UNIVERSITY OF CALIFORNIA

Los Angeles

The Dynamics of Collaboration in Knowledge-Based Work Processes

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy in Management

by

Morvarid Rahmani

2013

ABSTRACT OF THE DISSERTATION

The Dynamics of Collaboration in Knowledge-Based Work Processes

by

Morvarid Rahmani

Doctor of Philosophy in Management

University of California, Los Angeles, 2013

Professor Uday Karmarkar, Co-Chair

Professor Guillaume Roels, Co-Chair

The service sector represents the dominant part of the U.S. economy, accounting for over 80% of employment as well as GNP (Apte et al. 2011). Although much scholarly work has been done on services, research on professional or white collar services is still in its infancy. What makes these services distinctive is that they are knowledge-intensive and project-based. In addition, they are collaborative by nature; that is, their value is created from the combined effort of individuals (Hopp et al. 2009). For example, IBM Corporation discovered that their Request For Proposal (RFP) process for outsourcing services, instead of being linear, well sequenced, and deterministic, was information-intensive, collaborative, iterative, and stochastic. Improving their understanding of how the work actually took place led them to improve the efficiency of their work processes (Kieliszewski et al. 2010).

This dissertation focuses on the study of the dynamics of collaboration in knowledge-based work processes such as new product or service development, management and

IT consulting, technical projects, and education. It consists of three chapters respectively addressing the effect of contracting, team organization, and learning in collaboration, so as to generate insights for advancing strategic decision-making both across organizations and within them. In each chapter, collaboration is modeled as a dynamic double moral hazard game. Specifically, these models have the following features in common: (i) the collaborative project is subject to a tight deadline, (ii) the work dynamics are studied with respect to both time and project progress, and (iii) the transitions between states are stochastic and depend on the collective amount of effort exerted. Below is a summary of these three chapters.

The first chapter studies how contractual arrangements affect the work dynamics between a vendor and a client in finite-deadline collaborative projects that have been disambiguated and need to be executed. The analysis shows that it is optimal for the client and the vendor to both exert high effort when the project is near completion, i.e., either when the project has reached a high state or when there is limited time left until the deadline. Otherwise, only one of them needs to exert high effort. When efforts are not contractible, i.e., in cases of double moral hazard, the dynamics of collaboration depend on the contractual arrangement. The analysis shows that reward-sharing contracts yield suboptimal output and give rise to free-riding; fixed-fee completion bonuses make the vendor exert high effort only when the project has reached a high state; and with time-and-materials contracts, the vendor attempts to either shirk work or increase the project scope. Despite these shortcomings, the analysis reveals that simple contracts can perform well, but that they must be judiciously chosen based on project characteristics.

The second chapter considers knowledge-intensive projects such as new product or software development, in which the major resource is labor. The choice of team size is a crucial decision for project success since it balances effectiveness and efficiency.

In these knowledge-intensive projects, team size can be chosen either by the team members themselves or by an internal leader. This chapter studies how the choice of team size is affected by the leadership structure of the team. The project is modeled as a stochastic staffing game with a fixed deadline and the staffing decisions are examined relative to the best solution. The analysis shows that there is always under-staffing when members choose the team size. With an internal leader, there can be under- or over-staffing. In particular, there can be over-staffing because the team members' cost of effort is not internalized by the leader and there can be understanding because the leader can free ride. Comparing the efficiency of the two leadership structures, the analysis shows that letting the team members decide team size themselves generates a higher total surplus when the project reward is neither too small nor too large.

The third chapter analyzes the effect of learning on collaboration in knowledge-intensive projects. The ultimate success of such projects depends on effectiveness of collaboration. However at the outset, it may be challenging to predict how effective two parties may be at collaborating, especially if they have not worked together before. Accordingly, this chapter studies the dynamics of collaboration when parties are uncertain about the quality of their relationship, i.e., how effective they will be at working together. The problem is modeled as a stochastic game between two parties, in which the probability of success from individual work is known, but the probability of success from joint work is not. In particular, the parties start the project with the same prior belief about the effectiveness of their collaboration and update that belief through time, using Bayesian updating. The analysis shows that learning is favorable when the parties' expected prior belief is neither too strong nor too weak. In this case, the parties choose to collaborate early on to discover the effectiveness of their teamwork. If their collaboration fails for a certain number of trials, they stop collaborating and prefer to do the work individually. When the parties' expected prior belief is either too strong or too weak, learning may not be favorable, because the parties may postpone their collaboration until there is little time left before the deadline. This chapter can be extended in several directions. For example, what if the parties have the option to wait or delay the project? What if the parties make their decisions in a decentralized way, resulting in a situation of double moral hazard?

The dissertation of Morvarid Rahmani is approved.

Rakesh Sarin

Ichiro Obara

Uday Karmarkar, Committee Co-Chair

Guillaume Roels, Committee Co-Chair

University of California, Los Angeles

2013

In Memory of My Mother, Fariba

Contents

List of Figures

List of Tables

ACKNOWLEDGEMENTS

Completing my PhD program was one of the most rewarding challenges that I experienced in my life. It was a long journey full of hard work, joy, ups, and downs. I enjoyed every moment because I knew it was one of those experiences that will never get repeated. However, it would not have been possible without all those who supported me throughout this journey. I will try my best to acknowledge them here.

First and foremost, I would like to thank my advisors, Prof. Uday Karmarkar and Prof. Guillaume Roels. It was truly a pleasure for me to work under their supervision. Uday gave me tremendous guidance and invaluable advice not only on my dissertation but also for building my academic career. His encouraging comments helped me move forward and accomplish my educational goals. I truly admire his experience and his leadership ability. During the past three years, I interacted frequently with Guillaume and benefited from his sincere support. I cannot explain in words how much I learned from him and how patient he was through this learning process. His detailed advice and priceless coaching helped me succeed in my PhD program. I am indebted to him for my research ability and for my pleasant Ph.D. and job search experiences.

I would like to thank my dissertation committee members: Prof. Rakesh Sarin and Prof. Ichiro Obara. I think Rakesh was not just my committee member; he was like my third advisor. I always felt comfortable sharing my thoughts with him, for which he gave me precious advice. In addition, I was his teaching assistant for three years. During that time, he kindly allowed me to openly interact with MBA students and to improve my teaching and presentation skills. I would also like to thank Ichiro, who was always there to help me with my research and job search. It was a pleasure having him as part of my committee.

I would like to extend my special thanks to Prof. Reza Ahmadi, whom I would like to call my "Ph.D. mentor." He was the one who encouraged and supported me to start a Ph.D. program at UCLA. He guided me in all the stages of my Ph.D. program and taught me many lessons for my academic life and career. I would like to express my special thanks to Profs. Kevin McCardle, Felipe Caro, and Christiane Barz for their sincere support during my job search. I am also thankful to Profs. Corbett, Tang, Rajaram, Morrison, Lipman, Mamer, Bikhchandani, and Bendersky, who always willingly offered their insights for any question I asked. I also wish to acknowledge other UCLA faculty members, from whom I had the opportunity to learn: Profs. Rumelt, Rubin, Vandenberghe, Van Der Schaar, Riley, Zame, Matzkin, and Hahn.

I express my gratitude to Lydia Heyman, the PhD Director, for her extensive help in every step of my Ph.D. program. She always supported me and was there to answer any question that I had. I also wish to thank Prof. Mark Grinblatt, the Senior Associate Dean. His encouragement and support, especially during the last two years of my Ph.D. program, was very valuable to me. I sincerely thank the Easten Technology Leadership program for their generous financial support, which I received over the course of my Ph.D. program. I would also like to acknowledge my advisors for two of my masters degrees: Prof. Lieven Vandenberghe, who advised me during my Engineering master's program at UCLA, and Prof. Rasoul Haji, who advised me during my Engineering master's program at Sharif University of Technology.

In addition to all those that I named above, several other people made my Ph.D. years an enjoyable experience. I would like to acknowledge the presence and the support of my friends and officemates at UCLA. Firstly, I would like to mention my good friend and classmate George. We started the Ph.D. program together and we are now

finishing it together. We have many wonderful and unforgettable memories. Secondly, I would like to thank those Ph.D students who graduated before me, including Suresh, Steve, Dimitris, and Foad. They provided precious guidance to me during my Ph.D. program and job search. Finally, I would like to thank Ankur, Caitlin, Aparupa, Jay, Puru, Margaret, Wei, Boyoun, Paul, Sandeep, Priya, Christian, Elizabeth, Jim, Sara, and Vanessa with whom I had a great time inside as well as outside the office.

Last but not least, I would like to thank my family for the love and support that they gave me throughout my life and especially during my graduate studies. First and foremost, I would like to thank my mother, who did everything she could to help me succeed in my education. I lost her very early, but I will never forget what she has done for me. I am sure if she were alive, she would be elated about me getting my Ph.D. That is why I am dedicating my dissertation to her. Another person who would have felt the same is my maternal grandfather, whose dream it was to see me earn my Ph.D. Third, I am indebted to my kind father for his extensive support throughout my life. Fourth, I wish to thank my loving younger sister who has been my closest friend. Finally, I would especially like to thank my husband, Amir. He was next to me from the first day until the last day of my Ph.D. program. His encouragement, understanding, and patience helped me move forward throughout this journey. It would not have been possible for me to succeed without his support.

VITA

2005	B.S., Industrial Engineering University of Science and Technology, Tehran
2007	M.S., Industrial Engineering Sharif University of Technology, Tehran Thesis: *Mathematical modeling for selecting suppliers and batch quantity*
2009	M.S., Electrical Engineering University of California, Los Angeles Thesis: *Optimization problems with stochastic dominance constraints*
2010	M.A., Economics University of California, Los Angeles
2008-2013	Graduate Student Researcher UCLA Anderson School of Management, Los Angeles

PUBLICATIONS

1. Rahmani, M., 2012, Contracting and the Dynamics of Collaboration. The MSOM Society Student Paper Competition: Extended Abstracts of 2011 Winners, Manufacturing & Service Operations Management 14(2), pp. 344347.

2. Rahmani, M., G. Roels, and U. Karmarkar, 2012, Team Size in Collaborative Projects: The Effect of Leadership Structure Proceedings of MSOM, Columbia University, New York

3. Rahmani, M., G. Roels, and U. Karmarkar, 2011, Contracting and the Dynamics of Collaboration Proceedings of MSOM, University of Michigan, Ann Arbor

Chapter 1

Contracting and Work Dynamics in Collaborative Projects

*

1.1 Introduction

Many knowledge-intensive projects, such as research and development, high-technology, and management and information technology (IT) consulting, involve collaboration between a client and a vendor over a finite time horizon. For instance, a firm may hire Xoriant, a software product development and engineering company, to create a software application (e.g., a mobile application) in a specified time window (e.g., 3 months), with a deadline that is set by the client based on internal needs. Executing the project typically requires close collaboration between the client firm and Xoriant so as to address the firm's market needs, choose the adequate technology, and inte-

*This chapter is a version of Rahmani, M., Roels, G., and Karmarkar, U.S., 2013. Contracting and Work Dynamics in Collaborative Projects. *Submitted for review.*

grate the application within the firm's core business processes (Gaitonde 2013). As another example, a faculty member may hire a PhD student to work on a research paper. Similarly, industrial research and design is often done collaboratively (Hopp et al. 2009).

Although knowledge-intensive projects may be subject to a tight deadline (e.g., release of a new product, graduation of a PhD student), the content of such projects is often loosely defined at the outset, even after being disambiguated. Specifically, the choice of what components to include in a mobile application (e.g., applets to upload videos, traffic analysis tools) or how far a research idea should be pursued is often revised on the fly depending on the project progress, the potential benefit from such additions, and the time remaining until the deadline.

In such projects, how should collaboration be managed? In a software application development project, should the client outsource most of the work to the vendor, with the risk that it may take longer to integrate it with its internal processes? Or should the client bear more responsibility for the development of the application, with the risk that it may take more trial-and-error to make it run seamlessly? Or should they work together? Similarly, should an advisor let her PhD student work alone on a research project or should she be more involved in it? These decisions are often dependent on the state of completion of the project (e.g., when a research paper is about to be submitted) and on the deadline (e.g., the PhD student's graduation date).

The work dynamics typically depend on the parties' choice of contract. A common feature of knowledge-based work is indeed the joint nature of output, making it impossible to infer the respective inputs from the output (Fuchs 1968). For instance, it is difficult from the outside to attribute the quality of a software user interface to either the firm owning the software or its software developer. Similarly, a jointly authored research paper does not indicate the authors' relative contributions. This is in contrast to many physical projects for which output can be disentangled in proportion to the relative inputs. Such output jointness creates contractual difficulties (Karmarkar and Pitbladdo 1995) in case of effort unverifiability (Holmström 1982). In that case, which contract should be adopted? Can simple contracts perform well?

To answer these questions, we consider a knowledge-intensive project that has already been "disambiguated" and needs to be executed by a vendor and a client. Specifically, we consider a situation in which a vendor has already been hired and the scope of the project has already been defined (e.g., building a mobile application). What remains to be decided is the choice of components to be executed (e.g., video applets, traffic analysis tools) and who should execute them. Each component addition moves the project to an upper state, but not all components need to be completed before the deadline. Because low-hanging fruits tend to be executed first, the final reward from the project is assumed to be increasing concave in its completion state.

Consistent with the collaborative, iterative, and stochastic nature of knowledge-intensive processes (Kieliszewski et al. 2010), we model the collaborative work process as a finite-horizon stochastic game (Shapley 1953, Sobel 1971). That is, the state of the project evolves according to a Markov chain and the transition probabilities depend on the parties' choice of effort levels. In particular, the probability of reaching a higher-value state increases in the parties' effort level. For instance, the development of a software application will progress faster if the client and the vendor work together (Gaitonde 2013), or a research result may be established more quickly when two researchers work together.

We characterize the collaborative dynamics first when efforts are contractible and then when efforts are not contractible. When efforts are contractible, the first-best (FB) outcome can be attained. In that case, the players should both exert high effort when the project is near completion, i.e., either when it has reached a high state or when there is limited time left until the deadline. Otherwise, only one player needs to exert high effort.

When efforts are not contractible, the FB outcome is in general not achievable because of moral hazard (Holmström 1982); and the dynamics of collaboration depend on the type of contract adopted by the players. Reward-sharing (RS) contracts (e.g., sharing incremental revenue from a marketing campaign, co-authoring a research paper) are attractive because they align incentives well; yet, they can only be implemented when the outcome is measurable. By contrast, fixed-fee (FF) contracts, which pay the vendor a bonus upon the project completion, are often simpler to implement. However, they only provide incentives for the vendor to exert high effort when the bonus is within reach, i.e., in the high states of the project. In addition, they may give rise to client's scope creep. Specifically, the client may end up pushing the vendor to keep working on the project beyond reasonable levels. Similarly, time-and-materials (TM) contracts, which pay the vendor a per-period fee until project completion, are also simpler to implement, but they could give rise to vendor's scope creep. In that case, it is the vendor who may keep expanding the scope of the project. Can these simple contracts, despite their shortcomings, perform well? If the project output is not measurable, when are FF contracts preferable over TM contracts?

Although RS contracts are well known to achieve high efficiency, their adoption is limited to projects for which the reward is contractible, which is rather uncommon (Hopp et al. 2009, Gaitonde et al. 2013). When the reward is not contractible, only simple contracts, such as FF and TM contracts, are available. Although these simple contracts perform poorly in static settings (see, e.g., Roels et al. 2010), we find that they may perform well in dynamic settings. However, the contract choice must be judiciously made based on the project characteristics. Specifically, FF contracts tend to outperform TM contracts when the project reward is high; when the costs of joint work relative to individual work are low; and when the chances of success through joint work are significantly higher than those associated with individual work. Otherwise, TM contracts outperform FF contracts.

The remainder of this chapter is organized as follows. We review the related literature in the next section and present the model in §3.3. We characterize the FB solution in §3.4 and study the game dynamics for RS, FF, and TM contracts in §1.5. Benchmarking the equilibrium dynamics under RS, FF, and TM contracts to the FB work dynamics indicates when those contracts will perform well. We numerically compare the relative efficiency of the three contract types and validate our analytical prescriptions in §1.6. We present our conclusions in §1.7. All proofs appear in Appendix A (§4.1).

1.2 Literature Review

This chapter builds upon the literature on moral hazard in teams and contracting and upon the literature on new product development, which we review next.

Moral Hazard in Teams and Contracting. Knowledge-intensive projects often involve multiple players working toward a common output (Fuchs 1968, Karmarkar and Pitbladdo 1995), resulting in a situation of double moral hazard (Holmström 1982). The literature on double moral hazard and the dynamics of collaboration can be classified in four categories, depending on whether the project is deterministic or stochastic and on whether efforts are simultaneous or sequential. This chapter considers a stochastic game with simultaneous moves.

Studying deterministic team projects, Admati and Perry (1991) and Varian (1994) consider sequential moves, and Marx and Matthews (2000) and Demirezen et al. (2013) consider simultaneous moves. Marx and Matthews (2000) show that, with a linear reward function, the project is completed in equilibrium only if a bonus is granted upon completion of the project; otherwise no effort is exerted in equilibrium. We show that a less extreme equilibrium outcome, i.e., with positive effort, arises when the reward function is (strictly) concave. We consider a bonus in the context of fixed-fee contracts and show that the parties' contribution is decreasing over time, consistent with Marx and Matthews (2000). Besides FF contracts, we also study RS and TM contracts and compare their relative efficiency. More importantly, we consider a stochastic, and not deterministic, project. Demirezen et al. (2013) consider a deterministic double-moral hazard problem, in which the rate of reward is a function of the cumulative joint effort. While their model applies well to long-term projects, in which relationships must be built over time, we consider here a short-term project, in which the reward is only collected at the end of the project. Accordingly, the work dynamics are fundamentally different. With a cumulative reward, the equilibrium effort levels should decrease over time, whereas we find that, with discrete reward, efforts should increase over time as well as with the state of the project. We thus complement their study by considering a short-term, discrete (stochastic) project, as opposed to a long-term, cumulative (deterministic) project.

Studying stochastic team projects with simultaneous moves in static and dynamic settings, respectively, Teoh (1997) and Bonatti and Horner (2011) analyze informational issues in team projects, such as uncertain project outcome and duration. We contribute to this literature by adopting a finer view on collaborative dynamics: In contrast to Teoh (1997) and Bonatti and Horner (2011) who consider a one-state project, we consider here a multi-state project and therefore characterize the work dynamics with respect to both time and state. In particular, we show that time and state can in general not be aggregated because the optimal or equilibrium working mode is time- and state-dependent. Specifically, we find that the parties should increase their effort as deadline gets closer similar to Bonatti and Horner (2011), but also as the state of the project gets closer to completion, which are distinct events in a stochastic setting. Besides characterizing the work dynamics, we make contractual prescriptions for projects with dynamic double moral hazard.

How much efficiency can be generated with double moral hazard depends on the type of contract. In a static setting, Bhattacharyya and Lafontaine (1995) show that the second-best outcome can be achieved with a reward-sharing contract and Roels et al. (2010) study the trade-off between moral hazard and monitoring costs. Instead of characterizing the nature of second-best contracts in a dynamic setting, which may turn out to be very complex and therefore impractical, we study the dynamics of collaboration under such common contracts as fixed-fee, time-and-materials, and reward-sharing contracts (Czerniawska and Smith 2010, Sheedy 2010) and show these simple contracts can perform well. In operations management, Plambeck and Taylor (2006), Bhaskaran and Krishnan (2009), Kwon et al. (2010), Roels et al. (2010), and Kim and Netessine (2013) study the performance of similar contracts for static (or repeated) joint-production games. We complement these studies by opening the "black box" of their abstract production function and by characterizing the collaborative dynamics.

New Product Development Projects. Hopp et al. (2009) overview the key differences between blue- and white-collar processes, and highlight that the latter are inherently more knowledge-intensive and creative than the former, and often collaborative. New product development projects are a canonical example of knowledge-intensive projects (Ozkan et al. 2013). Two issues pertain to such projects: team organization and work scheduling. From the perspective of team organization, the key issues are how to allocate the work (Tripathy and Eppinger 2012), how to structure the team in terms of both hierarchy (Sting et al 2012) and diversity (Kavadias and Sommer 2009, LiCalzi and Surucu 2012), and how to provide incentives to the team members, either financial (Mihm 2010, Bayiz and Corbett 2005, Wu et al. 2013) or motivational (Huberman and Loch 1996). In addition, work must be scheduled to shorten development time (Ha and Porteus 1995, Loch et al. 2001), reduce project ambiguity (Terwiesch and Loch 2004, Loch and Terwiesch 2005), or leverage flexibility (Huchzermeier and Loch 2001, Santiago and Vakili 2005). Considering an exogenous team size and hierarchy (i.e., a client and a vendor) and an exogenous (stochastic) work process, this research bridges the gap between these two streams of literature by jointly characterizing how to dynamically structure the work process between two team members and what incentive mechanism to adopt as a function of the type of project. These two issues are indeed intrinsically linked given the impact of the contractual arrangement on the collaborative dynamics.

1.3 Model

In this section, we introduce a model of dynamic collaborative process. We consider a vendor (v) and a client (c), engaged in a multi-period, multi-stage stochastic project with finite deadline T. The deadline is exogenous and typically dictated by the client's

Table 1.1: Actions and work outcomes

Client\Vendor	Work W	Not work N
Work W	Duo	$Solo^c$
Not work N	$Solo^v$	$Finish$

internal constraints (Gaitonde et al. 2013). Let $\mathcal{N} = \{v, c\}$ denote the set of players and $\mathcal{T} = \{1, \ldots, T\}$ be the set of time periods. For each period $t \in \mathcal{T}$, the state of the project x_t is assumed to belong to a totally ordered set $\mathcal{X}_t$. Without loss of generality, we assume that $\mathcal{X}_t \subset \mathbb{Z}^+$. Once the project is stopped in state x_t, a common reward $R(x_t)$ is collected. The reward function $R(x_t)$ is assumed to be increasing concave, therefore making it optimal to stop the project beyond a certain state. For instance in a mobile application development project, x_t could represent the number of features included in the application (e.g., videos, traffic analysis tools). Having more features is obviously better, but if low-hanging fruits are tackled first, the benefit of additional features can be reasonably assumed to be decreasing. Time is discounted at a rate $\delta \leq 1$, making early project completion desirable.

In each period $t < T$ and state x_t, players must decide their effort levels. For simplicity, we consider binary effort levels, as is common in the project management literature (Tripathy and Eppinger 2012, Sting et al. 2012) and the principal-agent contracting literature (see, e.g., Laffont and Martimort 2002). Specifically, we assume that each party can either work (W) or not work (N). We thus consider an extreme situation in which low effort is essentially equivalent to not working. That is, exerting low effort entails virtually no cost and the project stalls if both parties choose to exert low effort. Although efforts may in reality be continuous, generalizing the model along that dimension generates limited additional insights, while being mathematically cumbersome.

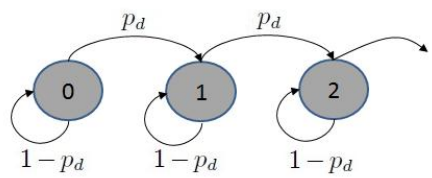

Figure 1.1: State dynamics when both players work (Duo)

Let $\mathcal{A} = \{W, N\}$ denote the set of possible actions, and let $a_t^i(x_t) \in \mathcal{A}$ denote player i's action in period t and state x_t, for $i \in \mathcal{N}$, $t < T$, and $x_t \in \mathcal{X}_t$. As a result of these binary action sets, four outcomes are possible, as depicted in Table 1.1. If both players work, either jointly or in parallel, i.e., $(a_t^c(x), a_t^v(x)) = (W, W)$, the outcome is referred to as Duo. If only player $i \in \mathcal{N}$ work, i.e., $(a_t^c(x), a_t^v(x)) = (N, W)$ or $(a_t^c(x), a_t^v(x)) = (W, N)$, the outcome is referred to as $Solo^i$. Finally, if neither player works, i.e., $(a_t^c(x), a_t^v(x)) = (N, N)$, the project is stopped and the outcome is referred to as $Finish$.

The evolution of the state of the project x_t is assumed to follow a homogenous pure-birth Markov process. Implicit in this model are the following three assumptions. First, task completion times are assumed to be exponentially distributed, e.g., because of repeated experimentation (e.g., Loch et al. 2001, Terwiesch and Loch 2004). See Kwon et al. (2010) for a justification of exponential distributions for modeling project completion times. Second, only same-state and one-step upward transitions are allowed. Although downward transitions could potentially happen in practice (e.g., a result believed to be true turns out to be false), progress is nevertheless expected, i.e., upward probabilities are likely dominating downward probabilities. Similarly, although multi-step transitions could happen in practice, they are presumably of second order relative to one-step transitions. Hence, our pure-birth Markov model can be interpreted as a first-order approximation of a general birth-and-death process in which upward one-step transitions dominate. Finally, assuming homogeneous transition probabilities is almost without loss of generality since the different states of the project are already differentiated by their reward upon project completion. Because Markov equilibria are known to be robust to model misspecifications (Maskin and Tirole 2001), we expect our insights to be robust to those modeling assumptions.

The evolution of the state of the project is governed by the work dynamics. Specifically, the upward transition probabilities are assumed to be increasing in each party's effort level. For instance, the state x_t evolves to state $x_t + 1$ with probability p_d or remains at state x_t with probability $1 - p_d$ if both players work in period t (*Duo*), as depicted in Figure 1.1. Similarly if only player i works ($Solo^i$), there is an upward one-state transition with probability p_s, and no transition with probability $1 - p_s$, for $i \in \mathcal{N}$. Because the chances of success are greater when more players are working (e.g., Huberman and Loch 1996), we assume that $p_d > p_s$. (Marx and Matthews (2000) and Bonatti and Horner (2011) made a similar, but stricter, assumption, namely that $p_d = 2p_s$. However if players are modeled as independent Bernoulli processes, $p_d = 1 - (1 - p_s)^2 = p_s(2 - p_s)$.)

Working is associated with some cost of effort. Although the cost of effort could be taken as independent of how many players work, as in Bonatti and Horner (2011) and Marx and Matthews (2000), this may not necessarily be the case in practice. Working together can indeed induce inefficiencies, e.g., scheduling meetings and travelings (Ha and Porteus 1995), or could in contrast be less costly, e.g., by making work more enjoyable (Maldonando et al. 2007). Accordingly, we denote by $c_d > 0$ the cost of working *Duo* and by $c_s^i > 0$ player i's cost of working $Solo^i$. A player who is not working incurs no cost. We assume that the cost of working alone is lower for the vendor than for the client, i.e., $c_s^v \leq c_s^c$, reflecting the vendor's greater efficiency, which may precisely be the reason for which the client wants to hire him. For instance, Xoriant can operate from offshore locations, giving them a labor cost advantage. We assume that players are symmetric on all other dimensions, that is, they both can do the work equally well (same p_s) but at different costs. Although similar results could have been obtained with identical costs but different probabilities of success, we adopt the above setup to simplify the mathematical expressions. Similarly, we assume identical costs of working *Duo* across players for simplicity. These costs are furthermore assumed to be stationary, which is a reasonable assumption over a short time horizon.

We assume perfect information about the project characteristics and the players' costs. When efforts are contractible, the players can write a contract to maximize the total surplus. We refer to that situation as first-best (FB), and study its dynamics of collaboration in §3.4. When efforts are not contractible, the first-best outcome is in general not achievable, resulting in a situation of double moral hazard (Holmström 1982), studied in §1.5.

1.4 First Best

In this section, we study the FB dynamics of collaboration, i.e., the work processes that maximize the total surplus. In practice, the FB outcome can be attained when the players' efforts are verifiable by writing a profit-sharing contract, i.e., sharing costs and rewards in fixed proportions.

The optimal policy can be identified by solving a finite-horizon dynamic program (DP). We denote by $\mathcal{E}_t^{FB}(x)$ the FB optimal policy in state x and period t. Under the assumption that $c_s^v \leq c_s^c$, $Solo^v$ (weakly) dominates $Solo^c$. Accordingly, $Solo^c$ is never optimal and we refer to $Solo^v$ as *Solo* in the sequel. Hence, only three outcomes are attainable in period t and state x, i.e., $\mathcal{E}_t^{FB}(x) \in \{Duo,\ Solo,\ Finish\}$. Let $V_t(x)$ denote the discounted total surplus in period t and state x, recursively defined as follows:

$$\begin{aligned} V_t(x) &= \max\{V_t(x \mid Duo), V_t(x \mid Solo), V_t(x \mid Finish)\} \quad t = 1, \ldots, T-1, \\ V_T(x) &= V_T(x \mid Finish), \end{aligned} \tag{1.1}$$

in which

$$\begin{aligned} V_t(x \mid Duo) &= -2c_d + \delta[p_d V_{t+1}(x+1) + (1-p_d)V_{t+1}(x)], \\ V_t(x \mid Solo) &= -c_s^v + \delta[p_s V_{t+1}(x+1) + (1-p_s)V_{t+1}(x)], \\ V_t(x \mid Finish) &= R(x). \end{aligned}$$

Throughout the chapter, we denote $\Delta V_t(x) \equiv V_t(x+1) - V_t(x)$ and $\Delta R(x) \equiv R(x+1) - R(x)$.

We present the characterization of the FB optimal policy in Proposition 1.1. We first show that there exists a time-independent state threshold x_f^{FB} above which it is optimal to stop the project and below which it is optimal to keep on working. Effectively, the $Finish$ state threshold x_f^{FB} can be viewed as a self-imposed completion state. Consequently, the DP recursion (1.1) simplifies to:

$$\begin{aligned} V_t(x) &= \max\{V_t(x \mid Duo), V_t(x \mid Solo)\} \quad \forall x < x_f^{FB} \text{ and } t \le T-1, \\ V_t(x) &= V_t(x \mid Finish), \quad \forall x \ge x_f^{FB} \text{ and } t \le T-1. \end{aligned}$$

Comparing $V_t(x \mid Duo)$ and $V_t(x \mid Solo)$ yields the following characterization of the

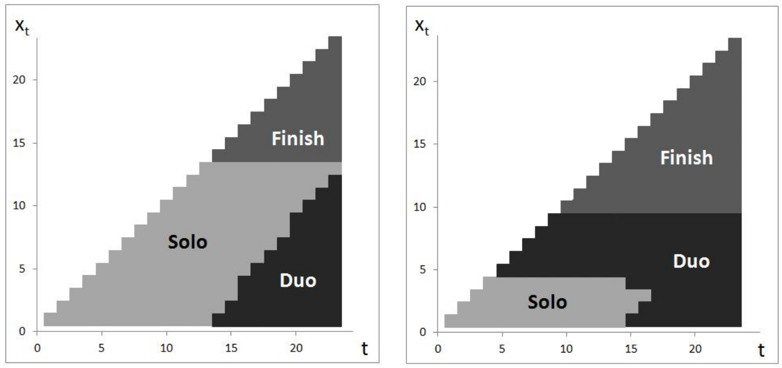

Figure 1.2: FB policy for the cases where $\Delta R(x_f^{FB}-1) \leq (2c_d - c_s^v)/[\delta(p_d - p_s)]$ (left) and $\Delta R(x_f^{FB}-1) > (2c_d - c_s^v)/[\delta(p_d - p_s)]$ (right)
The parameters are: $T = 24$, $c_d = 30$, $c_s^v = 10$, $p_d = 0.89$, $p_s = 0.5$, $R(x) = 950\sqrt{x}$, $\delta = 0.985$ (left) $\delta = 0.975$ (right). The state thresholds are: $x_f^{FB} = 14$, $x_\theta^{FB} = 15$, $x_\varphi^{FB} = 13$ (left) and $x_f^{FB} = 10$, $x_\theta^{FB} = 6$, $x_\varphi^{FB} = 3$ (right).

optimal policy in states $x < x_f^{FB}$ and periods $t \leq T-1$:

$$\mathcal{E}_t^{FB}(x) = \mathit{Duo} \text{ if } \Delta V_{t+1}(x) \geq \frac{2c_d - c_s^v}{\delta(p_d - p_s)}$$

$$\mathcal{E}_t^{FB}(x) = \mathit{Solo} \text{ if } \Delta V_{t+1}(x) \leq \frac{2c_d - c_s^v}{\delta(p_d - p_s)}. \tag{1.2}$$

The next proposition provides a full characterization of the FB dynamics. The proof consists in decomposing the space (t, x_t) into distinct regions in which the value function has such local properties as concavity or increasing differences in (t, x_t) and then applying a standard induction argument. The FB dynamics are illustrated in Figure 1.2. The horizontal axis denotes time, from the project start ($t = 0$) to the deadline ($t = T$), and the vertical axis represents the project state x_t. Because the state increases by at most one unit each period (Figure 1.1), the set of feasible states lies below the 45-degree line, i.e., $x_t \leq t$.

Proposition 1.1. *There exist state thresholds x_f^{FB}, x_θ^{FB}, and $x_\varphi^{FB} < \min\{x_\theta^{FB}, x_f^{FB}\}$ such that* *(i)* $\mathcal{E}_t^{FB}(x) = Finish$ *for all* t *if and only if* $x \geq x_f^{FB}$.

