The Optimal Throughput Order of Wireless Ad Hoc Networks and How To Achieve It

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***Abstract*—We show that, as the number of nodes in the network n tends to infinity, the *maximum concurrent flow* (MCF) and the *minimum cut-capacity* scale as $\Theta(n^2r^3(n)/k)$ for a random choice of $k \geq \Theta(n)$ source-destination pairs, where $r(n)$ is the communication range in the network. In addition, we show that it is possible to attain this optimal order throughput in interference-constrained networks if nodes are capable of multiple-packet transmission and reception. This result provides an improvement of $\Theta\left(nr^2(n)\right)$ over the highest achieved capacity reported to date. Furthermore, in stark contrast to the conventional wisdom that has evolved from the Gupta-Kumar results, our results show that the capacity of ad-hoc networks can actually *increase* with n while the communication range tends to zero!**

I. Introduction

Gupta and Kumar's seminal work [1] shows that the capacity of wireless ad-hoc networks does not scale with an increase in network size when nodes are static, transmit or receive one packet at a time, and the network traffic consists of unicast sessions. However, recent advances in many-to-one and many-to-many communication [2]–[4] and generalizations of routing (e.g., network coding (NC) [5]) are challenging the traditional view that avoiding multiple access interference (MAI) is the right approach to build wireless ad hoc networks. However, co-operative protocols that provide performance benefits in specific network configurations need not scale well with the network size. In particular, Liu et al. [6] showed that NC cannot increase the throughput order of wireless ad-hoc networks for multi-pair unicast applications under half-duplex communication. On the other hand, several techniques have been proposed aimed at improving the capacity of wireless ad hoc networks, and include taking advantage of mobility [7], changing physical-layer assumptions (e.g., using multiple channels or directional antennas [8]), or establishing different forms of cooperation between senders and receivers [9]. Cooperation can be extended to the simultaneous transmission and reception at various nodes in the network, and Garcia-Luna-Aceves et al. [3] have shown that, if nodes in the network are capable of multi-packet reception (MPR), then the order capacity of a network with n unicast sessions grows as $\Theta(r(n))$, where $r(n)$ is the communication range. This represents a gain of $\Theta\left(nr^2(n)\right)$ over the throughput order of $\Theta(1/nr(n))$ reported by Gupta and Kumar.

Interestingly, all the prior work on the capacity of wireless networks has focused on what is attainable with specific approaches to handle MAI. No prior work has focused on first establishing what is the optimal capacity of a wireless network in the absence of MAI, and then determining whether that capacity is attainable when MAI is present. This is precisely the focus and overall contribution of this paper.

Section II presents the network model assumed in this paper, which consists of a random network with n nodes, a homogeneous communication range of $r(n)$, and unicast traffic for k source-destination (S-D) pairs. In the absence of interference, such a network corresponds to a *random geometric graph* with an edge between any two nodes separated by a distance less than $r(n)$. We define a *combinatorial interference model* based on these graphs, and use it to introduce a protocol model in which nodes have the ability to decode correctly multiple packets transmitted concurrently from different nodes, and transmit concurrently multiple packets to different nodes. We refer to this as the multi-packet transmission and reception (MPTR) protocol model.

Section III characterizes the optimal interference-free capacity of a wireless network. The task of concurrently maximizing the data-rate for k S-D pairs is an instance of the multi-commodity flow problem. Hence, the *maximum concurrent multi-commodity flow-rate* (MCF) in a random geometric graph equals the interference free capacity (i.e., the optimal capacity) of the network. The max-flow min-cut theorem by Ford and Fulkerson [10] establishes that the MCF is tightly bounded by the minimum capacity of a multi-commodity cut for a single commodity. However, in general, the min-cut does not provide a tight bound on the max-flow [11], which is known to be tight only for special cases, and in general exhibits a gap of at least $\Theta(\log k)$ [12]. Leighton and Rao [11] showed that the gap between the max-flow and min-cut is at most $\Theta(\log n)$. We establish a tight max-flow min-cut theorem for random geometric graphs for the first time, and show that $\Theta\left(n^2r^3(n)/k\right)$ is a tight bound on the optimal capacity of a wireless network. Our work is inspired by the analysis of Leighton and Rao.

Section IV proves that the optimal capacity of wireless networks is attainable in the presence of MAI. We show that MPTR achieves the optimal capacity of $\Theta\left(n^2r^3(n)/k\right)$.

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Hence, MPTR provides a gain of $\Theta\left(nr^2(n)\right)$ over MPR and any previously reported feasible capacity. What is just as striking is that MPTR can achieve the dual objective of increasing capacity and decreasing the transmission range as n increases. This is in stark contrast to the commonly held view that the capacity of multihop wireless networks cannot increase as the number of nodes increases. Indeed, our results demonstrate that the capacity of ad-hoc networks can actually *increase* with n while the communication range tends to zero! Section V summarizes our results and points out future research directions.

II. Network Model and Preliminaries

For a continuous region R, $|R|$ denotes its area. The cardinality of a set S is denoted by $|S|$, and by $\|x-y\|$ the distance between nodes x and y. Whenever convenient, we utilize the indicator function $1_{\{P\}}$, which is equal to one if P is true and zero if P is false. $Pr(E)$ represents the probability of event E. An event E occurs with high probability (w.h.p.) as $n \to \infty$ if $Pr(E) > (1-(1/n))$. We employ the standard order notations $O, \Omega,$ and Θ.

We assume a random wireless network with n nodes distributed uniformly in a unit-square. As n goes to infinity, the density of the network also goes to infinity. Therefore, our analysis is applicable to dense networks. Furthermore, we assume a fixed transmission range $r(n)$ for all the nodes in the network. Thus, the network topology can be characterized using a random geometric graph, which we denote by G_r and define next.

