. In this level of activity, student's mathematical activity is tied tightly to the context. In the authors' analysis, student's definitions and concept images of triangles on the Euclidean plane formed the experientially real context in which they begin to work to develop a working definition of triangles on a sphere. In the second level of activity, referential, the activity is still tied to the context, but instead involves developing a model-of the mathematical activity that refers to the original experientially real situation. In this scenario, the students utilized their understandings of triangles on the plane in order to develop a definition of a triangle on the sphere. General activity then involves the use of the original model-of student activity as a model-for later activity. This allows students to develop interpretations and solve problems that are independent of the experientially real setting. In this case, students attempted to find the ramifications of having a triangle on the sphere, for example what the sum of all the angles in the triangle are. The authors emphasize that the difference between referential and general activity is the source for student activity. In referential activity, the source of student activity is the experientially real context. In general activity, the source is the new examples, images or definitions that students have created. Finally, formal activity refers to utilizing the new definitions or models in order to create new mathematics. In this study, the definitions and the ramifications of those definitions that students developed are then used to engage in further mathematical activity.

Rasmussen & Blumenfeld (2006) also references these levels of activity in describing emergent models. In both cases, the researchers were seeking to describe "the new mathematical reality" that emerged as students engaged in mathematical activity. In the case of the defining example, the new mathematical reality was the world of non-Euclidean geometry. In the differential equations example, the new reality was the general solution to the system of linear equations. These new mathematical realities allow for students to envision and work on mathematics that was otherwise not apparent before the development of the emergent model.

Taken as a whole, the studies that I highlighted demonstrate how argumentation and other forms of activities such as gesture further the mathematical meaning making in the classroom. Indeed, several studies focus specifically on the role of gestures in the meaning making process (Alaĉ & Hutchins, 2004; Goodwin, 2004; Rambusch & Ziemke, 2005; Rasmussen, Zandieh, & Wawro, 2009). For example, Marongelle (2007) analyzes how students used gestures in order to communicate meaning while creating the Euler algorithm for approximating first-order differential equations. Marongelle expands on the work of Rasmussen et al. (2005) by examining how the graphs that students created and the gestures that they displayed while using these graphs facilitated communication as they negotiated meanings within the classroom community. She conjectures that students gesturing and graphing changed as students moved from creating the algorithm to using the algorithm for different mathematical activity. In a similar vein, Rasmussen, Stephan, and Allen (2004) examine the function of gesturing and argumentation in the development of classroom mathematical practices. These researchers found that

gestures were aligned with argumentation as students negotiated the meaning for mathematical objects in the classroom. Furthermore, like the ideas and ways of reasoning that function-as-if-shared in the classroom, specific gestures could be utilized in more than one classroom mathematics practice. Both of these studies demonstrate what was earlier discussed in Roth and Lee (2003) and Goodwin (2004), where learning and communication in a given community of practice is multi-modal.

Why linear algebra?

Introductory linear algebra is one of students' first experiences with the system of symbolic forms populated with vectors and vector equations on a wide scale. Computation with such a system of symbols is fundamentally different from computation on the real number line. With vectors students must work in two or more dimensions, requiring new ways of interpreting the symbols and connecting them to geometric representations. In one dimension the mathematics on vectors is identical to the mathematics of the real line, however at higher dimensions the necessary mathematics is different. For example, multiplication needs to be redefined because analogies to stretching or iterating multiple times no longer suffice. Pickering (1995) has noted the issues that Hamilton encountered when trying to make this compromise. In his analysis, Pickering analyzes how Hamilton devised a fairly straightforward symbolic algebra with points in four-dimensional space, but when he needed to construct a geometric interpretation, conventional geometric approaches to multiplication in one-dimensional space, stretching or shrinking, were insufficient for a consistent geometric interpretation. Pickering notes that the creation of the

quarternions, Hamilton's solution to the geometric/algebraic dilemma, was the result of his engagement in practices that were common to the mathematical community. While the creation of these important mathematical objects seemed to be the product of particular imaginative insight, it was actually a process of continuously engaging in the normative behavior of the mathematics community.

A sizable amount of research in linear algebra has documented student difficulties in linear algebra, particularly as these difficulties relate to students' intuitive or geometric ways of reasoning and the formal mathematics of linear algebra (Dogan-Dunlap, 2010; Gueudet-Chartier, 2004; Harel, 1990). Researchers in linear algebra have noted that the field requires students to consider mathematics from a variety of perspectives, including formally as it relates to proof, geometrically, and computationally, among others. Hillel (2000) constructs a framework for understanding student reasoning in linear algebra. He identifies three modes of description in linear algebra: geometric, algebraic, and abstract. Hillel found that the geometric and algebraic modes of relating to vectors and vector spaces can become obstacles for understanding the abstract modes because they limited the amount of generality that a student can draw from either geometric or algebraic examples. However, even though Hillel makes a compelling argument that geometric and algebraic modes of reasoning can be obstacles to formal reasoning, geometric and algebraic reasoning are crucial to the overall understanding of linear algebra.

Wawro, Sweeney, and Rabin (2011) analyze the ways that students used different modes of representation in making sense of the formal notion of subspace. In this study, the researchers found that students employed geometric and algebraic

reasoning to shape and mold their understanding of formal notions of subspace. Specifically, the authors studied the relationship between student's understanding of the definition of subspace and student's concept images for the concept. In the study, students demonstrated a variety of ways of engaging with the formal definition and utilized geometric, algebraic and metaphoric ways of relating their concept image and the definition. The results of the study suggest that in generating explanations for the definition, students rely on their intuitive understandings of subspace. The study suggests that building on student intuitions to generate formal understandings of complex linear algebra concepts may be productive.

While many studies have focused on students dealing with the formal challenges of linear algebra, many researchers have also been tackling the problems that students deal with when encountering the new symbolic system of linear algebra. The results of these studies may be particularly important for my own research. Harel and Kaput (1991), for example, demonstrate that students had difficulties in generating relationships between many of the formal and algebraic symbols used in linear algebra and the conceptual entities that they are intended to represent. In examining student's decisions about whether a given set was in fact a vector space, the authors demonstrate that students who related to the vector space as a conceptual idea were better able to reason about whether a given set was a vector space than those who procedurally checked the axioms against the new set. Because symbols in advanced mathematics in general, and in linear algebra in particular, connect so many different ideas (e.g., formal notions, systems of equations, vector systems, etc.), developing an understanding of what a symbol represents conceptually is crucial to understanding

linear algebra as a whole. Further evidencing students' difficulties with symbols in linear algebra, Britton and Henderson (2009) demonstrate that students had difficulties in dealing with the notion of closure. Specifically, the students had problems in moving between a formal understanding of subspace and the algebraic mode in which a problem was stated. These authors argue that student difficulties stemmed from an insufficient understanding of the various symbols used in the questions and in the formal definition of subspace.

Dreyfus, Hillel, and Sierpinska (1998) postulated that a geometric but coordinate-free approach to issues such as transformations and eigenvectors may be helpful in coming to understand these concepts. These authors have further demonstrated that the understanding of abstract symbols of vectors and formal vector notation can be aided by a strong geometric understanding in two and three dimensions. The authors found that the use of a computer environment and tasks enabled students to develop a dynamic understanding of transformation, but that it hindered their ability to understand transformation as relating a general vector to its image under the transformation. In another study, Sierpinska, Dreyfus, and Hillel (1999) investigate how students determined if a transformation was linear or not using Cabri. They discovered that students made determinations about a transformation's linearity based upon a single example. Thus, they checked if for the vector, $\mathbf{v}$, a scalar k, and the transformation T, $T(k\mathbf{v})=kT(\mathbf{v})$. For this task the researchers found that the students checked only one image of $k\mathbf{v}$ under the transformation and did not vary $\mathbf{v}$ using the program's capabilities. Each of these studies has contributes to our understanding of student difficulties with the various symbolic expressions used in

linear algebra. Sierpinska et al.'s work with vector notation pointed to using geometric reasoning to better understand the formal notation of vector systems. However, she only asked students about vectors in two dimensions and explored only how students utilized these objects in relation to general vector spaces. In my study, I explore student's intuitive notions for vectors, their symbolic creations using vectors and vector equations, and how they utilize those intuitions and creations to develop a more formal notion of vectors and vector equations.

Vector notation and learning to work with vectors and vector equations also play a role in understanding the set theoretic underpinnings of linear algebra. Dorier, Robert, Robinet and Rogalski (2000a) express concern that the strong emphasis on algebraic concepts in linear algebra leaves little room for set theory and elementary logic. They contend that this absence leads to difficulty in working with the formal aspects of linear algebra. For example, students are often unable to reason with definitions and abstract concepts. Dorier, Robert, Robinet, and Rogalski (2000b) and Rogalski (2000) took an approach to dealing with these problems that involved teaching linear algebra as a long term strategy, having students revisit problems in a variety of different settings—geometric, algebraic, and formal. It also involved what the authors call the *meta-lever* in which students reflect on their activity in order to draw connections between the various settings and to build generalizations.

The examples cited so far have focused on linear algebra from a primarily formal standpoint. An assumption behind many of these approaches is that formal linear algebra is or should be one of the fundamental products of learning linear algebra. And while certain formal concepts can be very powerful for applying and

understanding linear algebra, some new approaches have used concrete examples and applications to teach linear algebra concepts. Klapsinou and Gray (1999), for example, studied a course in which students were first given concrete instantiations of linear algebra concepts and then used those concrete instantiations to generate understanding of the formal definitions of these concepts. The authors noted that students who were taught in this manner later had difficulty with understanding the definition and applying it to different situations. The authors argue that taking a computational approach and then developing the abstractions refines student's processes for doing computation in linear algebra, but not their understanding of certain concepts as objects. Portnoy, Grundmeier and Graham (2006), in a study of pre-service teachers in a transformational geometry course, demonstrate that students who had been utilizing transformations as processes that transformed geometric objects into other geometric objects had difficulty writing proofs regarding linear transformations. The authors argue that the process nature of student understanding of transformation inhibited the necessary object understanding for writing correct proofs.

Geometric and visual approaches to linear algebra have been effective in helping students develop understanding of the abstract symbolic notation of vectors and vector equations. Stewart and Thomas (2007) conducted a study of two groups of linear algebra students. They investigated student learning in a course in which the students were introduced to embodied, geometric representations in linear algebra along with the formal and the symbolic. The authors claimed that the embodied view enriched student understanding of the concepts and allowed them to bridge between concepts more effectively than employing just symbolic processes. In another study,

Stewart and Thomas (2010) demonstrate that students viewed basis from the perspective of the embodied, as a set of three non-coplanar vectors, symbolically, as the column vectors of a matrix with three pivot positions, and formally, as a set of three linearly independent column vectors. These studies demonstrate that in order to develop a nuanced and complete understanding of linear algebra, students need to consider linear algebra from a variety of different perspectives, with each perspective (geometric, embodied, symbolic or formal) informing the others in different ways.

Stewart and Thomas' studies are in line with a series of studies that indicate that teaching of linear algebra should begin with students' intuitive understandings. Dubinsky (1997) details how APOS theory could be used to analyze student thinking and develop linear algebra pedagogy from a constructivist perspective. Studies in linear algebra from an APOS perspective have focused on a variety of concepts including linear independence and dependence (e.g., Bogomolny, 2008). Recently, Stewart and Thomas (2009) used APOS theory in conjunction with Tall's (2004) three worlds of mathematics understanding (embodied, symbolic, and formal) to analyze student understanding of various concepts in linear algebra, including linear independence and dependence, span and basis. In a series of studies, the authors found that students did not think of many of these concepts from an embodied standpoint, but instead tended to rely upon an action/process oriented, symbolic way of reasoning.

In order to address students' difficulties in bridging the many representational forms and the variety of concepts present in linear algebra, some researchers have turned to computers to aid in teaching (e.g., Berry, Lapp, & Nyman, 2008; Dogan-

Dunlap & Hall, 2004; Hillel, 2001). Recently, Meel and Hern (2005) created a series of interactive applets using Geometer's Sketchpad and *JavaSketchpad* to teach linear algebra. More recently, different research teams have been spearheading innovations in the teaching and learning of linear algebra. Cooley, Martin, Vidakovic, and Loch (2007) developed a linear algebra course that combines the teaching of linear algebra with learning APOS. In Mexico, researchers have been working with Models and Modeling (Lesh & Doerr, 2003) and APOS to develop instruction that leverages students' intuitive ways of thinking to teach linear algebra. For example, Possani, Trigueros, Preciado, and Lozano (2010) utilized a genetic decomposition of linear independence and dependence and systems of equations in order to aid in the creation of a task sequence. The task sequence, which asked students to model the coordination of the traffic flow in a particular area of town, was designed to present students with a problem that they could first mathematize and then use to understand linear independence and dependence.

In the United States, another group of researchers is drawing on sociocultural theories (Cobb & Bauersfeld, 1995) and the instructional design theory of Realistic Mathematics Education (Freudenthal, 1973) to explore the prospects and possibilities for improving the teaching and learning of linear algebra. Using a design research approach (Kelly, Lesh, & Baek, 2008), these researchers simultaneously created instructional sequences and examined how student's reasoning about key concepts such as eigen-vectors and eigen-values, linear independence, linear dependence, span, and linear transformation (Henderson, Rasmussen, Zandieh, Wawro, & Sweeney, 2010; Larson, Zandieh, & Rasmussen, 2008; Sweeney, 2011). For example,

Henderson, et al. examined student's various interpretations with the equation A [x y] = 2 [x y], where [x y] is a vector and A is a 2 x 2 matrix prior to any instruction on eigen theory. They identified three main categories of student interpretation and argue knowledge of student thinking prior to formal instruction is essential for developing thoughtful teaching that builds on and extends student thinking.

Why symbolizing?

Understanding the collective production of meaning for vectors and vector equations is important because it provides insight into what Arcavi (1994) calls symbol sense. Linear algebra is replete with symbolic expressions for students to understand and use in order to solve problems. Just as the symbolization of high school algebraic symbols is problematic for students, so too is the symbolic expressions of linear algebra. Arcavi states eight different goals for developing symbol sense for high school algebra students. He states that students need to be able to read, interpret and manipulate symbols, so as to solve problems and interpret whether the conclusions drawn from the manipulation are reasonable. They should be able to create their own symbolic expressions for use in solving problems. They should be able to discern differing meanings from equivalent symbolizations and coordinate those different meanings in helpful ways. They should be able to choose between different symbols and expressions for different purposes and to be flexible in our choice of manipulations for different purposes. Finally, students should be able to reflect on different symbolizations and be able to contextualize symbolic expressions.

These goals do not just apply to the study of algebra. Instead, they extend to the understanding of any symbolic system. The students that I am studying are already somewhat mathematically sophisticated. Each of them has completed first semester calculus and hence has some level of mastery over the basic symbolic language of algebra. As linear algebra students they are asked to understand a new symbolic language, for example vectors and vector equations. Linear algebra, in particular, has a great deal of symbolic complexity, which is accompanied by its abstract nature and its difficult subject matter. Students are asked to understand and interpret vectors, vector equations, graphs of vectors and vector equations, graphs of new bases, etc. Furthermore, algebraic symbolizations that once had a certain meaning in the contexts of high school algebra and freshman calculus, are given new meaning as the symbols that are used to stand in for numbers or single unknowns are now used for matrices or vectors.

Some work has been done on student understandings on the difficulties that students have in moving from intuitive geometric symbolizations to theoretical representations like vector equations (Hillel, 2000; Sierpinska, 2000; Stewart & Thomas, 2009). Other work has been done on how students develop symbolic models of realistic situations in linear algebra (Larson, et al. 2008; Possani, et al., 2009). Larson, Zandieh, Rasmussen & Henderson (2009) examined how students interpret symbolizations that are common in high school algebra, but are now used within the new context of linear algebra. The authors explored student sense-making of the vector equation $A\textbf{x}=2\textbf{x}$. In this equation, the A represents a matrix and the bolded "x" represents a vector. As students tried to make sense of the symbolic expression, they

struggled with interpreting the 2**x** as a vector and dealing with A**x** as the product of a process of matrix multiplication. The researchers' analysis further concluded that student symbolic meanings for the equal sign made a significant impact on student's interpretation of the matrix equation. And conversely, their interpretation of the two sides of the equation had an impact on how the students interpreted the equal sign.

Duval (2008) has discussed some of the challenges that would be involved in developing a semiotic approach to mathematics education. He argues that the number of symbolic expressions involved in learning and understanding mathematics necessitates thinking about mathematics education in a way that is similar to reasoning about language and the acquisition of language. Duval covers eight research questions that can arise from studying the learning of mathematics from a semiotic perspective, two of which are particularly relevant to my study. The first is "How can we analyse the relationship between the variety of semiotic representations used in mathematics and what they stand for? (p. 40)." One of the primary problems that may arise in mathematics for students is the need to see all of the various symbols, graphs and tools in mathematics as part of a coherent system. Linear algebra requires students to navigate between a variety of mathematical symbols and to make sense of each one. The different symbolic relationships, whether they are vector equations or systems of equations, have uses that have been negotiated in the classroom and determined in the mathematical community. One of the challenges for linear algebra students is to understand where and when to use these symbols and what special insight can they give to understanding a situation or the mathematics.

The second problem that Duval (2008, p. 42) posits is, "How can we analyse the changes from one given semiotic representation to another?" A possible obstacle to understanding mathematics is understanding how two representations may on the surface seem to have nothing in common, for example a graph and a linear equation. Yet doers of mathematics are asked to relate to the two objects as if they are connected and virtually the same. A challenge for the study of these symbolic relationships is to understand how it is that students see these objects as the same either through their talk or through their use. A challenge in linear algebra is to see how a vector equation, a graph and a system of equations can represent the same situation and the same basic mathematical relationship. My first research question sheds some light on this important concern.

Research on how students develop meaning for symbolic expressions is necessary to accomplish both Arcavi (1994) and Duval's programs. Students need to develop symbol sense in order to become fully participating members of a linear algebra classroom and be able to communicate effectively in a mathematical world. Conversely, understanding what it means to use and interpret the symbolic systems of mathematics means that we as researchers need to consider how it is that students see different representations as the same and negotiate the wide variety of symbols present in modern mathematics.

Learning to symbolize in communities of practice

My dissertation develops some answers to Duval's (2008) two questions in the context of linear algebra. Other authors, in other disciplines, have posited some

explanations as to how the process of symbolizing occurs. The following section investigates a few of these explanations that are particularly relevant and powerful for understanding my research questions. In this section, I look at research on how members of a community of practice develop meanings for symbolic expressions and the ways that they use those meanings and expressions to solve problems and produce artifacts within the community.

A large body of work has been done in science studies on how scientists, science students and apprentice students come to develop meaning and understanding for inscriptions in the field. Latour (1987), for example, examines how the progress of science was facilitated by the invention of the printing press and hence the spread of scientific inscriptions. The invention of the printing press facilitated the easy spread and dissemination of graphical inscriptions to scientists across Europe. Hence, the inscriptions became objects of focus that could be scrutinized and critiqued by others. This strengthened the arguments of the original scientists and spread the ideas of those scientists for further refinement by their colleagues. Furthermore, this spread and dissemination allowed for what Latour calls a "cascade of inscriptions." Scientists could compare inscriptions and pull out relevant information from each, developing ways in which the different inscriptions told the "same" story, except from differing viewpoints. This juxtaposing of inscriptions allowed for scientists to cull meanings from the inscriptions that otherwise would not have been available. Furthermore, the use of inscriptions led to the "mathematizing" of scientific discourse. What Latour means by mathematizing is that experience of the world, the objects of scrutiny in science, could be measured and condensed into numerical and graphical inscriptions,

which could then be combined and recombined to give new perspectives and highlight underlying patterns in the natural world.

Researchers in science studies and science education have also examined how contemporary scientists and science students gather meaning from graphical inscriptions and become enculturated into the practice of being scientists. Kozma et al. (2000) has examined how scientists use graphical inscriptions to make sense of scientific phenomena. These researchers explain that scientific inscriptions do not give all of the information necessary in order to make conclusions. Instead, scientists cull from the inscription the desired information and combine that with information from other inscriptions to form conclusions. This process not only develops scientific knowledge, but transforms the inscription into a tool for arguing for the particular meaning that the scientist wants to convey. Kozma, et al. conclude that the process by which apprentice scientists become expert scientists is a process by which the novice learns to recognize the appropriate facets of a given inscription and then learns how to argue that those facets support their given argument. In related work, Roth & Bowen (2001) examined how professional scientists utilized graphs in order to structure the world that they are studying. This world is structured via interaction with others or with objects that are reified via both embodied and representational practices. Both of these studies illustrate that in the world of the professional scientist or mathematician, interpretations for graphs are constructed through their use within a community of practice. And the inscription is thus transformed into a tool for arguing what the professional sees as the appropriate interpretation of phenomena.

Goodwin (1994) examines specific practices that lead to being able to "see" or perceive in a manner that allows them to communicate and contribute to the community of practice for which they are a part. ." As in Kozma's et al. (2000), Kozma (2003), Latour (1984), and Goodwin (1994) argue that members of a community produce and articulate material representations, code events and objects in a specific setting into objects of knowledge for a given group, and highlight the facets of a given event in order to transform the event into something salient for further scrutiny. Goodwin (1994) states that these practices are some of the ones that transform an experience into an object for examination in a given discourse. He states:

> An event being seen, a relevant object of knowledge, emerges through the interplay between a domain of scrutiny (a patch of dirts, images made available by the King trial, etc.) and a set of discursive practices (dividing the domain of scrutiny by highlighting a figure against a ground, applying specific coding schemes for the constitution and interpretation of relevant events, etc.) being deployed within a specific activity (arguing a legal case, mapping a site, planting crops, etc.). (p. 606)

The first aspect of this statement that should be made salient is that an object of knowledge is emergent. This means that what a scientist or mathematician perceives is not already there in the world, but instead must be made noticeable and understandable through the practices of the discourse. Furthermore, the creation of an object is action-oriented. They are made with specific goals and intentions and are designed to facilitate the work of the discourse. Finally, this emergent knowledge is facilitated through practices that make salient the desired meanings that allow for the continuation of the discourse. One of the techniques for making these objects salient

is highlighting. Highlighting points out what objects in the field of scrutiny are necessary for a discussion of "what is happening here." A second technique, coding, places an interpretation of that highlighted object that allows for further work within the discourse. And then coded and highlighted objects of scrutiny can be combined into material representations for transport, combination or further scrutiny.

More recently, Goodwin (2004) adds specific social interactions to the list of discursive practices that are used by members of a community in order to produce and articulate objects of knowledge Although these interactions are implicit in the work of highlighting, coding, and material representations, he expands on highlighting, etc. as modes of communication. Of particular note was the role that body movement, hand gestures, and verbal argumentation played in the discourse.   Goodwin illustrated how specific ways that his subjects moved their gaze or pointed to objects highlighted for others what was to be looked at and what was important in a given interaction. Furthermore, importantly for my own work, Goodwin notes the role of gestures combined with arguments to develop what is important and salient within any given interaction or inscription.

Understanding that the development of knowledge is a product of these complex discursive processes has compelled some researchers to look outside of the individual to understand the meanings for symbolic expressions and graphs that individuals verbalize and produce.   Roth & Lee (2004), for example, have developed an analytic framework for understanding and interpreting student and professional interpretations of unfamiliar graphs in a science setting. Roth states that meaning is co-produced by the respondent and by the interviewer or other members of the

classroom. First, this is because the respondent is communicating in a manner that fits the discourse in which the interview or the classroom situation occurs. The production of an argument or an interpretation is the result of the individual's role in the interaction as a respondent. Furthermore, the kind of response or interpretation that is produced is the result of the normative guidelines of the environment that the individual is in. Second, the interpretation unfolds over time. The meaning given to any interpretation of the past experience of the individual and its creation and statement are the result of a moment in time. Encapsulated in their production is the past, the present existence, and future expectations of use and provision of meaning. Thus, analyzing any inscription requires considering how that object comes to be within the community and to what use it is intended. Finally, gesture, speech and the salient objects highlighted by gesture, speech or body movement are irreducible. Gesture, speech and body movement are part of the communicative and thinking process and hence understanding what they are communicating requires considering them as a cohesive whole.

Examining the construction of meaning for graphs, inscriptions and symbolic systems requires an examination of the activity system as a whole. The individuals in the linear algebra class that I am studying are not isolated cognizing units, but instead are part of an intricate and complex activity system. This activity system is made up of a variety of complementary and sometimes conflicting discourses. Communicating and generating knowledge within this activity system is multi-modal (Roth & McGinn, 2004), meaning that it incorporates speech, the inscriptions, the discourse, gestures and body movement to communicate and structure the environment. Consequently,

examination of the meanings developed in this discursive community requires examination of the activity system as a whole. Furthermore, the inscriptions that are created and interpreted are products of and reflect the goals and intentions of the community. Thus, meaning-making is action oriented and temporally bound, as it comes from a particular process of construction and reflects a desire for a future state of events. Finally, meaning is generated via social interaction as members of a community decide upon what objects are relevant for the discourse and in what ways they are relevant.

Understanding the discourse of the classroom

Understanding how the classroom community develops meaning for vectors and vector equations requires examining the concepts and symbolic expressions produced as a result of classroom activities and the normative activities that lead to these productions. Discourse studies have contributed greatly to our understanding of the normative activities of communities that lead to its knowledge production. Gee (1998) posited a distinction between the normative activities that provide the structure for a communities activities, called big "D" Discourse, and the actual activities that take place as a result of group interaction, called small "d" discourse. The Discourse of the classroom plays a large role in constituting what meanings get used in the community and hence by the individuals within the community. Understanding big "D" Discourse has recently become a focus in math education as math education researchers consider their students as members of their component communities and not as isolated cognizing units. Doing so necessitates examination of classroom small

"d" discourse, as this discourse gives us insight into the normative expectations and conventions of the big "D" Discourse. Classrooms are de facto communities whose existence is relatively short-lived and that lack their own histories (outside of their history as a part of an institution, for example the university). Consequently, I consider how the discourse comes to be. In this section, I look at some beneficial literature on normative Discourse inside of classrooms. I examine some work that has been done on more traditional classrooms, but I will focus primarily on the development of normative culture within inquiry-oriented classrooms.

Examination of classroom discourse is not new to the study of education. For example, the seminal work of Mehan (1979) examined the predominant discursive pattern within traditional classrooms. This pattern called Initiate-Respond-Evaluate (IRE) was common across geographical regions and was located across the spectrum of grade levels, K-14. The pattern first had the teacher initiate conversation with a student by posing a question that probes for particular answers. Then, the student would respond with either a correct or incorrect answer, or an answer of "I don't know." Then the teacher would evaluate the veracity of the response. The power of the IRE discursive pattern was not wholly in its pedagogical effectiveness, although for certain purposes it is highly effective, instead it was in its ability to set and regulate the discourse of the classroom. This pattern establishes the teacher as the arbiter of knowledge and the regulator of the discourse. The teacher initiates all conversations and determines the veracity of any statements made by the students. This places the onus of power over the discourse in the hands of an outside authority for the students. This in turn indicates for the student where the source of knowledge is in the class. It

also tells the student their role in the classroom. They are supposed to memorize answers or procedures and then recall them back for the teacher in a manner that is the same or similar to the one in which the teacher first gave it. This means that students are to listen for how to do procedures, what information could possibly be asked as a question, and in what manner answers should be appropriately given. Furthermore, because the IRE pattern has been a part of schooling from kindergarten onward and is located in many different environments, the IRE pattern establishes a way of acting that is comfortable, reassuring, and consistent with their previous experiences of schooling.

Researchers have challenged the efficacy of the IRE pattern for accomplishing certain goals in the classroom, most especially fostering beneficial beliefs and dispositions about the practice of mathematics (Wood, 1992; Lemke, 1990, Engeström, 1991). For example, Boaler (1998), in a study of two different high schools, examined how different discursive practices in the teaching of mathematics and the institutional disposition towards mathematics impacted student beliefs and attitudes towards mathematics. In the traditional school, Amber Hill, where the IRE pattern was prevalent, students were asked to listen to lectures for half of the class time and work silently in their seats for the other half of the time. Many of the students interviewed from this school stated that they believed mathematics to be procedurally bound and that there was only a right and wrong answer for any mathematical problem. Their views on the subject were rigid and they often expressed a dislike for the subject. Conversely, the second school, Phoenix Park, had a curriculum and institutional disposition that fostered inquiry. Students at this school were encouraged to find their

own problems and develop understanding of the mathematics via solving those problems. Students at this school found mathematics to be less cut and dry. As well, they believed that mathematics was about finding solutions to problems and discovering ways of interpreting the world. These students also expressed a greater willingness to continue in mathematics and mathematics-based pursuits.

In addition to shaping their attitudes and beliefs about mathematics, the institutional discourses facilitated how the students went about completing mathematical tasks. The discourse of Amber Hill led to students developing cue-based ways of reasoning about mathematics. Students would base their responses to mathematics questions on tests and in class based upon cues that were consistent across their books, classroom discussions, and exams. In order to find solutions to their problems, the students would assume that the solution methods were the same for every problem in a given section. In fact, this method was highly effective as each problem only tested a very specific solution method. Hence, these students would rarely have to make connections across mathematical ideas because where the problem appeared in the book, and not the mathematical concepts, determined the solution method. Conversely, the students at Phoenix Park made connections across mathematical ideas more readily and related to mathematics as being more cohesive. This was because the real world or realistic problems that they were working with did not focus on individual solution methods, but instead focused on understanding how mathematics and phenomena were connected.

One approach to transforming the discursive environment of the math classroom is to change the nature of the tasks that students are asked to do. The

students at Phoenix Park were asked to do problems that were outside the traditional curriculum and this change contributed to the difference in institutional discourses. Significant work has been done on transforming discourse via change in curriculum. One notable line of work in this direction was spearheaded by Freudenthal (1973), who characterized mathematics as a human activity that necessitates problem solving with symbolic tools. Mathematics is conceptualized and structured to deal with these real world, situated problems. Hence, tasks need to be constructed to deal with the real-world problems. This caused a change in the ways that curriculum was designed, as persons utilizing Realistic Mathematics Education (RME), the instructional design heuristic based upon Freudenthal's writings, developed new materials to reflect this real-world approach. Previous mathematics education was founded primarily upon abstract and solely symbolic approaches. Consequently, this switch caused a change in the discourse in these classrooms as math became tied to situations that were connected to students experience.

Even if there is a new curriculum with which teachers can work, students still enter into the classroom with expectations of what it means to be in a mathematics classroom. Continuation of IRE patterns of discourse or other markers of traditional schooling could lead to a continuation of the Discourse of the traditional classroom. Thus, other aspects of the classroom community need to be changed in order to enact changes in the overall discourse of the classroom.

The Phoenix Park school in Boaler's (1998) study utilized an inquiry driven curriculum. Furthermore, students were encouraged to discuss their solutions with one another and to explore topics that were personally meaningful to them. According

to Boaler, this allowed for a switch in the normative behavior of the students at the school. According to Cobb & Yackel (1996), the normative culture of the classroom can be characterized by the social and sociomathematical norms of the classroom. These two distinctions characterize what it means to be a student, to do mathematics and be a mathematics student in this classroom. An example of a social norm would be that students are asked to justify their answers to members of the classroom. An example of a sociomathematical norm would be what constitutes acceptable evidence to support conclusions (e.g, conclusions require mathematical data to back them up). A second example of a sociomathematical norm is what makes a particular mathematical solution different from a previously given solution. What separates a social norm from a sociomathematical norm is that the sociomathematical norm is specific to mathematics and the social norm could characterize any classroom. Hence, a normative behavior that has to do with having to justify your conclusions could be in any classroom. However, if the norm dictates that the quality of the justification is necessarily mathematical or governs the kinds of mathematical behavior in the classroom, then the norm can be said to be sociomathematical.

Since, Cobb & Yackel's initial formulation of sociomathematical and social norms, a significant amount of work has been done exploring the role of normative behavior in the classroom and the impact of a change in norms (Yackel & Cobb, 1996; Rasmussen, Yackel, & King, 2003; Lopez & Allal, 2007; Levinson, Tirosh, & Samir, 2009). For example, Yackel and Rasmussen (2003) examined the impact in changing normative behavior has on the beliefs of students within a mathematics classroom. The requirement that students justify their arguments and make sense of each other's

arguments changed the way that many students in the classroom viewed mathematics. Furthermore, this study pointed to ways that teachers might adapt their classroom discourse to facilitate this new normative behavior, including calling for justification from students. Other studies have also concluded that requiring students to engage in mathematical discussions that required higher level reasoning and talk provided for a change in the mathematical discourse. For example, Pierson (2009) demonstrated how teachers who required more cognitively demanding discourse had students who achieved higher levels of mathematical understanding. Pierson's study challenges the notion that students' levels of achievement are only the product of the individual student. Instead, this study demonstrates that teachers who require more of their students in discussion and hence transform the discourse of the classroom can have an impact on their understanding of mathematics.

Gresalfi, Martin, Hand & Greeno (2009) examined how the Discourse of the classroom dictates what it means to be a competent doer of mathematics. These researchers contend that competence is a product of the discourse as students and teachers define and refine their roles in interaction. Their findings reinforce Mehan's (1979) notion that the discourse dictates for the student and the teacher who they are within the interaction.

What it means to be a competent member of a classroom community speaks directly to what it means to identify as a member of a classroom community. Cobb, Gresalfi, and Hodge (2009) posit a framework for analyzing the relationship between student identities and the normative environment of the classroom. They posit that the relationship between the norms of the classroom and who one considers oneself to be

as a doer of mathematics and a member of the classroom community are connected. They analyzed the norms of the classroom and then interviewed students to see if their views of the normative behavior of the classroom were in line with the researchers observations. Then, they asked about the student's experience as members of the classroom. The researchers found that students in an inquiry oriented classroom felt freer to participate and talk about mathematics and felt that the meanings that they made in mathematics class belonged to them as individuals.

The research on social and sociomathematical norms and the previously discussed work on identity claim that classroom discourse plays a powerful role in shaping what it means to do mathematics and be a doer of mathematics in a classroom community. Furthermore, these studies examined the relationship between big "D" discourse and the small "d" discourse. Changes in discourse can have a significant impact on the Discourse of the classroom by establishing for students what gets valued in the classroom and how to conduct oneself in that classroom. The Discourse of the classroom establishes the student's roles in the classroom and what it means to do mathematics as a member of that community. In examining the classroom that I have chosen for my dissertation, I look carefully at what gets valued in the discourse in order to understand what it means to be a member of the classroom community and what sorts of meanings are considered valid and useful.

Conclusion

Meaning making in a community of practice is multi-modal and tied to the discourse of the community. The literature I reviewed in this chapter demonstrates

that when considering how meaning is negotiated in a community of practice, I can treat the classroom as an activity system. In that activity system there are many individuals who utilize tools, language, and their bodies in order to come to see the world in a consistent way so that they might come to consensus and produce knowledge. This literature also demonstrates that the discourse of the community has an impact upon the beliefs, attitudes and dispositions of the members of that community. Hence, when considering how we are to teach and what we are to teach there is more to pay attention to than just curriculum. Understanding classrooms and their meaning-making activities requires understanding and attention to all of these aspects in order to honor the true complexity of human knowledge production.

Chapter 3 : Methods

In this section I detail the methods that I used to answer the following two research questions:

1. What are the different meanings that students develop with regard to vectors and vector equations in this linear algebra classroom?

2. In what ways do individual students contribute to and take responsibility for the different meanings that this classroom community develops for vectors and vector equations?

First, I describe the setting and participants involved in my data collection, including a short discussion of how the classroom teaching experiment aided in my ability to answer my questions. Second, I discuss the tasks that were used during the classroom teaching experiment and how the instructional design theory of Realistic Mathematics Education played a role in the makeup of the tasks. Third, I address what data was collected. Finally, I address how I analyzed my collected data to answer my questions.

In the following table, I summarize the data and the methods used to analyze the two research questions:

Table 3.1: Table of Methods

Question	Data Sources	Methods
1	1. Whole Classroom Video and Transcripts	I utilized Rasmussen and Stephan's (2008) three phrase approach for identifying classroom mathematics practices. By analyzing the emergent ideas and ways of reasoning around vectors and vector equations that function as if shared, I established the ways that the symbolic expressions for vectors and vector equations had been reified for members of the classroom community.
2	1. Focus Group Video and Transcripts 2. Autobiographical Survey	I used grounded theory (Corbin & Strauss, 2008) in order to analyze the focus group videos. The autobiographical surveys gave me insight into the individuals who participate in the activities, along with a brief amount of history regarding their previous mathematical experience. In the course of developing the grounded theory, about students responsibility and contribution I identified instances of flexibility and collaboration in the course of working on the activities. I also coordinated these instances with student's self-reported impressions about their level of responsibility for developing knowledge in the classroom. This allowed me to then infer the roles that these students played in relation to knowledge production in the classroom community.

Table 3.2: Table of Methods Continued

Question	Data Sources	Methods
Reliability	1. Inter-Rater Reliability 2. Inter-Rater Reliability	Question 1: An independent coder was used to code day five of the data set. 76% inter-rater reliability was established and differences in coding were discussed and compromises on coding were reached. Eventually, 85% coder agreement was reached. Question 2: An independent coder was given the codes generated in the focus group analysis. The coder was then asked to code the second end-of-semester focus group. 87% rater agreement was achieved on first pass through the data.

Settings and participants

My dissertation study is a part of a larger research project in the teaching and learning of linear algebra. The projects goals of this larger project are three fold. The first is to develop theoretical means to understand and interpret student thinking and learning along the informal to formal continuum using four different analytic lenses. They are collective discipline practices, collective classroom practices, individual participation and individual acquisition. These lenses are an expansion of Cobb & Yackel's (1996) interpretive framework for analyzing student learning. The second goal of the project is to create methodologies to facilitate these kinds of analyses. And the final goal is to apply these analyses to the development of instructional design for the teaching of linear algebra.

In the spring of 2010, the research team working on this project conducted a semester-long classroom teaching experiment (Cobb, 2000) in an introductory course in linear algebra at a large, public research university in the Southwestern United

States. My dissertation data collection was conducted as a part of this classroom teaching experiment. There were 38 students in the class, with approximately 50 percent being math majors and 50 percent being computer science, computer engineering, and other mathematically intensive majors. There were 6 girls and 32 boys in the course, which is generally consistent with a sophomore level course required for engineering and mathematics majors. The linear algebra course was an introductory course and required students to have two semesters of calculus and is generally taken concurrently with a semester of vector calculus. The students were mathematically sophisticated students with extensive experience in working on the Cartesian plane and with systems of equations. From the standpoint of the study, this level of ability allowed students to engage with the tasks at a fairly high level. The course covered basic work with systems of equations, an introduction to vectors and vector equations, introductory work with linear transformations and basic eigen-theory.

The research setting was an inquiry-oriented classroom. An inquiry-oriented classroom is defined by the nature of student activity and teacher activity. On the one hand, students learn mathematics by inquiring into challenging problems. On the other hand, teachers regularly inquire into student thinking (Rasmussen & Kwon, 2007; Rasmussen, Marrongelle, & Kwon, 2009). Thus, the process of inquiry is two-fold. Teaching that engages in this two-fold process allows for classroom interaction that illuminates student thinking, action and overall classroom discourse. Because students are engaged in a process of active inquiry, they are participating in mathematics in a more authentic manner. Furthermore, because the teacher is inquiring into student

thinking, it provided me as a researcher a window into how students are thinking in situ.

The classroom teacher for this project was also a member of the research team. She was a third year doctoral student with extensive experience in teaching college level mathematics. She played a very active part in the design of the tasks for this and previous classroom teaching experiments. As well, she was committed to fostering particular social and sociomathematical norms that are conducive to authentic inquiry (Yackel & Cobb, 1996).

Tasks

Tasks that elicit student thinking and allow students to creatively develop and negotiate new mathematics are important to examining the development of discourse. In order to examine the negotiation of meaning students need to be given the opportunity to discuss and develop the mathematics for themselves. The particular instructional theory that informed the tasks used in the classroom is that of Realistic Mathematics Education (RME). A main goal of RME is to provide students an opportunity to reinvent important mathematics (Gravemeijer, 1999). From an RME point of view, mathematics is a human activity, hence by reinventing the mathematics students were given the opportunity to develop the mathematics for themselves and make that mathematics more meaningful (Freudenthal, 1973). These tasks which took place on six consecutive classroom days in association with inquiry-oriented teaching, provided me with the opportunity to observe students grappling with the symbolizations of vectors and vector equations and negotiating meaning of these

symbolizations for themselves. Below is a table that outlines the tasks that were given

to students for the days covered by my analysis.

Table 3.3: Magic Carpet Scenario Tasks

Task	Classroom Days	Description of the task
Riding on the Magic Carpet and Hoverboard	1	Students are asked to consider how movement on two modes of transportation, the hoverboard and the magic carpet can be symbolized using vector notation. They also explore vector addition and scalar multiplication
Can you get anywhere in R^2 using the magic carpet and the hoverboard?	2	Students explore the different points on the plane that they can reach using the two modes of transportation. In this activity students further explore vector addition and multiplication and are introduced to span.
Can you get back home using the magic carpet and hoverboard?	2, 3, and 4	Students examine whether or not there is a path that can be taken with the hoverboard and the magic carpet that will allow them to return to the origin. In this activity students are introduced to the concept of linear dependence and independence. This set of problems also introduces students to vectors and vector equations in R^3.
Worksheet identifying sets of linearly dependent and independent vectors	4, 5, and 6	Students are asked to generate sets of vectors that are either linearly dependent or independent.
Using Gaussian Elimination to solve systems of equations and identify span and linear independence or dependence	6, 7, and 8	Students are introduced to Gaussian elimination as a way of solving systems of equations. In addition they are asked to use row reduced echelon form to determine span and linear independence or dependence.

The first set of tasks that I collected data for center on students establishing

meaning for vectors and vector equations in an imaginary setting called Gauss' cabin

(see Appendix 1). Students were first asked to find out if an individual was able to get to a certain position using two forms of transportation, a hoverboard and magic carpet, with each form of transportation represented by a two-dimensional vector. This task was intended to give students the opportunity to create meaning for vectors and vector equations by generating meaning for the symbols for vectors, relating their symbolism to systems of equations, and making connections between the algebraic symbols and their geometric representations.

Students were then asked to find if they could get everywhere in the plane using the two forms of transportation. As the two transportation vectors were linearly independent, the person in the question was in fact able to get to everywhere in the plane, which was something that students concluded without a formal definition of linear independence . Students were then asked if they could get everywhere in the plane using a single vector. These questions offered the students the opportunity to explore notions of linear independence and span and how the symbols for vectors and vector equations could elucidate and provide record for this activity. Student reasoning was eventually formalized by introducing the definition of linear independence and span, both of which utilized vectors and vector notation.

This first set of tasks took approximately 5 1/2 days to complete in class and offered students various opportunities to engage with and build meanings for vectors and vector equations. The primary focus of these tasks was to have students reinvent meaning for vectors and vector equations and hence was ideal for answering the questions of my dissertation. In addition to these 5 ½ days, the subsequent two days

were also analyzed to ascertain the extent to which the functioning-as-if-shared meanings continued to be used for new activities and in new situations.

Data sources

Data collection for this dissertation was diverse and covered not only data collected during the teaching experiment, but also data collected in a series of focus group interviews. During the classroom experiment, there were four cameras working concurrently with an audio feed for each camera. Furthermore, still pictures were taken of board work and copies of in-class written work, homework, student reflections, portfolios and exams were also collected. Audio-recordings of debriefing meetings after each class session and weekly project meetings were also retained. In the paragraphs that follow, I describe in greater detail how each of these sources were collected.

Video-recordings were made of whole class discussion and also small group discussions for three specially selected groups. Students were placed within camera groups based upon their willingness to share their thinking and participate with the research group. Students self-selected tables to sit at after cameras had been placed. Students re-situated themselves on the first day if they did not want to be on camera. After, the second day of class, the groups were resituated in order to capitalize on key respondents.