(ii) If $\Delta R(x_f^{FB} - 1) \leq (2c_d - c_s^v)/[\delta(p_d - p_s)]$, *then*

(a) $\mathcal{E}_t^{FB}(x) = Solo$ *for all* $x_\varphi^{FB} \leq x < x_f^{FB}$;

(b) for $1 \leq x < x_\varphi^{FB}$, *there exists a time threshold* $\tau(x)$, *nondecreasing in* x, *such that* $\mathcal{E}_t^{FB}(x) = Duo$ *for all* $T > t > \tau(x)$ *and* $\mathcal{E}_t^{FB}(x) = Solo$ *for all* $x \leq t \leq \tau(x)$.

(iii) If $\Delta R(x_f^{FB} - 1) > (2c_d - c_s^v)/[\delta(p_d - p_s)]$, *then*

(a) if $x_\theta^{FB} < x_f^{FB}$, $\mathcal{E}_t^{FB}(x) = Duo$ *for all* $x_\theta^{FB} \leq x < x_f^{FB}$ *and all* t;

(b) for all $x_\varphi^{FB} \leq x < \min\{x_\theta^{FB}, x_f^{FB}\}$, *there exists a time threshold* $\tau(x)$, *nonincreasing in* x, *such that* $\mathcal{E}_t^{FB}(x) = Duo$ *for all* $T > t > \tau(x)$ *and* $\mathcal{E}_t^{FB}(x) = Solo$ *for all* $x \leq t \leq \tau(x)$;

(c) $\mathcal{E}_t^{FB}(x) = Duo$ *for any* $x \leq \overline{x}_{d,t}^{FB}$ *and* $t < T$, *in which* $\overline{x}_{d,t}^{FB} \equiv \max\{x \leq \overline{x}_{d,t+1}^{FB} - 1 \mid \mathcal{E}_t^{FB}(x) = Duo\}$ *for* $t < T - 1$ *and* $\overline{x}_{d,T-1}^{FB} \equiv x_f^{FB} - 1$.

Proposition 1.1 identifies two phases in which both players should work: when there is limited time left until the deadline (Figure 1.2, left and right) and when the state of the project has reached a high completion state (Figure 1.2, right only). The phase of collaboration induced by the project deadline is established in Proposition 1.1 (ii.b) and (iii.c). Although *Solo* is optimal in the early periods, *Duo* is optimal near the deadline, especially in low-value states. The rush to meet a deadline is a well-reported phenomenon in project management (Repenning 2001, Wu et al. 2013) and beyond. For instance, König and Kleinmann (2005) report that "researchers submit their papers for a conference just before the submission deadline ends, taxpayers in the United States queue in the post office on April 15 to send their tax forms, and students start to study just a few days before the exams." Although this "student syndrome" is often qualified as undesirable (Goldratt 1997, Mackenzie 1997), our analytical model shows that it can in fact be optimal, based on purely rational grounds.

The *Duo* phase when the project state is close to the completion state is established in Proposition 1.1 (iii.a) and (iii.b). Similar to a runner who sprints in the last mile, the players realize that they only need to work hard for a few more periods to reach the completion state and collect their reward sooner. (Interestingly, Scrum, an iterative and incremental agile software development method, extensively uses the concept of sprint; see, e.g., Schwaber 2004.)

The emergence of this latter *Duo* phase arises in particular when $R(x_f^{FB}) > (2c_d p_s - c_s^v p_d)/[(1-\delta)(p_d - p_s)]$. Because x_f^{FB} depends on $R(x)$ only through $\Delta R(x)$, this upper *Duo* phase turns out to arise when $R(x)$ is large, i.e., in high-reward projects. Moreover, for any given completion state x_f^{FB}, this upper *Duo* phase will tend to arise when the cost of joint work (c_d) is low and its success rate (p_d) is high, relative to the cost and success rate of individual work. In addition, this upper *Duo* phase will tend to arise when the discount factor δ is small. Naturally, impatient workers are more willing to sprint in the last stage of the project to collect their reward sooner. However, they are also less willing to push the project further, resulting in a lower completion state x_f^{FB}. The total effects are illustrated in Figure 1.2: As δ decreases from $\delta = 0.985$ (left) to $\delta = 0.975$ (right), the $Finish$ state threshold decreases, the *Duo* phase in the high states of the project appears, and the *Duo* phase induced by the deadline shrinks.

We next investigate how the equilibrium work dynamics under double moral hazard compare to these FB work dynamics.

1.5 Double Moral Hazard

In this section, we assume that efforts are not contractible, which typically happens when they are exerted in the back stage. For instance, one of the parties may come unprepared to a joint meeting and contribute only incrementally. Although one can verify that both parties attended the meeting, verifying their level of preparation is more challenging. See Bapna et al. (2010) for a discussion of effort non-verifiability in the context of joint-production IT projects. We may therefore face a situation of double moral hazard (Holmström 1982).

In addition to efforts, we assume that the state of the project x_t is not contractible at any point in time; otherwise, different contracts could be written for each state of the project. However, the reward $R(x)$ obtained at completion may be contractible. (Mathematically, this is ensured when $R(x)$ is a random variable stochastically increasing in x.) Hopp et al. (2009) take the example of a new product design, which value may be fully understood only after the product has been on the market for some time. Similarly, the quality of an unpublished research paper is rather subjective and hard to contract on, but contracts can be written on the ranking of the journal the paper is ultimately published in or on the number of citations the paper receives after publication.

Accordingly, the payment from the client to the vendor at the project completion, denoted by $g(R(x_t), t)$, can be a function of only the completion time (t) and the reward ($R(x_t)$) whenever it is contractible. Although, there may exist other sources of incentives in practice, such as career concerns and reputation (Bolton and Dewatripont 2005, Hopp et al. 2009), we do not consider them in our model and focus on monetary payments as the only source of incentive for the players. In addition to the payments at the project completion ($g(R(x_t), t)$), we assume that the client makes a

Table 1.2: Payoff matrix $(V_t^c(x), V_t^v(x))$ in period t and state x

Client\Vendor	Work W	Not work N
Work W	$\left(-c_d + \delta \mathbb{E}_d \left[V_{t+1}^c (x+\xi)\right],\right.$ $\left.-c_d + \delta \mathbb{E}_d \left[V_{t+1}^v (x+\xi)\right]\right)$	$\left(-c_s^c + \delta \mathbb{E}_s \left[V_{t+1}^c (x+\xi)\right],\right.$ $\left.\delta \mathbb{E}_s \left[V_{t+1}^v (x+\xi)\right]\right)$
Not Work N	$\left(\delta \mathbb{E}_s \left[V_{t+1}^c (x+\xi)\right],\right.$ $\left.-c_s^v + \delta \mathbb{E}_s \left[V_{t+1}^v (x+\xi)\right]\right)$	$(R(x) - g(R(x), t),$ $g(R(x), t))$

(potentially negative) upfront payment F to the vendor. It is thus possible for the client to capture the total surplus generated from the project after proper deduction of the vendor's reservation utility (Bolton and Dewatripont 2005).

We model the work process as a dynamic stochastic game with finite deadline (Shapley 1953). Let $V_t^i(x)$ be the discounted equilibrium payoff-to-go function of player $i \in \{c, v\}$ upon reaching state $x \in \mathcal{X}_t$ in period t. Table 1.2 depicts a normal form representation of the sub-game in period t and state x. Similar to (1.1), we define the payoff functions associated with the possible game outcomes as follows, for any $t < T, x \in \mathcal{X}_t$, and $i \in \mathcal{N}$:

$$V_t^i(x \mid Duo) = -c_d + \delta \mathbb{E}_d \left[V_{t+1}^i (x+\xi)\right],$$

$$V_t^i(x \mid Solo^i) = -c_s^i + \delta \mathbb{E}_s \left[V_{t+1}^i (x+\xi)\right] \text{ and } V_t^i(x \mid Solo^{-i}) = \delta \mathbb{E}_s \left[V_{t+1}^i (x+\xi)\right],$$

$$V_t^v(x \mid Finish) = g(R(x), t) \text{ and } V_t^c(x \mid Finish) = R(x) - g(R(x), t),$$

in which $\mathbb{E}_k[V_{t+1}^i(x+\xi)] := p_k V_{t+1}^i(x+1) + (1-p_k) V_{t+1}^i(x)$, for $k = d, s$.

Considering a feedback information structure (Başar and Olsder 1999), we focus on pure-strategy Markov-perfect equilibria (Maskin and Tirole 2001), as is common in dynamic games; see, e.g., Adlakha et al. (2012), Doraszelski and Satterthwaite (2010). Markov equilibria are the simplest form of behavior that is consistent with rationality and support the notion that bygones are bygones (Maskin and Tirole 2001). Moreover, they are sub game-perfect and always exist (Fudenberg and Tirole 1991, Theorem 13.1). Although the model could be enhanced to incorporate non-Markovian strategies, such as punishments and renegotiations, we show that there may not be a need to have recourse to such strategies because simple contracts can achieve high efficiency. (Consistent with our findings, less than 5% of Xoriant contracts are renegotiated, according to Gaitonde 2013.)

Similar to Marx and Matthews (2000), we analytically characterize the work dynamics only when a pure-strategy equilibrium exists. In our numerical analysis in §1.6, we however allow for mixed-strategy equilibria whenever a pure-strategy equilibrium does not exist. In addition, the game may admit multiple equilibria in pure strategies. A common situation with multiple equilibria is the chicken game (Osborne 2002), in which each player knows that someone should do the work, but neither of them wants to be the one working. Another common scenario is the stag-hunt game (Osborne 2002), in which players face a dilemma between collaboration (Duo) and safety ($Finish$), as in Marx and Matthews (2000). Although there exist many equilibrium selection rules (e.g., Pareto dominance, risk dominance), the collaborative dynamics turn out to be not significantly affected by the selection rule. For simplicity of exposition, we assume that the client has the authority, or decision right, to select the equilibrium to be played in case of multiple equilibria. We denote by $\mathcal{E}_t(x)$ the selected pure-strategy Markov equilibrium played in period t and state x. For instance, the notations $\mathcal{E}_t(x) = Duo$ and $\mathcal{E}_t(x) \neq Duo$ respectively mean that Duo is or is not the selected pure-strategy Markov perfect equilibrium in period t and state x.

1.5.1 Contractible Reward – Reward Sharing

In this section, we assume that the reward $R(x)$ is contractible. As a result, a reward-sharing (RS) contract can be adopted; that is, $g(R(x),t) = \alpha^v R(x_t)$ and $R(x) - g(R(x),t) = \alpha^c R(x_t)$ with $\alpha^c + \alpha^v = 1$, $\alpha^c \geq 0$, and $\alpha^v \geq 0$. There is also a transfer payment F at time $t = 0$. Reward sharing contracts have become more common in new product development projects (Bhaskaran and Krishnan 2009) and in consulting projects (e.g., Czerniawska and Smith 2010, Sheedy 2010). In addition, these contracts are known to be second-best in static games (Bhattacharyya and Lafontaine 1995).

We find that the collaborative dynamics under RS contracts are by and large similar to the FB solution. However, the players tend to stop the project at a lower state and to work less jointly than in the FB solution. Moreover, the work organization tends to be less structured. Proposition 1.2 characterizes the RS dynamics of collaboration and Figure 1.3 illustrates that result. For the purpose of analytical characterization, we assume that $\Delta R(x_f^{RS}-1) > c_d/[\delta p_d \min\{\alpha^c,\alpha^v\}]+(1-\delta)R(x_f^{RS}-1)/(\delta p_d)$, in which x_f^{RS} is defined in the proposition; our numerical simulations in §1.6 however show that the results of Proposition 1.2 are robust to that assumption.

Proposition 1.2. *There exist state thresholds x_f^{RS}, x_d^{RS}, and $x_{f,T-1}^{RS} \leq x_f^{RS}$ such that if $\Delta R(x_f^{RS} - 1) > c_d/[\delta p_d \min\{\alpha^c, \alpha^v\}] + (1 - \delta)R(x_f^{RS} - 1)/(\delta p_d)$,*

(i) $\mathcal{E}_t^{RS}(x) = Finish$ for all $x \geq x_f^{RS}$;

(ii) If $x_d^{RS} < x_{f,T-1}^{RS}$, $\mathcal{E}_t^{RS}(x) \neq Finish$ for all $t < T$ and $0 < x < x_f^{RS}$;

(iii) If $x_d^{RS} < x_{f,T-1}^{RS}$ and $\alpha^c \leq \alpha^v$, $\mathcal{E}_t^{RS}(x) = Solo^v$ for all $t < T$ and all $x_d^{RS} \leq x < x_f^{RS}$;

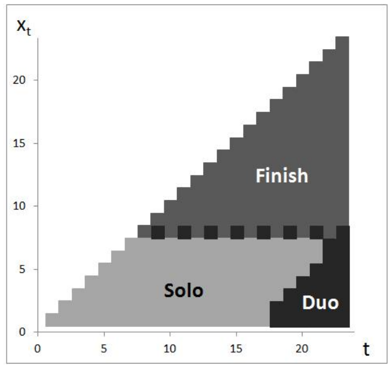

Figure 1.3: RS contract equilibrium outcome
Same parameters as in Figure 1.2 (right) with $\alpha^c = 0.495$. The state thresholds are: $x_f^{RS} = 9$, $x_{f,T-1}^{RS} = 8$, and $x_d^{RS} = 8$.

(iv) $\mathcal{E}_t^{RS}(x) = Duo$ *for any* $x \leq \overline{x}_{d,t}^{RS}$ *and* $t < T$*, in which* $\overline{x}_{d,T-1}^{RS} = \min\{x_d^{RS}, x_f^{RS} - 1\}$ *and* $\overline{x}_{d,t}^{RS} = \max\{x \leq \overline{x}_{d,t+1}^{RS} - 1 \mid \mathcal{E}_t^{RS}(x) = Duo\}$ *for* $t < T - 1$.

As shown in Proposition 1.2, the collaboration dynamics under an RS contract are in general similar to the FB solution (Proposition 1.1). However, the players turn out to stop the project at a lower state (i.e., $x_f^{RS} \leq x_f^{FB}$) because of double moral hazard when either $\alpha^c \leq \alpha^v$ or $x_d^{RS} < x_{f,T-1}^{RS}$. Also, the highest state in which they work *Duo* is lower than in the FB solution due to free-riding.

In addition, the work organization in an RS game appears less structured than in the FB solution. As illustrated in Figure 1.3, there may not exist a time-independent stopping threshold if $x_d^{RS} \geq x_{f,T-1}^{RS}$, preventing any clear expectation about the final state of the project if there was no deadline. Moreover, there may not exist an equilibrium in pure strategies or there may exist multiple equilibria, e.g., chicken games or stag-hunt games, which could lead to conflicts in practice. Hence, the complexity of managing collaboration under RS contracts may be another impediment to their adoption, besides the difficult of measuring outcomes.

1.5.2 Non-Contractible Reward – Fixed-Fee and Time-and-Materials

In many projects, the reward is not contractible (Hopp et al. 2009). For instance, it may be difficult to assess, and therefore to contract on, the value of a strategic management consultant report. In that case, the payment from the client to the vendor may only be a function of the completion time, i.e., $g(R(x), t) = g(t)$. We next consider the following two pure contractual forms:

1. Fixed-Fee (FF): $g(t) = b$, with $b \geq 0$.

2. Time-and-Materials (TM): $g(t) = \sum_{s=1}^{t-1} \delta^{s-t} f$, with $f \geq c_s^v$.

There is also a transfer payment F at time $t = 0$.

The FF contract pays the vendor a fixed bonus b upon project completion, as is common in IT consulting projects. Note that, since the project state is non-contractible, this bonus is not contingent upon reaching a certain state, but is rather paid when both players agree to stop the project. Although it is time-independent, it incentivizes the vendor to work so as to collect b sooner.

By contrast, the TM contract pays the vendor a constant fee f every period until the project is completed, as is common with independent contractors (e.g., in IT maintenance). Note that since efforts are not contractible, this fee is not contingent on the vendor's active work, but is instead paid every period until both players agree to stop the project. The incentives associated with TM contracts are thus opposite to those associated with FF contracts, in the sense that they reward the vendor for keeping on working on the project.

Although $g(t)$ could in principle be any nonlinear function of the completion time, we focus on fixed and linear payments for simplicity. In addition, contracts could have more stringent terms, such as giving the client the right to stop the project at any point in time, whereas in our model stopping must arise in equilibrium. Our first-order analysis allows us to derive clear insights onto the dynamics of collaboration that would arise under more complicated settings. More importantly, we show in §1.6 that these simple contracts can perform very well and that there is therefore limited value considering more complicated contracts or more stringent terms.

Fixed-Fee.

We first characterize the collaboration dynamics under FF contracts, i.e., when the client pays the vendor a fixed fee $b \geq 0$ upon project completion. We show that FF contracts may lead to a higher completion state than the FB solution and, although they do not induce joint work near the project deadline, they may give rise to joint work when the project is near its completion state and the deadline is still far.

The next proposition characterizes the equilibrium work dynamics under FF contracts and Figure 1.4 illustrates the results. For tractability, we assume that the client's net payoff when the project is completed $(R(x_f^{FF}) - b)$ is greater than the vendor's (b); our numerical simulations in §1.6 however show that our results are robust to that assumption.

Proposition 1.3. *There exist state thresholds x_f^{FF} and x_θ^{FF} such that when $R(x_f^{FF}) \geq 2b$,*

(i) $\mathcal{E}_t^{FF}(x) = Finish$ *for all t if and only if $x \geq x_f^{FF}$;*

(ii) If $x_f^{FF} < x_\theta^{FF}$, $\mathcal{E}_t^{FF}(x) = Solo^c$ for all $t < T$ and $x < x_f^{FF}$;

(iii) If $x_f^{FF} \geq x_\theta^{FF}$, for any $x < x_f^{FF}$ there exists a time period $\tau(x)$, increasing in x, such that $\mathcal{E}_t^{FF}(x) = Solo^c$ for $t \geq \tau(x)$ and it is Duo for $t < \tau(x)$.

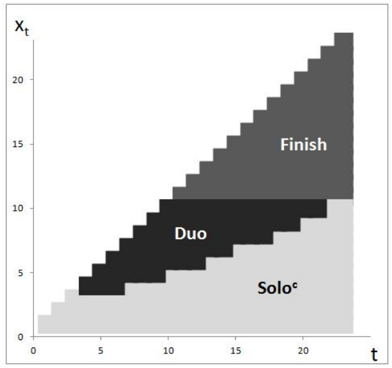

Figure 1.4: FF contract equilibrium outcome
Same parameters as in Figure 1.2 (right) with $p_s = 0.35$ and $b = 1500$. The state thresholds are: $x_f^{FF} = 11$ and $x_\theta^{FF} = 6$.

Proposition 1.3 (i) demonstrates that there exists a time-independent threshold x_f^{FF} above which the players stop working and below which they keep on working. Moreover Proposition 1.3 (ii) and (iii) show that there is no *Duo* work near the deadline. Specifically, in states below x_f^{FF} and near the project deadline, the equilibrium outcome is $Solo^c$, i.e., the client is the only one working. Intuitively, the vendor knows that the deadline will be reached soon, and therefore that he will receive b soon, irrespective of how much work he puts in. Although the vendor may appear to be reluctant to work at all, as would be the case in a static setting, joint work may still occur. In particular, Proposition 1.3 (iii) shows that the equilibrium outcome is *Duo* in the high states if there remains ample time before the deadline. In that case, the vendor realizes that the project is near its completion state and if he puts in some effort, he would receive his bonus sooner.

It turns out that the condition characterizing the upper *Duo* phase, i.e., $x_f^{FF} \geq x_\theta^{FF}$, is satisfied when the project has a high reward (high $R(x_f^{FF})$) and when the players are impatient (low δ). It is also satisfied when the success rate from joint work p_d is high relative to p_s and when the cost of joint work c_d is low. These conditions are thus similar to the conditions under which the upper *Duo* phase emerged in the

FB solution, described in Proposition 1.1 (iii) (see Figure 1.2, right). However unlike the FB solution, this *Duo* phase near the completion state tends to be larger in the earlier periods; the slope of that *Duo* phase is thus reversed compared to the FB solution.

In addition, comparing the completion state thresholds x_f^{FF} and x_f^{FB} reveals that FF contracts lead to effort under-provision when the discount factor δ is large and when the vendor's bonus b is small, and to effort over-provision otherwise. In particular, $x_f^{FF} \leq x_f^{FB}$ when $(1-\delta)b < c_s^c - c_s^v$ and $x_f^{FF} \geq x_f^{FB}$ when $(1-\delta)b \geq c_s^c - c_s^v$ and $x_\theta^{FB} \geq x_f^{FB}$. In contrast to RS contracts, the completion state can thus be higher than optimal with FF contracts. This is because the client wants to postpone the time she would need to pay the bonus to the vendor and, at the same time, increasing her chances of collecting a higher reward $R(x_f^{FF})$. Hence, the bonus does not offer only benefits, in terms of providing incentives to the vendor, it may also encourage the client to creep the scope of the project, as is commonly observed with FF contracts (Sheedy 2010).

Time-and-Materials.

We next characterize the collaboration dynamics under the TM contract, i.e., when the client pays the vendor a fixed fee $f \geq c_s^v$ every period until the project is stopped. We find that TM contracts never lead to joint work near the deadline or in the upper states of the project.

The next proposition characterizes the equilibrium work dynamics under the TM contract and Figure 1.5 illustrates the results. For the purpose of analytical characterization, we consider the case that $f \geq c_d$; our numerical simulations in §1.6 however show that the results of Proposition 1.4 are robust to that assumption.

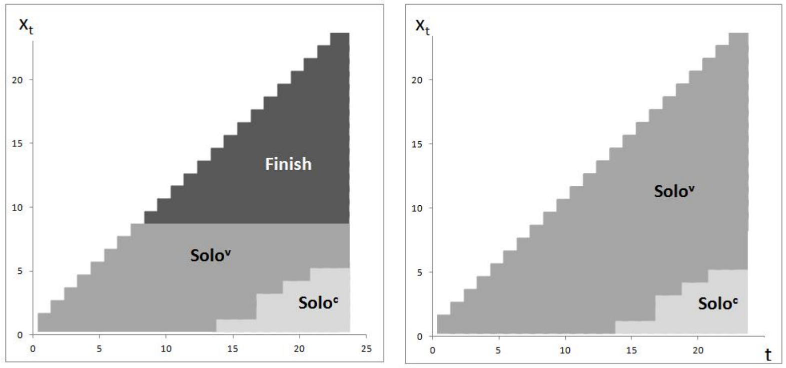

Figure 1.5: TM contract equilibrium outcome
Same parameters as in Figure 1.2 (left) with $p_d = 0.65$ and $f = 10$ (left) or $f = 11$ (right). The state thresholds are: $x_f^{TM} = 9$, $x_{s^v}^{TM} = 6$, and $x_{s^c}^{TM} = 5$.

Proposition 1.4. *There exist state thresholds x_f^{TM}, $x_{s^v}^{TM}$, and $x_{s^c}^{TM}$ such that if $f \geq c_d$*

(i) If $f = c_s^v$, then

(a) $\mathcal{E}_t^{TM}(x) = Finish$ if and only if $x \geq x_f^{TM}$;

(b) $\mathcal{E}_t^{TM}(x) = Solo^v$ for all t and $x_{s^v}^{TM} \leq x < x_f^{TM}$.

(ii) If $f > c_s^v$, then

(a) $\mathcal{E}_t^{TM}(x) = Solo^v$ for all t and $x \geq x_{s^v}^{TM}$;

(b) $\mathcal{E}_t^{TM}(x) \neq Finish$ for all t and $x < x_{s^v}^{TM}$.

(iii) $\mathcal{E}_t^{TM}(x) = Solo^c$ for any $x \leq \overline{x}_{s^c,t}^{TM}$ and $t < T$, in which $\overline{x}_{s^c,t}^{TM} = \max\{x \leq \overline{x}_{s^c,t+1}^{TM} - 1 \mid \mathcal{E}_t^{TM}(x) = Solo^c\}$ for $t < T - 1$ and $\overline{x}_{s^c,T-1}^{TM} = \min\{x_{s^v}^{TM}, x_{s^c}^{TM}\} - 1$.

Proposition 1.4 (i.a) and (ii.a) demonstrate the existence of thresholds (x_f^{TM} and $x_{s^v}^{TM}$) above which the equilibrium outcome is *Finish* if $f = c_s^v$ (Figure 1.5, left) and $Solo^v$ if $f > c_s^v$ (Figure 1.5, right). Moreover, Proposition 1.4 (i.b) shows that the equilibrium outcome is $Solo^v$ for states above $x_{s^v}^{TM}$ and below x_f^{TM}; that is, there is no joint work in the upper states of the project, i.e., when $x \geq x_{s^v}^{TM}$. Finally, Proposition

1.4 (iii) shows the equilibrium outcome is $Solo^c$ for states below $\min\{x_{s^v}^{TM}, x_{s^c}^{TM}\}$ and near the project deadline, which implies that there is no joint work near the deadline either, unlike the FB solution.

Intuitively, there is no incentive for the vendor to jointly work with the client. When there is limited time until the deadline, the client is willing to work if the project state is low so as to increase her reward, otherwise deemed too low. Seeing the client working hard, the vendor chooses to free ride since his fee is not affected by the ultimate state of the project. Although this outcome may appear caricatured, it raises the importance of free riding in TM contracts and therefore the potential benefits of setting up monitoring systems to verify the vendor's effort (Roels et al. 2010).

On the other hand, the vendor exerts high effort in the high states of the project, even though (or because) the client would like to stop the project, so as to keep being paid until the deadline. This equilibrium pattern is also consistent with the common belief that TM contracts can drag on forever because of vendor's scope creep (Sheedy 2010). This raises the importance of adopting a different structure of decision rights under TM contracts, such as letting the client unilaterally choose when to stop the project. Nevertheless, we show in §1.6 that TM contracts often perform well, despite these shortcomings.

Comparing the completion state thresholds x_f^{TM} and x_f^{FB} reveals that TM contracts lead to effort over-provision when $f > c_s^v$, because the players never stop the project before the deadline. In contrast when $f = c_s^v$, the project completion state is lower under the TM contract than in the FB solution ($x_f^{TM} \leq x_f^{FB}$); in that case, the TM contract leads to effort under-provision. Overall, the work dynamics under TM contracts are similar to the FB solution when joint work is neither efficient (c_d high) nor effective (p_d low), leading to predominantly *Solo* work.

1.6 Contract Efficiency

In this section, we numerically validate our analytical contractual prescriptions and show that simple contracts, such as FF and TM contracts, may perform well when they are chosen judiciously.

In our analytical characterization in §1.5, we obtained that RS contracts lead to similar dynamics to the FB solution, but with some inefficiencies due to moral hazard; FF contracts lead to joint work in the high states, similar to the FB solution, when $R(x)$ is large, δ is small, p_d/p_s is large, and c_d/c_s^v is small; and TM contracts primarily lead to individual work, similar to the FB solution when $R(x)$ is small, δ is large, p_d/p_s is small, and c_d/c_s^v is large. Based on this analytical characterization, we infer that (i) if the reward is contractible, RS contracts will tend to perform well, similar to static settings (Bhattacharyya and Lafontaine 1995) and that (ii) if the reward is noncontractible, FF tend to outperform TM contracts when $R(x)$ is large, δ is small, p_d/p_s is large, and c_d/c_s^v is small and vice versa.

To validate these contractual prescriptions, we compare the performance of the three contracts over randomly generated problem instances. We say that a contract outperforms another if it generates higher total surplus. Since we consider an upfront fixed transfer payment F when $t = 0$ for all contracts, the total surplus is a proxy for the client's total payoff (Bolton and Dewatripont 2005). Loss of efficiency is measured as the relative loss in total surplus from the FB solution.

We randomly generated 500 sets of parameter values over the ranges of parameters depicted in Table 1.3. For each contract type, we find the optimal payment parame-

Table 1.3: Ranges of parameters

Parameters	Ranges
Reward $R(x) = q + rx^k$	$q \sim U[0, 10, 000]$, $r \sim U[500, 1, 500]$, $k \sim U[0.35, 1]$
Costs	$c_d \sim U[0, 50]$, $c_s^c \sim U[0, 50]$, $c_s^v \sim U[0, c_s^c]$
Transition Probabilities	$p_d \sim U[0.5, 1]$, $p_s \sim U[0.2, p_d]$
Discount Factor	$\delta \sim U[0.955, 1]$

Table 1.4: Loss of efficiency

Loss of efficiency	RS	FF	TM	Best b/w FF and TM
Average	0.25%	6.43%	9.90%	5.81%
25% quantiles	0%	0%	1.37%	0%
50% quantiles	0%	1.59%	5.64%	0.72%
75% quantiles	0%	9.19%	13.73%	8.41%
Maximum	21.53%	56.30%	63.87%	56.30%

ters, i.e., the optimal reward share α for RS contracts ($\alpha^v \in [0, 1]$), the optimal bonus b for FF contracts ($b \geq 0$), and the optimal fee f for TM contracts ($f \geq c_s^v$). When a pure-strategy equilibrium did not exist, we considered the unique mixed-strategy equilibrium. With multiple pure-strategy equilibria, we selected the equilibrium that maximized client's payoff, consistent with our analytical characterization. Table 1.4 depicts the average, quartiles, and maximum loss of efficiency associated with each contract type, and the same statistics associated with the best contract between the FF and TM contracts.

Overall, RS contracts outperform FF and TM contracts. Consistent with our analytical results in §1.5.1, which showed that the collaborative dynamics under RS contracts are similar to the FB solution, our numerical analysis reveals that the total surplus generated under RS contracts is very close to the maximum total surplus. In fact, in 431 out of 500 instances, the RS contract attains 100%-efficiency; moreover the average loss of efficiency across all the problem instances is 0.25%. This high efficiency thus demonstrates the robustness of the second-best optimality of RS contracts in static settings (Bhattacharyya and Lafontaine 1995) to dynamic settings.

When the reward is not contractible, which is more the norm than the exception in knowledge-intensive work (Hopp et al. 2009), only FF and TM contracts are available. Although such simple contracts would perform very poorly in static settings without effort monitoring (Roels et al. 2010), Table 1.4 reveals that they can perform well in our dynamic setting: the median loss of efficiency is only 1.59% for FF contracts and 5.64% for TM contracts, which is remarkable despite their simplicity. This illustrates the importance of using time (an in particular deadlines) and the project state as a way to foster the incentives associated with these simple contracts.

Should the client be concerned about the type of contractual arrangement? Presumably, one of the reasons for which these simple contracts perform well could be that "anything goes," i.e., that as long as a contract is written, high efficiency can be achieved. As we shall demonstrate next, such presumption would be erroneous, i.e., the choice of contract matters. In fact, FF contracts tend to perform well precisely when TM contracts fail, and vice versa. The last column of Table 1.4 shows that, optimally choosing between FF and TM contracts can further reduce the median loss of efficiency to 0.72%. We next characterize when FF contracts outperform TM contracts.

As shown in §3.4, joint work in the high states of the project is more likely to arise in the FB solution when $R(x_f^{FB}) > (2c_d p_s - c_s^v p_d)/[(1-\delta)(p_d - p_s)]$. This condition holds when $R(x_f^{FF})$ is large, δ is small, the cost of joint work c_d is low and its success rate p_d is high, relative to the cost and success rate of individual work. As shown in Proposition 1.3, these are precisely the circumstances under which joint work will arise in the high states of the project under an FF contract (i.e., when

Table 1.5: Dominance of FF versus TM contracts

Subsets of ranked instances	Problem instances are ranked in increasing order of the following:					
	p_d/p_s			$1-\delta$		
	TM>FF	TM=FF	TM<FF	TM>FF	TM=FF	TM<FF
1-100	68	17	15	43	47	10
101-200	29	41	30	33	30	37
201-300	14	33	53	27	17	56
301-400	14	19	67	17	16	67
401-500	10	13	77	15	13	72

Subsets of ranked instances	Problem instances are ranked in increasing order of the following:					
	q			c_s^v/c_d		
	TM>FF	TM=FF	TM<FF	TM>FF	TM=FF	TM<FF
1-100	42	27	31	28	43	29
101-200	34	24	42	32	21	47
201-300	19	26	55	26	23	51
301-400	22	19	59	25	22	53
401-500	18	27	55	24	14	62

Note. 'TM>FF' ('TM<FF') denotes the number of instances, in which TM (FF) contract strictly outperforms FF (TM) contract. 'TM=FF' denotes the number of instances, in which FF and TM contracts generate the same total surplus.

$x_f^{FF} \geq x_\theta^{FF}$). Accordingly, we expect that FF contracts outperform TM contracts under those circumstances. Conversely, only *Solo* work is optimal in the FB solution when $R(x_f^{FF})$ is small, δ is large, the cost of joint work c_d is high and its success rate p_d is low, relative to the cost and success rate of individual work. Since TM contracts only lead to individual work, we thus expect that TM contracts outperform FF contracts precisely under those circumstances.