Definition 2.1: Random Geometric Graph G_r
We associate a *directed* graph $G_r(V_r, E_r)$ with a wireless network formed by distributing n nodes uniformly in a unit square. We represent the node-set by $V = \{1, \cdots, n\}$. Let the locations of these nodes be given by $\{X_1, \cdots, X_n\}$, the edge-set is then $E = \{(i,j) \mid \|X_i - X_j\| \le r(n)\}$.

While the results in this paper can be extended to undirected graphs, it is more convenient for us to use directed graphs due to the edge-coloring techniques used in our work. Note that, we permit two edges for a pair of connected vertices with possibly different capacity in each direction.

We assume that the network operates using a slotted channel and, in the absence of interference, the data rate in each time slot for every transmitter-receiver pair is a constant value of W bits/slot. Given that W does not change the order capacity, we normalize its value to 1. Hence, we say that the interference-free capacity of each edge in G_r is equal to 1.

Gupta and Kumar [1] have proved the following criteria for the connectivity of G_r.

Lemma 2.2: For a random distribution of n nodes in a unit-square, the graph G_r is connected w.h.p, if and only if (iff) $r(n) \ge r_c(n) = \Theta(\sqrt{\log n/n})$.

In a dense network, interference is the primary constraint on the capacity of the network. Like Madan et al. [13], we describe the interference of a network by the following generic model.

Definition 2.3: Combinatorial Interference Model
The interference model for the graph[1] $G(V,E)$ is determined by a function $I: E \to P(E)$, where $P(E)$ is the power set of E, i.e., the set of all possible subsets of E. For every $e \in E$, $I(e)$ represents an *interference set* such that, a transmission on edge e is successful iff there are no concurrent transmissions on any $\hat{e} \in I(e)$. An interference model can be restricted to a sub-graph $H(V_H, E_H)$ by defining a function $I_H : E_H \to P(E_H)$ such that $I_H(e) = I(e) \bigcap E_H$.

The various protocol models that have been proposed in the past can now be expressed as special cases on G_r.

Gupta and Kumar [1] studied a *single packet reception* (SPR) *protocol model* under which a transmission from node i to receiver j is successful iff $\|X_i - X_j\| \le r(n)$ and if $\|X_j - X_k\| \ge (1+\eta)r(n)$ for any other transmitter k. Here η is a guard-zone that is assumed to be constant for the entire network. Moreover, all the nodes operate in half-duplex mode.

Definition 2.4: Single-Packet Reception (SPR) Model:
Let $e = (e^+, e^-) \in E$, then the interference set for edge e is

$$\begin{aligned} I_{\mathrm{SPR}}(e) &= J(e) - e, \\ J(e) &= \{\hat{e} \in E_r \mid \|X_{\hat{e}^+} - X_{e^-}\| \le (1+\eta)r(n)\} . \end{aligned} \tag{1}$$

In this paper, we consider the case in which nodes have MPR and MPT capabilities, i.e., can decode multiple concurrent transmissions or can transmit concurrently multiple packets to different nodes, but operate strictly in a half-duplex manner. The following definition expresses this model in terms of the notation we have introduced.

Definition 2.5: Multi-Packet Transmission and Reception (MPTR) Model: The MPTR interference set for edge e is

$$\begin{aligned} I_{\mathrm{MPTR}}(e) &= J(e) - A(e) \quad \forall e \in E \\ A(e) &= \{\hat{e} \in E_r \mid \|X_{\hat{e}^+} - X_{e^-}\| \le r(n)\} \end{aligned} \tag{2}$$

The traffic in the network is generated by unicast communication between k source-destination (S-D) pairs. A rate vector $\lambda = [\lambda_1, \cdots, \lambda_k]$ is associated with these k pairs. The data rate for each S-D pair is considered non-zero. Hence, without loss of generality (w.l.g), the rate vector can be written as $\lambda = [fD_1, \cdots, fD_k]$ where $f \in \mathbb{R}_+$ and $D_i \in [1/2, 1]$ for $1 \le i \le k$.We refer to the parameter f as the *concurrent flow rate* and to $D = [D_1, \cdots, D_k]$ as the *demand vector*.

Definition 2.6: Feasible Flow Rate
Given k S-D pairs $\{(s(1), d(1)), \ldots, (s(k), d(k))\}$, a rate vector $\lambda = [fD_1, \cdots, fD_k]$ is feasible if there exists a spatial and temporal scheme for scheduling transmissions such that by operating the network in a multi-hop fashion, and buffering at intermediate nodes when awaiting transmission, every source $s(i)$ can send λ_i bits/sec on average to the chosen destination $d(i)$. A flow rate f is feasible for a demand vector $D = [D_1, \cdots, D_k]$ iff $\lambda = [fD_1, \cdots, fD_k]$ is a feasible.

Definition 2.7: Capacity of Random Networks
The capacity per commodity of a network is $\Theta(f(n))$ if under

[1]Here G_r represents a random geometric graph while G a general graph.

a random placement of n nodes, a random choice of k S-D pairs and for an arbitrary demand vector we have:

$$\lim_{n\to\infty} \Pr(cf(n)\text{is feasible flow rate}) = 1 \quad (3)$$
$$\liminf_{n\to\infty} \Pr(c'f(n)\text{is infeasible flow rate}) < 1 \quad (4)$$

for some $c > 0$ and $c < c' < +\infty$.

In the following sections we repeatedly utilize the well known Chernoff bounds.