In addition to the video data, written work was collected. Initially, students were asked to complete a survey about their mathematical habits and some of their school and mathematics history. As well, before each midterm, students were asked to

provide a portfolio that described their work from previous linear algebra classroom sessions and homework. For each portfolio entry, students were asked to identify which activities that they found most meaningful or fulfilling and why. These portfolios provide further data for student's responsibilities and contributions and also the particular meanings for vectors and vector equations that they developed throughout the course of the class. Student's written work on the tasks that were videotaped was collected. Pictures were taken of all class board work. And finally, in class reflections were assigned at the end of every class that asked students about their views on the day's materials.

As part of the classroom teaching experiment, weekly team meetings and debriefings after each class session were carried out. The weekly team meetings provided an opportunity for research team members to discuss aspects of student thinking that were observed during the week and to discuss upcoming activities. The teacher, as a member of the research team, had the opportunity to discuss her views on the week's activities and whether or not her goals for instruction were carried out. As well, after each class period a debriefing meeting was held with all members of the research team. At these meetings the teacher-researcher had the opportunity to reflect on the day's instruction, to hear the kinds of student thinking that were postulated by the other research team members and to discuss tasks for the next day's class. For both the weekly meetings and the teacher debriefings, I took field notes and audio-recordings documenting the meetings.

Focus group interviews (See Appendices 2-4,) were conducted at the beginning of the semester and after the final exam (Stewart & Shamdasani, 1990; Fern, 2001). I

conducted two sets of focus group interviews with four students in each interview

during the first three weeks of the semester. In the first and second interviews, there

were four different students in each interview. The second set of interviews, which

comprised a single interview, had students that were culled from the first set. In all,

eight students participated in the first two sets of focus group interviews. The same

students were then asked for another set of interview sessions at the end of the

semester. One of the students from the first two sets of interviews was unable to

attend the end-of-semester set of interviews. Hence, another student from a different

small group was selected to take his place. In addition, a ninth student was added to

make three focus group interviews with three students in each. These three focus

group interviews comprised nine total students. . For all three sets of focus group

interviews, a total of ten students participated, eight men and two women. Each of the

focus group interviews asked students to answer questions about their beliefs about

mathematics, their understanding of vectors and vector equations, applications using

vectors and vector equations, and their identities with regard to their mathematical

work. Focus group interviews were chosen because they afford students the

opportunity to talk about their views about being in mathematics classes in a group

environment. The group environment can allow for greater ease of communication

amongst the participants and for the participants to feed off of and react to each other's

comments and opinions (Kitzinger, 1994). Furthermore, I could observe how the

students interact with one another and observe their use of language in a more

naturalistic scenario (Morgan, 1988, 1996). Given my interest in students' use of the

symbols of vectors and vector equations and their role in discourse, this particular data

collection tool was particularly helpful. Focus groups are valuable when the goal of the interview is to understand how a group or groups of people react to situations or stimuli as they afford multiple perspectives in one setting (Fern, 2001).

The focus group interviews were videotaped. I acted as a moderator for the interviews and took field notes during the interviews. The students chosen for the focus group interviews were a representative sample from the classroom. I recruited members to the focus group that were representative of the classroom membership in terms of gender, two women and eight men, and declared major, three mathematics majors, one statistics major, 3 computer engineering majors, and three computer science majors. The 10 students came from six different small groups within the classroom.

Analysis

In this section I detail how I utilized the collected data in order to answer my research questions. The analysis began with transcription of the classroom sessions and the focus group interviews. After transcription, I employed two different kinds of coding in order to parse the data: a Toulmin analysis (Rasmussen & Stephan, 2008) and a grounded analysis (Corbin & Strauss, 2008) for use on the focus group interviews. As well, I used the other data sources, homework, exams, reflections, and portfolios, in order to triangulate the analysis (Mathison, 1988) and ensure its validity.

The analysis that I conducted is designed to deal with two of the central theoretical problems that this dissertation addresses. The first issue is the development of collective meaning. The second issue is the relationship between membership in

the classroom and mathematical community and their ability to shape and contribute to the meanings for vectors and vector equations.

Analysis for question 1

This analysis began with transcription of the video-recordings from each of the four cameras. Transcripts were made of the eight classroom sessions that comprised the sequence of tasks previously described. Selected portions of the debriefing sessions and weekly team meetings were transcribed depending on their relevance to the questions and the data analysis. These selections were chosen based upon the field notes taken. Furthermore, the two sets of focus group interviews were also transcribed .

Following transcription and thick descriptions, the data was analyzed using Toulmin's argumentation scheme (Toulmin, 1963). In particular, I used the argumentation scheme in a manner consistent with the methodology detailed by Rasmussen & Stephan (2008). Studying argumentation in the classroom can be an effective way to draw conclusions about the products of collective activity. Krummheur (1994) argues that learning takes place via argumentation and hence examining it is integral to understanding the meanings of the classroom. Furthermore, Rasmussen & Stephan (2008) add to that by stating that with regard to student interaction, "learning refers to the conceptual shifts that occur as a person participates in and contributes to the meaning that is negotiated in a series of interactions with other individuals (p. 196)." Argumentation is a key form of this kind of participation and negotiation and hence analysis of it can be fruitful in understanding the meanings that get reified in the classroom.

In analyzing argumentation as it applies to interaction in the mathematics classroom, Krummheur (1994) characterized argumentation as a process by which individuals or groups create consensus around the validity of a claim. Although, Toulmin's original work generally characterized argumentation with regard to utterances by individuals or in written documents, the scheme has been re-conceptualized to analyze discussions that generate arguments. Frequently, argumentation in the class consisted of interactions between members of a particular group or between members of the classroom community as a whole. Rasmussen and Stephan (2008) show how to use of the Toulmin scheme to document classroom discussions that generate arguments. The Toulmin scheme identifies three parts as the core of the argument: claim, data, and warrant. The claim is a statement whose veracity is being negotiated during an argumentation interchange. The data is the evidence that is given to support the claim. The warrant is the relationship between the data and the claim. Outside of the core of the argument are backings, qualifiers and rebuttals. Backings have an assortment of functions in the argument, including providing legitimacy for the core of the argument (See below).

```
┌──────────────┐                    ┌──────────────┐
│   Data       │━━━━━━━┳━━━━━━━━━━━━│   Claim      │
│              │       ┃            │              │
│  Evidence    │       ┃            │  Conclusion  │
└──────────────┘       ┃            └──────────────┘
                ┌──────┻──────────┐
                │   Warrant       │
                │                 │
                │ Explain how data│
                │ leads to claim or│
                │    warrant      │
                └──────┳──────────┘
                       ┃
                ┌──────┻──────────┐
                │   Backing       │
┌──────────────┐│                 │  ┌──────────────┐
│  Qualifier   ││  Explains why   │  │  Rebuttal    │
│              ││  warrant has    │  │              │
│  Degree of   ││  authority      │  │ Conditions under│
│ confidence in a│└─────────────┘  │   which the  │
│   claim      │                   │ conclusion doesn't│
└──────────────┘                   │    hold      │
                                   └──────────────┘
```
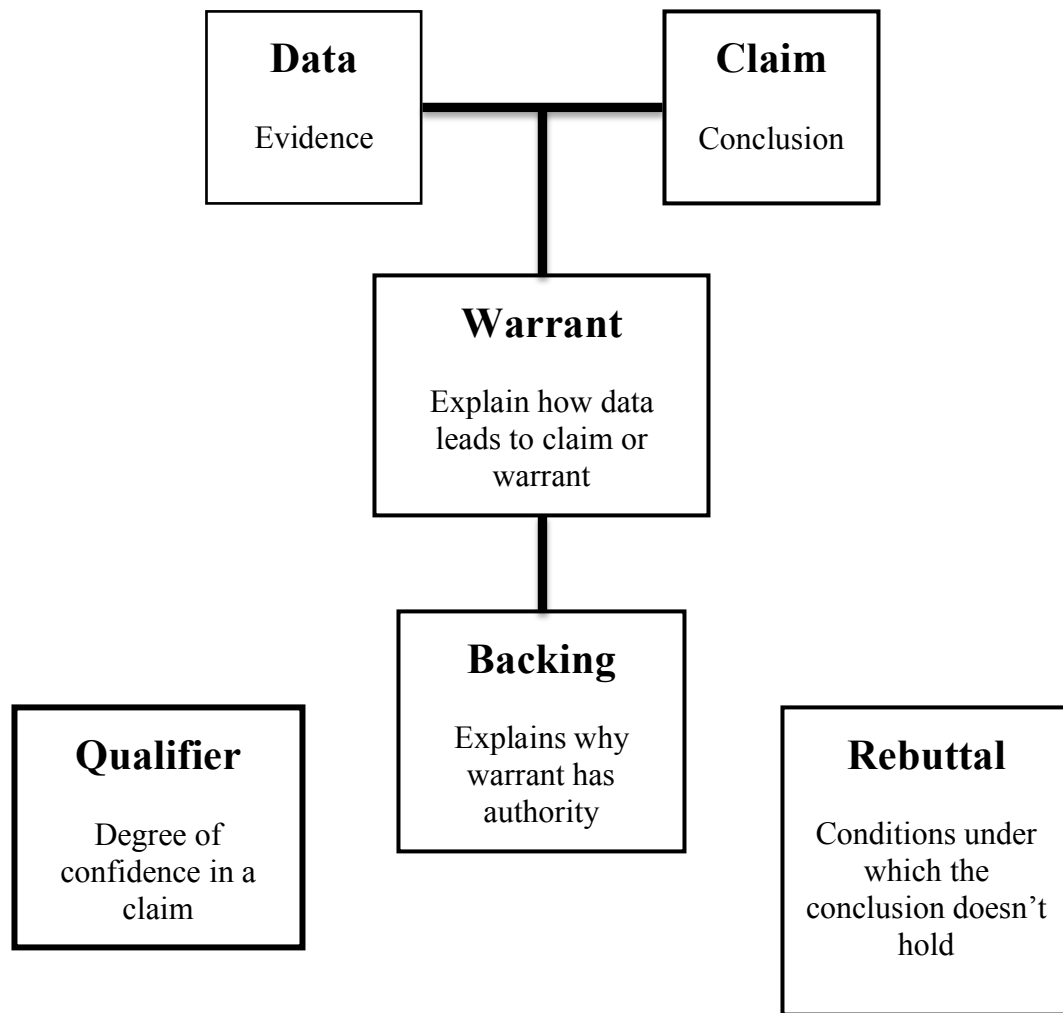
Figure 3.1: The Toulmin Scheme

A qualifier is a statement that amends the core of the argument and establishes for

those in the argument their degree of confidence in the claim by setting forth the

conditions on which the claim is valid. And a rebuttal is a statement that attempts to

invalidate the core of the argument either by questioning the validity of the claim, data

or the warrant. In analyzing the collective production of meaning in the classroom, I

utilized these definitions.   Use of the Toulmin scheme for analytic purposes is a form

of discourse analysis; because when analyzing argumentation in this manner, the focus is on the function of the utterance within the interaction.

Rasmussen and Stephan's (2008) use of the Toulmin scheme was developed in order to analyze collective mathematical activity.   They use their method in order to identify meanings, symbolizations, and interpretations that function as-if shared in the classroom.   Functioning as-if shared means that meanings developed during the course of classroom discourse may not be shared by every member of the classroom community. When meanings that function-as-if shared are present in the discourse they are used in the discourse as-if the members of the classroom are in agreement on their meaning. Consistent with Rasmussen and Stephan (2008), my analysis began with an analysis of the argumentation in the whole classroom discussion. I created an argumentation log in which I documented the claims that students made, along with their associated data, warrants, backings and any rebuttals or qualifiers. This was intended to be a functional analysis, where I focus on how the utterances function within an interaction. Hence, I took the ideas that are being negotiated in the class and examined how their functions change over time.

According to Rasmussen & Stephan, there are two primary ways of documenting if an idea is 'functioning as-if shared' in the classroom. The first criteria is whenever the backings or warrants for a claim ceased to be used explicitly by members of the classroom community. Early in the course of classroom argumentation, students may utilize a warrant or backing to justify the relationship between a claim and its data. The elimination of the warrant or backing for a similar argument in later discourse indicates that the relationship between the data and claim is understood

tacitly by members of the classroom community, and hence is functioning-as-if-shared. The second criteria is if an idea shifted function, for example from claim to warrant in a subsequent argumentation. When members of the classroom begin to use previously discussed claims as justifications for new claims, this implies that a particular idea or way of reasoning has become sufficiently agreed upon to be considered as a mathematical truth and hence can be used as data, for example, to support a new claim. A third criteria has recently been detailed in Cole, R., Towns, M., Rasmussen, C., Becker, N., Sweeney, G., & Wawro, M. (2011), which holds that ideas, concepts or utterances that appear repeatedly as data or warrants function as-if shared as they become key aspects of the classroom discourse. For each argument, I assessed if the data, claim, warrant, or backing had been utilized in previous arguments, and the extent to which that portion of the argument fulfilled the criteria. Finally, I clustered ideas that were functioning as-if shared by a common mathematical theme or form of activity. Each cluster represents a classroom mathematics practice. By examining how these students discussed the symbolic activities that were happening in the classroom and the meanings that were important to how they understood that activity I establish the set of meanings that were created in this linear algebra course.

Analysis for question 2

The second research question was answered by analyzing the focus group interviews using grounded theory (Corbin & Strauss, 2008). More specifically, I engaged in open coding where I created core categories for the ways that students took responsibility for classroom activity and the different kinds of contributions that the

individual members made to the discussion of vectors and vector equations. Corbin & Strauss (2008) define open coding as "the breaking data apart and delineating concepts to stand for blocks of raw data (p. 193)." In the creation of these open codes I was particularly interested in how the students talked about vector equations and their relationship to the use and production of vectors and vector equations. Furthermore, I created codes for how the students felt about their experiences in past and current mathematics classes and their interaction, contribution, and responsibility for the various symbolic expressions in the course.. I coded with the specific intent of providing as much conceptual density and breadth as possible for the given set of categories. Once the open codes were generated using the transcripts, videos and student's written work, I went back through the data and coded each utterance as to whether it demonstrated a code. An utterance as I define it is a turn or set of consecutive turns that focused on a particular idea or activity. In the course of this second round of coding, there were several instances in which a turn could have been coded with more than one of the codes. In this case, the utterance was coded using both codes.

After the first round of open coding, I moved to axial coding where I looked for relationships between the different kinds of responsibility and contribution. This allowed me to develop a cohesive analysis regarding the negotiability that these students encountered in the course of classroom activity. As constant comparison (Corbin & Strauss, 2008) is an important and necessary part of grounded theory, the act of axial coding also provided for revision and expansion of the original open codes. Once the codes were completed, I returned to the transcripts and identified each

utterance as being one of the codes. I then tabulated the number of utterances for each

of the codes, insuring that the codes represented a significant amount of the discussion

during the focus group interviews.

Because of the nature of the focus group interviews, paying attention to the

discourse of the focus group interviews was very important. The interactions between

the students, their ways of communicating with each other and the overall group

dynamics involved in the interview were important aspects of the analysis. While I

examined how particular utterances function in the interactions of the students, the

analysis focused on thematic elements. A thematic analysis (Lemke, 1994) focuses

primarily on the content of the utterances. Wells (1999) characterizes Lemke's notion

of thematic analysis as examining "the interrelationships of meaning between the

terms that are used in the talk (p. 175)."   Lemke theorizes that whenever people

engage in discussion, their talk centers on common sets of ideas, or themes, that

distinguish what the conversation is about. The patterns of meaning that are

developed within a theme establish for the members of the discourse what they are

talking about and what is meant by each other's utterances. A thematic analysis works

to parse out those different meanings and establish how they are connected to one

another to create meaning for the members of the group. From the results of both the

grounded theory and thematic analysis, I developed the answers to the second question

in my dissertation. Hence, when I completed the open coding and axial coding, I went

through all of the codes and grouped them thematically. There were five themes that

were developed as a result of this grouping, alienation, responsibility to oneself,

responsibility to the whole class discussion, responsibility to the small group discussion, and symbolic flexibility.

Reliability

There were two reliability activities that were conducted for this study. For the analysis of question 1, a second coder was asked to use the Toulmin analysis to code the arguments for the fifth day of classroom activity. I chose day five because it offered a significant subset of the arguments from whole class discussion (45 of the 186 arguments) and comprised 16.6% of the whole class time from the first six days. In addition, day one and day six would not have been appropriate for this activity, as day one contained significant amounts of teacher discourse in comparison to the other days of class and day six only spent 3/4 of the class time on issues dealing with vectors and vector equations. Then, the independent coders results were compared against my own codings to insure inter-rater reliability. This comparison led to a 76% agreement for all 45 of the arguments coded on day five. The second reliability activity involved an independent coder coding the second focus group using the codes that I generated. The focus group was chosen at random and the coder was given each of the codes and their descriptions. The independent coder and I agreed on 84% of the codes for the chosen day.

Conclusion

In this chapter I detailed the methods I used to answer my two research questions. The tasks and the interview protocol are included in Appendices 1-6. Data collection commenced on January 20 with video-recording of classroom sessions and

collection of written materials and reflections. The analysis of question 1 focused on the first six days of class.. During that portion of the data collection, I conducted the first set of focus group interviews. At the end of the semester, I conducted the final set of focus group interviews.

At the end of data collection, analysis began. Transcripts were created for each classroom episode and the focus group interviews. The classroom data was analyzed using the previously described Toulmin analysis derived from Rasmussen and Stephan (2008). Finally, the focus group interviews were analyzed using grounded theory (Corbin & Strauss, 2008). Inter-rater reliability was then established using independent coders.

Chapter 4 : Classroom Symbolizing Practices

In this chapter, I will answer the first research question of my dissertation:

1. What are the different meanings that this classroom community

develops for vectors and vector equations?

The primary focus of this chapter is the symbolic expressions that are used, created and understood by the members of this classroom community. However, I am not examining the multitude of different symbolic expressions and possible meanings that could be used by individuals in this classroom community could use for vectors and vector equations. The symbolic expressions and meanings that will be presented will necessarily be normative ways of reasoning about or using these expressions A normative way of reasoning is a way of reasoning or idea that functions-as-if-shared in the classroom community (Rasmussen & Stephan, 2008). I use the term functions-as-if-shared because the assumption is that not every member of the classroom community shares the same exact meaning for a particular way of reasoning; instead members of the classroom community treat the way of reasoning "as if" other members of the classroom understand the meaning that they are working with and the other members of the classroom act "as if" they understand the meaning that their fellow classmates are using. Normative ways of reasoning are the meanings that the classroom community develops for an idea.

Using whole class discussions, Rasmussen and Stephan (2008) created a methodology using Toulmin's scheme to document and analyze normative ways of

reasoning in the classroom community. There are three criteria for documenting that a way of reasoning is functioning-as-if-shared. The first two are detailed in Rasmussen and Stephan (2008) and the third is detailed in Becker et al. (2011). The first criterion is that the warrants and/or backings that initially accompany the presentation of the idea drop off in later arguments. When the backings and warrants drop off of the argument, this indicates that for the classroom community there is no longer a need for justification and the idea is then accepted. The second criterion is when a idea shifts function in later arguments. For example, when a way of reasoning that was previously used as a claim moves to data or warrant this indicates that the members of the classroom community feel sufficiently comfortable with the way of reasoning expressed in the claim to utilize it as a justification for another claim. The third criterion for documenting when an idea is functioning-as-if-shared is when the idea is used as data or warrant in multiple arguments (Becker, et al., 2011). When a way of reasoning is used as data in multiple different arguments, this evidences that the way of reasoning has been demonstrated to be particularly useful in multiple scenarios, and that it represents a valid justification across multiple claims.

These functioning-as-if-shared ideas are then grouped according to common mathematical activities into a classroom mathematics practice, which is defined as "a collection of as-if-shared ideas that are integral to the development of a more general mathematical activity (Rasmussen & Stephan, 2008, p. 201)." The more general mathematical activity may be the understanding of a concept, the development of meaning for particular symbolic expression, or a practice for finding solutions or drawing conclusions about mathematical activity. Since I am not analyzing all of the

ways of reasoning that get developed during the course of the days that I am analyzing,

the practices that I document will be a subset of all of the classroom mathematics

practice. I refer to the classroom mathematics practices that address my first research

question classroom symbolizing practices. These practices are a collection of as-if-

shared ways of reasoning that are integral to the development of student understanding,

creation, and use of symbolic expressions.

As Rasmussen and Stephan (2008) have noted, the appearance of particular

functioning-as-if-shared ways of reasoning does not necessarily follow a particular

temporal order and practices are made up of ideas that appeared on multiple class days

and in multiple tasks. In the case of vectors and vector equations, new concepts like

span and linear independence and dependence change the meanings that students have

for vectors and vector equations. Furthermore, classroom symbolizing practices

document how a classroom community makes sense of the symbolic expressions that

are used and created in the classroom. These meanings are in constant flux and in the

process of development. As the classroom community develops new concepts and

does new tasks, the meanings for the symbolic expressions for vectors and vector

equations expands. Instances in which this is the case will be documented in the

chapter.

In this chapter, I define eight classroom symbolizing practices that were

documented from the first six days of the classroom teaching experiment. These six

days dealt with four tasks (Appendix 1) from the beginning of the semester. These

tasks were crucially important to the development of these classroom symbolizing

practices. The argumentation that was generated by this classroom directly addressed

these tasks. The travelling language that became important for student's understanding of vectors, vector equations, linear dependence and independence and span. The tasks provided the impetus for using the language of "getting back home," "getting anywhere," "routes" and paths." I coded 186 separate arguments from whole class discussion. Arguments were only counted if they contained a claim and data. These arguments were made up of student and instructor utterances. At times arguments were coded wholly from instructor utterances, whereas at other times arguments came wholly from student utterances. In addition, there were times when instructor and student utterances were combined in a single argument. For example, the instructor would provide the claim, and the students would supply the data and the justification. Because this analysis came from classroom days that were at the very beginning of the semester, many of the social and socio-mathematical norms that were necessary for the coding of the argumentation needed to be developed. Hence, many of the early ideas that developed the symbolic expressions for vectors and vector equations were first introduced by the instructor. However, in each case, students gave the arguments that evidenced that the way of reasoning was functioning-as-if-shared. Furthermore, in those instances in which the instructor played a large role in the instantiation of a normative way of reasoning, I give additional evidence that the idea is functioning-as-if-shared, either in the form of later classroom argumentation or via a count of the number of times that the way of reasoning was used as either data, warrant or backing.

The use of the Toulmin analysis allowed me to examine the negotiation of meaning by examining the transformation of discourse through argumentation. The examination of discourse allows for the analysis of changes in meaning across

multiple discussions and days. While this kind of analysis makes examination of

moment-by-moment change difficult, it does illuminate meaning making for large

concepts and symbolic expressions. Furthermore, as I stated in the introduction, the

analysis of discourse and argumentation allows for me to characterize learning and

meaning making in a collective context. This implies examining how the collection of

individuals reasoning about the various ideas and engagement in practices, a more

micro-analytic lens would only allow me to make characterizations over a few

moments and would not give the full breadth and scope that is necessary to answer my

research questions. In addition, although this analysis is not a gestural analysis, the

role of gestures at times was important to understanding student's arguments. I will

not be analyzing the gestures, but at times I discuss them in order to illuminate student

arguments. These gestures are a crucial part of students ways of thinking and their

communication with each other. When necessary, I will discuss the gestures in the

results because without these gestures it would be difficult to understand the

arguments and the normative ways of reasoning.

Section 1

I have split the classroom symbolizing practices into two groups of four that I

believe naturally delineates two kinds of classroom symbolizing practices. The first

four practices dealt with the basic symbolic objects for vectors and vector equations:

vectors, scalars, the use of systems to find scalars for linear combinations, and linear

combinations. I analyze only the use of systems as it relates to finding scalar solutions

to vector equations, although students symbolizing of systems of equations could be

analyzed in its own right. The second four practices deal with the classroom

communities use, understanding and creation of the symbolic expressions for vectors

and vector equations as the class develops new concepts in linear algebra, including

span and linear independence and dependence. These practices deal largely with sets

of vectors, equivalence of vectors, and the definitions for span and linear

independence and dependence. The following table summarizes the first four of these

practices.

Summary Table of Section 1: Classroom Symbolizing Practices (CSP)

CSP #1 A vector in R^2 is defined as a path in two-dimensional space.
CSP #2 Scalars define the direction and amount that a vector is stretched.
CSP #3 Setting up and solving a system of equations allows one to solve for the
     scalars in the vector equation.
CSP #4 Linear combinations can be used to model relationships between sets of
     vectors

Figure 4.1: Summary of Classroom Symbolizing Practices 1-4

In each section of this chapter, I define the classroom symbolizing practice and the

normative ways of reasoning that were aggregated as a part of the practice as well as

the reasons for their grouping. For each symbolizing practice, I will present a vignette

or vignettes from the classroom discourse that demonstrates the method of the analysis.

For each of the vignettes that I present, I present both the argumentation that first

evidenced the idea in classroom discourse and the argumentation that evidenced that

the idea was functioning-as-if-shared. Unless otherwise noted, the normative ways of

reasoning will be presented in the order in which they were established during

classroom sessions.

I use the terms graphically, algebraically and theoretically to characterize

symbolic expressions for vectors and vector equations. An algebraic expression for a

vector or vector equation is one of the form:

$$a \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$$

where the components of the vector are explicitly represented in the symbolic

expression and vectors are represented as being multiplied by scalars. In some cases,

members of the classroom community will symbolize algebraic expressions without

explicit scalars. In these cases, there is the assumption that the vector has a scalar of 1.

A graphical expression of a vector or vector equation is one that has been drawn in or

is referenced on the Euclidean plane or 3-space. Students do not use graphical

expressions for vectors in more than 3-dimensions. In some cases, members of the

classroom community will use the term geometric to define graphical representations.

A theoretical expression of vectors and vector equations is one of the form:

$$c_1 \bar{v}_1 + c_2 \bar{v}_2 + \ldots c_n \bar{v}_n = \bar{v}_{n+1}$$

Where the c's are scalars and the $\bar{v}$'s are vectors. The use of the theoretical symbolic

expressions is primarily confined to students work in linear independence and

dependence. I also refer to situational expressions. These are not symbolic

expressions, but are instead verbal expressions that are tied directly to the problem

scenario that students are using. Situational ways of reasoning could include

referencing stretching a vector as "moving along a vector," "moving in the opposite

direction", and so forth. All of the situational expressions that will be included in this

analysis will be tied to traveling, which becomes a key way in which students

understand the symbolic expressions for vectors and vector equations.

CSP #1: A vector in R^2 is defined as a path in two-dimensional space.

The first classroom symbolizing practice deals with the use, creation and

understanding of individual vectors in their graphical and algebraic forms. The three

normative ways of reasoning that were collected for this CSP were grouped because

they established within the classroom community the graphical, algebraic and

situational expressions for individual vectors that allowed students to progress with the

Magic Carpet Ride Scenario and begin to address non-situational tasks. The

normative ways of reasoning for CSP #1 are listed below.

CSP #1: A vector in R^2 is defined as a path in two-dimensional space.

Associated Normative Ways of Reasoning (NWR)
      1.1 The components of the vector tell you the direction and the distance
            that you will be traveling.
      1.2 Traveling in a positive direction on the vector means moving in the
            direction of the components simultaneously.
      1.3 A vector is a path.

Figure 4.2: Classroom Symbolizing Practice #1

A path, as defined in this CSP, is a straight line direction that is determined by the

components of the vector, with the top value in the vector representing the east-west,

right-left directions and the second component of the vector defined as the north-south,

up-down directions. In addition, the classroom community's use of the term path is

not necessarily anchored to the origin, but may start and end anywhere on the

Euclidean plane or in three-space so long as it maintains its direction and magnitude. It

is important to note that the use of path in this way is not the traditional use of the term

path is not in the canonical sense of a curve in space, but is rather a use that is specific

to this classroom community. The normative ways of reasoning that I present in this

section deal with vectors in two dimensions and so this collection is also defined as

dealing with two-dimensional space. The classroom community would later

symbolize vectors in 3 or more dimensions. However, symbolizing vectors in three-

dimensions was not the subject of explicit arguments in whole class discussion in the

same manner that the two-dimensional Magic Carpet scenario vectors, and hence

could not be documented as functioning-as-if-shared in the classroom. The NWR's

that were established for two-dimensional vectors frequently carried over to classroom

reasoning about three-dimensional vectors, and as will be evidenced in later classroom

discussions the symbolic and verbal language that was developed in this CSP would

be prevalent in later arguments about vectors and vector equations.

The ways of reasoning for CSP #1 were introduced during student work on the

first task in the class when students were asked to find how long a rider would have to

ride on a magic carpet and a hoverboard to get to a particular point in space, called

Gauss's Cabin. According to the task, movement on the hoverboard was restricted by

the vector, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and movement on the magic carpet was restricted by the vector,

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The task then defined what the restriction on movement for a particular vector

means by stating that for the hoverboard you travel east on the hoverboard for 3 units,

while traveling simultaneously north for 1 unit in one hour. Students were then asked

to find how many hours it would take to get to Old Man Gauss's Cabin located at 107

miles east and 64 miles north, symbolized by the vector $\begin{bmatrix} 107 \\ 64 \end{bmatrix}$ (The actual task is

included in Appendix 1).

The idea that would become NWR 1.1 was first discussed in the class as the

instructor explained how the members of the classroom community were to symbolize

and think about task 1.

> NWR 1.1 The components of the vector tell you the direction and the distance that
> you will be traveling.

In Day 1 Argument 1 (D1A1), the instructor's first activity was to explain how to

symbolize a vector graphically and algebraically. The instructor had one of the

students read the task, and then began to give the students information that would be

helpful to them in solving task 1. As a part of setting up the task, the instructor stated:

> ...so this is what we're going to mean by a vector. It will have those
> components. So let's dive into it means by its movement. So the vector
> 3,1, it says if it traveled forward for an hour, it will move along a path
> that displaces you 3 miles east and 1 mile north.

The instructor began by stating that she was going to define what a vector meant and

what its movement will mean. The data for the argument is the notation that the

instructor provided, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, as this is what she was defining. Then she made a claim as

to what that notation means, by stating "it (the notation) says if it traveled forward for

an hour, it will move along a path that displaces you three miles east and one mile

north." The argument is summarized below (Note in each argument I may include text

from the actual discussion as a part of the argument structure. In these cases, the text

from the discussion will be italicized.):

Argument D1A1
Claim: *The vector has components, which indicate the direction that you will be traveling in and for what distance.* (Instructor)
Data: *The vector is notated as* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. (Instructor)

Figure 4.3: Day 1, Argument 1

This argument was similar to many arguments that defined the meaning of a particular

symbolic argument or a concept. In these cases, the notation or the definition is given

as the data and what that object means is defined as the claim. Arguments of this form

often lack warrants as the bridge between the claim and the data is the authority of the

mathematics community, the instructor, or the task scenario. The interpretation states

that the vector can be broken up into its individual components and the individual

components can have somewhat different meanings. But for the task that the students

are doing in this class, each component defined a different direction.

The instructor then continued with the setup of the task, first introducing the

ideas that would become both NWR 1.2 and 1.3.

NWR 1.2 Traveling in a positive direction on the vector means moving in the
   direction of the components simultaneously.
NWR 1.3 Geometrically, a vector is a path.

The argument that she made that mentioned both of these ideas followed immediately

after D1A1. The instructor continued the setup of the task by stating (For the

instructor's board work see Figure 4.4):

Let's say this is your house right here [draws a point on the board]. Can
you guys see from back there if I write that low? Okay, so if we travel

along a path $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, it says that eventually we ended up over here. That really says it traveled east three units and north one unit or one mile. [draws triangle]. So the hoverboard goes along this kind of path here. Let's just say first again if I went, so this is if I went 1 hour, I'd end up here.

The instructor's claim in this argument was that a rider riding on a transportation vector traveled along a particular path. In addition, she claimed, "If I went 1 hour, I'd end up here." There are two ideas that are being initiated in the claim for this data. The first was what a path means. A path in this case was the straight-line vector whose direction and magnitude are defined by the individual components in the algebraic expression for the vector. The second idea is that the mode of transportation rides along the entirety of the path drawn in one hour. The data, which supplies the interpretation for the path, is primarily graphical in her pictorial representation (See figure 4.4) and also tied to NWR 1.1 in that it uses the component-wise information in order to supply the data. This is also the data that is supplied for the meaning of traveling one hour on the board.
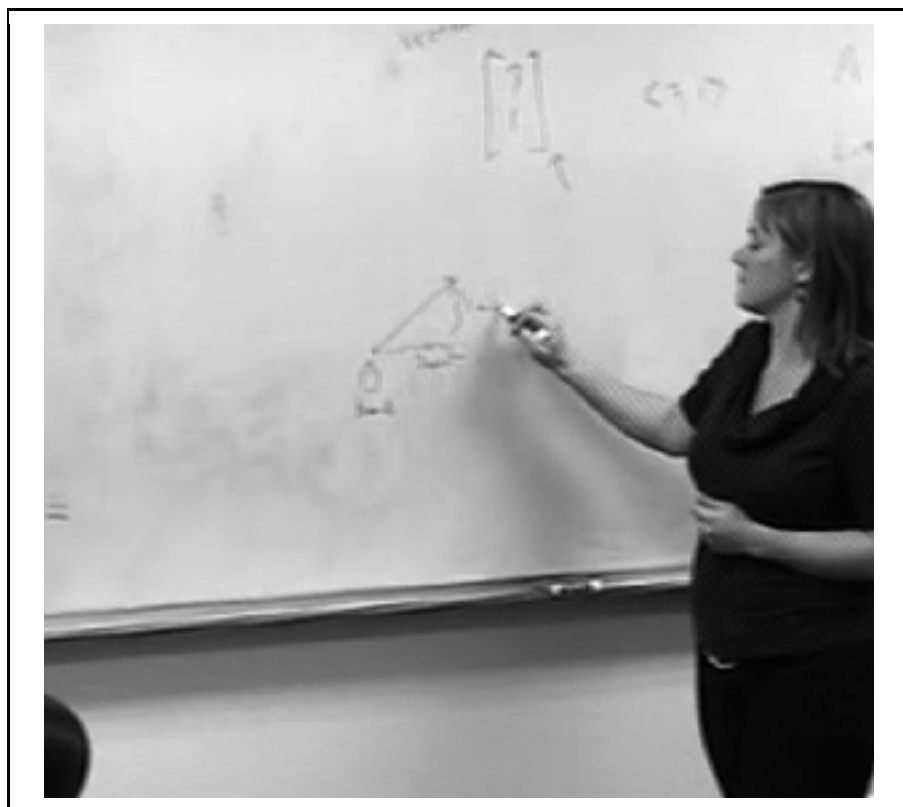
Figure 4.4: Instructor's Board Work for D1A2

Argument D1A2
Claim: The vector travels along a path.
So the hoverboard goes along this kind of path here. [Instructor draws a vector on the board and traces the line on the board] "Let's just say first again if I went, so this is if I went 1 hour, I'd end up here. (Instructor)
Data: The instructor's board work, The vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (Instructor)
Warrant: The destination of the vector and how using each of the components individually gets to the destination.
it says that eventually we ended up over here, that really says it traveled east three units and north one unit or one mile. (Instructor)

Figure 4.5: Day 1, Argument 2

D1A1 introduced the meaning of the components. D1A2 introduced what it meant to

travel in a positive direction as the teacher represented the whole vector as how far and

in what direction one would travel on the vector for one hour. D1A2 also introduced

what the conjoining of the components with the graphical representation of the vector

as a whole meant. The instructor connected the component-wise aspect of the vector

with the sides of the triangle, while establishing the vector as a holistic expression

with the hypotenuse. I use the term holistic to describe the coordinated and

simultaneous action of all the components of the vector. This became encapsulated in

the single line vector that is drawn with the hypotenuse. She then gave this

coordination the name "path". The term path is particularly important because it

established a key part of the metaphor that the classroom community would use to

reason about vectors and vector equations. I will refer to this metaphor as the

"traveling metaphor" as the language that is used in the metaphor has to do with

"getting to" places, traveling along vectors, and finding destinations or routes. The

term "path" happened to be the term she used in this argument, but I consider phrases

such as "route" as being equivalent to the term "path."

In order to evidence that these three ideas were functioning-as-if-shared, I have

chosen James' argument from D2A7. This is not the first time in which the idea were

evidenced as functioning-as-if-shared, but it did present a very clear instantiation of

the use of all three ideas acting as a warrant and data in later arguments, evidencing

criterion 2. James was asked to discuss his solution method and the graph (Figure

4.6) that he created to demonstrate his solution method.

> James: I just want to make sure I know what I'm doing. Alright, so
> pretty much the first thing I did was draw the lines. [inaud] We were
> just putting a point [draws a point]. This is the [stops talking, writes
> "DESTINATION"]. Basically we moved with these answers. It's going
> to take 2 ways. The fastest way will obviously be the straight line

[draws a line from the origin to the "DESTINATION"]. Then, yeah, didn't work that way, so that... [writes the vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ on the board] We have the first vector being the hoverboard, and the second vector being the carpet. And then we solved for the amount of hours. So that's going to be`, what are these? 17 and 30, so [puts the values 17 and 30 next to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and then switches the two scalars]. So what you get is, is that right, was it 30 on the first one?

Instructor: I think you did it right, I just sprung this on you, do you want a second to talk with your group or are you okay?

James: I don't want to. [writes $\begin{bmatrix} 90 \\ 30 \end{bmatrix}$ and $\begin{bmatrix} 17 \\ 34 \end{bmatrix}$ below the two scalar multiples] So we had the first vector would be 90 over 30 [draws a vector from the origin to a point on the board]. The 2nd vector again 17 over 34 [draws a vector from the end of the first vector to the "DESTINATION"], and you add those together and you get to .

Male Student: So you guessed it, right? How did you actually get to 30 and 17?

James: I pretty much, we didn't guess it, we pretty much just solved it, using the system of equations. It's just that before we even came, or before we even decided what equations to use, this would be a pretty much general idea of how we're going to get to it. Because we can't do a straight line, it would have to be through another route. A way being [inaud]. Any other questions?
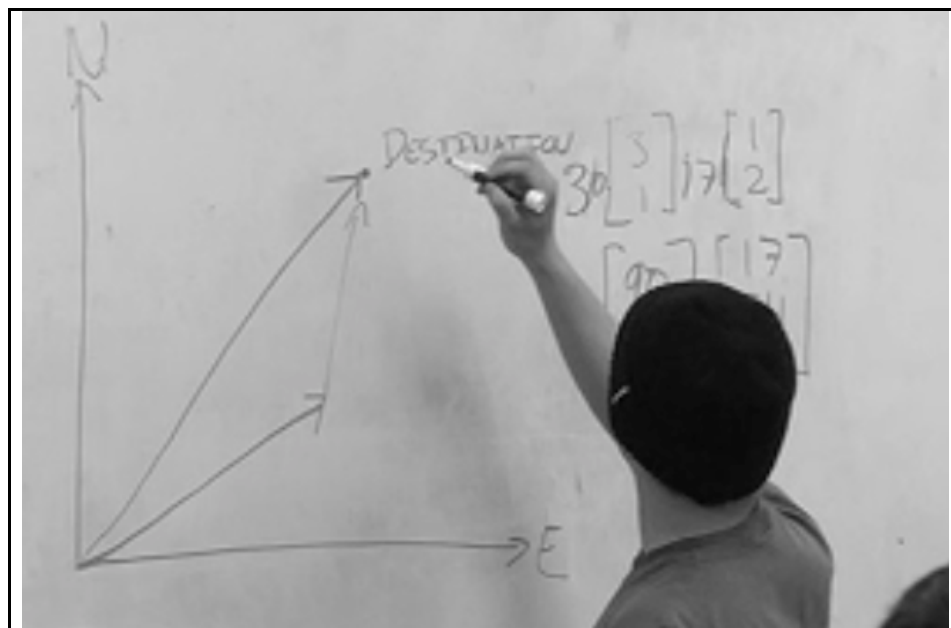
Figure 4.6: James' Drawing for D2A7

James's argument evidenced the introduction of a series of ways of reasoning within

the classroom community in addition to evidencing the ideas for NWR 1.1, 1.2, and

1.3 as functioning-as-if-shared. He actually began his argument with a qualifier, that

they could not find a straight-line route to their destination. This was coded as

qualifier because it provided the reasoning for finding another route because of the

failure of a single vector route. James's claim was that it would take 30 hours on the

hoverboard and 17 hours on the magic carpet to get to Gauss's Cabin. This was the

question that he was asked to answer, making it clearly his claim. He then wrote the

hoverboard and magic carpet vectors with the number of hours next to them and

continued by writing below those expressions, the resultant vectors from when the

vectors were multiplied by the scalars. These were coded as data as they provide the

information necessary to make his claim and the elements that he would work with as

he progressed through his explanation. Then he drew the scaled vectors on the board

from tip-to-tail and arriving at his destination. This was coded as a warrant because

these representations showed the graphical relationship between the data in the form

of the scaled vectors $30\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $17\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the resultant vectors $\begin{bmatrix} 90 \\ 30 \end{bmatrix}$ and

$\begin{bmatrix} 17 \\ 34 \end{bmatrix}$, and the claim that riding the modes of transportation for those number of

hours would have you arrive at the destination. Finally, he is asked for how he found

the scalars. He provided justification by stating that they used "the system of

equations" to solve for the scalars. The fact that this was in response to a call for

justification evidenced that it was a warrant. The argument is summarized in the table

below.

Argument D2A7
Claim: To get to Gauss's cabin, you needed to ride the hoverboard 30 hours and the magic carpet 17 hours. (James)
Data: Drawing of the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as the hoverboard and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as the magic carpet.   (James) (See Figure 4.14).   *Then we solved for the amount of hours, 17 and 30, (James)*   *The first vector would be 90 over 30 (drawing of a longer vector), the second vector would be 17 over 34 (James)*
Warrant: James' drawing of the two scaled vectors and the resultant vector using the tip-to-tail method.   The use of system of equations to solve for the hours that needed to be ridden on the magic carpet.
Qualifier: *We couldn't get to it using a straight line, it would have to be through another route.* (James)

Figure 4.7: Day 2, Argument 7

In James's picture, he used both the component-wise representation for the

vector, in the form of the both the scaled vectors and the scaled resultant vectors and

also in the form of the paths that were drawn in D1A1 and D1A2. This was coded as a

warrant. In addition, in James' qualifier and throughout the warrants and data for his argument he called the vectors that he was drawing "routes" and referred to the point at which he was trying to reach a destination. In addition, he drew a vector from the origin to the destination, further indicating that, although the two transportation vectors were not this vector, that such a vector might exist and would represent a path from the origin to the destination. Each evidenced the idea that the vector was a path with both the component-wise quality and the holistic quality. The use of the ideas was evidenced in the warrant for James's argument. Because these ideas were first made as claims and then shifted to warrants in later arguments, the ideas were evidenced as functioning-as-if-shared by criterion 2.

These three normative ways of reasoning were grouped together because they laid the groundwork for student's interpretations, use and creation of vectors and vector equations in later class periods at a very basic level. In particular, as students began to work on new tasks, deal with linear combinations, span and linear dependence and independence, they would continuously come back to these meanings for vectors. On the one hand, when students talked about the direction of the vector (CSP #2, 5 and 6), particularly when they are making graphical arguments and dealing with graphical ideas, the students would utilize the holistic properties of vectors. However, frequently when they needed to use systems of equations to find scalar solutions or to find linear combinations (CSP #3 and 4), particularly when they are making systems-based or algebraic arguments, they utilized the components-based interpretation of the vector. In both cases, the vector is a path. In the course of later arguments, students may not actually refer to the vector as a path, but the language

that they use, particularly language that has to do with traveling or "getting around" in R^2, the imagery that students are invoking is the imagery of the path.

CSP #2: Scalars define the direction and amount that a vector is stretched.

The normative ways of reasoning in CSP #2 incorporated scalars and demonstrated the kinds of activities that the classroom community participated in with regard to understanding what it meant to multiply a vector by a scalar graphically and algebraically. The following table details the normative ways of reasoning that were grouped together to create the second classroom symbolizing practice.

CSP # 2: Scalars define the direction and amount that a vector is stretched.

Associated Normative Ways of Reasoning

2.1 Multiplying the mode of transportation vector by the number of hours traveled (a resultant transportation vector) results in a new vector made up the total distance traveled east and the total distance traveled north.