To validate these prescriptions built upon our analytical characterizations of the work dynamics, we conduct the following experiment. We first rank the 500 problem instances in increasing order of p_d/p_s, $1-\delta$, q (i.e., the location parameter of $R(x)$), or c_s^v/c_d. For each successive set of 100 problem instances, Table 1.5 reports the number of instances in which TM outperforms FF, in which FF outperforms TM, and in which TM and FF generate the same total surplus. For example, when the problem instances are ranked in increasing order of p_d/p_s, the first 100 problem instances (i.e., for which p_d/p_s is small) are such that TM outperforms FF in 68 instances, TM and FF generate the same total surplus in 17 instances, and FF outperforms TM in 15 instances. Considering the following sets of 100 problem instances reveals that, as p_d/p_s increases, TM tends to outperform FF less often and FF tends to outperform TM more often, consistent with our predictions. Similar results hold when the problem instances are ranked in increasing order of $1-\delta$, q, or c_s^v/c_d. Overall, this experiment indicates that FF contracts tend to outperform TM contracts when collaboration is effective (i.e., high p_d/p_s) and efficient (i.e., high c_s^v/c_d), when players are impatient (i.e., high $1-\delta$), and when the project reward is high (i.e., high q), supporting our initial predictions based on our analytical characterization of the collaborative dynamics. Hence, contract choice matters, and that choice must be judiciously made based on the project characteristics.

These simple contracts could obviously be enhanced to mitigate undesirable behaviors, by contractually specifying a completion milestone (if feasible), by giving the client full decision rights to unilaterally stop the project, or by making payments a nonlinear function of the completion time. However, Table 1.4 reveals that the simple contracts studied here can be very efficient. In most cases, therefore, such contractual enhancements could only marginally improve efficiency. It is only when the performance of both FF and TM contract suffers, as shown in the last row of Table 1.4, that these more complex contract enhancements could be desirable.

1.7 Conclusions

In this chapter, we study how contractual arrangements affect the work dynamics between a client and a vendor in finite-horizon collaborative processes. We identify two phases in which the parties should both exert high effort: when there is limited time before the deadline and when the project has reached a high completion state. A practical implication of our characterization of the first-best (FB) work dynamics is that the end-of-project rush, be it either in the high states of the project or near the deadline, is optimal (and rational) and should be planned for.

When efforts are not contractible, i.e., with double moral hazard, the collaboration dynamics depend on the type of contract. We consider three contract types: Reward-sharing (RS), fixed-fee (FF), and time-and-materials (TM). RS contracts yield similar dynamics to the FB solution, but with less joint work and a lower completion state. RS contracts tend to perform very well, consistent with their second-best nature in static settings. However, their application is restricted to cases where the project reward is measurable. When neither efforts nor the project reward are contractible, only contracts with payments contingent on the completion time, such as FF and TM contracts, can be adopted. FF contracts, which pay the vendor a bonus upon project completion, only provide incentives for the vendor to exert high effort in the high states of the project, thereby leading to joint high effort in those states. Under a TM contract, only the client exerts high effort in the low states of the project to improve her potential reward, whereas only the vendor exerts high effort in the high states so as to prevent the project-and his contract–to be stopped too early. FF contracts tend to dominate TM contracts when the project reward is large, when the players are impatient, and when joint work is more efficient than individual work. Otherwise, TM contracts dominate FF contracts. Overall, our analysis reveals that

Table 1.6: Contractual prescriptions

		Effort: Contractible	Effort: Non-contractible
Reward	Contractible	FB	RS
	Non-contractible		FF if high p_d/p_s, low c_d/c_s^v, low δ, high R TM if low p_d/p_s, high c_d/c_s^v, high δ, low R

simple contracts can perform well, especially when they are chosen judiciously. Table 1.6 summarizes our contractual prescriptions based on the project characteristics.

This work can be extended in several directions. First, one can generalize several modeling assumptions (e.g., continuous efforts, birth-and-death Markov process, giving the client the right to unilaterally stop the project), although we have conducted robustness tests that indicate that the fundamental insights presented here remain unchanged. More fundamentally, this model can be used as a backbone model to address other managerially relevant questions:

- If the project structure is unknown at the outset, i.e., if the project has not been disambiguated, shall players collaborate upfront to jointly learn about it? This is likely, thus making *Duo* work desirable at the beginning and at the end of the project.

- If the project has been disambiguated, but the vendor has not been hired yet, when should the client offer the contract to the vendor? Our analysis of collaborative dynamics suggests that, since joint work occurs only in the later stages of the project, the client may be better off starting working alone and hiring the vendor if not enough progress has been made.

- In the few cases for which both FF and TM contracts fail to perform well, what contractual enhancement would be the most beneficial? Should the contract be renegotiable?

Collaboration also raises questions about team management and organization. Given the ubiquity of collaborative work in today's business processes and the need to adapt traditional process analysis tools to knowledge-based work (Karmarkar et al. 1995, Hopp et al. 2009, Kieliszewski et al. 2010, Staats and Upton 2011), investigating these questions would be worthwhile.

Chapter 2

Team Size in Collaborative Projects: The Effect of Leadership Structure

2.1 Introduction

According to the Bureau of Labor Statistics, 34% of jobs in the United States are white collar (Davenport et al. 2002). Many white-collar processes are inherently more knowledge-intensive, creative, and collaborative than blue-collar (manufacturing) processes (Hopp et al. 2009). Canonical examples of knowledge-intensive projects are new product and software development, which are often done collaboratively by members of an organization, i.e., a research group or an IT department. The major resource required for these collaborative projects is labor. According to Gratton and Erickson (2007), one out of the four traits that are crucial for a project success is the size of a project team. Small teams may not have enough workers to get the work done (effectiveness) and large teams may involve overwhelming cost of coordination (efficiency). One relevant question is: how should team size be chosen? Should one appoint a leader and let her assign workers from the organization to the team? Or should one let the members of the organization voluntarily choose to join the team? On the one hand, a project leader can monitor team members to prevent free-riding (Holmström 1982) or social loafing (Karau and Williams 19931993). On the other hand, project leaders often over-staff the projects. As stated by Hackman (2002), "probably the single biggest mistake team leaders make is believing that the bigger [the team], the better is the right approach [... i.e.,] if the project is a dozen person-months behind, perhaps assigning a dozen extra people to it for one month will get it back on track."

To answer above questions, we consider an *organization* responsible for a knowledge-intensive project, consisting of a group of individuals from which a *team* is formed. We model the staffing decision as a stochastic game over a finite time horizon and study the effect of leadership on the evolution of team size. The following example by Hackman (2002) illustrates the distinction between *organization* and *team*: a start-up company with a dozen founding officers wanted to come up with a reorganization plan. Because of such enormous personal stakes, the CEO could not expect the dozen funding officers to come up with a reorganization plan that they all could accept. As a result, the CEO selected a team of four officers to develop a proposal for the new structure. In this example the dozen founding officers constitute the organization and the four selected officers constitute the team. Although only four officers worked on the proposal, all members of the organization received benefits from the project.

Accordingly, we consider a project in which a fixed reward is collected when the project succeeds. For instance in research projects, the authors collect their reward when their paper or patent is published. Similarly, a department may receive a bonus upon a successful project completion. Moreover, we assume that the reward is equally shared among the members of the organization even if some members have never joined the team. Although this may seem unfair, it often happens in projects. For instance in a study group, all students receive a same grade even though only a few of them worked on the assignment.

We study two leadership structures: teams with no leader (NL) and teams with an internal leader (IL). In teams with no leader (NL), the team size is chosen by the project members themselves in a decentralized fashion, i.e., members voluntarily choose to join or not join the team. In contrast, in teams with an internal leader (IL), the team size is determined by a project leader in a centralized fashion, i.e., the leader assigns members from the organization to the team. The leader herself is an internal member of the organization and is therefore the only one free to decide whether to join the team or not. As a benchmark, we study the first-best solution (FB) in which the team size is set to maximize the total payoff of the organization. We compare the NL and IL equilibrium team sizes when the team size is chosen only once at the beginning of the project and when it is dynamically revised.

We show that there is always under-staffing in projects with no leader because of free riding. In contrast, there can be under- or over-staffing in projects with an internal leader. In particular, there can be over-staffing because the internal leader does not internalize the team members' cost of effort, and there can be under-staffing because she has the incentive to free ride. Comparing the efficiency of the leadership structures, we show that the NL structure generates a higher total surplus than the IL structure when the project reward is in an intermediate range. When the project reward is very large, all members join the team under both IL and NL structures and they both generate the same total surplus. When the project reward is smaller, but still large, it would be optimal that all members join the team. However, there is under-staffing in both IL and NL structure because of free-riding. The IL structure generates higher total surplus than the NL structure because only the leader free rides whereas all members are tempted to free ride in the NL structure. When the project reward is small, both the IL and NL structures are inefficient for opposite causes; i.e., under-staffing in the NL structure and over-staffing in the IL structure. Since free-riding under the NL structure is large with smaller rewards and over-staffing in the IL structure is large with larger rewards, the IL structure dominates the NL structure when the project reward is smaller than a threshold.

The chapter is organized as follows: We review the related literature in the next section. We present our model for the FB solution and for the IL and NL team structures in §2.3. We then characterize the team sizes under these structures and compare their efficiencies in §2.4 under static and dynamic decision-making. We present our conclusions in §2.5. All proofs are gathered in Appendix B (§4.2).

2.2 Literature Review

This chapter studies team size in collaborative knowledge-intensive projects (e.g., new product development). Therefore, it is related to the literatures on new product development, moral hazard, and staffing.

New product development (NPD) projects are canonical examples of knowledge-intensive projects (Ozkan et al. 2013). Loch and Terwiesch (1998) and Terwiesch and Loch (2004) study new product development projects where uncertainty in the project outcome results in rework. Similarly, we explicitly model project uncertainty in this chaper. Most of the work on new product development focus on team coordination, e.g., project overlap (Loch and Terwiech 1998), information sharing (Loch and Terwiech 2005, Marschak and Radner 1972, Garicano 2000), and search and problem solving (Mihm et al. 2010 and Sting, et al 2012, Kavadias and Sommer 2009 and LiCalzi and Surucu 2012). While we abstractly model coordination costs, our focus is on leadership structure similar to Dessein (2007), Sah and Stiglitz (1991) and Christensen and Knudsen (2010)). However, our perspective is different from theirs; i.e., they study the effect of leadership on decision making processes whereas we study the effect of leadership on choosing the size of a project team.

We consider knowledge-intensive projects that involve multiple players working toward a common output (Fuchs 1968, Karmarkar and Pitbladdo 1995), resulting in a situation of double moral hazard (Holmström 1982, Bonatti and Horner 2011). Unlike the literature on moral hazard which study compensation (Hutchison-Krupat and Kavadias 2012) and incentive design (Mihm 2010, Bayiz and Corbett 2005, Wu et al. 2013, and Georgiadis 2012), we assume members' compensation and incentives are determined exogenously. Instead, we model decentralized decision-making similar to Sah and Stiglitz (1988) and Alonso et al. (2008) and study the effect of leadership structure on team efficiency. More importantly, we focus on the allocation of decision rights about the size of a project team. That is, size of a project team is our decision variable, which those papers consider as exogenous.

Since our decision variable is the size of a project team, this chapter is also related to the staffing literature. Most of the literature on staffing decisions consider either manufacturing processes (Van Mieghem 2003, Luss (1982)) or repetitive service operations (Van Mieghem 2013, Pinker and Shumsky 2000). We complement this literature by studying capacity planning of workforce for collaborative projects with moral hazard. This also allows us to analyze the effect of project deadline and reward on staffing decisions.

2.3 Model

In this section, we introduce a dynamic model of a team project. We consider an organization of n members undertaking a project with finite deadline T. Let $\mathcal{N} = \{1, \ldots, n\}$ denote the set of members in the organization. Counting time backwards, we let $\mathcal{T} = \{T, T-1, \ldots, 0\}$ be the set of time periods until the deadline. Similar to Bonatti and Horner (2011), we consider a binary-state project with $x_t \in \{0, 1\}$, in which $x_t = 1$ if the project is completed and $x_t = 0$ otherwise. We assume that the project starts in state 0 at time T. Time is discounted with a discount rate of $\delta \leq 1$, making early project completion desirable. When the project is completed, a common reward R is collected and evenly shared among all n members of the organization, i.e., each individual receives R/n. Such contracts are commonly used to motivate engineers in high technology, software, or biotechnology companies in the form of stock options or cash bonuses; see e.g., Mihm (2010). Similarly co-authorship of research papers and patents in research and development projects effectively act as linear sharing rules. Such linear profit sharing rules turn out to be second-best (Bhattacharyya and Lafontaine 1995) and have been widely studied in the moral hazard literature and are often used in practice (Lambert 2001).

In any given period t, when the team size is k, the project is completed in that period with probability $p(k)$ and remains incomplete with probability $1 - p(k)$. Although the transition probabilities may also be a function of time in practice (e.g., members may learn about the quality of the project or their joint work), we consider memoryless probabilities for simplicity, resulting in a geometrically distributed project duration. For instance, problem solving through trial and error could be the source of this geometric distribution; see also Kwon et al. (2010) for a discussion of that assumption. We assume that the success probability $p(k)$ is increasing in the team size for $k \leq n$ with $p(0) = 0$, reflecting the notion that larger teams are more effective (Hackman et al. 2000, Katzenback and Smith 1993). In addition, we assume that $p(k)$ is concave; that is, the success probabilities exhibit diminishing marginal returns due to the necessity of reconciling diverse opinions and resolving potential conflicts (Becker and Murphy 1992, Huberman and Loch 1996). In fact, Hackman (1998) reports that "research evidence about team performance shows that teams usually do less well– no better– than the sum of their members' individual contributions."

We assume coordination costs of $kc(k) + g(k)$, and allocate the former among the set of active team members and the later among all members of the organization. That is for each team member, there is an individual cost of effort $c(k)$ with $c(0) = 0$. The cost $c(k)$ is assumed to be increasing convex in k, reflecting effort and collaboration costs which involve communications and logistical costs in meeting, scheduling, and traveling (Becker and Murphy 1992). Similarly, Brooks (1975) acknowledged that as each member is added to the project team, the complexity of communications goes up exponentially. In addition, there is a collective project cost $g(k)$ (with $g(0) = 0$) that is evenly shared among all n members of the organization, reflecting the loss of productivity for the organization when the team size is k. For instance in the Cisco ERP implementation project (Austin 2002), the IT group apparently "did nothing else" other than implementing the ERP system that year. This suggests that other projects must have been backlogged resulting in a loss for the whole organization. We also assume that the project cost $g(k)$ is an increasing convex function of k. Although $p(k)$ and $g(k)$ could in principe be dependent on the leadership structure, we take them here as identical for the sake of comparison.

As a benchmark, we first consider the first-best (FB) solution, in which the team size is chosen so as to maximize the total surplus. We denote by k_t^{FB} the FB team size in period t, by V_t^{FB} the project optimal total payoff-to-go in period t, and by $V_t^{FB}(k)$ the project expected total payoff-to-go in period t when the team size is equal to k. The dynamic programming formulation for the FB solution is as follows:

$$V_t^{FB} = \max_{k \leq n} V_t^{FB}(k) \equiv -kc(k) - g(k) + \delta p(k)R + \delta(1 - p(k))V_{t-1}^{FB} \tag{2.1}$$

with $V_0^{FB} = 0$.

We use the notation $\Delta h(i) = h(i+1) - h(i)$ for any discrete function $h(.)$. In addition, we assume that $-kc(k) - g(k) + \delta p(k)R \geq 0$ for $k \in \mathcal{N}$ to guarantee that the project total payoff is positive in any period t and for any team size k. Since the objective function in (2.1), $V_t^{FB}(k)$, is concave in k, k_t^{FB} can be obtained by solving

$$k_t^{FB} = \min\left\{ i \in \mathbb{Z}^+_{\leq n} \mid V_{t-1}^{FB} \geq R - \frac{\Delta[ic(i)] + \Delta g(i)}{\delta \Delta p(i)} \right\}, \tag{2.2}$$

in which $\mathbb{Z}^+_{\leq n}$ is the set of all positive integer values which are smaller than or equal to n. Accordingly, $k_t^{FB} = n$ if and only if $V_{t-1}^{FB} \leq R - \frac{\Delta[(n-1)c(n-1)] + \Delta g(n-1)}{\delta \Delta p(n-1)}$.

We next present the models of teams with and without an internal leader. We then study in §2.4 the effect of leadership structure on team size and project success.

2.3.1 No-Leader (NL) Structure

In the NL structure, the team size is chosen by members themselves, in a decentralized fashion, so as to maximize their individual payoffs. That is, members voluntarily choose in each period whether to join the team or not. We model the NL structure as a dynamic stochastic game with binary action set $a_i \in \{\text{Work}, \text{Not Work}\}$ for

$i \in N$ similar to Huberman and Loch (1996). We also focus on pure-strategy Markov equilibria for simplicity. These equilibria are common in dynamic games; see, e.g., Adlakha et al. (2012) and Doraszelski and Satterthwaite (2010).

Although we focus only on team size, and real projects involve other types of decisions, our NL structure is analogous to "self-managed performing units"(Hackman 1995, page 512), in which "members have responsibility not only for executing the tasks but also for monitoring and manging their own performance. [...] Self-managing units are common in managerial and professional work (e.g., a team of research assistants who share responsibility for collecting a set of data.)"

We denote by k_t^{NL} the equilibrium NL team size in period t, by $V_{t,i}^{NL}$ Member i's equilibrium expected payoff-to-go in period t, and by $V_{t,i}^{NL}(k)$ Member i's payoff-to-go in period t when the team size is k, $i \in \mathcal{N}$. All problem parameters are assumed to be common knowledge to all members. However, they cannot verify or contract on their actions; otherwise, they could contract on each others' participation to the team and attain the FB solution. The set $\mathcal{S}_t$ denote the set of team members, such that $|\mathcal{S}_t| = k_t^{NL}$. As a result, Member i's equilibrium payoff-to-go can be obtained by

$$V_{t,i}^{NL} = \begin{cases} -c(k_t^{NL}) - g(k_t^{NL})/n + \delta p(k_t^{NL})R/n + \delta(1 - p(k_t^{NL}))V_{t-1,i}^{NL} & \text{if } i \in \mathcal{S}_t \\ -g(k_t^{NL})/n + \delta p(k_t^{NL})R/n + \delta(1 - p(k_t^{NL}))V_{t-1,i}^{NL} & \text{if } i \notin \mathcal{S}_t, \end{cases} \tag{2.3}$$

with $V_{0,i}^{NL} = 0$ for $i \in N$. We define $V_{t,i}^{NL}(k \mid \text{Work}) = -c(k) - g(k)/n + \delta p(k)R/n + \delta(1 - p(k))V_{t-1,i}^{NL}$ and $V_{t,i}^{NL}(k \mid \text{Not Work}) = -g(k)/n + \delta p(k)R/n + \delta(1 - p(k))V_{t-1,i}^{NL}$. The equilibrium team size k_t^{NL} satisfies the following equilibrium conditions:

$$V_{t,i}^{NL}(k_t^{NL} \mid \text{Work}) \geq V_{t,i}^{NL}(k_t^{NL} - 1 \mid \text{Not Work}) \quad \text{for } i \in \mathcal{S}_t \tag{2.4}$$

$$V_{t,i}^{NL}(k_t^{NL} \mid \text{Not Work}) \geq V_{t,i}^{NL}(k_t^{NL} + 1 \mid \text{Work}) \quad \text{for } i \notin \mathcal{S}_t, \tag{2.5}$$

That is, the members who joined the team prefer working over not working, whereas the others prefer not working over working. We therefore obtain:

$$V_{t-1,i}^{NL} \leq \frac{R}{n} - \frac{nc(k_t^{NL}) + \Delta g(k_t^{NL} - 1)}{n\delta\Delta p(k_t^{NL} - 1)} \quad \text{for } i \in \mathcal{S}_t, \tag{2.6}$$

$$V_{t-1,i}^{NL} \geq \frac{R}{n} - \frac{nc(k_t^{NL} + 1) + \Delta g(k_t^{NL})}{n\delta\Delta p(k_t^{NL})} \quad \text{for } i \notin \mathcal{S}_t. \tag{2.7}$$

At the deadline, $V_{0,i}^{NL} = 0$ for all $i \in N$ by (2.3). As a result when $t = 1$, conditions (2.6) and (2.7) are true for any set $\mathcal{S}_1$ of size k_1^{NL}. Without loss of generality, we assume that $\mathcal{S}_1 = \{1, 2, \ldots, k_1^{NL}\}$. We show by induction in Lemma B-1 in Appendix B (§4.2) that without loss of generality, we can assume $\mathcal{S}_t = \{1, 2, \ldots, k_t^{NL}\}$ for all $t \in \mathcal{T}$. We then show in Lemma B-2 and B-3 in Appendix B (§4.2) that k_t^{NL} is unique and can be obtained by

$$k_t^{NL} = \min\left\{ i \in Z_{\leq n}^{+} \mid V_{t-1,i+1}^{NL} \geq \frac{R}{n} - \frac{nc(i+1) + \Delta g(i)}{n\delta\Delta p(i)} \right\} \tag{2.8}$$

and $k_t^{NL} = n$ if and only if $V_{t-1,n}^{NL} \leq \frac{R}{n} - \frac{nc(n)+\Delta g(n-1)}{n\delta\Delta p(n-1)}$.

2.3.2 Internal Leader (IL) Structure

Under the IL structure, the team size is centrally chosen by an internal leader who maximizes her own payoff (Dessein 2007 and Alonso et al. 2008). We assume that the internal leader can unilaterally assign members to the team. Specifically, the leader can verify the members' effort, potentially at some monitoring cost, which we assumed to be zero for simplicity (with a constant monitoring cost, all our results remain the same albeit the IL structure is less attractive.) According to Stroebe et al. (1996) "When group members were led to believe that their individual output could be monitored," free-riding behaviour disappears. As a result, the internal leader is the only one who can free-ride under the IL structure.

Although we only consider staffing decisions, the IL structure is analogous to "manager-led performing units" (Hackman 1995, page 512), in which "members have authority only for executing the tasks; managers monitor and manage performance processes [. . .] In this view, managers manage, workers work, and the two functions are kept distinct." Similarly in "project education system" (Ruel et al, 2003), teachers are not the formal providers of knowledge; they fulfil the role of the students' coach by supporting the students in their learning process. Students have to work actively together in teams on cases and projects.

Under the IL structure, the internal leader assigns the members to the team to maximize her individual payoff and, naturally, would be the last one to join the team, consistent with Hackmann's distinction between upper management and operational work (Hackman 1995). We denote by k_t^{IL} the IL team size, by $V_{t,n}^{IL}$ the internal leader's optimal expected payoff-to-go in period t, denoting the leader as the nth member, and by $V_{t,n}^{IL}(k)$ the leader's expected payoff-to-go in period t when the team size is k. Given that the internal leader joins the team last, she incurs a cost of effort $c(n)$ if $k_t^{IL} = n$ and no cost of effort otherwise. The dynamic programming formulation for the IL structure is as follows:

$$V_{t,n}^{IL} \quad = \quad \max_{k \leq n} -1\{k = n\}c(k) - \frac{g(k)}{n} + \delta p(k)\frac{R}{n} + \delta(1 - p(k))V_{t-1,n}^{IL} \qquad (2.9)$$

with $V_{0,n}^{IL} = 0$. We obtain

$$k_t^{IL} = \min\left\{ i \in Z_{\leq n-1}^{+} \mid V_{t-1,n}^{DL} \geq \frac{R}{n} - \frac{\Delta g(i)}{n\delta\Delta p(i)} \right\}. \qquad (2.10)$$

and $k_t^{IL} = n$ if and only if $V_{t-1,n}^{IL} \leq \frac{R}{n} - \frac{nc(n)+\Delta g(n-1)}{n\delta\Delta p(n-1)}$.

2.4 Team Size Characterization

In this section, we characterize and compare team sizes under the NL and IL structures with respect to the FB solution. We present our results under static decision-making (i.e., team size is chosen only once at the beginning of the project) in §2.4.1 and under dynamic decision-making (i.e., team size is dynamically revised) in §2.4.2.

2.4.1 Static Staffing Decision

We consider a situation where the team size is chosen only once at the beginning of the project and is not revised subsequently i.e., $k_t^l = k_{t-1}^l$ for $l \in \{FB, NL, IL, SL\}$. To simplify the mathematical exposition, we here assume that $T = 1$. We characterize the team sizes under NL and IL structures and compare them with respect to the FB solution. Using (2.2), (2.8), and (2.10), the team sizes in the static model where $T = 1$ can be expressed as:

$$k_1^{FB} = \min\left\{i \in Z^+_{\leq n} \mid \frac{\Delta ic(i) + \Delta g(i)}{\delta\Delta p(i)} \geq R\right\}, \tag{2.11}$$

$$k_1^{NL} = \min\left\{i \in Z^+_{\leq n} \mid \frac{nc(i+1) + \Delta g(i)}{\delta\Delta p(i)} \geq R\right\}, \tag{2.12}$$

$$k_1^{IL} = \min\left\{i \in Z^+_{\leq n-1} \mid \frac{\Delta g(i)}{\delta\Delta p(i)} \geq R\right\} \tag{2.13}$$

such that $k_1^{FB} = n$ if and only if $\frac{\Delta c(n-1)+\delta\Delta g(n-1)}{\delta\Delta p(n-1)} \leq R$ and $k_1^{NL} = k_1^{IL} = n$ if and only if $\frac{nc(n)+\delta\Delta g(n-1)}{\delta\Delta p(n-1)} \leq R$.

We show that if the staffing decision is made only once, there is always under-

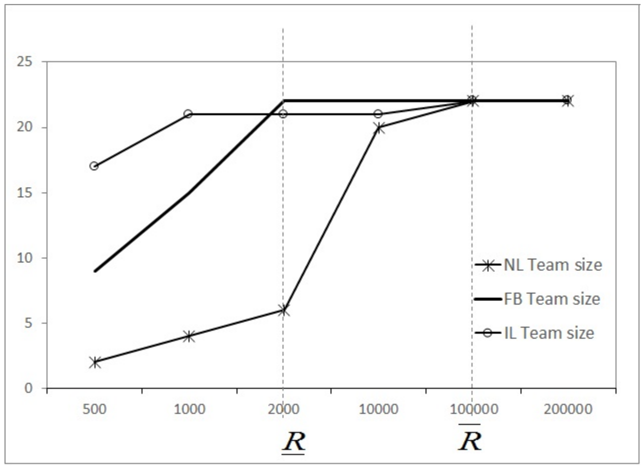

Figure 2.1: Team sizes with a static staffing decision
The parameters are: $n = 22$, $p(k) = (k/n)^{0.5}$, $c(k) = 0.5k$, $g(k) = 2k^{1.5}$, and $\delta = 0.98$.

staffing in the NL structure. However, there could be under- or over-staffing in the IL structure depending on whether the project reward is large or not. The next lemma characterizes equilibrium team sizes with respect to the project reward. Figure 2.1 illustrates the results. The horizontal axis denotes the project reward and the vertical axis represents the team sizes under FB, NL, and IL structures.

Lemma 2.1. *There exist thresholds $\underline{R}$ and $\overline{R}$ on the project reward such that*

(i) If $R > \overline{R}$, then $k_1^{IL} = k_1^{FB} = k_1^{NL}$.

(ii) If $\underline{R} < R \leq \overline{R}$, then $k_1^{FB} > k_1^{IL} \geq k_1^{NL}$.

(iii) If $R \leq \underline{R}$, then $k_1^{IL} \geq k_1^{FB} \geq k_1^{NL}$.

There is always under-staffing in the NL structure because of free-riding (Holmström 1982) or social loafing (Karau and Williams 1993). In addition, under-staffing in the NL structure increases with the the size of the organization consistant with Alchian and Demsetz (1972) or equivalently decreases with the project reward. Similarly, Ringelmann (1913) discovered in a classical rope pulling experiment that when more people pull a rope, the average force exerted by the group members declines.

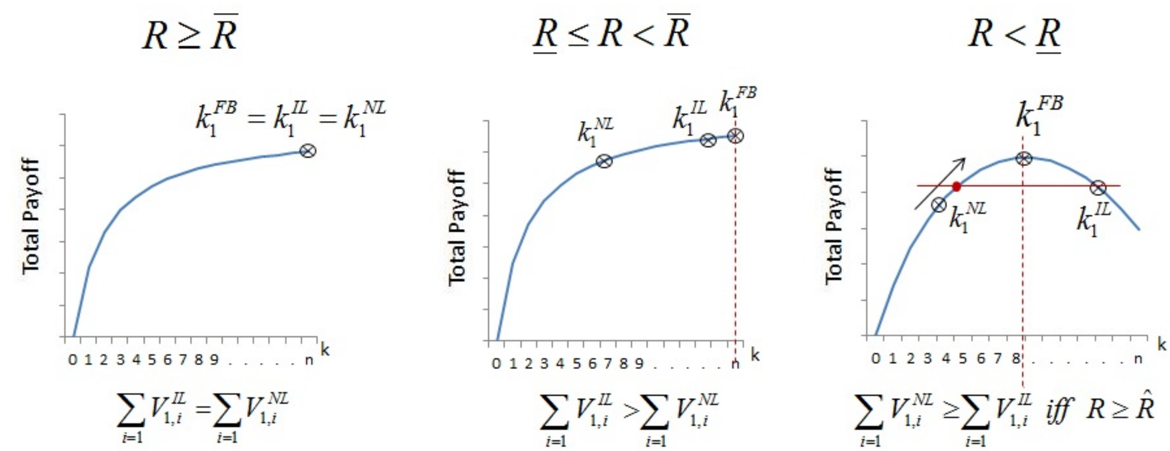

Figure 2.2: Team payoffs with a static staffing decision

Under the IL structure, there is under-staffing whenever the project reward is small. When the project reward is smaller than $\underline{R}$, the optimal team size is smaller than the size of the organization (i.e., $k_1^{FB} < n$). Similarly, the internal leader does not assign all members to the project. However, since the leader does not internalize the members' individual cost of effort, she tends to assign more members to the project than optimal. In fact, it often happens in new product development or software development teams that the leader overstaffs the project by asking too many members to attend unnecessary meetings or to work on simple tasks. When the project reward is larger than $\underline{R}$, it is optimal in the FB solution that everybody in the organization joins the team. Similarly, the internal leader assigns all members to the project. The internal leader herself may also participate in the team if the project reward is larger than $\overline{R}$, and she free rides otherwise.

We next compare the efficiency of the two structures. Figure 2.2 illustrates the total surplus with respect to the project reward. The horizontal axis denotes the team sizes, from $k = 0$ to n, and the vertical axis represents the total payoff $V_1^{FB}(k) = -kc(k) - \delta g(k) + \delta p(k)R$. The next proposition shows that the IL structures generates higher total surplus than the NL structure unless the project reward lies in an intermediate range.

Proposition 2.1. *There exist thresholds $\underline{R}$, $\overline{R}$ and $\hat{R} \leq \underline{R}$ such that*

(i) If $R > \overline{R}$, then $\sum_i V_{1,i}^{NL} = \sum_i V_{1,i}^{IL}$.

(ii) If $\underline{R} < R \leq \overline{R}$, then $\sum_i V_{1,i}^{IL} \geq \sum_i V_{1,i}^{NL}$.

(iii) If $R \leq \underline{R}$, then $\sum_i V_{1,i}^{NL} \geq \sum_i V_{1,i}^{IL}$ if and only if $R \geq \hat{R}$.

It sometimes believed that leaderless organizations perform poorly. For instance, a senior executive of BMG international stated that "If you don't have a dictator, you won't be successful. Show me a company run by democracy, and I'll show you a loser. There's always got to be one chief and plenty of Indians." (Weber and Hill 1994, page 12) Contrary to this belief, Proposition 2.1 part (c) shows that, as far as staffing decisions are concerned, the NL structure could dominate the IL structure when project reward lies in an intermediate range.

2.4.2 Dynamic Staffing Decision

In many projects, staffing decisions may be revised dynamically. For instance in the IBM OS/360 project (Hackman 2002), the project ran behind schedule and the management added staff to the team to make up time. As another example, the Jaguar project at Teradyne Corporation (Gino and Pisano 2006), the senior management kept adding software engineers to the project as they were getting closer to the deadline. Similarly in the last three months of the Cisco Viking project (Shao and Lee 2009), all the workers from the engineering organization were working full time to meet the deadline for the product delivery. Consistent with these evidences, we show that the optimal and equilibrium team sizes increase over time under the FB

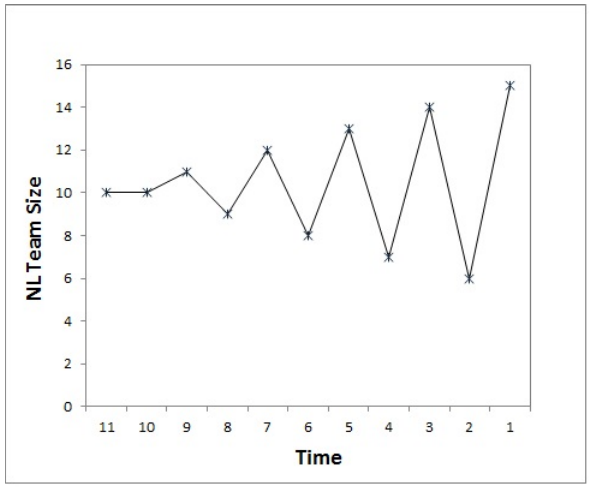

Figure 2.3: The NL team size over time

The parameters are: $T = 11$, $c(k) = 1.69$, $R = 2500$, $n = 15$, $g(k) = 5k^{1.2}$, $p(k) = \frac{k^{0.8}}{n^{0.8}}$, $\delta = 0.8$.

solution and IL structure, because the chance of ultimate success becomes smaller as the deadline becomes closer. It is therefore optimal to have more members join the team to increase the success rate.