Lemma 2.8: Chernoff Bounds: Consider N i.i.d random variables $Y_i \in \{0,1\}$ with $p = \Pr(Y_i = 1)$. Let $Y = \sum_{i=1}^{N} Y_i$. Then for every $c > 0$ there exist $0 < \delta_1 < 1$ and $\delta_2 > 0$ such that

$$\Pr(Y \le (1-\delta_1)Np) < e^{-cNp} \quad (5)$$
$$\Pr(Y \ge (1+\delta_2)Np) < e^{-cNp}. \quad (6)$$

We review some defintions from graph theory. In particular, note that the task of identifying a feasible flow rate can be posed as a multi-commodity flow problem, specifically the k-commodity flow problem.

Definition 2.9: k-Commodity Flow Problem
Consider a directed graph $G(V,E)$ with a capacity function $c : E \to [0,1]$. Let $\{(s(1),d(1)),\ldots,(s(k),d(k))\}$ be k S-D pairs, with a demand vector $D \in [1/2,1]^k$. Let $f \in \mathbb{R}_+$ be a concurrent flow rate. Find flow functions $f_i : E \to \mathbb{R}_+$ for $1 \le i \le k$, which satisfy the following flow constraints:
Capacity Constraint: $\sum_{1\le i\le k} f_i(e) \le c(e)\ \forall e \in E$
Flow Conservation: $\sum_{e:e^+=v} f_i(e) = \sum_{e:e^-=v} f_i(e)\ \forall v \neq s(i), d(i)$
Demand Satisfaction: $\sum_{e:e^+=s(i)} f_i(e) = \sum_{e:e^-=d(i)} f_i(e) = fD_i$ for $1 \le i \le k$

Flow functions that satisfy the above constraints are called feasible. Other inputs to the problem being fixed, a flow rate f is said to be feasible iff the above problem has a solution. Furthermore, let f^* be the MCF such that the above problem has a feasible solution. A wireless network can be represented by an equivalent graph with capacity functions determined by the interference. Thus, the MCF in an equivalent graph can be perceived as the maximum flow that can be routed in a network. Additionally, if w.h.p. f^* is the MCF for any graph formed by a random distribution of nodes, sources and destinations, then the capacity of the wireless network is also f^*. Consider the following additional definitions.

Definition 2.10: Vertex Cut
Given a node set V , a cut is the separation of the vertex set V into two disjoint and exhaustive sets (S, S^C). We shall often reference a cut just by the set S .

Definition 2.11: Multi-commodity Cut Capacity
Given a graph $G(V,E)$, a capacity function $c : E \to [0,1]$ and a cut (S, S^C). The multi-commodity cut capacity is defined as

$$\Upsilon_{G,S} = \frac{\sum_{e\in E} 1_{[e^+\in S, e^-\in S^C]} c(e)}{\sum_{i:s(i)\in S, d(i)\in S^C} D_i}. \quad (7)$$

Definition 2.12: Minimum Cut Capacity
Given a graph $G(V,E)$, a capacity function $c : E \to [0,1]$ and a cut (S, S^C). The minimum multi-commodity cut capacity is defined as

$$\Upsilon_G = \min_{S\subset V} \frac{\sum_{e\in E} 1_{[e^+\in S, e^-\in S^C]} c(e)}{\sum_{i:s(i)\in S, d(i)\in S^C} D_i}. \quad (8)$$

It is well-known that the minimum cut-capacity provides an upper bound on the maximum flow rate.

Lemma 2.13: For any k-commodity flow $f^* \le \Upsilon_G$

III. Optimal Capacity

We show that for random geometric graphs, the MCF provides a tight approximation of the minimum-cut capacity. This relationship implies a tight characterization of the interference-free capacity of wireless ad-hoc networks with a homogenous transmission range. Our approach can be summarized as follows: For a particular demand vector, we provide an upper bound by showing that there exists a multi-commodity cut in G_r of order $O(g(n))$ and a lower bound by constructing a flow of order $\Omega(g(n))$ in a sub-graph $H_r \subseteq G_r$. These results along with the following Lemmas prove that the capacity of H_r and G_r has a tight bound $\Theta(g(n))$.

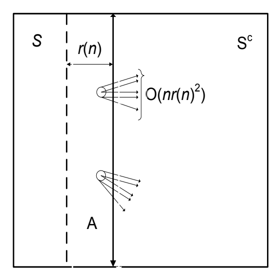

Fig. 1. A bi-partitioning of the unit square

Lemma 3.1: A graph $G(V,E)$ and a sub-graph $H(V_H, E_H)$ satisfy the following two properties: (a) If f is a feasible flow-rate in H then f is feasible in G; and (b) the capacity of a cut (S, S^C) in G is always greater than or equal to the capacity of the cut in H.

Proof: To prove Part (a) of the lemma, let $f_{H,i}$ for $1 \le i \le k$ be the flow functions associated with the feasible flow of rate f in H. Note that these flow functions satisfy the constraints in Definition 2.9. We construct a flow in G of rate f with the following flow functions: For $1 \le i \le k$ let $f_{G,i} : E \to \mathbb{R}_+$ such that $f_{G,i}(e) = f_{H,i}(e)$ if $e \in E_H$ and 0 otherwise. Now, if we show that the functions $f_{H,i}$ satisfy the flow constraints, then the flow rate f is feasible in G. Definition 2.9 states that $c(e) \ge 0$ for all the edges. As a result, $\forall e \in E - E_H$ the capacity constraints are satisfied trivially, given that $f_{G,i}(e) = 0$ for such edges. Furthermore, $\forall e \in E_H$ we have $\sum_{1\le i\le k} f_{H,i}(e) = \sum_{1\le i\le k} f_{G,i}(e) \le c(e)$. Therefore, $f_{G,i}$ satisfy the capacity constraint. In addition, note that the following equations hold $\forall v \in V$:

