Visibility Laboratory
University of California
Scripps Institution of Oceanography
San Diego 52, California

UNIFIED IRRADIANCE EQUATIONS

R. W. Preisendorfer

22 August 1957
Index Number NS 714-100

Bureau of Ships
Contract NObs-72092

SIO REFERENCE 58-43

Approved:

Seibert Q. Duntley

Seibert Q. Duntley, Director
Visibility Laboratory

Approved for Distribution:

Roger Revelle

Roger Revelle, Director
Scripps Institution of Oceanography

Unified Irradiance Equations †

by Rudolph W. Preisendorfer

Scripps Institution of Oceanography, University of California, La Jolla, Californi

ABSTRACT

The necessary structure of the coefficient functions occurring in the Schuste equations is found in order that they be consistent with the scattering functions of general radiative transfer theory. The general procedure followed yields a bas for the unification of the manifold forms of the equations used in practice and provides an objective means for their evaluation. Necessary and sufficient condit are given in order that the Schuster equations be exact. In illustration of the theory, an extension, based on recent experimental evidence, is made of the classi equations to the case of two flows whose radiance distributions have distinct angu structure. Finally, the n-flow non steady state Schuster equations are rigorously derived from the equation of transfer for an arbitrary optical medium with sources

† Contribution from the Scripps Institution of Oceanography, New Series No.______
This paper represents results of research which has been supported by the Bureau of Ships, U. S. Navy.

INTRODUCTION

Our purpose is to derive the Schuster equations for irradiance from the equation of transfer for radiance with particular emphasis on the resultin radiometric structure of the coefficient functions in the equations and on their relations to the scattering functions of general radiative transfer theory. This procedure provides an objective means of evaluation of the various forms of the Schuster equations that have been used in practice and affords a means of their unification under one general form.

The principal results are: an exact delineation of the intrinsic structure of the Schuster equations; the necessary and sufficient conditions under which they become exact differential equations with known constant coefficients; a generalization of the classical two-flow equations, based on recent experiment evidence, such that each flow has its own distinctive fixed geometrical structur and finally, a generalization of the Schuster equations to arbitrary geometries and arbitrary numbers of flows:

It is generally agreed that the history of the Schuster equations begins with the classical paper by Schuster[1]. The differential equations derived dealt with a pair of irradiance functions representing two antiparallel flows of radiant energy in a steller atmosphere. In the hands of Schwarzschild[2], King[3],

---

[1]A. Schuster, Astrophys. J. 21, 1 (1905).

[2]K. Schwarzschild, Nachr. Akad. Wiss. Göttingen, Math.-physik.Kl. 41(1906).

[3]L. V. King, Trans. Roy. Soc. (London) A212, 375 (1913).

and Milne[4], Schuster's approach was developed into a relatively more complete description of the light field by means of the equation of transfer for radiance (specific intensity). Under Hopf[5], Ambarzumian[6], and Chandrasekhar[7], the mathematical problems of radiative transfer were subsequently crystallized into forms generally used today, such as general integral equation approaches along with the principles of invariance.

On the other hand, there followed from Schuster's work another chain of studies which dwelled almost exclusively on his original pair of equations for irradiance, reshaping them, successively generalizing them, and applying them to all manners of optical media from paint and paper to the atmosphere and the sea. The industrial researchers and the geophysicists took alternate turns in the formulations and applications, the results being typified by the papers of Channon, Renwick and Storr[8], Mecke[9], Dietzius[10], Silberstein[11], Ryde[12], and

---

[4] E. A. Milne, "Thermodynamics of the Stars," Handbuch der Astrophysik (Springer, Berlin, 1930), Vol.3, Chap.2.

[5] E. Hopf, Mathematical Problems of Radiative Equilibrium (Cambridge Tracts in Math. and Math. Physics. No. 31, University Press, Cambridge, 1934).

[6] V. A. Ambarzumian, Compt. rend. (Doklady) Acad. Sci. U.R.S.S. 38, 229 (1943).

[7] S. Chandrasekhar, Radiative Transfer (Clarendon Press, Oxford, 1950).

[8] H. J. Channon, F. F. Renwick and B. V. Storr, Proc. Roy. Soc. (London) A94, 222(191

[9] R. Mecke, Ann. Physik. 65, 257(1921).

[10] R. Dietzius Beitr. Phys. freien Atm. 10, 202(1922).

[11] L. Silberstein, Phil. Mag. 4, 129(1927).

[12] J. W. Ryde, Proc. Roy. Soc. (London) A131, 451(1931).

Duntley[13]. Concurrently certain Russian authors notably Gurevic[14], Boldyrev[15], Gershun[16], and Alexandrov[17], made lasting contributions to the Schuster theory. The latter papers are curious mixtures of the archaic forms of the equations during that period along with some brilliant innovations which only much later came into widespread use.

With the formulation of neutron diffusion problems there arose a certain amount of mutually profitable cross-fertilization of techniques between the neutron diffusion and radiative transfer theories which stems principally from the papers of Wick[18], and Chandrasekhar[19]. In these papers the Schuster equations were extended to handle n-flows with particular emphasis on the form of the coefficients most suitable to numerical analysis. Some relatively recent works based on or related to the Schuster theory are contained in the papers of Whitney

---

13 S. Q. Duntley, J. Opt. Soc. Am. 32, 61 (1942).

14 M. M. Gurevic, Trans. Opt. Inst. Leningrad. 6, No. 57, 1 (1931).

15 N. Boldyrev, Trans Opt. Inst. Leningrad, 6, No. 59. 1(1931).

16 A. Gershun, Trans. Opt. Inst. Leningrad. 11, No. 99, 43(1936).

17 N. Boldyrev and A. Alexandrov, Trans. Opt. Inst. Leningrad. 11, No. 99, 56(1931).

18 G. C. Wick, Z. Physik, 121, 702(1943).

19 S. Chandrasekhar, Astrophys. J. 100, 76(1944).

20 L. V. Whitney, J. Opt. Soc Am. 31, 714(1941).

Hulburt[21], Kubelka[22], Middleton[23], and a report by Sliepcevitch[24]. A fairly exhaustive bibliography of the Schuster theory may be compiled from the references in the preceding papers.

In view of this immense array of works on the Schuster equations, it may be felt that relatively little more of importance can be said about them. Perhaps as far as their practical ramifications are concerned this is true. Further, by being instrumental in the introduction of modern mathematical techniques into the disciplines of radiative transfer and neutron diffusion theory, it appears that the Schuster equations as ground-breaking theoretical tools may now be respectfully laid to rest. Despite these facts, the Schuster equations persistently reappear along with an occasional novel twist, and continue to remain to this day as a rough and ready tool of great practical interest. Thus the continuing use (and abuse) of the Schuster equations appears to justify a study of their intrinsic structure and the development of a means of unifying the various forms they have taken in the past, principally in the studies the industrial researchers and the geophysicists.

---

[21] E. O. Hulburt, J. Opt. Soc. Am. 33, 42, (1943).

[22] P. Kubelka, J. Opt. Soc. Am. 38, 448, (1948); 44, 330, (1954).

[23] W. E. K. Middleton, J. Opt. Soc. Am. 44, 793, (1954).

[24] C. M. Sliepcevitch and others. Confidential report (Army Chem. Corps. Contract No. DA18-108-CML-4695. AFSWP-749, ERI-2089-2-F. Eng. Res. Inst. Univ. of Mich., Ann Arbor, Michigan, (1954).

TWO-FLOW ANALYSIS FOR THE SLAB GEOMETRY

A slab of depth z_1 is a subset X of Euclidean three space E_3 defined as the set of all points between and including two planes parallel to the x-y plane and separated a distance z_1. Using the usual vector notation for E_3, (Fig. 1), a point in E_3 is denoted by a vector $\underset{\sim}{x} = (x, y, z)$, and for the present discussion X may be defined as $X = \{\underset{\sim}{x} : 0 \leq z \leq z_1\}$. The plane z = 0 is the upper boundary of X, and the plane z = z_1 is the lower boundary of X. Let Ξ denote the collection of all unit vectors $\underset{\sim}{\xi}$ in E_3. The radiance at time t at $\underset{\sim}{x}$ into the direction $\underset{\sim}{\xi}$ is denoted by $N(\underset{\sim}{x}, \underset{\sim}{\xi}, t)$. The function N and all the other functions introduced below refer to a given fixed wavelength of radiant flux. The light field in X is the vector-valued function $\underset{\sim}{H}$ defined at each point $\underset{\sim}{x}$ of X by:

$$\underset{\sim}{H}(\underset{\sim}{x}, t) = \int_{\Xi} \underset{\sim}{\xi} N(\underset{\sim}{x}, \underset{\sim}{\xi}, t)\, d\Omega(\underset{\sim}{\xi}). \tag{1}$$

$\underset{\sim}{H}(\underset{\sim}{x},t)$ is the vector irradiance at $\underset{\sim}{x}$ at the time t. The scalar irradiance $h(\underset{\sim}{x},t)$ is defined by

$$h(\underset{\sim}{x}, t) = \int_{\Xi} N(\underset{\sim}{x}, \underset{\sim}{\xi}, t)\, d\Omega(\underset{\sim}{\xi}). \tag{2}$$

Ω is the solid angle measure on Ξ $(d\Omega = \sin\theta\, d\theta\, d\phi = -d\mu\, d\phi, \mu = \cos\theta)$.