Lemma 2.2. *(i) k_t^{FB} and k_t^{IL} are non-increasing in t. (ii) If $\delta = 1$, k_t^{NL} is non-increasing in t.*

When members are infinitely patient, the equilibrium team size under the NL structure is also increasing over time. However when members are impatient ($\delta < 1$), the equilibrium NL team size may oscillate. Figure 2.3 illustrates the equilibrium team sizes over time. The reason of this non-monotone behavior is the following. With the FB and IL structures, the total payoff-to-go (under the FB solution) and the leader's payoff-to-go (under the IL structure) are decreasing over time. Accordingly as time goes by, there are greater incentives to put in more effort because the fall-back option, if the current period effort fails, becomes less attractive. By contrast under the NL structure, the members' payoff may not evolve dynamically. This is because members are competing against each other from period to period. To see this, suppose that

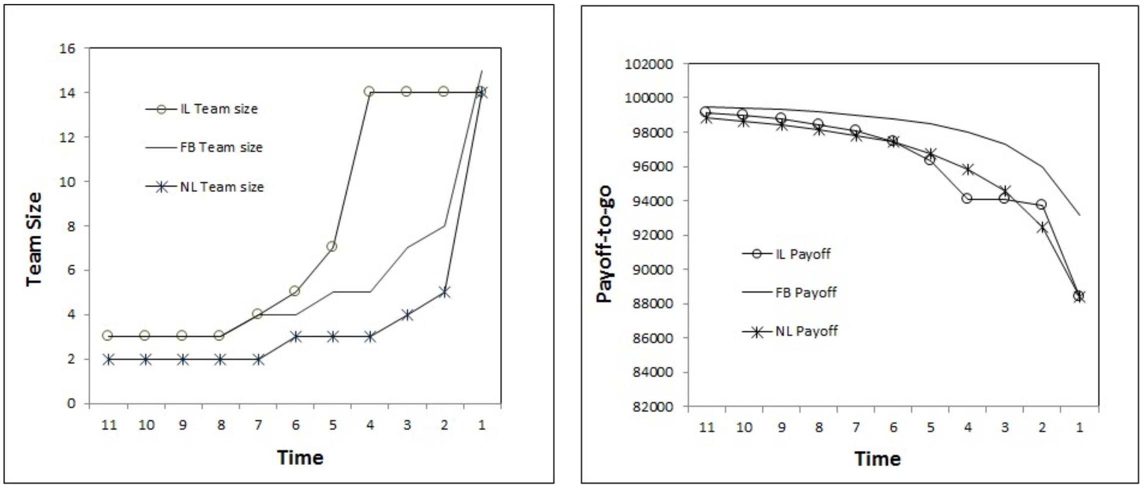

Figure 2.4: Equilibrium team sizes and total payoffs with a dynamic staffing decision The parameters are: $T = 11$, $c(k) = 2k^2$, $R = 500$, $n = 15$, $g(k) = 5k$, $p(k) = (k/n)^{0.9}$, $\delta = 1$.

a large pool of members decide to join the team when there remains $t-1$ periods until the deadline. Anticipating this burst of work in period $t-1$, fewer members would then decide to join the team when there remains t periods until the deadline. Similarly, anticipating that few members join the team in period t, more members would then choose to join the team when there are $t+1$ periods left to deadline. This oscillatory pattern is reminiscent of the performance oscillation reported by Mihm et al. (2003) although it translates into staffing decisions here.

Figure 2.4 (Left) illustrates the optimal team sizes under the FB solution and the equilibrium team sizes under the NL and IL structures. Similar to static staffing decisions (Lemma 2.1), there is always under-staffing in the NL structure, but there could be under- or over-staffing in the IL structure. In particular, there is under-staffing when there is limited time left to the deadline and over-staffing when the project is far from the deadline. Differentiating between reward and the fall back option, this is in fact analogues to Lemma 2.1. That is when $t = 1$, the fall back option is fixed at zero, whereas here, reward remains fixed at R and fall back option changes over time. In particular as the deadline becomes closer, the fall back option becomes smaller and therefore the marginal gain through success incraeses. Accordingly, the early periods of the project can be interpreted as having a small reward and the late periods can be interpreted as having a large reward.

Figure 2.4 (Right) illustrates the corresponding total payoffs obtained under the FB, NL, and IL structures. We observe that these payoffs may cross each other several times as time elapses. That is, the NL structure dominates the IL structure in intermediate periods and the reverse holds when the project is either far from the deadline or very close to deadline. Similar to the above discussion, the change in the remaining time to deadline can be interpreted as the change in the project reward. Therefore, the intermediate periods can be interpreted as having an intermediate reward (neither small nor large.) In that case, our observation in Figure 2.4 (right) is consistent with our findings in Proposition 2.1. To compare the staffing decisions and payoffs under the NL and IL structures to the FB solution over time, we next consider two special cases: (i) linear costs and probabilities with infinitely patient members and (ii) stationary staffing decisions, i.e., as $T \to \infty$.

Linear Costs and Probabilities.

As is common in the literature (e.g., Marx and Matthews 2000, Bonatti and Horner 2011 and Weinschenk 2011), we assume that the probability of success and project costs are all linear functions of the team size; i.e., $p(k) = pk$, $g(k) = gk$, and $c(k) = c$. We also assume that $\delta = 1$. Similar to the static staffing decision, we show that there is always under-staffing in the NL structure. However under the IL structure, there is over-staffing in the early periods (far from deadline) and under-staffing in the late periods (close to deadline). The leader assigns more workers to the project early on,

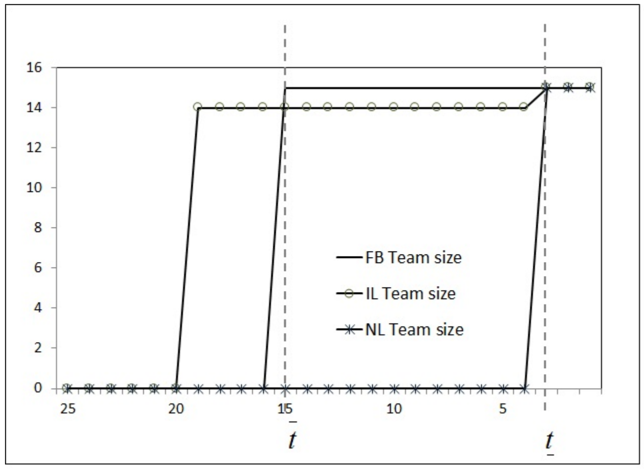

Figure 2.5: Dynamic team size with linear costs and probabilities
The parameters are: $T = 25$, $c(k) = 0.5k$, $R = 19000$, $n = 15$, $g(k) = k$, $p(k) = 0.06k$, $\delta = 1$.

so that she herself does not work in the late periods. The results are presented in Lemma 2.3 and illustrated in Figure 2.5.

Lemma 2.3. *Suppose $p(k) = pk$, $g(k) = gk$, $c(k) = c$, and $\delta = 1$. There exist time thresholds $\underline{t}$ and $\bar{t}$, such that*

(i) If $t \leq \underline{t}$, then $k_t^{FB} = k_t^{IL} = k_t^{NL}$.

(ii) If $\underline{t} < t < \bar{t}$, then $k_t^{FB} > k_t^{IL} \geq k_t^{NL}$.

(iii) If $t \geq \bar{t}$, then $k_t^{IL} \geq k_t^{FB} = k_t^{NL}$.

Comparing the efficiency of the NL and IL structures, we show that the IL structure dominates the NL structure in all periods. Although the internal leader makes suboptimal decisions early on by potentially over-staffing the project, she generates a higher total surplus than if the team was decentralized because she makes better decisions later on.

Proposition 2.2. *Suppose $p(k) = pk$, $g(k) = gk$, $c(k) = c$, and $\delta = 1$. Then, $\sum_{i=1}^{n} V_{t,i}^{IL} \geq \sum_{i=1}^{n} V_{t,i}^{NL}$.*

Although Proposition 2.2 shows that it is beneficial to appoint an internal team leader when efforts are additive, efforts are often non-additive in Practice (Brooks 1975) and this result may not hold in general (see e.g., Figure 2.4 Right).

Stationary Staffing Decision.

We denote by k_∞^l the stationary team size under structure $l \in \{FB, NL, IL\}$. We show in Lemma B-6 in Appendix B (§4.2) that the stationary equilibrium team sizes are unique provided that the NL equilibrium team size converges, which by Lemma 2.2 is guaranteed when $\delta = 1$. The next lemma shows that the IL stationary team size is the largest and the NL stationary team size is the smallest. That is, when there in no deadline for the project, there is always under-staffing in the NL structure and over-staffing in the IL structure.

Lemma 2.4. *Suppose that k_t^{NL} converges when $T \to \infty$. Then, there exists thresholds $\overline{r}$ and $\underline{r}$ such that*

(i) If $R > \overline{r}$, then $k_\infty^{IL} = k_\infty^{FB} = k_\infty^{NL}$.

(ii) If $\underline{r} < R \leq \overline{r}$, then $k_\infty^{FB} > k_\infty^{IL} \geq k_\infty^{NL}$.

(iii) If $R \leq \underline{r}$, then $k_\infty^{IL} \geq k_\infty^{FB} \geq k_\infty^{NL}$.

We next compare the efficiency of the NL and IL structures. Similar to the static staffing decision, we show that when there is no deadline for the project, the IL structure generates higher surplus than the NL structure unless the project reward lies in an intermediate range.

Proposition 2.3. *Suppose that k_t^{NL} converges when $T \to \infty$. There exist thresholds $\overline{r}$, $\underline{r}$, and $\hat{r}$ such that*

(i) If $R > \overline{r}$, then $\sum_i V_{\infty,i}^{IL} = \sum_i V_{\infty,i}^{NL}$.

(ii) If $\underline{r} < R \leq \overline{r}$*, then* $\sum_i V_{\infty,i}^{IL} \geq \sum_i V_{\infty,i}^{NL}$.

(iii) If $R \leq \underline{r}$*, then* $\sum_i V_{\infty,i}^{NL} \geq \sum_i V_{\infty,i}^{IL}$ *if and only if* $R \geq \hat{R}_\infty$.

Overall Proposition 2.3 supports our findings in Proposition 2.1. That is, we show that under two extreme cases, namely (i) very tight deadline ($t = 1$) and (ii) very loose deadline ($t = \infty$), the NL structure dominates the IL structure when the project reward lies in an intermediate range, because the over-staffing in the IL becomes larger than the under-staffing in the NL structure.

2.5 Conclusions

In knowledge-intensive projects, the choice of team size is a crucial decision for the project success, because it should balance effectiveness and efficiency. In this chapter, we study the effect of leadership structure on the choice of team size. We consider two types of leadership structures: teams with an internal leader (IL) and teams with no leader (NL). We show that there is always under-staffing in the NL structures because of free-riding. In contrast, there could be under- or over-staffing in the IL structure. The under-staffing is because the internal leader is tempted to free ride and the over-staffing is because the leader does not internalize the member's individual cost of effort.

Comparing the efficiency of the IL and NL structures, we show that projects with an internal leader are more efficient when the project reward is either very small or very large, because free-riding is controlled. However, the IL structure becomes less attractive when the project reward is in an intermediate range. In that case, the project over-staffing may be too costly (e.g., take time away from other projects) and voluntary participation of the members is more efficient. Therefore, top-down staffing decisions perform better in projects with very small or vary large rewards and decentralized staffing decisions work better in projects with an intermediate reward.

This model can be extended in different directions. For example, what happens if the members receive nonequal shares of reward? What if the members are asymmetric in terms of cost and skills? Or, what if the project has multiple phases? Investigating these questions would be worthwhile given the ubiquity of team work in today's business processes.

Chapter 3

Learning in Collaborative Work Processes

3.1 Introduction

Knowledge-intensive projects such as research and development often involve collaboration between different parties. The ultimate success of such projects therefore depends on effectiveness of collaboration. However at the outset, it may be challenging to predict how effective two parties may be at collaborating, especially if they have not worked together before. Accordingly, some projects start with an exploration phase (Thompson 2013). For instance in research projects, two co-authors who have never worked together on previous projects may spend the early phases of a project to learn how they can complement each other by combining knowledge and expertise. As another example of industrial research, Cisco Systems Incorporation learned about Foxconn Technology Group over the exploration phase of a Viking project (Shao and Lee 2009). This learning process encouraged Cisco to work jointly with Foxconn that resulted in completing a successful project. The senior management of the product operations said "over the assessment period, we noted the high executive level commitments, excellent team, and a demonstration of processes and technical capability. Foxconn proved they were committed and ready to take on this challenge."

Exploring collaboration effectiveness raises many questions. For instance, what is the effect of learning on future collaborative work dynamics? Does learning induce collaboration in early phases of the project? When is it favorable to explore working with a new collaborator? What is the effect of deadline and prior beliefs on work dynamics?

To answer these questions, we consider two parties who work jointly on a knowledge-intensive project. While they are certain about their own capabilities, they only have a prior probabilistic belief about the effectiveness of their joint work. Similar to Chapter 1, we consider a project for which the scope has already been defined. However, the parties still need to decide about the choice of components to be executed. Each component addition moves the project to a higher state. Because low-hanging fruits tend to be executed first, the final reward from the project is assumed to be increasing concave in its completion state. Consistent with the collaborative, iterative, and stochastic nature of knowledge-intensive processes (Kieliszewski et al. 2010), we model the collaborative work process as a finite-horizon stochastic program. That is, the state of the project evolves according to a Markov chain and the transition probabilities depend on whether the parties work individually or jointly. If they work jointly on the project, the success rate also depends on the effectiveness of their collaboration, about which they learn gradually in a Bayesian process. Moreover, there is a cost associated with their effort.

We show that if the parties' expected prior belief about their collaboration effectiveness is neither too high nor too low, it is optimal for them to work jointly in early periods of the project to learn about the effectiveness of their work. If they don't succeed while collaborating for certain number of periods, it is optimal for them to switch to individual work and never revert back to collaboration. When the parties' expected prior belief is low, it is optimal for them to work individually in early periods and to collaborate when few periods remain before the deadline, because collaboration is costly and not very efficient. However, it is also optimal for the parties to postpone their collaboration when their expected prior belief is high. In that case, it is very likely that they will finish the project if they collaborate; therefore, it is optimal for them to postpone their collaboration and give themselves a chance to make progress at smaller costs. In such cases, learning may not be beneficial for the parties, because they do not have enough time to adjust their future working modes based on their updated beliefs.

The chapter is organized as follows. We review the related literature in the next section and present the model in §3.3. We characterize the optimal work dynamics in §3.4. We present our conclusions in §3.5. All proofs are gathered in Appendix C (§4.3).

3.2 Literature Review

This chapter studies learning in collaborative projects and therefore it is related to two strands of research: experimentation in teams and new product development.

This chapter is related to Bonnati and Horner (2011) since they also study learning in collaborative teams. While their focus is on learning about the quality and duration of the project, ours is on learning about the effectiveness of parties' joint work. In particular, we study the impact of collaboration effectiveness on project success as emphasized by Thompson (2013). Bonnati and Horner (2011) find that collaboration increases as the parties get closer to the project deadline. However, we show that parties's collaboration depends on their updated belief, i.e., if the parties' collaboration fails for many periods, they would rather do the work individually even if the deadline is close.

There is a growing literature in economics on experimentation in teams. For instance, Keller et al. (2005) and Klein and Rady (2008) study a two-armed bandit problem in research teams where there is no learning. Bolton and Harris (1999) also study a two-armed bandit problem but in a situation where an agent can learn from the current experimentation of other agents. They show that learning derives encouragement in a team. Similar to Bonnati and Horner (2011), these papers also study exploration about the project outcome. Although our focus is on learning about the effectiveness of collaboration and not project outcome, we also show that learning increases collaboration in the early phases of a project.

This chapter is also related to the literature on new product development (NPD) projects since they are canonical examples of knowledge-intensive projects (Ozkan et al. 2013). Building on previous work by Thomke (1998), Thomke (2003), and Loch et al. (2001), Erat and Kavadias (2008) study learning in sequential product design. They propose an optimal policy for deciding about future design tests based on the result of pervious tests. Instead of characterizing design tests, we characterize the optimal work structure, i.e., collaboration versus individual working as a function of the current state of a project. We therefore complement the NPD literature by studying the effect of learning on collaborative work dynamics.

3.3 Model

In this section, we introduce a model of dynamic learning in collaborative processes. We consider two parties engaged in a multi-state, multi-period project with finite deadline T. Let $\mathcal{T} = \{0, 1, \ldots, T\}$ be the set of time periods and $x_t \in \{0, \ldots, S\}$ be the set of project states. The parties start the project at state 0. Once the project is stopped in state x_t, a common reward $R(x_t)$ is collected. The reward $R(x_t)$ is assumed to be increasing concave, therefore making it optimal to stop the project beyond a certain state. Time is discounted with a discount rate $\delta \leq 1$, making early project completion desirable.

In each period, the parties choose among three working modes: collaboration (Duo), individual working ($Solo$), and stopping the project ($Finish$). In any period t, the problem state is three dimensional and consists of the project state (x), the number of failures from collaboration (m), and the number of successes from collaboration (n). The evolution of the state (x, m, n) is assumed to follow a stochastic process. In any time t if the parties collaborate and succeed, the state (x, m, n) evolves to state $(x + 1, m, n + 1)$ with probability $f(m, n)$ or it evolves to state $(x, m + 1, n)$ with probability $1 - f(m, n)$. However, if only one party works $Solo$, the state evolves to state $(x+1, m, n)$ with probability p_s or it remains at state (x, m, n) with probability $1 - p_s$ as depicted in Figure 3.1.

The parties start with a prior on the probability of success if they collaborate (p_d), which follows a Beta distribution with parameters (a, b). Using Bayesian updating, the updated probability of success through collaboration after m unsuccessful trials and n successful trials can be estimated by $f(m, n) = \frac{a+n}{a+b+m+n}$. We also assume that the prior probability p_d is larger than the known success probability of their individual work p_s (i.e., $p_d = f(0, 0) \geq p_s$). Effort is costly and depends on the working modes.

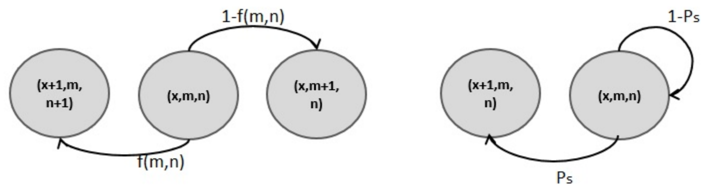

Figure 3.1: State dynamics under collaboration (left) and individual work (right)

We denote the costs of working *Duo* and *Solo* by c_d and c_s, respectively and assume that $2c_d \geq c_s$. Our assumptions on success rates and costs (i.e., $p_d \geq p_s$ and $2c_d \geq c_s$) capture the trade offs between *Duo* and *Solo* working modes. Without these assumptions, one working mode (*Duo* or *Solo*) dominates the other one. In particular, *Duo* is more efficient and effective than *Solo* if $2c_d \leq c_s$ and $p_d \geq p_s$, and the reverse holds true if $2c_d \geq c_s$ and $p_d \leq p_s$.

We next present our model of dynamic learning in collaborative projects. The notation $\mathcal{E}_t(x, m, n)$ denotes the optimal working mode in period t and state (x, m, n), i.e., $\mathcal{E}_t(x, m, n) \in \{Duo,\ Solo, Finish\}$. We denote by $V_t(x, m, n)$ the discounted total payoff in period t and state (x, m, n). The finite horizon DP formulation is as follows:

$$\begin{aligned} V_t(x, m, n) &= \max\{V_t(x, m, n \mid Duo), V_t(x, m, n \mid Solo), V_t(x, m, n \mid Finish)\} \\ V_T(m) &= 0, \end{aligned} \tag{3.1}$$

in which

$$\begin{aligned} V_t(x, m, n \mid Duo) &= -2c_d + \delta[f(m, n)V_{t+1}(x + 1, m, n + 1) \\ &+ (1 - f(m, n))V_{t+1}(x, m + 1, n)], \\ V_t(x, m, n \mid Solo) &= -c_s + \delta\left[p_s V_{t+1}(x + 1, m, n) + (1 - p_s)V_{t+1}(x, m, n)\right], \end{aligned}$$

$$V_t(x, m, n \mid Finish) \quad = \quad R(x).$$

3.4 Optimal Policy

In this section, we characterize the optimal work dynamics under two special cases: (i) projects with two periods and multiple states and (ii) projects with multiple periods and two states.

3.4.1 Two Periods and Multiple States

We consider a project that consists of two periods, i.e., $\mathcal{T} = \{0, 1\}$ with the deadline $T = 2$. By (3.1), we have

$$\begin{aligned} V_0(x, m, n) &= \max\{-2c_d + \delta\left[f(m, n)V_1(x+1, m, n+1) + (1 - f(m, n))\, V_1(x, m+1, n)\right], \\ &\quad -c_s + \delta\left[p_s V_1(x+1, m, n) + (1 - p_s)\, V_1(x+1, m, n)\right], R(x)\}, \qquad (3.2) \\ V_1(x, m, n) &= \max\{-2c_d + \delta\left[f(m, n)R(x+1) + (1 - f(m, n))\, R(x)\right], \\ &\quad -c_s + \delta\left[p_s R(x+1) + (1 - p_s)\, R(x)\right], R(x)\} \qquad (3.3) \\ V_2(x, m, n) &= R(x). \qquad (3.4) \end{aligned}$$

We first characterize the work dynamics in period $t = 1$. To simplify the notation, we define the following quantities:

$$\theta_{fs}(x) \doteq \frac{(1-\delta)R(x) + c_s}{\delta p_s}, \theta_{fd}(x, m, n) \doteq \frac{(1-\delta)R(x) + 2c_d}{\delta f(m, n)}, \theta_{sd}(m, n) \doteq \frac{-c_s + 2c_d}{\delta(f(m, n) - p_s)}.$$

The thresholds θ_{kl} allow for a pairwise comparison of policies k and l, for any $k \in$

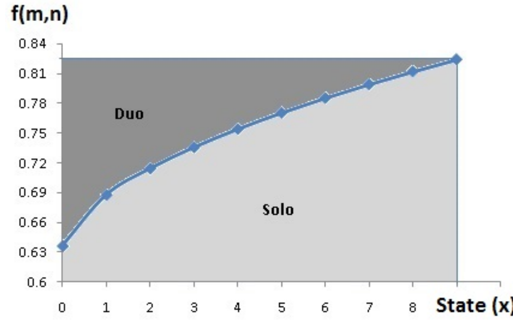

Figure 3.2: *Duo*-*Solo* state-threshold as a function of $f(m,n)$
The solid line is obtained by $f(n,m) = \frac{2c_d - c_s}{\delta[R(x+1)-R(x)]} + p_s$ for $R(x) = 200\sqrt{x}$, $p_s = 0.6$, $\delta = 0.95$, $c_d = 5$, $c_s = 3$

$\{f, s\}$ and $l \in \{s, d\}$. Specifically,

$$\mathcal{E}_1(x,m,n) = Finish \quad \Leftrightarrow \quad R(x+1) - R(x) \leq \min\{\theta_{fs}(x), \theta_{fd}(x,m,n)\}$$

$$\mathcal{E}_1(x,m,n) = Solo \quad \Leftrightarrow \quad \theta_{fs}(x) \leq R(x+1) - R(x) \leq \theta_{sd}(m,n)$$

$$\mathcal{E}_1(x,m,n) = Duo \quad \Leftrightarrow \quad R(x+1) - R(x) \geq \max\{\theta_{fd}(x,m,n), \theta_{sd}(m,n)\}.$$

Accordingly, when the project marginal return $(R(x+1)-R(x))$ is large, the optimal policy is either *Duo* or *Solo* and when the project marginal return is small, it is either *Duo* or $Finish$.

Not surprisingly, collaboration is more desirable when its success rate is high. In particular, the parties prefer to work *Duo* rather than *Solo* if and only if $R(x+1) - R(x) > \frac{-c_s + 2c_d}{\delta(f(m,n) - p_s)}$ which is equivalent to $f(m,n) \geq \frac{2c_d - c_s}{\delta(R(x+1) - R(x))} + p_s$. Therefore, the state threshold between *Solo* and *Duo* increases as $f(m,n)$ increases. Figure 3.2 illustrates the threshold between *Duo* and *Solo*.

In Figure 3.3, we numerically compare the optimal policy in period $t = 1$ (right) and in period $t = 0$ (left) to analyze the effect of learning. We present the work

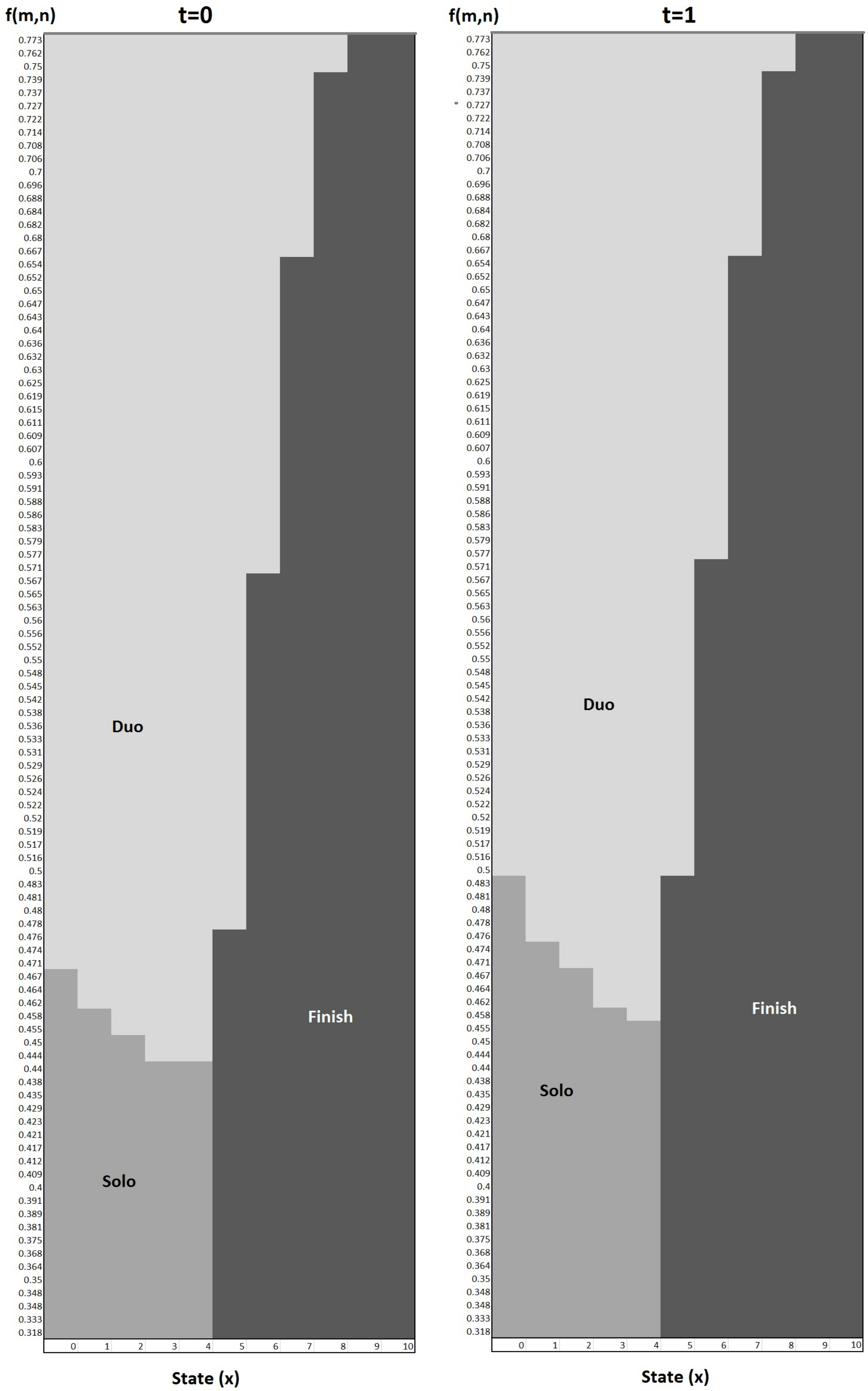

Figure 3.3: Optimal policy in a two-period and multi-stage project
The parameters are: $R(x) = 700\sqrt{x}$, $p_s = 0.5$, $\delta = 0.96$, $c_d = 7$, $c_s = 14$, $a = 7$, $b = 5$.

dynamics as a function of $f(m,n)$ and x. We observe that in both periods when $f(m,n)$ is small, it is optimal for the parties to work *Solo* if the state is low and *Finish* if the state is high. As $f(m,n)$ increases, the *Solo* region shrinks and it is optimal for the parties to work *Duo* if the state is low and to *Finish* if the state is high.

As shown in Figure 3.3, the *Duo* region in period $t = 0$ is larger than in period $t = 1$ while its *Solo* and *Finish* regions are smaller than in $t = 1$. The parties explore the effectiveness of their joint work in the early period; if their collaboration fails, $f(m,n)$ decreases and the parties may decide to continue collaborating (e.g., $x_0 = 3$ and $f(m,n) = 0.474$), work individually (e.g., $x_0 = 3$ and $f(m,n) = 0.467$), or even stop the project (e.g., $x_0 = 5$ and $f(m,n) = 0.5$) in the next period. However if they succeed, $f(m,n)$ increases and they either decide to stop the project (e.g., $x_0 = 5$ and $f(m,n) = 0.529$) or continue collaborating (e.g., $x_0 = 4$ and $f(m,n) = 0.533$) in the next period. The intuition is as follows: when collaboration is effective in period $t = 0$, the belief about the collaboration success rate increases and therefore it is optimal to continue collaborating unless the state is high, in which case it is optimal to stop the project. If collaboration fails, the belief about the collaboration success rate decreases. If the belief is very low, it may be optimal to work *Solo* or stop the project. However, the belief might be high enough to still collaborate one more period.

Hence, learning induces early collaboration. Consequently, the project stopping state might be time-dependent; that is, it might be optimal for the parties to work jointly on the project in state (x,m,n) and period $t = 0$, but to stop the project in state $(x,m+1,n)$ and period $t = 1$. However, this situation does not occur when the optimal policy is *Solo* in period $t = 0$. In particular, we show in Proposition 3.1 that

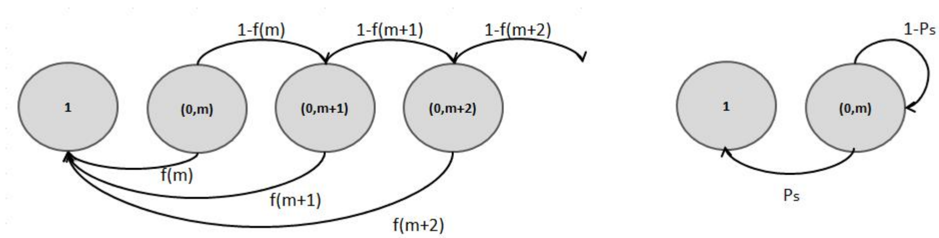

Figure 3.4: State dynamics in a two-stage project under collaboration (left) and individual work (right)

when it is optimal for the parties to stop the project in state (x, m, n) and period $t = 1$, it is not optimal for them to work *Solo* in state (x, m, n) and period $t = 0$.

Proposition 3.1. *Suppose* $\mathcal{E}_1(x, m, n) = Finish$*. Then* $\mathcal{E}_0(x, m, n) \neq Solo$*.*

In summary, we observe that learning induces early collaboration and results in a time-dependent *Finish* threshold. These results are in contrast to our findings in Chapter 1, in which there is no learning. We show in Chapter 1 that in a situation without learning, the optimal *Finish* threshold is time-independent and it is optimal for the parties to collaborate either close to deadline or close to the *Finish* threshold (see Proposition 1.1).

3.4.2 Multiple Periods and Two States

In this section, we consider a project that consists of two states ($x_t = 0$ and $x_t = 1$) similar to Bonatti and Horner (2011). The project starts at state 0 and it is completed if it reaches state 1, at which point a common reward R is collected and shared. If the project is not completed by the deadline, there is no reward to collect.

Given that the project is completed in state $x = 1$, the *Finish* working mode is beside the point. In particular, if the parties decide to stop the project in state $x = 0$, the project will never get started and there is no need for further analysis.