2.2 The scalar stretches the vector by a factor indicated by the scalar value.

2.3 Multiplying by a negative scalar moves you in the reverse direction

2.4 The vector notation (multiplying by a negative or positive scalar) allows for the ability to move in the positive or negative direction.

2.5 Changing a vector computationally by multiplying by a scalar is the only way to change the vector graphically.

Figure 4.8: Classroom Symbolizing Practice #2

The five NWR's for this CSP were grouped together because they demonstrated how students utilized scalars in both vector equations and in reasoning about stand-alone vectors. This CSP also demonstrated how students began to talk about direction and its relationship to vectors and scalars. The idea of direction as it is first established with this CSP and then later as it is developed in CSP 5,6, and 7 was crucial to student

reasoning. In this CSP, the classroom established normative ways of reasoning that

explained how scalars affected the distances and directions of vectors for a variety of

scenarios. This CSP also established the algebraic and graphical foundation for

students later work with scalar multiples and linear combinations. Four of the five

NWR's, 2.1, 2.2, 2.3, and 2.5, that were established in this CSP will be presented

along with the criteria that established that they were functioning-as-if-shared. I will

present the arguments that first established that the ideas and the arguments that

evidenced that the NWR's were functioning-as-if-shared. For NWR 2.4, I define the

NWR and discuss the role that it plays in the development of the CSP. Finally, I detail

how the different NWR's come together to establish CSP #2.

The first NWR for CSP #2 details how this classroom community came to

understand what happened graphically, situationally, and algebraically when a vector

was multiplied by a scalar.

> NWR 2.1 Multiplying the mode of transportation vector by the number of hours
> traveled (a resultant transportation vector) results in a new vector made
> up the total distance traveled east and the total distance traveled north.

This NWR brings together three ideas that formed the normative way of reasoning.

The first was that the algebraic representation for a vector could be multiplied by a

scalar transforming the vector by changing its components to multiples of the scalar.

Second, that graphically a scalar multiple is the original vector stretched. And third,

that in the context of the Magic Carpet Ride tasks, the action of multiplying a vector

by the scalar gives you the total distance traveled in the east and west direction by

multiplying the number of hours traveled on the mode of transportation by the distance

each mode travels in the east and north direction for one hour. As I discussed in for

CSP #1, the argument D2A7 evidenced the ideas in CSP#1 as functioning-as-if-shared, and it also introduced the idea for NWR 2.1 to the classroom community. The idea was demonstrated as functioning-as-if-shared via criterion 3, when it was used to justify three different arguments with three different claims. James's argument for D2A7 is revisited below (for the original analysis of James's argument see Figure 4.6 and 4.7).

Argument D2A7
Claim: To get to Gauss's cabin, you needed to ride the hoverboard 30 hours and the magic carpet 17 hours. (James)
Data: Drawing of the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as the hoverboard and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as the magic carpet.   (James) (See Figure 4.14).   *Then we solved for the amount of hours, 17 and 30, (James)*   *The first vector would be 90 over 30 (drawing of a longer vector), the second vector would be 17 over 34 (James)*
Warrant: James' drawing of the two scaled vectors and the resultant vector using the tip-to-tail method.   The use of system of equations to solve for the hours that needed to be ridden on the magic carpet.
Qualifier: *We couldn't get to it using a straight line, it would have to be through another route.* (James)

Figure 4.9: Day 2, Argument 7 Argumentation revisited

As I discussed for NWR's 1.1, 1.2, and 1.3, James's argument demonstrated a great deal regarding vectors and it also incorporated a significant amount of information about scalars as well. He begins his argument by drawing four vectors, $30\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $17\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the resultant vectors $\begin{bmatrix} 90 \\ 30 \end{bmatrix}$ and $\begin{bmatrix} 17 \\ 34 \end{bmatrix}$. He placed the scalar 30 next to the vector, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and the 17 next to the vector, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. He then continued by drawing

each of the stretched vectors on his board and connecting them to the destination

vector. The graphical vectors as they were drawn on his board were coded as the

warrants and the algebraic vectors were coded as the data. In addition, because James

stated that the scalars were the number of hours ridden for each mode of transportation

and the original vectors, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, were each the mode of transportation, this

created the link between the algebraic, situational, and graphical expressions for the

vectors as detailed in the NWR.

The second time this idea was used as data and warrant was later in that class

period when the teacher asked the classroom community if they could reach Gauss's

cabin with just one of the vectors. Debbie demonstrated use of this idea in the

argument that she created as justification for her group's argument as to why you

could not reach Gauss's cabin with just one vector. Debbie stated:

> So what our group did, we decided to see if we could use either the
> hovercraft or the magic carpet in one go. So we started with the
> hovercraft, and we knew it would have to go north at least 107 miles.
> So by dividing 107 by 3, we discovered that it did not go into it evenly,
> so you would always either go over your destination or go right before
> it. So then we decided you couldn't use the hovercraft. And inversely,
> we did the same thing with going 64 miles east. And by saying that 64
> times 1 is 64, which is the same as this, so then we also have to times
> 64 by 3, which is 192, which is way over our projected destination. So
> then we did the same process with the magic carpet. So 1 goes into 32,
> oh, 64 divided by 2 is 32, so then we used that 32, to times that by 1,
> and you'd only go 32 miles north, so that didn't work. And then we
> also tried that 107 times 1, so you went 107 degrees north. But then if
> you times that by 2, you end up going 214 miles east.

Debbie was asked by the instructor to explain why her group did not believe that you

could find Gauss's cabin with just one of the transportation vectors. The argument

was structured using three sets of data and warrants. Each datum referenced the

components of the vectors and the particular scalars that would allow the hoverboard

or magic carpet to get to one of the components of the Gauss's cabin vector. She then

used division of the destination by the one of the components of the transportation

vector to find the scalar. This allowed her to find the scalar. She then multiplied the

other component by the scalar. When that multiplication did not end up with the

second component of the destination vector, she concluded that you could not get to

the destination with that transportation vector. This was coded as a warrant because it

provided a relationship between the transportation and destination vectors and her

conclusion for why you could not get to the destination using one of the transportation

vectors.

Argument D2A16
Claim: You can't reach Gauss's cabin with just one of the modes of transportation. (Debbie)
Data: The components of the vector and the values that those components would be multiplied by *So we started with the hovercraft, and we knew it would have to go north at least 107 miles.* (Debbie) *With the hoverboard, "And inversely, we did the same thing with going 64 miles east.* (Debbie) *The vector for the magic carpet* (Debbie)
Warrant: Division of each of the resultant vectors values by the component of the mode of transportation and then multiplying that value by the second component. *So by dividing 107 by 3, we discovered that it did not go into it evenly, so you would always either go over your destination or go right before it.* (Debbie) *And by saying that 64 times 1 is 64, which is the same as this, so then we also have to times 64 by 3, which is 192, which is way over our projected destination.* (Debbie) *So 1 goes into 32, oh, 64 divided by 2 is 32, so then we used that 32, to times that by 1, and you'd only go 32 miles north, so that didn't work. And then we also tried that 107 times 1, so you went 107 degrees north. But then if you times that by 2, you end up going 214 miles east.* (Debbie)

Figure 4.10: Day 2, Argument 16

In her response, Debbie first stated that they tried to figure out if you could get to Gauss's cabin using just the hoverboard. She then stated that you could not because 3 did not go evenly into 107. This was incorrect reasoning because if she had chosen a non-whole number then she could have found the number of hours. However, what is important is that in the second part of her argument, Debbie then moves to the 1 in the hoverboard vector and multiplies that by 64. The 64 would be the number of hours necessary for the hoverboard to travel to get to its northern endpoint. When she multiplied the 3 in the hoverboard by the hoverboard, she finds that it has gone "way over our destination" at 192.

Her claim was that "you cannot get to Gauss's cabin" with just one of the vectors, echoing the qualifier in James's argument. In giving her reasoning for why this was not possible she provided the values of each of the components of the vector. Then as warrants she provided the fact that you would need to multiply each of the components by the same value. Since multiplying each component by the same value did not supply the necessary destination vector, she concluded that you couldn't get to the destination. She used the phrase, "you are way over your projected destination" evidencing the use of the traveling metaphor in her reasoning.

Debbie's warrants both discussed the relationship between the original vectors, the number of hours that are multiplied by those vectors and the resultant vectors that are a product of the multiplication. In her warrants, Debbie referred to the Gauss's cabin vector as being the destination and that the resultant vectors would go past the destination if they were multiplied by a single number of hours. This demonstrated the use of the idea that multiplying a traveling vector by a number of hours resulted in

a new vector that had a different distance. This same idea is used as data or warrant in 12 other arguments with different claims throughout the first six days of class. By criterion 3, this evidenced that the idea is functioning-as-if-shared.

This particular idea was tied to the task situation through the use of the words "destination," "north" and "east". It represented an advance in how students thought about scalars and vectors. In a general sense, members of the classroom community frequently created new vectors from multiplying vectors by scalars, and they also factored out values from each component of a vector to create scalar multiples. Converting a vector to a possible scalar multiple would be a frequent symbolic activity that students engaged in and demonstrated an added level of symbolic complexity.

The next two ideas were first instantiated as part of different arguments, but they were evidenced as functioning-as-if-shared in the same argument, D3A18. The second NWR provides an additional meaning for the scalar as a value that stretches or shrinks a vector.

NWR 2.2 The scalar stretches the vector by a factor indicated by the scalar value. The instructor revoiced the conclusions of the classroom community for the first task. The instructor presented several meanings for the scalar and what the scalar does to the vector when the vector is multiplied by one. The instructor was answering her own question as to how the classroom community can see the scalars, 30 and 17, in the graph of the vector equation

$$\begin{bmatrix} 90 \\ 30 \end{bmatrix} + \begin{bmatrix} 17 \\ 34 \end{bmatrix} = \begin{bmatrix} 107 \\ 64 \end{bmatrix}.$$

The interpretation that she presented is that the 30 and the 17 are the amount by which each vector is being stretched.

The third normative way of reasoning was introduced as a part of a classroom discussion that settled a dispute that had taken place during the third day of class.

NWR 2.3 Multiplying by a negative scalar moves you in the reverse direction Brad presented his argument that there was a place that Gauss could hide in the plane. He presented his argument that the places that you could go using the magic carpet and the hoverboard were restricted by the line of vectors that were in the same direction as $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Although, Brad's argument did not introduce the idea that becomes NWR 2.3, I present the argument in it's entirety because the discussion generated many of the discussions that initiated the ideas that made up the NWR's presented in this analysis. Furthermore, I reference his argument extensively and it provides context for the discussions that follow in days 3 and 4. Brad was asked by the instructor to present his findings from task 2 in the Magic Carpet scenario (See Appendix 1). The task asked students to find if there was anywhere that Gauss could hide his cabin such that you could not find him by using either or both of the modes of transportation. Brad concluded that there were places that Gauss could hide. Figure 4.11 gives Brad's board work. Figure 4.12 demonstrates Brad's gestures.
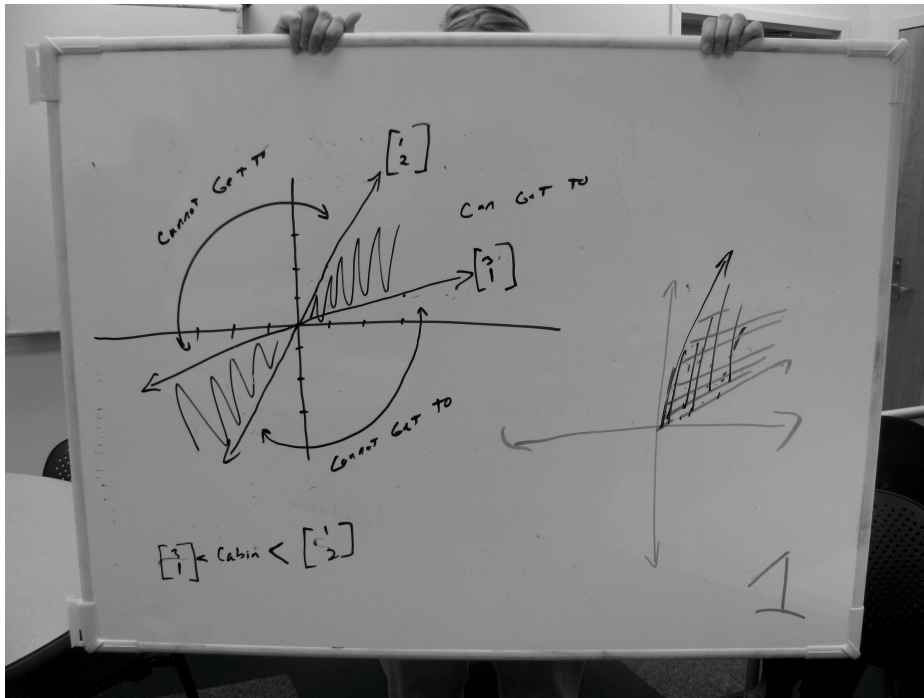
Figure 4.11: Brad's Board Work for D3A2

> Brad: I didn't get too far with it. I was just thinking logically that any places you could go to no matter what on here, would be a combination of this vector and this vecr. And this vector pulls you this way, this vector pulls you this way. So if you use only this vector, which is the farthest left you could go, this would be it right here, drew a curve from the top vector to the x-axis]. So if there's no combination that would get you, this angle that's bigger than this angle, no combination of these 2 things will get you that. Or an angle that's smaller than that angle. So I deducted that the only places you could get on the graph were in between these 2 vectors.

Brad's claim was the last statement in his argument, "The only places that you could get on the graph were in between these two vectors." The data for his claim was that the vectors that he drew on his board and his contention that any places that you could get to would be a combination of the two vectors. In addition, he added the ways that those particular vectors would "pull" the vectors. He supplied the bridge between his claim and his data by stating, "So if you use only this vector, which is the farthest left you could go, this would be it right here. So if there's no combination that would get

you, this angle that's bigger than this angle, no combination of these 2 things will get

you that. Or an angle that's smaller than that angle."

D3A2 Argument
Claim: There is a place that Gauss can hide on the plane if you are only riding the two transportation vectors. *The only places that you could get on the graph were in between these two vectors."* (Brad)
Data: The places that you are able to get to are a product of a linear combination of the two transportation vectors. The vectors pull the modes of transportation in specific directions. *"Any places you could go to no matter what on here, would be a combination of this vector and this vector. And this vector pulls you this way, this vector pulls you this way. [Makes gesture indicating pulling in either direction using his hands in the directions of the vectors.]"*(Brad)
Warrant: His gestures and graphical work demonstrating that any area outside of those vectors would be impossible to get to. *"So if you use only this vector, which is the farthest left you could go, this would be it right here. So if there's no combination that would get you, this angle that's bigger than this angle, no combination of these 2 things will get you that. Or an angle that's smaller than that angle."* (Brad)

Figure 4.12: Day 3, Argument 2

Brad's argument in D3A2 (as I will refer to it throughout the analysis) introduced or

expanded a variety of foundational ideas that would be used by students throughout

the next four days of class for linear combinations, directions of vectors, the effect of

scalars on vectors, and the restrictions on movement that a particular vector would

cause. Brad's argument did have some problems and these problems became the

focus of discussions throughout the next 30 minutes of class in both small group and

whole class discussion. Brad did not consider traveling in a reverse direction on one

of the modes of transportation. As a result of this deficiency in his argument, the

classroom community would discuss if movement in a negative direction was possible,

if you can use negative scalars and if a scalar meant more than just the number of

hours. The idea that would become NWR 2.3 is one of the ideas that was generated from these discussions.

Aziz in response to Brad's argument posited that a negative scalar could be thought of as moving in the reverse direction. As was stated earlier, multiple arguments presented a relationship between the scalar and the direction of the vector. The idea that multiplying by a negative scalar could be thought of as moving in the reverse direction is an advance over what it meant to be a negative direction because it explicitly connects the existence of the negative scalar with the change in direction. This idea was further evidenced as functioning-as-if-shared by being used in 8 different arguments throughout the next two days of class.

NWR 2.2 was not evidenced as functioning-as-if shared until later in the third day of class. Several groups presented their findings and Jason was asked to explain his reasoning for why he felt that you could get anywhere using either or both of the Gauss's cabin vectors. In addition to evidencing that idea was functioning-as-if-shared, this argument was also one of the arguments that evidenced the idea for NWR 2.3 was functioning-as-if-shared.
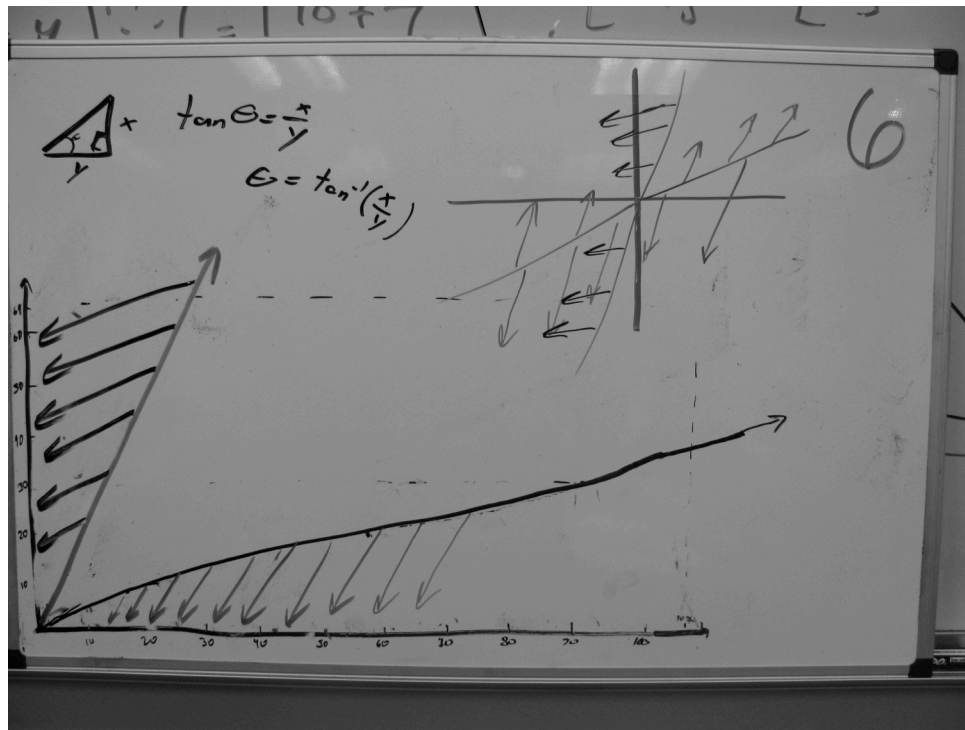
Figure 4.13: Jason's Group's Board Work for Task 2

4.14.1	4.14.2	4.14.3
4.14.4	4.14.5	4.14.6
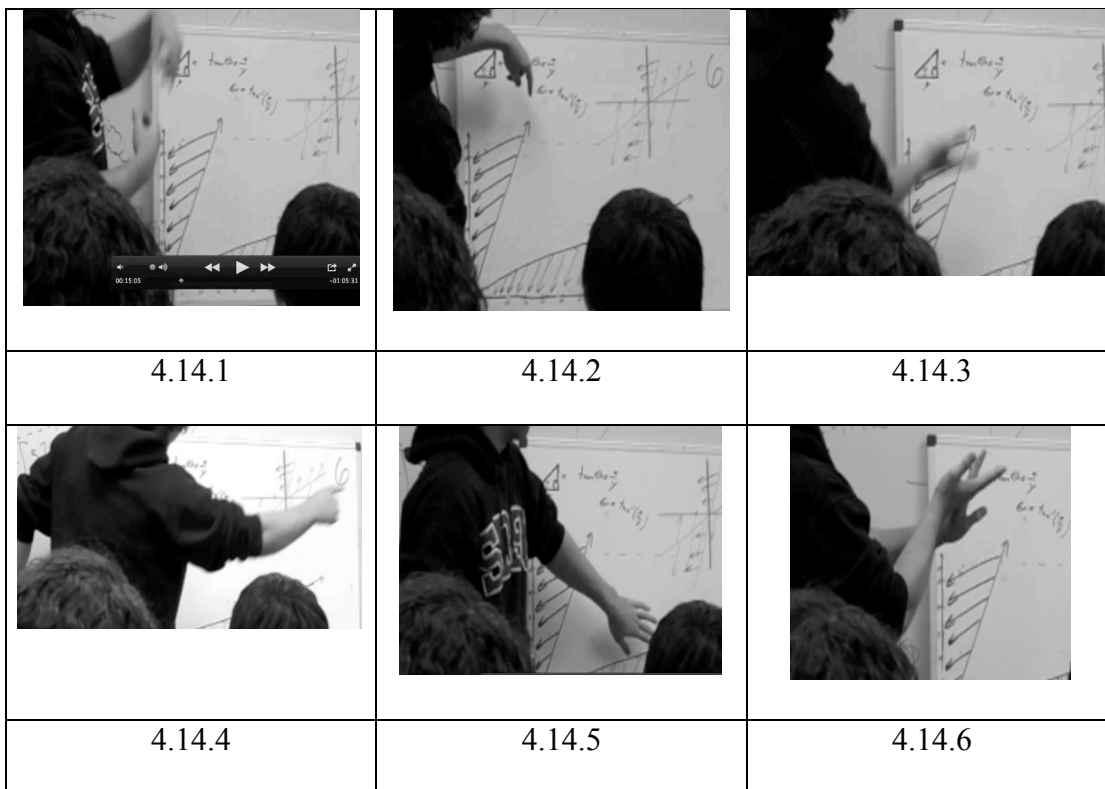
Figure 4.14: Jason's Gestures for D3A18

4.14.7	4.14.8	4.14.9
	4.14.10	
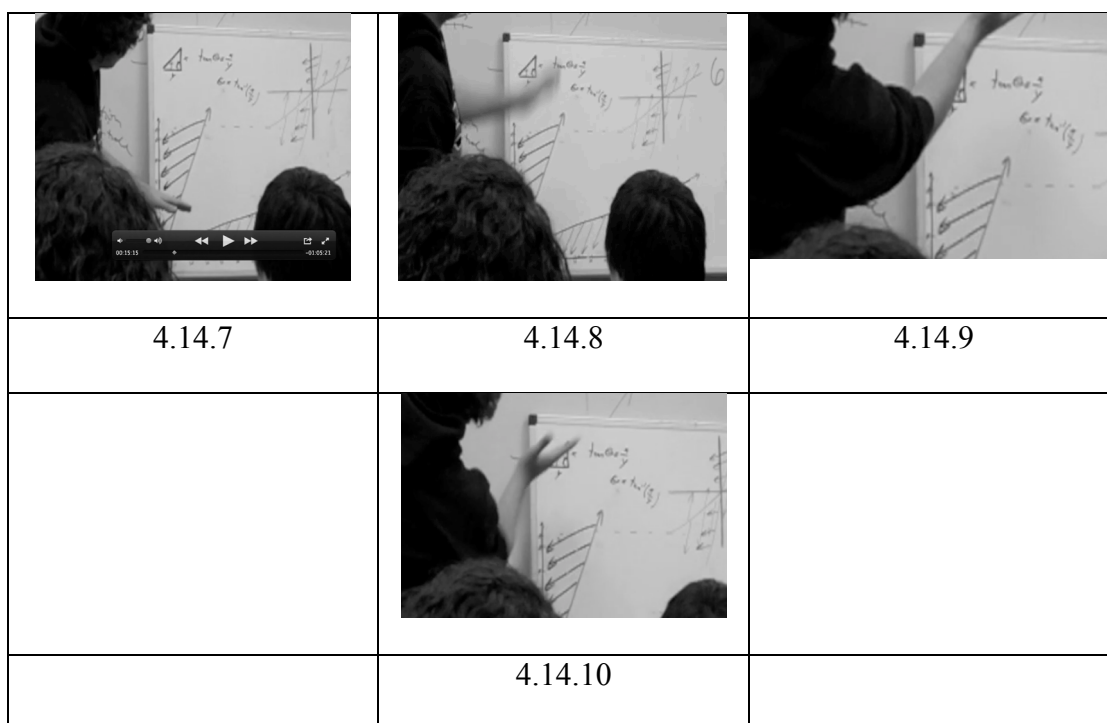
Figure 4.15: Jason's Gestures for D3A18 continued

Jason: So the way our group thought about it is, if we just take one of the vectors, so we take this one [4.14.1], and we can multiply it by any scalar, so we can extend it all the way to infinity [4.14.2] and negative infinity [4.14.3] along this angle. Right? So now this vector[4.14.4] is intersecting it [4.14.5], but it's not...since it's not parallel [4.14.6], we can slide [4.14.7] where we start riding our other mode of transportation, anywhere up and down. We can extend this one to infinity each way, and then we just set the initial conditions of this one, anywhere we want on here. So if we're trying to get back here[4.14.8], we slide it way up [4.14.9], and then our line comes down and crosses our point [4.14.10]. And that gives us infinity in the positive on both x and y.

Jason's claim was that you can get anywhere in the plane using the two modes of

transportation. As data for this he used the data that you can use any scalar that means

that you can stretch either vector infinitely in the positive or negative direction. He

gestured the infinite stretching with his hands in Figure 4.14.4 and 4.14.5. Then, he

added the data that "we can slide [4.14.7] where we start riding our other mode of

transportation, anywhere up and down." He gestured the sliding in 4.14.6-4.14.9. He

then provided the warrant for this data by stating, "since they are not parallel." He

then provided a warrant for the entire argument by stating, "So if we're trying to get

back here[4.14.8], we slide it way up [4.14.9], and then our line comes down and

crosses our point [4.14.10]." The argument is summarized in the following table:

D3A18 Argument
Claim: You can get anywhere in the plane with the hoverboard and the magic carpet. *"Unanimous, anywhere."* (Jason)
Data: You can start and stop anywhere on the vector and begin riding the other mode of transportation from the destination point to the intersection of the first vector. *we can slide where we start riding our other mode of transportation, anywhere up and down. We can extend this one to infinity each way, and then we just set the initial conditions of this one, anywhere we want on here.* (Jason) *(I)f we just take one of the vectors, so we take this one* (See gestures in Figure 4.46)*, and we can multiply it by any scalar, so we can extend it all the way to infinity and negative infinity along this angle. Right?* (See also Figure 4. for Jason's group's board work.)
Warrant: The vectors can be stretched to any length and in any direction. The vectors are in different directions, they are not parallel. *So now this vector is intersecting it, but it's not...since it's not parallel,"* *So if we're trying to get back here, we slide it way up, and then our line comes down and crosses our point. And that gives us infinity in the positive on both x and y. And infinity negative on both y and x.* (Jason)

Figure 4.16: Day 3, Argument 18

The imagery that was associated with this data and warrant was that Jason selected an

arbitrary point on the plane that he wanted to get to using the two vectors. He then

extended the first vector infinitely along its direction. He then started at the point that

he wanted to get to in the plane and extended a vector in the direction of the second

mode of transportation from the point to the first mode of transportation vector. The

fact that he could extend both vectors infinitely in either direction allowed him to

extend the vectors in the manner that was necessary to get to any point on the plane using this method.

The idea that would become NWR 2.2 is evidenced as functioning-as-if-shared when Jason gestured the stretching of the vector as he discussed the vector extending to infinity (Figure 4.14.3 and 4.14.4). In addition, the stretching from negative infinity demonstrates that he was considering negative direction. He then said that you can "multiply by any scalar" evidencing that the mechanism for the stretching and the moving in positive and negative direction is a result of the multiplication by scalars. The argument regarding the reverse direction evidenced the idea for NWR 2.3 functioning-as-if-shared. In this argument, Jason's contention was regarding whether or not one can get anywhere on the plane using the two vectors. In this case, the data for the argument was the intersection of the two vectors. As justification for his argument, he referenced his conclusion that a vector multiplied by "any scalar" could move in the negative and positive infinity along the line of the vector and the fact that the vectors were not parallel. This justification was as a warrant. The move of the idea for NWR 2.2 from Data-Claim to Warrant evidenced that the idea was functioning-as-if-shared. In addition, this was one of the arguments that had the idea for NWR 2.3 in the data or warrant.

A valuable note about these ways of reasoning is that the ideas were not just verbal, but also gestural and graphical. Jason's gestured where the vector stretched in both the negative and positive directions along the line of the particular vector. Students in multiple arguments used this kind of reasoning both gesturally and verbally. In addition, his drawing where the multiple parallel lines of vectors

intersected more parallel lines of vectors was used on two other boards as a response

to this question. The idea that a vector stretches, can be stretched to infinite length,

can move in an opposite direction, and the direction and amount of stretching depend

upon the scalar that the vector is being multiplied by became crucial to students

understanding of span and linear independence and dependence.

The final two normative ways of reasoning in CSP #2 related restrictions on

graphing vectors and vector equations to the algebraic expressions for those objects.

NWR 2.4 related how the classroom community viewed the interplay between the

algebraic expressions for scalars and vectors and graphical restrictions on vectors and

scalars.

NWR 2.4 The vector notation (multiplying by a negative or positive scalar) allows for the ability to move in the positive or negative direction.

The idea for this NWR was first introduced in the classroom during the discussion that

students had about whether or not you could use negative scalars to move the magic

carpet and hoverboard when trying to find if there was anywhere that Gauss could

hide. Brad made an argument that negative scalars could not be used because the

scalars represented hours was challenged by several of the groups as a follow-up to his

argument in D3A2. Aziz presented a counterargument to Brad's argument that

employed the vector notation as a justification for his reasoning. The warrant for

Aziz's argument stated that the "matrices" allow for the use of positive or negative

scalars, which would in turn allow for a negative direction. Aziz's warrant

specifically coordinated the algebraic representation of the vector times the scalar with

the ability to go positive or negative. In this case, the fact that a scalar can be positive

or negative implied that it could be possible to find an interpretation for a positive or negative scalar as dictating both stretching and direction-switching. The idea that he was drawing upon was one in which there was a connection between the algebraic expression and its possibilities (having a positive or negative scalar) and how that could be interpreted graphically.

The idea for NWR 2.4 was evidenced as functioning-as-if-shared by being used as data or warrant in two other arguments including D4A21 and D4A24 and 4 additional arguments in Days 5 and 6, fulfilling criteria 3. Students began to use negative and positive scalars to deal with questions that were not tied to the magic carpet scenario and questions dealing with three-dimensional vectors. In D4A24, the class was asked if three vectors, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ were linearly dependent. The third instance of this idea being used by students as data or warrant arose in a later argument on that same task as the students clarified the vector equation that they would use to prove that the three vectors were linearly dependent. In D4A24, Gary argued that there is more than just one vector that is a solution to the vector equation . Gary's warrant for this argument used the language of using the negative scalar and the language of getting back in order to evidence movement on the scalar as one of the ways that he was thinking about working with vectors and scalars. He mentioned that you use the negative form of the vector in conjunction with his solutions that have negative scalars. The tying of the negative scalars with the negative directions and the algebraic form that he used throughout his argument evidenced the use of this idea as a

warrant. When three different arguments with three different claims are evidenced as having the same way of reasoning as data or warrant, the idea is said to function-as-if-shared by criterion 3.

The idea for NWR 2.5 was first introduced during the classroom discussion on whether you could use a negative scalar to find a place where Gauss could hide.

NWR 2.5 Changing a vector computationally by multiplying by a scalar is the only way to change the vector graphically.

James, a member of Brad's group, contended that interpreting a negative scalar as a change in direction was problematic because there was no reason to think that a 180 degree turn was the appropriate change in direction necessitated by multiplying the vector by a negative scalar.

James: I had a problem with the negative. (Walks up to the board) So let's say you have whatever you're on, it doesn't matter, it has a predefined direction in which it goes. So you have a vector up there, and to say that you're going in reverse, you pretty much flipped the vector,, and now you're going this way, right? But whatever you're riding, it would also require you to flip it a full 180 degrees. So what's stopping you from flipping it only half way? In which case, you could pretty much get anywhere. Why not flip it ninety degrees.

James began by drawing his theoretical vector. He then drew the interpretation that students had of the change in direction. But then, he questioned the choice of 180 degrees, while he moves his body 180 degrees. He then turned another 180 degrees back to his original position. He concluded his argument by asking why it is that the direction-switching had to be 180 degrees. Why not a direction-switch of 90 degrees? And wouldn't that particular switch allow you to get anywhere in the plane with just one vector? He argued that the choice of multiplication by a negative scalar being a

180 degree flip was arbitrary, and that there was nothing to prevent an individual from choosing to turn the vector in a ninety degree direction or whatever direction you would choose. Being able to change the direction of the vector arbitrarily would mean that you could get anywhere in the plane using a single vector because you could simply multiply by a negative and change the direction. This would make the question "Can you could get anywhere in the plane using two vectors?" trivial because you could in fact get anywhere in the plane using any single vector.

As a rebuttal to James' argument, Nate made an argument that connects the algebraic notation of the vector and scalar to the task scenario. This is the first instantiation of the way of reasoning. Nate claims that the algebraic notation restricts multiplication by a negative to a 180-degree flip.

> Nate: Is it specified as a vector so we can't, we ask can it go backwards? You can't just...if you say flip it 90 degrees, we would change its specification vector to do that, and we can't do that. We can only multiply it by a negative scalar. That's what we do when we go backwards.

Nate's argument began with the claim that you could not flip the vector 90 degrees because that was the question that Nate was responding to. The data that he provided was that "it is a specified vector," by which he meant that the vector was specified with a group of components. He also added that you could only multiply the vector by a negative scalar. Then in order to tie his data to his claim, he added the warrant that "If we flip it 90 degrees, we would change its specification of the vector to do that, and we can't do that," implying that a flip of a vector 90 degrees would have changed the components in the vector. He concluded this by reasoning that multiplying by a negative scalar would still change the vectors components, but would change them

only by the same number. Otherwise, this would change the vector by definition

(NWR 1.1). He added to his warrant that multiplying by a negative scalar caused the

mode of transportation to move backwards implying that that move was allowed.

Nate's argument is summarized below.

D3A16 Argument
Claim: A 90 degree flip of the vector by scalar multiplication is not allowed. *You can't just flip the vector ninety degrees*. (Nate)
Data: The components of the vector. Only multiplication by a negative number is allowed. *It is specified as a vector." "We can only multiply it by a negative scalar*.(Nate)
Warrant: Flipping a vector by ninety degrees changes the components of the vector by a value that is not the same for both components. *If we flip it 90 degrees, we would change its specification of the vector to do that, and we can't do that....That's what we do when we go backwards.* (Nate)

Figure 4.17: Day 2, Argument 16

Nate argued that if you could change the direction of the vector arbitrarily that would

require a value for the scalar that you could multiply the vector by to get the new

vector with the 90 degree flip. He contended that such a scalar does not exist and that

multiplying by the negative scalar gave a resultant vector that was graphed as a 180

degree flip. Nate's argument was algebraic in nature as it depended on the creation of

a new vector algebraically, by multiplying by a negative, and not graphically by

arguing that you could move in reverse. He then tied the ability to create a kind of

vector algebraically to what is possible geometrically.

In later classroom arguments, relating the algebraic construction of a vector to

what it will be as a geometric object became common.   I have chosen one of the

arguments from a later class period as evidence of criterion 2, where the idea used in

Nate's argument is used in the warrant of a later argument. The argument came from

the discussion on whether or not the set $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ was linearly

dependent. To give context, I begin the transcript for the argument:

> Instructor: So let's consider a similar set, let's just make it a little bit different. So we still have, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ and then say maybe $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. And the task, 'Is this a linearly dependent set?' Does anyone have an idea how we could start going about figuring this out.

> Robert: It's just an idea, the third vector is just a multiple of the first. So is there any point in doing it? [inaud] just the same.

> Instructor: Oh, interesting, say more about that.

> Robert: I was just saying the first vector if multiplied by four is the third vector. So just the first one stretched is the same thing, so I don't see any point in even having a third vector if it's going to be a multiple of the first.

Robert's claim was that the set of vectors was linearly dependent; this was in response to the instructor's question. His data was that the third vector was a multiple of the first. And then connecting the data to the claim was his contention that first vector multiplied by four was the third vector, and that the "first one stretched is the same thing."

D4A19 Argument
Claim: The set $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ and then say maybe $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ is linearly dependent. (Robert)

Figure 4.18: Day 4, Argument 19

> Data: The first and third vectors are scalar multiples of each other.
> *It's just an idea, the third vector is just a multiple of the first. So is there any point in doing it? (Robert)*
>
> Warrant: The first vector multiplied by four is the third vector. The first vector stretched is the same as the third vector.
> *I was just saying the first vector if multiplied by four is the third vector. So just the first one stretched is the same thing, so I don't see any point in even having a third vector if it's going to be a multiple of the first. (Robert)*

Figure 4.19: Day 4, Argument 19 Continued

Robert argued that the set was linearly dependent because algebraically you could change the first vector into the third vector by multiplying by 4. In addition, he added

that by multiplying by 4 would stretch $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$, tying the geometric change of the vector to the algebraic multiplying of the vector by the scalar. The shift of the way of reasoning from the data-claim to the warrant demonstrated criterion 2 and evidenced the idea as functioning-as-if-shared.

Each of the normative ways of reasoning presented for CSP #2 dealt with how students understood, created and used scalars in understanding vectors and vector equations. From these NWR's I determined a meaning for scalars that students developed throughout the course of the first 6 days of class. The conclusion that I came to was that students viewed scalars as having algebraic, graphical and situation specific attributes. As was evidenced by the NWR's presented in this section, these three different kinds of attributes are inextricable for the members of the classroom community. Students came to their understanding of graphical, algebraic and situational ways of reasoning by interplaying with these different conceptual and symbolic interpretations. For this classroom community, scalars were multiplied by

vectors to produce new vectors, to stretch vectors or to change the vectors direction.

In addition, the kinds of directions and stretching that were allowed by scalar

multiplication were constrained by the algebraic nature of the symbols that the

students were using. Furthermore, the meanings that students constructed for scalars

and scalar multiplication were informed by the task situation, as students dealt with

movement as a central metaphor for their work with vectors and scalars.   In fact, the

concepts of stretching and changing of direction being synonymous with

multiplication by positive or negative scalars arose from students activity with tasks

that asked them to find ways of finding locations on the plane, places where an

individual could hide on the plane, and whether or not an individual could get back to

their original starting point using particular modes of transportation.

CSP # 3: Setting up and solving a system of equations allows one to solve for the scalars in a vector equation

The grouping for Classroom Symbolizing Practice #3 was based upon student's

use of systems of equations as a solution method for the scalars to linear combinations.

While students used systems of equations to find many other kinds of solutions to

various problems and situations in their homework and during exams, I concentrate on

the use of systems in relation to the meanings that are developed for vectors and vector

equations.

CSP # 3: Setting up and solving a system of equations allows one to solve for the scalars in the vector equation.

Normative Ways of Reasoning (NWR)

> 3.1 The solution to the system of equations tells you how long you need to ride each mode of transportation in order to get to Gauss's cabin.
> 3.2 The solutions as represented by the systems of equations, the graphs and the vector equations are the same.

Figure 4.20: Classroom Symbolizing Practice #3

CSP #3 was made up of two normative ways of reasoning that were evidenced as functioning-as-if-shared over the first three days of class. The arguments that first introduced the idea that became NWR 3.1 was first presented in the first day of class, and the idea that became NWR 3.2 was presented during the third day of class. In the following section, I detail the normative ways of reasoning, the arguments in which these ideas were first discussed in class, and the arguments that evidenced that the ideas were functioning-as-if-shared.

The first way of reasoning was introduced early in class discussion in D1A9. The three NWR's had to do with what a solution to the system of equations meant and how the symbolic expression for the system of equations was related to the vector equation. This NWR was tied to the situation and how the system of equations presented solutions in the task situation.

> NWR 3.1 The solution to the system of equations tells you how long you need to ride each mode of transportation in order to get to Gauss's cabin.

The introduction of this idea was in response first task in the Magic Carpet Scenario, where students were asked to find how many hours one would need to ride the

hoverboard and the magic carpet to get to Gauss's cabin. George was asked to present

his argument for why it would take 17 hours on the magic carpet and 30 hours on the

hoverboard. (George's gestures and board work are presented below the transcript in

Figure 4.21 and 4.22).

> George: We wrote down that x [4.31.1] is basically the hoverboard, it's going to go 3 east and 1 north [4.31.2], y [4.31.3] is going to go 1 east and 2 north. And the final destination is 170 [4.31.4] east and 64 north here [4.31.5]. So I broke that down into 2 separate equations [4.31.6] to where it's at 3x , so 3 [4.31.7] times the hoverboard, 3 east and 1 north, you go 170[4.31.8], which is our final destination [4.31.9]. Then it's 1 east, and 2, x + 2y is 64, our final destination. So we broke that down, we separated it, the separation where that y equals 170 minus 3x, we have y by itself. And then the same here in this equation, you plug it right back, so we solved for that, we get x = 30 here. From that, we plug that right back into the equation to give us what our y would equal, we got 17. So we end up with x = 30, and y = 17. Is that clear?
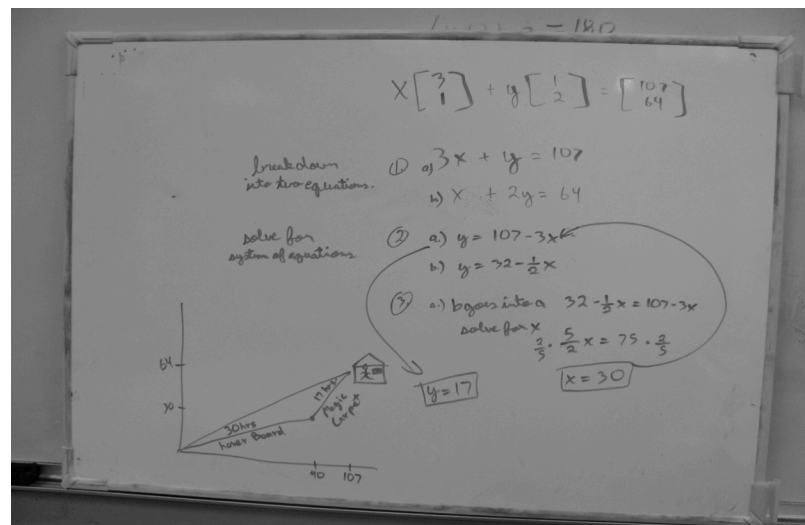
Figure 4.21: George's group's Board work for D1A9

4.22.1	4.22.2	4.22.3
4.22.4	4.22.5	4.22.6
4.22.7	4.22.8	4.22.9
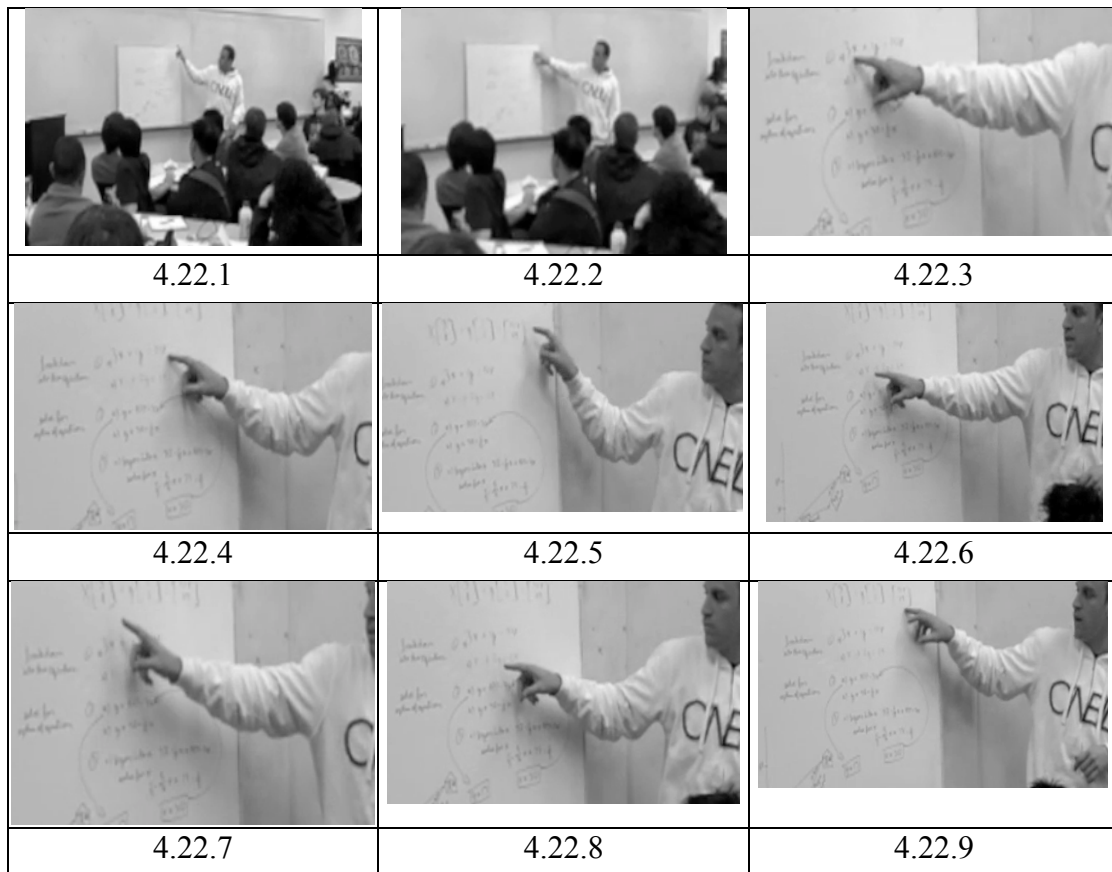
Figure 4.22: George's gestures for D1A9

George's claim was that it will take 30 hours on the hoverboard and 17 hours on the

magic carpet to get to Gauss's cabin. The data that he used was multi-faceted. The

answer that he arrived at x=30 and y=17 as the solutions to the systems of equations.