The radiance distribution at $\underset{\sim}{x}$ and t is a function on Ξ obtained from the radiance function N by fixing $\underset{\sim}{x}$ and t, and is denoted by $N(\underset{\sim}{x}, \cdot, t)$. Let $\underset{\sim}{n}$ be a unit vector, then the radiance distribution $N(\underset{\sim}{x}, \cdot, t)$ gives rise to an irradiance $H(\underset{\sim}{x}, \underset{\sim}{n}, t)$ on a unit area normal to $\underset{\sim}{n}$:

$$H(\underset{\sim}{x}, \underset{\sim}{n}, t) = \int_{\underset{\sim}{\xi} \cdot \underset{\sim}{n} \geq 0} \underset{\sim}{\xi} \cdot \underset{\sim}{n}\, N(\underset{\sim}{x}, \underset{\sim}{\xi}, t)\, d\Omega(\underset{\sim}{\xi}). \tag{3}$$

$H(\underset{\sim}{x}, \underset{\sim}{n}, t)$ is the radiant flux at time t across a unit area at $\underset{\sim}{x}$ in the direction $\underset{\sim}{n}$. $\underset{\sim}{H}(\underset{\sim}{x}, t)$ has the following property:

$$\underset{\sim}{n} \cdot \underset{\sim}{H}(\underset{\sim}{x}, t) = H(\underset{\sim}{x}, \underset{\sim}{n}, t) - H(\underset{\sim}{x}, -\underset{\sim}{n}, t). \quad (4)$$

The structure of the radiance function is governed by the equation of transfer:

$$[n^2(\underset{\sim}{x},t)/v(\underset{\sim}{x},t)]\, D[N(\underset{\sim}{x},\underset{\sim}{\xi},t)/n^2(\underset{\sim}{x},t)]/Dt = -\alpha(\underset{\sim}{x},t) N(\underset{\sim}{x},\underset{\sim}{\xi},t) + N_*(\underset{\sim}{x},\underset{\sim}{\xi},t) + N_\eta(\underset{\sim}{x},\underset{\sim}{\xi},t), \quad (5)$$

where n is the index of refraction function, v is the velocity of light function, α is the volume attenuation function, and N_* is the path function defined by

$$N_*(\underset{\sim}{x},\underset{\sim}{\xi},t) = \int_{\Xi} \sigma(\underset{\sim}{x};\underset{\sim}{\xi},\underset{\sim}{\xi}',t)\; N(\underset{\sim}{x},\underset{\sim}{\xi}',t)\, d\Omega(\underset{\sim}{\xi}'), \quad (6)$$

where σ is the volume scattering function. Finally, N_η is the emission function. The following discussion will require that n be constant on X and independent of t. In this case (5) reduces to:

$$\underset{\sim}{\xi} \cdot \nabla N(\underset{\sim}{x},\underset{\sim}{\xi},t) + (1/v)\, \partial N(\underset{\sim}{x},\underset{\sim}{\xi},t)/\partial t = -\alpha(\underset{\sim}{x},t) N(\underset{\sim}{x},\underset{\sim}{\xi},t) + N_*(\underset{\sim}{x},\underset{\sim}{\xi},t) + N_\eta(\underset{\sim}{x},\underset{\sim}{\xi},t), \quad (7)$$

where:

$$\nabla = \underset{\sim}{i}\, \partial/\partial x + \underset{\sim}{j}\, \partial/\partial y - \underset{\sim}{k}\, \partial/\partial z\,.$$

While for a great part of the present discussion it is not actually necessary to do so, we shall in the interests of brevity make the customary assumption that X is <u>stratified</u>, which means that N, α, σ (Hence N_*) and N_η depend spatially only on z. Thus (7) reduces to the relatively more familiar form:

$$-\mu\, dN(z,\mu,\phi,t)/dz + (1/v)\,\partial N(z,\mu,\phi,t)/\partial t = \\ = -\alpha(z,t)N(z,\mu,\phi,t) + N_*(z,\mu,\phi,t) + N_\eta(z,\mu,\phi,t), \tag{8}$$

where $\underset{\sim}{x}$ has been replaced by z, and $\underset{\sim}{\xi}$ by the pair (μ,ϕ), $\mu = \cos\theta$.
Fig. 1

The following definitions are necessary prerequisites to the derivation of the general Schuster equations. First, the collection of all <u>outward directions</u> is defined as $\Xi_+ = \{\underset{\sim}{\xi} : \underset{\sim}{\xi}\cdot\underset{\sim}{k} \geq 0\}$, and the collection of all <u>inward directions</u> is defined as $\Xi_- = \{\underset{\sim}{\xi} : \underset{\sim}{\xi}\cdot\underset{\sim}{k} < 0\}$. An <u>outward radiance distribution</u> is the restriction of a radiance distribution to the collection of outward directions and is denoted by $N(z,+\cdot,\cdot,t)$, so that $N(z,+\mu,\phi,t)$ is an outward radiance, $0 \leq \mu \leq 1$, $0 \leq \phi < 2\pi$. An <u>inward radiance distribution</u> is defined analogously and is denoted by $N(z,-\cdot,\cdot,t)$, so that $N(z,-\mu,\phi,t)$ is an inward radiance, $0 < \mu \leq 1$, $0 \leq \phi < 2\pi$. Irradiances associated with the special direction $\underset{\sim}{k}$ play a central role in the sequel. From (3) with now $\underset{\sim}{n} = \underset{\sim}{k}$, define:

$$H(z,+,t) \equiv H(z,\underset{\sim}{k},t), \qquad H(z,-,t) \equiv H(z,-\underset{\sim}{k},t). \tag{9}$$

These irradiances are induced by the outward and inward radiance distributions at z, at time t. The pair of functions $(H(\cdot,+,t), H(\cdot,-,t))$ is called the

Two-flow Schuster Analysis of the light field, or Analysis for short. The light field is analyzed by this pair of functions in the sense of (4):

$$\underset{\sim}{k} \cdot \underset{\sim}{H}(z,t) = H(z,+,t) - H(z,-,t) . \qquad (10)$$

The outward and inward radiance distributions also give rise to two scalar irradiances:

$$h(z,+,t) = \int_{\Xi_+} N(z,\mu,\phi,t)\, d\mu\, d\phi , \qquad (11)$$

$$h(z,-,t) = \int_{\Xi_-} N(z,\mu,\phi,t)\, d\mu\, d\phi . \qquad (12)$$

If N is replaced by N_η in (11) and (12), we have $h_\eta(,+,t)$ and $h_\eta(,-,t)$ in analogy to the functions $h(,+,t)$ and $h(,-,t)$.

Derivation of the Equations for the Analysis

The derivation of the equations for the Analysis proceeds as follows: holding z fixed, (8) is integrated over Ξ in two steps : once over Ξ_+ and once over Ξ_-. The resulting pair of equations is a conglomeration of irradiance, scalar irradiance, and radiance functions. The immediate goal is to arrive at a pair of equations explicitly involving only the members of the Analysis. An attempt to reach this goal supplies the motivation for the introduction of the so-called forward and backward scattering functions f and b and the important distribution function D.

Holding z fixed, integrate (8) over Ξ_+:

$$-dH(z,+,t)/dz + (1/v)\,\partial h(z,+,t)/\partial t = -\alpha(z,t)\,h(z,+,t) + \int_{\Xi_+} N_*(z,\mu,\phi,t)\, d\mu\, d\phi + h_\eta(z,-,t) , \qquad (13)$$

and then over Ξ_- :

$$dH(z,-,t)/dz + (1/v)\,\partial h(z,-,t)/\partial t = -\alpha(z,t)\,h(z,-,t) + \int_{\Xi_-} N_*(z,\mu,\phi,t)\,d\mu\,d\phi + h_\eta(z,-,t). \tag{14}$$

Definitions:

$$D(z,\pm,t) = h(z,\pm,t)/H(z,\pm,t). \tag{15}$$

D(,+,t) is the distribution function for the outward radiance distribution,

D(,−,t) is defined similarly.

Definitions:

$$f(z,\pm,t) = \frac{1}{H(z,\pm,t)} \int_{\Xi_\pm} \left[\int_{\Xi_\pm} \sigma(z;\mu,\phi;\mu',\phi';t)\, N(z,\mu',\phi',t')\, d\mu'\, d\phi' \right] \tag{16}$$

$$b(z,\pm,t) = \frac{1}{H(z,\pm,t)} \int_{\Xi_\mp} \left[\int_{\Xi_\pm} \sigma(z;\mu,\phi;\mu',\phi',t)\, N(z,\mu',\phi',t)\, d\mu'\, d\phi' \right] \tag{17}$$

f(,±,t) and b(,±,t) are the forward and backward scattering functions of the Analysis. Each member of the Analysis has associated with it an f and a b function. By observing that the integral for N_* can be written as the sum of two integrals: one over Ξ_+ and the other over Ξ_-, (13) and (14) can be writt in the required forms:

$$\mp dH(z,\pm,t)/dz + (1/v)\,\partial[D(z,\pm,t)\,H(z,\pm,t)]/\partial t = \tag{18}$$

$$= -D(z,\pm,t)\,\alpha(z,t)\,H(z,\pm,t) + f(z,\pm,t)\,H(z,\pm,t) + b(z,\mp,t)\,H(z,\mp,t) + h_\eta(z,\pm,t).$$

(18) is the sought-for general pair of equations for the Analysis of the light field.