Therefore, we consider a dynamic formulation with two working modes: *Duo* and *Solo*. Accordingly, the state variables can be simplified from (x, m, n) to a single state variable m. Since the parties collect their reward (R) when the project reaches state $x = 1$, we have $n \in \{0, 1\}$ and $V_t(1, m, 1) = R$. We also define $V_t(m) \equiv V_t(0, m, 0)$ for simplicity of exposition. We therefore have $f(m) = \frac{a}{a+b+m}$. Figure 3.4 illustrates the corresponding state dynamics. Consequently, the DP recursion (3.1) simplifies to:

$$\begin{aligned} V_t(m) &= \max\{-2c_d + \delta f(m)R + \delta(1 - f(m))V_{t+1}(m + 1), \\ &\quad -c_s + \delta p_s R + \delta(1 - p_s)V_{t+1}(m + 1)\} \qquad (3.5) \\ V_T(m) &= 0. \end{aligned}$$

Accordingly, *Solo* working mode is preferred over *Duo* if and only if

$$\delta(1 - p_s)V_{t+1}(m) - \delta(1 - f(m))V_{t+1}(m + 1) \geq -2c_d + c_s - \delta p_s R + \delta f(m)R \quad (3.6)$$

For simplicity of exposition, we define $H_t(m) = \delta(1-p_s)V_{t+1}(m)-\delta(1-f(m))V_{t+1}(m+1) - \delta f(m)R$. Similar to (3.6), $\mathcal{E}_t(m) = Solo$ if and only if $H_t(m) \geq -2c_d + c_s - \delta p_s R$.

We first characterize the optimal policy in period $T - 1$. Using (3.6), we have that $\mathcal{E}_{T-1}(m) = Solo$ if and only if

$$-2c_d + c_s - \delta p_s R + \delta f(m)R \leq 0 \Leftrightarrow f(m) \leq \frac{2c_d - c_s + \delta p_s R}{\delta R}. \quad (3.7)$$

By definition, $f(m)$ is decreasing in m and therefore if $V_{T-1}(m) = Solo$, then $V_{T-1}(m + 1) = Solo$. Let us define the threshold m_s above which *Solo* is an optimal policy in time $T-1$:

$$m_s = \min\left\{m \in \mathbb{Z}^+ \mid f(m) \leq \frac{2c_d - c_s + \delta p_s R}{\delta R} := \theta_{sd}\right\}. \tag{3.8}$$

In the next proposition, we generalize the above result for any period $t \leq T-1$. We show that it is optimal for the parties to work *Solo* if their number of failures exceeds the threshold m_s. This phase of individual working is absorptive; that is, it is optimal for the the parties to continue working *Solo* above the threshold m_s and to not revert back to *Duo* until they reach the deadline.

Proposition 3.2. *If $m \geq m_s$, $\mathcal{E}_t(m) = Solo$ for all t.*

Next, we characterize the optimal policy region for states $m < m_s$. We first define state thresholds $\hat{m}$ and m_d as follows:

$$\hat{m} \;=\; \min\left\{m < m_s \mid f(m+1) \leq \frac{(1-\delta)R + \delta p_s R + 2c_d}{\delta R}\right\}, \tag{3.9}$$

$$m_d \;=\; \max\left\{m \mid f(m) \geq \frac{-2c_d + (1-\delta)[c_s - \delta p_s R] + 2c_d\delta(1-p_s)}{\delta(\delta R - c_s - R)}\right\}. \tag{3.10}$$

The characterization of the work dynamics depends on whether $\hat{m}$ and m_d are zero or not. We say the expected belief about collaboration success rate is low if $\hat{m} = m_d = 0$, is neither low nor high if $\hat{m} = 0$ and $m_d > 0$, and is high if $\hat{m} > 0$.

We first characterize the work dynamics when $\hat{m} = 0$. In that case, the prior belief about collaboration success rate is low and it is therefore optimal to explore the effectiveness of collaboration in the early phases of the project. We show that the optimal policy is to work *Duo* when either the number of failures from collaboration (m) is lower than a threshold or when there is limited time left until the deadline. To this end, we define the following thresholds: (i) time threshold $t_s(m)$ (with $t_s(m_s) =$

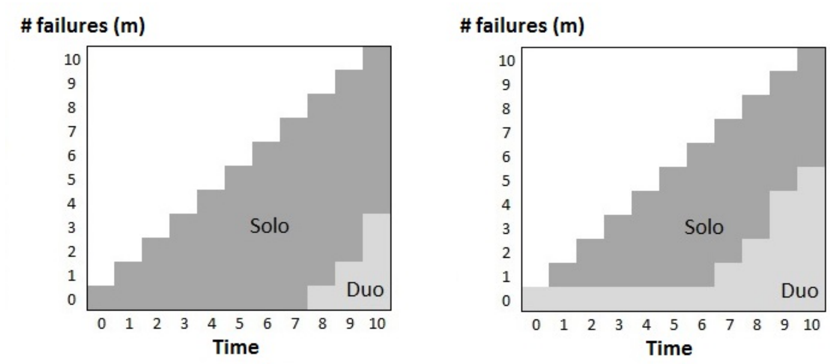

Figure 3.5: Optimal policy for the case where $\hat{m} = 0$
Left figure: $T = 10$, $R = 350$, $\delta = 1$, $c_s = 5$, $c_d = 4.5$, $p_s = 0.45$, $a = 3.5$, $b = 1$.
Right figure: $T = 10$, $R = 2000$, $\delta = 1$, $c_s = 8$, $c_d = 4.5$, $p_s = 0.45$, $a = 5$, $b = 1$.
State thresholds: $m_s = 4$, $m_d = 0$ (left) and $m_s = 6$, $m_d = 1$ (right).

$T - 1$), which denotes the maximum time period in which *Solo* is optimal after m failures; (ii) state threshold $m_s(t)$ (with $m_s(T-1) = m_s - 1$), which denotes the maximum number of failures for which the optimal policy is *Duo* in period t.

$$t_s(m) \quad = \quad \max\{t \mid H_t(m) \geq -2c_d + c_s - \delta p_s R\} \tag{3.11}$$

$$m_s(t) \quad = \quad \max\{m \in \mathbb{Z}^+ \mid H_t(m) \leq -2c_d + c_s - \delta p_s R\}. \tag{3.12}$$

The next proposition presents the characterization of the optimal work dynamics for $m < m_s$ when $\hat{m} = 0$.

Proposition 3.3. *Suppose $\hat{m} = 0$. The following statements hold true:*

(a) For $m_d < m \leq m_s$, $\mathcal{E}_t(m) = Solo$ for $t \leq t_s(m)$, $\mathcal{E}_t(m) = Duo$ for $m < m_s(t)$, threshold $t_s(m)$ is increasing in m, and threshold $m_s(t)$ is strictly increasing in t.

(b) For $m \leq m_d$, $\mathcal{E}_t(m) = Duo$ for all $t < T$.

Figure 3.5 illustrates the optimal policy for the case where $\hat{m} = 0$ characterized in Proposition 3.3. The horizontal axis denotes time, from the project start ($t = 0$)

to the deadline ($t = T$), and the vertical axis represents the number of failures from collaboration until time t (m_t). The feasible region lies below the 45-degree line because number of failures m increases by at most one unit each period. Figure 3.5 (left) illustrates the case with $m_d = 0$ and Figure 3.5 (right) illustrates the case with $m_d > 0$.

When $m_d = 0$ (Figure 3.5 left), early collaboration is not optimal and the parties should only work *Duo* when few periods remain before the deadline. This happens when the expected prior belief about the collaboration success rate is lower than a threshold; that is, when $p_d = f(0) \leq [-2c_d + (1-\delta)[c_s - \delta p_s R] + 2c_d\delta(1-p_s)]/[\delta(\delta R - c_s - R)]$. In that case, *Duo* work is not very effective in comparison to *Solo* work. As a result, the parties should work *Solo* to save on the cost of collaboration and they may work *Duo* only if the deadline is very close. We show in Proposition 3.3 that $m_s(t)$ is strictly increasing in t and therefore this *Duo* phase close to the deadline is absorptive.

When $m_d > 0$ (Figure 3.5 right), it is optimal for the parties to work *Duo* early on to learn about the effectiveness of their joint work. If collaboration fails m_d times, it is optimal for them to work *Solo*. However, they may revert back to *Duo* working a few periods before the deadline. Since $m_s(t)$ is strictly increasing in t, when the parties revert back to *Duo* a few periods before the deadline, it is optimal for them to continue working *Duo* until the deadline is reached.

We next study the optimal policy for the case where $\hat{m} > 0$. We show in Proposition 3.3 that when $\hat{m} = 0$, the parties should work *Duo* when $m \leq m_s(t)$ and work *Solo* when $m > m_s(t)$ in any period t. Therefore, when $\hat{m} = 0$, the optimal policy only changes once in each period. However when $\hat{m} > 0$, the optimal policy is less structured. In particular, in any period $t < T$, it might be optimal for the the parties

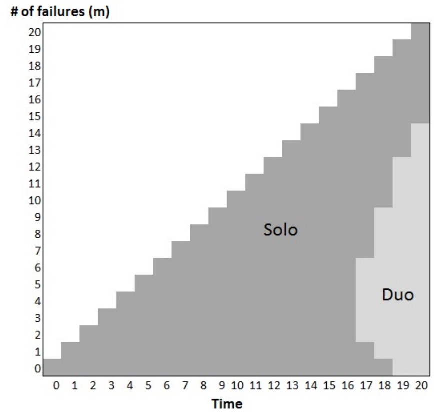

Figure 3.6: Optimal policy for the case where $\hat{m} > 0$
The parameters are: $T = 20$, $R = 25$, $\delta = 1$, $c_s = 0.2$, $c_d = 0.8$, $p_s = 0.2$, $a = 5$, $b = 0.1$.

to switch from *Duo* to *Solo* more than once.

Figure 3.6 illustrates the work dynamics for the case where $\hat{m} > 0$ (i.e., $f(m) \geq \frac{(1-\delta)R+\delta p_s R+2c_d}{\delta R}$ for $m < \hat{m}$). In that case, we observe that it is optimal for the parties to work *Duo* close to the deadline. However unlike when $\hat{m} = 0$, the threshold $t_s(m)$ may not be increasing. In particular, $t_s(m)$ is decreasing when m is small and it becomes increasing as m gets larger. The intuition is as follows: the optimal policy is *Solo* when m is large; i.e., the parties have low estimate for $f(m)$. However, the optimal policy can also be *Solo* when m is very small. In that case, the parties know that they will do *Duo* in the next period if they cannot make progress in the current period. Because their estimate of $f(m)$ is very high (it is very likely that they will finish the project if they collaborate), it may be optimal for the parties to give themselves a chance to make progress at smaller cost, by working *Solo*, so as to avoid paying the high cost of collaboration.

3.5 Conclusions

In this chapter, we studied how learning affects collaborative work dynamics between parties that are engaged in a knowledge-intensive project. We show that learning induces early collaboration when the parties' expected prior belief about the effectiveness of their joint work is neither too high nor too low. Not surprisingly, when the parties expected prior belief is low, there is no incentive for them to collaborate. When the parties' expected prior belief is high, they know that it is very likely that they will complete the project if they collaborate. Therefore, it may be optimal for them to first work individually at smaller cost and collaborate when there are a few periods left until the deadline. When the parties' expected prior belief is in an intermediate range, it is optimal for them to collaborate in early phases of the project to explore the effectiveness of their joint work. If their collaboration fails for a certain number of periods, it is then optimal for them to do the work individually. We therefore conclude that learning can be favorable when the parties' expected prior belief is in an intermediate range, in which case early collaboration allows the parties to adjust their future decisions accordingly.

This model can be extended in several directions. For example, what causes the non-monotonic collaboration dynamics in Figure 3.6? How can we reduce the three-dimensional state (x, m, n) to two-dimensional state $(x, f(m, n))$? When is early collaboration desirable is a multi-period, multi-stage project? What would be the dynamics of collaboration if the parties have the option to wait or delay the project? What if the parties can make their decisions in a decentralized way, resulting in a situation of double moral hazard? What is the effect of contracting on learning in collaborative projects? Investigating these questions would be worthwhile given the ubiquity of team work in today's business processes.

Chapter 4

Appendix

4.1 Appendix A : Proofs for Chapter 1

4.1.1 FB Solution

Preliminaries

We first define the following thresholds:

$$\theta_{fs}^{FB}(x) \doteq \frac{(1-\delta)R(x)+c_s^v}{\delta p_s}, \theta_{fd}^{FB}(x) \doteq \frac{(1-\delta)R(x)+2c_d}{\delta p_d}, \theta_{sd}^{FB} \doteq \frac{-c_s^v+2c_d}{\delta(p_d-p_s)}. \quad \text{(A1-1)}$$

The thresholds $\theta_{kl}^{FB}(x)$ allow for a pairwise comparison of policies k and l, for any $k \in \{f,s\}$, $l \in \{d,s\}$, $k \neq l$. Specifically, policy k is preferred over policy l in period $T-1$ if and only if $R(x+1)-R(x) \leq \theta_{kl}^{FB}(x)$. It turns out that for any state x, either $\theta_{fs}^{FB}(x) \leq \theta_{fd}^{FB}(x) \leq \theta_{sd}^{FB}$ or $\theta_{sd}^{FB} \leq \theta_{fd}^{FB}(x) \leq \theta_{fs}^{FB}(x)$. Let

$$x_\theta^{FB} \quad := \quad \min\left\{x \in \mathbb{Z}^+ \mid \theta_{fd}^{FB}(x) \geq \theta_{sd}^{FB}\right\} \quad \text{(A1-2)}$$

$$= \quad \min\left\{x \in \mathbb{Z}^+ \mid R(x) \geq \frac{2c_d p_s - c_s^v p_d}{(1-\delta)\,(p_d - p_s)}\right\}.$$

By the definition of x_θ^{FB}, we have that,

$$\theta_{fs}^{FB}(x) < \theta_{fd}^{FB}(x) < \theta_{sd}^{FB} \text{ if } x < x_\theta^{FB} \text{ and } \theta_{fs}^{FB}(x) \geq \theta_{fd}^{FB}(x) \geq \theta_{sd}^{FB} \text{ if } x \geq x_\theta^{FB}. \quad \text{(A1-3)}$$

We also define

$$x_f^{FB} := \min\left\{x \in \mathbb{Z}^+ \mid \Delta R(x) \leq \min\left\{\theta_{fs}^{FB}(x), \theta_{fd}^{FB}(x)\right\}\right\}. \quad \text{(A1-4)}$$

For each state $x < \min\{x_\theta^{FB}, x_f^{FB}\}$, let us define $\tau(x) := \max\{t \in \mathcal{T} \mid \Delta V_{t+1}(x) \leq \theta_{sd}^{FB}\}$ if there exists a t such that $\Delta V_{t+1}(x) \leq \theta_{sd}^{FB}$; otherwise, $\tau(x) := x - 1$. We define

$$x_\varphi^{FB} := \max\{x < \min\{x_\theta^{FB}, x_f^{FB}\} \mid \tau(x) > \tau(x-1)\}, \quad \text{(A1-5)}$$

such that $\tau(x)$ is nondecreasing for all x, $x_\varphi^{FB} \leq x < \min\{x_\theta^{FB}, x_f^{FB}\}$.

Finally, we denote, for any t, the highest state smaller than x_φ^{FB} where $\mathcal{E}_t^{FB}(x) = Duo$, as

$$x_{d,t}^{FB} := \min\left\{\max\left\{x \in \mathbb{Z}^+ \mid V_{t+1}(x+1) - V_{t+1}(x) \geq \theta_{sd}^{FB}\right\}, x_\varphi^{FB}\right\}. \quad \text{(A1-6)}$$

Proofs

Lemma A1-1. *For any x, $V_t(x)$ is nonincreasing in t.*

Proof. We prove the lemma by induction on t. When $t = T-1$, $V_T(x) = R(x) \leq$

$V_{T-1}(x)$ by (1). Fix $t < T-1$ and suppose that $V_{t+1}(x) \leq V_t(x)$, $\forall x$. Then,

$$\begin{aligned} V_t(x) &= \max\left\{R(x), -2c_d + \delta\mathbb{E}_d\left[V_{t+1}(x+\xi)\right], -c_s^v + \delta\mathbb{E}_s\left[V_{t+1}(x+\xi)\right]\right\} \\ &\leq \max\left\{R(x), -2c_d + \delta\mathbb{E}_d\left[V_t(x+\xi)\right], -c_s^v + \delta\mathbb{E}_s\left[V_t(x+\xi)\right]\right\} = V_{t-1}(x).\square \end{aligned}$$

▮

Lemma A1-2. *$\mathcal{E}_t^{FB}(x) = Finish$ for all t if and only if $x \geq x_f^{FB}$.*

Proof. The proof uses Lemma A1-1 in the e-companion. Using (1.1) and (A1-1), we obtain $\mathcal{E}_{T-1}^{FB}(x) = Finish \Leftrightarrow \Delta R(x) \leq \min\left\{\theta_{fs}^{FB}(x), \theta_{fd}^{FB}(x)\right\}$. Because $\Delta R(x)$ is decreasing and both $\theta_{fs}^{FB}(x)$ and $\theta_{fd}^{FB}(x)$ are increasing, $\mathcal{E}_{T-1}^{FB}(x) = Finish$ if $x \geq x_f^{FB}$ and $\mathcal{E}_{T-1}^{FB}(x) \neq Finish$ if $x < x_f^{FB}$ by (A1-4).

We next show by induction on t that if $\mathcal{E}_{T-1}^{FB}(x) = Finish$, then $\mathcal{E}_t^{FB}(x+k) = Finish$ $\forall k \geq 0$ and $t \leq T-1$. Fix $x \geq x_f^{FB}$. When $t = T-1$, $\mathcal{E}_{T-1}^{FB}(x+k) = Finish$ $\forall k \geq 0$. For any $t < T-1$, suppose that $\mathcal{E}_{t+1}^{FB}(x+k) = Finish$ $\forall k \geq 0$. Applying the conditions $V_{t+1}(x) = R(x)$ and $V_{t+1}(x+1) = R(x+1)$ to (1.1) shows that $V_t(x) = V_{T-1}(x) = R(x)$, i.e., $\mathcal{E}_t^{FB}(x) = Finish$, completing the induction step.

We next show the converse: if $\mathcal{E}_t^{FB}(x) = Finish$ for some t, then $\mathcal{E}_{T-1}^{FB}(x) = Finish$. By Lemma A1-1 and (1.1), we have $R(x) = V_t(x) \geq V_{T-1}(x) \geq R(x)$; therefore, $V_{T-1}(x) = R(x)$.$\square$ ▮

Lemma A1-3. *If $x_\theta^{FB} < x_f^{FB}$, $\mathcal{E}_t^{FB}(x) = Duo$ for all $x_\theta^{FB} \leq x < x_f^{FB}$ and all t.*

Proof. The proof uses Lemma A1-1 in the e-companion. Because $x_f^{FB} - 1 \geq x_\theta^{FB}$, $\theta_{sd}^{FB} \leq \theta_{fd}^{FB}(x_f^{FB} - 1) \leq \theta_{fs}^{FB}(x_f^{FB} - 1)$ by (A1-3). Therefore for any $x \leq x_f^{FB}$, $\Delta R(x) \geq \Delta R(x_f^{FB} - 1) \geq \min\{\theta_{fd}^{FB}(x_f^{FB} - 1), \theta_{fs}^{FB}(x_f^{FB} - 1)\} \geq \theta_{sd}^{FB}$ given that $R(x)$

is nondecreasing concave in x and by (A1-4). As a result, $\mathcal{E}_{T-1}^{FB}(x) = Duo$ for all $x < x_f^{FB}$ by (1.2) and (A1-1).

Fix x, $x_\theta^{FB} \leq x \leq x_f^{FB}$. Suppose that $\mathcal{E}_{t+1}^{FB}(x) = Duo$, i.e., $V_{t+1}(x) = -2c_d + \delta\mathbb{E}_d\left[V_{t+2}(x+\xi)\right]$. By (1.1), $V_{t+1}(x+1) \geq R(x+1)$ and by Lemma A1-1, $V_{t+1}(x+1) \geq V_{t+2}(x+1)$. Hence, $V_{t+1}(x+1) \geq (1-\delta)R(x+1) + \delta V_{t+2}(x+1)$, and $\Delta V_{t+1}(x) \geq (1-\delta)R(x+1) + \delta V_{t+2}(x+1) + 2c_d - \delta\mathbb{E}_d\left[V_{t+2}(x+\xi)\right] = (1-\delta)R(x+1) + \delta(1-p_d)\Delta V_{t+2}(x) + 2c_d$. Because $\mathcal{E}_{t+1}^{FB}(x) = Duo$, $\Delta V_{t+2}(x) \geq \theta_{sd}^{FB}$ by (1.2) and (A1-1). Similarly, because $\mathcal{E}_{T-1}^{FB}(x) = Duo$, $\Delta V_T(x) = \Delta R(x) \geq \theta_{sd}^{FB}$. Combining these results yields the following lower bound: $\Delta V_{t+1}(x) \geq (1-\delta)R(x+1) + \delta(1-p_d)\theta_{sd}^{FB} + 2c_d \geq (1-\delta)R(x) + (1-\delta p_d)\theta_{sd}^{FB} + 2c_d$. Finally, because $x \geq x_\theta^{FB}$, $\theta_{fd}^{FB}(x) \geq \theta_{sd}^{FB}$ by (A1-3), i.e., $(1-\delta)R(x) + 2c_d \geq \delta p_d \theta_{sd}^{FB}$. As a result, $\Delta V_{t+1}(x) \geq \theta_{sd}^{FB}$. By (1.2), $\mathcal{E}_t^{FB}(x) = Duo$.□ ∎

Lemma A1-4. *If $\mathcal{E}_{t+1}^{FB}(x+1) = \mathcal{E}_{t+1}^{FB}(x) = Solo$, then $\mathcal{E}_t^{FB}(x) = Solo$.*

Proof. Because $\mathcal{E}_{t+1}^{FB}(x) = \mathcal{E}_{t+1}^{FB}(x+1) = Solo$, $\Delta V_{t+2}(x) \leq \theta_{sd}^{FB}$ and $\Delta V_{t+2}(x+1) \leq \theta_{sd}^{FB}$ by (1.2) and (A1-1). Moreover, $\Delta V_{t+1}(x) = -c_s^v + \delta\mathbb{E}_s\left[V_{t+2}(x+1+\xi)\right] + c_s^v - \delta\mathbb{E}_s\left[V_{t+2}(x+\xi)\right] = \delta p_s \Delta V_{t+2}(x+1) + \delta(1-p_s)\Delta V_{t+2}(x)$. As a result, $\Delta V_{t+1}(x) \leq \delta p_s \theta_{sd}^{FB} + \delta(1-p_s)\theta_{sd}^{FB} \leq \theta_{sd}^{FB}$, and $\mathcal{E}_t^{FB}(x) = Solo$ by (1.2) and (A1-1).□ ∎

Lemma A1-5. *If $\mathcal{E}_{t+1}^{FB}(x+1) = Finish$ and $\mathcal{E}_{t+1}^{FB}(x) = Solo$, then $\mathcal{E}_t^{FB}(x) = Solo$.*

Proof. The proof uses Lemma A1-1 in the e-companion and Lemma A1-2 in this appendix. When $\mathcal{E}_{t+1}^{FB}(x+1) = Finish$, $V_{t+1}(x+1) = V_{t+2}(x+1) = R(x+1)$ by Lemma A1-2. By Lemma A1-1, $V_{t+1}(x) \geq V_{t+2}(x)$. Moreover, because $\mathcal{E}_{t+1}^{FB}(x) = Solo$, $\Delta V_{t+2}(x) \leq \theta_{sd}^{FB}$ by (1.2) and (A1-1). Hence, $\Delta V_{t+1}(x) = R(x+1) - V_{t+1}(x) \leq R(x+1) - V_{t+2}(x) = \Delta V_{t+2}(x) \leq \theta_{sd}^{FB}$, and $\mathcal{E}_t^{FB}(x) = Solo$ by (1.2).□ ∎

Lemma A1-6. *Fix $\hat{t}$, $\tilde{t}$ and $\hat{x}$, with $\hat{t} < \tilde{t} < T-1$. Suppose that $\mathcal{E}_t^{FB}(x) = Duo$ for all $t \geq \hat{t}$, $\hat{x} < x < x_f^{FB}$, $\mathcal{E}_t^{FB}(\hat{x}) = Solo$ for all $\hat{t} \leq t \leq \tilde{t}$ and $\mathcal{E}_t^{FB}(\hat{x}) = Duo$ for all $t > \tilde{t}$. Then, for all $t \geq \hat{t}$ and all $x \geq \hat{x}$, $V_t(x)$ has increasing differences in (x,t), i.e., $\Delta V_t(x) \leq \Delta V_{t+1}(x)$.*

Proof. The proof uses Lemmas A1-1 and A1-2 in Appendix 4.1.1. Fix (x,t) such that $t \geq \hat{t}$ and $x > \hat{x}$. By Lemma A1-2, $\mathcal{E}_t^{FB}(x) = Finish$ for all $x \geq x_f^{FB}$. Because $\mathcal{E}_t^{FB}(x) = Duo$ for all $t \geq \hat{t}$, $\hat{x} < x < x_f^{FB}$, we thus need to consider three cases: (i) $\mathcal{E}_t^{FB}(x+1) = \mathcal{E}_t^{FB}(x) = Finish$, (ii) $\mathcal{E}_t^{FB}(x+1) = Finish$ and $\mathcal{E}_t^{FB}(x) = Duo$, and (iii) $\mathcal{E}_t^{FB}(x+1) = \mathcal{E}_t^{FB}(x) = Duo$. We prove the results by induction on t. Consider first period $T-1$.

(i)-(ii) $\Delta V_{T-1}(x) = R(x+1) - V_{T-1}(x) = V_T(x+1) - V_{T-1}(x) \leq \Delta V_T(x)$ by Lemmas A1-1 and A1-2.

(iii) By concavity of $R(x)$, $\Delta V_{T-1}(x) = -2c_d + \delta\mathbb{E}_d[R(x+1+\xi)] + 2c_d - \delta\mathbb{E}_d[R(x+\xi)] = \delta p_d \Delta R(x+1) + \delta(1-p_d)\Delta R(x) \leq \delta p_d \Delta R(x) + \delta(1-p_d)\Delta R(x) = \delta \Delta R(x) \leq \Delta V_T(x)$.

Inductively applying the same argument as in period $T-1$, shows that $\Delta V_t(x) \leq \Delta V_{t+1}(x)$, completing the induction step.

Consider next (x,t) such that $x = \hat{x}$ and $\tilde{t} \geq t \geq \hat{t}$. To initialize the proof, we first consider time $\tilde{t}$. Using (1) we obtain: $\Delta V_{\tilde{t}}(\hat{x}) \leq V_{\tilde{t}}(\hat{x}+1) - V_{\tilde{t}}(\hat{x} \mid Duo) = \delta p_d \Delta V_{\tilde{t}+1}(\hat{x}+1) + \delta(1-p_d)V_{\tilde{t}+1}(\hat{x}) \leq \Delta V_{\tilde{t}+1}(\hat{x})$.

Consider next any time t, $\hat{t} \leq t < \tilde{t}$. Because $\mathcal{E}_t^{FB}(\hat{x}) = \mathcal{E}_{t+1}^{FB}(\hat{x}) = Solo$ and $\mathcal{E}_t^{FB}(\hat{x}+1) = \mathcal{E}_{t+1}^{FB}(\hat{x}+1) = Duo$, $\Delta V_t(\hat{x}) = c_s^v - 2c_d + \delta p_d \Delta V_{t+1}(\hat{x}+1) + \delta(1-p_s)\Delta V_{t+1}(\hat{x}) \leq c_s^v - 2c_d + \delta p_d \Delta V_{t+2}(\hat{x}+1) + \delta(1-p_s)\Delta V_{t+2}(\hat{x}) = \Delta V_{t+1}(\hat{x})$, in which the inequality follows by the induction hypothesis. This completes the induction step.□ ∎

Lemma A1-7. *For all* $x_\varphi^{FB} \le x < \min\{x_\theta^{FB}, x_f^{FB}\}$*, there exists a time threshold* $\tau(x)$*, nonincreasing in* x*, such that* $\mathcal{E}_t^{FB}(x) = Duo$ *for all* $T > t > \tau(x)$ *and* $\mathcal{E}_t^{FB}(x) = Solo$ *for all* $x \le t \le \tau(x)$*.*

Proof. The proof uses Lemmas A1-2, A1-3, A1-4, and A1-5 and Lemma A1-6 in this appendix. Suppose first that $x_\theta^{FB} > x_f^{FB} - 1$. When $x = x_f^{FB} - 1$, we have $\mathcal{E}_{T-1}^{FB}(x) = Solo$ for all $x_\varphi^{FB} \le x < x_f^{FB}$ by (1.2) and (A1-3); that is, $\tau(x) = T - 1$ for all $x_\varphi^{FB} \le x < x_f^{FB}$. Then, $\mathcal{E}_t^{FB}(x_f^{FB} - 1) = Solo$ for all $t \le T - 1$ by Lemma A1-5 and $\mathcal{E}_t^{FB}(x) = Solo$ for all t and all $x_\varphi^{FB} \le x < x_f^{FB}$ by Lemma A1-4.

Suppose next that $x_\theta^{FB} \le x_f^{FB} - 1$. Fix $\hat{x}$, $x_\varphi^{FB} \le \hat{x} < x_\theta^{FB}$, and consider first $\hat{t} > \tau(\hat{x} + 1)$. Suppose that $\mathcal{E}_t^{FB}(\hat{x}) = Solo$ for all $\hat{t} \le t \le \tau(\hat{x})$; we will show that $\mathcal{E}_{\hat{t}-1}^{FB}(\hat{x}) = Solo$.

Because $\hat{t} > \tau(\hat{x} + 1)$ and $\tau(x)$ is nonincreasing when $x \ge x_\varphi^{FB}$, $\hat{t} > \tau(x)$ for all $\hat{x} + 1 \le x \le x_\theta^{FB}$. By Lemma A1-2, $\mathcal{E}_t^{FB}(x) \ne Finish$; therefore, by (1.2) and (A1-1), $\mathcal{E}_t^{FB}(x) = Duo$ for all $t \ge \hat{t}$ and $\hat{x} < x < x_\theta^{FB}$. Moreover, by Lemma A1-3, $\mathcal{E}_t^{FB}(x) = Duo$ for all t and all $x_\theta^{FB} \le x < x_f^{FB}$. Thus, $\mathcal{E}_t^{FB}(x) = Duo$ for all $t \ge \hat{t}$ and $\hat{x} < x < x_f^{FB}$. Also, by definition of $\tau(\hat{x})$, $\mathcal{E}_t^{FB}(\hat{x}) = Duo$ for all $t > \tau(\hat{x})$. Hence, by Lemma A1-6, $V_t(x)$ has increasing differences in (x, t) for all $x \ge \hat{x}$ and $t \ge \hat{t}$; in particular, $\Delta V_{\hat{t}}(\hat{x}) \le \Delta V_{\hat{t}+1}(\hat{x})$. Because $\mathcal{E}_{\hat{t}}^{FB}(\hat{x}) = Solo$, $\Delta V_{\hat{t}+1}(\hat{x}) \le \theta_{sd}^{FB}$ by (1.2); therefore, $\Delta V_{\hat{t}}(\hat{x}) \le \theta_{sd}^{FB}$. This implies that $\mathcal{E}_{\hat{t}-1}^{FB}(\hat{x}) = Solo$ by (1.2). As a result, for any $x_\varphi^{FB} \le x < x_\theta^{FB}$, $\mathcal{E}_t^{FB}(x) = Solo$ for all $\tau(x + 1) \le t \le \tau(x)$.

Consider next $\hat{t} \le \tau(\hat{x} + 1) - 1$. The proof proceeds by induction. Fix $\hat{x}$ and suppose that $\mathcal{E}_t^{FB}(\hat{x} + 1) = Solo$ for all $t \le \tau(\hat{x} + 1)$ and $\mathcal{E}_t^{FB}(\hat{x} + 1) = Duo$ for all $t > \tau(\hat{x} + 1)$. By the previous argument, $\mathcal{E}_t^{FB}(\hat{x}) = Solo$ for all $\tau(\hat{x} + 1) \le t \le \tau(\hat{x})$. Applying Lemma A1-4 yields that $\mathcal{E}_t^{FB}(\hat{x}) = Solo$ for all $t \le \tau(\hat{x} + 1) - 1$.□ ∎

Lemma A1-8. *For any* t, $x^{FB}_{d,t}$, *defined in (A1-6), is nondecreasing in* t, *i.e.,* $x^{FB}_{d,t} \leq x^{FB}_{d,t+1}$.

Proof. The proof uses Lemmas A1-2, A1-3, A1-4, A1-5, and A1-7 in Appendix 4.1.1. By definition, $\mathcal{E}^{FB}_t(x) \neq Duo$ for all $x^{FB}_{d,t} < x \leq x^{FB}_{\varphi}$; hence, by Lemma A1-2, $\mathcal{E}^{FB}_t(x) = Solo$ for all $x^{FB}_{d,t} < x \leq x^{FB}_{\varphi}$. Suppose for contradiction that $x^{FB}_{d,t} > x^{FB}_{d,t+1}$ for some t. Then, there must exist some x, $x^{FB}_{d,t} \geq x > x^{FB}_{d,t+1}$, such that $\mathcal{E}^{FB}_t(x) = Duo$ and $\mathcal{E}^{FB}_{t+1}(x) = Solo$. If either $\mathcal{E}^{FB}_{t+1}(x+1) = Solo$ or $\mathcal{E}^{FB}_{t+1}(x+1) = Finish$, we should then have had $\mathcal{E}^{FB}_t(x) = Solo$ by Lemmas A1-4 and A1-5, a contradiction. Hence, we must have $\mathcal{E}^{FB}_{t+1}(x+1) = Duo$, which can only happen when $x+1 = x^{FB}_{d,t}+1 = x^{FB}_{\varphi}+1$. Moreover, given that $\mathcal{E}^{FB}_{t+1}(x) = Solo$, $x < x^{FB}_f$ by Lemma A1-2 and $x < x^{FB}_{\theta}$ by Lemma A1-3. By Lemma A1-7, we should then have had $\mathcal{E}^{FB}_t(x) = Solo$ given that $\mathcal{E}^{FB}_{t+1}(x) = Solo$, a contradiction.□ ∎

Proof of Proposition 1.1.