In addition, he used as data that x is the hoverboard (by this he means the number of

hours you travel on the hoverboard), that it has a specified vector. And that y was the

magic carpet (implying the number of hours travelled on the magic carpet) and it had a

specified vector. His warrant was that he converted the vector equation into the

system of equations, this conversion was evidenced via his gestures where he

systematically pointed to the system of equations and then the parts of the vector

equation that corresponded to the system. Finally, he finished the warrant by stating

that his group solved the system for the required values. In this argument, the vectors

for the hoverboard and the magic carpet and the scalars, as represented by x and y, for

the number of hours along with his solutions provided the data for the argument. The

transforming of the vectors into the system and the solving of the system provided a

bridge between the data and the claim that he made by providing the means by which

he transformed the data into his claim and also how the solutions related his final

solution. His argument is summarized below. (Figure 4.22)

D1A9 Argument
Claim: In order to get to Gauss' s cabin, you need to take the hoverboard 30 and the magic carpet 17. (George)
Data: The solutions to the system of equations. x is defined as the hoverboard, y is defined as the magic carpet. The particular vectors that represent the magic carpet and hoverboard. The final destination vector.   *So we end up with x = 30, and y = 17. (George)*   *x is the hoverboard, which goes 3 east and 1 north. [Points at the vector]*   *y is the magic carpet and that travels 1 east and 2 north. [Points at the vector]*   *The final destination is 170 east and 64 north. [Points at the resultant vector]*   *On the hoverboard, you go 3 east and 1 north to go 170, which is our final destination. Then it's 1 east and 2, x+2y is 64 [Underlines with his hand the system of equations.](See Figure 4.32 for George's board work and Figure 4.33 for his gestures)(George)*
Warrant: Separating the vector equation into the system of equations.   *So we broke that down, we separated it, the separation where that y equals 170 minus 3x, we have y by itself. And then the same here in this equation, (Points to the system of equations) you plug it right back, so we solved for that, we get x = 30 here. From that, we plug that right back into the equation to give us what our y would equal, we got 17. (George)*

Figure 4.23: Day 1, Argument 9

In his verbal explanation, he misidentified the relationship between the vector

equation and the system of equations by stating that the 3x+1y=170 represented the 3

north and the 1 east that needed to be traveled on the hoverboard, when in fact it

should have been the 3 east on the hoverboard and the 1 east on the magic carpet

giving the total displacement east of 170. However, he did connect the symbolic representations for vector equations and systems of equations using his gestures. Although George's interpretation of the relationship between the symbolic representation of the system of equations and the algebraic representation of the vector equations was incorrect verbally his gestures did operate correctly. In addition, he did properly identify the scalars and the meanings for the solutions to the systems. This was the connection that was classified as the normative way of reasoning. The idea detailed the relationship between the solution to the systems of equations, the number of hours ridden on the hoverboard, and the scalars in the vector equation. ,.

The evidence for the idea that became NWR 3.1 functioning as-if-shared in the classroom community came on day 2 as students continued to discuss their conclusions for the number of hours that it would take to get to Gauss's cabin. Gary presented his argument (D2A4) for why he felt it would take 30 hours on the hoverboard and 17 hours on the magic carpet to get to Gauss's cabin. He used the system of equations to argue his conclusion. When he was asked by the instructor for justification for his solution method, Gary presented the number of hours that the board was ridden and connected the x and y scalars via a separation of the vectors into systems of equations. His warrant identified the meaning of the x's and y's as the hours that the hoverboard and magic carpet would be ridden respectively.. In D1A9, George presented the x and y and their meaning as hours as data, whereas here Gary used the idea as a warrant, evidencing criterion 2.

In addition to presenting the system of equations as being a valid way of solving for the scalars in the vector equations, the class also discussed whether or not

the solutions to the vector equation and the system of equations were the same. NWR

3.2 summarizes the normative way of reasoning that arose from this discussion:

> NWR 3.2 The solutions as represented by the systems of equations, the graphs and the vector equations are the same.

The classroom community was explicitly asked whether or not the representations for

the graphs, the systems and the vector equations were the same. The members of the

community had already done activities that established this connection and the

question was raised in order to find out to what extent the members of the classroom

community saw the representations as the same. Jason's argument, the board work

that he referenced are given in Figure 4.23.
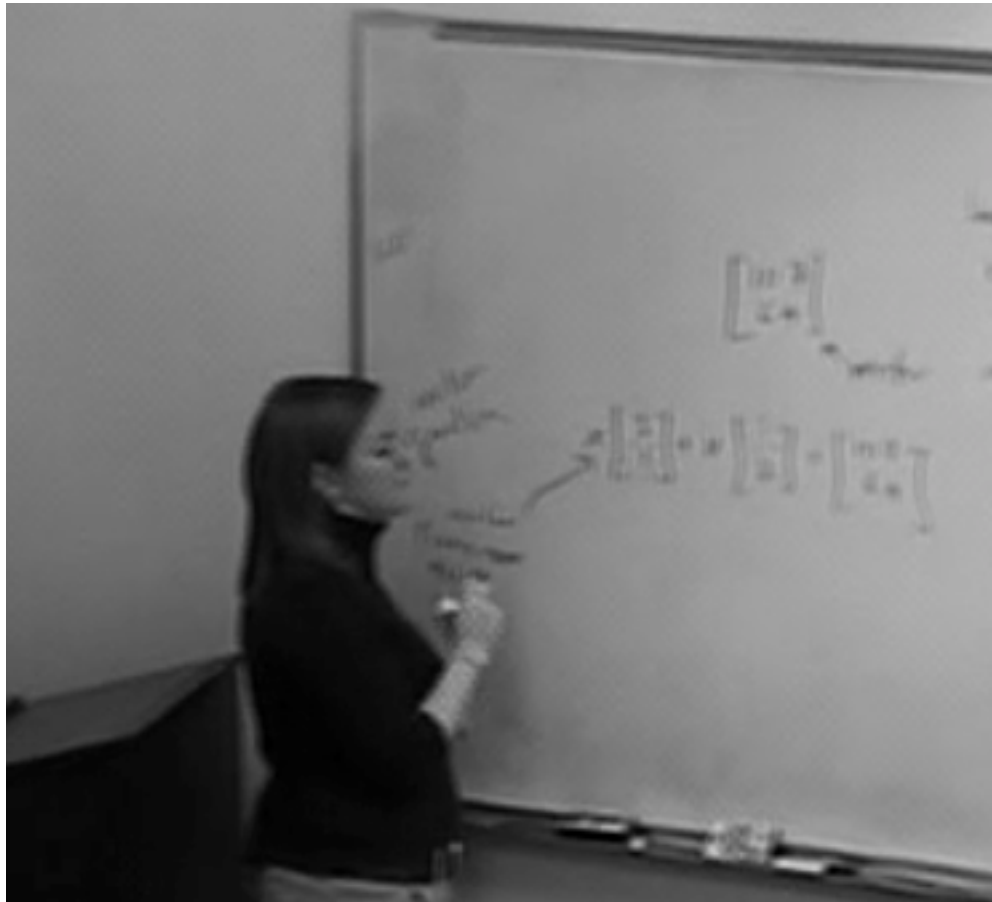
Figure 4.24:Board Work for Argument D2A13

Jason: The way we talked about it, we thought they were all the same, because with the linear combination, so $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with the scalars on the front is just like with matrices, when you add them, it's like across each.I don't know what it's called, with the 3 and the 1 and the 107 right there, they all, you add those together, on the top, you know. And then the bottom is separate. And so the system of equations is just doing the same action but separating it outside of the matrix. So we saw those 2 as the same. And then the graphs are, just the graphs of those systems of equations with constraints on certain areas.

Jason's argument referenced the drawing in 4.23 as he gestures and talks. The picture on the board was the vector equation, but he made the argument that the two expressions were the same and he detailed ways that those two expressions are the same. Then Jason discussed how the 3, 1 and the 107 were then separated in the system of equations. The separation that he gestured in the air mimics the separation of the components of the vector, which he referenced as working "across." The "across" reference was important because it contrasted with the up and down nature of the vector representation and clearly evidenced that he was talking about the systems. He then discussed how the "action" of the vector equation was doing the same thing as the "action" of the systems of equations except "outside of the matrix." The action that Jason referenced when he discussed the similarity of the vector equation and the system of equation was the solving for the scalars, as this was the only discussion that the classroom community had had up to this point. As well, his breaking up of the vector equation in the air with his gestures demonstrated that he felt that the differences between the vector equation and the system of equation was in how they were represented when solving for those scalars.

The idea was evidenced as functioning-as-if-shared in D3A28, when Jason gave both data and warrant to the instructors claim that solving the system of equations for general scalars c_1 and c_2 will prove that you can "get everywhere" using the hoverboard and the magic carpet.

D3A28 Argument
Claim: The fact that there is an algebraic expression that presents a relationship between any resultant vector and the scalars chosen to find that resultant vectors implies that there is an algebraic proof that you "can get anywhere" using the vectors.   *If c_1 is 2a - b over 5, and c_2 is 3b - a over 5, convinces us that we can get everywhere with these two vectors.* (Instructor)
Data: The choice of destination vectors is arbitrary and then a can be chosen and b can then be solved for.   *Because you know where you want to go, the vector a,b that you want to go to. You just plug in your a and your b, that gives you the coefficient.* (Jason)
Warrant: Jason's translation of the instructors use of the system of equations form in terms of the algebraic vector equation form and the graphical form.   *I think c_1, yeah, so c_1 is how long you ride the 1st vector, when you get to c1, which is given by your a. Then you hop off and ride it a distance c_2.* (Jason)

Figure 4.25: Day 3, Argument 28

The instructor first used the system of equations:

$$3c_1 + 1c_2 = a$$
$$1c_1 + 2c_2 = b$$

as an equivalent form for the vector equation:

$$c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

She then used algebraic manipulation to arrive at the equations:

$$c_1 = \frac{2a - b}{5}$$

$$c_2 = \frac{3b - a}{5}$$

Jason's data includes aspects of the graphical and situational expressions as he

discussed, "how far you want to go" as represented by the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. He then

referenced c_1 as being "how far you ride the first vector" connecting the situational

references to the vector equation and the system of equations. The connection

between the scalars in the vector equation and the coefficients in the system of

equations was also functioning-as-if-shared in the classroom community, being

evidenced in D1A9 and D2A7. In Jason's warrant, he treated the various algebraic

and system of equations expressions as if they were the same. In D2A13, the idea that

the vector equations and the systems of equations generated the same solutions was

part of the data-claim of the argument. In D3A28, the idea moved to the warrant,

evidencing criterion 2.

The NWR's that were collected in CSP #3 both revolved around the use of

systems of equations to solve for the scalars in a vector equation. These two NWR's

were necessary to progress with the other activities in the Magic Carpet Scenario

because they provided the algebraic mechanisms to find scalars that were not

immediately obvious from inspection of the vectors. In addition, by concluding that

the solutions to the systems, the graphs and the scalars in the vector equation were the

same, the students could proceed confidently with later work and know that they had a

computational technique that was effective for a variety of situations.

CSP # 4: Linear combinations can be used to model relationships between sets of vectors

CSP #4 collected basic normative ways of reasoning for linear combinations. The normative ways of reasoning for linear combinations incorporated CSP #1 and #2 as individual vectors and scalars make up the symbolic expressions for linear combinations. The normative ways of reasoning for CSP #4 are summarized in Figure 4.24:

CSP # 4: Linear combinations can be used to model relationships between sets of vectors

Normative Ways of Reasoning (NWR)

     4.1 A linear combination is a scalar times a vector plus a scalar times another vector.
     4.2 The tip-to-tail method graphs the addition of scaled vectors.
     4.3 You can perform operations on vector equations by multiplying each component in a vector by its scalar and adding it to the other vectors times their scalars and making them equal to some resultant vectors.
     4.4 A combination of a set of vectors determines a path using the set of vectors to "get to" a resultant vector.
     4.5 Multiplying an entire vector equation by a constant does not change the existence of a solution.

Figure 4.26: Classroom Symbolizing Practice #4

In CSP #1, the vector was imagined as a path. The idea of a path can be expanded beyond a straight-line path by imagining the linear combination as a non-straight-line path made up of multiple straight-line vector paths. The introduction of a new non-straight line path meant that there needed to be new ways of creating these paths in space. The mechanisms for creating these paths are the subject of CSP #4. However, in addition to relating to the linear combination as paths, the classroom developed meanings for linear combinations that were not necessarily tied to the Magic Carpet Scenario. Similar to CSP #2, the meanings that were developed in situational,

algebraic or geometric contexts were tightly connected to each other. In fact,

frequently student's understandings for what the algebraic symbols meant was tied to

their graphical work and vice-versa.

The first normative way of reasoning defined a resulting vector as a

combination of scalars and vectors as a "linear combination."

NWR 4.1 A linear combination is a scalar times a vector plus a scalar times
another vector.

This NWR was the basic definition that was used for the term linear combination. The

NWR did not necessarily imply equaling a set of scalars and vectors to another vector.

Instead, students could think of the linear combination as representing a graphical

representation. In addition, the NWR did not specify the actual value for the scalar,

instead it left open the possibility that scalars could be variables. This was key in that

students could flexibly create linear combinations for a variety of circumstances. The

instructor first used the term, linear combination, as she tagged the resulting vector

equation from task 1 in D2A2. The instructor referred to the linear combination as the

addition of vectors that are multiplied by a scalar. Similar to the tagging of scalar, the

classroom community quickly began using the term and its associated symbolic

expression interchangeably without having to define the term for other members of the

classroom community. D2A13 introduced the idea that systems of equations, vector

equations and graphs are interchangeable from certain perspectives. In his argument,

Jason includes the term linear combination and he references the symbolic expression

that the instructor has written on the board (Figure 4.23). Jason does not reference the

instructor's definition, which was used as the warrant in D2A2. In Jason's argument,

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there was no need to again define what a linear combination was. This definition

would have been supplied as backing for his warrant. The backing dropping off for

this argument evidenced that criterion 1 had been fulfilled.

The second normative way of reasoning in this classroom community dealt

with how these students graphed linear combinations.

NWR 4.2 The tip-to-tail method graphs the addition of scaled vectors.

The "tip-to-tail" method as the way of graphing linear combinations was called in the

classroom community was first introduced in James' explanation of his graphical

representation of the solution to Task 1 in D2A7 (Figure 4.6 and 4.7). The drawing

was James' drawing, but it was the instructor who pointed out the drawing and

provided the data and warrant to establish the NWR in the classroom community.

> Instructor: So I hear him say this vector here, the blue one, was the 90 over 30, and then he added it, that 2nd vector, which was this one here, the 17, 34 and he knew that he would end up at your found location. And one of the things that he wrote up here was that this vector here is actually that vector, 107,64. I kind of held your hand, I didn't just let you go straight in that direction, you had to go along the route of these other 2 vectors. So this drawing method, it might often work, it's often called 'tip to tail'. We have the 1st vector, and at its tip, the tail of the next one, you add together and that final location will be the endpoint of the resultant vector.

The argument was for the most part a revoicing of James's argument, however, the

new information was that his method of graphing was called "tip-to-tail." Hence, this

was the claim in the argument. The data immediately followed, when she said, "We

have the 1st vector, and at its tip, the tail of the next one, you add together and that

final location will be the endpoint of the resultant vector." This described verbally the

process that James used to graph the vector. The warrant that she supplied, as

evidenced by the word "so," she related how the more general activity that she discussed in the data was related to the work that Jhoel was doing in D2A7. Hence, she could conclude that the method of drawing used in James's drawing was called "tip-to-tail."

D2A8 Argument
Claim: The drawing method that James used to visualize how to get to Gauss's cabin is called "tip-to-tail" (Instructor) (See figure 4.46)
Data: *We have the 1st vector, and at its tip, the tail of the next one, you add together and that final location will be the endpoint of the resultant vector.* (Instructor)
Warrant: *So I hear him say this vector here, the blue one, was the 90 over 30, and then he added it, that 2nd vector, which was this one here, the 17, 34 and he knew that he would end up at your found location.* (Instructor)

Figure 4.27: Day 2, Argument 8

The instructor tagged the method as "tip-to-tail" in the claim and the explanation for the drawing of the vector is demonstrated in the warrant.

The idea that the instructor introduced in D2A8 introduced the name for the graphing technique for linear combinations, but also supplied the validity for the technique. Using the tip-to-tail method allowed for students to visualize linear dependence and independence and span as students could then imagine adding a vector in a particular direction to another vector that was in a different direction. In addition, this way of reasoning may have also played a role in visualizing why vectors that were scalar multiples of each other were linearly dependent or did not have an impact on span. Two vectors that were scalar multiples of each other when added tip-to-tail would not have given any new information or provided for any new places in the plane for you to go with the vectors.

The members of the classroom community did not discuss graphing of

particular solutions or particular linear combinations again, but they did have the

opportunity to discuss potential graphs or graphical expressions for linear

combinations. D3A18 was an argument that culminated the discussion about "Is there

anywhere on the plane that Gauss can hide?" The instructor had the students break

back up into groups in order to answer if the members of the group could get

anywhere on the plane with the hoverboard and magic carpet. In D3A18, Jason

summarized his group's argument (For a full treatment of Jason's argument see NWR

2.2 and 2.3).

D3A18 Argument
Claim: You can get anywhere in the plane with the hoverboard and the magic carpet. *"Unanimous, anywhere." (Jason)*
Data: You can start and stop anywhere on the vector and begin riding the other mode of transportation from the destination point to the intersection of the first vector. *we can slide where we start riding our other mode of transportation, anywhere up and down. We can extend this one to infinity each way, and then we just set the initial conditions of this one, anywhere we want on here.* (Jason) *"if we just take one of the vectors, so we take this one* (See gestures in Figure 4.46)*, and we can multiply it by any scalar, so we can extend it all the way to infinity and negative infinity along this angle. Right?* (See also Figure 4. for Jason's group's board work.)
Warrant: The vectors can be stretched to any length and in any direction. The vectors are in different directions, they are not parallel. *So now this vector is intersecting it, but it's not...since it's not parallel,*(Jason) *So if we're trying to get back here, we slide it way up, and then our line comes down and crosses our point. And that gives us infinity in the positive on both x and y. And infinity negative on both y and x.* (Jason)

Figure 4.28: Day 3, Argument 18 Revisited

Jason utilized a visual approach to explain why his group came to the conclusion that

they could get anywhere in R^2 with the hoverboard and the magic carpet. His picture

demonstrated a cross-hatched pattern that resembled multiple tip-to-tail drawings. He

discussed how his drawing related to the ability to get everywhere in the plane with

the two vectors. Jason gestured in the air a single vector and he extends his arms

outward while he says, "infinity." In conjunction with his statement that "we can

multiply it by any scalar" this implied that the multiplication by any scalar allowed the

creation of any vector along that line of vectors. Then, he said that "we ride our other

mode of transportation" while gesturing a different direction. The gestures that he used

in conjunction with the pictures that he drew evidenced the tip-to-tail method in

helping him find all of the places that he could get using the magic carpet and

hoverboard. Jason's use of the "tip-to-tail" method came as a part of his data and his

warrant, whereas in the instructor's argument the NWR was used as the claim. This

evidenced that the NWR was functioning-as-if-shared by criterion 2.

NWR 4.3 built on NWR 4.1 by including the resultant vector as a potential part

of the linear combination.

> NWR 4.3 You can perform operations on vector equations by multiplying
> each component in a vector by its scalar and adding it to the other
> vectors times their scalars and making them equal to some
> resultant vector.

The inclusion of the resultant vector to linear combination allowed for the creation of

the vector equation. Students used the vector equation form from the beginning of

task 1, when they tried to find Gauss's cabin. But the idea of this being a kind of

linear combination and the resultant vector being the product of the process of creating

a linear combination was not established until day 2. D2A13 was introduced earlier in

this chapter when Jason discussed how the system of equations, vector equations and

the graphs were the same.   In his argument, Jason referenced the vector equation as

the linear combination and pointed to the vector equation on the board as he discussed

it. He also explicitly referenced the vectors and scalars and how they were combined

to equal the resultant vector. He does this in the data for his argument for why the

three symbolic expressions are the same. The picture concluded his argument by

setting the scalar times the vector equal to the resultant vector.

The second instance in which the algebraic expression for linear combinations

was explicitly used as data came later in that same class period in D3A51. The

instructor asked the classroom community if they could get back home using the set of

vectors, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$. Eddie responded by creating a linear combination

and setting that linear combination equal to the zero vector.
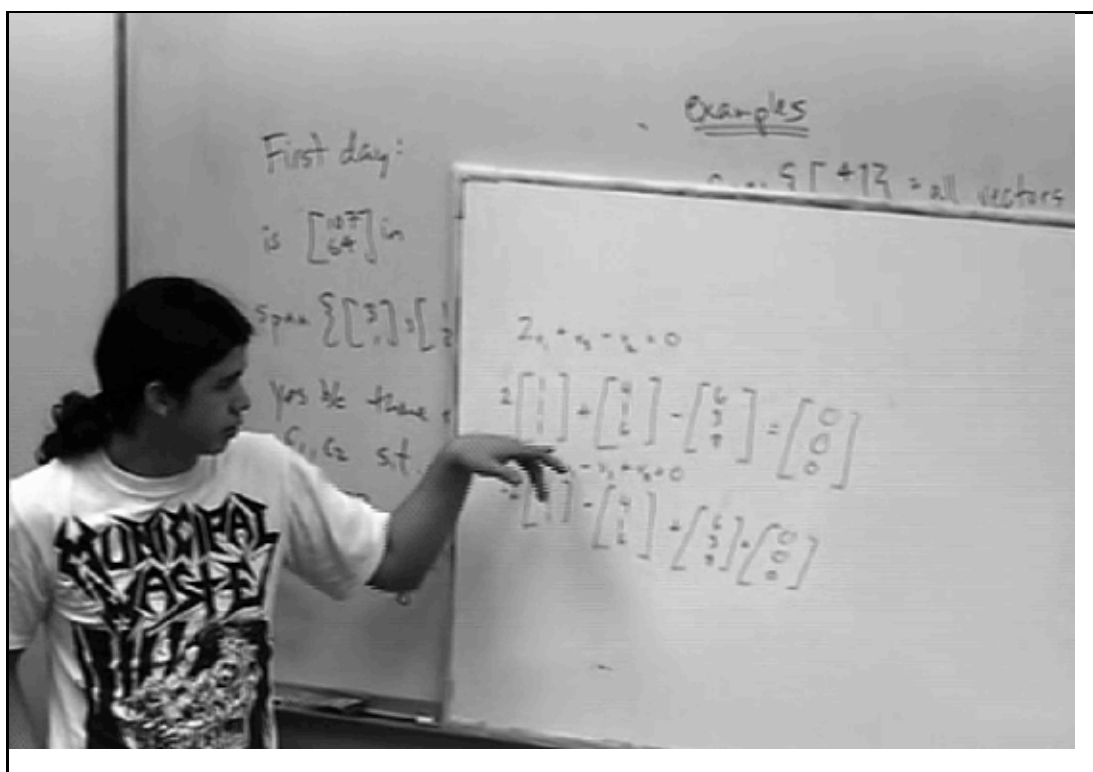
Figure 4.29: Eddie's Board Work for D3A51

Eddie's board work demonstrated the idea that a vector equation can be created by multiplying scalars times vectors, adding them together and then setting them equal to the a resultant vector, in this case the zero vector. The expression that he supplied demonstrated his conclusion and was used as data in the argument.

The third instance in which NWR was established as data or warrant for a different claim occurred in D4A21. The classroom community was asked if the set of vectors, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ were linearly dependent. Brad's response to the question evidenced the idea when he stated that the three vectors would be linearly dependent as his claim. He then set up a linear combination with $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as the resultant vector and argued that you could choose any scalar in front of $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ and take that scalar and multiply it by -4 and put that in front of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. He finally concluded that as long as the scalar in front of $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ was a zero, then he could get back home. The linear combination complete with the resultant zero-vector was thus used as data in his argument. In these three arguments, the idea was used as

justification in an argument for task 1; a claim where the three vectors were linearly

dependent, but none of them were scalar multiples; and a third claim where the set of

vectors were linearly dependent, but were also scalar multiples. Hence, the idea was

used as data in three separate arguments with three different claims, fulfilling criterion

3. This idea was used as data or justification in 5 other arguments coded from the six

class days, further evidencing that the way of reasoning was functioning-as-if-shared.

The fifth NWR used in CSP #4 connected the traveling language used in the

Magic Carpet Ride tasks and the graphical expression for the linear combinations.

NWR 4.5 A combination of a set of vectors determines a path using the set of
vectors to "get to" a resultant vector.

The reason I included this NWR in the analysis was because this was the imagery that

was used for the activities that covered span and linear dependence and independence.

The notion of "getting to" a particular vector established the resultant vector as a

destination. This language had been used earlier in evidencing NWR 2.2 when James

annotated $\begin{bmatrix} 107 \\ 64 \end{bmatrix}$ as the destination. He even put this vector into the vector equation,

however he does not state that this resultant vector is where he wanted to get. This

language was used so frequently throughout classroom discussion that it was in and of

itself a normative way of reasoning.

In D3A2, Brad presented an argument for why he believed that where you

could "get to" using the Magic Carpet and the hoverboard. In his warrant, Brad

described how the path of each vector restricted the movement of the hoverboard and

the magic carpet to between the two vectors. In discussing his conclusion, he used the

"get to" language in his claim, referencing the task situation. He then supplied data with a graphical approach that demonstrated that given his argument you could not get everywhere with the two vectors. . As a rebuttal to Brad's argument, Aziz made the claim that you can get "everywhere" in R^2 using the magic carpet and the hoverboard if you reconsider moving in negative directions on the hoverboard and the magic carpet (Figure 4.31).

D3A3 Argument
Claim as Rebuttal to Brad's Warrant: You can get anywhere with the magic carpet and the hoverboard. (Aziz)
Data/Warrant to Rebuttal: Moving backwards on the magic carpet or the hoverboard allows you to get outside of the bounds of the two transportation vectors. *If you go backwards with the magic carpet and then go up with the hoverboard.* (Aziz)
Warrant: There are no restrictions on the movement of a particular vehicle. (Aziz)

Figure 4.30: Day 3, Argument 3

Aziz's Data/Warrant regarding going "backwards" on the magic carpet and then "up" on the hoverboard coordinated the graphical expressions of the two vectors in a combination. In addition, he referenced moving towards some point in the plane not covered by Brad's drawing. Brad would gesture as Aziz was talking tracing a path outside of the picture that he drew. However, Aziz's statement did not include the gestural arguments that Brad made, nor did it require an explicit reference to a set of paths. Both of these ideas were used as warrants in Brad's argument. This indicated that the warrants, formerly present in Brad's argument, in Aziz's argument had dropped off, fulfilling criterion 1.

The final normative way of reasoning, NWR 4.5 connected the two sides of the equal sign of a vector equation with the possible scalars that could make the vector equation true.

NWR 4.5 Multiplying an entire vector equation by a constant does not change the existence of a solution.

This normative way of reasoning connected the algebraic and graphical expressions for vectors and vector equations by establishing that their exists a set of scalars that would allow for one set of vectors to be combined into another vector. Instead each of the scalars in the linear combination, including the scalar in front of the resultant vector, could be multiplied by a common value and the equality would remain the same. In D3A52, Eddie first evidenced this idea. The classroom community had been working on developing understanding for linear independence and dependence. Eddie had just given his argument (Figures 4.28) as to why the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ were a linearly dependent set. The instructor then asked him if his linear combination was the only linear combination that would demonstrate the relationship he had just given. Eddie and Nate argued that it was not in fact the only set of scalars that would demonstrate the linear dependence of the set of vectors.

Nate: Yeah, basically this would be the signs, multiply the entire thing by the scalar product, and you still end up the same solution.

Instructor: I think I know what you guys mean, but give me an example, what do
you mean, 'multiply the whole thing by a scalar solution?'

Eddie: In other words, if you were to multiply this entire left side, this entire
equation rather by 3, and make this side 6 times v1, 3v2 -3v3 you'd
still end up with zero, the same difference, same as if you switched the
signs. Any other questions?

The solution that he referenced was that no matter what value you multiplied the entire

vector equation by you would still get the zero vector. The data was that multiplying

the vector equation with the scalars by another number would still have the right side

of the equation equal the zero vector. Then Eddie supplied the warrant by stating, "if

you were to multiply this entire left side, this entire equation rather by 3, and make

this side 6 times v_1, 3 v_2 -3 v_3 you'd still end up with zero, the same difference, same

as if you switched the signs." The equivalent algebraic expression would look like:

$$3\left(2\begin{bmatrix}1\\1\\1\end{bmatrix}+1\begin{bmatrix}4\\1\\6\end{bmatrix}-1\begin{bmatrix}6\\3\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$$

The argument is summarized below:

D3A52 Argument
Claim: We can multiply the whole system by another scalar. (Eddie)
Data: *Yeah, basically this would be the signs, multiply the entire thing by the scalar product, and you still end up with the same solution. (Nate)*
Warrant: *In other words, if you were to multiply this entire left side, this entire equation rather by 3, and make this side 6 times v_1, $3v_2$ -$3v_3$ you'd still end up with 0, the same difference, same as if you switched the signs. (Nate)*

Figure 4.31: Day 3, Argument 52

Nate supplied the data that you could multiply each of the scalars by the same value.

His use of the term "entire thing" referenced multiplying both sides of the vector

equation by a common value. Eddie then supplied a warrant in which he demonstrated

that multiplying all of the scalars by 3 would not change the result. Implicit in his argument was that multiplying each of the components in the vectors by 3 times the scalars would still allow you to add each of the components to zero.

The way of reasoning that would become NWR 4.6 was evidenced as functioning-as-if-shared when the classroom community began having a discussion about the general solutions to the vector equation:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The classroom community had already established in D3A51 that there was at least one way to get the zero vector using a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$.

Norton's argument argued how you could actually have any number of solutions to the vector equation.

D4A13 Argument
Claim; The general solution to the vector equation is a relationship between c_1, c_2 and c_3 such that $c_1 = -c_3 = (1/2)c_2$. (Norton)
Data: Solving the system of equations gives you a relationship between the three scalars. (Norton)
Warrant: Any set of scalars that has this relationship will necessarily solve the system of equations. *As long as these relationships hold, you know your variable.* (Norton)

Figure 4.32: Day 4, Argument 13

Figure 4.33: Norton's Board Work for D4A13

Norton used a system of equations approach to demonstrate what the general solutions would be. However, this way of reasoning follows NWR 3.3, where the classroom community treated the solutions to the vector equations and the systems of equations as the same. Hence, when Norton drew the systems of equations on the board it could be treated by the members of the classroom community as a stand-in for the vector equation. Each of the c_i's in his board work (Figure 4.33) was the scalars for the vector equation. The equations that he pointed to on the board were the relationships that he referenced when he talked about "as long as the relationships hold." The implication that he was making here is that each of the c_i's can be any value so long c_1 is -2 c_2 and 2 c_3. In D3A52, the way of reasoning was the Data-Claim. Here in D4A13, the way of reasoning was used as warrant, fulfilling criterion 2.

This NWR established the idea that there was a multiplicity of solutions to any vector equation. It expanded the classroom community's understanding of the linear combination as a path because they could reason about multiple solutions to vector equations. In addition, it set the focus of their discussion on the scalars having to be a common ratio rather than a set value. When students would reason about linear dependence and independence, they would use this NWR to determine graphically that there were multiple ways to "get to" a particular vector because they stretch all of the vectors in a set by a common value, expanding the length of each of the vectors on the path but maintaining a path nonetheless.

Section 2

In the first section of Chapter 4, I detailed the set of Classroom Symbolizing Practices that established the meanings for the basic symbolic expressions for vectors and vector equations. These practices established in the classroom community what the algebraic, graphical and situational expressions for vectors and vector equations meant, including the fundamental meanings for vectors, scalars, linear combinations and the use of systems of equations for solving vectors and vector equations. In Section 2, I analyzed a set of more advanced meanings for these symbolic expressions. These practices are summarized in table 5.1

Summary of Section 2 Classroom Symbolizing Practices

> CSP #5. Scalar multiples or vectors along the same line are linearly dependent.
> CSP #6. The directions of the vectors in a set determine the span of the set of vectors.
> CSP #7. Linear Dependence is determined when there exists a path of vectors from and back to the origin.
> CSP #8. Finding a non-trivial solution to the vector equation proves that a set of vectors is linearly dependent..

Figure 4.34: Summary of Section 2 Classroom Symbolizing Practices

These practices dealt with the symbolic expressions as they were used by the classroom community to deal with the theoretical issues of span, linear independence and linear dependence. Many of the normative ways of reasoning (NWR) that the classroom community developed for the symbolic expressions in this section also established meanings for span, linear independence and dependence. As the community developed meanings for these theoretical ways of reasoning, their use, creation, and understanding of vectors and vector equations also developed. This evidenced a reflexive relationship between the development of theoretical knowledge about mathematical content, in this case linear algebra, and the development of the symbolic expressions that are used in the development of that mathematical content.

The majority of the NWR's that make up the four CSP's were instantiated and evidenced as functioning-as-if-shared during the second through fourth tasks that the classroom community undertook to understanding span, linear independence and dependence. The second task began on day 2 of the course and involved finding if there was any place in the plane, R^2, that Gauss if you could only use the hoverboard,

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and the magic carpet, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, to find him (See Appendix 1). This task dealt

with developing all of the possible linear combinations that could be created using the

two traveling vectors. In addition, the instructor introduced and defined the rigorous

term, span, and students were also asked about the span of a single vector and vectors

in R3.

The third task asked the students if there was a way that one could "get back

home" using a set of traveling vectors (See Appendix 1). The idea of "getting back

home" would become synonymous with determining if the set of vectors were linearly

dependent or independent as will be demonstrated in NWR 7.1. Students would also

be asked if you could "get back home" using only the two original Gauss's cabin

vectors, if scalar multiples were linearly dependent or independent, and if a set of

vectors in R^3 were linearly dependent or independent. There was a significant

increase in the use of three-dimensional vectors during the course of arguments and

discussions regarding this task.

The fourth task asked students to create a chart of sets of vectors that were

linearly dependent or independent (See Appendix 1). The classroom community was

asked to find sets of two or three vectors in R^2, and sets of two, three, or four vectors

in R^3 that were linearly independent, and the same sized sets of vectors that were

linearly dependent. If they were unable to create a set that fit the category then they

were asked to state to explain why such a set was impossible.. For example, a set of

four vectors in R^3 would never be linearly independent and hence would be

categorized as impossible. Groups within the classroom community were then asked

to present their arguments and discuss why they chose the sets they did or respond with the conclusion that the set was impossible.

In this section, I detail the four CSP's that expanded upon the classroom community's use, creation and understanding of vectors and vector equations. I define the CSP and the NWR's that were grouped according to that CSP. In addition, similar to section 1, I present vignettes for selected arguments that evidenced when an idea that would become a normative way of reasoning was first instantiated and those arguments that evidenced that the idea was functioning-as-if-shared. The CSP's in this section are presented based upon the number of vectors that were under consideration first, scalar multiple reasoning precluding reasoning about sets of vectors, and then will be presented based upon reasoning about span and then linear indpendence and dependence. The NWR's for each of the CSP's are presented in the order in which.

CSP #5: Scalar multiples or vectors along the same line are linearly dependent

The fifth CSP was grouped based upon the classroom community's development of symbolic reasoning for scalar multiples. Although the CSP deals with the linear dependence of scalar multiples, the NWR's that make up this practice deal with span and linear dependence and independence. As classroom members developed an understanding of scalar multiples, they used this understanding to reason about these ideas. Scalar multiples would become a major reasoning tool that student would use to create examples for linear independence and dependence and to create arguments regarding larger sets of vectors. The NWR's for CSP #5 are summarized in table 5.2.

Classroom Symbolizing Practice # 5 (CSP #5)

Scalar multiples or vectors along the same line are linearly dependent

Normative Ways of Reasoning (NWR)

> 5.1 You cannot reach Gauss's cabin using just one of the transportation vectors because the Gauss's cabin vector is not on the same line as either of the two transportation vectors.
> 5.2 In order to reach one vector with another vector, you need to be able to multiply the one vector by a single number to get the other vector
> 5.3 Each vector has a pre-determined direction that tells where you can get to with that vector.
> 5.4 No matter what scalar you multiply by, the vector will always stay along the same line.
> 5.5 All the vectors that are on the same line as the vector v_1 can be written as cv_1.
> 5.6 Two vectors that lie on the same line have the same span as just one of the vectors.
> 5.7 Two vectors are linearly dependent if there exists a c such that $v_1=cv_2$.

Figure 4.35: Classroom Symbolizing Practice #5

The NWR's for CSP #5 were used very frequently by the classroom community to work on a variety of tasks and problems. This was a product of the classroom community's frequent use of the scalar multiple as a reasoning tool. In addition, as evidenced in the NWR's, scalar multiples can be used, created and understood from a variety of different symbolic perspectives. The classroom community developed functioning-as-if-shared ideas that incorporated algebraic, graphical, theoretical and situational meanings for scalar multiples and moved between these meanings with fluidity. As will be evidenced in this section, multiple symbolic expressions were frequently used in the instantiation and evidencing of the functioning-as-if-shared ways of reasoning.

The development of the CSP progressed from primarily situational ways of reasoning to algebraic and graphical ways of reasoning to theoretical ways of reasoning. The CSP was defined as being about the linear dependence of scalar multiples because the classroom community used their reasoning about span and the Gauss's cabin scenario to develop their ways of reasoning about the linear dependence of scalar multiples. However, the CSP should be regarded as having to do with the classroom community's understanding with regard to the creation, use and understanding of vectors and vector equations as they participated in developing meaning for both span and linear independence and dependence. The classroom community would develop NWR's for what it would mean to travel on and symbolize a single line of vectors. In the development of these meanings, members of the classroom community would come to symbolize the scalar multiple in a variety of ways including writing one vector as a scalar multiple of another and determining if two vectors were scalar multiples. The creation of these symbolic expressions would then lead to reasoning about the linear dependence of scalar multiples once the task changed and students were asked to consider questions regarding that topic.

The first NWR for scalar multiples dealt with situational reasoning and the algebraic and graphical expressions for vectors and scalar multiples.

> NWR 5.1 You cannot reach Gauss's cabin using just one of the transportation vectors because the Gauss's cabin vector is not on the same line as either of the two transportation vectors.

> NWR 5.2 In order to reach one vector with another vector, you need to be able to multiply the one vector by a single number to get the other vector.

The idea for NWR 5.1 would be first instantiated during the second day of class as students were summarizing their arguments for the first task, where they were asked to find a linear combination that would get them to Gauss's cabin. Both D2A7 (Figures 4.6 and 4.7) and D2A16 (Figure 4.10) have been discussed in section 1 in the evidencing of the establishment of NWR 2.1. D2A7 arose from James's explanation of his solution for task 1.

D2A7 Argument
Claim: To get to Gauss's cabin, you needed to ride the hoverboard 30 hours and the magic carpet 17 hours. (James)
Data: Drawing of the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as the hoverboard and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as the magic carpet. The first vector would be 90 over 30 (drawing of a longer vector), the second vector would be 17 over 34" "Then we solved for the amount of hours, 17 and 30, (James)
Warrant: Solved for 17 and 30 using the systems of equations[See Figure 4.37 for James' board work]. "The fastest way will obviously be the straight one. Then, yeah, didn't work that way, so that" "Because we can't do a straight line, it would have to be through another route." (James)

Figure 4.36: Day 2, Argument 7 Revisited

James's warrant in D2A7, clearly stated that his group did not believe that Gauss's cabin could be reached using a single vector because as he said, "we can't do a straight line." Given his drawing on the board (Figure 4.14), I concluded that he was referring to a straight line path of the vector, $\begin{bmatrix} 107 \\ 64 \end{bmatrix}$. His argument was graphical in nature as demonstrated by his drawing of the scaled vectors for $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 107 \\ 64 \end{bmatrix}$.

He used the line of reasoning that the Gauss's cabin vector was not on the paths of the

other two vectors, hence another route would need to be taken. I coded his graphical

argument as backing for his contention that you could not use a single vector to get to

Gauss's cabin. The evidence that this way of reasoning was functioning-as-if-shared

came about later in the class period when Debbie was asked explicitly if you could

reach Gauss's cabin using a single mode of transportation.

Argument D2A16
Claim: You can't reach Gauss's cabin with just one of the modes of transportation. (Debbie)
Data: So we started with the hovercraft, and we knew it would have to go north at least 107 miles. (Debbie) With the hoverboard, "And inversely, we did the same thing with going 64 miles east. (Debbie) The vector for the magic carpet (Debbie)
Warrant: So by dividing 107 by 3, we discovered that it did not go into it evenly, so you would always either go over your destination or go right before it. (Debbie) And by saying that 64 times 1 is 64, which is the same as this, so then we also have to times 64 by 3, which is 192, which is way over our projected destination. (Debbie) So 1 goes into 32, oh, 64 divided by 2 is 32, so then we used that 32, to times that by 1, and you'd only go 32 miles north, so that didn't work. And then we also tried that 107 times 1, so you went 107 degrees north. But then if you times that by 2, you end up going 214 miles east. (Debbie)

Figure 4.37: Day 2, Argument 16 Revisited

Debbie responded that she did not feel that such a trip was possible and supplied

support in the form of data and warrants for why she felt that using a single mode of

transportation was not sufficient for finding Gauss's cabin. However, her data and

backing came in the form of an algebraic argument. She did not express her argument

in terms of the graphical approach of the fact that the vector was not on the same line

that James used. Hence, the straight line backing that James used as justification for

his argument drops off. This evidenced that the NWR was functioning-as-if-shared,

fulfilling criterion 1.

Debbie's argument for why you could not reach Gauss's cabin using a single mode of transportation first introduced NWR 5.1. Debbie's data and warrants dealt with dividing a component of the Gauss's cabin vector by the corresponding component in the mode of transportation and then seeing if that number would divide evenly into the Gauss's cabin vector. For example, the 107 in the Gauss's cabin vector corresponded to the 3 in the hoverboard. But that value did not go evenly into 107. She then divided 64 by 1, but when she multiplied 64 times 3 that gave her 192, which she stated was "way over" her desired destination. Implicit in her argument is the fact that she is looking for a single scalar to multiply the hoverboard by in order to reach Gauss's cabin. The fact that she cannot do as such means that such a scalar does not exist. From this I concluded that for her in order to reach one vector using a different vector, you needed to be able to multiply the first vector by a single number and have its result be where you want to go.

The idea of NWR 5.2 was evidenced as functioning-as-if-shared during the third day of the course. The classroom community was asked to describe the span of a pair of scalar multiples. D3A46 is a continuation of a series of arguments in which members of the classroom debated ways in which to categorize the span of various vectors. By the time, Robert made his conclusion, the members of the classroom community had come to the conclusion that "parallel or reversible vectors", scalar multiples would have the same span. In D3A46, the instructor revoiced their conclusions.