The transient case has been carried along up to this point to show the generality of the present mode of derivation. With regard to the purposes of this paper, however, no essential loss of generality will be engendered if the steady state form of (18) is considered instead:

$$\mp dH(z,\pm)/dz = -D(z,\pm)\alpha(z)H(z,\pm) + f(z,\pm)H(z,\pm) + b(z,\mp)H(z,\mp) + h_\eta(z,\pm) \, . \tag{19}$$

<u>Some Properties of the Coefficient Functions</u>

From this point on, the main purpose of the discussion will be to relate (19) by successive stages to the classical Schuster equations with special emphasis on the structure of the coefficient functions. The first term of (19) suggests the

<u>Definitions</u>:

$$\alpha(z,\pm) = D(z,\pm)\alpha(z) \, . \tag{20}$$

Now the <u>total</u> (volume) <u>scattering</u> <u>function</u> s is defined as:

$$s(z) = \int_{\Xi} \sigma(z;\mu',\phi';\mu,\phi)\,d\mu\,d\phi \, , \tag{21}$$

and if a denotes the <u>volume absorption function</u>, we have from general radiative transfer theory the relation:

$$\alpha(z) = a(z) + s(z) \, . \tag{22}$$

In analogy to (20) we make the

<u>Definitions</u>:

$$a(z,\pm) = D(z,\pm)\,a(z) \, , \tag{23}$$

$$s(z,\pm) = D(z,\pm)\,s(z) \, . \tag{24}$$

From (16) and (17) it follows that

$$f(z,\pm) + b(z,\pm) = D(z,\pm)\,s(z) = s(z,\pm), \tag{25}$$

and (19) may then be written

$$\mp\, dH(z,\pm)/dz = -\left[a(z,\pm) + b(z,\pm)\right] H(z,\pm) + b(z,\mp) H(z,\mp) + h \tag{26}$$

In certain contexts, notably in hydrological and meteorological optics, it is useful to introduce into the equation of transfer the <u>equilibrium radiance</u> N_q defined as:

$$N_q(\boldsymbol{x},\boldsymbol{\xi},t) = N_*(\boldsymbol{x},\boldsymbol{\xi},t)/\alpha(\boldsymbol{x},t), \tag{27}$$

and which is analogous to the source function used in astrophysics. Thus (8) may be written:

$$-\mu\, dN/dz = \alpha(N_q - N) + N_\eta . \tag{28}$$

In the absence of any emissive sources ($N_\eta \equiv 0$) in X, N_q serves as a criterion for the test of whether N is locally increasing or decreasing along a path of length r. For if $N_q > N$, then $dN/dr > 0$ ($dz = -\mu\, dr$) and if $N_q < N$, then $dN/dr < 0$. This points up the meaning of the term <u>equilibrium radiance</u>. In a similar manner the notion of <u>equilibrium irradiance</u> H_q can be associated with each member of the Analysis:

$$H_q(z,\pm) = \frac{b(z,\mp)\, H(z,\mp)}{\left[a(z,\pm) + b(z,\pm)\right]}, \tag{29}$$

so that in analogy to (28), (26) may be written:

$$\mp\, dH(z,\pm)/dz = [a(z,\pm) + b(z,\pm)][H_{\xi}(z,\pm) - H(z,\pm)] + h_{\eta}(z,\pm), \quad (30)$$

and in a similar way we have a criterion for the local increase or decrease with depth of each member of the Analysis.

The similarity in structure between the equation of transfer (29) and the equations (30) of the Analysis only begins to lay bare the deeper lying connections which must naturally exist between the two. Even at this stage of the exposition, it is perhaps evident that the study of these connections is most profitably pursued by riveting attention on the comparatively little studied coefficient functions a, f, b, et cetera of the Analysis.

In previous studies of the system (26) the main object was, of course, to solve it and apply the results to problems of immediate interest in the particular field concerned. To attain this end the system (26), or some minor variant, was considered as a pair of differential equations with constant coefficients a, f, b, and h_{η} was assumed known or absent. As to the constancy of these coefficient functions, what conditions are necessary and sufficient that this be true? Is the requirement that σ be independent of z sufficient? Even without the help of the definitions (16) and (17) the negative answer would perhaps be easily and correctly reached. But with their help it is at once clear that a <u>sufficient condition that the forward and backward scattering functions be independent of depth is that both σ and the radiance distributions be independent of depth</u>. The radiance distributions are defined to be <u>independe</u>

of depth if $N(z, \ ,)/N(z,1,0) = N(z', \ , \ )/N(z', 1,0)$ for all z and z' in the slab. Such a condition on the radiance function implies that there is a multiplicative uncoupling of the depth and direction dependences, i.e., N is of the form $N(z,\mu,\phi) = g(z)\nu(\mu,o)$. According to (16) and (17), a radiance function with this property, along with a depth independent σ, results in depth independent forward and backward scattering functions. (A slight generalization of the preceding condition is effected if in addition to N, σ has its depth and direction dependences multiplicatively uncoupled. Then once again, after suitable modifications, the f and b functions can be made independent of depth.)

But what of the necessity of these conditions? That is, if f and b are independent of depth, is it necessarily true that N must be factorable and that σ is independent of depth? The answer, which depends upon some relatively intricate mathematical analysis, is a qualified yes (exceptions can occur only on the physically unimportant sets of z of zero measure).

The necessity and sufficiency of these conditions are extendable to the functions $a(\ ,\pm)$, $s(\ ,\pm)$ and $\alpha(\ ,\pm)$. In view of (22), (23), and (24) attention in these cases is naturally directed toward the distribution function $D(\ ,\pm)$. It turns out that in the homogeneous slab the functions $f(\ ,\pm)$, $b(\ ,\pm)$, $a(\ ,\pm)$, $s(\ ,\pm)$ and $\alpha(\ ,\pm)$ are independent of depth if and only if the radiance distributions are independent of depth, and this in turn is true if and only if the distribution functions $D(\ ,\pm)$ are independent of depth.

So far only mathematical interrelations among the coefficient functions and the optical and radiometric properties of the medium have been drawn. It remains to ask, is the major premise, namely the depth independence of radiance distributions, actually realizable in a given optical medium with the slab geom[etry?] The answer is: in general, no. However, certain numerical calculations [25, 26] and experimental results [20, 27, 28] bear evidence in favor of a limiting--or asymptotic--form of the radiance distributions in certain optically deep scatteri[ng] media. In such media these asymptotic radiance distributions are, according to some preliminary mathematical investigations, independent of the external lighting conditions and dependent only on the inherent optical properties of the media. Hence, under such circumstances, the coefficient functions would be sensibly constant below a certain depth, and the system (26) may be considered as a pair of differential equations.

The net conclusion is that the system (26) as a pair of differential equations with constant coefficients is at best a good approximation. Some recent experimental evidence [28] (summarized in Table II) has verified a particular form of (26) which yields a theory of maximal accuracy for a two-flow Analysis of the light field.

---

[25] J. Lenoble, Rev. Optique. **35**, 1(1956).

[26] J. Lenoble, Opt Acta. **4**, 1(1957).

[27] J. Lenoble, Ann. Geophysique. **12**, 16(1956)

[28] The Lake Pend Oreille experiments conducted in the Spring of 1957 by J. E. Tyler of the Visibility Laboratory of the Scripps Institution of Oceanography, La Joll[a] California. Publication of these results is planned.

ANALYSIS OF THE DECOMPOSED LIGHT FIELD

The classical Schuster equations were customarily written in terms of the diffuse flux component of the light field. This procedure will now be clarified and extended. In order to draw out the full symmetry of the following formulations, it will be assumed initially that there exist incident radiance distributions at both the upper and the lower boundaries of the slab, whose values will be designated by $N^o(0,-\mu,\phi)$ and $N^o(z_1,+\mu,\phi)$, $0<\mu \leq 1$, $0 \leq \phi < 2\pi$.

The incident radiance distributions and the emission function N_η generate the light field $\underset{\approx}{H}$ in X. Now the radiance function N from which $\underset{\approx}{H}$ is derived may be decomposed into the sum $N^o + N^*$ of two functions. These functions are such that N^o represents radiance which, relative to $N^o(0,-\ ,\ )$, $N^o(z_1,+\ ,\ )$, and $N_\eta(\ ,\ ,\ )$ has zero scattering order. N^* represents radiance which, relative to $N^o(0,-\ ,\ )$, $N^o(z_1,+\ ,\ )$, and $N_\eta(\ ,\ ,\ )$, has scattering orders one, two, and higher. The existence of these two functions follows immediately from the scattering-order decomposition of the equation of transfer:

$$\underset{\approx}{\xi} \cdot \nabla N^o + (1/v)\, \partial N^o/\partial t = -\alpha N^o + N_\eta \qquad ($$

$$\underset{\approx}{\xi} \cdot \nabla N^j + (1/v)\, \partial N^j/\partial t = -\alpha N^j + \int_{\Xi} \sigma N^{j-1}\, d\Omega\ , \quad j=1,2,\ldots \qquad ($$

in which the two incident radiance distributions and the emission function have been assigned scattering order zero. The components N^j of the radiance function N consisting of scattering order $j \geq 1$ are defined inductively by means of (32). Hence the solution N of the equation of transfer may be formally written as

$$N = \sum_{j=0}^{\infty} N^j\ , \qquad ($$

and by defining $$N^* = \sum_{j=1}^{\infty} N^j \,, \tag{34}$$

we have $$N = N^o + N^* . \tag{35}$$

This decomposition of N in turn gives rise to the decomposition H° + H* of the light field, and in general any radiometric quantity derived from or related to N. N* is referred to as the diffuse component of N, and N° as the reduced component.