(i) This is shown in Lemma A1-2 in this appendix.

(ii.a) The proof uses Lemmas A1-2, A1-4, A1-5, and A1-7 in this appendix. Given that $\Delta R(x^{FB}_f - 1) \leq \frac{2c_d - c^v_s}{\delta(p_d - p_s)} = \theta^{FB}_{sd}$, $\mathcal{E}^{FB}_{T-1}(x^{FB}_f - 1) = Solo$ by (1.2). By Lemmas A1-2 and A1-5, $\mathcal{E}^{FB}_t(x^{FB}_f - 1) = Solo$ for all $t < T$. Because $\tau(x)$ is nondecreasing when $x^{FB}_{\varphi} \leq x < x^{FB}_f$ by (A1-3), $\tau(x) \geq \tau(x+1) = T-1$, i.e., $\tau(x) = T-1$. Therefore, $\mathcal{E}^{FB}_{T-1}(x) = Solo$ for all $x^{FB}_{\varphi} \leq x < x^{FB}_f$ by Lemma A1-7. As a result by Lemma A1-4, $\mathcal{E}^{FB}_t(x) = Solo$ for all $x^{FB}_{\varphi} \leq x < x^{FB}_f$ and $t < T$.

(ii.b) The proof uses Lemma A1-8 in this appendix. Let $x^{FB}_{d,t}$ be defined as in (A1-6). By (ii.a), $x^{FB}_{d,t} < x^{FB}_{\varphi}$. Because $x^{FB}_{d,t}$ is nondecreasing in t by Lemma A1-8, (1.2) implies that if $\mathcal{E}^{FB}_t(x) = Duo$ then $\mathcal{E}^{FB}_{t+1}(x) = Duo$; hence, $\mathcal{E}^{FB}_t(x) = Duo$ for all $t > \tau(x)$ and $\mathcal{E}^{FB}_t(x) \neq Duo$, or equivalently $\mathcal{E}^{FB}_t(x) = Solo$ by (1.2), for all $t \leq \tau(x)$.

We show next that $\tau(x)$ is nondecreasing; i.e., for $0 < x \leq x^{FB}_{\varphi}$, if $\mathcal{E}^{FB}_t(x) = Duo$

then $\mathcal{E}_t^{FB}(x-1) = Duo$. To do so, we show by induction that $\Delta V_t(x-1) \geq \Delta V_t(x)$ for all $x \leq x_{d,t}^{FB}$, and that $\mathcal{E}_t^{FB}(x) = Duo$ for all $x \leq x_{d,t}^{FB}$. When $t = T$, $V_t(x)$ is concave because $V_T(x) = R(x)$, which is concave by assumption.

Fix t and suppose that $\Delta V_{t+2}(x-1) \geq \Delta V_{t+2}(x)$, $\forall 0 < x \leq x_{d,t+2}^{FB}$. Because $x_{d,t+1}^{FB} \leq x_{d,t+2}^{FB}$ by Lemma A1-8, $\Delta V_{t+2}(x) \geq \Delta V_{t+2}(x_{d,t+1}^{FB})$ for all $x \leq x_{d,t+1}^{FB}$. Moreover, by (A1-6), $\Delta V_{t+2}(x_{d,t+1}^{FB}) \geq \theta_{sd}^{FB}$. As a result, $\Delta V_{t+2}(x) \geq \theta_{sd}^{FB}$ for all $x \leq x_{d,t+1}^{FB}$, which, by (1.2), implies that $\mathcal{E}_{t+1}^{FB}(x) = Duo$ for all $x \leq x_{d,t+1}^{FB}$.

For any x, $0 < x \leq x_{d,t}^{FB}$, suppose that $\mathcal{E}_t^{FB}(x) = Duo$. By Lemma A1-8, $x \leq x_{d,t+1}^{FB}$ and therefore by the above, $\mathcal{E}_{t+1}^{FB}(x) = \mathcal{E}_{t+1}^{FB}(x-1) = Duo$. We next show that $\Delta V_{t+1}(x-1) \geq \Delta V_{t+1}(x)$ in the following three cases: (i) $\mathcal{E}_{t+1}^{FB}(x+1) = Solo$, (ii) $\mathcal{E}_{t+1}^{FB}(x+1) = Finish$, and (iii) $\mathcal{E}_{t+1}^{FB}(x+1) = Duo$.

(i) When $\mathcal{E}_{t+1}^{FB}(x+1) = Solo$, $\mathcal{E}_{t+1}^{FB}(x) = \mathcal{E}_{t+1}^{FB}(x-1) = Duo$, using the induction hypothesis, given that $x \leq x_{d,t}^{FB} \leq x_{d,t+2}^{FB}$, Equation (2), and the identity $-c_s^v + 2c_d = \delta(p_d - p_s)\theta_{sd}^{FB}$ by (A1-1), we obtain: $\Delta V_{t+1}(x-1) - \Delta V_{t+1}(x) = -2c_d + c_s^v + \delta(2p_d - 1)\Delta V_{t+2}(x) + \delta(1-p_d)\Delta V_{t+2}(x-1) - \delta p_s \Delta V_{t+2}(x+1) \geq -2c_d + c_s^v + \delta(2p_d - 1 + 1 - p_d)\Delta V_{t+2}(x) - \delta p_s \Delta V_{t+2}(x+1) \geq -\delta(p_d - p_s)\theta_{sd}^{FB} + \delta(p_d - p_s)\theta_{sd}^{FB} = 0$.

(ii) When $\mathcal{E}_{t+1}^{FB}(x+1) = Finish$, $\mathcal{E}_{t+1}^{FB}(x) = \mathcal{E}_{t+1}^{FB}(x-1) = Duo$, using the induction hypothesis, given that $x \leq x_{d,t}^{FB} \leq x_{d,t+2}^{FB}$, Equation (2), the inequality $\theta_{sd}^{FB} > \theta_{fd}^{FB}(x+1)$ obtained from (A1-3), and given that $x \leq x_{d,t}^{FB} < x_{\varphi}^{FB} < x_{\theta}^{FB}$, we obtain: $\Delta V_{t+1}(x-1) - \Delta V_{t+1}(x) = -2c_d - (1-\delta)R(x+1) + \delta(2p_d - 1)[R(x+1) - V_{t+2}(x)] + \delta(1-p_d)\Delta V_{t+2}(x-1) \geq -2c_d - (1-\delta)R(x+1) + \delta p_d[R(x+1) - V_{t+2}(x)] \geq -\delta p_d \theta_{sd}^{FB} + \delta p_d \theta_{sd}^{FB} = 0$.

(iii) When $\mathcal{E}_{t+1}^{FB}(x+1) = \mathcal{E}_{t+1}^{FB}(x) = \mathcal{E}_{t+1}^{FB}(x-1) = Duo$, we must have that $x + 1 \leq x_{d,t+2}^{FB}$. Suppose, for contradiction, that $x + 1 > x_{d,t+2}^{FB}$. Because $x \leq x_{d,t}^{FB} \leq x_{d,t+1}^{FB} \leq x_{d,t+2}^{FB}$ by Lemma A1-8, we must have that $x = x_{d,t+1}^{FB}$ and therefore

$x = x_{\varphi}^{FB}$ by definition of $x_{d,t+1}^{FB}$ given that $\Delta V_{t+2}(x+1) \geq \theta_{sd}^{FB}$ by (2). Since by assumption $\Delta R(x_f^{FB}-1) \leq \frac{2c_d - c_s^v}{\delta(p_d - p_s)}$, we have $\mathcal{E}_{T-1}^{FB}(x_f^{FB}-1) = Solo$ and therefore $\tau(x_{\varphi}^{FB}) = T-1$. We then should have that $\mathcal{E}_t^{FB}(x) = Solo$, a contradiction. Hence, $x+1 \leq x_{d,t+2}^{FB}$, and one can apply twice the induction hypothesis to obtain: $\Delta V_{t+1}(x-1) - \Delta V_{t+1}(x) = \delta p_d \left[\Delta V_{t+2}(x) - \Delta V_{t+2}(x+1)\right] + \delta(1-p_d)\left[\Delta V_{t+2}(x-1) - \Delta V_{t+2}(x)\right] \geq 0$.

As a result, $\Delta V_{t+1}(x-1) \geq \Delta V_{t+1}(x)$ for all $x \leq x_{d,t}^{FB}$. Therefore if $\mathcal{E}_t^{FB}(x) = Duo$, i.e., $\Delta V_{t+1}(x) \geq \theta_{sd}^{FB}$ by (2), then $\Delta V_{t+1}(x-1) \geq \theta_{sd}^{FB}$, i.e., $\mathcal{E}_t^{FB}(x-1) = Duo$.

(iii.a) This is shown in Lemma A1-3 in this appendix.

(iii.b) This is shown in Lemma A1-7 in this appendix.

(iii.c) The proof proceeds by induction on t. Because $\Delta R(x_f^{FB}-1) > \theta_{sd}^{FB}$, $\mathcal{E}_{T-1}^{FB}(x_f^{FB}-1) = Duo$ by (1.2). Because $\Delta R(x) \geq \Delta R(x_f^{FB}-1) > \theta_{sd}^{FB}$, $\mathcal{E}_{T-1}^{FB}(x) = Duo$ for all $1 \leq x < x_f^{FB}$. Hence, $\Delta V_{T-1}(x)$ is nonincreasing in x for all $1 \leq x < x_f^{FB}-1$ because $\Delta V_{T-1}(x) = \delta p_d \Delta R(x+1) + \delta(1-p_d)\Delta R(x) \leq \delta p_d \Delta R(x) + \delta(1-p_d)\Delta R(x-1) = \Delta V_{T-1}(x-1)$. Therefore, if $\mathcal{E}_{T-2}^{FB}(\overline{x}_{d,T-2}^{FB}) = Duo$, $\Delta V_{T-1}(x) \geq \Delta V_{T-1}(\overline{x}_{d,T-2}^{FB}) \geq \theta_{sd}^{FB}$, and thus $\mathcal{E}_{T-2}^{FB}(x) = Duo$ for all $x < \overline{x}_{d,T-2}^{FB}$.

Fix $t < T-1$ and $x = \overline{x}_{d,t}^{FB}$. As an induction hypothesis, suppose that $\mathcal{E}_{\tau}^{FB}(\xi) = Duo$ for all $\xi \leq \overline{x}_{d,\tau}^{FB}$ and $\tau > t$, and $\Delta V_t(\xi)$ is nonincreasing in ξ for all $\tau > t$ and $\xi < \overline{x}_{d,\tau}^{FB}$. If $\mathcal{E}_t^{FB}(\overline{x}_{d,t}^{FB}) = Duo$, then for all $x < \overline{x}_{d,t}^{FB}$, $\Delta V_t(x) \geq \Delta V_t(\overline{x}_{d,t}^{FB}) \geq \theta_{sd}^{FB}$ and therefore $\mathcal{E}_t^{FB}(x) = Duo$. To complete the induction step, we show that $\Delta V_t(x)$ is nonincreasing in x for all $x < \overline{x}_{d,t}^{FB}$. Because $\Delta V_{t+1}(\xi)$ is nonincreasing in ξ for all $\xi < \overline{x}_{d,t+1}^{FB}$ and because $\overline{x}_{d,t}^{FB} < \overline{x}_{d,t+1}^{FB}$, we obtain for any $x < \overline{x}_{d,t}^{FB}$, $\Delta V_t(x) = V_t(x+1 \mid Duo) - V_t(x \mid Duo) = \delta p_d \Delta V_{t+1}(x+1) + \delta(1-p_d)\Delta V_{t+1}(x) \leq \delta p_d \Delta V_{t+1}(x) + \delta(1-p_d)\Delta V_{t+1}(x-1) = V_t(x \mid Duo) - V_t(x-1 \mid Duo) = \Delta V_t(x-1)$, completing the induction step.□

4.1.2 RS Contract

Preliminaries

We first define the following thresholds for $i \in \{c, v\}$:

$$\begin{aligned} \theta^{RS}_{s^i d} &:= \frac{c_d}{\delta\alpha^i(p_d - p_s)} \\ \theta^{RS}_{fs^i}(x) &:= \frac{c^i_s}{\delta\alpha^i p_s} + \frac{(1-\delta)R(x)}{\delta p_s} \\ \theta^{RS}_{fd^i}(x) &:= \frac{c_d}{\delta\alpha^i p_d} + \frac{(1-\delta)R(x)}{\delta p_d} \end{aligned}$$

The thresholds $\theta^{RS}_{kl}(x)$ allow for a pairwise comparison of equilibrium outcomes k and l, for any $k \in \{f, s^i\}$, $l \in \{d, s^i\}$, $k \neq l$. Specifically in period $T-1$, player i prefers $Solo^{-i}$ over Duo if and only if $\Delta R(x) \leq \theta^{RS}_{s^i d}(x)$; she prefers $Finish$ over $Solo^i$ if and only if $\Delta R(x) \leq \theta^{RS}_{fs^i}(x)$; and she prefers $Finish$ over Duo (in case of multiple equilibria, as an equilibrium selection rule) if and only if $\Delta R(x) \leq \theta^{RS}_{fd^i}(x)$. We define

$$x^{RS}_d := \max\left\{x \in \mathbb{Z}^+ \mid \Delta R(x) \geq \max\left\{\theta^{RS}_{s^c d}, \theta^{RS}_{s^v d}\right\}\right\}, \tag{A2-1}$$

$$x^{RS}_{f,T-1} := \min\left\{x \in \mathbb{Z}^+ \mid \Delta R(x) \leq \min\left\{\theta^{RS}_{fs^c}(x), \theta^{RS}_{fs^v}(x)\right\}\right\}. \tag{A2-2}$$

We also define $\overline{x}^{RS}_f$ such that $\overline{x}^{RS}_f = x^{RS}_{f,T-1}$ when $x^{RS}_d < x^{RS}_{f,T-1}$ and $\overline{x}^{RS}_f :=$ $\min\left\{\min\left\{x \in \mathbb{Z}^+ \mid \Delta R(x) \leq \theta^{RS}_{fd^c}(x)\right\}, x^{RS}_d + 1\right\}$ otherwise. Finally, we define

$$x^{RS}_f = \max\{x^{RS}_{f,T-1}, \overline{x}^{RS}_f\}. \tag{A2-3}$$

Proofs

Lemma A2-1. *Suppose that, for all $t \geq \hat{t}$ and $\hat{x} \leq x < x_f^{RS}$, $\mathcal{E}_t^{RS}(x) = Solo^i$. Then, $V_t^i(x) \geq V_{t+1}^i(x)$ and $V_t^{-i}(x) \geq V_{t+1}^{-i}(x)$ for all $t \geq \hat{t}$ and $\hat{x} \leq x < x_f^{RS}$.*

Proof. The proof proceeds by induction. Consider period $T-1$. In state x, $\hat{x} \leq x < x_f^{RS}$, because $\mathcal{E}_{T-1}^{RS}(x) = Solo^i$, $V_{T-1}^i(x) = V_{T-1}^i(x \mid Solo^i) \geq V_{T-1}^i(x \mid Finish) = V_T^i(x)$. This implies $R(x) \leq -c_s^i/\alpha_i + \delta p_s R(x+1) + \delta(1-p_s)R(x) < \delta p_s R(x+1) + \delta(1-p_s)R(x)$. Multiplying both sides of the inequality by α^{-i} yields $V_{T-1}^{-i}(x) = V_{T-1}^{-i}(x \mid Solo^i) \geq V_{T-1}^{-i}(x \mid Finish) = V_T^{-i}(x)$.

Consider now any period t, $\hat{t} \leq t < T-1$, and suppose that $V_{t+1}^i(x) \geq V_{t+2}^i(x)$ for $x \geq \hat{x}$. Then, $V_t^i(x) = V_t^i(x \mid Solo^i) = -c_s^i + \delta \mathbb{E}_s[V_{t+1}^i(x+\xi)] \geq -c_s^i + \delta \mathbb{E}_s[V_{t+2}^i(x+\xi)] = V_{t+1}^i(x \mid Solo^i) = V_{t+1}^i(x)$. Similarly, suppose that $V_{t+1}^{-i}(x) \geq V_{t+2}^{-i}(x)$ for $x \geq \hat{x}$. Then, $V_t^{-i}(x) = V_t^{-i}(x \mid Solo^i) = \delta \mathbb{E}_s[V_{t+1}^{-i}(x+\xi)] \geq \delta \mathbb{E}_s[V_{t+2}^{-i}(x+\xi)] = V_{t+1}^{-i}(x \mid Solo^i) = V_{t+1}^{-i}(x)$, completing the induction step.□ ▮

Proof of Proposition 1.2. (i) The proof proceeds by induction. In period $T-1$, $Finish$ is an equilibrium in state x if $x \geq x_{f,T-1}^{RS}$ (see (A2-2)). When $x_d^{RS} \geq x_{f,T-1}^{RS}$, both Duo and $Finish$ are equilibria in period $T-1$ and states $x_{f,T-1}^{RS} \leq x < x_d^{RS}$. By our equilibrium selection rule, $\mathcal{E}_{T-1}^{RS}(x) = Finish$ for all $x \geq \overline{x}_f^{RS}$. Therefore by the definition of x_f^{RS}, $\mathcal{E}_{T-1}^{RS}(x) = Finish$ for all $x \geq x_f^{RS}$.

Suppose that $\mathcal{E}_{t+1}^{RS}(x) = Finish$ for all $x \geq x_f^{RS}$. Then, for any $i \in \{c, v\}$ and $x \geq x_f^{RS}$, $V_t^i(x \mid \mathcal{E}) = V_{T-1}^i(x \mid \mathcal{E})$ for any equilibrium $\mathcal{E} \in \{Duo, Solo^c, Solo^v, Finish\}$, completing the induction step.

(ii) The proof proceeds by induction. Given that $x_d^{RS} < x_{f,T-1}^{RS}$, $\overline{x}_f^{RS} = x_{f,T-1}^{RS}$ and therefore $x_f^{RS} = x_{f,T-1}^{RS}$ by (A2-3).

By (A2-2), $\Delta R(x_f^{RS}-1) > \min\left\{\theta_{fs^c}^{RS}(x_f^{RS}-1), \theta_{fs^v}^{RS}(x_f^{RS}-1)\right\}$. Given that $R(x)$ is increasing concave, for all $x < x_f^{RS}$, $\Delta R(x) \geq \Delta R(x_f^{RS}-1) > \theta_{fs^i}^{RS}(x_f^{RS}-1) \geq \theta_{fs^i}^{RS}(x)$, i.e., $V_{T-1}^i(x \mid Solo^i) > \alpha^i R(x)$. Therefore, $\mathcal{E}_{T-1}^{RS}(x) \neq Finish$ for all $x < x_f^{RS}$. Moreover, $V_{T-1}^i(x \mid Solo^{-i}) > V_{T-1}^i(x \mid Solo^i) > \alpha^i R(x)$ and $V_{T-1}^i(x \mid Duo) > \alpha^i R(x)$ since $\Delta R(x) > \Delta R(x_f^{RS} - 1) > c_d/[\delta \min\{\alpha^c, \alpha^v\} p_d] + (1-\delta)R(x)/[\delta p_d] = \max\{\theta_{fd^c}^{RS}(x), \theta_{fd^v}^{RS}(x)\}$. Thus, $V_{T-1}^i(x) > \alpha^i R(x)$, for all $x < x_f^{RS}$.

Fix $t < T-1$ and consider state $x < x_f^{RS}$. Suppose that $V_{t+1}^i(x \mid Solo^i) > \alpha^i R(x)$ and $V_{t+1}^i(x \mid Duo) > \alpha^i R(x)$. Similar to above, $V_{t+1}^i(x \mid Solo^{-i}) > V_{t+1}^i(x \mid Solo^i)$. Therefore, $V_{t+1}^i(x) > \alpha^i R(x)$. By induction on the above states, $V_{t+1}^i(x+1) > \alpha^i R(x+1)$. As a result, $V_t^i(x \mid Solo^i) = -c_s^i + \delta p_s V_{t+1}^i(x+1) + \delta(1-p_s)V_{t+1}^i(x) \geq -c_s^i + \delta p_s \alpha^i R(x+1) + \delta(1-p_s)\alpha^i R(x) = V_{T-1}^i(x \mid Solo^i) > \alpha^i R(x)$. Similarly, $V_t^i(x \mid Duo) \geq V_{T-1}^i(x \mid Duo) > \alpha^i R(x)$. This completes the induction step.

(iii) The proof uses Lemma A2-1 in this appendix. We only need to show that $Solo^v$ is an equilibrium since in case of multiple equilibria, $Solo^v$ is selected as the client always prefers $Solo^v$ over $Solo^c$. We prove the result by induction. Given that $x_d^{RS} < x_{f,T-1}^{RS}$ and $\alpha^c \leq \alpha^v$, $\theta_{fs^v}^{RS}(x) < \theta_{s^c d}^{RS}$ by (A2-1) and (A2-3). Also, $x_d^{RS} < x_{f,T-1}^{RS}$ implies $\overline{x}_f^{RS} = x_{f,T-1}^{RS}$ and therefore $x_f^{RS} = x_{f,T-1}^{RS}$ by (A2-3). As a result, $\mathcal{E}_{T-1}^{RS}(x) = Solo^v$ for all $x_d^{RS} \leq x < x_f^{RS}$.

Fix x and suppose that $\mathcal{E}_{t'}^{RS}(x) = Solo^v$ for all $t' \geq t$; we show that $\mathcal{E}_t^{RS}(x) = Solo^v$. By Lemma A2-1 and because $\mathcal{E}_{t+1}^{RS}(x) = Solo^v$, $V_t^v(x \mid Solo^v) \geq V_{t+1}^v(x \mid Solo^v) \geq V_{t+1}^v(x \mid Finish) = V_t^v(x \mid Finish)$.

Also, it can be shown that $V_t^c(x \mid Solo^v) \geq V_t^c(x \mid Duo)$, i.e., $\Delta V_{t+1}^c(x) \leq \alpha^c \theta_{s^c d}^{RS}$. under the two possible cases of (i) $\mathcal{E}_{t+1}^{RS}(x+1) = Solo^v$ (using a similar argument to the proof of Lemma A1-4) and (ii) $\mathcal{E}_{t+1}^{RS}(x+1) = Finish$ (since $V_{t+1}^c(x) \geq \alpha^c R(x)$ by Lemma A2-1 and $\alpha^c \Delta R(x) \leq \alpha^c \theta_{s^c d}^{RS}$). Because $V_t^c(x \mid Solo^v) \geq V_t^c(x \mid Duo)$ and $V_t^v(x \mid Solo^v) \geq V_t^v(x \mid Finish)$, $\mathcal{E}_t^{RS}(x) = Solo^v$ for all $t \leq T-1$ and all

$x_d^{RS} \leq x < x_f^{RS}$.

(iv) The proof proceeds by induction on t. First consider the case in which $x_d^{RS} < x_{f,T-1}^{RS}$. By part (ii), $\mathcal{E}_t^{RS}(x) \neq Finish$ for all $t < T$ and $x < x_f^{RS} = x_{f,T-1}^{RS}$. Therefore, it is enough to show $\Delta V_t^i(x) \geq \theta_{sd}^{RS} := \alpha^i \theta_{s^i d}^{RS}$ for $i \in \{c, v\}$, $t < T$, and $x \leq \overline{x}_{d,t}^{RS}$.

Consider period $T-1$. By (A2-1), $\mathcal{E}_{T-1}^{RS}(x_d^{RS}) = Duo$ since $x_d^{RS} < x_{f,T-1}^{RS}$. Therefore, $\mathcal{E}_{T-1}^{RS}(x) = Duo$ for all $x \leq \overline{x}_{d,T-1}^{RS} = \min\{x_d^{RS}, x_f^{RS} - 1\} = \min\{x_d^{RS}, x_{f,T-1}^{RS} - 1\} = x_d^{RS}$, because $\Delta R(x) \geq \Delta R(x_d^{RS}) > \theta_{s^i d}^{FB}$. Moreover, $\Delta V_{T-1}^i(x)$ is nonincreasing in x for all $x \leq \overline{x}_{d,T-1}^{RS}$ as in the proof of Proposition 1.1 (iii.c). Similar to that proof, it can be shown that $\Delta V_t^i(x)$ is nonincreasing in x for all t and $x < \overline{x}_{d,t}^{RS}$.

Next, consider the case in which $x_d^{RS} \geq x_{f,T-1}^{RS}$. By (A2-3), $\overline{x}_{d,T-1}^{RS} = \min\{x_d^{RS}, x_f^{RS} - 1\} = x_f^{RS} - 1$. Similar to the above, one can show that $\Delta V_t^i(x) \geq \theta_{sd}^{RS}$ for $t < T-1$ and $x \leq \overline{x}_{d,t}^{RS}$. Hence, *Duo* is an equilibrium. Potentially, *Finish* is also an equilibrium and we then need to show that *Duo* will be selected by the client. We show this result by induction on t. Since $x_d^{RS} \geq x_{f,T-1}^{RS}$, $\mathcal{E}_{T-1}^{RS}(x_f^{RS} - 1) = Duo$, i.e., $\Delta R(x_f^{RS} - 1) \geq \theta_{fd^c}^{RS}(x_f^{RS} - 1)$. Given that $R(x)$ is nondecreasing concave, $\Delta R(x) \geq \Delta R(x_f^{RS} - 1) > \theta_{fd^c}^{RS}(x_f^{RS} - 1) \geq \theta_{fd^c}^{RS}(x)$, i.e., $V_{T-1}^c(x \mid Duo) > \alpha^c R(x)$, and therefore $\mathcal{E}_{T-1}^{RS}(x) \neq Finish$ for all $x < x_f^{RS}$. Fix $t < T-1$ and suppose that $\mathcal{E}_t^{RS}(\xi) = Duo$ and that $V_t^c(\xi \mid Duo) > \alpha^c R(x)$ for all $\xi \leq \overline{x}_{d,\tau}^{RS}$ and $\tau > t$. We obtain $V_t^c(x \mid Duo) = -c_d + \delta \mathbb{E}_d[V_{t+1}^c(x + \xi)] \geq V_{T-1}^i(x \mid Duo) > \alpha^c R(x)$, i.e., $\mathcal{E}_t^{RS}(x) \neq Finish$, completing the induction step.□

4.1.3 FF Contract

Preliminaries

We first define the following thresholds:

$$\theta_{sd}^{FF} := \frac{c_d}{\delta(p_d - p_s)}, \theta_{fs^c}^{FF}(x) := \frac{c_s^c + (1-\delta)\left[R(x) - b\right]}{\delta p_s}, \theta_{fd}^{FF}(x) := \frac{(1-\delta)\left[R(x) - b\right]}{\delta p_d}. \tag{A3-1}$$

The thresholds $\theta_{kl}^{FF}(x)$ allow for a pairwise comparison of policies k and l, for any $k \in \{s, f\}$, $l \in \{s^c, d\}$, $k \neq l$. Specifically in period $T-1$, the client prefers $Solo^v$ over Duo if and only if $\Delta R(x) \leq \theta_{sd}^{FF}$; she prefers $Finish$ over $Solo^c$ if and only if $\Delta R(x) \leq \theta_{fs^c}^{FF}(x)$; and she prefers $Finish$ over Duo (in case of multiple equilibria, as an equilibrium selection rule) if and only if $\Delta R(x) \leq \theta_{fd}^{FF}(x)$. It turns out that $\theta_{fd}^{FF}(x) \geq \theta_{sd}^{FF}$ implies that $\theta_{fs^c}^{FF}(x) \geq \theta_{fd}^{FF}(x)$. Accordingly, let us define

$$x_{\theta}^{FF} := \min\left\{x \in \mathbb{Z}^+ \mid \theta_{fd}^{FF}(x) \geq \theta_{sd}^{FF}\right\}, \tag{A3-2}$$

such that $\theta_{fs^c}^{FF}(x) \geq \theta_{fd}^{FF}(x) \geq \theta_{sd}^{FF}$ for all $x \geq x_{\theta}^{FF}$, given that $R(x)$ is increasing.

We also define

$$x_f^{FF} := \min\left\{x \in \mathbb{Z}^+ \mid \Delta R(x) \leq \theta_{fs^c}^{FF}(x)\right\}. \tag{A3-3}$$

Proofs

Lemma A3-1. *For all t and all x, $V_t^v(x \mid Solo^v) < V_t^v(x \mid Finish)$, i.e., $\mathcal{E}_t^{FF}(x) \neq Solo^v$.*

Proof. Because $V_t^v(x) \leq \max_t \max_x g_t^v(x) = b$ for all t, $V_t^v(x \mid Solo^v) = -c_s^v + \delta\mathbb{E}_d\left[V_{t+1}^v(x+\xi)\right] \leq -c_s^v + \delta b < b = V_t^v(x \mid Finish)$; hence, $\mathcal{E}_t^{FF}(x) \neq Solo^v$ for all t

and all x.□ ∎

Lemma A3-2. *For any* $t \in \mathcal{T}$*,* $\mathcal{E}_t^{FF}(x) = Finish \; \forall x \geq x_f^{FF}$*.*

Proof. The proof proceeds by induction on t and uses Lemma A3-1 in this appendix. Fix period $T-1$ and state $x \geq x_f^{FF}$. By the definition of x_f^{FF}, $V_{T-1}^c(x \mid Solo^c) < V_{T-1}^c(x \mid Finish)$ and by Lemma A3-1, $V_t^v(x \mid Solo^v) < V_t^v(x \mid Finish)$. Therefore, $Finish$ is an equilibrium outcome. Also, Duo is not an equilibrium outcome, because $V_{T-1}^v(x \mid Duo) = -c_d + \delta p_d b + \delta(1-p_d)b < \delta p_s b + \delta(1-p_s)b = V_t^v(x \mid Solo^c)$. As result, $\mathcal{E}_{T-1}^{FF}(x) = Finish$ for $x \geq x_f^{FF}$.

Fix $t < T-1$. Suppose that $\mathcal{E}_{t+1}^{FF}(x) = Finish$ for $\forall x \geq x_f^{FF}$. Then for any $i \in \{c, v\}$ and $x \geq x_f^{FF}$, $V_t^i(x \mid \mathcal{E}) = V_{T-1}^i(x \mid \mathcal{E})$ for $\mathcal{E} \in \{Duo, Solo^c, Solo^v, Finish\}$. As a result, $\mathcal{E}_t^{FF}(x) = \mathcal{E}_{T-1}^{FF}(x) = Finish$.□ ∎

Lemma A3-3. *For all* $x < x_f^{FF}$*,* $\mathcal{E}_{T-1}^{FF}(x) = Solo^c$*.*

Proof. By Lemma A3-1 in this appendix, $\mathcal{E}_{T-1}^{FF}(x) \neq Solo^v$. Moreover for any $x < x_f^{FF}$, $\Delta R(x) \geq \Delta R(x_f^{FF} - 1) > \theta_{fs^c}^{FF}(x_f^{FF} - 1) \geq \theta_{fs^c}^{FF}(x)$, i.e., $V_{T-1}^c(x \mid Solo^c) > V_{T-1}^c(x \mid Finish)$. Finally, $V_{T-1}^v(x \mid Duo) < V_{T-1}^c(x \mid Solo^c)$ since $-c_d + \delta b < \delta b$.□ ∎

Lemma A3-4. *For any* $x < x_f^{FF}$*, if* $\mathcal{E}_{t+1}^{FF}(x+1) = \mathcal{E}_{t+1}^{FF}(x) = Solo^c$*, then* $\mathcal{E}_t^{FF}(x) = Solo^c$*.*

Proof. By Lemma A3-1 in this appendix, $\mathcal{E}_t^{FF}(x) = Solo^c$ if (i) $V_t^v(x \mid Solo^c) > V_t^v(x \mid Duo)$, i.e., $\Delta V_{t+1}^v(x) < \theta_{sd}^{FF}$, and (ii) $V_t^c(x \mid Solo^c) > V_t^c(x \mid Finish)$, i.e., $-c_s^c + \delta \mathbb{E}_d\left[V_{t+1}^c(x+\xi)\right] > R(x) - b$.

(i) When $\mathcal{E}_{t+1}^{FF}(x+1) = \mathcal{E}_{t+1}^{FF}(x) = Solo^c$, $\Delta V_{t+2}^v(x+1) \leq \theta_{sd}^{FF}$ and $\Delta V_{t+2}^v(x) \leq \theta_{sd}^{FF}$. As a result, $\Delta V_{t+1}^v(x) < \theta_{sd}^{FF}$ similar to the proof of Lemma A1-4.