$$\sum_{e\in E:e^-=v} f_{G,i}(e) = \sum_{e\in E_H:e^-=v} f_{G,i}(e) + \sum_{e\in E-E_H:e^-=v} f_{G,i}(e)$$
$$= \sum_{e\in E_H:e^-=v} f_{H,i}(e) + 0 \quad (9)$$

$$\sum_{e\in E:e^+=v} f_{G,i}(e) = \sum_{e\in E_H:e^+=v} f_{G,i}(e) + \sum_{e\in E-E_H:e^+=v} f_{G,i}(e)$$
$$= \sum_{e\in E_H:e^+=v} f_{H,i}(e) + 0 \tag{10}$$

Eqs. (9) and (10) imply that the net in-flow and the net out-flow, under $f_{G,i}$ and $f_{H,i}$, is identical for all nodes and commodities. Therefore, $f_{G,i}$ satisfies the flow conservation and demand constraints.

To show Part (b), observe that

$$\Upsilon_{G,S} = \Upsilon_{H,S} + \frac{\sum_{e\in E-E_H} 1_{[e^+\in S, e^-\in S^C]} c(e)}{\sum_{i:s(i)\in S, d(i)\in S^C} D_i} \tag{11}$$

Because $c(e) \geq 0$ for all edges, we have $\Upsilon_{G,S} \geq \Upsilon_{H,S}$. ■

A. Upper Bound

We utilize the following properties of G_r.

Lemma 3.2: If $r(n) \geq r_c(n)$, then w.h.p. graph G_r is such that: (a) The minimum vertex degree $\nabla \geq \Theta\left(nr^2(n)\right)$, and (b) the maximum vertex degree $\Delta \leq \Theta\left(nr^2(n)\right)$.

Proof: We first show that $\nabla \geq \Theta\left(nr^2(n)\right)$. A node v in G_r is connected to all the nodes in a disk of radius $r(n)$ centered at v . The area of this disk is $\pi r^2(n)$. Given a uniformly random distribution of nodes, the probability of another node u lying within this disk is $\pi r^2(n)$. Consider a random variable $Y_{v,u} \in \{0, 1\}$ which is equal to one iff node u is connected to node v. The degree of node v can be written as $deg(v) = \sum_{u\in V-\{v\}} Y_{v,u}$. Therefore, the Chernoff Bound implies that $\forall c > 0$ there exists a $0 \leq \delta \leq 1$ such that

$$\Pr\left(deg(v) \leq (1-\delta)n\pi r^2(n)\right) < e^{-cn\pi r^2(n)} \tag{12}$$

From the union bound we obtain

$$\Pr\left(\nabla \leq (1-\delta)n\pi r^2(n)\right) < n\Pr\left(deg(v) \leq (1-\delta)n\pi r^2(n)\right) \tag{13}$$

Given that $r(n) \geq r_c(n)$ we have $r(n) \geq c_1\sqrt{\log n/n}$ for some $c_1 > 0$. Therefore, Eqs. (12) and (13) imply that $\Pr\left(\nabla \leq (1-\delta)\pi nr^2(n)\right) < ne^{-c\pi c_1 \log n} = 1/n^{c\pi c_1 - 1}$ Now, Lemma 2.8 tell us that we can choose $c \geq 2/(\pi c_1))$ and corresponding $0 < \delta_1 < 1$ such that $\Pr\left(\nabla \geq (1-\delta)\pi nr^2(n)\right) > 1 - (1/n)$ We use similar arguments to show that $\Delta \leq \Theta\left(nr^2(n)\right)$ w.h.p.. This fact follows from Eq. (8), which implies that $\forall c > 0$ there exists a $0 \leq \delta \leq 1$ such that $\Pr\left(deg(v) \geq (1+\delta_1)n\pi r^2(n)\right) < e^{-cn\pi r^2(n)}$ The union bound and the fact that $r(n) \geq c_2\sqrt{\log n/n}$ implies that $\Pr\left(\Delta \geq (1+\delta)\pi nr^2(n)\right) < ne^{-c\pi c_2 \log n}$ Therefore, there exists a $\delta_2 > 0$ such that $\Pr\left(\Delta \leq (1+\delta_2)\pi nr^2(n)\right) > 1 - (1/n)$ ■

Consider the cut S described by Figure 1. The cut consists of all the nodes in the rectangular region of a constant area.

Lemma 3.3: If the network consists of $k \geq \Theta(\log n)$ S-D pairs, then w.h.p. a region R of constant area $|R|$ contains $\Theta(k)$ sources with destinations outside region R.

Proof: Let $Y_i \in 0, 1$ be a random variable that is equal to one iff the ith S-D pair is such that $s(i)$ belongs to region R and $d(i)$ does not. Under a uniformly random placement of nodes, $\Pr(Y_i = 1) = |R|(1-|R|)$. The total number of S-D pairs satisfying the required condition can be represented by $Y = \sum_1^k Y_i$. If $k \geq c\log n$ the Chernoff Bounds imply the existence of constants $c_1 = 1/(c|R|(1-|R|))$, $0 \leq \delta_1 \leq 1$ and $\delta_2 > 0$ such that

$$\Pr(Y \geq (1-\delta_1)|R|(1-|R|)k) > 1 - e^{-c_1 k|R|(1-|R|)}$$
$$\geq 1 - e^{-c_1 c|R|(1-|R|)\log n} = 1 - (1/n) \tag{14}$$
$$\Pr(Y \leq (1+\delta_2)|R|(1-|R|)k) > 1 - e^{-c_1 k|R|(1-|R|)}$$
$$= 1 - e^{-c_1 c|R|(1-|R|)\log n} = 1 - (1/n) \tag{15}$$

■

Furthermore, consider the subset A in S defined by a strip of dimension $1 \times r(n)$. The total number of vertices in A is $\Theta(nr(n))$ because of the uniform distribution of nodes in the network.