Instructor: So what about #2 then, Robert, would you mind sharing what you said about #2?

Robert: I basically, the result vectors, and they lie on the same line. You can just stretch the 1st vector, it becomes the 2nd vector, use the scalar.

Instructor: So they lie on the same line, so we could actually say they don't add anything, we could say it's the same as the span if I only have 1 of them.

The claim that Robert was making is echoed by the instructor, so although it was coded in the argument as the instructor making the claim it was actually a joint construction by both of them. His data for why this is the case was that the two vector lie on the same line. Although frequently the use of the word "so" indicated a warrant, in this case he led his argument with the information that they "lie on the same line" indicating that this is the primary information that is necessary to determine that they have the same span. He then gave additional information that stretching the first vector with a scalar would give you the second vector indicating that this was the connection that he saw between the vectors being on the same line and the fact that the two vectors have the same span. The argument is summarized below:

D3A46 Argument
Claim: "we could say it's the same as the span if I only have 1 of them" (Instructor)
Data: "I basically, the result vectors, and they lie on the same line." (Robert)
Warrant: "You can just stretch the 1st vector, it becomes the 2nd vector, use the scalar."

Figure 4.38: Day 3, Argument 46

Robert's warrant for why the vector's had the same span incorporated Debbie's reasoning about two vectors lying on the same line. He specifically stated that "stretching" the one vector had it "become" the second vector and the mechanism by which this occurred was via the scalar. In Debbie's argument, the idea was data and

claim, whereas in Robert's argument, the idea is used as a warrant, fulfilling criterion 2.

NWR 5.2 presented an advance on earlier reasoning about scalars and vectors from CSP #2 in that it incorporated an equivalency between two vectors. This equivalency has to do with the creation of one vector using another vector. The fundamental idea would be used in NWR 5.5, 5.6 and 5.7. Establishing equivalency between vectors was a key way that students began to reason about the qualities of sets of vectors, which distinguished the CSP's in section 2 from the CSP's in section 1.

NWR's 5.3 and 5.4 deal with similar issues to NWR 5.1 and 5.2 as they established the meaning for the direction of the vector and how multiplying by a scalar changed the vector. Unlike 5.1 and 5.2, the way in which the scalar affects or does not affect the direction of the vector is the focus of these two ideas.

> NWR 5.3 Each vector has a pre-determined direction that tells where you can get to with that vector.

> NWR 5.4 No matter what scalar you multiply by, the vector will always stay along the same line.

Both ideas were first instantiated during the course of classroom discussion for task 2. The arguments that established these ideas as functioning-as-if-shared are presented in Appendix 7. Both of these ideas arose out of Brad's argument in D3A2 that you could not get everywhere on the plane using the modes of transportation vectors. Brad and James's arguments that you could not use a negative scalar because they represented negative time, coupled with the negative scalar's interpretation as moving backwards were both debated for more than 20 minutes of class time. Out of this discussion,

many of the ideas that were established for scalars, operations on vectors with scalars, and scalar multiples were first seen in the data. These ideas included NWR 5.3 and 5.4.

NWR 5.3 and 5.4 established within the classroom community that the direction of a vector was set and could not be changed by multiplying by a scalar or any other way. This idea was an offshoot of the ideas (NWR 2.5 and 2.6) that restricted the ways that a scalar could affect the vector. The NWR's were included in this CSP because it was used by the classroom community to reason about sets of vectors and because it established an immutable quality for a particular vector or vectors. It expanded the meaning of direction in this classroom community as well in that direction could now be used as a relationship between vectors. 5.3 established direction of the vector as an immutable quality and 5.4 established that multiplication by a scalar did not change that immutable quality.

The normative ways of reasoning that made up CSP#2 informed to a great degree the NWR's that make up CSP #5. The direction of the vector became a key way of reasoning about scalars as presented in CSP #2. In CSP#5, the NWR's are specifically tied to reasoning about multiple vectors and the equivalence of scalar multiples. Two vectors are equivalent in this case if one vector can be multiplied by a scalar and then come up with the other vector. Transforming a vector using scalars, as discussed in NWR 2.3 through 2.6 are used in the arguments and ideas from NWR 5.1 through 5.4. For the following NWR's I will reference the NWR's from CSP #2 when necessary.

NWR 5.5 was used by students to symbolize equivalent vectors, first algebraically and then theoretically. The instructor used the algebraic notation to transition to the theoretical notation by annotating the scalar as a c.

NWR 5.5 All the vectors that are on the same line as the vector v_1 can be written as cv_1.

This way of reasoning integrated the graphical expressions that students created to symbolize Task 2 with an algebraic expression that could relate any possible vector on the line of vectors. It also established a way to determine equivalency of different vectors based upon the ability to write one of the vectors as a scalar multiple of another vector. Whereas in NWR 5.3 and 5.4, students developed graphical imagery to reason about the equivalence of vectors, NWR 5.5 set forward a symbolic equality, which allowed students to generate algebraic and theoretical proofs about linear independence and dependence. This idea was first instantiated as a part of D3A23. The classroom community has just finished the discussion on whether or not you can get anywhere using the modes of transportation and the instructor introduced the algebraic representation for scalar multiples. The instructor added to the discussion by bringing in the c and multiplying it by a vector.

D3A23 Argument
Claim: "so we could say they're all of the form $c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$." (Instructor)
Data: "So for instance, if we were right here, if we went this way -2, that would be this vector . If we went farther down, if we went down -100, -200." (Instructor)
Warrant: "So all the vectors along this line are of what form? They're all some scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$." (Instructor)

Figure 4.39: Day 3, Argument 23

The instructor's argument connected the representation $c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with the idea that all of the scalar multiples of the vector and that no matter which value you chose to multiply the vector by you would always end up with a vector that was on the line of the vectors.

The way of reasoning that would become NWR 5.5 was evidenced as functioning-as-if-shared using criterion 3. Students made frequent reference to being able to write a vector as a scalar multiple as reason for the equivalency of the two vectors. In D4A19, the classroom community was aked if the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ were linearly dependent.

D4A19 Argument
Claim: The set $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ and then say maybe $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ is linearly dependent. (Robert)
Data/Warrant: It's just an idea, the 3rd vector is just a multiple of the 1st. So is there any point in doing it? (Robert)
Data/Warrant: I was just saying the 1st vector if multiplied by 4 is the 3rd vector. So just the 1st one stretched is the same thing, so I don't see any point in even having a 3rd vector if it's going to be a multiple of the 1st. (Robert)

Figure 4.40: Day 4, Argument 19

When Robert stated is there "any point in doing it?" he was referencing setting up a linear combination with a resultant vector equal to the zero vector. He stated that the

first vector, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and the third vector, $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$, are scalar multiples of each other and

that the third vector was just the first vector stretched and that they lay on the same

line. Each of these ways of reasoning established an equivalency between the two

vectors and utilized the reasoning from NWR 5.5. The way of reasoning in this case

was used as data. The third argument in which the equivalency of two scalar multiples

was used as data came in D5A12. The classroom community was asked to create a set

of generalizations for why a set of vectors would be linearly dependent or independent.

Karl argued that two vectors that are on the same line were linearly dependent.

D5A12 Argument
Claim: To be linearly dependent, two vectors have to be on the same line. (Karl)
Data: Multiplying a vector by a scalar stretches the vector into the other vector. *You just take one vector and you add a scalar to it, and it just stretches it.* (Karl)
Warrant: *They lie on the same linearly dependent line.* (Karl)

Figure 4.41: Day 5, Argument 12

Karl's argument for the generalization used NWR 5.5 as a justification for the linear

dependence of two scalar multiples. He stated that you would add a scalar to one of

the vectors and that would stretch the scalar. In addition, the two vectors would be on

the same line. The establishment of an equivalency between two vectors based upon

being scalar multiples was used as data. The three separate arguments each had

different claims, but all used the idea as data. Hence the way of reasoning was

functioning-as-if-shared by criterion 3.

NWR 5.6 expanded how the classroom community reasoned about lines of vectors, and it expanded their understanding of the symbolic expressions for vectors and vector equations by factoring in span.

> NWR 5.6 Two vectors that lie on the same line have the same span as just one of the vectors and are linearly dependent.

The way of reasoning was first introduced in D3A23 (Figure 4.38), when the instructor asked what the span of two scalar multiples was. Students had presented their arguments for why they felt you could get everywhere using the Gauss's Cabin vectors, and the instructor had introduced the term span. The instructor made the claim that two scalar multiples have the same span. Robert completed the argument by stating that the data for this claim was that the vectors lay on the same line and providing the warrant that the first vector would be the second vector stretched.

The way of reasoning was evidenced as functioning-as-if-shared in D6A11. The argument that Jason made in D6A11 relied heavily upon ways of reasoning that were discussed in D5A43. At the end of Day 4, students were given the fourth task, constructing a table of sets of vectors that were linearly dependent or independent. The instructor also asked the class to develop a set of generalizations for determining when any randomly selected set might be linearly dependent or independent. Jason's argument detailed why he believed that a set of three vectors in R^2 would always be linearly dependent.

> Jason: So we're still in R^2. So basically, let's just start with any random vector, let's call it that one. Now after we have one vector down, there's only basically two situations we could have. We can either have a vector that is parallel with this one, either another multiple or going the other way or whatever. Or we can have one that is not parallel, it doesn't have to be perpendicular, it can be anywhere. But

it's either parallel or not. So if it's parallel, we already said that if we have two vectors that are parallel, we have a, they're dependent. But when we did our magic carpet-hoverboard, we had two that weren't parallel, and we said the span of any two that aren't parallel, is all of R^2. So if we have two that aren't parallel, and we can get anywhere in R^2, no matter where we throw in our 3rd vector, we can get there with a combo of these two and make it back on that third one. So there can't be any solution, so there's no, as long as we have three vectors in R^2, it has to be linearly dependent. Does that make sense, any questions?

D5A43 Argument
Claim: A set of three vectors in R^2 is always linearly dependent. (Jason)
Data: "let's just start with any random vector, let's call it that one." "Now after we have 1 vector down, there's only basically 2 situations we could have. We can either have a vector that is parallel with this one, either another multiple or going the other way or whatever. Or we can have one that is not parallel, it doesn't have to be perpendicular, it can be anywhere. But it's either parallel or not." If the two vectors are not parallel, "no matter where we throw in our 3rd vector, we can get there with a combo of these 2 and make it back on that 3rd one." (Jason)
Warrant: "So if it's parallel, we already said that if we have 2 vectors that are parallel, we have a, they're dependent. (Jason)
Backing for the entirety of the argument: "So there can't be any solution." (Jason)

Figure 4.42: Day 5, Argument 43

Jason's argument contained several sets of data and warrants to support his claim. He began his argument with a single vector. He then put forward a situation in which there were two different possibilities, one was that you have a vector that is parallel (a term used frequently to describe a scalar multiple) or the second vector is not parallel. If the second vector is parallel, the set is linearly dependent (this argument will be presented fully in D6A11). Jason continued by supposing that the two vectors are not parallel and then stated that they span the entirety of R^2. Then when a third vector is added to the set, you can "get back home" using the third vector because the first two vectors allow you to get anywhere in R^2. If you can get back home using each of the vectors only once, then the set is linearly dependent.

The idea that would become NWR 5.6 was not evidenced in this argument. But this argument and its associated ways of reasoning played a key role in Jason's argument in D6A11. At the beginning of Day 6, the classroom community was asked about the generalizations that they developed for any set of vectors. Jason expanded his argument from D5A43 to explain why if you have had more vectors than you had components in the vectors, then the set would have to be linearly dependent.

> Jason: So if you start in any R^n, and you just start with one vector and keep adding more. So let's do R^3, just for an example. So we start with one vector. So either, we have two choices: The next vector we add can either be on the same line, which means it's already linearly dependent, so we don't want that, so we're going to put it off somewhere else. Now the span of that is a plane in three dimensions. So now we're going to add another vector in. Our third vector, now it can either be in that span or out of that span. And we want it to be linearly independent, so we're going to put it out of that span. But now that we have that going off of that plane, we just extended our span to all of R^3. So our fourth vector, when we put it in, no matter where we put it, it's going to get us back home. Because just like in this case, we have to have the last one to get back home, we can get anywhere with those first three that we put in, but we have to have to have that fourth one to come back. And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back. Unless you have two vectors that are lying on the same line, then you won't have the span of all of your dimension, but it's negligible because those two will give you a linearly dependent set. Does that make sense?

In this argument, like the argument that he made in D5A43, Jason makes a conditional argument that is very similar to the earlier argument. In this argument, though, Jason is discussing any set of vectors where the number of vectors exceeds the number of dimensions. And he also incorporated the language of span into his argument indicating that he has incorporated the idea that scalar multiples have the same span.

Jason's claim was that if you have more vectors than you have dimensions for any number of dimensions then the set of vectors is linearly dependent. He then proceeds to include data on a case by case basis. He then provided an example from R^3 to discuss his argument.

His claim was that a when the number of vectors in a set of vectors exceeded the number of dimensions that those vectors was in then the set of vectors was linearly dependent. He began with the use of an example from R^3 to argue his case. Jason's argument in D6A11 proceeds via a sub-argument structure. I first present his example for R3 as a sub-argument and then I show how the sub-argument is connected to the primary argument via a warrant for why the example applied to the primary argument. In his sub-argument, Jason claimed that if you had four vectors in R^3, then the set of vectors was linearly dependent. For this argument, he presented a case-based argument similar to his argument in D5A43. He began by stating his first case, with the data being that you are given two vectors in the set with "the next vector we add can either be on the same line, which means it's already linearly dependent." However, this only gave partial evidence to his claim, so he added the warrant, "so we don't want that, so we're going to put it off somewhere else." By this he meant that the second vector will be in a different direction than the first. Because the two vectors were on a plane, this gave a second piece of data that the two vectors give a plane in R^3. To which he added another piece of data, a third vector that can be in the span or out of the span. He then added the warrant that if the third vector was in the span of the other two vectors, the set will be linearly dependent. Since again this did not fully complete his argument, he added the warrant that you choose a vector that is not in the

span of the other two vectors. This brought a third data, "But now that we have that going off of that plane, we just extended our span to all of R^3." Then there was additional datum, of the addition of a fourth vector. This was then accompanied by the warrant, "so our fourth vector, when we put it in, no matter where we put it, it's going to get us back home." To this he added the backing, "Because just like in this case, we have to have the last one to get back home, we can get anywhere with those first three that we put in, but we have to have that fourth one to come back." This completed the argument that he made for four vectors in R^3. Jason made a series of data and warrants to support his claim by establishing that by including additional vectors after the first vector, you are left with either a linearly dependent set already or an increase in the span of the vectors. This occurred until you are left with a set of vectors that spans three dimensions. At this point, any additional vectors that were added would necessarily be in the span of the first three and hence the set would be linearly dependent.

In order to extend his example from R^3 to any number of dimensions, Jason generalized his argument by adding a warrant. This warrant is coded as a warrant to in the original argument, whereas the example for R^3 is coded as the data for the original argument. Jason stated "the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back." This argument was interpreted by reasoning that continuing the process of determining if the addition of a vector made the set linearly dependent and then assuming that the additional vector did not make the set linearly dependent would allow for the individual to eventually get the span of the entire dimension that the

vectors were a part of. Once the set of vectors had the maximal span, then any additional vector would necessarily allow him to "get back home." Jason finally finished his argument emphasizing that two vectors that are on the same line would not allow you to get the entirety of the span of the dimensions. He concluded that with the data that two vectors on the same line would make the set linearly dependent. Jason's argument is summarized in the table below:

D6A11 Argument
Claim: If you have more vectors than you have dimensions for any number of dimensions than the set will be linearly dependent. (Jason)

R^3 example as data for the primary argument:
 	
    Claim: Four vectors in R^3 are linearly dependent.
 	
    Data: The addition of a second vector. "the next vector we add can either be on the same line, which means it's already linearly dependent,"
    Warrant: "so we don't want that, so we're going to put it off somewhere else."
 	
    Data: "Now the span of that is a plane in three dimensions. So now we're going to add another vector in. Our third vector, now it can either be in that span or out of that span."
    Warrant: "And we want it to be linearly independent, so we're going to put it out of that span."
 	
    Data: "But now that we have that going off of that plane, we just extended our span to all of R^3."
    Warrant: So our fourth vector, when we put it in, no matter where we put it, it's going to get us back home.
 	
    Backing for the R^3 example: Because just like in this case, we have to have the last one to get back home, we can get anywhere with those first three that we put in, but we have to have to have that fourth one to come back. (Jason)

Figure 4.43: Day 6, Argument 11

Warrant for the primary claim: And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back." (Jason)

Qualifier for the primary claim:
    Claim for qualifier: A set of two vectors that are already linearly dependent will not allow you to span."
    Data for the qualifier: The two vectors lie on the same line. (Jason)

Figure 4.44: Day 6, Argument 11 Continued

I present the entirety of the argument that he made here because it presented a culmination of many of the ways of reasoning from the entirety of the first six days of class. In relation to NWR 5.6, the idea that the two vectors that were on the same line were linearly dependent came as part of the qualifier. In D3A46, the idea that any two vectors that were on the same line had the same span was a part of the data-claim. In addition, there was a warrant for this argument.   In D6A11, the idea shifts to the qualifier and the warrant for the qualifier dropped off, evidencing that the way of reasoning functioned-as-if-shared by criteria 1 and 2.

In NWR 5.5, the classroom community reasoned that scalar multiples could be written as a scalar c times the vector. In NWR 5.6 the classroom community established the idea that scalar multiples (vectors that were on the same line) were linearly dependent. NWR 5.7 integrated these two ideas to create a theoretical expression that related scalar multiples to linear dependence.

NWR 5.7 Two vectors are linearly dependent if there exists a c such that $v_1 = cv_2$. The way of reasoning was first introduced in D5A18 by Karl and was later evidenced as functioning-as-if-shared in D5A27, when the warrants for the idea as presented in D5A18 dropped off, these arguments and the evidencing for the idea functioning-as-if-

shared are presented in Appendix 7. NWR 5.7 incorporated the definition for linear

dependence into the classroom community's reasoning as to why scalar multiples

would be linearly dependent. The inclusion of the definition for linear dependence

allowed for the theoretical justification. This completed the development of a whole

chain of reasoning for why scalar multiples were linearly dependent.

CSP #5 was an expansion of CSP #1-4, particularly CSP#1 and 2. As the

classroom community developed notions for span and linear independence and

dependence through the lens of scalar multiples, they expanded what it meant for

scalar multiples to be equivalent and the impact of that equivalency on the span and

linear dependence on a set of vectors. In addition, the NWR's for CSP #5 contributed

new symbolic expressions, particularly meaning for the theoretical expressions for

vectors and vector equations, and expanded on the meanings that the classroom

community developed for the algebraic expressions. NWR 5.1 and 5.2 supplied

connections between the graphical expressions and the situational reasoning necessary

to begin to discuss equivalency of vectors and sets of vectors. Then NWR 5.3 and 5.4

expanded on the graphical reasoning and supplied more algebraic underpinnings for

why scalar multiples could be equivalent. NWR 5.5 encapsulated the reasoning from

5.1-5.4 in an algebraic expression that would allow for the classroom community to

deal with the theoretical expressions and express equivalency of two scalar multiples

algebraically. NWR 5.6 established the language of span and solidified the idea that

scalar multiples were in fact linearly dependent. And finally, 5.7 established the

theoretical expressions and justification for why scalar multiples were linearly

dependent.

CSP #6: The directions of the vectors in a set determine the span of the vectors.

The sixth classroom symbolizing practice dealt exclusively with span, focusing on the span of a set of vectors that were not scalar multiples. The NWR's that were enumerated in CSP's #2, 3, and 5 would play significant roles in the development of this functioning-as-if-shared idea as students incorporated previous NWR's in their arguments and the ways of reasoning that were developed during those arguments. The NWR for CSP #6 is summarized below.

Classroom Symbolizing Practice #6 (CSP #6)

The directions of the vectors in a set determine the span of the vectors.

Normative Ways of Reasoning (NWR)

> 6.1 Two vectors need to be in different directions in order to be able to get everywhere in R^2.

Figure 4.45: Classroom Symbolizing Practice #6

There is only one NWR in CSP #6 as many of the ideas that are incorporated in the NWR were already addressed in earlier sections. However, I chose to create a separate CSP for this idea because it differed significantly from CSP #5 as the classroom used it to reason about vectors that were not scalar multiples, and as its focus was exclusively on span. This idea would be central to arguments about linear dependence, independence and span.

The focus of the CSP is on direction. The direction of the vector was first defined in CSP #1. However, the concept of direction expanded beyond where the

vector was pointing to mean much more. Members of the classroom community would speak of the direction of the vector as referencing anywhere upon the line in R^2 on which a particular vector was located or to all of the possible scalar multiples of a particular vector. In addition, as was evidenced in CSP #2, direction would also come to include a 180 degree turn of the original vector. And as was evidenced in CSP#5, direction became a key idea for reasoning about sets of vectors as the classroom community concluded that vectors that were in the same direction did not add to the span and were linearly dependent. NWR 6.1 and consequently CSP #6 would add to the classroom community's idea of direction as it established the role that two vectors with differing directions would have on the span of the set of vectors. Hence, the role that direction played in the classroom community's understanding of sets of vectors expanded as well.

]In this functioning-as-if-shared idea members of the classroom community began to reason about vectors beyond the individual vectors and even the result of multiplying the vector by a scalar. These new ways of reasoning incorporated reasoning about all of the possible scalars that a vector could be multiplied by and the possible linear combinations that could be created using different scalars for multiple vectors.

> NWR 6.1 Two vectors need to be in different directions in order to be able to get everywhere in R^2.

This way of reasoning was first introduced by Jason in whole class discussion by Jason in D3A18. D3A18 has already been discussed extensively in section 1(NWR 2.2 and 2.3; NWR 4.2). In Jason's argument for why you can get everywhere using

the transportation modes, he mentioned that you can "slide anywhere" along one of

the vectors and then stop and then begin moving along the second vector towards

whichever place in the plane that you want to go to. In his second data, he states that

this is because the two vectors are not "parallel." Jason's use of the term "parallel"

was his way of saying that two vectors were scalar multiples as this had been

established in earlier classroom discussions. His contention that you can "get

everywhere" within the plane using the two transportation modes was part of the data-

claim.

The reasoning that Jason used to justify that vectors in different directions

allowed you to get everywhere in R^2 was interesting in the way it utilized linear

combinations and scalar multiples to consider movement along the entire plane.

Jason's imagery was an important advance for the classroom community as it

considered all of the possible linear combinations that could be created with the

transportation vectors. In some ways, his argument echoed Brad's argument in D3A2

(Figure 4.10 and 4.11) in that Brad was also considering the addition of possible scalar

multiples of the two vectors and as his gestures evidenced, moving the modes of

transportation infinitely in the directions that the vector was pointing and adding them

in positive directions. Jason's argument incorporated the possibility of multiplying by

negative scalars, and the effects of multiplying by those scalars on the possible

directions. His gestures and his language demonstrated that he was stretching each of

the vectors infinitely in positive and negative directions along the line of vectors. This

NWR also evidenced NWR 4.2; the graphing of linear combinations is done from "tip-

to-tail". The advance that he was presenting in this argument is to consider all of the

possible linear combinations graphically and to include the mechanism(multiplying by positive and negative scalars) adding vectors from tip-to-tail, and extending the vectors to desired lengths to "get to" any point in the plane that you want to go to.

The way of reasoning was evidenced as functioning-as-if-shared in D3A42. The classroom community had spent a significant amount of time discussing the span of the transportation modes and come to the conclusion that there span was everything. The instructor had introduced a definition for span, indicating that it was everywhere that you could get with a particular set of vectors. Nate was asked if his classmates had come up with a rule for determining the span of a set of vectors in R^2.

D3A42 Argument
Claim: "Then the span of those 2 vectors is all real numbers." (Nate)
Data: If there are 2 vectors, and one of them is not 0,0. And the other rule is that, they intersect at 1 point that is not 0,0, or the same thing saying they're not reversible. So it's not like $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so they do not copy each other."(Nate)

Figure 4.46: Day 3, Argument 42

Nate's argument states that the span of two non-scalar multiples was all of R^2. His argument was similar to Jason's in that he states that as long as the vectors are not "reversible", then the span is all real numbers. Nate's language was idiosyncratic in that he was the only student who used the term "reversible." However, he consistently used the term in situations in which it was appropriate to discuss scalar multiples and it is clear that he was referring to scalar multiples in this argument as well. While Jason had warrants to justify his argument, Nate's argument lacks the same justification, indicating that NWR 6.1 is functioning-as-if-shared by criterion 1.

CSP #7: Linear Dependence is determined when there exists a path of vectors from and back to the origin.

CSP #7 and #8 both deal explicitly with linear dependence and independence. From the middle of day three until the end of day six, the tasks that students were given dealt primarily with the topics of linear independence and dependence. The instructor would ask questions regarding span, and students were frequently making arguments that incorporated span, but the classroom community's activity always revolved around these key concepts. My choice to split the normative ways of reasoning between CSP #7 and #8 relied upon the different symbolic expressions that were primary for the two CSP's. CSP #7 relied primarily upon the graphical, situational and algebraic expressions that were developed with regard to linear dependence and independence. The normative ways of reasoning for CSP #7 are summarized in Figure 4.44:

Classroom Symbolizing Practice #7 (CSP #7)

Linear Dependence is determined when there exists a path of vectors from and back to the origin.

Normative Ways of Reasoning (NWR)

> 7.1 'Can we get back home with a group of vectors' is asking if there is a way that we can ride out on two of the vectors and then ride the third vector back home.
> 7.2 The addition of a vector to a set makes the set dependent if there exists a set of scalars that allows the new vector to be written as a linear combination of the other vectors.
> 7.3 If a vector is contained in the span of 2 other vectors, then you can get back to the origin with the first vector.
> 7.4 If you have more vectors than you have dimensions, then the set is linearly dependent.

Figure 4.47: Classroom Symbolizing Practice #7

As I stated previously, the NWR's for CSP #7 are primarily situational, graphical and algebraic in their nature. For many of the NWR's students reasoned using the language of "getting back home," which became synonymous with a set being linearly dependent. In task three (Appendix 1), students were asked if there was a way to "get back home" using the set of vectors, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$. The set of vectors is linearly dependent and hence there is a way to "get back home" using them. The students conceptualized getting back home by reasoning about riding out on two of the vectors and then coming back to the origin on the third vector. CSP #7 traced the development and expansion of this way of reasoning through days three through six.

Unlike in other sections, I chose to begin the section somewhat out of order. The idea for NWR 7.1 was actually introduced after the idea for NWR 7.2 as a part of whole class discussion. However, because the idea of "getting back home" was such a key idea in the progression of this CSP, I began with it instead of NWR 7.2. In addition, the language of "getting back home" was introduced as a part of the task.

> NWR 7.1 "Can we get back home with a group of vectors" is asking if there is a way that we can ride out on two of the vectors and then ride the third vector back home.

NWR 7.1 was explicitly stated in D4A9 as the instructor began class with a re-introduction of task 3.

D4A9
Claim: Getting back home (Instructor)
Data: "that we could go out on 1 of them, add on a direction with the 2nd and then the 3rd can take us straight back home. " (Instructor)

Figure 4.48: Day 4, Argument 9

The instructor presented the idea of getting back home as riding out on one of the vectors, moving in the direction of the second vector and then taking the third vector back to the origin. She introduced the idea of getting back home after discussions that happened on the third day of class when Eddie was asked if you could get back home

with the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$. His response was that you could get back home, and he justified his argument with the vector equation:

$$2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

His reasoning didn't include the language about "getting back home" as he instead

chose to express his reasoning as "the two times the first vector plus the second vector

gives us the third vector." In D4A9, the instructor revised this reasoning to provide

the situational expression that connected Eddie's solution to the task. The idea was

then evidenced as functioning-as-if-shared in D4A21, when Brad gave his argument

for why the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ was a linearly dependent set of vectors.

At this point in the task, the definition for linear dependence had been expressed as

being able to get back home.

D4A21 Argument
Claim: The set $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ is a linearly dependent set of vectors. (Brad)
Data: Can we just use 0 for w_2 and then say 4 for w_1, -1 for w_3? (Brad)
Warrant: Because as long you don't use w_2, you're going to make it back. But as soon as you use w_2, you're done, you can't get back. Unless you use w_2 twice, positive and negative. (Brad)

Figure 4.49: Day 4, Argument 21

Brad expressed in his warrant that as long you don't use w_2, you're going to make it

back. But as soon as you use w_2, you're done, you can't get back." Brad's argument

used the language of "getting back" as he discussed using different vectors. Implicit

in the argument is language of riding out vectors in specific directions and then riding

other vectors to get back home. He then restricted his argument to not being able to use all of the vectors because then you would not be able to get back. Brad tied "getting back home" to riding a member of the set in one direction and then using the others to get back home as a part of his warrant. The instructor's use of that idea was made as a claim; hence the way of reasoning is evidenced as functioning-as-if-shared by criterion 2.

In Eddie's argument from Day 3, he argued that the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ allowed you to get back home because 2 of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and 1 of $\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$ equaled $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$. This idea was an important way of reasoning about linear dependence in the course and was coded as NWR 7.2.

> NWR 7.2 The addition of a vector to a set makes the set dependent if there exists a set of scalars that allows the new vector to be written as a linear combination of the other vectors.

The expression "getting back home" meant finding if a member of the set of vectors could be written as a linear combination of the other vectors seemed to be functioning-as-if-shared from late in Day Three and early in Day Four as evidenced by Eddie's argument. Consequently, idea for NWR 7.2 could not be officially evidenced as functioning-as-if-shared until Day Five. In D5A20, Robert is asked to give a generalization about when a set of vectors is linearly dependent.

Robert: Oh, yeah, for independent. For independent, I put, 'No 2 vectors can add to be another.' So say you have $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then the last vector can't be $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$, because you would have added those 2 vectors to get another one, and that would have allowed us to get back to 0, if one of them [inaud]. Or one of those was negative.

Robert was asked for a generalization for when a set of vectors is linearly independent. His claim was that for a set of vectors to be linearly independent, no 2 vectors can add to be another vector in the set. The data that he provided is an example, "So say you have $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then the last vector can't be $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$, because you would have added those 2 vectors to get another one." Then he provided a warrant that a set of scalars can then be used that would allow to get back to zero using a negative scalar in front of the third vector.

D5A20 Argument
Claim: For a set of vectors to be linearly independent, "No 2 vectors can add to be another." (Robert)
Data: "So say you have $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then the last vector can't be $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$, because you would have added those 2 vectors to get another one, (Robert)
Warrant: "and that would have allowed us to get back to 0, if one of them [inaud]. Or one of those was negative." You add two vectors to get the third vector and then use a negative scalar in front of the third to get back to zero. (Robert)

Figure 4.50: Day 5 Argument 20

In D5A20, Robert argued that if you had a set of vectors that had one vector be a linear combination of the other vectors, then you could "get back to 0" using the third vector. The line of reasoning that he needed to get back to zero was the warrant for this argument and provided the connection between his contention and the data that he provided about the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$. The warrant provided a link between the chain of reasoning and the definition for linear dependence that had been established in class.

The idea was then evidenced as functioning-as-if-shared in D6A3. In the argument, Lance was asked if you had three vectors in R^2 would the set be linearly dependent.

> Lance: If you have 3 nonparallel vectors, if you graphically think about it. You say you have a vector over here, and then plus a vector that's not parallel to it [draws board]. And we have just any other random vector, say it goes this way. If you add any 2 nonparallel vectors that at some point these vectors are going to come across this other vector, and hit it at a point. So if that happens, then you can just take the last vector and multiply it by some scalar to get you back to the origin. That's how graphically we got it.
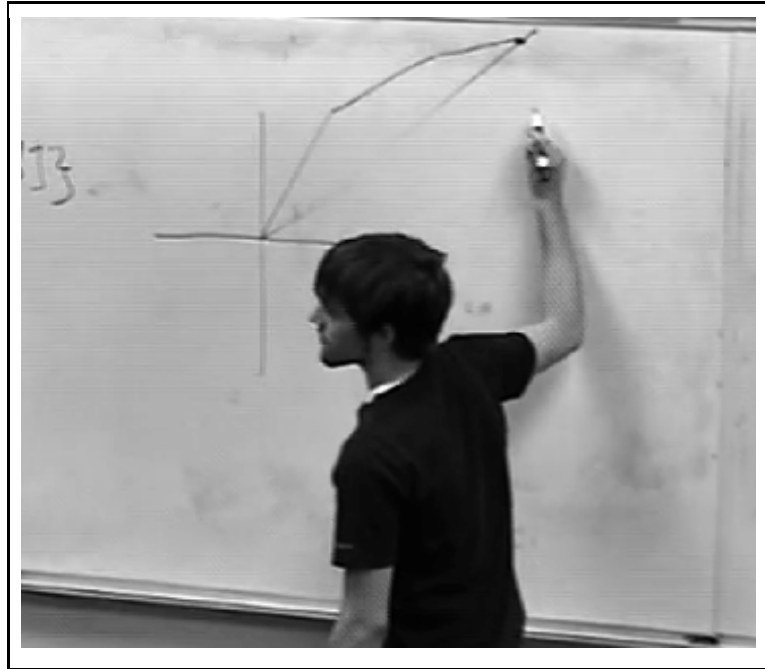
Figure 4.51: Lance's Board Work for D6A3

Lance's argument was similar to Jason's in D5A43 and D6A11, however, he did not

make the case-based argument in this argument. He began with the claim that if you

have three non-parallel vectors then three vectors in R^2 are linearly dependent. His

data was the drawing that he created (Figure 4.51). He continued with the warrant that

echoed the claim and data pair that Robert made. He added the two parallel vectors

on the board using the tip-to-tail method (which was established as functioning-as-if-

shared in NWR 4.2) and his warrant also stated in a similar manner to Robert when he

discussed multiplying the last vector by a negative scalar to get back to the origin.

D6A3 Argument
Claim: "If you have three nonparallel vectors" then "Three vectors in R2 are linearly dependent." (Lance)
Data: "You say you have a vector over here, and then plus a vector that's not parallel to it [draws board]. And we have just any other random vector, say it goes this way. " (Lance)
Warrant: "If you add any 2 nonparallel vectors that at some point these vectors are going to come across this other vector, and hit it at a point. " So if that happens, then you can just take the last vector and multiply it by some scalar to get you back to the origin. " (Lance)

Figure 4.52: Day 6 Argument 3

Lance's claim was that three vectors in R^2 were linearly dependent. He used the idea that a set of vectors that added to one another would be a linearly dependent set as the data and warrant for his argument. Hence, the idea shifts from data-claim to warrant, evidencing that that the idea was functioning-as-if-shared by criterion 2.

The next ideas for the next two NWR's were introduced in D6A11, which was discussed in CSP #5. The arguments that evidenced these NWR's being established and functioning-as-if-shared are contained in Appendix 7. NWR 7.3 was an idea that was used in arguments as justification for the arguments that introduced the ideas for NWR 7.4.

NWR 7.3 If a vector is contained in the span of 2 other vectors, then you can get back to the origin with the first vector and the set of vectors is linearly dependent.

NWR 7.4 If you have more vectors than you have dimensions, then the set is linearly dependent.

NWR 7.3 connected span and linear dependence by establishing a connection between span and linear dependence and independence. The functioning-as-if-shared idea used the situational expression of getting back home as a way of thinking about linear

dependence and independence. However, as was evidenced in D6A11, Jason and other students in the classroom community used the language of "getting back home" and other elements of the traveling metaphor in conjunction with more theoretical language such as linear dependence. The NWR evidenced the fact that the two ways of expressing linear dependence and independence were used interchangeably. As was evidenced in D6A11, the idea that a third vector that is in the span of two other vectors was featured prominently when reasoning about why having more vectors than dimensions would imply linear dependence.

The imagery used for both of these functioning-as-if-shared ideas used the traveling metaphor prominently. This is consistent with each of the NWR's in CSP #7. "Getting back home" and the associated ideas of moving out on a particular set of vectors and moving back on another vector were crucial to reasoning about linear dependence and independence. The four normative ways of reasoning established algebraic, graphical and situational expressions for reasoning about getting back home.

CSP #8: Proving that a set of vectors is linearly dependent means finding a non-trivial solution to the vector equation .

Whereas the NWR's that were established as a part of CSP # 7 relied heavily on imagery derived from the traveling metaphor, the ideas in CSP #8 did not necessarily rely on the "getting back home" language. Instead, the reasoning that was foundational for these NWR's had to do with the definition of linear dependence or independence. The first NWR established a connection between the "getting back home" language and the definition for linear dependence and independence. The

remaining functioning-as-if-shared ideas expand on the meaning of the definition and

its implications for sets of vectors. CSP#8 is summarized below.

Classroom Symbolizing Practice #8

Proving that a set of vectors is linearly dependent means finding a non-trivial solution
to a vector equation .

Normative Ways of Reasoning

       8.1 In order to ascertain if you can go out and get back home, all of the
           scalars in a vector equation that has been set to the zero vector cannot
           equal zero.
       8.2 A zero in front of the vector implies that the vector is not being used.
       8.3 Finding a non-zero solution to the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

           makes the set linearly dependent.
       8.4 For a set to be linearly dependent, the only solution to the vector
           equation:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + ... c_n\vec{v}_n = \vec{0}$$

           is the trivial solution.
       8.5 If you include the zero vector in any set of vectors all of the other
           vectors can have scalars equal to zero and the scalar in front of the zero
           can be anything and the sum will be zero.

Figure 4.53: Summary of Classroom Symbolizing Practice #8

As would be consistent with the wording and nature of the definition for linear

dependence and independence, the role of zeroes and the zero-vector was prominently

featured in this CSP. In addition, what it meant for a solution to be a non-zero

solution or to be the trivial solution was also prominently featured. In addition, one of

the generalizations for a set of vectors to be linearly dependent was arrived at using

these NWR's and so is included in CSP #8 as well.

NWR 8.1 established the link between the traveling metaphor, the imagery of "getting back home" and the definition of linear dependence. The definition of linear dependence that was used in class was:

> A set of vectors, $\vec{v}_1, \vec{v}_2, \vec{v}_3, ... \vec{v}_n$, is linearly dependent if there exists a set of non-zero scalars, c_1, c_2, c_3,...c_n such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + ...c_n\vec{v}_n = \vec{0}$.

NWR 8.1 established part of why it was important for the set of scalars to be non-zero and what that meant from the perspective of the Magic Carpet tasks.

> NWR 8.1 In order to ascertain if you can go out and get back home, all of the scalars cannot equal zero.

The arguments for that introduced as evidenced the idea are functioning-as-if-shared are summarized in Appendix 7.   The classroom community had to come to a conclusion as to what it meant to have a zero scalar from the perspective of the Magic Carpet tasks. The zero-scalar came to be defined as not actually moving on the mode of transportation. The idea that having all of the scalars in the linear combination being zero implied in the traveling metaphor that the individual would not have actually moved anywhere. This was established as NWR 8.2 (the arguments for NWR 8.2 are summarized in Appendix 7):

> NWR 8.2 A zero in front of the vector implies that the vector is not being used.

The classroom community decided that in order to determine linear dependence, the individual actually had to move out from the origin and hence in conjunction with NWR 8.1, at least one of the vectors had to be used. The two normative ways of reasoning established in the classroom community an interpretation of the definition as stating that a zero in front of the vector meant that the vector did not actually move

and that in order for the set of vectors to be linearly dependent, at least one of the

vectors had to move. This was a truly normative way of reasoning, as members of

different classroom community in which there were no Magic Carpet task would not

come to this idea as being equivalent to linear dependence. In fact, the entire NWR

was established by the classroom community as a way of creating consistency

between the situational expressions that they had been using and the formal definition

of linear dependence.

NWR 8.3 formalized the relationship between NWR 8.1 and 8.2 and the

definition. Once students had already decided that they could create a mathemnatical

and situational relationship between the Magic Carpet tasks and the definition, they

quickly moved away from the reasoning situated in the task and instead started

reasoning primarily using the definition as justification.

NWR 8.3 Finding a non-zero solution to the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

makes the set linearly dependent.

This idea was evidenced as functioning-as-if-shared by being used as data or warrant

in at least 3 different arguments with three different claims. The first use of the idea

came about in D4A16 by the instructor as she discussed the set of all vectors in the

three-dimensional Magic Carpet task were linearly dependent. She was adding

information to Norton's argument in D4A13 (Figure 4.33 and 4.24). The instructor

stated:

Instructor: The original question was, is there a solution, or can you get back
home? So can we get back home, which we kind of symbolize as, does
there exist a non-0 solution c_1, c_2, c_3 to the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_n\vec{v}_n = \vec{0}$$.

Alright, I want to pause for a second, because the way the question is worded on your paper, and the way I wrote it here, are not the same. So I want to make sure the terminology writing it more as a mathematical sentence makes more sense. So trying to get back home would be the same, we're saying, as trying to figure out if there's a solution to this equation. And we found out that yes, there was. Any solution of the form m like you had originally [2,-1,1] is the solution.

In this case, a different way of representing the solutions to the vector equation in question using a single vector. The instructor made the claim that any scalar multiplied by the vector $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ would be a solution to the vector equation :

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

as was presented in Norton's argument. The data that she used for this was the actual solution that Norton used. She then connected the solution to the definition of linear dependence by stating, "So trying to get back home would be the same, we're saying, as trying to figure out if there's a solution to this equation

$$c_1 \bar{v}_1 + c_2 \bar{v}_2 + ... + c_n \bar{v}_n = \bar{0} \quad ?$$

And we found out that yes, there was." She also added earlier in her argument that in order for the set to be linearly dependent, "we kind of symbolize as, does there exist a non-0 solution c_1, c_2, c_3 to the equation." The warrant tied together the idea of "getting back home" and the definition for linear dependence as it was presented in class. The argument is summarized in the table below.

D4A16 Argument

Claim: *Any solution of the form like you had originally* $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ *is the solution to the*

vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. (Instructor)

Data: Norton's solution to the vector equation
as presented in D4A13 (Figure 4.29)

$$c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + c_3\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Warrant: *So trying to get back home would be the same, we're saying, as trying to figure out if there's a solution to this equation? And we found out that yes, there was.*
So can we get back home, which we kind of symbolize as, does there exist a non-0 solution c1,c1,c3 to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ *."*
(Instructor)

Figure 4.54: Day 4, Argument 16

The second instance of this idea being used as data-warrant was in D5A9, when the

classroom community was discussing what it meant for a set of vectors to be linearly

dependent. Nate was asked to present one of the generalizations that his group had

come to regarding what made a set of vectors linearly dependent.

> Instructor: Let's take this opportunity to go through and try to figure out a
> generalization from this that you guys were most likely using as you
> were coming up with this set.

> Nate: They have to be multiples of each other.

> Instructor: Yeah, so Nick is saying it for me. So let's say it a little more completely.
> Say better.

> Nate: They have to be multiples of each other. So there has to exist a
> constant c that makes the equation equal to the zero vector.

Nate put forward the claim that for two vectors in R^2 would be linearly dependent if they were scalar multiples. He then added the data/warrant that "there has to exist a constant c that makes the equation equal to the zero vector."

D5A9 Argument
Claim: A linearly dependent set of 2 vectors in R2 is one where the vectors have to be multiples of each other.
Data: If two vectors are multiples of each other, then there exists a constant c that make the vectors equal to the zero vector.

Figure 4.55: Day 5, Argument 9

Nate's argument used the idea that for a set of vectors to be linearly dependent, then the their has to exist a scalar such that one of the vectors becomes the opposite of the other vector. This would then allow for the two vectors to be added together to get the zero vector.

On Day 5 and Day 6, the classroom community worked on the fourth task in the Magic Carpet scenario. This task asked students to find sets of various sizes of linearly dependent and independent vectors in various dimensions. The next two arguments, which used the idea as data, all came from this activity or from the related activity when they had to create generalizations. In D5A30, Nate was asked for three linearly dependent vectors in R^2. The transcript that established his argument follows:

Instructor: The 2nd row is asking us for 3 vectors in r2, that are dependent and independent. So let's just broadly look at this. This table up here has a solution, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$; $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. I'll write it here, the photocopier made it a little blurry. And this table down here, for the exact same row, 3 vectors in R^2 and linear independent put 'no solution,' meaning it was impossible. Can both tables be correct?