For the steady state case in the slab geometry, (31) becomes

$$-\mu\, dN^o/dz = -\alpha N^o + N_\eta \,. \tag{36}$$

Summing each side of (32) over the range $1 \leq j < \infty$, we have (for the steady state case, slab geometry)

$$-\mu\, dN^*/dz = -\alpha N^* + \int_{\Xi} \sigma N^* \,d\Omega + \int_{\Xi} \sigma N^o \,d\Omega \,. \tag{37}$$

Equation (36) may be solved immediately:

$$N^o(z,-\mu,\phi) = T_r(z,-\mu,\phi)\, N^o(0,-\mu,\phi) + \int_0^r T_{r-r'}(z',-\mu,\phi)\, N_\eta(z',-\mu,\phi) \tag{38}$$

where

$$T_r(z,-\mu,\phi) = \exp\left\{-\int_0^r \alpha(z')\, dr'\right\},\ r' = z'/\mu,\ 0<\mu\leq 1 . \tag{39}$$

A similar expression exists for $N^o(z,+\mu,\phi)$.

Hence the values $N^o(z,\mu,\phi)$ of the reduced component of N are known for all depths and directions. Under the present decomposition of N, it follows that the boundary conditions for the diffuse component N* are

$$\left.\begin{array}{l} N^*(0,-\mu,\phi) = 0 \,, \\ N^*(z_1,+\mu,\phi) = 0 \,, \end{array}\right. \quad 0<\mu\leq 1,\quad 0\leq\phi<2\pi . \tag{40}$$

To solve (37), suppose for the moment that the radiance distributions at the upper and lower boundaries are collimated:

$$N^\circ(0,-\mu,\phi) = N^\circ \delta(\mu-\mu_0)\,\delta(\phi-\phi_0)\,, \qquad 0<\mu_0 \le 1\,, \quad 0 \le \phi_0 < 2\pi\,. \qquad (41)$$
$$N^\circ(z_1,+\mu,\phi) = N^\circ \delta(\mu-\mu_0)\,\delta(\phi-\phi_0)\,,$$

Then, extending a general procedure initiated by Ambarzumian[6] and developed by Chandrasekhar[7], the solution of (37) subject in turn to the boundary condtion (40) and each of the incident lighting conditions in (41), yields two pairs (R_-,T_-), $(R_+,$ of functions with the general properties:

$$N^*(0,+\mu,\phi) = (1/\mu)\int_{\Xi_+} R_-(z_1;\mu,\phi;\mu',\phi')\,N(0,-\mu',\phi')\,d\mu'\,d\phi'$$
$$+ (1/\mu)\int_{\Xi_+} T_+(z_1;\mu,\phi;\mu',\phi')\,N(z_1,+\mu',\phi')\,d\mu'\,d\phi'\,,$$

$$N^*(z_1,-\mu,\phi) = (1/\mu)\int_{\Xi_+} T_-(z_1;\mu,\phi;\mu',\phi')\,N(0,-\mu',\phi')\,d\mu'\,d\phi'$$
$$+ (1/\mu)\int_{\Xi_+} R_+(z_1;\mu,\phi;\mu',\phi')\,N(z_1,+\mu',\phi')\,d\mu'\,d\phi'\,. \qquad (4$$

The functions R_- and T_- are the diffuse reflectance and diffuse transmittance functi for radiance incident at the upper boundary of the slab. A similar designation hold for R_+ and T_+. If the slab is homogeneous (or separable, i.e., s/α is a const function) then the two pairs (R_-,T_-) and (R_+,T_+) are identical. However, in the eve of a general inhomogeneity, the pairs are distinct[29]. The functions R and T are clo akin to σ. This is illustrated by observing that the volume scattering function has the property that

$$N_*(z,\mu,\phi) = \int_{\Xi} \sigma(z;\mu,\phi;\mu',\phi')\,N(z,\mu',\phi')\,d\mu'\,d\phi'\,, \qquad (4$$

[29] Partial evidence for this may be found in the irradiance context (ref. 22). A pro that $R_- \neq R_+$ in the case of isotropic scattering may be based on the results in R.Be and R. Kalaba, Proc. Nat. Acad. Sci. 42, 629(1956). In lieu of a general direct pro the assertion, $R_- = R_+$, $T_- = T_+$, may be countered by the following example: consider t contiguous homogeneous slabs in which $\alpha \neq 0$ but $\sigma = 0$ in one and $\sigma \neq 0$ in the other.

and that $N_*(z,\mu,\phi)$ is the scattered radiance generated per unit length in the direction (μ,ϕ). Hence $N_*(z,\mu,\phi)/|\mu|$ is the corresponding radiance generated p unit depth in the slab. If the above integration is carried out explicitly ov Ξ_+ and Ξ_- then:

$$N_*(z,\mu,\phi)/\mu = (1/\mu)\int_{\Xi_+}\sigma(z;\mu,\phi;\mu',\phi')\,N(z,\mu',\phi')\,d\mu'd\phi'$$
$$+ (1/\mu)\int_{\Xi_+}\sigma(z;\mu,\phi;-\mu',\phi')\,N(z,-\mu',\phi')\,d\mu'd\phi' \qquad ($$

Since Ξ_+ and $-\Xi_-$ differ only by a set of Ω-measure zero, Ξ_+ may replace Ξ the second integral. The similarity between (45) and either one of (42) or (4 goes deeper than these superficial appearances. For example, if we define* (read upper signs together and lower signs together):

$$\sigma_+(z;\mu,\phi;\mu',\phi') \equiv \sigma(z;\pm\mu,\phi;\pm\mu',\phi') = \sigma(z;\mp\mu',\phi';\mp\mu,\phi), \qquad (4$$
$$\sigma_-(z;\mu,\phi;\mu',\phi') \equiv \sigma(z;\pm\mu,\phi;\mp\mu',\phi') = \sigma(z;\pm\mu',\phi';\mp\mu,\phi),$$
$$0\le\mu\le 1,\ 0\le\phi<2\pi;\ 0\le\mu'\le 1,\ 0\le\phi'<2\pi.$$

then the functions σ_+ and σ_- have the properties

$$\lim_{z_1\to 0} R_+(z;\mu,\phi;\mu',\phi')/z_1 = \lim_{z_1\to 0} R_-(z_1;\mu,\phi;\mu',\phi')/z_1 = \sigma_-(0;\mu,\phi;\mu';$$
$$\lim_{z_1\to 0} T_+(z_1;\mu,\phi;\mu',\phi')/z_1 = \lim_{z_1\to 0} T_-(z_1;\mu,\phi;\mu',\phi')/z_1 = \sigma_+(0;\mu,\phi;\mu',\phi'$$

(47) emphasizes the fact that R and T play the same role for a slab of finite thickness as does the volume scattering function for a slab of infinitesimal thickness. Further relations between the functions R, T, and σ_+, σ_- may be exhibited, such as the differential forms of the first four principles of inva iance, but these matters will not be pursued here.

---

*(46) summarizes the following assumed property of the medium: (i) isotropy of scattering, i.e., $\sigma(z;\boldsymbol{\xi}_1;\boldsymbol{\xi}_2) = \sigma(z;\boldsymbol{\xi}_3;\boldsymbol{\xi}_4)$ if $\boldsymbol{\xi}_1\cdot\boldsymbol{\xi}_2 = \boldsymbol{\xi}_3\cdot\boldsymbol{\xi}_4, 0 \le z \le z_1$, from which follows, (ii) reciprocity of scattering, i.e., $\sigma(z;\boldsymbol{\xi}_1;\boldsymbol{\xi}_2) = \sigma(z;-\boldsymbol{\xi}_2;-\boldsymbol{\xi}_1)$. procity of reflection processes will also be tacitly assumed for r_o, r_1 (equation (82)). Clearly, of the two, isotropy is the more restrictive. From (47) it appears that σ_- acts like a reflectance, σ_+ like a transmittance function.

To derive the Schuster equations for the decomposed light field, we begin with (37). The derivation follows the procedure outlined earlier for the case of the undecomposed light field. The only novel feature in the present derivation is the introduction of a battery of coefficient functions for each of the two components of the members of the Analysis:

$$(H^\circ(z,+)+H^*(z,+),H^\circ(z,-)+H^*(z,-)).$$

These coefficient functions are defined by the general definitions (16), (17), and are summarized in Table I. The result of the derivation is:

$$\mp\, dH^*(z,\pm)/dz = -[a^*(z,\pm)+b^*(z,\pm)]\,H^*(z,\pm) + b^*(z,\mp)\,H^*(z,\mp) + f^\circ(z,\pm)\,H^\circ(z,\pm) + b^\circ(z,\mp)\,H^\circ(z,\mp)\,. \tag{48}$$

THE CLASSICAL EQUATIONS FOR THE TWO-FLOW ANALYSIS

The classical equations associated with the two-flow Analysis as studied by Schuster, Silberstein, Ryde, Duntley, et cetera were in each study derived *de novo* for the case of the decomposed light field. The geometrical setting of the optical medium was the slab geometry; homogeneity was assumed. The boundaries were non-reflecting, the usual plan being that the equations were first solved for this case, and an interreflection study was to be taken into account subsequently if desired. The light field was generated by an incident radiance distribution at the upper boundary which was either uniform, collimated, or a combination of both. The diffuse component of the radiance distribution was invariably assumed to be uniform at all depths, thus :

$D^*(z,\pm) = 2\,, \quad 0 \leqslant z \leqslant z_1.$

Each coefficient function was therefore to be constant. Ryde gave the first detailed description of the coefficient functions, under the above incident and internal lighting conditions, and related them to the volume scattering function, but in a manner which neglected the general effect on the coefficients of the angular structure of the inward and outward radiance distributions. In the extension of Ryde's work by Duntley, some of the conditions imposed on the coefficients by Ryde were relaxed, but the basic definitions remained unaltered. For the purposes of comparison, Table I exhibits the coefficient functions of the present work with those used by Ryde and Duntley.