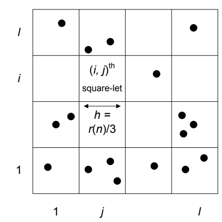

Fig. 2. Decomposition of network area into l^2 squarelets

Theorem 3.4: If $r(n) \geq r_c(n)$ and $k \geq \Theta(\log n)$, then the capacity of the cut S is $\Upsilon_{G_r,S} = O(n^2r^3(n)/k)$ w.h.p.

Proof: According to the definition of G_r two nodes are connected iff they are separated by a distance less than $r(n)$. Consequently, if an edge cuts across S then it has to be incident upon a node at a distance less than $r(n)$ from the boundary separating S and S^C , i.e. the head of the edge should lie in the subset A of dimension $r(n) \times 1$. Furthermore, for each node in A the maximum number of edges cutting across the cut is bounded by Δ, i.e., $\sum_{e\in E} 1_{[e^+\in S, e^-\in S^C]} \leq |A|\Delta$. In the absence of interference $c(e) = 1$ for all the edges. Hence,

$$\Upsilon_{G_r,S} = \frac{\sum_{e\in E_r} 1_{[e^+\in S, e^-\in S^C]}}{\sum_{i:s(i)\in S, d(i)\in S^C} D_i} \leq \frac{|A|\Delta}{\sum_{i:s(i)\in S, d(i)\in S^C} D_i}$$

According to Lemma 3.3, there exists $c_1 > 0$ s.t. the total number S-D pairs across the cut is $c_1 k$. Furthermore, by Definition 2.9 the demand for each pair is at least 1/2. Hence, $\Upsilon_{G_r,S} \leq (2|A|\Delta)/c_1 k$. Lemma 3.2 implies that there exists a $c_2 > 0$ s.t. $\Delta < c_2 nr^2(n)$, while uniform distribution of nodes in the network implies that there exists a $c_3 > 0$ s.t. $|A| \leq c_3 nr(n)$. Hence, $\Upsilon_{G_r,S} \leq 2c_2c_3n^2r^3(n)/(c_1 k)$. ■

Any cut in G_r has a capacity greater than the minimum cut capacity Υ_{G_r}. Consequently Theorem 4.4 implies the following Corollary.

Corollary 3.5: If $r(n) \geq r_c(n)$ and $k \geq \Theta(\log n)$, then Υ_{G_r} is upper bounded as $O(n^2r^3(n)/k)$.

B. Lower Bound

To describe a capacity-achieving flow in a more generic setting, we use an important result from parallel and distributed computing. Consider a mesh of l^2 processing units with l processors in each row and column. Let each processor be a source and destination of exactly h packets. The problem of routing the hl^2 packets to their destinations is known as $h \times h$ permutation routing and can be characterized by the following result [14].

Lemma 3.6: If in a single slot, each processor can transmit one packet each to its immediate horizontal and vertical neighbors, then an $h \times h$ permutation routing in a $l \times l$ mesh can be performed deterministically in $hl/2 + o(hl)$ steps.

We utilize the following corollary that can be readily deduced from the above Lemma.

Corollary 3.7: If a processor is capable of transmitting at least η packets to each of its neighbors in each slot, then an $h \times h$ permutation routing in a $l \times l$ mesh can be performed deterministically in $O(hl/\eta)$ steps.

Now consider a sub-graph $H_r \subseteq G_r$ obtained by employing location based constraints on the edge-set. In order to describe these constraints, we first define a location dependent hash function.

Definition 3.8: Index Function ζ
Divide the network area into l^2 squarelets [15] of side-length $a = r(n)/3$, as shown in Figure 2. Let ζ be a function that associates an index (i, j) with a squarelet in the i^{th} column and j^{th} row. Furthurmore, the index assigned to each squarelet is associated with each vertex in the squarelet.

We obtain H_r by removing all edges, except those connecting two nodes in vertically or horizontally adjacent squarelets. We do not necessarily have to consider H_r in order to obtain a lower bound on the interference-free capacity. However, the performance bounds for H_r play an important role when we analyze interference constrained networks in Section IV.

Definition 3.9: Geographically Restricted Sub-Graph H_r
The graph $H_r(V_r, E_{r,H})$ is a sub-graph of G_r with an identical node-set and an edge-set defined as

$$E_{r,H} = \{e \in E \mid \zeta(e^-) = (a, b) \Rightarrow \zeta(e^+) = (a \pm 1, b \pm 1)\}. \quad (16)$$

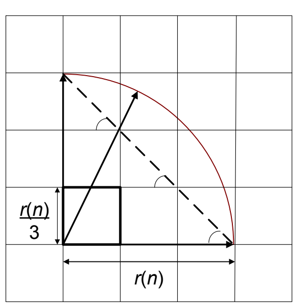

Fig. 3. A geometric proof to show that any two points in adjacent square-lets are within a distance r(n) of each other. The proof follows from the fact that the chord of a circle lies within it.

Consider some of the properties of the squarelets and H_r.