(ii) When $\mathcal{E}^{FF}_{t+1}(x+1) = \mathcal{E}^{FF}_{t+1}(x) = Solo^c$, $V^c_{t+1}(x+1 \mid Solo^c) \geq R(x+1) - b$ and $V^c_{t+1}(x \mid Solo^c) \geq R(x) - b$. Moreover given that $x < x^{FF}_f$, $\mathcal{E}^{FF}_{T-1}(x) = Solo^c$. We obtain $-c^c_s + \delta \mathbb{E}_d \left[V^c_{t+1}(x+\xi)\right] \geq -c^c_s + \delta p_s \left[R(x+1) - b\right] + \delta(1-p_s)\left[R(x) - b\right] = V^c_{T-1}(x \mid Solo^c) > R(x) - b.\square$

∎

Lemma A3-5. *Fix $\hat{x}$. Suppose that, for all $\hat{x} \leq x < x^{FF}_f$, there exists a time period $\tau(x)$ such that $\mathcal{E}^{FF}_t(x) = Solo^c$ for $t \geq \tau(x)$ and $\mathcal{E}^{FF}_t(x) = Duo$ for $t < \tau(x)$. Then for all $x \geq \hat{x}$, $V^c_t(x)$ is decreasing in t.*

Proof. The proof uses Lemmas A3-2 and A3-3 in Appendix 4.1.3. By Lemma A3-2, $V^c_t(x) = V^c_{t+1}(x) = R(x) - b$ when $x \geq x^{FF}_f$. Suppose next that $x < x^{FF}_f$. The proof proceeds by induction on t. In period $T-1$ and state $x < x^{FF}_f$, $V^c_{T-1}(x) > R(x) - b = V^c_T(x)$ because $\mathcal{E}^{FF}_{T-1}(x) = Solo^c$ by Lemma A3-3. Fix t and suppose that $V^c_t(x) > V^c_{t+1}(x)$. If $t > \tau(x)$, $V^c_{t-1}(x) = -c^c_s + \delta\mathbb{E}_s[V^c_t(x+\xi)] > -c^c_s + \delta\mathbb{E}_s[V^c_{t+1}(x+\xi)] = V^c_t(x)$. If $t = \tau(x)$, $V^c_{t-1}(x) = V^c_{t-1}(x \mid Duo) \geq V^c_{t-1}(x \mid Solo^v)$ given that $\mathcal{E}^{FF}_{t-1}(x) = Duo$; moreover, $V^c_{t-1}(x \mid Solo^v) > V^c_{t-1}(x \mid Solo^c)$ because $c^c_s > 0$; as a result, $V^c_{t-1}(x) > V^c_{t-1}(x \mid Solo^c) = -c^c_s + \delta\mathbb{E}_s[V^c_t(x+\xi)] > -c^c_s + \delta\mathbb{E}_s[V^c_{t+1}(x+\xi)] = V^c_t(x)$. Finally, if $t < \tau(x)$, $V^c_{t-1}(x) = -c_d + \delta\mathbb{E}_d[V^c_t(x+\xi)] > -c_d + \delta\mathbb{E}_d[V^c_{t+1}(x+\xi)] = V^c_t(x)$, completing the induction step.$\square$ ∎

Lemma A3-6. *Fix a period $\hat{t}$ and a state $\hat{x} < x^{FF}_f$. Suppose that, for all $\hat{x} \leq x < x^{FF}_f$, there exists a time period $\tau(x)$ such that $\mathcal{E}^{FF}_t(x) = Solo^c$ for $t \geq \tau(x)$ and $\mathcal{E}^{FF}_t(x) = Duo$ for $t < \tau(x)$. Then, $V^c_{\hat{t}}(\hat{x} \mid Solo^c) > V^c_{\hat{t}}(\hat{x} \mid Finish)$.*

Proof. By Lemma A3-5 in this appendix, $V^c_{\hat{t}+1}(\hat{x}+1) > V^c_T(\hat{x}+1)$ and $V^c_{\hat{t}+1}(\hat{x}) > V^c_T(\hat{x})$. Hence, $V^c_{\hat{t}}(\hat{x} \mid Solo^c) = -c^c_s + \delta[p_s V^c_{\hat{t}+1}(\hat{x}+1) + (1-p_s)V^c_{\hat{t}+1}(\hat{x})] > -c^c_s + \delta[p_s V^c_T(\hat{x} +$

$1) + (1-p_s)V_T^c(\hat{x})] = V_{T-1}^c(\hat{x} \mid Solo^c) > R(\hat{x}) - b$, in which the last inequality is because $\mathcal{E}_{T-1}^{FF}(\hat{x}) \neq Finish$ when $\hat{x} < x_f^{FF}$.□ ∎

Lemma A3-7. *Suppose $x_f^{FF} < x_\theta^{FF}$ and $R(x_f^{FF}) \geq 2b$. Then, $\mathcal{E}_t^{FF}(x) = Solo^c$ for all t and $x < x_f^{FF}$.*

Proof. The proof uses Lemmas A3-1, A3-3, A3-4, and A3-6 in this appendix. By Lemma A3-1, $\mathcal{E}_t^{FF}(x) \neq Solo^v$, hence it suffices to show that there exists an equilibrium $Solo^c$. We prove the result by induction on t. Consider first state $x_f^{FF} - 1$. By Lemma A3-3, $\mathcal{E}_{T-1}^{FF}(x_f^{FF}-1) = Solo^c$. Fix $t < T-1$ and suppose that $\mathcal{E}_{t+1}^{FF}(x_f^{FF}-1) = Solo^c$, i.e., $\Delta V_{t+2}^v(x_f^{FF} - 1) < \theta_{sd}^{FF}$. Applying the induction hypothesis, we show that $\Delta V_{t+1}^v(x_f^{FF} - 1) < \theta_{sd}^{FF}$, i.e., $V_t^v(x_f^{FF} - 1 \mid Duo) < V_t^v(x_f^{FF} - 1 \mid Solo^c)$. Given that $x_f^{FF} < x_\theta^{FF}$, $\theta_{fd}^{FF}(x_f^{FF}) < \theta_{sd}^{FF}$. We then have: $\Delta V_{t+1}^v(x_f^{FF} - 1) = b - \delta p_s V_{t+2}^v(x_f^{FF}) - \delta(1-p_s)V_{t+2}^v(x_f^{FF} - 1) = (1-\delta)b + \delta(1-p_s)\Delta V_{t+2}^v(x_f^{FF} - 1) \leq (1-\delta)(R(x_f^{FF}) - b) + \delta(1-p_s)\theta_{sd}^{FF} = \delta p_d \theta_{fd}^{FF}(x_f^{FF}) - c_d + \delta(1-p_s)\theta_{sd}^{FF} < \delta(1 - p_s + p_d)\theta_{sd}^{FF} - c_d = \delta\theta_{sd}^{FF} + c_d - c_d < \theta_{sd}^{FF}$, in which the first inequality holds by the assumption that $R(x_f^{FF}) \geq 2b$. Therefore, $V_t^v(x_f^{FF} - 1 \mid Duo) < V_t^v(x_f^{FF} - 1 \mid Solo^c)$. Also by Lemma A3-6, $V_t^c(x_f^{FF} - 1 \mid Solo^c) > V_t^c(x_f^{FF} - 1 \mid Finish)$. As a result, $\mathcal{E}_t^{FF}(x_f^{FF} - 1) = Solo^c$.

Given that $\mathcal{E}_t^{FF}(x_f^{FF}-1) = Solo^c$ for all t and $\mathcal{E}_{T-1}^{FF}(x) = Solo^c$ for all $x < x_f^{FF}$ by Lemma A3-3, $\mathcal{E}_t^{FF}(x) = Solo^c$ for all states $x < x_f^{FF} - 1$ and all period t by Lemma A3-4.□ ∎

Lemma A3-8. *Fix state $\hat{x} < x_f^{FF}$ and period $\hat{t}$. Suppose that there exists a pure-strategy equilibrium with $a_t^c(x) = W$ in every period $\hat{t} \leq t < T$ and every state $\hat{x} \leq x < x_f^{FF}$. Then, for all $\hat{x} \leq x < x_f^{FF}$ and all $t \geq \hat{t}$, $V_t^v(x)$ is increasing in t.*

Proof. The proof uses Lemmas A3-2 and A3-3 in Appendix 4.1.3. We prove the result by induction. Consider first state $x_f^{FF}-1$. By Lemma A3-3, $\mathcal{E}_{T-1}^{FF}(x_f^{FF}-1)=Solo^c$. Hence, $V_{T-1}^v(x_f^{FF}-1)=\delta b<b=V_T^v(x_f^{FF}-1)$. Fix t and suppose that $V_{t+1}^v(x_f^{FF}-1)<V_{t+2}^v(x_f^{FF}-1)$. By Lemma A3-2, $V_t^v(x_f^{FF})=b$ for all t. Hence, similar to the proof of Lemma A1-5, we obtain that $V_t^v(x_f^{FF}-1)=\max\left\{-c_d+\delta p_d b+\delta(1-p_d)V_{t+1}^v(x_f^{FF}-1),\delta p_s b+\delta(1-p_s)V_{t+1}^v(x_f^{FF}-1)\right\}<V_{t+1}^v(x_f^{FF}-1)$, completing the induction step.

Consider next any state $x<x_f^{FF}-1$ and suppose that the result holds for $x+1$, i.e., $V_t^v(x+1)<V_{t+1}^v(x+1)$ for all t. Because $\mathcal{E}_{T-1}^{FF}(x)=Solo^c$ by Lemma A3-3, $V_{T-1}^v(x)=\delta b<b=V_T^v(x)$. Fix t and suppose that $V_{t+1}^v(x)<V_{t+2}^v(x)$. Applying the induction hypothesis and following the same argument as above establish that $V_t^v(x)<V_{t+1}^v(x)$. This completes the induction step.□ ∎

Lemma A3-9. *Suppose that $R(x_f^{FF})\geq 2b$. Fix period $\hat{t}$ and state $\hat{x}<x_f^{FF}$. Suppose that, for all $\hat{x}\leq x<x_f^{FF}$, there exists a time period $\tau(x)\leq T-1$, nondecreasing in x, such that $\mathcal{E}_t^{FF}(x)=Solo^c$ for $t\geq\tau(x)$ and $\mathcal{E}_t^{FF}(x)=Duo$ for $t<\tau(x)$. Then, $\Delta V_{\hat{t}}^c(\hat{x})>\Delta V_{\hat{t}}^v(\hat{x})$.*

Proof. We prove the result by induction. Consider first state $x_f^{FF}-1$. In period T, $\Delta V_T^c(x_f^{FF}-1)=R(x_f^{FF})-R(x_f^{FF}-1)\geq 0=\Delta V_{T-1}^v(x_f^{FF}-1)$. Fix t and suppose that $\Delta V_{t+1}^c(x_f^{FF}-1)\geq\Delta V_{t+1}^v(x_f^{FF}-1)$. Then, for $t\geq\tau(x_f^{FF}-1)-1$, we have $\Delta V_t^c(x_f^{FF}-1)=(1-\delta)[R(x_f^{FF})-b]+c_s^c+\delta(1-p_s)\Delta V_{t+1}^c(x_f^{FF}-1)\geq(1-\delta)b+\delta(1-p_s)\Delta V_{t+1}^v(x_f^{FF}-1)=\Delta V_t^v(x_f^{FF}-1)$ using the induction hypothesis and the fact that $R(x_f^{FF})\geq 2b$. Similarly, for $t<\tau(x_f^{FF}-1)-1$, we have $\Delta V_t^c(x_f^{FF}-1)=(1-\delta)[R(x_f^{FF})-b]+c_d+\delta(1-p_d)\Delta V_{t+1}^c(x_f^{FF}-1)\geq(1-\delta)b+\delta(1-p_d)\Delta V_{t+1}^v(x_f^{FF}-1)=\Delta V_t^v(x_f^{FF}-1)$.

Consider next any state $x<x_f^{FF}-1$ and suppose that $\Delta V_t^c(x+1)\geq\Delta V_t^v(x+1)$ for all t. In period T, $\Delta V_T^c(x)=\Delta R(x)\geq 0=\Delta V_{T-1}^v(x)$. Fix $t<T$ and suppose

that $\Delta V_{t+1}^c(x) \geq \Delta V_{t+1}^v(x)$. Given that $\tau(x)$ is assumed to be nondecreasing, we consider the following three cases: (i) $\mathcal{E}_\sqcup^{\mathcal{FF}}(\S) = \mathcal{E}_\sqcup^{\mathcal{FF}}(\S + \infty) = \mathcal{S}\wr\updownarrow\wr^\rfloor$, (ii) $\mathcal{E}_\sqcup^{\mathcal{FF}}(\S) = \mathcal{E}_\sqcup^{\mathcal{FF}}(\S + \infty) = \mathcal{D}\sqcap\wr$, and (iii) $\mathcal{E}_\sqcup^{\mathcal{FF}}(\S) = \mathcal{S}\wr\updownarrow\wr^\rfloor$ and $\mathcal{E}_\sqcup^{\mathcal{FF}}(\S + \infty) = \mathcal{D}\sqcap\wr$.

(i) $\Delta V_t^c(x) = \delta p_s \Delta V_{t+1}^c(x+1) + \delta(1-p_s)\Delta V_{t+1}^c(x) \geq \delta p_s \Delta V_{t+1}^v(x+1) + \delta(1-p_s)\Delta V_{t+1}^v(x) = \Delta V_t^v(x)$.

(ii) $\Delta V_t^c(x) = \delta p_d \Delta V_{t+1}^c(x+1) + \delta(1-p_d)\Delta V_{t+1}^c(x) \geq \delta p_d \Delta V_{t+1}^v(x+1) + \delta(1-p_d)\Delta V_{t+1}^v(x) = \Delta V_t^v(x)$.

(iii) $\Delta V_t^c(x) = -c_d + c_s + \delta p_d \Delta V_{t+1}^c(x+1) + \delta(1-p_s)\Delta V_{t+1}^c(x) \geq -c_d + \delta p_d \Delta V_{t+1}^v(x+1) + \delta(1-p_s)\Delta V_{t+1}^v(x) = \Delta V_t^v(x)$.

Hence, $\Delta V_t^c(x) \geq \Delta V_t^v(x)$, completing the induction step.$\square$ ∎

Lemma A3-10. *Suppose that $x_f^{FF} \geq x_\theta^{FF}$ and $R(x_f^{FF}) \geq 2b$. There exists a time threshold $\tau(x_f^{FF}-1)$ such that $\mathcal{E}_t^{FF}(x_f^{FF}-1) = Duo$ for all $t < \tau(x_f^{FF}-1)$ and $\mathcal{E}_t^{FF}(x_f^{FF}-1) = Solo^c$ for all $t \geq \tau(x_f^{FF}-1)$.*

Proof. The proof uses Lemmas A3-1, A3-2, A3-3, A3-6, A3-8, and A3-9 in Appendix 4.1.3. Let $\tau(x_f^{FF}-1) := \max\{t | \Delta V_t^v(x_f^{FF}-1) \geq \theta_{sd}^{FF}\}$.

We first show by induction that $\mathcal{E}_t^{FF}(x_f^{FF}-1) = Solo^c$ for all $t \geq \tau(x_f^{FF}-1)$. In period $T-1$, $\mathcal{E}_{T-1}(x_f^{FF}-1) = Solo^c$ by Lemma A3-3. Fix $t \geq \tau(x_f^{FF}-1)$ and suppose that $\mathcal{E}_{t+1}(x_f^{FF}-1) = Solo^c$. By Lemma A3-6, $V_t^c(x_f^{FF}-1 \mid Solo^c) > V_t^c(x_f^{FF}-1 \mid Finish)$. Moreover, $V_t^v(x_f^{FF}-1 \mid Solo^v) < V_t^v(x_f^{FF}-1 \mid Finish)$ by Lemma A3-1. Furthermore, $V_t^v(x_f^{FF}-1 \mid Solo^c) > V_t^v(x_f^{FF}-1 \mid Duo)$ when $t \geq \tau(x_f^{FF}-1)$ by (A3-1). Hence, $\mathcal{E}_t^{FF}(x_f^{FF}-1) = Solo^c$.

We next show by induction that $\mathcal{E}_t^{FF}(x_f^{FF}-1) = Duo$ for all $t < \tau(x_f^{FF}-1)$. Consider first period $\tau(x_f^{FF}-1)-1$. By definition, $\Delta V_\tau^v(x_f^{FF}-1) \geq \theta_{sd}^{FF}$. Moreover,

$\Delta V_\tau^c(x_f^{FF}-1) \geq \theta_{sd}^{FF}$ because $\Delta V_\tau^c(x_f^{FF}-1) \geq \Delta V_\tau^v(x_f^{FF}-1)$ by Lemma A3-9. Hence, $V_{\tau-1}^v(x_f^{FF}-1 \mid Solo^c) \leq V_{\tau-1}^v(x_f^{FF}-1 \mid Duo)$ and $V_{\tau-1}^c(x_f^{FF}-1 \mid Solo^v) \leq V_{\tau-1}^c(x_f^{FF}-1 \mid Duo)$. Furthermore, $V_{\tau-1}^c(x_f^{FF}-1 \mid Solo^c) > V_{\tau-1}^c(x_f^{FF}-1 \mid Finish)$ by Lemma A3-6. Hence, $\mathcal{E}_{\tau-1}^{FF}(x_f^{FF}-1) = Duo$.

Next, fix $t < \tau(x_f^{FF}-1)$ and suppose that $\mathcal{E}_{t+1}(x_f^{FF}-1) = Duo$. From Lemmas A3-1 and A3-6, $\mathcal{E}_t^{FF}(x_f^{FF}-1) \neq Finish$ and $\mathcal{E}_t^{FF}(x_f^{FF}-1) \neq Solo^v$; hence, $a_t^c(x_f^{FF}-1) = W$. Accordingly, using Table 1.2, $\mathcal{E}_t^{FF}(x_f^{FF}-1) = Duo$ if (i) $V_t^v(x_f^{FF}-1 \mid Duo) > V_t^v(x_f^{FF}-1 \mid Solo^c)$ and (ii) $V_t^c(x_f^{FF}-1 \mid Duo) > V_t^c(x_f^{FF}-1 \mid Solo^v)$.

(i) Because $\mathcal{E}_{t+1}^{FF}(x_f^{FF}-1) = Duo$, $V_{t+1}^v(x_f^{FF}-1 \mid Duo) \geq V_{t+1}^v(x_f^{FF}-1 \mid Solo^c)$, or equivalently, using Lemma A3-2, $b - V_{t+2}^v(x_f^{FF}-1) \geq \theta_{sd}^{FF}$. By Lemma A3-8, $V_{t+1}^v(x_f^{FF}-1) < V_{t+2}^v(x_f^{FF}-1)$ since $a_t^c(x_f^{FF}-1) = W$. As a result, $b - V_{t+1}^v(x_f^{FF}-1) > \theta_{sd}^{FF}$, i.e., $V_t^v(x_f^{FF}-1 \mid Duo) > V_t^v(x_f^{FF}-1 \mid Solo^c)$.

(ii) By Lemma A3-9, $\Delta V_t^c(x_f^{FF}-1) > \Delta V_t^v(x_f^{FF}-1) \geq \theta_{sd}^{FF}$. Therefore, $V_t^c(x_f^{FF}-1 \mid Duo) > V_t^c(x_f^{FF}-1 \mid Solo^v)$.□

∎

Lemma A3-11. *Fix state $\hat{x}$ and period $\hat{t}$. Suppose that for all $\hat{x} \leq x < x_f^{FF}$, $\mathcal{E}_t^{FF}(x) = Solo^c$ for all $t \geq \tau(x)$ and $\mathcal{E}_t^{FF}(x) = Duo$ for all $\hat{t} \leq t < \tau(x)$; moreover, suppose that $\tau(x)$ is nondecreasing. Then, $\Delta V_t^v(x) > \Delta V_{t+1}^v(x)$ for all $\hat{t} \leq t \leq \tau(x)$.*

Proof. The proof uses Lemmas A3-2 and A3-4 in Appendix 4.1.3 and Lemma A3-8 in this e-companion and proceeds by induction. To initialize the induction, consider state $x_f^{FF}-1$ and assume that $x_f^{FF}-1 > \hat{x}$. By Lemma A3-2, $V_t^v(x_f^{FF}) = V_{t-1}^v(x_f^{FF}) = b$. Moreover, given that $a_t^c(x_f^{FF}-1) = W$ in every equilibrium, Lemma A3-8 yields $V_{t-1}^v(x_f^{FF}-1) < V_t^v(x_f^{FF}-1)$. Hence, $\Delta V_{t-1}^v(x_f^{FF}-1) > \Delta V_t^v(x_f^{FF}-1)$.

Fix $x < x_f^{FF} - 1$. Consider first period $\tau(x)$. Because $\mathcal{E}_{\tau(x)}^{FF}(x) = Solo^c$, $V_{\tau(x)}^v(x \mid Duo) \leq V_{\tau(x)}^v(x \mid Solo^c)$, i.e., $\Delta V_{\tau(x)+1}^v(x) \leq \theta_{sd}^{FF}$. On the other hand, because $\mathcal{E}_{\tau(x)-1}^{FF}(x) = Duo$, $V_{\tau(x)-1}^v(x \mid Duo) > V_{\tau(x)-1}^v(x \mid Solo^c)$, i.e., $\Delta V_{\tau(x)}^v(x) > \theta_{sd}^{FF}$. Hence, $\Delta V_{\tau(x)}^v(x) > \Delta V_{\tau(x)+1}^v(x)$.

Consider any particular period $t < \tau(x)$ and suppose that $\Delta V_{t+1}^v(x+1) > \Delta V_{t+2}^v(x+1)$. Because $t < \tau(x) \leq \tau(x+1)$, $\mathcal{E}_t^{FF}(x) = \mathcal{E}_t^{FF}(x+1) = Duo$. Moreover, $\mathcal{E}_{t+1}^{FF}(x+1) \neq Solo^c$, for otherwise we would have had $\mathcal{E}_{t+1}^{FF}(x) = Solo^c$ given that $\tau(x) \leq \tau(x+1)$ and therefore $\mathcal{E}_t^{FF}(x) = Solo^c$ by Lemma A3-4, a contradiction. Applying the induction hypothesis, we obtain: $\Delta V_t^v(x) = V_t(x+1 \mid Duo) - V_t(x \mid Duo) = \delta p_d \Delta V_{t+1}^v(x+1) + \delta(1-p_d)\Delta V_{t+1}^v(x) > \delta p_d \Delta V_{t+2}^v(x+1) + \delta(1-p_d)\Delta V_{t+2}^v(x) = V_{t+1}^v(x+1 \mid Duo) - V_{t+1}^v(x \mid Duo)$. Given that $\mathcal{E}_{t+1}^{FF}(x+1) = Duo$ and that either $\mathcal{E}_{t+1}^{FF}(x) = Duo$ or $\mathcal{E}_{t+1}^{FF}(x) = Solo^c$, $V_{t+1}^v(x) \geq V_{t+1}^v(x \mid Duo)$ and $V_{t+1}^v(x+1) = V_{t+1}^v(x+1 \mid Duo)$. Therefore, $V_{t+1}^v(x+1 \mid Duo) - V_{t+1}^v(x \mid Duo) \geq \Delta V_{t+1}^v(x)$, completing the induction step. $\square$ ▮

Lemma A3-12. *Suppose that $x_f^{FF} \geq x_\theta^{FF}$ and $R(x_f^{FF}) \geq 2b$. For all $x < x_f^{FF}$, there exists a time period $\tau(x)$, increasing in x, such that $\mathcal{E}_t^{FF}(x) = Solo^c$ for $t \geq \tau(x)$ and $\mathcal{E}_t^{FF}(x) = Duo$ for $t < \tau(x)$.*

Proof. The proof uses Lemmas A3-1, A3-3, A3-4, A3-6, A3-9, A3-10, and A3-11 in this appendix and proceeds by induction on x. When $x = x_f^{FF} - 1$, the result holds by Lemma A3-10.

Consider now any state $x < x_f^{FF} - 1$ and suppose that the result holds for state $x+1$. Because $\mathcal{E}_{T-1}^{FF}(x) = Solo^c$ by Lemma A3-3 and because $\mathcal{E}_t^{FF}(x+1) = Solo^c$ for all $t \geq \tau(x+1)$, we obtain that $\mathcal{E}_t^{FF}(x) = Solo^c$ for all $t \geq \tau(x+1) - 1$ by Lemma A3-4.

Let $\tau(x) := \max\{t | \Delta V_t^v(x) \geq \theta_{sd}^{FF}\}$. Because $\mathcal{E}_t^{FF}(x) = Solo^c$ for all $t \geq \tau(x+$

1) − 1, we have $\tau(x) \geq \tau(x+1) - 1$. Consider period $\tau(x) - 1$. By definition, $\Delta V^v_\tau(x) \geq \theta^{FF}_{sd}$. Moreover, $\Delta V^c_\tau(x) \geq \theta^{FF}_{sd}$ because $\Delta V^c_\tau(x) \geq \Delta V^v_\tau(x)$ by Lemma A3-9. Hence, $V^v_{\tau-1}(x \mid Solo^c) \leq V^v_{\tau-1}(x \mid Duo)$ and $V^c_{\tau-1}(x \mid Solo^v) \leq V^c_{\tau-1}(x \mid Duo)$. Furthermore, $V^c_{\tau-1}(x \mid Solo^c) > V^c_{\tau-1}(x \mid Finish)$ by Lemma A3-6. As a result, $\mathcal{E}^{FF}_{\tau-1}(x) = Duo$.

Consider next any period $t < \tau(x) - 1$ and suppose that $\mathcal{E}^{FF}_{t+1}(x) = Duo$, and therefore, given that $t < \tau(x+1)$, that $\mathcal{E}^{FF}_{t+1}(x+1) = Duo$. By Lemma A3-1, $\mathcal{E}^{FF}_t(x) \neq Solo^v$ and by Lemma A3-6, $\mathcal{E}^{FF}_t(x) \neq Finish$. Accordingly, if $\mathcal{E}^{FF}_t(x) = Duo$, we must have that (i) $V^v_t(x \mid Duo) > V^v_t(x \mid Solo^c)$, and (ii) $V^c_t(x \mid Duo) > V^c_t(x \mid Solo^v)$.

(i) Because $\mathcal{E}^{FF}_{t+1}(x) = Duo$, $V^v_{t+1}(x \mid Duo) \geq V^v_{t+1}(x \mid Solo^c)$, which is equivalent to $\Delta V^v_{t+2}(x) \geq \theta^{FF}_{sd}$. By Lemma A3-11, $\Delta V^v_{t+1}(x) > \Delta V^v_{t+2}(x)$. Hence, $\Delta V^v_{t+1}(x) > \theta^{FF}_{sd}$.

(ii) Using Lemma A3-9, $\Delta V^c_{t+1}(x) \geq \Delta V^v_{t+1}(x) > \theta^{FF}_{sd}$. This shows that $V^c_t(x \mid Duo) > V^c_t(x \mid Solo^v)$.□

∎

Proof of Proposition 1.3. (i) By Lemma A3-2 in this appendix, $\mathcal{E}^{FF}_t(x) = Finish \ \forall x \geq x^{FF}_f$ and for any $t \in \mathcal{T}$. By Lemmas A3-7 and A3-12, $\mathcal{E}^{FF}_t(x) \neq Finish \ \forall x < x^{FF}_f$ and for any $t \in \mathcal{T}$.

(ii) This is shown in Lemma A3-7 in this appendix.

(iii) This is shown in Lemma A3-12 in this appendix.□

4.1.4 TM Contract

Preliminaries

We first define the following thresholds:

$$\theta_{fs^c}^{TM}(x) := \frac{c_s^c + f + (1-\delta)R(x)}{\delta p_s}, \theta_{sd}^{TM} := \frac{c_d}{\delta(p_d - p_s)}. \tag{A4-1}$$

The thresholds $\theta_{kl}^{TM}(x)$ allow for a pairwise comparison of policies k and l, for any $k \in \{s, f\}$, $l \in \{s^c, d\}$, $k \neq l$. Specifically in period $T-1$, the client prefers $Finish$ over $Solo^c$ if and only if $\Delta R(x) \leq \theta_{fs^c}^{TM}$; she prefers $Solo^v$ over Duo if and only if $\Delta R(x) \leq \theta_{sd}^{TM}$.

We also define

$$x_f^{TM} := \min\left\{x \in \mathbb{Z}^+ \mid \Delta R(x) \leq \frac{f + (1-\delta)R(x)}{\delta p_s}\right\}, \tag{A4-2}$$

$$x_{s^c}^{TM} := \max\left\{x \in \mathbb{Z}^+ \mid \Delta R(x) \geq \theta_{fs^c}^{TM}(x)\right\}, \tag{A4-3}$$

$$x_{s^v}^{TM} := \min\left\{x \in \mathbb{Z}^+ \mid \Delta R(x) \leq \theta_{sd}^{TM}\right\}. \tag{A4-4}$$

Proofs

Lemma A4-1. *If $f = c_s^v$, $\mathcal{E}_t^{TM}(x) = Finish$ for all $x \geq x_f^{TM}$ and $t < T$.*

Proof. The proof proceeds by induction on t. Because $V_{T-1}^v(x \mid Duo) = f - c_d < f = V_{T-1}^v(x \mid Solo^c)$, $\mathcal{E}_{T-1}^{TM}(x) \neq Duo$. In addition, $V_{T-1}^v(x \mid Solo^v) = f - c_s^v = 0 = V_{T-1}^v(x \mid Finish)$. Also, given that $x \geq x_f^{TM}$, $\Delta R(x) \leq \frac{f+(1-\delta)R(x)}{\delta p_s} < \theta_{fs^c}^{TM}(x)$ and therefore $\mathcal{E}_{T-1}^{TM}(x) \neq Solo^c$. Hence, $Finish$ is always an equilibrium and $Solo^v$ is also potentially an equilibrium. In case of multiple equilibria, $V_{T-1}^c(x \mid Solo^v) \leq V_{T-1}^c(x \mid Finish)$ because $\Delta R(x) \leq \frac{f+(1-\delta)R(x)}{\delta p_s}$; therefore $\mathcal{E}_{T-1}^{TM}(x) = Finish$ for all $x \geq x_f^{TM}$.

Fix $t < T-1$ and suppose that $\mathcal{E}_{t+1}^{TM}(x) = Finish$ for all $x \geq x_f^{TM}$. Then, for any $i \in \{c, v\}$ and $x \geq x_f^{TM}$, $V_t^i(x \mid \mathcal{E}) = V_{T-1}^i(x \mid \mathcal{E})$ for $\mathcal{E} \in \{Duo, Solo^c, Solo^v, Finish\}$. As a result, $\mathcal{E}_t^{TM}(x) = \mathcal{E}_{T-1}^{TM}(x) = Finish$.□ ▮

Lemma A4-2. *Suppose $f \geq c_d$. If $f = c_s^v$, then $\mathcal{E}_t^{TM}(x) \neq Finish$ for all t and all $x < x_f^{FF}$.*

Proof. The proof proceeds by induction on t. In period $T-1$, $V_{T-1}^v(x \mid Solo^v) = f - c_s^v = 0 = V_{T-1}^v(x \mid Finish)$ and $V_{T-1}^c(x \mid Solo^v) > V_{T-1}^c(x \mid Finish)$ for all $x < x_f^{TM}$ by (A4-2). Therefore, $\mathcal{E}_{T-1}^{TM}(x) \neq Finish$ for all $x < x_f^{FF}$. Also, $V_{T-1}^v(x) \geq 0$ because $V_{T-1}^v(x \mid Solo^v) = f - c_s^v = 0$, $V_{T-1}^v(x \mid Solo^c) = f > 0$, and $V_{T-1}^v(x \mid Duo) = f - c_d \geq 0$.