Jason: Yes.

Instructor: No, it can't.

Nate: No, but for the top one, if you use the definition of linear dependent, it says that not all of them has to equal 0. So if you set the 1st 2 constants to equal 0 and the last one can equal anything, and you'll still stay at that point, so it's actually linearly dependent.

The instructor and Jason's comments are added to provide context to the argument.

The instructor was asking the class if it was possible that one table that had the vectors

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and another that said there was no solution for three linearly

independent vectors in R^2. There was some disagreement as evidenced by Jason and

the instructor's follow-up. Then Nate presented his conclusion as to why the set was

linearly dependent. He provided data for his argument by stating for the particular set

of vectors, "So if you set the 1st 2 constants to equal 0 and the last one can equal

anything, and you'll still stay at" zero." At the beginning of his argument, he provided

the warrant that connected the actual values for the scalars and the conclusion that the

set was linearly dependent by stating that, "if you use the definition of linear

dependent, it says that not all of them has to equal 0."

D5A30 Argument
Claim: The set $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is linearly dependent. (Nate)
Data: "So if you set the 1st 2 constants to equal 0 and the last one can equal anything, and you'll still stay at" zero." (Nate)
Warrant: "if you use the definition of linear dependent, it says that not all of them has to equal 0." (Nate)

Figure 4.56: Day 5, Argument 30

Nate's argument stated directly that the definition for linear dependence stated that not all of the scalars had to equal zero. This was a second instance of the idea being used as data or warrant.

The third instance that I am documenting of the idea as data or warrant occurred during the same task, as the classroom community discussed whether or not a set with the zero vector in it was actually linearly dependent. The following transcript presents the argument:

> Nate: Yeah, but at the same time, if you say 0,0 is a vector, by linear dependent definition, we say that you can use it if one of them, if one constant is not equal to 0, so we set c1 and c2 as 0's and when c3 is set to anything, and it's still linear dependent set.
>
> Gary: So the definition says that they all have to be 0.
>
> Jason: And the last one can be any number you want.

Nate first presented that if the zero vector is included in the set, the set is linearly dependent. He then gave the data for his claim that if you have two vectors that are non-zero vectors, then you can choose c_1 and c_2 to equal zero, and then c_3 can be set to anything. Gary added a warrant to Nate's argument when he stated that the definition said that all of the scalars had to be zero. Jason echoed Nate when he said that the last number could be anything.

D5A32 Argument
Claim: Even if zero is a vector, then the set with zero in it is linearly dependent. (Nate)

Figure 4.57: Day 5, Argument 32

> Data: if one constant is not equal to 0, so we set c_1 and c_2 as 0's and when c_3 is set to anything, and it's still linear dependent set. (Nate)
> Data: "The last number can be anything." (Jason)
>
> Warrant: By definition, To be linearly dependent all of the scalars have to be zero. (Gary)

Figure 4.58: Day 5, Argument 32 Continued

Nate's argument was that any set of vectors that contains the vector would be linearly dependent. Whereas, his earlier argument was that the particular set of vectors that was under discussion was linearly dependent.

The idea that became NWR8.3 was used in five other arguments as data or warrant in different arguments. This idea was a key way of justifying arguments, especially as students created generalizations for why a set of vectors might be linearly dependent or independent. The idea was tightly tied to the definition and reasoning about the possible scalars in a linear combination.

The way of reasoning that became NWR 8.4 was established in the same set of arguments that introduced and evidenced the idea for NWR 8.3 as functioning-as-if-shared. The instructor set up the definition for linear independence as when a set of vectors was not linearly dependent. Hence, whereas NWR 8.3 implied that there was non-zero solution to the vector equation $c_1 v_1 + c_2 v_2 + \ldots c_n \, _n = 0$, NWR 8.4 established that a linearly independent set can only have the trivial solution.

NWR 8.4 For a set to be linearly independent, the only solution to the vector equation:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots c_n \vec{v}_n = \vec{0}$$

is the trivial solution.

The arguments that established that the way of reasoning for NWR 8.4 was functioning-as-if-shared are summarized in Appendix 7. The majority of the classroom community's arguments that evidenced this idea presented linear independence as the negation of linear dependence. In addition, the classroom community would determine linear independence when a linear combination was set to zero and the only solution that they arrived at was a set of zero scalars.

The final NWR for CSP #8 had to do with the existence of the zero-vector in a set of vectors and the linear dependence of that set. The reason that I included the NWR was that this idea was a key point of contention throughout all of the fifth day that I analyzed.

> NWR 8.5 If you include the zero vector in any set of vectors all of the other vectors can have scalars equal to zero and the scalar in front of the zero can be anything and the sum will be anything.

The two arguments that established this idea as functioning-as-if-shared have already been presented as evidence that NWR 8.3 was functioning-as-if-shared. At the end of Day 6, the idea was presented without warrant as a generalization for determining if a set of vectors was linearly dependent. The addition of the zero-vector to the set of vectors was actually necessary to fully understand the definition for linear dependence. The use of the Magic Carpet scenario allowed for the "getting back home" idea for linear dependence. However, the classroom community needed to come up with an explanation for the zero-vector and the meaning for zero scalars. The classroom community's reasoning about why the inclusion of the zero-vector always made a set linearly dependent depended almost entirely on the definition.

CSP #8 was a practice where students used the definition of linear dependence to prove that vectors were linearly dependent. In the activity of proving that a set of vectors was linearly dependent, the students gained a greater understanding of what it meant to be linearly dependent and also how to use formal definitions. The NWR's that make up the practice were all established and used in the process of drawing conclusions about sets of vectors. The act of proving that a set was linearly dependent allowed for the members of the classroom community to have confidence in the conclusions that they drew about vectors. The definition provided authority from the mathematical community about the conclusions that students had drawn about sets of vectors. This also set up the opportunity to use theoretical expressions and mathematics to draw further conclusions about vectors and vector equations.

Conclusion

I began with the assumption that the process of meaning-making is fundamentally social and context dependent. In this chapter, I demonstrated that the classroom symbolizing practices that were developed in this community were a product of their situated activity with a set of tasks that had traveling as its central metaphor. The development and establishment of what it means to travel on a vector, get to a destination, or find a path to a destination became central to the classroom community's reasoning about vectors and vector equations. I evidenced this development in the eight practices and their associated normative ways of reasoning. The practices are summarized in the table below:

Summary of Chapter Four Classroom Symbolizing Practices
  CSP #1 A vector in R^2 is defined as a path in two-dimensional space.
  CSP #2 Scalars define the direction and amount that a vector is stretched.
  CSP #3 Setting up and solving a system of equations allows one to solve for the
    scalars in the vector equation.
  CSP #4 Linear combinations can be used to model relationships between sets of
    vectors
  CSP #5 Scalar multiples or vectors along the same line are linearly dependent.
  CSP #6 The directions of the vectors in a set determine the span of the vectors.
  CSP #7 Linear Dependence is determined when there exists a path of vectors
    from and back to the origin.
  CSP #8 Proving that a set of vectors is linearly dependent means finding a non-
    trivial solution to the vector equation.

Figure 4.59: Summary of Chapter Four Classroom Symbolizing Practices

Through its activities with the Magic Carpet Ride Scenario, the classroom community

established normative ways of reasoning for what a vector, scalar, linear combination,

and a set of vectors. These practices summarize the products of these activities and

the ways that the members of the classroom community established for dealing with

the ways of reasoning presented in the first six days of class.

The analysis in this chapter also demonstrated how the meaning for the

fundamental objects of linear algebra are under a constant process of adaptation as

members of the community participated in new activities and developed new ways of

reasoning. While the idea of the vector as a path with direction and magnitude (CSP

#1) remained a central meaning throughout the first six days, the community

established ideas for what it meant for two vectors to be equivalent (CSP #5) and

what the properties for a set of vectors (CSP #6 and 7) were. In addition, the

classroom developed a robust set of operations on vectors (CSP #2 and #4) and a

fundamental equivalency between systems of equations and vector equations (CSP #3).

These practices were essential to their continued reasoning throughout the course as evidenced in the analysis of CSP's #5-8. As members established new normative ways of reasoning for different concepts, they consistently used the previously developed ways of reasoning as a means of providing shared reference for their activities.

For example, in NWR 6.1, " Two vectors need to be in different directions in order to be able to get everywhere in R^2," the normative way of reasoning and the justifications for its validity were tied to many of the ideas and practices that were developed in earlier tasks. Jason's reasoning about the span of two vectors with different directions clearly referenced an advanced notion of direction that was beyond the meaning of direction presented in CSP #1. Direction in his argument was a general property of vectors that included the entire line in two-dimensional space that the vector was on moving in both positive and negative directions. In addition, in order to delineate what he meant by different directions, he used ways of reasoning about vectors in the same direction as a contrast. And his reasoning about why the different directions would allow one to get the span of the entire dimension played on earlier ideas of what a linear combination was and the allowed operations on vectors that had been established in class. He constructed his argument using the language of traveling and the associated meanings of the traveling metaphor that had been established through participation with the Magic Carpet Ride Scenario in service of talking about linear independence and dependence and while establishing a new normative way of reasoning for that concept.

The normative way of reasoning that was evidenced in Jason's argument was just one example of how, in this classroom, meaning became a process of establishing

new ideas, participating with those ideas, and generating normative ways of reasoning for working with vectors and vector equations.   This process of constant refinement and expansion of previous ideas is the process of meaning making. One of the major meanings that was demonstrated in the analysis of the six days of classroom activity was the meaning for the concept of direction. I use the term concept here instead of idea because the concept of direction employed a collection of ideas that were graphical, algebraic, theoretical and situational to create a cohesive notion of what the direction of a particular vector was. This concept also extended across multiple practices (CSP #1 and 2, 4-8) as members of the classroom community employed, refined and expanded it to play a key role in their activity. Furthermore, clearly the meaning of direction transformed as students used it in new and different activities. It began with the instructor's definition for a vector. It expanded as the classroom community established symbolic meanings for scalars and linear combinations, as the community reasoned about what the effect of scalars were upon vectors and what occurred when you added together vectors that had different directions and were multiplied by scalars. These ideas employed a variety of symbolic resources including the graphical and algebraic symbolizations for scalar multiples and linear combinations. When the classroom began to engage with span and linear dependence and independence, the concept of direction took on new meaning. The introduction of linear independence, dependence and span and the tasks used to produce meaning for them brought to bear what it meant for two vectors to have the same direction and what was possible for vectors in different directions. Furthermore, classroom activity on the development of linear dependence established what it meant to have three or

more vectors with truly different directions, e.g. when one vector in the set could be written as a linear combination of the other vectors in the set. In addition to expanding the classroom's algebraic and graphical repertoire for direction, participation in these activities introduced theoretical ways of symbolizing and talking about differences and similarities in direction.

Most of the normative ways of reasoning that established the meaning for direction were intimately tied to the situational expressions for the traveling metaphor. This seems natural as the word "direction" has with it cultural meanings that imply movement and paths. However, whereas the introduction of a theoretical definition of direction and its relation to vectors and vector equations would provide one touchstone for understanding and using direction, the use of the travel metaphor offered students the opportunities to gesture, talk, and draw pictures that had an already familiar set of meanings. Onto this set of meanings could be grafted the algebraic, graphical, and theoretical symbolizations for vectors and vector equations that would then expand the notion of direction and filter it into the context of linear algebra. The outcome of this process was a robust notion of a key concept in linear algebra.

The story of this chapter is of the development and establishment of meaning for the symbolic expressions for vectors and vector equations. The establishment of functioning-as-if-shared ways of reasoning demonstrated the situated and participatory nature of meaning as students engaged with each other and the tasks to generate symbolic expressions and meanings that furthered their academic activity. The analysis also demonstrated that meaning is context and time dependent.   To ask the question, what does the symbolic expression mean for a particular individual implies

asking to what community is that individual addressing, in what task are they

employing the symbol, and at what time are you asking the question.

Chapter 5 : Responsibility and Contribution

The analysis for this chapter came from the focus group interviews that were conducted at the beginning and end of the semester of Spring 2010. I used open coding (Corbin & Strauss, 2007) in order to generate a set of codes that answered the second question in my dissertation:

> In what ways did individual students contribute to and take responsibility for the different meanings that this classroom community developed for vectors and vector equations?

In this section, I discuss the sets of codes that I generated from my initial analysis on the focus group interviews. I conducted 6 total focus groups. In the following table, there are three sets of focus group interviews. The first set consisted of two focus groups. The second set of focus groups was conducted in order to be able to compare the two focus groups in set 1 more accurately given that both interviews would have been conducted after the first 6 days of the class. The third set was conducted at the end of the semester.

Table 5.1: Table of Focus Group Participants

Focus Group Set 1	
After Classroom Period 3	Norton, Nico, Nancy, and Carol
After Classroom Period 6	Karl, Lance, Dan, and James

Table 5.2: Focus Group Participants Continued

Focus Group Set 2	
After Classroom Period 8	Norton, Nico, Nancy, and Carol
Focus Group Set 3	
After Final Classroom Period	Nancy, Carol, and Karl
After Final Classroom Period	Lance, Abraham, and Nico
After Final Classroom Period	James, Norton, and George

The members of these focus came from different small groups in the whole class. Nico and Dan were in one small group. Carol and Karl were in a second small group. James and Nancy were in a third small group. And Lance and Norton were each in different small groups. During the course of the semester, many of the small groups changed and so there were some difference in the representativeness of the small groups. By the end of the semester,, Abraham and George were in one small group. Karl, James, and Carol were part of a second small group. And Norton, Nico, Lance, and Nancy were all in different small groups during the linear algebra class.

Using the first two focus group interviews, I generated a set of codes that related how these students related to the activities and knowledge constructed in previous mathematics classes and also in their linear algebra class. When appropriate, the codes that were generated to describe their experience in previous math classes were compared and contrasted against the codes that were generated when they spoke about their linear algebra class. The remainder of the codes in this analysis was drawn from all six of the focus groups. When I appropriate I detail the particular focus group interviews that the codes were drawn from. The codes I constructed to capture their expressed experience in linear algebra were based upon to whom the responsibility is

directed and the various kinds of symbolic flexibility that were related by students doing activities in the focus group interviews. These codes were then grouped in terms of two collections of themes, *responsibility towards oneself and the community* and *symbolic flexibility*.

An important note needs to be made about the coding for responsibility and contribution. The responsibility and contribution that I was coding for were the responsibility and contribution that members of the linear algebra classroom community demonstrated towards themselves, their in-class small groups, and the whole class discussion as was expected within the classroom. Within the focus groups, the students did demonstrate some of the same tendencies towards responsibility and contributing to the focus group community. However, I did not code these instances as instances of responsibility and contribution as the normative expectations of the focus group could be different from those of the classroom.

How students related to their previous mathematics classes

During the first two beginning-of-semester focus groups students were asked about their past experiences in mathematics classrooms, their immediately previous experience with mathematics, and their opinions about their current linear algebra classroom. I first examined the early semester interviews with a view towards understanding the relationships between their roles in mathematics classrooms, how these students saw themselves as doers of mathematics and how they saw themselves as members of the classroom community in both past mathematics classrooms and the linear algebra class.

As background for understanding student roles in their mathematics classrooms, I asked them about their previous mathematical experiences and how they envisioned mathematics functioning in their future careers. Their responses led to the creation of codes that illuminated their roles as doers of mathematics. Students frequently described feeling dissatisfied with their activities in their previous mathematics courses and believed that mathematics would primarily function as a series of solution methods and facts that they could use in their jobs. There were three different codes that emerged in my analysis, *resignation, frustration* and *dislocation*. I collected these codes under the common theme of alienation. Alienation is the experience of feeling that one is "outside" of the community and not connected to the activities or practices of a community. The experience of being "outside" of a community is not literal in this case, as the students were physically present in the classroom and were aware that they were doing mathematics. Instead, the "outside" that is referenced here is feeling that one cannot directly make an impact on the community, in this case the mathematics community, and that the activities that they are engaged in have no relationship to their own goals.

In the following table, I have categorized each of the codes with the number of utterances that were made during the focus group interviews. In addition, I have added the students who made the utterances that were coded as such. An utterance is defined as a discursive turn taken by a student. In some cases, turns had to be collected to make up a single utterance, but this only occurred when students made short utterances that were broken up by discussion from other students or the interviewer. In the tables that follow in this chapter, I break the themes into the codes that

generated those themes and the students that made those utterances. In addition, I

provide the number of utterances that were coded with a particular code. In some

cases, individual students made more than one utterance that could be interpreted as a

particular code.

Table 5.3: Summary of Alienation Codes

Alienation: How students responded to previous math classes		
Code	Number of Utterances Coded	Students making utterances
Resignation	6	Carol, Karl, Nico, Lance, Dan
Frustration	8	Carol, Karl, Nico, Dan,
Dislocation	2	Norton

In the following section, I describe what each of these codes mean and how they relate

to the research question. In order to illuminate the meaning of these codes, I have

included examples from the focus groups and analysis of these examples.

During the first two interviews, students were asked to "describe what being in

your last math class was like." Seven of the eight students in the first focus group

interview had taken the same class, Discrete Mathematics, in their previous semesters

and the remaining student, Nancy, had taken the class in a different semester. All of

the students in this focus group were either computer science, computer engineering,

or math majors, and each expressed the opinion that they felt they were strong in

mathematics. Seven of the eight students expressed some level of dissatisfaction with

this course, but the reasons for that dissatisfaction varied between students and focus

groups. Five of the students in these two focus groups felt that the methods that the

teacher used to assess how well the students learned the material was arbitrary. Two

of the students stated that there was a noticeable lack of continuity between the lecture, the textbook, and the homework. And one of the students stated that he did not feel that the class was a math class at all, but rather was a logic class. Each of these statements was made despite the fact that each of the students described the instructors as dynamic lecturers, who made class interesting.

The first code developed was *resignation.* I use the term *resignation* to mean the feeling that one cannot make a significant contribution to the class or have control of their own understanding and/or success in the class. In one of the classes, the instructor chose whose quizzes he would grade at random. The students felt that this method of grading was arbitrary and unfair. In fact, for four of the students, this led to them expressing resignation. For example, one of the students expressed that the members of the class quit studying for the quizzes and that he felt no responsibility to the members of the class or himself for doing well on the quizzes.

> So pretty much what happened was, a lot, a big percentage of class did not study for it. And then when they would get picked, most of the people wouldn't do so well. This one time I remember, he collected everybody's, the dice rolled that everybody gets to turn it in. And I came to give him my paper, which was probably a 0 again, most of the people I saw were 0's. (Nico)

Nico's statement exemplifies resignation by expressing how he stopped trying to score well on his quizzes, and he felt that other members of the class were doing the same. In relation to the other members of the classroom, this sense of resignation clearly demonstrated a lack of responsibility on the part of Nico, in particular, for the understanding of the material and being able to apply that understanding to new questions or tasks.

The second code that I developed was *frustration.* I define *frustration* as the feeling that the activities that one is engaging in are not making an impact on achieving one's goals. Frustration differs from resignation in that students experiencing frustration may or may not stop taking responsibility for the activities of the classroom. For utterances coded as resignation there was a clear expression that the student had stopped participating and felt that any continued activity would be non-productive. Conversely, while frustration may lead to resignation, students did not express the desire to quit working on the mathematics. Five of the eight students expressed that they did not know what recourse they had to solve problems for themselves. These students expressed situations in whole class discussions, during quizzes, and when doing homework in which they felt that they had no idea of where to begin when solving a problem or proving a theorem. Without having tools to solve the problem for themselves, they looked to the instructor to describe how they were to solve the problems that they were given. Unfortunately, students expressed that they felt that the instructor expected them to process what he was saying in class quickly and then be able to regurgitate the methods from memory and apply them to new situations.

> He would have us each day come into class, and he would just randomly write a question up on the board, 'prove this.' And maybe the question had come from these 2 lines of text in the book. And you're like, 'I didn't even understand it in the book, how am I supposed to solve it up there?' (Carol)

Carol's expression of frustration at being asked to prove something that she had no relationship to or expectation of understanding was common to other focus group members. Five of the 8 group members expressed similar sentiments. As Nico stated:

> And during class he did not really address, he addressed the problem for 2-3 minutes, he explained the problem again and kept going. He did not care that most of the people did not understand anything that he says. And that was, it was just a bad experience for math.

This statement demonstrated a disconnect between what these students expected from themselves and what the instructor expected from them. Students expressed that a single line of text or the request to "prove" something was insufficient to provoke students to think about the mathematics concepts, but that for the instructor it was sufficient.

The third coding, *dislocation,* is defined as the inability to relate to one's activities as being engaged in the practices and processes of mathematics. Only one of the students made utterances that were linked to this coding, but these utterances could not be coded as either resignation or frustration. This code differs significantly from either resignation or frustration in that students expressing either resignation or frustration still expressed that the activities they were engaged in were mathematical in nature. For both the resignation and frustration codes, students may have felt that they had no ability to develop solutions or to contribute to the class, but they did express that the activities were mathematical in nature. In utterances coded with dislocation, students stated that they no longer related to the activities of the classroom as being mathematics. For example, Norton's way of dealing with the problematic nature of the class was to no longer relate the class to mathematics. Instead, he thought of the class as a logic class.

> There's good and bad to everything. I saw it a lot more logically, that class as more of a logic class than a math class. So I liked it, because I like thinking logically, too, so I liked that part.

As a response to the difficulties that he faced in the class, Norton began to no longer relate to the activities in the class as relating to math. Unlike his other focus group members, Norton did not express resignation regarding his activity in the course, but instead he shifted his view of the class. Realizing that logic plays an important role in mathematics could be very productive as he could begin to expand his impressions of math. However, his stated belief that such activities are not a part of mathematics or are somehow separate mathematics could be problematic as Norton entered into new math classes that expect a higher level of logical rigor. While this was one only two instances of an utterance coded this way, it demonstrated that a student can feel positively about a course or set of activities, and still not relate productively to the field of mathematics.

Resignation, frustration, and dislocation can be powerful actors in shaping the roles that students inhabit within a classroom. In this case, students felt that the teacher expected them to understand the material, but they had no expectation of that for themselves, did not feel that their contributions would make a difference to the larger community, or that their activities were not mathematical. For these three codes, I developed the theme of *alienation.* I define *alienation* as the experience of not being connected in meaningful or productive ways to the activities of the classroom or the mathematics community as a whole. Being connected means that one expresses a relationship to a set of activities or practices that is beyond conceptual or procedural knowledge. Some examples of this might be contributing the classroom community, finding relevance beyond the classroom, or feeling part of

a larger community. And meaningful and productive ways are those that could potentially expand and solidify the students relationship to the classroom community, such as feeling that one does contribute to the classroom body of knowledge, one can extend their knowledge beyond the classroom's activities into homework activities and to other classes, and that the activities that one is engaged in in the classroom are legitimately mathematical in nature.

Responsibility to oneself, the whole class discussion, and small group

The codes that were generated from the analysis of the six focus groups were initially separated into two different groups, codes regarding responsibility and codes regarding student contribution to symbolizing vectors and vector equations. Once they were separated into the two groups, the codes were separated by theme. In the case of codes regarding responsibility, the codes were separated into three themes based upon to whom the responsibility was directed: responsibility to oneself, the whole class discussion and the small group. In this section, I discuss each of these themes and define the codes that make them up.

Responsibility to oneself

In contrast to how students viewed their expectations and responsibilities in their previous classes, the students in the first three focus groups expressed the responsibility to themselves and other members of the class was an expectation in the linear algebra class. The members of the focus groups expressed three kinds of responsibility. The first of these was a responsibility

to themselves as individual class members to understand the material presented in the class. In discussing earlier classes, these students expressed a lack of responsibility to themselves to understand the underlying concepts and structure in the mathematics that was presented to them in the classroom. Instead, they expressed a desire to find ways to come to solutions and attempt to follow the instructors' lectures and directions in order to get a passing grade in the course. When these goals failed, these students expressed alienation. Instead of working to understand the material on their own, they expressed an inability to utilize their own abilities to generate meaningful solutions. In contrast, students expressed a very different relationship to problem solving and their study of linear algebra. Three codes were developed that demonstrated responsibility to oneself. The first code for responsibility was validating conclusions. Utterances coded thusly demonstrated the students desire and expectation to determine if other members of the classroom community agreed with their solutions and justifications.

The second code was pre-eminence. Pre-Eminence is defined as the desire to come to one's own conclusions before exploring the solutions and justifications of others. Utterances that were coded as pre-eminence were also related to student's responsibility to the whole class as they developed justifications that might have to be presented. The third code was testing conclusions. Testing conclusions is defined as the desire of students to invalidate and find flaws in the conclusions of others in order to either test their own conclusions or strengthen the conclusions of others. The following table provides a breakdown of the codings and the utterances that students made with regard to these codes.

Table 5.4: Summary of Responsibility to Oneself Codes

Responsibility to oneself		
Code	Number of Utterances Coded	Students Making utterances
Validating Conclusions	6	Karl, Norton, Lance, Abraham, Nancy, Carol
Pre-Eminence	6	Lance, Nico, Abraham, James, George, Norton
Testing Conclusions	4	Norton, Nico, Abraham, Karl

In the following section, I describe each of the codes in this table and how they relate to answering my research question. In addition, I contrast these codes with the codes that were developed to characterize their experience in discrete mathematics.

A major theme that stretched across the early and the end-of-semester interviews was student's search for the validity of their solution methods. In all six of the focus group interviews, students expressed the frequent difficulty and uncertainty that they had in determining if their solution methods were correct. All of the students expressed that one of the values of whole class discussion was in challenging their assumptions and solidifying their conclusions. Utterances that specifically referenced the desire to strengthen their own conclusions by listening to others' justifications and reasoning were coded as validating conclusions. For example, in one of the end-of-semester interviews, Karl was asked what value he saw in the whole class discussions.

> Think about a box, if all you ever saw was the front of the box, could you really describe what a box was? Until you really look at it from all sides and really understand it, you can't tell someone else how to do it. So when you're in class, and people question your theories and bouncing things off of each other, just like in the discussion sessions, it's just, until you can really argue your theory or support your theory,

all those challenging questions make your understanding just that much stronger.

Karl's statement discussed the value that he saw in listening to whole class discussion by stating that it gave him the opportunity to "look at it from all sides." His use of the personal pronoun "you" seems to dislocate his statement from his own experience, whereas the use of the term "I" in place of "you" would be less ambiguous. However, this may have been a by-product of the way that the question was asked given that he was asked to describe the value of whole discussion and not a question that would elicit the use of the personal pronoun "I", like if he were asked, "What value did you find in participating in whole discussion?" However, Karl's expressions do seem personal. He talked about "bouncing ideas off each other" and "people question your theories" each expressing a certain level of personal experience. I concluded that he was discussing what value he saw in whole class discussion, and that he expected that he would be testing his theories and bouncing ideas off of other students.

Each of the groups expressed valuing the whole class discussion because it offered the opportunity to determine if their own conclusions and understanding were correct. 6 of the 9 students in the end-of-semester focus group expressed the need to examine other people's conclusions to develop confidence and establish the validity of their own conclusions. As was evidenced in Karl's statement above, there were many ways in which students would interact to attain validity for their conclusions. But in each case, students expressed that there was an expectation that they would in fact have valid conclusions and this desire for validity was not directly linked to their membership in the classroom community.

The extent to which each of the students felt the whole class discussions were productive sometimes depended upon how much they themselves understood the problem they were dealing with. Utterances that expressed that students were looking for their own conclusions before engaging with the whole class or small group discussion were coded as pre-eminence. Pre-eminence is defined as the responsibility to oneself to come to one's own conclusions before testing theories in whole class or small group discussion. In an end-of-semester focus group, Lance, Nico, and Abraham, expressed that they would sometimes stop paying attention to the whole class discussions when they felt confident of their conclusions. However, when they did not feel confident of their conclusions, they spent more time listening to the conversations and testing their own theories. In two of the three end-of-semester focus group interviews, the interviewees discussed how they wanted to continue working on their own work after whole class discussion began because their focus and concentration was on finding a solution for themselves before they engaged with the classroom community. When the group was asked why, if they did care about coming to conclusions and testing their own theories, there were blank stares for periods of time when the teacher asked for questions? James responded:

> You could say probably unfinished thought process, given time that we have. Because a lot of the questions, she sets us off on our own to figure things out our own way. And sometimes by the time we're expressing or trying to show how we were thinking, a lot of times I'm still wanting to finish what I started, I want to do this before I, so I'm still in my little, my own mind. And everything else I don't really care about until I finish off. (James)

James's statement that he wanted to "finish what I started" and "that he was still in my little, my own mind" demonstrate that he felt that he needed to come to his own conclusions first before he engaged with the classroom. He expressed that this occurrence sometimes happened after whole class discussion had started and that he did not have the opportunity to consider what others had said or to formulate questions regarding their solutions or justifications. James's sentiment was echoed by five different focus group participants. Each of these participants felt that sometimes, whole group discussion was an interruption to their thought processes and the work that they were currently doing. Rather than deal with the interruption, they continued to work on the problem on their own.

Norton expressed a differing relationship to whole class discussions. In the first beginning-of-semester interview, Norton expressed his desire to challenge other people's conclusions by attempting to prove them wrong. For him, trying to prove other people's conclusions wrong allowed him to strengthen his own understanding and confirmed or denied his own conclusions. I coded utterances of this type as testing other's conclusions. While this code exhibited engagement with members of the classroom community, this responsibility was expressed as a responsibility to oneself. Individuals who made utterances of this type expressed that trying to invalidate the conclusions of others allowed them to confirm the validity of their own conclusions. Similarly, three of the other focus group members would listen to whole class discussion attempting to figure out if the conclusions of other class members fit their own interpretations or represented a new way of approaching the same problem. That the students could see new ways to view the same problem situation and recognize

those methods as valid or invalid is an advance over their activity in their earlier mathematics classes. This code differs from validating conclusions in that validating conclusions leads to an affirmation of one's own justifications and solutions by seeking common points of agreement.

These students' relationship to discussions and arguments during whole class discussion demonstrated their responsibility to themselves as learners of the material. The students saw value in whole class discussion when they could test their own theories and solutions against the theories and solutions of other groups and classroom members. Conversely, they did not express this as a responsibility to listen to others during whole class discussion. Participation in whole class discussion was required and was viewed favorably by all of the focus group participants except for Nancy. However, many of the members expressed a lack of concern about not listening during whole class discussion, preferring instead to focus on their own thought processes and develop understanding for themselves. From this I concluded that one of the major responsibilities that the members of this classroom had was to develop understanding for themselves. They sought validity for their conclusions, sought to understand alternate viewpoints, and struggled to develop a more comprehensive view of the mathematics by attempting to invalidate the conclusions of others.

Responsibility to the whole class discussion

The second theme that was generated regarding student's responsibility in the linear algebra class was *responsibility to the whole class discussion.* There were two codes that were generated in the course of open coding for this theme: preparing for

questioning and assuming understanding. The following table lists the codes that will

be discussed in this section.

Table 5.5:Summary of Responsibility to Whole Class Discussion Codes

Responsibility to the Whole Class Discussion		
Code	Number of Utterances Coded	Students Making Utterances
Preparing for Questioning	3	George, Karl, Carol
Assuming Understanding	4	Abraham, Lance, Nico,

In this section, I will define and illustrate each of these codes. I will also discuss the

relationship between this set of codes and those codes that related to students

responsibility to their own understanding.

Both the first theme for responsibility (Responsibility to oneself) and this

theme (Responsibility to the whole class discussion) demonstrated the responsibility

to generate conclusions that are valid and for understanding of problem solutions and

mathematical concepts. This theme differed from the first theme in that the

responsibility that is demonstrated by codes in this theme was directed towards the

other members of the class. Furthermore, unlike the first responsibility codes

students were not consistent in the whether or not they felt an affirmative or negative

sense of responsibility towards the whole class discussion.   The first code in this

theme: *preparing for questioning,* expressed a student's need to develop clear,

cohesive and correct solution methods and justifications for the purposes of

presenting in whole class discussion. The second code, assuming understanding,

expressed student's belief that once their presentations were made that the members

of the classroom community understood their statements and required no further

clarification. The primary difference between these two groups of students extended

from their expressed confidence in their explanations and understanding. The first

group, made up of two students- George and Carol, expressed a lack of confidence in

their understanding of the material, and that their efforts at understanding were

targeted at making sure that they could answer questions. Karl did express a

responsibility to prepare for questioning, but the lack of confidence was absent. The

second group, made up of three students- Lance, Nico, and Abraham. felt that they

were confident in their knowledge and took for granted that the other members of the

classroom understood the material well. A third group, Nancy, James, Norton, and

David, did not address their responsibility to whole group discussion. Nancy also

stated that she did not like to talk in whole class discussions and so would avoid these

interactions.

As was detailed in Figure 5.3, 3 of the 9 end-of-semester focus group members

made utterances that were coded as desiring a firm understanding of their

justifications and conclusions in order to make valid arguments and answer questions

if they were offered. For example, George stated that when he was asked to present

in front of the class, he felt the need to understand the material for himself in order to

be prepared to present his work and answer others questions about it.

> I think yes, because there are points in the semester where I do
> something, I knew to do it, but I don't know why. When you
> get in front of the class, you have to explain why, this is what
> we did. Why? And I was like [shrugs]. Not that I, it's just the
> way we would talk but I saw that, I know to do this. And so
> that's where I got stuck, I got caught up and I was intimidated
> by that, explaining my reasons why I did that. Which is
> something that I should know, I understand that, why I'm doing

it. But it's something I struggle with.

George's professed belief that when he gets in front of the class he should know why he has done what he has done is clearly a source of tension for him. He feels that it is his responsibility to answer questions if necessary and to feel confident in his understanding of the material. He needed to know "why" he came to his conclusions because he felt that the other members of the community expected him to know why. His response connects between his responsibility to himself to understand the material and the responsibility to the community to understand what he is talking about when he is in front of the class.

George frequently stated that he did not feel entirely confident in his understanding of the material, and this probably lent considerably to the tension that he felt and how aware he was to the other members of the community. By contrast, 3 of the participants, Nico, Lance & Abraham stated that they did not feel a responsibility to the class to make sure that the members of the community understood their responses. These utterances demonstrated that these students felt that when they presented to the class that the presentation was to give another view on the activities and their request for questions was an expression of some norms that were modeled by the instructor. But these students rarely expressed a lack of confidence in what they were presenting. For example:

> Sometimes it's because, I felt at some points in the case,
> everybody did understand. I presented a couple of times, and I
> felt like everybody was just looking at me like, 'Yeah, I got it.' I
> was looking at everybody else, they're looking at me like, 'yeah,
> I understand it,' so they have nothing to say. (Abraham)

This was consistent with the utterances of Lance and Nico. They felt that by the time they were presenting to the group, they had confidence in their answers and that the explanations that they were making were correct. These students assumed that the members of the classroom community shared their understanding and hence there was no sense of responsibility to confirm the other classroom community members' understanding. Consequently, the tension that was present in George's response was not a present for them and they felt that their responsibility to the whole class discussion had been fulfilled. The utterances that were coded amongst this group expressed the student's lack of responsibility towards the whole class to help other students understand the material.

James echoed the relationship between confidence in their own explanations and their responsibility to the group during the end-of-semester focus group. When students did not feel that their explanations were correct, they did not feel as responsible to the community to make sure that the community understood their explanations.

> Actually, I don't feel responsible, because there's this one point, mainly because she does make us go off and try to figure out our own way. So when we're presenting, we actually don't know if we're right or not, so I can't be that totally confident in trying to get everybody else to learn my way, because I'm not sure, until she clarifies it. She'll go over it after we've discussed, maybe change some terminologies here and there. But for the most part, sometimes we're not totally sure on our method. (James)

All of the students expressed the opinion that the teacher was the final arbiter of whether or not their conclusions were correct. James expressed that if he did not get

an indication from the instructor that his conclusion or approach was correct, he felt that he was not responsible for the members of the community understanding of his approach. During each of the end-of-semester focus groups, each group discussed their dependence upon the instructor for finally telling them that their conclusions were correct and that it was appropriate for them to proceed.

Responsibility to the small group discussions

The relationship between confidence in their interpretations and conclusions and the kind of responsibility to the whole class community differed markedly from how these students viewed their responsibility to the members of their small groups. On the first day of class, the students were broken up into small groups in order to allow them to work on various problems and to discuss the topics that were presented in whole class discussions. As was evidenced by students in all 6 focus group interviews, there was a difference in the expectations and kinds of responsibility that students felt towards members of their small group. The third theme, *responsibility to the small group discussions,* reflects this change in responsibility. All of the members of the focus groups made utterances that were coded as responsibility to their small groups. However, there was some variability in the ways that this responsibility was expressed. There were three different codes for this theme that summarized this kind of responsibility: developing consensus, voicing one's opinions, collaborating on conclusions. The following table summarizes these codes and the frequency with which they were coded in the focus group data.

Table 5.6:Summary of Responsibility to the Small Group Discussion Codes

Responsibility to the Small Group Discussions		
Code	Number of Utterances Coded	Students Making Utterances
Developing Consensus	5	Norton, George, Carol
Voicing One's Opinion	7	Karl, Carol, James, Abraham
Collaborating on conclusions	3	Karl

In this section, I will define each of these codes and discuss their relationship to previously coded forms of responsibility. I will also discuss how these codes address my research question and provide examples that illuminate the codes' meanings.

The first code, developing consensus, expressed students responsibility towards the members of their small groups to come to a group conclusion that could be then presented in whole class discussion. Students made these responses during questioning that occurred after a survey given in the end-of-semester interviews that asked them about what sorts of things they were responsible for in small groups and also in whole class. The survey was given in order to spark conversation about how students felt about whole class and small group discussions. Four of the nine participants made utterances that were coded this way. These utterances expressed the need to make sure that their groups came to a consensus and had arrived at an understanding of the material. Frequently in the focus group, the students expressed that there was a greater need to understand the material in the course of small group work then during whole class discussion. For example, Norton expressed his belief

that in the small group discussions, more discussions took place and these

opportunities were especially important in understanding the mathematics.

> So it's a lot easier to talk in your small group and understand
> each other, and then to communicate your ideas, than it is in the
> bigger one. And if you understand it in the small group, as long
> as you understand it, you don't really care what really happens
> out there, because she's going to say it anyway.

Norton expressed the importance of understanding what conclusions were made in the

small group because as long as he understood in the small group, he felt that whole

class discussions were not as important and he could more clearly understand the

material that was presented. He expressed that he felt that, "as long as you understand

it, you don't really care what really happens out there, because she is going to say it

anyway." This comment demonstrated Norton's belief that much of the understanding

of the material came during the course of small group discussions. He also expressed

that it was "easier" to discuss ideas in small group. This kind of comment, that small

group afforded the opportunity to come to and discuss conclusions in a way that was

more comfortable or markedly different from whole class discussion, was a similar to

many of the other utterances coded with this code.

Three of the end-of-semester focus group participants, Karl, Carol and James

were all part of the same small group during the linear algebra class. The kinds of

responsibility that they felt towards each other as members of that group were

apparent in their discussions, even though they were members of different focus

groups. These students expressed the final two codes in this theme, whereas only

Abraham made utterances that could be coded with the second code outside of this

group. This may have to do with the particular dynamics of the group, James, Carol,

and Karl's expressed that they felt that the members of the small group were equals

and that they all contributed their fair share to the small group discussions. In addition,

Karl, Carol, and James were active contributors to the classroom discussion (as was

their fourth small group member, Brad), and this active role in the whole class

discussion could have contributed to their feelings of solidarity and closeness.

The second code, voicing one's opinions, was made by 4 of the participants,

Karl, Carol, James, and Abraham. This code is defined as the responsibility to the

small group to express one's opinions for the purpose of strengthening the conclusions

and justifications that the small group generates. These participants expressed that

regardless of whether they had developed a full conclusion or strategy for solving,

they still needed to express their views on the task at hand. This kind of expression

represents a level of confidence and security that the members of the small group feel

when interacting with one another. As was discussed earlier, Karl, Carol, and James

all expressed a high level of comfort when working with each other and it seemed that

that level of comfort allowed them to feel comfortable in generating their opinions.

For example, Karl expressed how he felt the responsibility to the small group

members to understand the material.

> The way the class was structured for me personally made me feel
> like all you guys are my friends, that we can always collaborate
> on stuff. And I don't feel set back or anything to just voice an
> opinion or throw something in. And I feel that that same thing
> goes the same way, we could all discuss this stuff after class, we
> don't hate each other, I want you to understand as much as I do.

Karl expressed how much he felt that he had a responsibility to the members of his

group to help them understand. He felt that the members of the group were all

working towards a similar goal and that, whether or not he is correct or not, he should voice his opinion. Abraham also expressed this kind of responsibility, but his responses differed from the other three in that he did not express the same kind of closeness or equality with the members of his group. Instead, it seemed that the responsibility to express his opinions was his own and related to his desire to make sure that his voice was heard in small group discussions.

The third kind of responsibility that was coded for this theme had to do with collaborating on solutions. While other students expressed the desire to aid their fellow classmates in the development of solutions, only Karl expressed that collaboration was an integral responsibility in his work with his small group. Karl's statements about his desire to collaborate differ in his expressions to aid other group members in that he felt that working with the members of his group was a goal in and of itself. For example, Karl stated:

> But still, if Carol's unclear on something, or James or Brad, I'll be the first one to say, 'What's going on, can I help clarify, or what are you thinking?' I really want to know, because it's all part of the learning process. The same thing goes back to #2. If he's got something that he's not sure about, and he bounced that off me, either mine's wrong and his is right, in which case I'm close to the answer. Or it reaffirms my solution. So it's always a win-win for me. So definitely responsible.

In this statement, Karl's desire for collaboration did not extend from being right or wrong, as he said, "either mine's wrong and his is right." He also stated that "I really want to know, because it is all part of the learning process," indicating that his work with the group was valuable on its own. He felt a responsibility to the members of his group to ask for their conclusions, to test those conclusions against his own

conclusions, and to seek clarity from the other members of his group. This contrasts with the other two codes in this theme where the explicit goal is to come to correct conclusions and to develop agreement amongst the members of the group.

Symbolic flexibility

The second theme that I coded from the focus group interviews was *developing flexibility in drawing conclusions about vectors and vector equations*. By flexibility I mean the ability to utilize and understand a variety of symbolic expressions. Utilizing a symbolic expression means to be able to identify times when a specific symbolic expression is useful, explain clearly why they chose a specific expression, and to apply different symbolic expressions when their application is useful. Understanding a symbolic expression means being able to explain one's use of the expression, identify when another has used a particular expression, and relate to why another's use of that expression might be useful or valid. Each of these activities is a contribution to meaning making in that they allow members of a small group or focus group to communicate about the mathematics and contribute to each other's understanding. Without the flexibility to relate to each other's work, a group's attempt to solve problems or understand concepts can be stalled, eliminating the group members' ability to contribute. In the following section, I discuss the three codes that made up this theme, and provide examples that demonstrate each of the codes.

In the first two beginning of the semester focus group interviews, students were asked to examine 4 different symbolic representations for vectors and vector equations and rate them based upon their comfort with those forms and how useful

they felt those forms were to doing linear algebra. The codes for this theme were generated from this series of discussions, but there were other instances in the focus group interviews in which students made utterances that were coded with these codes. The theme that was generated was symbolic flexibility. This theme was generated from three codes: switching between vector equations and systems of equations, visualizing using graphical expressions, and transforming real-world situations to symbolic forms. The following table summarizes the theme and the codes that were generated. It also describes the number of utterances made for each codes and the students who made those utterances.

Table 5.7:Summary of Symbolic Flexibility

Symbolic Flexibility		
Code	Number of Utterances Coded	Students Making Utterances
Switching between vector equations and systems of equations	19	Karl, Nico, Lance, Abraham, George, Norton, Dan, Carol, Nico, Nancy
Visualizing using graphical representations	7	Carol, Karl, Norton, Lance, Dan
Transforming real-world Situations into Symbolic Expressions	13	Abraham, Karl, Nico, Norton, Lance, George

In the following section, I will define each of the codes and describe how they relate to the theme. In addition, I will discuss how these codes contribute to answering the research question and provide examples of many of the codes for the purposes of illuminating their meaning.