Lemma 3.10: [15] If $r(n) \geq r_c(n)$, then w.h.p. the total number of nodes in any squarelet is $\Theta(nr^2(n))$.

Proof: The area of a squarelet is equal to $\Theta(r^2(n))$. Hence, the proof is identical to that of Lemma 3.2. ■

Lemma 3.11: If $r(n) \geq r_c(n)$ and $k \geq \Theta(n)$, then w.h.p. the total number of sources in any squarelet are $\Theta(kr^2(n))$ and the total destinations in any squarelet are $\Theta(kr^2(n))$.

Proof: For $1 \leq i \leq k$ let $Y_{i,m} \in 0, 1$ be a random variable that is equal to one iff source $s(i)$ belongs to the m^{th} squarelet. Let $Y_m = \sum_1^k Y_{i,m}$ represent the total number of sources in the squarelet. Because $r(n) \geq r_c(n)$, Eq.(11) implies

$$\Pr\left(Y_m \leq (1 - \delta)kr^2(n))\right) < e^{-(ck \log n)/n} \quad (17)$$

The total number of squarelets in a unit square is equal to $(3/r(n)) \times (3/r(n)) \leq c_1 n/\log n$. Therefore, the union bound implies that

$$\begin{aligned} \Pr \quad & \left(\textit{min. no. of nodes in a squarelet} < \Theta(kr^2(n))\right) \\ \leq \quad & (\textit{total no. of squarelets}) \times e^{-(ck \log n)/n} \\ \leq \quad & (c_1 n/\log n) \times e^{-(ck \log n)/n} = c_1/(n^{(ck/n)-1} \log n) \end{aligned}$$

Thus, $k \geq \Theta(n)$ guarantees the required convergence and hence we can say that each squarelet has at least $\Theta(kr^2(n))$ sources. The upper bound on the number of sources and the bounds on the number of destinations can be calculated similarly. ■

Theorem 3.12: If $r(n) \geq r_c(n)$ and $k \geq \Theta(n)$, then w.h.p. the maximum flow rate f_H^* in H_r is at least $\Theta\left(n^2 r^3(n)/k\right)$.

Proof: The proof follows from mapping various components of the above defined problem to the $h \times h$ permutation routing problem. Let us map each squarelet to a processor. Consequently, for the chosen size of squarelets, the network equates to a mesh of l^2 processors with $l = 3/r(n)$. Assume that each source intends to transmit D_i as the ith element of the demand vector. Because $D_i \leq 1$, Lemma 3.11 implies that the total number of bits to be transmitted to and from each squarelet are at most $h \leq ckr^2(n)$. Finally, note that any two nodes in adjacent squarelets are within a distance $r(n)$. Fig. 3 provides a geometric proof for this fact; an alternative proof can be easily obtained by employing the Pythagoras theorem. In each slot, we can send one packet along each edge between two adjacent squarelets. Therefore, Lemma 3.10 implies that $\eta = (\textit{min. no. of edges between adjacent squarelets}) \leq (\textit{min. no. of nodes per squarelet})^2 \leq c_1 n^2 r^4(n)$. Hence, the total number of slots γ required to complete the desired routing is $\gamma \leq (c_2 hl/\eta) \leq c_2 \times (ckr^2(n)) \times (3/r(n)) \times (1/c_1 n^2 r^4(n)) = (3c_2 ck/c_1 n^2 r^3(n))$. We can repeat the above routing periodically to guarantee a flow rate of $f = (1/\gamma) \geq \Theta(n^2 r^3(n)/k)$. By definition, the max-flow rate is greater than any other feasible flow rate, and the theorem follows. ■

Aggregating the above results we have the following conclusion.

Theorem 3.13: If $r(n) \geq \Theta\sqrt{\log n/n}$ and $k \geq \Theta(n)$, then the max- flow f_G^* in G_r can be approximated tightly by the min-cut capacity $\Upsilon^*_{G_r}$ in G_r. Moreover, the f^*_G and $\Upsilon^*_{G_r}$ scale as $\Theta\left(n^2 r^3(n)/k\right)$.

Proof: Lemma 3.1 implies that $f^*_G \geq f^*_H$. Hence, the result follows from the lower bound provided by Theorem 3.12 and the upper bound provided by Corollary 3.5 ■

The following corollary follows for the case in which $k = \Theta(n)$, which has been studied in the literature.

Corollary 3.14: Consider an ad-hoc network described by a random placement of n nodes in a unit square, with $\Theta(n)$ S-D pairs and a homogenous transmission range of $r(n) \geq \Theta(\sqrt{\log n/n})$. The interference-free capacity of the network scales as $\Theta\left(nr^3(n)\right)$.

IV. Interference-Limited Capacity

A. General Results on Interference Models

Interference can severely limit the network capacity. In this section we obtain scaling laws for the MPTR interference model to show by example that the optimal capacity of wireless networks can be attained.

Definition 4.1: Dual-Interference-Set
Consider an edge set E and an interference set $I(e)$ for an edge $e \in E$, as defined in Definition 2.3. The dual interference-set for e is defined by $F(e) = \{\hat{e} \in E \mid e \in I(\hat{e})\}$, which is the set of edges that experience a collision on account of a transmission on edge e.

Definition 4.2: Dual Conflict Graph
Given a graph $G(V, E)$ and an interference function I, we define the *dual conflict graph* as $G_D(E, E_D)$, where $E_D = \{(e, \hat{e}) \mid \hat{e} \in I(e))\}$.

Definition 4.3: Total Degree in Dual Conflict Graph
The total degree of each node in a dual conflict graph is equal to $|M(e)|$ where $M(e) = I(e) \bigcup F(e)$.