After applying the present definitions of the coefficient functions to the classical assumptions given above, we will compare some of the results with those found by the earlier methods. Of the incident lighting conditions discussed, the collimated radiance distribution is the most basic. Accordingly, we will assume that

$$N^\circ(0,-\mu;\phi) = N^\circ\delta(\mu-\mu_0)\,\delta(\phi-\phi_0),\ 0<\mu_0\leq 1,\ 0\leq\phi_0<2\pi.$$

TABLE I Comparison of the general coefficient functions for reduced and diffuse flux with those occuring in the works of Ryde and Duntley.

Undecomposed flux	Reduced flux		Diffuse flux			
$H(\ ,\pm)$	$H^\circ(\ ,-)$	I'_z	$H^*(\ ,+)$	s	$H^*(\ ,-)$	t
$\alpha(\ ,\pm)$	$\alpha^\circ(\ ,-)$	$\mu'+B'+F'$	$\alpha^*(\ ,+)$	$\mu+B+F$	$\alpha^*(\ ,-)$	$\mu+B+F$
$a(\ ,\pm)$	$a^\circ(\ ,-)$	μ'	$a^*(\ ,+)$	μ	$a^*(\ ,-)$	μ
$b(\ ,\pm)$	$b^\circ(\ ,-)$	B'	$B^*(\ ,+)$	B	$b^*(\ ,-)$	B
$f(\ ,\pm)$	$f^\circ(\ ,-)$	F'	$f^*(\ ,+)$	F	$f^*(\ ,-)$	F
$s(\ ,\pm)$	$s^\circ(\ ,-)$	S'	$s^*(\ ,+)$	S	$s^*(\ ,-)$	S
$D(\ ,\pm)$	$D^\circ(\ ,-)$		$D^*(\ ,+)$		$B^*(\ ,-)$	

For the reduced component of the light field:

$$N^{\circ}(z,-\mu,\phi) = N^{\circ} e^{-\alpha z/\mu_0}\,\delta(\mu-\mu_0)\,\delta(\phi-\phi_0).$$

It follows from (15) that

$$D^{\circ}(z,-) = 1/\mu_0\,, \quad 0 \le z \le z_1\,.$$

Further, from (16)

$$f^{\circ}(z,-) = (1/\mu_0)\int_{\Xi_+} \sigma_+(\mu,\phi;\mu_0,\phi_0)\,d\mu\,d\phi \equiv (1/\mu_0)\,\sigma_+(\mu_0)\,,$$

and from (17)

$$b^{\circ}(z,-) = (1/\mu_0)\int_{\Xi_+} \sigma_-(\mu,\phi;\mu_0,\phi_0)\,d\mu\,d\phi \equiv (1/\mu_0)\,\sigma_-(\mu_0)\,.$$

For the diffuse component of the light field, we have

$$D^{*}(z,\pm) = 2\,, \quad 0 \le z \le z_1\,,$$

and
$$f^{*}(z,-) = f^{*}(z,+) = (1/\pi)\int_{\Xi_+}\left[\int_{\Xi_+}\sigma_+(\mu,\phi;\mu',\phi')\,d\mu'\,d\phi'\right]$$

$$= (1/\pi)\int_{\Xi_+}\sigma_+(\mu')\,d\mu'\,d\phi' = 2\int_0^1 \sigma_+(\mu')\,d\mu' \equiv 2\,\bar{\sigma}_+\,.$$

Similarly

$$b^{*}(z,-) = b^{*}(z,+) = (1/\pi)\int_{\Xi_+}\sigma_-(\mu')\,d\mu'\,d\phi' = 2\int_0^1 \sigma_-(\mu')\,d\mu' \equiv 2\,\bar{\sigma}_-\,.$$

From the general properties of the f and b functions, or directly from above, it follows that

$$s^{\circ}(z,-) = f^{\circ}(z,-) + b^{\circ}(z,-) = (1/\mu_0)\,s\,,$$

$$s^{*}(z,\pm) = f^{*}(z,\pm) + b^{*}(z,\pm) = 2\,s\,.$$

Ryde's[12] conclusion was that B + F = B' + F', i.e. that S = S', a disagreement with the present conclusion $s^* = 2\mu_o s^\circ$, which apparently arises from different definitions of B and b, F and f. According to the present formulations, there is agreement when and only when $\mu_o = 1/2$, i.e., when $\theta_o = 60°$. If $\theta_o = 0°$, $\mu_o = 1$ and $s^* = 2s^\circ$, a conclusion correctly reached, for example, by Hulbert.[21]

In the extension of Ryde's results, Duntley[13] introduced a new constant μ' which corresponds, according to Table I, to $a^\circ(\ ,-)$. Under the present assumptions, it follows from (23) that

$$a^\circ(z,-) = (1/\mu_o)a\ , \tag{5'}$$

and

$$a^*(z,-) = 2a\ . \tag{5}$$

Duntley rightly concluded that μ and μ' differ by virtue of the difference in angular structure of the reduced and diffuse radiance distributions. However the simple relation

$$a^* = 2\mu_o a^\circ \tag{5}$$

between the two that existed by virtue of the assumed character of the light field was not given. If $\theta_o = 0°$, then $a^* = 2a^\circ$, another observa correctly made by Hulburt.[21] The preceding relations are special examples of the general relations

$$a^*(z,\pm) = [D^*(z,\pm)/D^\circ(z,\pm)]\,a^\circ(z,\pm), \tag{6}$$

$$s^*(z,\pm) = [D^*(z,\pm)/D^\circ(z,\pm)]\,s^\circ(z,\pm). \tag{6}$$

The preceding discussion was concerned with radiance distributions of restricted angular structure and a volume scattering function of arbitrary angular structure. Below we examine the consequences of reversing this situation: the angular structure of the radiance distributions will be arbitrary and in fact allowed to assume their natural forms in a medium which exhibits isotropic scattering. Then, by virtue of the general definitions,

$$\sigma = (1/4\pi)\,s \ , \tag{62}$$

and

$$f(z,\pm) = \tfrac{1}{2} D(z,\pm)\, s \ , \tag{63}$$

$$b(z,\pm) = \tfrac{1}{2} D(z,\pm)\, s \ . \tag{64}$$

Further,

$$a(z,\pm) = D(z,\pm)\, a \ , \tag{65}$$

$$s(z,\pm) = D(z,\pm)\, s \ , \tag{66}$$

so that in this case the burden of depth dependence is carried by the distribution functions. Thus (26), the general equations for the undecomposed light field take the form:

$$\mp\, dH(z,\pm)/dz = -\tfrac{1}{2}\left[2a + s\right] D(z,\pm) H(z,\pm) + \tfrac{1}{2} D(z,\mp)\, s\, H(z,\mp) + \ldots \tag{67}$$

Since D(,+) and D(,-) clearly depend upon the unknown structure of the radiance distributions, equation (67), as it stands, has unknown variable coefficients. If the usual assumption is now made that D(z,+) and D(z,-) are known constants (or that they vary in some relatively innocuous manner) then the preced ing system is solvable. By initially decomposing the light field and allowing (to take its appropriate form, such assumptions invariably lead to relatively use approximate descriptions of the Analysis of the light field. A recent paper by Kubelka[22] presents a pair of differential equations which are related in structu to (67) (with $h_\eta \equiv 0$). The derivation of the pair proceeded in the usual manner by means of conservation arguments.

It is of interest to observe that (67) is just two steps away from a steady state diffusion equation for photons. By adding the members of (67), the left side becomes the divergence $(\nabla \cdot \underset{\sim}{H})$ of the light field. Assuming for the moment t Fick's law of diffusion is valid for photons,

$$\underset{\sim}{H}(z) = -C(z)\nabla h(z),$$

where C is a diffusion function, (67) leads to

$$\nabla(C(z)\nabla h(z)) = \alpha h(z) - h_\eta .$$

If C is a constant, the more familiar form involving $C\nabla^2 h$ is obtained. By decomposing the light field, a pair of equations in h^0 and h^* is obtained. The equation involving h^0 is readily solved. Under the isotropic sca assumption, it may be shown that Fick's law holds rigorously for the diffuse com of the light field, so that in this case an exact diffusion theory discussion of the light field is possible.