Part of understanding symbolic flexibility has to do with finding out the different purposes that students perceive for the different symbolic expressions. How did a change in symbolic expression change the way that students perceived vectors or vector equations? During the first three focus groups, each group was asked to determine the uses for a set of representations given to them by the interviewer and to discuss which representations they found to be the most useful for the purposes of doing classwork and homework. By useful, I mean for which purposes would students utilize the particular expression and in what ways were those expressions utilized for that purpose. Students discussed for each symbolic representation the circumstances in which they would or would not use the particular representations, and the limitations of the representations if they were asked to use them in different situations. The following table summarizes each of the different representations that students were given, the places where they would find them most useful and areas where they found them less than useful and the overall rating that they gave each representation. The rating that was generated was called a comfort rating as students were asked which representation that they felt most comfortable using. I chose the use of the term comfort because the feeling of comfort denotes strength and usability. Hence, an individual who feels very comfortable with a symbolic expression feels that they both are strong in their understanding of the expression and that they can use it in a variety of settings. Whereas, a person who does not feel comfortable with a symbolic expression would either feel that they did not understand the expression well, that they did not know when to use it, or both. In the first two beginning-of-semester focus groups, students were asked to rank each of the representations according to the

highest to lowest level of comfort. After analyzing the first focus group's responses, it became clear that there needed to be a more precise way to identify the differing levels of confidence or comfort that the students had for each representation.. Students were asked to rank the four representations based upon their comfort in using those representations to do linear algebra. In the table below, I have listed the kind of representation, the areas that the students found to be useful or not useful for that expression, and the comfort rating that they agreed on for the expression. The areas that coded as useful or less useful were drawn from the students' answers for "in what situations did you find the expression useful." The numbers are from lowest to highest, with 1 representing the most comfortable and 4 representing the least comfortable. For a visual of the graphical representation, see Appendix 2..

Table 5.7 Continued: Usefulness Response Tables

Focus Group 1: Symbolic Representation Usefulness Table			
Representation	Useful Areas	Less Useful Areas	Comfort Rating
Graphical Representation	4. Visualizing solutions 5. Solving problems that did not require large scalars	5. Solving problems that had fractional or large scalars	3
$2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$	3. Organizing information 4. Relating the system of equations to graphical representation	5. Finding linear combinations 6. Visualizing linear combinations	2

Table 5.8: Usefulness Response Tables Continued

Focus Group 1: Usefulness Response Table			
Representation	Useful Areas	Less Useful Areas	Comfort Rating
$2x+3y=9$ $2x+3y=8$	1. Finding solutions and linear combinations	1. Visualizing linear combinations	1
Student generated verbal representation: The first vector <3,1> tells you to go up 3 and to the right 1, the 2 in front of that vector tells you to draw the vector twice. The second vector <1,2> tells you to go up 1 and to the right 2 and the 3 in front of the vector tells you to draw the vector three times. Putting those vectors together from tip to tail puts you at <9,8>	6. Graphing Solutions 7. Visualizing the relationship between vectors		4

Focus Group 2: Symbolic Representation Usefulness Table			
Representation	Useful Areas	Less Useful Areas	Comfort Rating
Graphical Representation	2. Visualizing Problem Situations 3. Visualizing Linear Combinations 4. Understanding and visualizing span	2. Finding Linear Combinations	3
$2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$	2. Organizing information 3. Expressing linear combinations 4. Finding Linear Combinations		1

Table 5.9: Usefulness Response Tables Continued

Focus Group 2: Symbolic Representation Usefulness Table			
Represenation	Useful Areas	Less Useful Areas	Comfort Rating
$2x+3y=9$ $2x+3y=8$	2. Finding Linear Combinations in 3 or more dimensions 3. Expressing the solutions to modeled problems	2. Expressing Linear Combinations 3. Understanding and visualizing span	2
Student generated verbal representation: The first vector <3,1> tells you to go up 3 and to the right 1, the 2 in front of that vector tells you to draw the vector twice. The second vector <1,2> tells you to go up 1 and to the right 2 and the 3 in front of the vector tells you to draw the vector three times. Putting those vectors together from tip to tail puts you at <9,8>	1. Generating Visual Representations	2. Expressing Linear Combinations 3. Finding Linear Combinations	4

Many of the differences that occurred in the two focus groups usefulness coding can be attributed to the differences in time in between the two focus groups. For example, the students in the first beginning-of-semester focus group as demonstrated in table 5.6, students expressed that the vector equation representation was not useful for finding linear combinations, whereas the second beginning-of-semester focus group as presented in table 5.7 did not have this same problem area. Conversely, the second focus group did not find the graphical representation as being useful in finding linear combinations in all situations, the first focus group only expressed that the graphical were problematic when the scalars were large or fractional. The first focus group

interview was conducted after the third day of instruction. Students in this focus group had only been introduced to the various expressions they were asked to categorize in the interview and the tasks that they had been doing related mostly to finding and graphing linear combinations of vectors. They had also begun to work on activities relating to span, but conclusions regarding span and a definition of the term had yet to be introduced. Much of their activity in the class had been with graphical representations and understanding linear combinations from this perspective. They had been using the system of equations to find scalars for linear combinations, but the connection between systems and vector equations had yet to be solidly established within the classroom community. In the first focus group, students may have found the graphical representation less problematic in general because it had been central to their work up to this point.

The interview originally only had the first three representations. The fourth representation, which was verbally stated by Norton in the course of the first interview, was added impromptu because Norton, Nico, and Carol felt that that was a way that they often thought about the vector equation. Given the limited exposure that they had gotten from the class, their answers focused on using the representations to find solutions to systems of equations and to generate linear combinations, a term that had been introduced in a previous class. As well, students only conjectured about the lack of usefulness of graphing for three-dimensions as they had not had experiences with three-dimensional vector representations.

The second focus group was interviewed after the sixth period of instruction and so they had been exposed to span and linear independence and dependence. They

had also done significant work with finding linear combinations of vectors in three or

more dimensions and had been introduced in the previous class period to augmented

matrices. This focus group differed from the first focus group in that they saw a

higher level of usefulness for vectors and vector equations, particularly as it related to

expressing linear combinations and visualizing the mathematics that they were doing

in class. They also expressed an additional useful component to the graphs as they felt

that the graphs helped them visualize span and other theoretical considerations, such

as their nascent notions of linear dependence and independence. In addition, the

second focus group still felt that systems of equations were useful, especially when

relating to solutions to modeled problems, but they did not feel that this representation

necessarily gave them insight into linear combinations. This group added a fifth

representation, that of the augmented matrix, which had been recently introduced in

class. These students saw this representation as being another way of representing

systems of equations, and that the particular representation was useful for solving

systems in four or more variables. Because their responses regarding usefulness were

generally the same with regard to the systems and the augmented matrix, I did not

include a fifth table element. There was a consensus among the focus group members

that the system of equations became unwieldy when they needed to solve systems that

had more than three variables. This group specifically stated that they had difficulty

visualizing graphically three or more dimensions and saw this as a limitation to the

graphical representation.

Switching between systems of equations and vector equations

The first code: switching between systems of equations and vector equations, was generated from utterances made in four of the focus group interviews, the first two beginning-of-semester and the last two end-of-semester focus groups. Students expressed a variety of situations in which transforming a system of equations or a real world problem situation into a vector equation allowed them to organize information in a way that allowed them to draw more conclusions about the system, the graph or the real-world situation. Utterances categorized using this code detailed times in which transforming one symbolic representation into a vector equation allowed students to discuss the relationship between solutions to systems and graphical representations and efficiently come to conclusions about linear independence and dependence. Students also discussed the vector equation as an intermediary between the graphical representation and the system primarily in the first two focus group interviews. These utterances arose from questions in these focus groups that asked students to describe the usefulness of vector equations, graphical representations and systems of equations when doing linear algebra.

Differences between the first two focus groups were identified in relation to this code. Nico, Carol, Norton, and Nancy, the four members of the first beginning-of-semester focus group, were split in the way that they understood the relationship between the vector equation and the system of equations. While Nico and Norton strongly preferred the vector equation as a way of organizing information about systems, Nancy and Carol had reservations. The following exchange exemplified this tension. The focus group was asked, which of the forms they found to be most comfortable.

Nico: Then you break them up like that [draws lines around the values in the system to construct vectors], but at the same time, this method could make it more confusing, because you have your, you start being confused, you could start try to group this.

Carol: The bracket is kind of confusing a little bit, so I think pretty much we're agreeing that this is 1st [system of equations], 2nd [graph] and 3rd [vectors]?

Nico: Yeah, but at the same time, I feel that this way [systems of equations], if you have a lot of brackets, you could start getting lost- which one goes to which- brackets make it more organized.

Carol: It does make it more organized but it kind of…

Nico: Confuses at the same time.

Carol: Logically in your head and like every other math teacher tells you, all the top rows go together, the 2nd row goes together, the 3rd row goes together.

Norton: I like that way [points to the vector equation] for a final answer.

As was common during the conversations about their level of comfort with the various representations, students discussed the usefulness that they saw from using each of the representations. Nico initially tried to explain to the members of the group that the vector equation was the most useful way to organize information. Carol countered that for her the brackets that made up the vector equation made solving the system more confusing, and Nico countered with his contention that as the system has more equations, the addition of the brackets organizes the information. Rather than acknowledging Nico's point, Carol related the vector equation back to the system. In both the early semester focus group interview and the end-of-semester focus group interview, Carol expressed her preference to relate to any problem

situation in terms of the system of equations. However, in the end-of-semester focus group, she specifically stated that she understands what the other students are doing with the vector equations but that she prefers the systems approach.

In the second beginning-of-semester focus group, students did not share this tension, and instead they were more unified in their preference of the vector equation as a way to organize information. Unlike the first focus group, these students included an augmented matrix in their discussions of various representations. The students in this focus group acknowledged that in the augmented matrix there were no variables (the variables are implied) and hence this was deficiency in that form. The vector equation described the relationship between variables and coefficients or between the solution to the system and the columns of the augmented matrix. In first two beginning-of-semester focus groups, all 8 of the students expressed their preference for using systems over vector equations when performing computations. They expressed that while vectors and vector equations were a good way to model situations or to determine relationships between vectors, their use was problematic when trying to solve for scalars.

Students did not express the problem of using system of equations to solve for scalars in the end-of-semester interviews, but they did display the preference of using systems of equations or augmented matrices to solve for scalars. This method was also preferred when determining if a set of three vectors in R^2 were linearly dependent. In the end-of-semester focus group interviews, students were asked to develop a symbolic expression that would model a set of three floor plans that would eventually be used to create an apartment building with a given number of one, two

and three bedroom apartment. Each of floor plans was given in pictorial form (See Appendix 4). Floor plan A had three one-bedrooms, five two-bedrooms, and one three-bedroom apartment. Floor plan B had six one-bedrooms, two two-bedrooms, and two three-bedroom apartments. And floor plan C had one one-bedroom, three two-bedrooms, and three three-bedroom apartments. At least one participant in all three end-of-semester created vector equations to model an apartment building with a particular number of floor plans. For example, the students were asked to tell how many 1,2, and 3 bedroom apartments could be created with 6 floors of floor plan A, 11 floors of floor plan B, and 17 floors of floor plan C. Norton stated:

> Norton: Okay, so pretty much I counted all the rooms, we all had the same numbers, so that's what these are [points, closeup]. And then, so for Plan A, there's going to be 3 1-bedroom, 5 2-bedrooms, and then 1 3-bedroom. So it's like vector, so I have all 3 for A,B and C. And the question asks for how many 6 A, 11 B and 17 C, which is what these scalars are. Multiplied by the A, B and C matrix, and that's the answer.

As a response to the question, Norton created the vector equation:

$$6\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} + 11\begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} + 17\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 101 \\ 103 \\ 89 \end{bmatrix}$$

Norton's creation of the vector equation was echoed in the other two end-of-semester focus group interviews. This model would later be transformed by students into a general vector equation for later problems when students were asked how many floors of each floor plan would be needed to create a building that would have a given number of bedrooms. For example, Karl created the vector equation:

$$A \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} + B \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} + C \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 23 \\ 17 \end{bmatrix}$$

to model the possible number of floor plans that would make up a building that had

11 one-bedrooms, 23 two bedrooms, and 17 three-bedroom apartments. The students

in each of the focus groups then transformed the expression into a system of linear

equations or in the case of one of the focus groups an augmented matrix in order to

determine the numbers of each floor plan that were necessary to create the building.

Each of the groups would then utilize a calculator to row reduce the matrix to come

to their conclusions. In a later question, the students were asked if a building with

any number of apartments would be possible using the three floor plans. After a

preliminary analysis, the students would then create a matrix with the three floor-plan

vectors and then row-reduce the matrix to determine linear dependence or

independence. In each case, students first created vector equations to model the

problem and then utilized the system to determine the values of the scalars or the

linear independence or dependence of the set of vectors.

The code for switching between vector equations and systems of equation

described the fluidity with which students moved between vector equations and

systems of equations and the reasons for why they made that transition. Students

switched from vector equations to systems of equations when they needed to find

scalars for a particular linear combination or when they needed to determine

definitively if a set of vectors was linearly dependent or independent. On the other

hand, vector equations were preferred when students needed expressions that were

easily relatable to problem scenarios and needed to group information into easily

identifiable units. As well, vector equations allowed for easy reference for the

meaning of scalars and how scalars operated within a linear combination.

Visualizing using graphical expressions

The second code: visualizing using graphical representations was generated

from utterances drawn from the focus group interviews from the beginning-of-

semester interviews. Visualizing using graphical representations expresses how the

usefulness or lack of usefulness that students found in using graphical representations.

Students stated that while graphical representations were valuable in visualizing the

solutions to linear combinations or concepts like span, whereas they were not useful

in finding solutions to vector equations or in generating a resultant vector from a

linear combination of two vectors. They also expressed that the graphical

representation was not useful in situations that were in more than three dimensions.

This code defines the ways that the students viewed the use of graphs and how they

related to their understanding of linear algebra. In this case, I included the caveat that

the forms could aid or hinder because students in these groups expressed ambivalence

about graphs and their usefulness.

In the first focus group, the students were split in how they saw the usefulness

of graphic representations. Nowhere in the transcript is it stated that this split is

based upon their work inside of the class. Instead, Nico and Norton both expressed a

desire to consider their work graphically because that was how they chose to

approach problems in general, not just linear algebra problems. Conversely, Nancy

expressed a general dislike of graphing. Carol felt that the use of the graphical

approach was helpful when it allowed her to "visualize what I am doing." The

students in this focus group maintained those positions even as they discussed the

usefulness of the representations. Members of the second group saw specific uses

for graphical ways of interpreting linear algebra problems. For example, Lance

discussed the different kinds of usefulness that he saw from using different symbolic

representations.

> I would say this is [systems of equations] useful for quickly calculating
> something. This [vector equation] is more visually easier to understand,
> so that's useful in that aspect. And as far as graphically goes, I'm sure
> you wouldn't want to calculate anything using graphs, that would be far
> too time-consuming.

Graphs were cited as being inefficient for doing computation, finding linear

combinations, and solving systems by members of the focus groups. However,

students frequently stated that they did see using the graphs as visualization tools.

Students decided that graphical representations were useful in checking a solution

based upon if the two sides of the vector equation matched.

Participants in the third end-of-semester focus group also found that the

graphical representations were useful in dealing with theoretical ideas like span where

they had to determine if a pair of vectors would allow them to "get everywhere" in $\mathbf{R^2}$,

a phrase that was directly related to whether or not a set of vectors spanned from their

work in the class. A possible reason for this may have had to do with when the focus

group occurred in relation to the first 6 days of the class. During the Gauss's cabin

tasks, graphical representations were the primary tool for working with and expressing

span. In whole class, as was demonstrated in Chapter 4, the span of a group of non-

scalar multiples was generally dealt with in regard to the directions of the vectors and whether those directions were different. The concept of direction, although it did have meanings that related to the algebraic expressions for vectors and vector equations, was primarily graphical in its conception and frequent use. This preference for the graphical representation when dealing with span was different from responses from the end-of-semester focus groups, when students interpreted row-reduced matrices to determine span. However, even in these focus groups, the language that you could "get everywhere" using the three vectors in the Apartments-R-Us task was still used to determine if any number of floor plans was possible indicating at least some residue of graphical ways of dealing with span.

Transforming real-world problems into symbolic expressions

The third code: Transforming real-world problems into vectors and vector equations, demonstrated how students draw conclusions about the relationship between objects in the situation was drawn from utterances produced in the two end-of-semester interviews. This code was separated into two sub-codes associated with it that demonstrated the different ways that students created information using vectors and vector equations and then used the symbolic expressions to inform their decisions about the real-life situation presented in the Apartments-R-Us scenario. After the creation of this code, I broke the code into two sub-codes which distinguished between the creation of vectors and vector equations when doing real-world scenarios and interpreting between the task situation and the algebraic expressions for vectors and vector equations.

Utterances coded with the first sub-code: creating vector equations when working with real world scenarios expressed students' ability to generate conclusions after they had created vector equations or their inability to generate conclusions if the vector equations had not been constructed. In these focus group interviews, students were asked to draw conclusions about a fictional set of apartment floor plans. Students were given the leeway to develop any symbolic representation that they felt would help them best answer the questions that were presented to them. Lance, Abraham, Nico, Norton, and Karl created vector equations to describe the problem scenario. George did not develop a recognized symbolic expression. Carol developed a hybrid between a system of equations and the vector equation. And Nancy developed a system to deal with the problem. When students were then asked if they could get any number of rooms from the three floor plans given, the five students who symbolized vector equations expressed that they felt that they could, based upon their belief that the vectors were linearly independent. The other three students did not express this belief initially, and instead waited for the other focus group members to come to conclusions before expressing their agreement or disagreement. This appeal to linear independence and dependence arose in each case from an examination of the vectors in the vector equation. In this case, the students demonstrated the value in transforming the situation into a vector equation for drawing new conclusions.

Utterances coded with the second sub-code: distinguishing between the uses of symbolic expressions for theoretical mathematics and the uses of them for modeled, realistic scenarios were drawn from the same set of questions. These utterances demonstrated student's ability to draw conclusions about vectors and sets of vectors

and then compare those conclusions to the real-world scenario. In the course of answering these questions, students spent a significant amount of time discussing the relationship between the real-world scenario and the theorems and definitions of linear algebra. Utterances were coded with this code when participants decided that a particular conclusion that they made based upon their understanding of linear algebra was or was not valid based upon their understanding of the real world scenario. Conversely, utterances could also have been coded in a similar way if students had determined that their understanding of the real-world scenario needed amending based upon their understanding of the mathematics. However, no students made utterances to this effect.

In the course of this activity, students expressed how using linear algebra to determine if "any number of 1,2, and 3 bedroom apartments could be constructed using the given floor plans" was problematic from the standpoint of the real world. While the vectors that represented each of the floor plans were linearly independent, and so any number of floors could algebraically be determined, the scalars that expressed how many of each floor plan was necessary to build that number of apartments did not make sense from a real world standpoint. Nico expressed his problem with answering the question that was presented to him. Abraham, one of the other members of the focus group, showed him that the matrix that was created from using the 3 vectors that represent the different floor plans row reduces to the identity matrix. Theoretically, this row reduction demonstrated that any solution can be found to the system of equations represented by the three floor plans. Nico challenged

Abraham's conclusion, but not by invalidating Abraham's justification, but rather by bringing in a differing viewpoint that dealt with the reality of the problem situation:

> The identity matrix, this matrix spans this point, but in real situation, we can't subtract a floor. So just because the identity matrix tells us we can span it, this linear independent spans everything, in a real situation, it's not going to work.

Nico expressed that row reduction to the identity did in fact indicate that the three vectors span all of R^3 because they row-reduce to the identity, however, the scalars that would express the number of floor plans does not make sense because you "can't subtract a floor." Other members of this focus group and other focus groups also stated that fractional numbers of floors were also invalid and hence could not contribute to a particular linear combination. For Nico, and each of the other members of the focus group, the scalars had to be positive whole numbers, because each of the components of the vectors that represented the floors was made up of whole numbers. So while the mathematics allowed for any linear combination and hence spanned $\mathbf{R^3}$, the existence of an actual solution for a particular number of apartments was problematic.

Nico's appeal to the "real-world," in the statement above was consistent across all of the focus groups. The members of these focus groups expressed this problematic situation as the relationship between the "real world" in which the apartments existed, and the theoretical world in which the vector equations, matrices and systems of equations existed. Karl called this world the R^3 or the idealistic world:

> Idealistically, assuming that you could use, I can't say for sure, but assuming you could use negatives in an idealistic world, or

> in a R^3 type of world, you could get anywhere, you could get
> any number of floors with those numbers. But what if you
> wanted 5 of 1 bedrooms and 4 of 2 bedrooms, with this set up
> you couldn't get there. Because with that specific combination
> of that floor plan plus that floor plan, there is going to be a gray
> area. But if you could have a 3rd of a floor or half of a floor,
> then you could get there. But for the example, I don't think,
> there are definitely some gray areas, but theoretically, if it
> wasn't specifically floor plans and apartment units, it was just
> numbers, heck yeah.

One interpretation of Karl's statement was actually an alienation of the mathematical

and theoretical world and the real world in which the problem situation lives. Karl's

language did not suggest that this is the case. Instead, his statement that you could

"get anywhere" (a reference to the Gauss's Cabin activity) if you could use negatives

or fractions acknowledged the theoretical linear algebra, while his acknowledgement

that that interpretation would only be appropriate if you could have "half a floor"

demonstrated that such a conclusion invalidates his first conclusion. Karl

demonstrated flexibility in his interpretation by moving seamlessly between the

theoretical conclusions that his group made and the problem of actually finding a

specific number of floors.

Conclusion

The focus group interviews illuminated a great deal about how students in this

classroom community viewed their responsibility to themselves and to their fellow

classmates. From the beginning-of-semester interviews, it was apparent that students'

previous experiences in mathematics classrooms could frequently be frustrating and

alienating as they struggled to reconcile their expectations for mathematics with the

expectations of those classes and instructors. *Frustration, resignation, and dislocation*

were each responses that students had to these classroom experiences. From student

responses, I concluded that these experiences coincided with students not feeling

responsible to themselves to engage with the concepts of the course. In addition, these

students did not feel a responsibility to the other members of the classroom, as they

frequently responded that their frustration and resignation led them to actions that

damaged themselves and other members of the class.

This can be contrasted with the productive forms of responsibility that they

related from their activity in the linear algebra class. Responsibility to themselves, to

the whole discussion, and to their small groups all were instantiated in their attempts

to test the validity of their conclusions against other students' conclusions, come to

consensus within their small groups, contribute their opinions to the small groups, and

develop conclusions and justifications that could be presented ably in whole class

discussion.

However, there were also other forms of responsibility that could be seen as

more ambiguous or even non-productive from certain perspectives. More confident or

able students would assume that the other students understood their conclusions and

not look for affirmation of the validity for their presentations in whole class discussion.

These students expressed that their conclusions and justifications were sufficient for

all members of the class. This could be potentially problematic, as students who have

significant contributions to make to the class fail to take advantage of those

opportunities and as the classroom community misses the opportunity to challenge

potentially provocative ideas that could move student understanding further forward

via discussion. In addition, some students stated that they would not listen to whole

class discussion if they had not finished their line of thinking from individual work or small group discussions. While on the one hand, this exhibited a lack of engagement in the community's activity, it also represented a larger expectation that the student would understand the material for themselves and develop their own conclusions as a paramount priority.

What students express they are responsible for as members of the classroom community are a reflection of the expectations that they have for themselves and for others. In this community, these students expressed that they were expected to provide justifications for their reasoning for a variety of purposes, including creating cohesive presentations in whole discussion, forwarding the activity of their small groups, and coming to understand the material on their own. In addition, I conclude that these students expected to do the majority of their reasoning and preliminary testing of hypotheses and conjectures during small group. During these small group sessions they also expected to present their opinions to the members of the group and come to a consensus of their groups findings. From these conclusions, I conclude that small group discussion was a highly collaborative time and that many of the students felt closeness and camaraderie with the members of their groups. This was in contrast to their expectations of work in whole class, where they reported that the majority of their individual time was spent testing the validity of their conclusions, solidifying their understanding of the material, and waiting for confirmation from the instructor that their conclusions and solutions were correct.

The analysis of student's responsibility also revealed some kinds of contribution that students made in this classroom community. The expectation that

one would voice their opinions and come to consensus in the small group community

does not just express a kind of responsibility, but also a kind of contribution that

students expected to make within the class. In addition, students were expected to

contribute to the classroom community by challenging each other's conclusions and

preparing their justifications and reasoning for the purposes of whole class discussion.

Symbolic Flexibility deals with a kind of contribution that students made or

could make as members of a classroom community. Wenger (1998) has noted that for

individuals to successfully contribute to the community that they are a part of or want

to be a part, they need to feel that they have some ability to create, interpret and use

the reified objects of the community. In the absence of this ability, individuals will

resort to rote and mechanical applications of tools in an attempt to accomplish the

tasks that they have been given. This leads to alienation from the community that they

are a part of.

In my analysis of symbolic flexibility, the students demonstrated the ability to

move flexibly between the different symbolic expressions for vectors and vector

equations, particularly those for systems of equations and the algebraic expressions for

vector equations. In addition, they had a clearly defined set of uses for graphical

expressions that aided them in visualizing theoretical concepts from the course. And

they exhibited the ability to move between the symbolic expressions for vectors and

vector equations and the realistic situations from which the expressions were created.

Each of these kinds of symbolic flexibility can aid in students abilities to communicate

with one another about linear algebra and problem situations. Consequently, when

students needed to generate conclusions using the multiple expressions for vectors and

vector equations, students displayed the ability to communicate about these objects and the situations and problems that led to their creation. This ability to communicate about, create and interpret flexibly with vectors and vector equations could expand the kinds of contribution that they can make to their small groups and whole class discussion.

The three different kinds of responsibility and symbolic flexibility also demonstrate the possible relationships between contribution and responsibility. As I already discussed earlier in the conclusion there was an overlap between the kinds of contribution and the kinds of responsibility exhibited in the classroom. Students were expected to make contributions to the whole class discussion and to their small groups, and these expectations were then exhibited in student's classroom behavior. However, these kinds of contributions may not have been possible without student's ability to create, interpret and use the symbolic expressions of the course flexibly. One possible hypothesis is that because students felt empowered to define what the symbols meant for themselves and to create justifications and conclusions that they were then able to contribute more fully. This reinforced the responsibility that they felt towards each other by giving them the tools to voice their opinions and to ascertain if their conclusions were valid or their classmates' conclusions were invalid. Without this ability students may have experienced the same kinds of frustration and resignation that they exhibited in their statements about earlier classes, as there could be a disconnect between what is expected of them as members of the classroom community and their ability to fulfill those expectations

Chapter 6 : Conclusions

This dissertation explored the ways in which meaning was generated in a classroom community of practice. This exploration took place via the answering of two research questions:

1. What are the different meanings that this classroom community develops for vectors and vector equations?

2. In what ways do individual students contribute to and take responsibility for the different meanings that this classroom community develops for vectors and vector equations

The lens that I sought to explore these questions was through Wenger's *Communities of Practice* (1998) and Lave & Wenger's *Legitimate Peripheral Participation*. These two works have generated a kind of situated perspective on learning in which learning is defined as participation and non-participation within communities of practice. From this perspective, what a student learns is a product of their activity within communities of practice that they encounter during the course of their lives. Furthermore, living is a constant process of creating meaning for the objects, activities and relationships that they interact with or become a part of as they navigate through the world.

Fundamental to this notion and to this dissertation is the process by which students create meaning for the reified objects that they participate with. In the case of this dissertation, the reified objects that I am concerned with are the symbolic

expressions for vectors and vector equations. Through Wenger's lens, the process of creating meaning is a negotiation in which members of a community engaged in a common goal develop and establish knowledge to accomplish the tasks with which they are assigned or that are a normative part of life within the community. This definition for negotiation of meaning fits well with what it means to be a part of a classroom community. The individual members of the linear algebra classroom share a common goal, learning linear algebra. The Magic Carpet Scenario and its associated tasks set forward the activities that the members of the classroom in which the members would participate. And a set of norms became established within the classroom that further defined their participation as students were required to provide arguments, present their arguments, validate their own understanding, and be responsible for how the tasks aligned with the task of learning linear algebra. It is in this environment that the process of the negotiation of meaning took place for the meaning for vectors and vector equations. As was demonstrated in my analysis, each of these factors within the community had an impact on the meaning that got developed for the symbolic expressions.

The second, but no less important aspect of the creation of meaning is who the individual is in relation to the community. Wenger labels this identity. Identity, as he defines, is in itself a negotiation of meaning, but it is not a negotiation around concepts or activities, instead it is a negotiation of self. Identity sets forward who the individual is in relation to classroom community. In this case, the classroom community is not only the people in the class, but also the objects, activities, and expectations that are a part of classroom life. Who the individual is in relation to the classroom community is

no less important than the individual's understanding and facility with the reified objects of the community. This is because the process of the negotiation of meaning is a process of dealing with reified objects via participation. Without participation, reified objects have no meaning. Identity constrains and expands participation by setting forward the expectations and normative ways of being that are at play within the community. As was evidenced in the first two beginning-of-semester focus group interviews, when students felt resignation or frustration with the activities in their previous mathematics classrooms, they curtailed their participation and this limited their activities to finding rote solutions and proofs. They did not express a desire to create justifications or to even fully understand the material that was presented.

Conversely, in the linear algebra classroom, students did express the desire to create justifications, to validate their conclusions for themselves, and to challenge the conclusions of others within the community. In addition, they expressed the desire to contribute to their small groups in various ways. These ways of participating could have potentially enhanced their understanding of the symbolic expressions for vectors and vector equations as they chose to dig more deeply into the conclusions that were put forward in class, to add to other's conclusions by bringing forward their own opinions, and to come up with new and innovative ways to work with the expressions. This freedom was reflected in the flexibility that these students demonstrated in their interpretations and understanding of the symbolic expressions for vectors and vector equations. And theoretically, the high level of flexibility in the use of these symbolic expressions could lead to greater contribution by the members of the classroom to the classroom discourse.

In Wenger's formulation of learning, these kinds of responsibility and contribution are meanings in and of themselves. To consider the knowledge that the student has garnered as a product of their own activity, to feel that they have contributed to the generation of knowledge, and that they are responsible for their own learning changes the nature of what the student has learned. It is a different dimension of meaning and it transforms the nature and level of participation that the student engages in during the course of the negotiation of meaning. In addition, the level of responsibility and contribution that the student is capable of can potentially expand the number of learning opportunities and change their relationship to the classroom community and the community of professional mathematics.

In the following sections, I detail some of the conclusions that I have drawn from my work with the negotiation of meaning for vectors and vector equations. In this section, I will summarize the conclusions that were drawn from each of my analysis chapters. I then discuss the relationship between the flexibility that students demonstrated in Chapter 5 and my analysis of classroom symbolizing practices in Chapter 4, specifically the theoretical conclusions that these chapters contribute. Next, I discuss the implications for teaching as they relate to the use of realistic scenarios and robust metaphors for developing complex algebraic, graphical and theoretical understanding in linear algebra. Finally, I detail some of the open questions regarding responsibility and negotiability that arose from my analysis and some potential opportunities for continuing research.

Summary of analysis: Chapter 4

In chapter 4, I analyzed the normative ways of reasoning that were developed in the course of the first six days of the linear algebra classroom. In particular, I was interested in the symbolic expressions for vectors and vector equations. I used Rasmussen and Stephan's (2008) approach to documenting classroom mathematics practices. In addition to their two criteria of documenting that an idea was functioning-as-if-shared, I also included Becker, et al's (2011) third criterion. Then, when all of the functioning-as-if-shared ideas were documented, I collected subsets of them into classroom symbolizing practices. These eight CSP's are summarized in the table below.

Table 6.1: Summary of Chapter Four Classroom Symbolizing Practices

Summary of Chapter Four Classroom Symbolizing Practices
CSP #1 A vector in R^2 is defined as a path in two-dimensional space.
CSP #2 Scalars define the direction and amount that a vector is stretched.
CSP #3 Setting up and solving a system of equations allows one to solve for the scalars in the vector equation.
CSP #4 Linear combinations can be used to model relationships between sets of vectors
CSP #5 Scalar multiples or vectors along the same line are linearly dependent.
CSP #6 The directions of the vectors in a set determine the span of the vectors.
CSP #7 Linear Dependence is determined when there exists a path of vectors from and back to the origin.
CSP #8 Finding a non-trivial solution to the vector equation proves that the set is linearly dependent.

The eight CSP's encapsulated student activity with graphical, algebraic, theoretical and situational expressions for vectors and vector equations that were a part of classroom activity. These CSP's summarize the mathematical meanings that students developed for the symbolic expressions for vectors and vector equations.

The first four CSP's focused on the basic symbolic expressions for vectors and vector equations. In the first CSP, vectors were initially defined as paths in two-dimensional space with both a component and holistic quality that allowed students to perform algebraic computations and create graphical representations of vectors. The second CSP defined what a scalar was in this classroom community and what occurred when scalars were multiplied by vectors. In the course of establishing the NWR's for this CSP, the classroom community developed a set of rules for multiplying by negative scalars, stretching vectors, and performing algebraic operations via scalar multiplication. The third CSP set forth the mechanism by which vector equations were transformed into systems of equations and established the validity for the use of this activity to find scalar solutions to vector equations. And the fourth CSP defined linear combinations, where the symbolic expressions for vectors, scalars, and algebraic operators like addition and the equal sign were recruited to model the relationships between sets of vectors and resultant vectors. In this CSP, normative ways of reasoning from CSP's#1-3 all played a role to develop meaning for linear combinations.

The second four CSP's established meanings for sets of vectors and concentrated primarily on student activity with span and linear dependence and independence. And although these CSP's concentrated on sets of vectors, they did evidence advances in the classroom community's participation with the fundamental elements discussed in CSP #1-4. In the fifth CSP, I detailed a set of normative ways of reasoning that established meaning for scalar multiples of vectors. Scalar multiples were a primary tool of reasoning for all of the activities that the classroom community

engaged in during the first six days of class, and they were used as a contrast in reasoning about span and linear dependence and independence when the vectors in question were not scalar multiples. The sixth CSP had to do with determining span when vectors were not scalar multiples. Although it only consisted of a single functioning-as-if-shared way of reasoning, this way of reasoning was particularly useful in its presence in multiple ways of reasoning about linear dependence and independence. The seventh and eighth CSP's both concerned classroom symbolizing activity with regard to linear dependence and independence. In CSP #7, the foundational reasoning of linear dependence as getting back home was detailed. Whereas, in CSP #8, the use of the definition to prove that a set of vectors was linearly dependent was the focus. On the one hand, CSP #7 provided the conceptual foundation for understanding linear dependence and independence and CSP #8 provided the classroom community with a method to validate a set of vectors as linearly dependent or independent.

The analysis of the normative ways of reasoning that were established in this classroom led to a series of interesting conclusions about the nature of symbolizing in a classroom community of practice. The graphical, algebraic, theoretical and situational expressions are in many ways inextricable from each other as members of the classroom community use them to make sense of linear algebra. The diversity of symbolic expressions used in each of the CSP's evidenced the role of each of them in the process of meaning making.

Summary of analysis: Chapter 5

Chapter 5 was the analysis used to answer question 2 of my dissertation. In order to examine student's responsibility for and contribution to the symbolic expressions for the symbolic expressions for vectors and vector equations, I used a grounded theory approach (Corbin & Strauss, 2007), developing a series of open codes and grouping those codes based upon a set of common themes (Lemke, 1998). These codes were generated from the three beginning-of-semester and three end-of-semester focus group interviews.

The analysis of student's kinds of responsibility and contribution was separated into two main sections. The first section detailed the analysis of students varying kinds of responsibility for the symbolic expressions for vectors and vector equations. Responsibility was defined as the expectations that students had for themselves and others as members of the classroom community. In the following table, the responsibility codes have been broken into the four themes and the codes that made up those themes.

Table 6.2: Summary of Chapter 5 Responsibility Codes

Responsibility Themes and Codes	
Alienation: How students responded to previous math classes	
1.	Resignation
2.	Frustration
3.	Dislocation
Responsibility to Oneself	
1.	Validating Conclusions
2.	Pre-Eminence

Table 6.3: Summary of Chapter 5 Responsibility Codes Continued

3.	Testing Conclusions
Responsibility to Whole Class Discussion	
1.	Preparing for Questioning
2.	Assuming Understanding
Responsibility to the Small Group Discussions	
1.	Developing Consensus
2.	Voicing One's Opinions
3.	Collaborating on Conclusions

The first theme, alienation, was generated from codes that were drawn from student's discussions about their previous math classes and expressed how their experiences in previous math classes led to a lack of responsibility to themselves and others for developing justification for their reasoning and solutions. The questions that I analyzed to generate these codes were drawn from the first two focus groups. The next three themes were drawn from student's discussions in all six of the focus group interviews and detailed the kinds of responsibility to themselves and others that students felt as members of the linear algebra classroom community.

The second section dealt with an aspect of student's contribution to the symbolic expressions for vectors and vector equations that I called symbolic flexibility. Symbolic Flexibility is defined as the ability to utilize and understand a variety of symbolic expressions across different situations and tasks. In the following table, I summarize the three codes that made up the theme of symbolic flexibility.

Table 6.4: Summary of Chapter 5 Symbolic Flexibility Codes

Symbolic Flexibility	
1.	Switching between vector equations and systems of equations
2.	Visualizing Using Graphical Representations
3.	Transforming real-world Situations into Symbolic Expressions

Each of these kinds of symbolic flexibility allowed students to work interchangeably with various kinds of symbolic expressions. The ability to move seamlessly between different symbolic expressions expanded student's ability to understand one another and arrive at new conclusions more easily. This expanded the student's ability to contribute to whole class and small group discussion.

Theoretical contributions: Flexibility and meaning making

In the course of my analysis of the beginning and end-of-semester focus groups, the flexibility in the use, creation, and interpretation of vectors and vector equations was analyzed. The construct of flexibility was developed from three different codes that were generated from the discussions of the members of the focus groups. These three codes, *switching between vector equations and systems of equations, visualizing using graphical representations, and transforming real-world situations into symbolic expressions* related how the students in the six focus group interviews related to the graphical and algebraic expressions for vectors and vector equations, as well as their relationship to systems of equations. The flexibility construct detailed some of the ways that students moved between these different kinds of expressions.

In this section, I describe the relationships that were culled between the two analyses and what these mean for teaching and learning of linear algebra. I make three major points in this discussion. The first is that switching between vector equations and systems of equations was directly related to the development of CSP #3. The second is that graphical representations may be very productive for students when creating arguments and attempting to understand complicated theoretical concepts. And finally, transforming real world situations into symbolic expressions may have been the product of a sociomathematical norm developed in the course of classroom discussion and established a relationship between understanding from a mathematical perspective and developing arguments about realistic situations.

The first of the codes developed for symbolic flexibility was established as normative in the classroom community in CSP #3 (NWR 3.1 and 3.2). The students expressed their belief that the systems of equations were useful in finding solutions for scalars in vectors and vector equations. In addition, they expressed their belief that the vectors and vector equations allowed for better organization of information and allowed them to model situations more easily. In other words, the algebraic expressions for vectors and vector equations allowed them to keep the values for the components of vectors and the relationship between vectors organized. CSP #4, *linear combinations can be used to model relationships between vectors*, established these organizing principles in the classroom community.

The second code, *visualizing using graphical representations,* expressed student's belief that graphical expressions were valuable for visualizing different concepts in linear algebra, but they didn't feel that they were valuable for finding

solutions to vector equations. This was corroborated in CSP #3 when students used systems of equations for their solutions to vectors and vector equations. But the students also ranked graphical expressions as being the expression that they found the least comfortable or useful with which to work. This particular conclusion doesn't seem to be consistent with whole class discussion. Graphical expressions and the situational language that accompanied them, like getting back home, getting to a destination, and routes or paths, were used throughout the first six days of class in all but one of the CSP's (all except for CSP #3). This attests to the fact that graphical expressions played a significant role in all of the classroom community's reasoning. A possible reason for their perception of the lack of comfort or usefulness that they saw for these representations could be that the students perceived that the most important use for a symbolic expression was for solving something, whether it be the scalars in a vector equation, a destination, or whether or not a set of vectors was linearly dependent. They may not have seen the value and usefulness in visualizing linear algebra concepts and hence did not see the impact of graphical expressions and approaches on their understanding of vectors and vector equations or other concepts in linear algebra.

This contrast also seemed to highlight an issue that arises in the literature around the use of graphical approaches for doing linear algebra. In both Dorier (2000) and Gueudet-Chartier (2004), each author contended that graphical approaches can be problematic for students in understanding of theoretical expressions for vectors and vector equations. The basis of their conclusions came from a set of interviews in which students were asked to interpret and draw conclusions from theoretical

expressions. When students attempted to use graphical expressions to answer these questions, they had difficulties drawing conclusions about the relationships between the vectors. Graphical approaches to these questions were clearly not appropriate or at least problematic. But the evidence from this analysis suggests that concluding that graphical approaches can be problematic in the teaching of linear algebra may be incorrect. There are two reasons for my contention. One, the questions came in an individual interview (in the case of Guedet-Chartier) or a written test (in the case of Dorier). Both of these individual situations do not allow for the give and take that is so prevalent in small group or whole class discussion. Individual situations rely frequently upon information in memory and students already established ways of dealing with mathematics. Hence, if students are unable to deal with problems from the graphical perspective, they would not have had access to the classroom community's collective understandings to deal with their issues. In my analysis of the normative ways of reasoning in the classroom, members of the classroom community incorporated their graphical approaches into their arguments as a way of strengthening their conclusions or justifying their reasoning in the language of the classroom discourse. When the graphical approach raised issues or proved insufficient as a justification, then the other members of the class could present counter-arguments or point out issues with the argument. The members of the classroom community could use the graphical approach as a contrast for their arguments. This would actually increase the flexibility that students would have with the symbolic expressions for vectors and vector equations, as they could then understand why the graphical approach is inappropriate and why another approach might prove more useful. The

relationship between the graphical expression and the theoretical or algebraic expression could also be emphasized further deepening student understanding of the different symbolic expressions and the concepts. The students in these studies did not have that opportunity. Given the constraints of the interview setting, though, Gueudet-Chartier and Dorier's conclusions about the viability of the graphical approach for those questions seem appropriate.

Second, the graphical approach was actually demonstrated to be beneficial for students to draw conclusions where the graphs may not have seemed appropriate. This was because the graphical approach and its attendant language became ingrained in a great deal of the classroom discourse. For example, in D6A11 (Figure 4.44) Jason argued why having more vectors in a set of vectors than dimensions meant that the set of vectors was linearly dependent. Jason's argument provided an example in R^3 that was developed using normative ways of reasoning that were predominantly graphical. He then used his theoretical understanding of span or linear dependence or independence to extend his graphical argument to the general case of R^n. Without the graphical meanings that had been established as normative in the classroom, Jason's conclusions and justifications would not have been possible. Hence, rather than concluding that the graphical approaches are problematic or that they can be a hindrance to student reasoning about theoretical concepts, it could be more appropriate to consider the ways in which graphical expressions can be leveraged to develop theoretical reasoning.