Similar to the work in [13], we have the following Lemma.

Lemma 4.4: Consider a graph $G(V, E)$ and interference I. Let $\kappa = \max_{e \in E} |M(e)|$, the maximum vertex degree of the dual conflict graph G_D. If f is a feasible flow rate in the absence of interference, then flow rate $f_I = f/(1 + \kappa)$ is feasible in presence of interference I.

Proof: In the absence of interference the capacity of each edge is assumed to be one. However, because of interference, all edges cannot be activated simultaneously. Let σ be the minimum frequency with which each edge is activated without causing any interference conflicts. Then, for each edge we have $c(e) \geq 1/\sigma$. It is well known that, if κ is the maximum vertex degree, then $\kappa + 1$ colors are sufficient to provide a proper vertex coloring [16]. Thus, by providing a vertex coloring for the dual conflict graph, we can partition the edge-set E into $1+\kappa$ subsets such that no two edges in the same subset interfere. Consequently, we can periodically activate these subsets to realize $c(e) \geq 1/(1 + \kappa)$ for each edge. Thus, a feasible flow rate $f_I = f/(1 + \kappa)$ can be obtained by scaling the flow functions associated with f by a factor of $1/(1 + \kappa)$. ■

The maximum vertex degree does not provide a tight bound on the minimum number of colors required to provide a proper vertex coloring. Hence, in order to analyze a wider variety of protocol models, we introduce the concept of *interference clones*.

Definition 4.5: Interference Clone
Two edges e_1, e_2 are said to be interference-clones under function I if they satisfy the conditions that $M(e_1) = M(e_2)$.

Lemma 4.6: Clone Piggy-backing
Consider a graph $G(V, E)$ along with interference functions I_A and I_B, then I_A and I_B are such that: (a) $\kappa = \max_{e \in E} |M_A(e)|$, and (b) there exists a set $M_{A,\bar{B}}(e) \subseteq M_A(e)\ \forall e \in E$ such that every edge belonging $M_{A,\bar{B}(e)}$ is an interference-clone of e under I_B . Further, let $\mu = \min_{e \in E} |M_{A\bar{B}}(e)|$. If f is a feasible flow rate in G without any interference, $f_{I_A} = f/(1 + \kappa)$ is a feasible flow rate in G under the I_A interference function and κ as its correponding parameter, then $f_{I_B} = f(1+\mu)/(1+\kappa)$ is feasible in presence of interference defined by I_B.

Proof: Consider the interference defined by I_A. From Lemma 4.4 we know that there exists a conflict free periodic schedule which can activate each edge at least once every $(1 + \kappa)$ slots. Let us represent this schedule by an indicator function $\alpha(e, \tau)$ which equals one iff edge e is active in slot τ. Note that the capacity of each edge under schedule α is given by $c_\alpha(e) = \Sigma_\tau \alpha(e, \tau) = 1/(1 + \kappa)$. Now let us use this schedule in the presence of interference I_B. Observe that, for every e, α allocates a distinct slot for each edge in $M_{A,\bar{B}}(e) \bigcap \{e\}$. Consequently, every edge has $|M_{A,\bar{B}}(e)|$ interference clones scheduled in slots distinct from each other and the edge itself. In addition, note that if an edge is activated in a time slot meant for one of its interference clones, then it does not lead to any conflict. Therefore, we can define a new conflict-free schedule β such that $\beta(e, \tau) = 1$ iff there exits an $e_1 \in M_{A,\bar{B}}(e) \bigcap \{e\}$ such that $\alpha(e_1, \tau) = 1$. Given that $\mu = \min_{e \in E} |M_{A\bar{B}}(e)|$, the capacity of each edge under schedule β for interference I_B, is given by $c_\beta(e) = \Sigma_{e_1 \in M_{A,\hat{B}}} \Sigma_\tau \alpha(e_1, \tau)$ $\leq (1 + \mu) \times (1/(1 + \kappa))$. Accordingly, a feasible flow of $f_{I_B} = f(1 + \mu)/(1 + \kappa)$ can be obtained by scaling all the flow functions associated with the inference-free flow by a factor of $(1 + \mu)/(1 + \kappa)$. ■

In the subsequent discussion, we find it convenient to deduce a bound for a particular interference model and then show that it applies to a wider set of models. In order to facilitate such arguments, we define the following partial order.

Definition 4.7: Partial Order of Interference Models
An interference function I_A is said to be more restrictive than I_B, represented as $I_A \preceq I_B$, iff every edge satisfies the conditions that $I_B(e) \subseteq I_A(e)$.

Lemma 4.8: Consider a graph $G(V, E)$ along with interference I_A and I_B. If $I_A \preceq I_B$, then a feasible flow rate under I_A remains feasible under I_B.

Proof: A conflict free schedule under I_A remains conflict free under I_B. Hence we can say that

$$c_A(e) \leq c_B(e) \tag{18}$$

where $c_A(e)$ and $c_B(e)$ represent the edge capacities under each interference model. Therefore, if a particular flow satisfies the capacity constraints under I_A, it necessarily satisfies those same constraints under I_B. ∎

B. Lower Bound

For mathematical convenience, we define a restrictive interference model for MPTR that introduces more restrictions (i.e., collisions) on the interference set for each edge than those strictly dictated by the original interference model. We show that, under this restrictive model, the order of the lower bound capacity achieves the upper bound under the original (non-restrictive) interference model. Lemma 4.8 allows us to utilize this performance limit to indirectly bind the capacity under the interference model MPTR.