THE TWO-D THEORY

The chain of successive generalizations of the two-flow theory from Schuster's original work in 1905 to Duntley's work in 1942 increased the number of optical constants used in the theory from two to six. Aside from certain academic sophistications to which the classical two-flow theory can be subjected (e.g., extension to the transient case, to more general geometries to n flows, and to the inclusion of arbitrary emission functions) there remains one final extension of some practical importance, namely the endowment of each of the two flows with a distinct geometrical structure. That is, the inward and the outward radiance distributions are assigned arbitrary but fixed shapes. Equivalently, to each member of the Analysis is assigned an arbitrary but fixed distribution factor.

This extension was made some time ago[30] but the result remained only as an idle curiosity of academic interest. However, some recent experimental work[28] on the measurement of radiance distributions in natural hydrosols has supplied some evidence in favor of the two-D hypothesis. This evidence is summarized in Table II. In the course of the experimental work, radiance distributions were measured from the surface down to depths of about 200 feet. The medium was found to be homogeneous in this depth interval. Further, the measurements were taken under a variety of incident radiance distributions varying from sunny to completely overcast skies. The presently available data was kindly put at the disposal of the author by J. E. Tyler and his staff prior to their publication of the experimental results.

---

[30] R. W. Preisendorfer, Lectures on Radiometry and Geophysical Optics, unpublished lecture notes (Visibility Laboratory, Scripps Institution of Oceanography, Fall, 1954).

Table II Experimentally Determined Distribution Functions

Clear Sunny Sky			Completely Overcast Sky		
Depth z, ft.	D(z,+)	D(z,-)	Depth z, ft	D(z,+)	D(z,-)
13	2.67	1.25	10	2.75	1.22
33	2.70	1.26	40	2.82	1.32
53	2.79	1.28	80	2.85	1.31
93	2.76	1.31	120	2.93	1.33
133	2.78	1.31	160	2.86	1.33
173	2.77	1.30			

Before embarking on the details of the two-D theory, it should be noted that a slight additional generalization can be incorporated in the present extension if one assumes that the medium is inhomogeneous in such a way that α and σ vary in t same manner with depth, so that s/α is a constant function. Such a generalizat is inessential to the structure of the resulting equations since the equations are immediately reducible to the homogeneous case, for example, by a transformation fr geometrical depth z to optical depth $\tau = \int_0^z \alpha(z')\,dz'$. On the other hand, th assumption of a general type of inhomogeneity introduces essential modifications which vitiate the customary utility of the Schuster equations arising from the presence of constant coefficient functions. For these reasons the medium is assumed homogeneous at the outset with α and σ otherwise arbitrary.

Basic Properties

We begin by agreeing that, (i) the incident radiance distribution at the upper boundary is of the form: $N^\circ(0,-\mu,\phi) = N^\circ\,\delta(\mu-\mu_0)\,\delta(\phi-\phi_0)$, $0<\mu_0$: $0 \le \phi_0 < 2\pi$, (ii) $N^\circ(z,+\ ,\ ) \equiv 0$, (iii) the upper and lower boundaries are non reflecting, (iv) $M_\eta(\;,\;,\;) \equiv 0$, (v) $D^*(z,\pm) = D^*(\pm)$, two generally different constants. If the response of the medium to a collimated incident radiance dis tion can be determined, the response of the medium to an arbitrary incident rad distribution is readily synthesized from the results developed below.

The requisite equations for the two-flow Analysis follow from (48), (50), and (51):

$$\mp\, dH^*(z,\pm)/dz = -[a^*(\pm) + b^*(\pm)]\, H^*(z,\pm) + b^*(\mp)\, H^*(z,\mp) + N^\circ e^{-\alpha z/\mu_o}\, \sigma_\mp(\mu_o), \tag{70}$$

where, in view of the homogeneity assumption, the depth dependence of the coeff functions has been dropped from the notation.

The general solution of the system (70) is readily obtained and may be expressed in the form

$$H^*(z,\pm) = m_+ g_+(\pm)\, e^{k_+ z} + m_- g_-(\pm)\, e^{k_- z} - N^\circ C(\mu_o,\pm)\, e^{-\alpha z/\mu_o}, \tag{7}$$

where m_+ and m_- are two constants (for given μ_o and z_1) which are determined by using the boundary conditions (which follow from (40)):

$$H^*(0,-) = H^*(z_1,+) = 0\,. \tag{72}$$

It follows that

$$m_\pm = N^\circ \left[\, g_\mp(-)\, C(\mu_o,+)\, e^{-\alpha z_1/\mu_o} - g_\mp(+)\, C(\mu_o,-)\, e^{k_\mp z_1} \right] / \Delta \tag{73}$$

where

$$\Delta(z_1) = g_+(+)\, g_-(-)\, e^{k_+ z} - g_+(-)\, g_-(+)\, e^{k_- z}, \tag{74}$$

and where

$$g_+(\pm) = 1 \pm \frac{a(\mp)}{k_+}, \quad g_-(\pm) = 1 \pm \frac{a(\mp)}{k_-} \tag{75}$$

The two constants k_+, k_- are obtained during the solution procedure and are defined by

$$k_\pm = \frac{1}{2}\left\{\left[a^*(+)+b^*(+)-a^*(-)-b^*(-)\right] \pm \left[\left(a^*(+)+b^*(+)+a^*(-)+b^*(-)\right)^2 - 4b^*(+)\,b^*(-)\right]^{1/2}\right\}. \tag{76}$$

These constants have the property that

$$k_+ > 0 > k_- \text{ if } a > 0; \text{ and } k_+ = k_- = 0 \text{ if } a = 0.$$

Finally,

$$C(\mu_0, \pm) = \frac{\sigma_\pm(\mu_0)\,b^*(\mp) + \sigma_\mp(\mu_0)\left[a^*(\mp) + b^*(\mp) \mp (\alpha/\mu_0)\right]}{\left(k_+ + \frac{\alpha}{\mu_0}\right)\left(k_- + \frac{\alpha}{\mu_0}\right)}. \tag{77}$$

The above expressions reduce readily to those of the one-D theory by assuming D*(+) = D*(-) = 2. It follows that $a^*(+) = a^*(-) = 2a$, $b^*(+) = b^*(-) = 2\sigma_-$, so that $k_+ = -k_- = 2[a(a+\sigma_-)]^{1/2} = k$. Further $g_+(+) = g_-(-) = 1+(2a/k)$, $g_+(-) = g_-(+) = 1-(2a/$ Under these conditions, and setting $\mu_0 = 1$, (71) reduces to the equations originally considered by Ryde if in addition the identifications $s^* \equiv s^0$, $a^* \equiv a^0$ are made.

Diffuse Reflectance and Transmittance Functions

The diffuse reflectance and transmittance functions $R(z_1; \mu_0)$ and $T(z_1; \mu_0)$ for irradiance are defined by the relations

$$N^0 R(z_1; \mu_0) = H^*(0, +), \tag{78}$$

$$N^0 T(z_1; \mu_0) = H^*(z_1, -). \tag{79}$$

Using (71), these functions are readily found:

$$R(z_1;\mu_0) = C(\mu_0,-)g_+(+)g_-(+)\left[e^{k_+z_1} - e^{k_-z_1}\right]/\Delta(z_1) + \\ + C(\mu_0,+)\left[(\Delta(0)/\Delta(z_1))e^{-\alpha z_1/\mu_0} - 1\right], \tag{80}$$

$$T(z_1,\mu_0) = C(\mu_0,+)g_+(-)g_-(-)\left[e^{k_+z_1} - e^{k_-z_1}\right]/\Delta(z_1) + \\ + C(\mu_0,-)\left[(\Delta(0)/\Delta(z_1))e^{(k_++k_-)z_1} - e^{-\alpha z_1/\mu_0}\right]. \tag{81}$$

To see $R(z_1;\mu_0)$ and $T(z_1;\mu_0)$ in their proper perspective, it is instructive to retu to the exact solutions of the standard problem as given in (42), (43). Under the present assumptions,

$$N^*(0,+\mu,\phi) = (1/\mu)\, N^\circ R(z_1;\mu,\phi;\mu_0,\phi_0),$$

$$N^*(z_1,-\mu,\phi) = (1/\mu)\, N^\circ T(z_1;\mu,\phi;\mu_0,\phi_0).$$

Since

$$H^*(0,+) = \int_{\Xi_+} N^*(0,+\mu,\phi)\,\mu\, d\mu\, d\phi,$$

$$H^*(z_1,-) = \int_{\Xi_-} N^*(z_1,-\mu,\phi)\,\mu\, d\mu\, d\phi,$$

we have from the exact theory

$$H^*(0,+) = N^\circ \int_{\Xi_+} R(z_1;\mu,\phi;\mu_0,\phi_0)\, d\mu\, d\phi,$$

$$H^*(z_1,-) = N^\circ \int_{\Xi_+} T(z_1;\mu,\phi;\mu_0,\phi_0)\, d\mu\, d\phi.$$

In the exact theory one may make the definitions

$$R(z_1;\mu_0) \equiv \int_{\Xi_+} R(z_1;\mu,\phi;\mu_0,\phi_0)\, d\mu\, d\phi,$$

$$T(z_1;\mu_0) \equiv \int_{\Xi_+} T(z_1;\mu,\phi;\mu_0,\phi_0)\, d\mu\, d\phi.$$