The third code for symbolic flexibility, *transforming real-world situations into symbolic expressions*, had its roots in the discussions that students had about the

Magic Carpet Scenario. The flexibility that they demonstrated between the real-world situation in the Apartments-R-Us task and the symbolic expressions for vectors and vector equations could be foreshadowed by their activity with the Magic Carpet Scenario. During classroom discussion of the Magic Carpet Scenario, the classroom community debated the meaning for the scalars, scalar multiples and the linear combinations that allowed them to "get to" particular destinations in the plane. Although it was not discussed in the course of analyzing the normative ways of reasoning there was discussion as to whether or not fractions of hours were allowed. Because there was such a thing as fractional hours, this interpretation was allowed in the Magic Carpet Scenario. In addition, in the establishment of NWR 2.3 and 2.4, the classroom community discussed the situational reasoning for negative scalars and the algebraic and graphical representations for scalars and scalar multiples of vectors. The discussions for the meanings of scalars, scalar multiples of vectors, and linear combinations allowed members of the classroom community to develop justifications for their conclusions that were mathematically consistent and valid in the particular situation. When I say mathematically consistent, I am referring to the theoretical, algebraic and graphical meanings for vectors and vector equations that are "situation-free." I acknowledge that doing the work of mathematics is a situated activity, but I would like to make a distinction between general use of vectors and vector equations and those that are tied to an applied situation. Hence a kind of socio-mathematical norm (Cobb & Yackel, 1996) was developed in which the meanings for scalars, vectors and linear combinations had to make sense from the perspective of the mathematical meanings and the situation. Without agreement on the validity of the

conclusions from the situational and mathematical perspective, the conclusions could not be considered correct. For example, students concluded that because there was no such thing as a negative floor or a fraction of a floor, then all possible numbers of apartments for a particular building was not possible. When asked about the Apartments-R-Us task in which they needed to interpret the meanings for vectors and vector equations from a situational and mathematical perspective, they operated within this norm.

In chapter 5, I contended that high levels of flexibility leads to a high level of contribution by allowing individuals the opportunity to communicate in multiple ways and to see multiple symbolic expressions as the same depending upon the situations in which the expressions were used. The connections that were demonstrated in the analysis of the whole class and the focus groups showed that this kind of flexibility is at least partially generated from the normative ways of reasoning established in the classroom community. The development of flexibility with symbolic expressions extended from participation in classroom activities. And as their flexibility developed they could contribute more to classroom discourse and make connections between new concepts and their established ways of reasoning, giving them the opportunity to develop greater flexibility with the symbolic expressions.

Implications for teaching: The traveling metaphor and classroom discourse

In chapter 4, I demonstrated that the traveling metaphor, which was generated from the classroom community's engagement with the Magic Carpet Scenario, played a significant role in the development of normative ways of reasoning for vectors and

vector equations. The language of the traveling metaphor; routes, paths, getting to a destination, traveling in a particular direction, getting anywhere in the plane, or getting back home; played a key role in the establishment of the meaning for all eight of the classroom symbolizing practices. Even as students argued about symbolic expressions and concepts that were not directly tied to the scenario, they continued to use language related to the magic carpet ride scenario.

Early in the first six days, the members of the classroom community used gestures and graphs that demonstrated traveling or moving along vectors in different directions extensively. In the analysis in chapter 4, this use was demonstrated most clearly in the arguments demonstrated in CSP #1-4. In understanding the meanings for vectors, scalars, linear combinations and span (all which took place during the first three days of class), gestures and graphical displays were featured prominently. However, as the classroom community progressed towards discussing linear dependence and independence (Task 3) and generating sets of linearly dependent and independent vectors (Task 4), the gestures and graphical expressions that were generated for discussing the Magic Carpet Scenario were not used for the majority of arguments. The language, though, remained.

The traveling metaphor's contribution to the normative ways of reasoning demonstrated that the language of the Magic Carpet Scenario had become a part of the discourse of the classroom community. The normative ways of reasoning for CSP #1-4 had gestural and graphical components that demonstrated meaning for the symbolic expressions for vectors and vector equations. But according to Wenger, the meaning for the symbolic expressions for vectors and vector equations and the participation that

led to the creation of these meanings are inextricable. The gestures and the graphs are the meanings as much as the verbal expressions. Furthermore, the language that was used for later normative ways of reasoning was consistent with the normative ways of reasoning from earlier discussions. This demonstrated that the meanings for the traveling metaphor, complete with their gestural and graphical expressions, had become a part of the classroom discourse.

The use of the Magic Carpet Scenario generated a variety of ways of thinking about the symbolic expressions for linear algebra. These meanings extended to all of the tasks and to situations that did not directly relate to the scenario. For example, in discussing the Apartments-R-Us task during the end-of-semester focus group interviews, students continued to use the language generated from the task to discuss solutions that had to do with span and linear dependence/independence. This evidenced the robustness of the traveling metaphor and its attendant meanings.

The robustness of the traveling metaphor demonstrates the value of using experientially real tasks for the development of complex mathematical concepts. For the purposes of the development of new, more effective instructional tasks, the research team developed the Magic Carpet Scenario inspired by the instructional design theory of Realistic Mathematics Education (RME). RME hypothesizes that the use of experientially real tasks, like the Magic Carpet scenario, can provide students with ways of using their own knowledge of either mathematics or their lives to develop new mathematical understanding. Student's previous experience in mathematics allowed them to relate to the graphs on the Cartesian plane. And their experience with traveling through space, which was frequently evidenced in their

gestures, allowed a second experientially real starting point for their activities. As new algebraic and theoretical symbolic expressions were created to represent these experientially real activities were created and used, students gained access to new concepts and ways of dealing with travelling, the Cartesian plane, and linear algebra. However, rather than supplanting or eliminating the need for the experientially real language generated through the traveling metaphor, the language of the traveling metaphor became incorporated into these new ways of reasoning. These ways of reasoning continued even in the absence of the gestures and graphs that created them. Furthermore, the presence of these ways of reasoning in the end-of-semester focus group interviews evidenced the robustness of these normative ways of reasoning and consequently the value of the use of the RME heuristics for instructional design.

In addition, other implications for teaching arose from the analysis. The responsibility that students feel towards themselves and other members of the classroom community has an impact on how they feel about what they learn inside of the class. In the focus group interviews, the students demonstrated that they felt responsible to validate their conclusions, test their conclusions against the conclusions of other classroom members, and to justify their reasoning. Each of these kinds of responsibility lead to behaviors that were productive in the classroom as they expanded student understanding of classroom concepts and their confidence in their ability to generate mathematical knowledge for themselves.

Further questions: Responsibility and the ownership of meaning

As was stated at the beginning of this dissertation, each of these kinds of contribution are characterizations not only of individuals, but a characterization of what it means to be a member of this community. The initial attempt to characterize identity from the perspective of individuals in a classroom community led to the development of a set of codes and themes that actually characterized the set of responsibilities of individuals within the classroom community, but did not characterize the identities of individuals within the community. Hence, what was generated was a key aspect of an individual's identity if they were to be a part of this classroom community. Hence, I could say that a member of this classroom community would be expected to have some responsibility for the justifications for their reasoning, would look for consensus within their small group, and would seek their own understanding of the material by testing their conclusions against others within the classroom. This, however, does not say whether or not the individuals in the study actually did these things, what kinds of variability was exhibited in carrying out these responsibilities and contributions, or the reasons for their engagement or non-engagement in them.

My working definition for identity, the collection of roles, responsibilities and expectations that an individual has for themselves as a product of their membership within their various communities necessarily implies that the responsibilities that I have noted in my analysis are not the sum total of the student's identity. These students are all members of various communities outside of the mathematics classroom, and their identity is shaped by their membership in these communities as well. It is possible that these memberships play a decisive role in how the students

actually relate to being members of this classroom community and their responsibilities as well. I did not look at the place of the classroom community as a part of these students overall identities. This is a limitation of my study and also an area for further research. Are there conflicting aspects of a student's identity that may actually inhibit them from actively participating fully in the classroom community? For example, Nancy's reluctance to speak in class or to offer her opinions despite her understanding that it is an expectation of the community implies that there is some kind of conflicting message or set of expectations that are not allowing her to fulfill her role within the linear algebra classroom community. One of the areas of further study would be to seek to better understand what are the forces that compel individual students, with individual identities, to choose not to engage in the tacitly or explicitly understood expectations of the community.

In examining the kinds of responsibility and contribution that were present for students within the linear algebra class, several questions arose. In the course of the analysis, the students demonstrated three primary kinds of responsibility: *responsibility to themselves, responsibility to the whole class discussion, and responsibility to the small group*. There were several connections between these three kinds of responsibility. Members of the focus groups expressed a responsibility to the whole class to develop valid justifications for presentations and to understand the material to answer questions in whole group discussion. In addition, they expressed a responsibility to the small group to express their opinions and come to consensus. The responsibility to themselves that they expressed was to validate their own conclusions and to test their conclusions against the conclusions of those of the classroom.

These kinds of responsibility seem reflexive in that the students, in describing their responsibility to themselves, expressed the desire to listen to others and consider the opinions of others. Conversely, in expressing their responsibility to the whole class discussion students frequently discussed the need for them to create justifications and to understand the ideas of the classroom fully in order to prepare presentations. This reflexive quality between the responsibility to others and the responsibility to others could potentially be a productive avenue to explore in greater depth and detail. We want students to not only validate and justify their conclusions for themselves, but theoretically this can begin as a responsibility to others. The process by which responsibility to others shifts to being a responsibility to the student themselves is one of the open questions that arose from this analysis. If a responsibility to justify our reasoning, contribute to classroom discussions, validate our conclusions, or develop consensus for our opinions can become internalized, then they can potentially extend beyond the classroom that a student is in at the moment and extend to other mathematics classrooms, eventually becoming a part of what they believe to be the actual practice of mathematics.

Wenger (1998) has posited that the construct of identity relates the individual to the community in ways that constrict and enhance participation with the reified objects of the community. Identification and negotiability make up the primary duality that Wenger employs for explaining and constructing his notion of identity. On the one hand is identification, which is the process by which we come to understand ourselves as part and not part of a community. The second aspect, negotiability, is more central to my dissertation and to the notions of responsibility and contribution.

Negotiability refers to the degree to which the member of the community has control over the meanings that get generated within a particular community of practice. Negotiability is defined using two constructs: economies of meaning and ownership of meaning. The second open question that arose from this analysis has to do with ownership of meaning. Ownership of meaning is defined as the degree to which we can make use of, affect, control, modify, or in general, assert as ours the meanings that we negotiate. The responsibility that individuals feel towards the meanings that are negotiated in the classroom community dictate ownership of meaning by establishing whether or not the individual is expected to have control of or modify or even assert their meanings in the community. If the individual does not feel responsible for these activities, then, as was evidenced by student feelings of alienation, there is the possibility that they feel a low level of the ownership of meaning in the community. Hence, it would be productive to understand to what extent did the development of these kinds of responsibility lead to an increase in the student's ownership of meaning.

Chapter 7 : Appendices

Appendix 1: Magic Carpet Ride Tasks

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 units East and 1 unit North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 unit East and 2 units North of its starting location.

SCENARIO ONE: THE MAIDEN VOYAGE

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

TASK:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

As a group, state and explain your answer(s) on paper in such a way that allows you to present it to the class using the document camera. Use the vector notation for each mode of transportation as part of your explanation. Use a diagram or graphic if it helps illustrate your point(s).

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 units East and 1 unit North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 unit East and 2 units North of its starting location.

SCENARIO TWO: HIDE-AND-SEEK

Each week Old Man Gauss moves his cabin to a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation? Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include an argument for why you are correct.

27 January 2010   Name _____

Table # _____

Suppose you are now in a three-dimensional world for the carpet ride problem, and

you have three modes of transportation: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$.

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\mathbf{v}_1$, c_2 on $\mathbf{v}_2$, c_3 on $\mathbf{v}_3$). Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home OR explain why it is not possible to do so.

After you have completed the part above, answer the questions on the back of the page:

1. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?

2. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?

3. What is $span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}\right\}$?

Create examples according to the following stipulations:

1. A linearly dependent set that contains 2 vectors in R^3

2. A linearly independent set that contains 2 vectors in R^3

3. A linearly dependent set that contains 3 vectors in R^3

4. A linearly independent set that contains 3 vectors in R^3

5. A linearly dependent set that contains 4 vectors in R^3

6. A linearly independent set that contains 4 vectors in R^3

****Are there qualitatively different ways to make sets that satisfy the same requirements? (Ex: multiple ways to answer #3?)

Appendix 2: Beginning-of-Semester Focus Group Interviews

Beginning-of-Semester Focus Group Interview
George Sweeney
January 11, 2010

1. What role do you see mathematics playing in your major or your future career?
2. Describe what being in the last math class you were in was like.
   a. If it was a good experience: What made it a good experience?
   b. If it was a bad experience: What made it a bad experience?
   c. What were you expected to do in the class?
   d. What were you expected to accomplish on tests and homework?
   e. How did you know that the answers you came up with were valid and correct?
   f. Do you expect the linear algebra class to be the same way?
3. Suppose you all have read a text in your English class and your teacher begins a discussion with the group:
   a. How comfortable would you feel getting involved in the discussion
      i. What would make you feel more comfortable?
   b. How confident would you feel in the arguments that you made about the meanings in the book?
      i. What would make you feel more confident?
4. Suppose that you are given a math assignment to do outside of your linear algebra class and the teacher begins a discussion with the classroom on what you learned and the results that you got:
   a. How comfortable would you feel getting involved in the discussion?
      i. What would make you feel more comfortable?
   b. How confident would you feel in the arguments that you made about your results?
      i. What would make you feel more confident?
5. Suppose that you are given the algebraic expression: $2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$

   a. What are some possible meanings of this expression?
      i. For each meaning, probe if necessary:
         1. How certain are you of that?
         2. What would make you more certain?
         3. Why do you feel so certain?
         4. Did your classmates say anything to sway your opinion?
         5. Did your teacher say anything to sway your opinon?
         6. Does anyone here disagree?
         7. On a scale of 1 to 5, with 1 being the least and 5 being the most, how certain are you of the validity of this meaning?

b. Which meaning do you think is the most accurate? Why?
c. Which meaning do you think is the least accurate? Why?
d. Is the system of equations: $\begin{array}{l} 2x + 3y = 9 \\ 2x + 3y = 8 \end{array}$, the same or different from the vector equation above? Why do you think that? (This question would be if the students did not come up with a system of equations on their own?

6. Suppose that you are given the picture below:
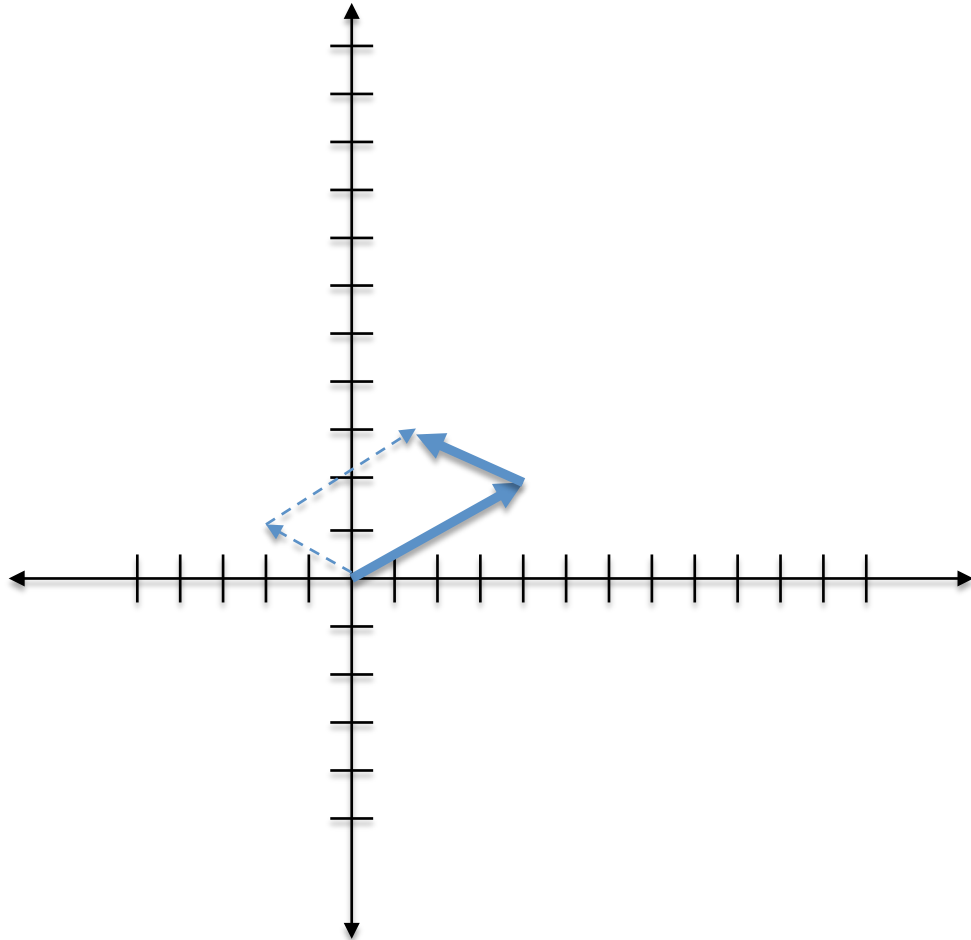
a. What are some possible meanings of this expression?
   i. For each meaning, probe if necessary:
      1. How certain are you of that?
      2. What would make you more certain?
      3. Why do you feel so certain?
      4. Did your classmates do anything to sway your opinion?
      5. Did the teacher say anything to sway your opinion?
      6. Does anyone here disagree?

7. On a scale of 1 to 5, with 1 being the least and 5 being the most, how certain are you of the validity of this meaning?

b. Which meaning do you think is the most accurate? Why?

c. Which meaning do you think is the least accurate? Why?

Appendix 3: Mid-Term Focus Group Interview

Mid-Term Focus Group Interviewc
George Sweeney
February 25, 2010

7. Explain in your own words what a vector is.
    a. Why do you think that way?
    b. Write down an example of a vector.
    c. What are the geometric ways of thinking about vectors that you use?
    d. What are the uses of vectors inside of your classroom?
    e. In your opinion, why is it that mathematicians in linear algebra use vectors and vector equations?

8. Consider the algebraic equation: $3\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + -2\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$

    a. What are some possible meanings or interpretations of this equation?
        i. For each meaning, probe if necessary:
            1. How confident are you of that interpretation as being valid?
            2. How did you come to that conclusion?
            3. Was there a particular classroom event or activity that helped you think that way?
            4. Did your classmates say anything to help you think this way?
            5. Did your teacher say anything to help you think this way?
            6. What did you use that particular meaning for?
            7. Does anyone here disagree?
            8. On a scale of 1 to 5, with 1 being the least and 5 being the most, how confident are you of the validity of this meaning?
    b. What do you see as the use for this particular kind of symbol or representation?
        i. Was there a particular activity that led you to feel that way?
        ii. What influenced your thinking in that way?
    c. Are there any ways that you see these meanings being the same?
    d. Are there any ways that you see these meanings as being different?
    e. Consider the expression:

$$x \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} + z \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix}$$

Can this expression be rewritten as:

$$x + 3y - z = -1$$
$$2x + 2y - 2z = 5$$
$$-x + 2z + 1 = -7$$

Why or why not?
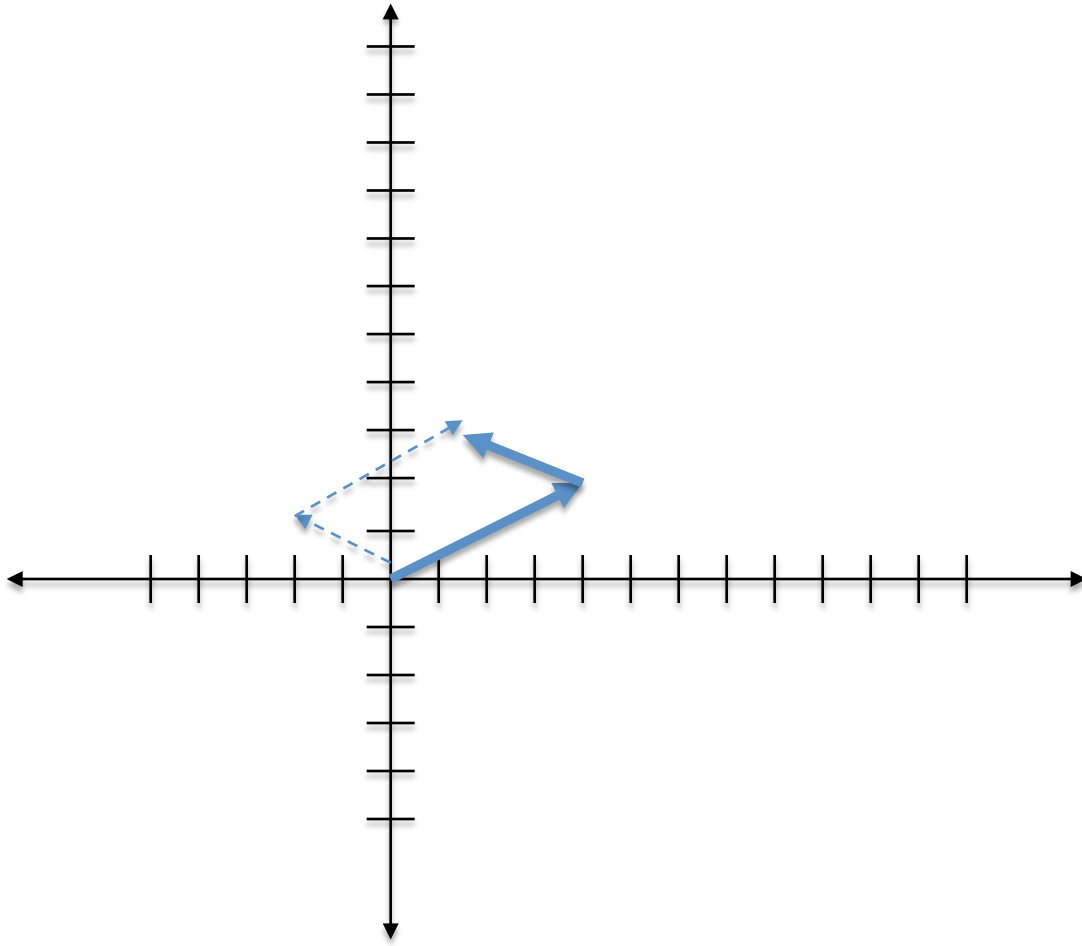-How confident are you of your conclusion?
-What would make you more confident?
(If students feel that a kind of compelling argument from a peer would make them more confident, offer the student an opportunity to give that argument.)
     -If this is done: Did that argument convince you?
     -Would anything have made it more convincing?

3. Consider the picture below:

    f. What are some possible meanings or interpretations of this equation?
        i. For each meaning, probe if necessary:
               1. How confident are you of that interpretation as being valid?
               2. How did you come to that conclusion?
               3. Was there a particular classroom event or activity that helped you think that way?
               4. Did your classmates say anything to help you think this way?
               5. Did your teacher say anything to help you think this way?
               6. What did you use that particular meaning for?
               7. Does anyone here disagree?

8. On a scale of 1 to 5, with 1 being the least and 5 being the most, how confident are you of the validity of this meaning?

g. What do you see as the use for this particular kind of symbol or representation?
   i. Was there a particular activity that led you to feel that way?
   ii. What influenced your thinking in that way?
h. Are there any ways that you see these meanings being the same?
i. Are there any ways that you see these meanings as being different?
j. Write the relationship between the two vectors pictured as a vector expression or equation?
   i. Are there any other ways you might write this relationship?

9. The definition for linear dependence is given:

   A subset S of a vector space V is called *linearly dependent* if there exists a finite number of distinct vectors v_1, v_2, ..., vn in S and scalars c_1, c_2, ..., cn, not all zero, such that $c_1 v_1 + c_2 v_2 + ... c_n v_n = 0$
   a. definition mean to you?
   b. Give me an example of a set of linearly dependent vectors?
      a. How do you know that that set is linearly dependent?
      b. How would you prove that it is linearly dependent
   c. What do you see as the value of knowing if a set is linearly dependent or not?
   d. Possible follow ups:
      a. What does the c represent?
      b. What does the v represent?
      c. When it says that they add to zero what does that mean to you?

10. Once the students are complete with these particular prompts, I will ask the following questions:
    a. For the representation from question 1:
       i. What does this representation tell you anything about the span of

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

       ii. What does this representation tell you anything about the span of

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix} \right\}$$

iii. What does this representation tell you about the linear dependence of:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

iv. What does this representation tell you about the linear dependence of:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -7 \end{bmatrix} \right\}$$

v. Follow ups:
   1. If the representation does not actually tell them anything about the properties in question.
      a. What can you tell me about the span of:
      b. What can you tell me about the linear dependence/independence of:
      c. How do you know that to be true?
      d. Was there anything said or done in class to lead you to believe that is true?

Appendix 4: End-of-Semester Focus Group Interview Protocol

Focus Group- End of Semester v3

Apartments 'R Us is a modular construction company that has been contracted to design and build a number of high-rise apartment buildings in cities across the United States. The company uses the following floor plans, where a single square indicates a 1-bedroom apartment, a 1x2 rectangle indicates a 2-bedroom apartment and a 1x3 rectangle or a 3 square L shape indicates a 3-bedroom apartment.

| Floor Plan A | Floor Plan B | Floor Plan C |
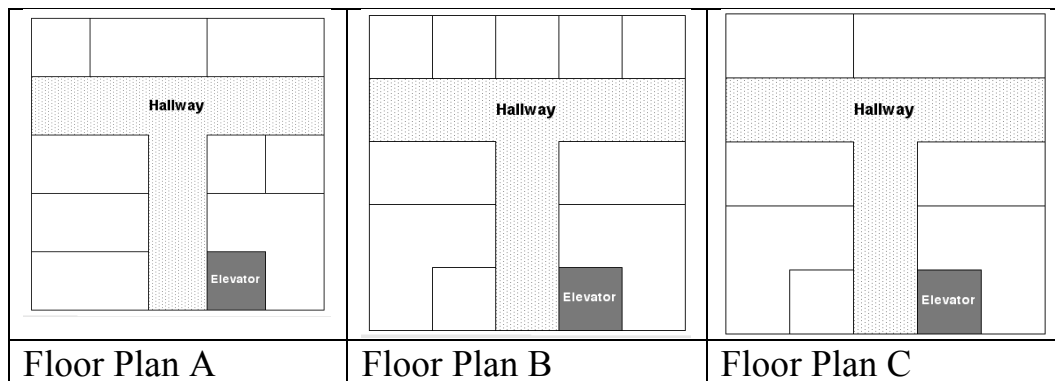
1. Suppose that you are the contractor and you need to relate how many one, two and three bedroom units there are to a realtor who is selling these apartments? In particular, how many one, two and three bedroom units would you have if a building has 6 floors of Plan A, 11 floors of Plan B, and 17 floors of Plan C?
   a. Do you think that you could get any number of one, two and three bedrooms from some group of these three floor plans?
2. What if you wanted to find the total number of one, two or three bedroom apartments for a variety of different building plans, how would you go about figuring this out?
3. Apartments 'R Us has been contracted to construct a building with 11 three-room units, 23 two-room units, and 17 one-room units. As employees of Apartments 'R Us, your team has been given the task of determining how many of each floor plan will be needed to meet this specification.
   a. Is there more than one group of floor plans that would give you the same number of one, two and three bedroom units for a building? Assume that where each floor plan will be placed in the building is not relevant.
   b. If they have not written the plan as a vector equation: I would ask them to write it as a vector equation.
   c. If they have not brought up linear independence or dependence: I would ask them if the vectors were linearly independent or dependent and how they would know.

4. A fourth floor plan which has 2 one bedrooms, 1 two bedroom and 4 three bedrooms was added to the original three floor plans. Suppose that the real estate agent wants to have 48 one bedrooms, 74 two bedrooms, and 24 three bedrooms? What set of floor plans would you have to use?
   a. If they haven't brought up linear independence or dependence:
   b. If they haven't brought up the linear dependence or independence: How does the change in linear dependence/indpendence of the set of room vectors change the nature of the floor plans that you can use?

5. The matrix A is a 4x4 matrix.

   a. What if anything can you say about A if $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ is not zero and if

   $$A\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ?$$

      i. What could you tell me about the basis made up of the column vectors?
         1. How did you know?
         2. What is a basis?
      ii. What could you tell me about the column space of A?
         1. What is the column space?
      iii. What could you tell me about the null space of A?
         1. What is the null space?

   b. Suppose that A was invertible, what can you say about the matrix

      equation: $A\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} ?$

      i. How do you know?

      ii. What can you say about $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} ?$

         1. How did you figure that out?

c. What, if anything, could you say about $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$, if I said that A $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = 4$

$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$?

d. Let A= $\begin{bmatrix} 3 & 6 & 1 & 2 \\ 5 & 2 & 3 & 4 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

   i.

      What is the column space of A?

   ii.

      What is the null space of A?

   iii.   A $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, what could you say about

      Suppose again that $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$?

         1. Why?

Use the following key to answer the questions below. All questions refer to your experience in this class.

6	5	4	3	2	1
Strongly Agree	Agree	Mildly Agree	Mildly Disagree	Disagree	Strongly Disagree

1. There were opportunities in whole class for me to contribute to the discussions about the mathematics that we learning.

    6   5   4   3   2   1

    a. Why did you respond the way that you responded?
    b. Were there some times when you felt comfortable contributing and others when you did not? When were the times that you felt comfortable contributing? When were the times when you were not comfortable contributing?

2. The discussions that we had in whole class contributed to my understanding of the mathematics.

    6   5   4   3   2   1

    a. In what ways did these discussions contribute to your understanding of the mathematics?
    b. In what ways did these discussions detract from your understanding of the mathematics?

3. In class I felt responsible for helping other students understand the material.

    6   5   4   3   2   1

    a. Did you feel (or not feel) this responsibility both in small group and in whole class discussions?
    b. In what ways did you contribute to other class members understanding in small gourps? In whole class discussions?
    c. In what ways did others contribute to your understanding of the mathematics? In small groups? In whole class discussions?

4. One of the benefits of trying to understand how other students thought was that it furthered my own understanding.

    6   5   4   3   2   1

  a. Why did you give the score that you gave?
  b. Did you find small group discussions more helpful than whole class
   discussions? Why is this the case?
    a. Do you feel that the other members of the class helped you to
     understand more about linear algebra.

5. The task, discussions and teaching style were effective in helping me learn
 linear algebra.

    6   5   4   3   2   1

  a. What about the tasks helped you to learn the material more
   effectively?
  b. What about small group discussions helped you learn the material?
  c. What about whole class discussions helped you learn the material
  d. What would have made the experience more effective?

Appendix 5: End-Of-Semester Focus Group Survey Tool

Use the following key to answer the questions below. All questions refer to your experience in this class.

6	5	4	3	2	1
Strongly Agree	Agree	Mildly Agree	Mildly Disagree	Disagree	Strongly Disagree

6. There were opportunities in whole class for me to contribute to the discussions about the mathematics that we learning.

   6   5   4   3   2   1

7. The discussions that we had in whole class contributed to my understanding of the mathematics.

   6   5   4   3   2   1

8. In class I felt responsible for helping other students understand the material.

   6   5   4   3   2   1

9. One of the benefits of trying to understand how other students thought was that it furthered my own understanding.

   6   5   4   3   2   1

10. The task, discussions and teaching style were effective in helping me learn linear algebra.

    6   5   4   3   2   1

Appendix 6: Mathematics Background Survey

Mathematics Background Survey
January 20, 2010

I would like to find out some information about why you are in this class and some of your background in mathematics classes. Please spend some time considering your answers to these questions. There are no right or wrong answers, and any thoughtful effort will constitute full credit for this assignment. Please type your answers, double-spaced.

1. What is your major?
2. What are the previous mathematics classes that you took?
3. What are the previous classes that you have taken where math played a significant role?
4. Why are you taking this math class? Is this class required by your major?
5. Are you planning on taking more mathematics classes after this semester?
6. In a paragraph, briefly describe your favorite experience inside of a math class. Be sure to include what made it your favorite experience.
7. In a paragraph, briefly describe your least favorite experience inside of a math class. Again, be sure to include what made it your least favorite experience.
8. Other than a grade, what do you expect to gain from your experience in this mathematics class?

Appendix 7: Summary of Arguments that Demonstrate Ideas Functioning-as-if-shared not presented in Chapter 4

NWR 2.4

> NWR 2.4 The vector notation (multiplying by a negative or positive scalar) allows for the ability to move in the positive or negative direction.

D3A9
Claim: You can move on the hoverboard or magic carpet in the south and west directions. (Aziz)
Data: The algebraic notation for multiplication of a scalar by a vector allows for multiplication by a negative scalar
It's the matrices, you can go positive or negative. (Aziz)
Warrant: You need to interpret multiplication by a negative scalar as moving in the opposite direction for a positive amount of time.
But if you go negative in that direction, you're still going positive time but negative in that direction. (Aziz)

D4A21 Argument

Claim: The set $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ is a linearly dependent set of vectors. (Brad)

Data: Setting the scalar in front of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to 4, $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ to -1 and $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ to zero results in the zero-vector.
Can we just use 0 for w_2 and then say 4 for w_1, -1 for w_3? (Brad)

Warrant: Multiplying $\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$ by a scalar other than zero will not result in the zero vector, unless you use the vector twice by multiplying it by a scalar and then multiplying it again by the negative of that scalar.
Because as long you don't use w_2, you're going to make it back. But as soon as you use w_2, you're done, you can't get back. Unless you use w_2 twice, positive and negative.
(Brad)

NWR 5.3

NWR 5.3 Each vector has a pre-determined direction that tells where you can get to with that vector.

D3A15
Claim: The direction of the vector dictates where you are can go with the vector.
It's still going going to get from point a to point b, no matter what course it goes. (Norton)
Data: Scalar multiplication does not allow for movement off of the line of vectors.
You think of it as a train, on a train track. (Norton)
Warrant: The definition of the vector allows for only a predetermined direction. Rotation changes how the vector is defined in terms of its components.
It has a predetermined direction that it's going to go, how it's going to get there. And you can't rotate it. (Norton)

D3A22
Claim: We can get anywhere using the two vectors. (Jason)
Data: "It's 1,2, start with 1,2. And then if we set it up as lines to negative infinity to infinity, it covers the height along that line. And then using our other vector which is different angles, 3,1 vector, we had set it to infinity and negative infinity." (Jason)
Warrant: "But then choose initial conditions for when we change modes of transportation," (Jason)
[Instructor writes on the board as Jason talks]

NWR 5.4

NWR 5.4 No matter what scalar you multiply by, the vector will always stay along the same line.

Claim: A vector can change magnitude, but it will always stay along the same line of vectors.
If you multiply it by a scalar, it might change its magnitude, or you multiply it by a different scalar, i might change the direction it's pointing, but it's still along that line. Like a train track. (Instructor)
Data: The vector is defined as having a particular direction and a particular magnitude.
Part of the vector is specified in its direction and its magnitude. (Instructor)
Warrant: As a class, we stated that a vector was defined as having a specified set of components that dictate its magnitude and direction.
It obeys this rule, this is the direction that it goes. (Instructor)

Claim: You can get anywhere in the plane using the vectors, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
We can get anywhere using the two vectors. (Jason)
Data: The line of vectors for both $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ extends from positive infinity to negative infinity along that line of vectors.

It's $\begin{bmatrix}1\\2\end{bmatrix}$ *start with* $\begin{bmatrix}1\\2\end{bmatrix}$. *And then if we set it up as lines to negative infinity to infinity, it covers the height along that line. And then using our other vector which is different angles,* $\begin{bmatrix}3\\1\end{bmatrix}$ *vector, we had set it to infinity and negative infinity.* (Jason)

Warrant: Once you have traveled out on $\begin{bmatrix}1\\2\end{bmatrix}$ for a certain amount of time, you can change directions on the vector $\begin{bmatrix}3\\1\end{bmatrix}$ to arrive at your desired destination.

But then choose initial conditions for when we change modes of transportation, (Jason)
[Instructor writes on the board while Jason talks]

NWR 5.7

NWR 5.7 Two vectors are linearly dependent if there exists a c such that $v_1 = cv_2$.

D5A18
Claim: *You write that two vectors are scalar multiples of each other by writing v1=cv2* (Karl)
Data: To set the two vectors equal to each other, one of the vectors needs to be multiplied by a scalar to get the second vector.
Since they're multiples of each other, one has got to be a multiple of the other, so a multiple of that scalar is c. (Karl)

D5A27
Claim: The set $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$ is a linearly independent set of two vectors in R^2. (Max)

Data/Warrant: There is no scalar that you can multiply $\begin{bmatrix}1\\2\end{bmatrix}$ by in order to get $\begin{bmatrix}3\\1\end{bmatrix}$

You would have to multiply, the only way to get to 0 would be to multiply both of those by 0. There's nothing you could multiply $\begin{bmatrix}1\\2\end{bmatrix}$ *and* $\begin{bmatrix}3\\1\end{bmatrix}$ *by to equal something together.* (Max)

Warrant: The instructor multiplies the components of $\begin{bmatrix}1\\2\end{bmatrix}$ by 3 and demonstrates that that multiplication does not come up with $\begin{bmatrix}3\\1\end{bmatrix}$.

We tried to come up with a c to multiply $\begin{bmatrix}1\\2\end{bmatrix}$ *by to get to* $\begin{bmatrix}3\\1\end{bmatrix}$, *you're not going to get it, right? Like* $\begin{bmatrix}3\\1\end{bmatrix}$ *times 3 will give you 3, but does 2 x 3 give you 1, can you use the same c throughout?* (Instructor)

NWR 7.3

NWR 7.3 If a vector is contained in the span of 2 other vectors, then you can get back to the origin with the first vector and the set of vectors is linearly dependent.

D6A5

Claim: If you have three vectors in R^2, then the set is linearly dependent. (Aziz)
Data/Warrant: If two vectors span R^2, then any third vector in R^2 must be within the span of the first two vectors.
If 2 vectors span R^2 if the 3rd one is contained in R^2, then you should be able to reach the origin back. (Aziz)
Warrant: You ride out on the first two vectors and then ride the third vector back home because the third vector can be written as a linear combination of the other two vectors.
Them 2 making the span makes the 3rd one be able to reach back to the origin. (Aziz)
Qualifier: If the two original vectors are scalar multiples of each other, then the set is already linearly dependent.
But if the 2 vectors are multiples of each other, then it already makes the set linearly dependent. (Aziz)

D6A11

Claim: If you have more vectors than you have dimensions for any number of dimensions than the set will be linearly dependent. (Jason)
Data for Primary Claim: R^3 example as data for the primary argument:

Claim: Four vectors in R^3 are linearly dependent.

Data: The addition of a second vector. "the next vector we add can either be on the same line, which means it's already linearly dependent,"
Warrant: "so we don't want that, so we're going to put it off somewhere else."

Data: "Now the span of that is a plane in three dimensions. So now we're going to add another vector in. Our third vector, now it can either be in that span or out of that span."
Warrant: "And we want it to be linearly independent, so we're going to put it out of that span."

Data: "But now that we have that going off of that plane, we just extended our span to all of R^3."
Warrant: So our fourth vector, when we put it in, no matter where we put it, it's going to get us back home.

Backing for the R^3 example: Because just like in this case, we have to have the last one to get back home, we can get anywhere with those first three that we put in, but we have to have to have that fourth one to come back. (Jason)

Warrant for the primary claim: And so it works like that in any dimension, because the more you, if you keep adding, eventually you're going to get the span of your dimensions, and then you're going to have that extra one bringing you back." (Jason)
Qualifier for the primary claim:
> Claim for qualifier: A set of two vectors that are already linearly dependent will not allow you to span."
> Data for the qualifier: The two vectors lie on the same line. (Jason)

NWR 7.4

> NWR 7.4 If you have more vectors than you have dimensions, then the set is linearly dependent.

D6A10
Claim: A set of vectors that spans R^2 does not necessarily have to be linearly independent.
Data: *The set of three vectors in R^2 and the set of two vectors in R^2 both span all of R^2, but the set of 3 is linearly dependent.* (Instructor)
Warrant: A set of two linearly independent vectors in R^2 will span all of R^2, and the addition of a third vector would allow you to get back home. However, if you removed the third vector, there would be no way back home.
I was just saying you can go anywhere with the 1st 2 vectors in R^2, but your way home, if you took away $\begin{bmatrix} -9 \\ 7 \end{bmatrix}$, there's no way back. So it's independent because you can go anywhere, but there's no home. (Robert)

D6A15
Claim: If you have more vectors than dimensions, then the set is *"going to come back to the origin."* (Alex)
Data: *Whenever you have the same amount of vectors as dimensions, then the set will just span.*
(Alex)

NWR 8.1

> NWR 8.1 In order to ascertain if you can go out and get back home, all of the scalars cannot equal zero.

D4A7
Claim:
To find if you can get back home you need to assume that c_1, c_2, and c_3 are not 0.
(Male Student)
Data: *If any of them are 0, then essentially the only way you can achieve this problem is by not moving at all.* (Male Student)

Warrant: *Because in order to get to where you need to be, you have to move all of them at least once.* (Male Student)

D4A11
Claim: If a set is linearly dependent, there are an infinite number of ways to get the zero-vector using a linear combination of the vectors in the set.
There are an infinite number of ways to get back home. (Jason)
Data: Multiplying a vector equation constructed from a set of linearly dependent vectors that has been set to the zero-vector will give a linear combination with a different set of scalars that will still be equal to the zero-vector.
We said you could take that equation and multiply it by a constant that's any real number, besides 0. (Jason)
Warrant:
It's just you go farther on each one, each mode of transportation, by the same constant, that's any real number, but not 0. (Jason)

NWR 8.2

NWR 8.2 A zero in front of the vector implies that the vector is not being used.

D5A4
Claim: For a set of vectors to be linearly dependent, "*at least one of the modes of transportation has to be used.*" (Brad)
Data: In the traveling metaphor, a scalar of zero implies that the vector has not moved at all.
When c is 0, I think of that vector as not being used. Brent, (9:25)
Warrant: In the definition for linear dependence, the definition states that at least one of the scalars in the linear combination must be non-zero.
The condition on c (in the definition) is saying that he has to use at least one of the modes of transportation. (Brad)
There has to be like, one of the modes there has to be a determinate amount of time that you travel on it, with at least, with only, with none of them being, or with not all of them being 0.
(Corey)

D5A7
Claim: *Linear independence means that have zero for all of the scalars this is the only way to get the zero vector.* (Instructor)
Data: A vector times a zero scalar plus another vector times a zero scalar will always result in the zero vector.
You can always add a vector times 0 to another one times 0 and that will always be zero. (Instructor)
Warrant: A linear combination set to zero will always have a set of scalars equal to zero as a solution. According to the definition of linear dependence, there has to be one solution that does not have all of the scalars equal to zero.

Linear dependence, it will have a zero solution, but it will have other ones. Having c's be 0 is always going to be true as a solution to a linear combination equaling the zero vector.
(Instructor)
Backing: For a set to be linearly independent, you must have at least one non-zero scalar in the linear combination. Linear independence can thus be expressed as the only way to get the zero-vector as a result is to not use any of the modes of transportation, or the set of scalars has to be all zeros.
Linear independence is like "*you can't use any of the modes of transportation.*"
(Karl)

NWR 8.4

NWR 8.4 For a set to be linearly independent, the only solution to $c_1v_1+c_2v_2+...=0$ is the trivial solution.

D5A27
Claim: The set $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$ is a linearly independent set of two vectors in R^2. (Max)

Data/Warrant: There is no scalar that you can multiply $\begin{bmatrix}1\\2\end{bmatrix}$ by in order to get $\begin{bmatrix}3\\1\end{bmatrix}$
You would have to multiply, the only way to get to 0 would be to multiply both of those by 0. There's nothing you could multiply $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$ by to equal something together.
(Max)

Warrant: The instructor multiplies the components of $\begin{bmatrix}1\\2\end{bmatrix}$ by 3 and demonstrates that that multiplication does not come up with $\begin{bmatrix}3\\1\end{bmatrix}$.
We tried to come up with a c to multiply $\begin{bmatrix}1\\2\end{bmatrix}$ by to get to $\begin{bmatrix}3\\1\end{bmatrix}$, you're not going to get it, right? Like $\begin{bmatrix}3\\1\end{bmatrix}$ times 3 will give you 3, but does 2 x 3 give you 1, can you use the same c throughout?
(Instructor)

D5A32
Claim: Any set with the zero vector in it is linearly dependent.
Even if zero is a vector, then the set with zero in it is linearly dependent. (Nate)
Data: The definition of linearly dependent. (Nate)
Warrant: By setting the first two scalars in the linear combination to zero and using any other scalar multiplied by the zero vector, the result of the linear combination will be the zero vector.
If one constant is not equal to 0, so we set c_1 and c_2 as 0's and when c_3 is set to anything, and it's still linear dependent set. (Nate)
Backing: *By definition, To be linearly dependent all of the scalars have to be zero.*
(Gary)

Chapter 8 :References

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