Definition 4.9: Restricted SPR (RSPR) Model:

$$\begin{aligned} I_{\text{RSPR}}(e) &= W(e) - \{e\} \quad \forall e \in E_{r,H} \\ \textit{where} \;\; W(e) &= \bigcup_{\hat{e}:\zeta(\hat{e}^-)=\zeta(e^-)} M_{SPR}(\hat{e}) \end{aligned} \tag{19}$$

Definition 4.10: Restricted MPTR (RMPTR) Model

$$\begin{aligned} I_{\text{RMPTR}}(e) &= W(e) - V(e) \quad \forall e \in E_{r,H} \\ V(e) &= \bigcup_{\hat{e}:\zeta(\hat{e}^-)=\zeta(e^-)} U(\hat{e}) \\ U(e) &= \left\{\hat{e} \in E_{r,H} \mid \hat{e}^- = e^-\right\} \end{aligned} \tag{20}$$

Consider the following properties of the restricted model.

Lemma 4.11: For the graph H_r, we have a partial order defined by $I_{\text{RSPR}} \preceq I_{\text{RMPTR}} \preceq I_{\text{MPTR}}$

Proof: (Sketch) The left side of the partial order follows directly from definition. Meanwhile the right side follows from the fact that that $A(e) \subseteq V(e)$. ∎

Lemma 4.12: If $r(n) \geq r_c(n)$, then all edges $e \in E_{r,H}$ have $|M(e)| = \Theta\left(n^2 r^4(n)\right)$ under interference described by I_{MPTR}, I_{RMPTR} or I_{RSPR}.

Proof: (Sketch) Due to Lemma 4.11 it suffices to prove

$$\gamma_{max} = max_{e\in E_{r,H}} |I_{\text{RSPR}}(e)| = O(n^2 r^4(n)) \tag{21}$$

$$\gamma_{min} = min_{e\in E_{r,H}} |I_{\text{MPTR}}(e)| = \Omega(n^2 r^4(n)) \tag{22}$$

Recall that a node in H_r is connected to all and only those nodes that are placed in adjacent squarelets. Hence Lemma 3.10 tells us that the degree of each vertex $v \in H_r$ is bounded as $4c_1 nr^2(n) \leq deg(v) \leq 4c_2 nr^2(n)$. We can prove the lower bound by considering the model MPTR. According to Definition 2.5, the transmission on edge e experiences interference from any transmission by a node v such that $r(n) < \|X_{e^-} - X_v\| \leq (1+\eta)r(n)$. Therefore, there exists an annular ring around e^- of width $\eta r(n)$ such that any transmission from a node in this ring interferes with e. The area of this annular ring is given by $\eta(2+\eta)\pi r^2(n)$. We have already seen (Lemma 3.2) that an area of $\Theta(r^2(n))$ contains at least $\Theta(nr^2(n))$ nodes. Hence, there exists a c_3 such that $\gamma_{min} \geq c_3 nr^2(n) \times 4c_1 nr^2(n)$, which proves the lower bound. The proof for the upper bound is obtained with a similar argument, and the proof is ommitted due to space limitations. ∎

Lemma 4.13: Consider the graph H_r with $r(n) \geq r_c(n)$ and $k \geq \Theta(n)$. In such a graph, each edge e has at least $\Theta\left(n^2 r^4(n)\right)$ clones under interference I_{RMPTR}. such that these clones interfere with each other and e, under the interference I_{RSPR}.

Proof: According to Definition 4.10, $V(e)$ represents the desired set of clones for I_{RMPTR}. Lemma 3.10 implies that there exist c_1 and c_2 such that $\mu_{RMPTR} = min_{e\in E_{r,H}}|V(e)| \geq [min_{e\in E_{r,H}}|U(e)|] \times$ *min. nodes per squarelet* $\geq c_1 nr^2(n) \times c_2 nr^2(n)$. ∎

Theorem 4.14: For $r(n) \geq r_c(n)$ and $k \geq \Theta(n)$, the capacity of random geometric network is at least $\Theta(n^2 r^3(n)/k)$ under the MPTR model.

Proof: Recall that the capacity of the random network is greater than the feasible flow rate in H_r. Theorem 3.12 shows that a rate of $f = c_1 n^2 r^3(n)/k$ is feasible in H_r. If we take into consideration the interference clones, then Lemma 4.6 further implies that the rate $f \times (1/(1+\kappa_{RSPR})) \times (1+\mu_{RMPTR}) \geq (c_3/r(n)k) \times (c_5 n^2 r^4(n)) = (c_3 c_5 n^2 r^3(n)/k)$ is feasible under the RMPTR model. Finally, note that a feasible rate under a restricted model is necessarily feasible under the original model. Hence, the result proven in Lemma 4.12 completes the proof. ∎

The interference-free capacity provides an upper bound on the capacity under any model, and Theorem 4.14 already shows that the MPTR model achieves this capacity, thus we have a tight bound of $\Theta(n^2 r^3(n)/k)$ on the capacity under the MPTR model.

V. Conclusions and Future Work

We have shown that the optimal capacity that *any* protocol architecture can attain in a wireless network is $\Theta\left(n^2 r^3(n)/k\right)$. In addition, we demonstrated that this capacity can indeed be attained when nodes embrace MAI as transmitters and receivers, and that a non-vanishing capacity is attainable per S-D pair even when information must be disseminated over multiple hops. While these results provide a completely new outlook on the design of wireless ad hoc networks from the traditional view based on avoiding MAI, much work remains to be done to fully understand the fundamental limits of wireless networks! In particular, the cases of multicast and broadcast information dissemination must be considered. We also hope that this paper motivates research on protocol architectures that combines multi-packet reception and transmission to attain massively scalable ad hoc networks.

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