Hence if the two-D hypothesis were to hold exactly, then the latter functions wo be identical with those introduced in (78) and (79). In any event, the diffuse reflectance and transmittance functions introduced in (78) and (79) have the properties that, for an arbitrary incident radiance distribution,

$$H^*(0,+) = \int_{\Xi_+} R(z_1;\mu)\, N^\circ(0,-\mu,\phi)\, d\mu\, d\phi,$$

$$H^*(z_1,-) = \int_{\Xi_+} T(z_1;\mu)\, N^\circ(0,-\mu,\phi)\, d\mu\, d\phi.$$

The similarity of the R and T functions of the two-D theory with those of the exa theory is strengthened by noting that

The above sets of simultaneous equations are statements in the exact theory, are just as difficult to solve as the general equation of transfer itself. However they have been deliberately formulated so that their appearance is that of a set of simultaneous linear algebraic equations with the irradiances as unknowns, and if they are to be solved as such, the various coefficients R^*, T^*, r_0^*, and r_1^* must be assumed known. Thus, the parallel with the general two-flow equations for the Analysis is complete: in order to solve the above sets of equations as algebraic equations, it follows from (83)-(86) that some assumption must be made about the angular structure of the diffuse radiance distributions, and the reflected radiance distributions at the boundaries. As far as the diffuse radiance distributions are concerned, one may adopt a one-D or a two-D theory; and for the reflected radiance distributions, matte or specular reflecting characteristics of the boundaries are the customary concessions to complexity. If the one-D and specular assumptions are made, (88) will yield, upon solution, the correct forms of the transmittance and reflectance of the slab with reflecting boundaries which will reduce to the classical results of Ryde[12], for example, after adopting the appropriate assumptions made in each case. Though we shall not do so here, it would be of interest to apply the two-D theory to the systems (88) and (89) to complete the generalizations begun in the preceding section.

GENERALIZED SCHUSTER ANALYSIS

We now indicate briefly the generalization of the classical two-flow Analysis to geometries other than the slab geometry, and then finally the two-flow Analysis is generalized, in the spirit of the preceding sections, to n-flows.

Fig. 2

Let X now be an arbitrary subset of Euclidean three space, $\underset{\sim}{x} = (x_1, x_2, x_3)$ a point of X, and Ξ the collection of all unit vectors in E_3. In the slab geomet the vector $\underset{\sim}{k}$ was used to partition Ξ into Ξ_+ and Ξ_-. In the present case, select at each x a fixed unit vector $\underset{\sim}{n}(\underset{\sim}{x})$, or $\underset{\sim}{n}$ for short. Then at $\underset{\sim}{x}$, partition Ξ into $\Xi_+(\underset{\sim}{n}) = \{\underset{\sim}{\xi} : \underset{\sim}{\xi}\cdot\underset{\sim}{n} \geq 0\}$ and $\Xi_-(\underset{\sim}{n}) = \{\underset{\sim}{\xi} : \underset{\sim}{\xi}\cdot\underset{\sim}{n} < 0\}$ (Figure 2(a)). In analo to $\underset{\sim}{H}(\underset{\sim}{x}, t)$, define

$$\underset{\sim}{H}(\underset{\sim}{x}, \pm\underset{\sim}{n}, t) = \int_{\Xi\pm(\underset{\sim}{n})} \underset{\sim}{\xi}\, N(\underset{\sim}{x}, \underset{\sim}{\xi}, t)\, d\Omega(\underset{\sim}{\xi}), \tag{9}$$

then

$$|\underset{\sim}{n}\cdot\underset{\sim}{H}(\underset{\sim}{x}, \pm\underset{\sim}{n}, t)| = H(\underset{\sim}{x}, \pm\underset{\sim}{n}, t). \tag{9}$$

Similarly, define

$$h(\underset{\sim}{x}, \pm\underset{\sim}{n}, t) = \int_{\Xi\pm(\underset{\sim}{n})} N(\underset{\sim}{x}, \underset{\sim}{\xi}, t)\, d\Omega(\underset{\sim}{\xi}). \tag{9}$$

A corresponding definition exists for $h_\eta(\underset{\sim}{x}, \pm\underset{\sim}{n}, t)$. Holding $\underset{\sim}{x}$ fixed, (7) is now integrated over $\Xi_+(\underset{\sim}{n})$ and $\Xi_-(\underset{\sim}{n})$, which supplies the general analogy to (26):

$$\begin{aligned} &\nabla\cdot\underset{\sim}{H}(\underset{\sim}{x}, \pm\underset{\sim}{n}, t) + (1/v)\,\partial[D(\underset{\sim}{x}, \pm\underset{\sim}{n}, t)\,H(\underset{\sim}{x}, \pm\underset{\sim}{n}, t)]/\partial t = \\ &= -[a(\underset{\sim}{x}, \pm\underset{\sim}{n}, t) + b(\underset{\sim}{x}, \pm\underset{\sim}{n}, t)]\,H(\underset{\sim}{x}, \pm\underset{\sim}{n}, t) + b(\underset{\sim}{x}, \mp\underset{\sim}{n}, t)\,H(\underset{\sim}{x}, \mp\underset{\sim}{n}, t) \\ &\quad + h_\eta(\underset{\sim}{x}, \pm\underset{\sim}{n}, t). \end{aligned} \tag{9}$$

The a's and b's are defined as in (16) and (17), the integrations now being taken $\Xi_+(\underset{\sim}{n})$ and $\Xi_-(\underset{\sim}{n})$. The definition of $D(\underset{\sim}{x}, \pm\underset{\sim}{n}, t)$ parallels that in (15). If X is re presented by spherical, cylindrical, or generally some curvilinear coordinate syst the divergences $\nabla\cdot \underset{\sim}{H}(\underset{\sim}{x}, \pm\underset{\sim}{n}, t)$ take their characteristic form in that system The divergences reduce to the familiar derivatives in the slab geometry. For th generalized Schuster analysis to be most effective, one must choose the coordina system in such a way that the members of the analysis are constant over each sur of some space-filling one-parameter family of surfaces (e.g., spheres, cylinders planes, etc.).

The general n-flow equations are obtained by partitioning Ξ into n mutua exclusive subsets whose union is Ξ (Figure 2(b)). As before, let $\underset{\sim}{n}(\underset{\sim}{x})$ be som chosen unit vector at $\underset{\sim}{x}$. Then with respect to $\underset{\sim}{n}$, partition Ξ into n subsets Ξ_j, j=1,...,n, in some well-defined manner (e.g., in the slab geometry, let th partition be n equiangular concentric zones about $\underset{\sim}{n} = \underset{\sim}{k}$; if n = 2, the usual par tion is obtained). Define

$$\underset{\sim}{H}_j(\underset{\sim}{x}, \underset{\sim}{n}, t) = \int_{\Xi_j} \underset{\sim}{\xi}\, N(\underset{\sim}{x}, \underset{\sim}{\xi}, t)\, d\Omega(\underset{\sim}{\xi}), \tag{94}$$

along with

$$|\underset{\sim}{n}\cdot\underset{\sim}{H}_j(\underset{\sim}{x}, \underset{\sim}{n}, t)| = H_j(\underset{\sim}{x}, \underset{\sim}{n}, t), \tag{95}$$

which is the irradiance at time t on a unit area at $\underset{\sim}{x}$ normal to $\underset{\sim}{n}$ induced by the radiant flux in the directions Ξ_j. $(H_1, H_2, \ldots, H_n)$ is the <u>n-flow Schuster Analysis of the light field</u>, or <u>n-flow Analysis</u>, for short. Further, set

$$h_j(\underset{\sim}{x}, \underset{\sim}{n}, t) = \int_{\Xi_j} N(\underset{\sim}{x}, \underset{\sim}{\xi}, t)\, d\Omega(\underset{\sim}{\xi}), \tag{96}$$

with $h_{\eta,j}(\underset{\sim}{x}, \underset{\sim}{n}, t)$ defined analogously, and agree to define the jth distribution factor by

$$D_j(\underset{\sim}{x}, \underset{\sim}{n}, t) = h_j(\underset{\sim}{x}, \underset{\sim}{n}, t) / H_j(\underset{\sim}{x}, \underset{\sim}{n}, t). \tag{97}$$

Finally, the general counterparts to (16) and (17) must be of the form

$$s_{jk}(\underset{\sim}{x}, \underset{\sim}{n}, t) = \frac{1}{H_k(\underset{\sim}{x}, \underset{\sim}{n}, t)} \int_{\Xi_j} \left[\int_{\Xi_k} \sigma(\underset{\sim}{x}, \underset{\sim}{\xi}, \underset{\sim}{\xi}', t) N(\underset{\sim}{x}, \underset{\sim}{\xi}', t)\, d\Omega(\underset{\sim}{\xi}') \right] d\Omega(\underset{\sim}{\xi}) \quad (9$$

$$j = 1, \ldots, n, \quad k = 1, \ldots, n.,$$

and α_j, a_j, and s_j are defined analogously to (22), (23), and (24). Then holding $\underset{\sim}{x}$ fixed, integrate (7) over Ξ via the n partitions. With the above definitions, the result is reducible to

$$\nabla \cdot \underset{\sim}{H}_j + (1/v)\, \partial[D_j H_j]/\partial t = -\alpha_j H_j + \sum_{k=1}^{n} s_{jk} H_k + h_{\eta, j} \quad (9$$

$$j = 1, \ldots, n.$$

Despite the generality of the partition it is still possible to define "forward" and "backward" scattering functions f_j and b_j by adopting the following device: let $f_j = s_{jj}$, for $j = 1, \ldots, n$; and if $\sum_{k \neq j}$ denotes summation over all k from 1 to n excluding j, then let $b_j = \sum_{k \neq j} s_{kj}$. Consequently, $f_j + b_j = s_j = D_j s$. Final. set $b_{jk} \equiv s_{jk}$ for $j \neq k$. Then (99) becomes

$$\nabla \cdot \underset{\sim}{H}_j + (1/v)\, \partial[D_j H_j]/\partial t = -[a_j + b_j] H_j + \sum_{k \neq j} b_{jk} H_k + h_{\eta, j} \quad (1$$

$$j = 1, \ldots, n,$$

which establishes the final generalization. By letting $n \to \infty$ such that $\max\{\Omega(\Xi_j), j = 1, \ldots, n\} \to 0$, (100) returns to the equation of transfer (7), and the circle is complete.

CAPTIONS

Figure 1. Illustrating the slab geometry. $\underset{\sim}{k}$ denotes the basic outward direct. $-\underset{\sim}{k}$, the basic inward direction. The z-coordinate increases as one progresses into the medium from the upper boundary (the x-y plane). The origin is at 0. Ξ denotes the sphere of unit directions about the point $\underset{\sim}{x}$.

Figure 2. (a) Illustrating the partition of the unit sphere as used in the derivation of the two-flow equations for an arbitrary coordinate system.

(b) Illustrating the partition of the unit sphere as used in the derivation of the n-flow equations.

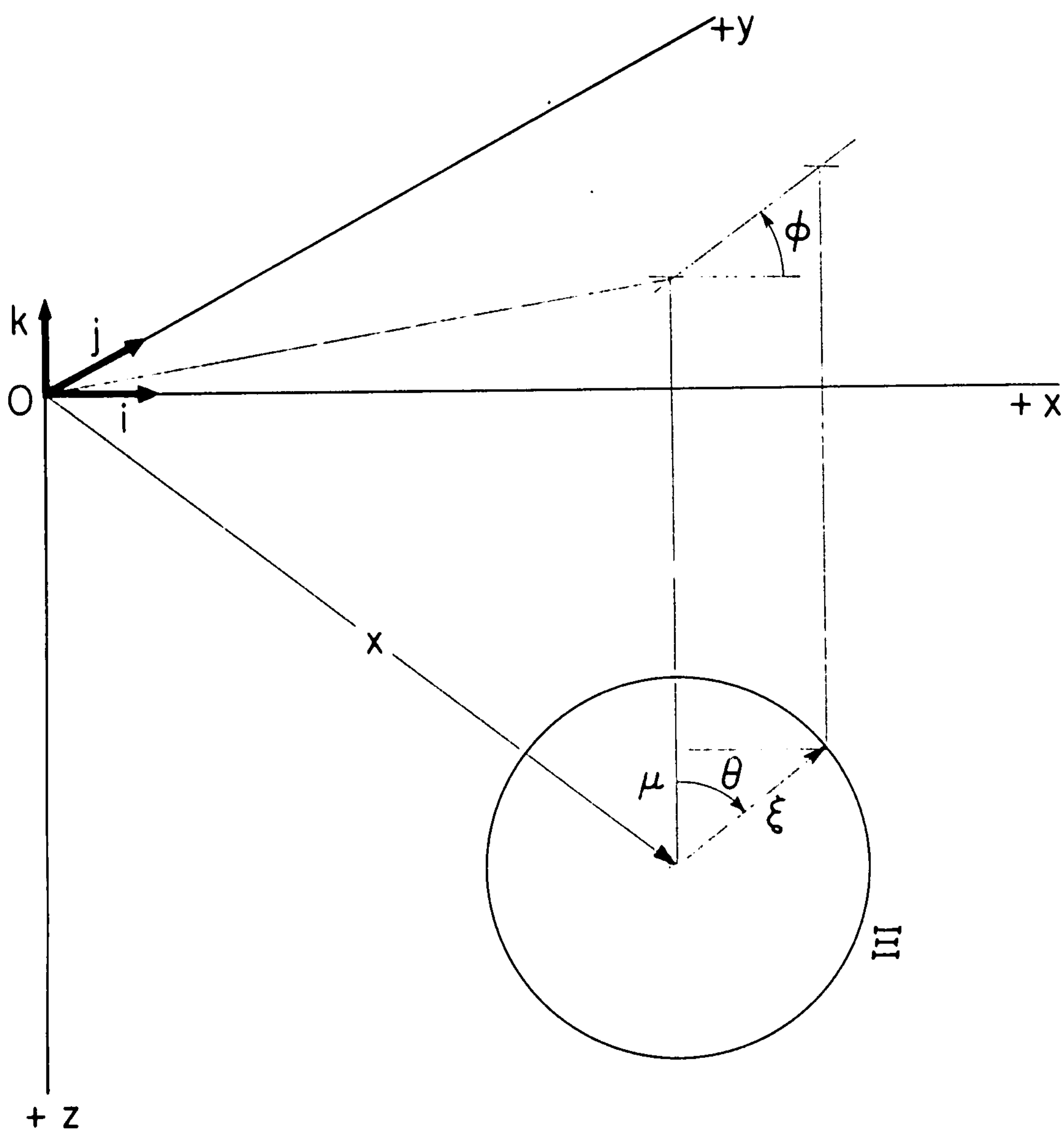

Figure 1. Rudolph W. Preisendorfer

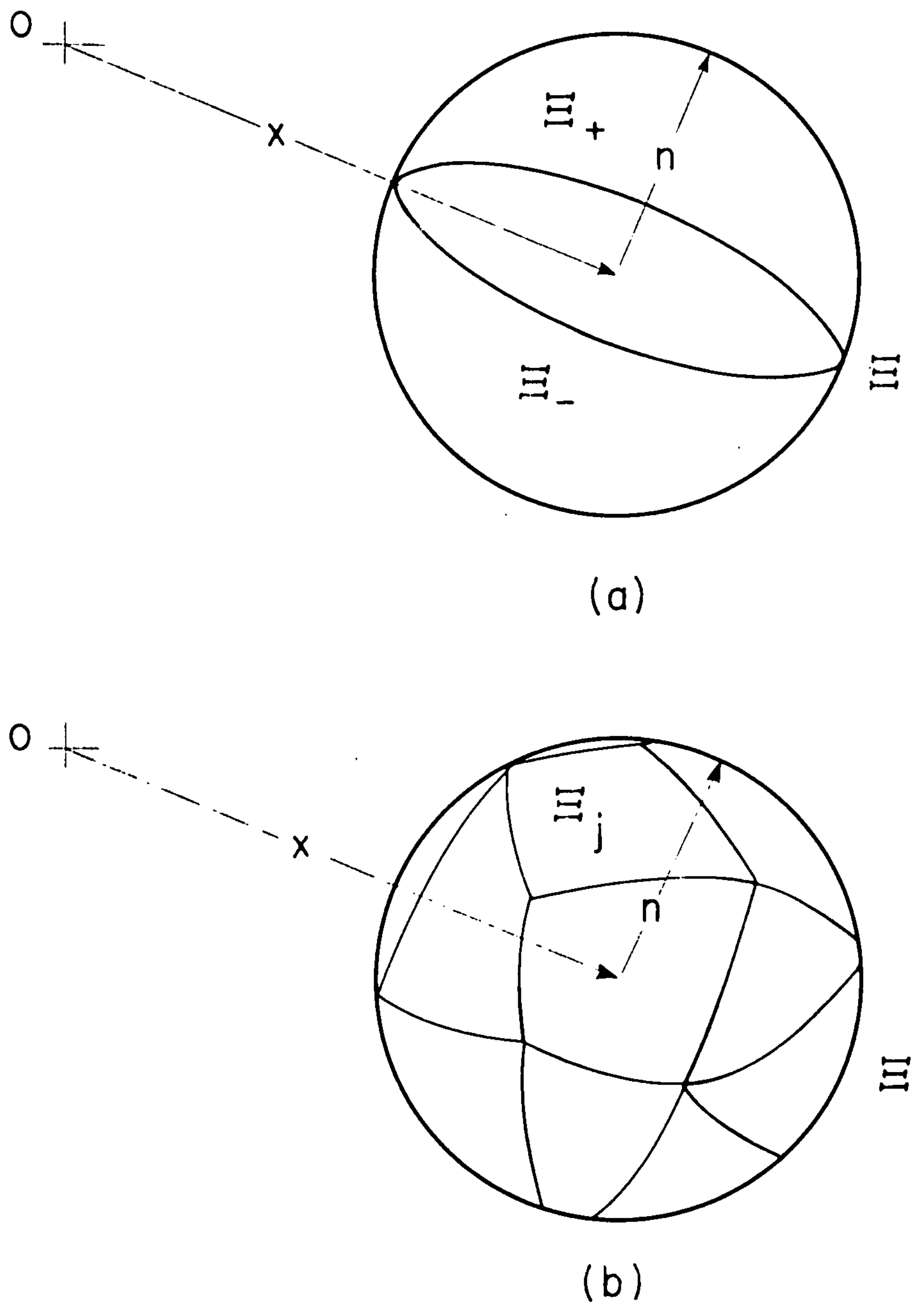

Figure 2. Rudolph W. Preisendorfer