Fix $t < T$ and suppose $V_\tau^c(x \mid Solo^v) > V_\tau^c(x \mid Finish)$ for all $x < x_f^{TM}$ and $\tau > t$, and $V_\tau^v(x) \geq 0$ for all x and $\tau > t$. We then have that $V_t^c(x \mid Solo^v) = -f + \delta\mathbb{E}_s[V_{t+1}^c(x+\xi)] \geq -f + \delta\mathbb{E}_s[V_{T-1}^c(x+\xi)] > V_{T-1}^c(x \mid Finish) = V_t^c(x \mid Finish)$. Moreover, $V_t^v(x) \geq 0$ because $V_t^v(x \mid Solo^v) = f - c_s^v + \delta\mathbb{E}_s[V_{t+1}^v(x+\xi)] \geq 0$, $V_t^v(x \mid Solo^c) = f + \delta\mathbb{E}_s[V_{t+1}^v(x+\xi)] > 0$, and $V_t^v(x \mid Duo) = f - c_d + \delta\mathbb{E}_d[V_{t+1}^v(x+\xi)] \geq 0$. As a result, the vendor either prefers $Solo^v$ over $Finish$, or if she is indifferent between the two, the client prefers $Solo^v$ over $Finish$, i.e., $\mathcal{E}_t^{TM}(x) \neq Finish$.□ ▮

Lemma A4-3. *Fix state $\hat{x}$. Suppose that $f = c_s^v$ and for all $\hat{x} \leq x < x_f^{TM}$, $\mathcal{E}_t^{TM}(x) = Solo^v$. Then for all $x \geq \hat{x}$, $V_t^c(x)$ is decreasing in t.*

Proof. By Lemma A4-1 in this appendix, $V_t^c(x) = V_{t+1}^c(x) = R(x)$ when $x \geq x_f^{FF}$. Suppose next that $x < x_f^{FF}$. In period $T-1$ and state $x < x_f^{FF}$, $V_{T-1}^c(x) > R(x) = V_T^c(x)$ by (A4-2). Fix t and suppose that $V_{t+1}^c(x) > V_{t+2}^c(x)$. Then, $V_t^c(x) = -f + \delta\mathbb{E}_s[V_{t+1}^c(x+\xi)] > -f + \delta\mathbb{E}_s[V_{t+2}^c(x+\xi)] = V_{t+1}^c(x)$, completing the induction step.□ ▮

Lemma A4-4. *Suppose* $x_{s^v}^{TM} < x_f^{TM}$ *and* $f = c_s^v$*. Then,* $\mathcal{E}_t^{TM}(x_f^{TM} - 1) = Solo^v$ *for all* $t < T$*.*

Proof. The proof uses Lemmas A4-1 and A4-3 in this appendix, and proceeds by induction on t. Because $V_{T-1}^v(x_f^{TM} - 1 \mid Duo) = f - c_d < f = V_{T-1}^v(x_f^{TM} - 1 \mid Solo^c)$, $\mathcal{E}_{T-1}^{TM}(x_f^{TM} - 1) \neq Duo$. As in the proof of Lemma A4-2, $V_{T-1}^v(x_f^{TM} - 1 \mid Solo^v) = V_{T-1}^v(x_f^{TM} - 1 \mid Finish)$. Because $x_{s^v}^{TM} < x_f^{TM}$, $\frac{f+(1-\delta)R(x_f^{TM}-1)}{\delta p_s} < \Delta R(x_f^{TM} - 1) \leq \theta_{sd}^{TM}$, i.e., $V_{T-1}^c(x_f^{TM} - 1 \mid Solo^v) > V_{T-1}^c(x_f^{TM} - 1 \mid Finish)$ and $V_{T-1}^c(x_f^{TM} - 1 \mid Solo^v) > V_{T-1}^c(x_f^{TM} - 1 \mid Duo)$. As a result, $\mathcal{E}_{T-1}^{TM}(x_f^{TM} - 1) \neq Finish$ and $Solo^v$ is an equilibrium in period $T-1$ and state $x_f^{TM} - 1$. Because $V_{T-1}^c(x_f^{TM} - 1 \mid Solo^v) > V_{T-1}^c(x_f^{TM} - 1 \mid Solo^c)$, $\mathcal{E}_{T-1}^{TM}(x_f^{TM} - 1) = Solo^v$ by our equilibrium selection rule.

Fix $t < T$ and suppose $\mathcal{E}_\tau^{TM}(x_f^{TM} - 1) = Solo^v$ for all $\tau > t$. Applying the induction hypothesis shows that $V_t^v(x_f^{TM} - 1 \mid Duo) < V_t^v(x_f^{TM} - 1 \mid Solo^c)$ and $V_t^v(x_f^{TM} - 1 \mid Solo^v) = V_t^v(x_f^{TM} - 1 \mid Finish)$. Also, $V_t^c(x_f^{TM} - 1 \mid Solo^v) > V_t^c(x_f^{TM} - 1 \mid Duo)$ using a similar argument to the proof of Lemma A1-5, and $V_t^c(x_f^{TM} - 1 \mid Solo^v) > V_t^c(x_f^{TM} - 1 \mid Finish)$ given that $V_{t+1}^c(x)$ is decreasing in t by Lemma A4-3. This completes the induction step.□ ∎

Lemma A4-5. *If* $f = c_s^v$, $\mathcal{E}_t^{TM}(x) = Solo^v$ *for all* $x_{s^v}^{TM} \leq x < x_f^{TM}$ *and* $t < T$*.*

Proof. The proof uses Lemma A4-4 in this appendix and proceeds by induction on t. By Lemma A4-4, $\mathcal{E}_t^{TM}(x_f^{TM} - 1) = Solo^v$ for all $t < T$. Using a similar argument to the proof of Lemma A4-4, we obtain that $\mathcal{E}_{T-1}^{TM}(x) = Solo^v$ for all $x_{s^v}^{TM} \leq x < x_f^{TM}$.

Fix $t < T$ and suppose that $\mathcal{E}_\tau^{TM}(x) = \mathcal{E}_\tau^{TM}(x+1) = Solo^v$ for all $\tau > t$. We then have $V_{t+1}^v(x+1) = V_{t+1}^v(x) = 0$. Similar to period $T-1$, $V_t^v(x \mid Solo^v) = V_t^v(x \mid Finish)$ and $V_t^v(x \mid Duo) < V_t^v(x \mid Solo^c)$. We thus need to show that $V_t^c(x \mid Solo^v) > V_t^c(x \mid Duo)$, i.e., $\Delta V_{t+1}^c(x+1) < \theta_{sd}^{TM}$, and $V_t^c(x \mid Solo^v) > V_t^c(x \mid Finish)$. By the induction hypothesis, $\Delta V^c_{t+2}(x) < \theta^{TM}_{sd}$ and $\Delta V^c_{t+2}(x+1) < \theta^{TM}_{sd}$. Thus with the same argument as in Lemma A1-4, we find that $\Delta V^c_{t+1}(x+1) < \theta^{TM}_{sd}$. Moreover, we have $V^c_{t+1}(x \mid Solo^v) > V^c_{t+1}(x \mid Finish)$ and $V^c_{t+1}(x+1 \mid Solo^v) > V^c_{t+1}(x+1 \mid Finish)$, which imply $V^c_t(x \mid Solo^v) = -f + \delta\mathbb{E}_s[V^c_t(x+\xi)] > -f + \delta\mathbb{E}_s[R(x+\xi)] = V^c_{T-1}(x \mid Solo^v) > V^c_{T-1}(x \mid Finish) = V^c_t(x \mid Finish)$. This completes the induction step.□ ▮

Lemma A4-6. *Suppose $f \geq c_d$. If $f > c^v_s$, then $V^v_t(x \mid Solo^v) > V^v_t(x \mid Finish) = 0$ for all t and all x.*

Proof. Similar to the proof of Lemma A4-2 in this appendix with strict inequalities. □ ▮

Lemma A4-7. *Suppose that $f > c^v_s$. Then, $\mathcal{E}^{TM}_t(x) = Solo^v$ for all t and all $x \geq x^{TM}_{s^v}$.*

Proof. The proof uses Lemma A4-6 in this appendix is similar to the proof of Lemma A4-5 with strict inequalities.□ ▮

Proof of Proposition 1.4. (i.a) By Lemma A4-1 in this appendix, $\mathcal{E}^{TM}_t(x) = Finish$ for all $x \geq x^{TM}_f$, and by Lemma A4-2 in the e-companion, $\mathcal{E}^{TM}_t(x) \neq Finish$ for all $x < x^{TM}_f$.

(i.b) This is shown in Lemma A4-5 in this appendix.

(ii.a) This is shown in Lemma A4-7 in this appendix.

(ii.b) This is shown in Lemma A4-6 in this appendix.

(iii) The proof uses Lemmas A4-2 and A4-6 in the e-companion and proceeds by induction. In period $T-1$, $V^c_{T-1}(x \mid Duo) > V^c_{T-1}(x \mid Solo^v)$, i.e., $\Delta R(x) > \theta^{TM}_{sd}$ for all $x < \min\{x^{TM}_{s^v}, x^{TM}_{s^c}\}$ by (A4-4) and therefore $\mathcal{E}^{TM}_{T-1}(x) \neq Solo^v$. Also by Lemmas A4-2 and A4-6, $\mathcal{E}^{TM}_{T-1}(x) \neq Finish$ for all $x < \min\{x^{TM}_{s^v}, x^{TM}_{s^c}\}$. In addition,

$V^c_{T-1}(x \mid Solo^c) > V^c_{T-1}(x \mid Finish)$ for all $x < \min\{x^{TM}_{s^v}, x^{TM}_{s^c}\}$ by (A4-3). Finally, $V^v_{T-1}(x \mid Solo^c) = f > f - c_d = V^v_{T-1}(x \mid Duo)$. As a result, $\mathcal{E}^{TM}_{T-1}(x) = Solo^c$ for all $x < \min\{x^{TM}_{s^v}, x^{TM}_{s^c}\}$. Moreover, $V^c_{T-1}(x)$ is concave for all $x < x^{TM}_{s^v}$, because $R(x)$ is concave.

Fix $t < T$ and suppose $\mathcal{E}^{TM}_{\tau}(x) = Solo^c$ and $V^c_{\tau}(x)$ is concave for all $x \leq \overline{x}^{TM}_{s^c,\tau}$ and $\tau > t$. Fix $x < x^{TM}_{s^c}$. Using the concavity of $V^c_{t+1}(x)$ and the fact that $\mathcal{E}^{TM}_{t}(\overline{x}^{TM}_{s^c,t}) = Solo^c$, $\Delta V^c_{t+1}(x) \geq \Delta V^c_{t+1}(\overline{x}^{TM}_{s^c,t}) \geq \theta^{FF}_{sd}$. This implies that $V^c_t(x \mid Duo) > V^c_t(x \mid Solo^v)$. In addition, $V^v_t(x \mid Solo^c) = f + \delta \mathbb{E}_s[V^c_{t+1}(x+\xi)] = f + \sum_{s=1}^{t-1} \delta^{s-t} f > f - c_d + \sum_{s=1}^{t-1} \delta^{s-t} f = V^v_t(x \mid Duo)$. Also by Lemmas A4-2 and A4-6, $\mathcal{E}^{TM}_t(x) \neq Finish$ for all $x < \min\{x^{TM}_{s^v}, x^{TM}_{s^c}\}$. Finally, by the induction hypothesis, $V^c_t(x \mid Solo^c) = f - c^c_s + \delta \mathbb{E}_s[V^c_{t+1}(x+\xi)] \geq f - c^c_s + \delta \mathbb{E}_s[R(x+\xi)] = V^c_{T-1}(x \mid Solo^c) > V^c_{T-1}(x \mid Finish) = V^c_t(x \mid Finish)$. Therefore, $\mathcal{E}^{TM}_t(x) = Solo^c$. In addition, $V^c_t(x)$ is concave in x for all $x \leq \overline{x}^{TM}_{s^c,t}$ because $V^c_{t+1}(x)$ is concave for all $x \leq \overline{x}^{TM}_{s^c,t+1}$. This completes the induction step.□

4.2 Appendix B : Proofs for Chapter 2

Lemma B-1. *Conditions (2.6) and (2.7) hold true for* $\mathcal{S}_t = \{1, 2, \ldots, k_t^{NL}\}$ *for all* $t \in \mathcal{T}$ *and* $V_{t,i}^{NL}$ *are increasing in* $i \in \mathcal{N}$.

Proof. When $t = 1$, $V_{t-1,i}^{NL} = 0$ for all $i \in \mathcal{N}$ and therefore conditions (2.6) and (2.7) hold true for any $\mathcal{S}_1$ with $|\mathcal{S}_1| = k_1^{NL}$. Without loss of generality, we consider $\mathcal{S}_1 = \{1, 2, \ldots, k_1^{NL}\}$. Accordingly, $V_{1,i}^{NL} = -c(k_1^{NL}) + \delta p(k_1^{NL})R/n - \delta g(k_1^{NL})/n$ for $i \leq k_1^{NL}$ and $V_{1,i}^{NL} = \delta p(k_1^{NL})R/n - \delta g(k_1^{NL})/n$ for $i > k_1^{NL}$ by (2.3). Therefore,

$$V_{1,1}^{NL} = V_{1,2}^{NL} = \ldots = V_{1,k_1^{NL}}^{NL} < V_{1,k_1^{NL}+1}^{NL} = V_{1,k_1^{NL}+2}^{NL} = \ldots = V_{1,n}^{NL}.$$

Hence, $V_{1,i}^{NL}$ is increasing in i. Suppose $\mathcal{S}_{t-1} = \{1, 2, \ldots, k_{t-1}^{NL}\}$ and $V_{t-1,i}^{NL}$ is increasing in i. According to (2.6) and (2.7), the members with smaller payoff in period $t-1$ will join the team in period t. We consider two cases: (a) $k_t^{NL} \leq k_{t-1}^{NL}$ or (b) $k_t^{NL} > k_{t-1}^{NL}$.

(a) $\mathcal{S}_t^{NL} = \{1, 2, \ldots, k_t^{NL}\} \subseteq \mathcal{S}_{t-1}^{NL}$.

(b) Suppose $\mathcal{S}_{t-1}^{NL} \subseteq \mathcal{S}_t^{NL}$. Since $V_{t-1,i}^{NL}$ is increasing in i, we should have $\mathcal{S}_t - \mathcal{S}_{t-1} = \{k_{t-1}^{NL} + 1, \ldots, k_t^{NL}\}$; that is, $\mathcal{S}_t = \{1, 2, \ldots, k_t^{NL}\}$.

Hence, $V_{t,i}^{NL} = -c(k_t^{NL}) + \delta p(k_t^{NL})R/n - \delta g(k_t^{NL})/n + \delta(1 - p(k_t^{NL}))V_{t-1,i}^{NL}$ for $i \leq k_t^{NL}$ and $V_{t,i}^{NL} = \delta p(k_t^{NL})R/n - \delta g(k_t^{NL})/n + \delta(1 - p(k_t^{NL}))V_{t-1,i}^{NL}$ for $i > k_t^{NL}$. Given that $V_{t-1,i}^{NL}$ is increasing in i, we obtain

$$V_{t,1}^{NL} \leq V_{t,2}^{NL} \leq \ldots \leq V_{t,k_t^{NL}}^{NL} < V_{t,k_t^{NL}+1}^{NL} \leq V_{t,k_t^{NL}+2}^{NL} \leq \ldots \leq V_{t,n}^{NL}.$$

This completes the induction step.□ ∎

Lemma B-2. *Suppose* $\mathcal{S}_t = \{1, 2, \ldots, k_t^{NL}\}$ *is an equilibrium that satisfies (2.4) and*

(2.5). Then, k_t^{NL} is obtained by (2.8).

Proof. The proof uses Lemma B-1. When $\mathcal{S}_t = \{1, 2, \ldots, k_t^{NL}\}$, the equilibrium conditions (2.4) and (2.5) can be rewritten as

$$\delta\Delta p(k_t^{NL}-1)\left[\frac{R}{n} - V_{t-1,i}^{NL}\right] \geq c(k_t^{NL}) + \delta\Delta g(k_t^{NL}-1)\frac{1}{n} \quad \text{for } i \leq k_t^{NL} \tag{B-1}$$

$$\delta\Delta p(k_t^{NL})\left[\frac{R}{n} - V_{t-1,i}^{NL}\right] \leq c(k_t^{NL}+1) + \delta\Delta g(k_t^{NL})\frac{1}{n} \quad \text{for } i > k_t^{NL}. \tag{B-2}$$

Taking $i = k_t^{NL}$, we obtain

$$\delta\Delta p(k_t^{NL}-1)\left[\frac{R}{n} - V_{t-1,k_t^{NL}}^{NL}\right] \geq c(k_t^{NL}) + \delta\Delta g(k_t^{NL}-1)\tfrac{1}{n}$$

$$\delta\Delta p(k_t^{NL})\left[\frac{R}{n} - V_{t-1,k_t^{NL}+1}^{NL}\right] \leq c(k_t^{NL}+1) + \delta\Delta g(k_t^{NL})\tfrac{1}{n}.$$

By Lemma B-1, $V_{t-1,i}^{NL}$ is increasing in i. In addition, $p(i)$ is concave, $g(i)$ is convex, and $c(i)$ is increasing. Therefore, if $\delta\Delta p(i-1)\left[\frac{R}{n} - V_{t+1,i}^{NL}\right] \geq c(i) + \delta\Delta g(i-1)\frac{1}{n}$ for $i = k_t^{NL}$, it should also be true for all $i \leq k_t^{NL}$. On the other hand, if $\delta\Delta p(i-1)\left[\frac{R}{n} - V_{t+1,i}^{NL}\right] \leq c(i) + \delta\Delta g(i-1)\frac{1}{n}$ for $i = k_t^{NL}+1$, it should also be true for all $i > k_t^{NL}$. Therefore, equilibrium conditions (B-1) and (B-2) imply

$$\delta\Delta p(i-1)\left[\frac{R}{n} - V_{t+1,i}^{NL}\right] \geq c(i) + \delta\Delta g(i-1)\frac{1}{n} \quad \text{for } i \leq k_t^{NL} \tag{B-3}$$

$$\delta\Delta p(i-1)\left[\frac{R}{n} - V_{t+1,i}^{NL}\right] \leq c(i) + \delta\Delta g(i-1)\frac{1}{n} \quad \text{for } i > k_t^{NL}. \tag{B-4}$$

The left-hand sides of (B-3) and (B-4) are decreasing in i and the right-hand sides are increasing in i. As a result, k_t^{NL} is unique in any period t.

Rewriting (B-3) and (B-4), we obtain

$$V_{t+1,i}^{NL} \leq \frac{R}{n} - \frac{nc(i) + \delta\Delta g(i-1)}{n\delta\Delta p(i-1)} \quad \text{for } i \leq k_t^{NL} \tag{B-5}$$

$$V_{t+1,i}^{NL} \geq \frac{R}{n} - \frac{nc(i) + \delta\Delta g(i-1)}{n\delta\Delta p(i-1)} \quad \text{for } i > k_t^{NL}. \tag{B-6}$$

Thus using Lemma B-1, we can obtain k_t^{NL} by using (2.8).□ ∎

Lemma B-3. *Suppose that k_t^{NL} is obtained by (2.8). Then $\mathcal{S}_t = \{1, 2, \ldots, k_t^{NL}\}$ is an equilibrium that satisfies (2.4) and (2.5).*

Proof. The proof uses Lemmas B-1 and proceeds by induction. Because $V_{0,i}^{NL} = 0$ for all i, (2.8) yields, $k_1^{NL} = \min\left\{i \mid \frac{R}{n} - \frac{nc(i+1)+\delta\Delta g(i)}{n\delta\Delta p(i)} \leq 0\right\}$. This implies

$$-c(k_1^{NL}) + \delta p(k_1^{NL})\frac{R}{n} - \frac{g(k_1^{NL})}{n} \geq \delta p(k_1^{NL}-1)\frac{R}{n} - \frac{g(k_1^{NL}-1)}{n} \quad \text{for } i \leq k_1^{NL}$$

$$-c(k_1^{NL}+1) + \delta p(k_1^{NL}+1)\frac{R}{n} - \frac{g(k_1^{NL}+1)}{n} \leq \delta p(k_1^{NL})\frac{R}{n} - \frac{g(k_1^{NL})}{n} \quad \text{for } i > k_1^{NL}.$$

consistent with (2.4) and (2.5). We also have that $V_{1,i}^{NL}$ is increasing in i by Lemma B-1.

Next, suppose k_t^{NL} obtained by (2.8) is an equilibrium team size and $V_{t,i}^{NL}$ is increasing in i for all $t < t'$. Then $k_{t'}^{NL} = \min\left\{i \mid V_{t'-1,i+1}^{NL} \geq \frac{R}{n} - \frac{nc(i+1)+\delta\Delta g(i)}{n\delta\Delta p(i)}\right\}$ implies $V_{t'-1,k_{t'}^{NL}+1}^{NL} \geq \frac{R}{n} - \frac{nc(k_{t'}^{NL}+1)+\delta\Delta g(k_{t'}^{NL})}{n\delta\Delta p(k_{t'}^{NL})}$ and $V_{t'-1,i}^{NL} \leq \frac{R}{n} - \frac{nc(k_{t'}^{NL})+\delta\Delta g(k_{t'}^{NL}-1)}{n\delta\Delta p(k_{t'}^{NL}-1)}$ for all $i \leq k_{t'}^{NL}$. Given that $V_{t'-1,i}^{NL}$ is increasing in i, we have

$$V_{t'-1,i}^{NL} \geq \frac{R}{n} - \frac{nc(k_{t'}^{NL}+1) + \delta\Delta g(k_{t'}^{NL})}{n\delta\Delta p(k_{t'}^{NL})} \quad \text{for } i > k_{t'}^{NL}$$

$$V_{t'-1,i}^{NL} \leq \frac{R}{n} - \frac{nc(k_{t'}^{NL}) + \delta\Delta g(k_{t'}^{NL}-1)}{n\delta\Delta p(k_{t'}^{NL}-1)} \quad \text{for } i \leq k_{t'}^{NL}.$$

which are consistent with (2.4) and (2.5). This completes the induction step.□ ▮

Proof of Lemma 2.1. (i) By (2.12) and (2.13), $k_1^{NL} = k_1^{IL} = n$ if and only if $R \geq \frac{nc(n)+\delta\Delta g(n-1)}{\delta\Delta p(n-1)}$. We therefore define $\overline{R} = \min\{R \mid R \geq \frac{nc(n)+\delta\Delta g(n-1)}{\delta\Delta p(n-1)}\}$, which is unique given that $c(k)$ and $g(k)$ are convex, and $p(k)$ is concave in k.

We next show that for $R > \overline{R}$, $k_1^{FB} = n$. By (2.11), $k_1^{FB} = n$ if and only if $R \geq \frac{\Delta(n-1)c(n-1)+\delta\Delta g(n-1)}{\delta\Delta p(n-1)}$. We therefore define $\underline{R} = \min\{R \mid R \geq \frac{\Delta(n-1)c(n-1)+\delta\Delta g(n-1)}{\delta\Delta p(n-1)}\}$. Given that $(n)c(n) > \Delta(n-1)c(n-1)$, $\overline{R} \geq \underline{R}$. As a result, $k_1^{FB} = n$ for $R > \overline{R}$.

(ii) For $\underline{R} < R \leq \overline{R}$, $k_1^{FB} = n$. However, $k_1^{NL} < n$ and $k_1^{IL} < n$. In addition, we have that $R \geq \frac{\Delta(n-1)c(n-1)+\delta\Delta g(n-1)}{\delta\Delta p(n-1)} \geq \frac{\Delta g(n-1)}{\Delta p(n-1)}$. Therefore by (2.13), $k_1^{IL} = n-1 \geq k_1^{NL}$.

(iii) For $R \leq \underline{R}$, $k_1^{FB} < n$, $k_1^{NL} < n$, and $k_1^{IL} < n$. In addition, since $nc(i+1) \geq \Delta ic(i) \geq 0$, we have $k_1^{IL} \geq k_1^{FB} \geq k_1^{NL}$ by (2.11), (2.12), and (2.13).□

Proof of Proposition 2.1. (i) When $R > \overline{R}$, $k_1^{FB} = k_1^{IL} = k_1^{NL}$ by Lemma 2.1. Therefore $\sum_i V_{1,i}^{IL} = \sum_i V_{1,i}^{NL} = V_1^{FB}$.

(ii) When $\underline{R} < R \leq \overline{R}$, $k_1^{FB} > k_1^{IL} \geq k_1^{NL}$ by Lemma 2.1. Therefore $\sum_i V_{1,i}^{IL} \geq \sum_i V_{1,i}^{NL}$.

(iii) When $R \leq \underline{R}$, $k_1^{IL} \geq k_1^{FB} \geq k_1^{NL}$ by Lemma 2.1. Please see Figure 2.2 (left). By (2.12) and (2.13), k_1^{NL} and k_1^{IL} are both increasing in R. In addition, $V_1^{FB}(k)$ and k_1^{FB} are also increasing in R by (2.1) and (2.11).

We denote by $V_1^{FB}(k, R)$ the FB payoff to-go when team size is k and project reward is R. By (2.1), (2.3), and (2.9), $\sum_{i=1}^{n} V_{1,i}^{NL}(k, R) = \sum_{i=1}^{n} V_{1,i}^{IL}(k, R) = V_1^{FB}(k, R)$ for any team size k and reward R. Suppose that R increases to $R+m \leq \underline{R}$. We have $k_1^{NL}(R+m) \geq k_1^{NL}(R)$ and $k_1^{IL}(R+m) \geq k_1^{IL}(R)$. Since $k_1^{NL} \leq k_1^{FB}$ for $R \leq \underline{R}$, having $k_1^{NL}(R+m) > k_1^{NL}(R)$ implies $V_1^{FB}(k_1^{NL}(R+m), R+m) - V_1^{FB}(k_1^{NL}(R), R+m) \geq 0$. However since $k_1^{IL} \geq k_1^{FB}$ for $R \leq \underline{R}$, having $k_1^{IL}(R+m) > k_1^{IL}(R)$ implies

$V_1^{FB}(k_1^{IL}(R+m), R+m) - V_1^{FB}(k_1^{IL}(R), R+m) \leq 0$. As a result, there exists a threshold $\hat{R}$ above which $\sum_i V_{1,i}^{NL} = V_1^{FB}(k_1^{NL}) \geq V_1^{FB}(k_1^{IL}) = \sum_i V_{1,i}^{IL}$.□

Lemma B-4. *V_t^{FB}, $V_{t,n}^{IL}$ are increasing in t.*

Proof. The proof proceeds by induction. We first show the result for V_t^{FB}. We have $V_1^{FB} \geq V_0^{FB} = 0$. Suppose $V_{t-1}^{FB} \geq V_{t-2}^{FB}$. By (2.1),

$$\begin{aligned} V_t^{FB} &= \max_{k \leq n} -kc(k) - g(k) + \delta p(k)R + \delta(1-p(k))V_{t-1}^{FB} \\ &\geq \max_{k \leq n} -kc(k) - g(k) + \delta p(k)R + \delta(1-p(k))V_{t-2}^{FB} = V_{t-1}^{FB}. \end{aligned}$$

Next, we show the result for $V_{t,n}^{IL}$. We have $V_{1,n}^{IL} \geq V_{0,n}^{IL} = 0$. Suppose $V_{t-1,n}^{IL} \geq V_{t-2,n}^{IL}$. By (2.9),

$$\begin{aligned} V_{t,n}^{IL} &= \max_{k \leq n} -1\{k=n\}c(k) - g(k)/n + \delta p(k)R/n + \delta(1-p(k))V_{t-1,n}^{IL} \\ &\geq \max_{k \leq n} -1\{k=n\}c(k) - g(k)/n - g(k)/n + \delta p(k)R/n + \delta(1-p(k))V_{t-2,n}^{IL} \\ &= V_{t-1,n}^{IL}.\square \end{aligned}$$

∎

Lemma B-5. *$V_{1,i}^{NL}$ is nonnegative for all $i \in \mathcal{N}$.*

Proof. Because $V_{0,i}^{NL} = 0$ for all $i \in N$, $V_{1,i}^{NL} = -c(k_1^{NL}) - g(k_1^{NL})/n + p(k_1^{NL})R/n$ for $i \leq k_1^{NL}$ and $V_{1,i}^{NL} = -g(k_1^{NL})/n + p(k_1^{NL})R/n$ for $i > k_1^{NL}$. By (2.4), k_1^{NL} members join the team if $-c(k_1^{NL}) - g(k_1^{NL})/n + \delta p(k_1^{NL})R/n \geq -g(k_1^{NL}-1)/n + \delta p(k_1^{NL}-1)R/n$. Similar to (B-3), we have that $-c(i) - g(i)/n + \delta p(i)R/n \geq -g(i-1)/n + \delta p(i-1)R/n$

for $i \leq k_1^{NL}$. We therefore obtain the following:

$$-c(k_1^{NL}) - g(k_1^{NL})/n + \delta p(k_1^{NL})R/n \geq -g(k_1^{NL}-1)/n + \delta p(k_1^{NL}-1)R/n$$

$$\geq \quad c(k_1^{NL}-1) - g(k_1^{NL}-2)/n + \delta p(k_1^{NL}-2)R/n$$

$$\geq \quad \ldots \geq \sum_{i=0}^{k_1^{NL}-1} c(i) - g(0)/n + \delta p(0)R/n \geq 0.$$

As a result, $V_{1,i}^{NL} = -c(k_1^{NL}) - g(k_1^{NL})/n + \delta p(k_1^{NL})R/n \geq 0$ for $i \leq k_1^{NL}$ and $V_{1,i}^{NL} = -g(k_1^{NL})/n + \delta p(k_1^{NL})R/n \geq 0$ for $i > k_1^{NL}$.$\square$ ∎

Proof of Lemma 2.2. (i) We show in Lemma B-4 that V_t^{FB} and $V_{t,1}^{SL}$ are increasing in t. Therefore, k_t^{FB} and k_t^{IL} are non increasing in t (i.e., the team sizes increase as the project is getting closer to deadline) by (2.2) and (2.10) respectively.

(ii) The proof uses Lemmas B-3 and B-5. We have $V_{1,i}^{NL} \geq V_{0,i}^{NL} = 0$ by Lemma B-5 and therefore $k_2^{NL} \leq k_1^{NL}$ by (2.8). We also have that $V_{1,1}^{NL} = V_{1,i}^{NL}$ for $i \leq k_1^{NL}$.

Fix $t = t'$. Suppose $V_{t-1,i}^{NL} \geq V_{t-2,i}^{NL}$ for all $i \leq k_{t-1}^{NL}$ and $k_t^{NL} \leq k_{t-1}^{NL}$ for all $t \geq t'$. By definition in (2.3), $V_{t',i}^{NL} = -c(k_{t'}^{NL}) - g(k_{t'}^{NL})/n + p(k_{t'}^{NL})\left[R/n - V_{t'-1,i}^{NL}\right] + V_{t'-1,i}^{NL}$ for $i \leq k_{t'}^{NL}$. Therefore, $V_{t',i}^{NL} \geq V_{t'-1,i}^{NL}$ for $i \leq k_{t'}^{NL}$ if and only if $-c(k_{t'}^{NL}) - g(k_{t'}^{NL})/n + p(k_{t'}^{NL})\left(R/n - V_{t'-1,i}^{NL}\right) \geq 0$.

Rewriting (B-5) from Lemma B-3, we obtain the following for $i \leq k_{t'}^{NL}$

$$-c(i) - \frac{g(i)}{n} + p(i)\left[\frac{R}{n} - V_{t'-1,i}^{NL}\right] \geq -\frac{g(i-1)}{n} + p(i-1)\left[\frac{R}{n} - V_{t'-1,i}^{NL}\right] \qquad \text{(B-7)}$$

Using (B-7) recursively, for $i \leq k_{t'}^{NL}$ we have

$$-c(k_{t'}^{NL}) - \frac{g(k_{t'}^{NL})}{n} + p(k_{t'}^{NL})\left(\frac{R}{n} - V_{t'-1,i}^{NL}\right)$$

$$\geq \quad p(k_{t'}^{NL}-1)(\frac{R}{n}-V_{t'-1,i}^{NL})-\frac{g(k_{t'}^{NL}-1)}{n}$$

$$\geq \quad c(k_{t'}^{NL}-1)-\frac{g(k_{t'}^{NL}-2)}{n}+p(k_{t'}^{NL}-2)(\frac{R}{n}-V_{t'+1,i}^{NL})$$

$$\geq \quad \ldots \geq \sum_{i=0}^{k_{t'}^{NL}-1} c(i)+p(0)(\frac{R}{n}-V_{t'+1,i}^{NL})-g(0)\geq 0,$$

and therefore, $V_{t',i}^{NL} \geq V_{t'-1,i}^{NL}$, which implies $k_{t'+1}^{NL} \leq k_{t'}^{NL}$ by (2.8).□

Proof of Lemma 2.3. (i) By (2.8) and (2.10), $k_t^{NL} = k_t^{IL} = n$ if and only if $V_{t-1,n}^{S} \leq \frac{R}{n} - \frac{cn+g}{np}$ for $S \in \{NL, IL\}$. By Lemma 2.2, k_t^{NL} and k_t^{IL} are non-increasing in t. Therefore, if $k_t^S = n$, then $k_{t'}^S = n$ for all $t' < t$, $S \in \{NL, IL\}$. In addition, $V_{t,i}^{NL} = V_{t,n}^{IL} = (1-(1-pn)^{T-t-1})\frac{p-c-g}{np}$, $i \in \mathcal{N}$. We therefore define, $\underline{t}$ as follows:

$$\underline{t} = \max\{t \in Z^+ \mid (1-pn)^{T-t-1} \geq 1 - \frac{pR-cn-g}{p-c-g}\},$$

such that $k_t^{NL} = k_t^{IL} = n$ if and only if $t \leq \underline{t}$.

By (2.2), $k_t^{FB} = n$ if and only if $V_{t+1}^{FB} \leq R - \frac{c+g}{p}$. By Lemma 2.2, k_t^{FB} is non-increasing in t. Therefore, if $k_t^{FB} = n$, then $k_{t'}^{FB} = n$ for all $t' < t$. In addition, $V_t^{FB} = (1-(1-pn)^{T-t-1})\frac{p-c-g}{p}$. We therefore define, $\bar{t}$ as follows:

$$\bar{t} = \min\{t \in Z^+ \mid (1-pn)^{T-t-1} \geq 1 - \frac{pR-c-g}{p-c-g}\},$$

such that $k_t^{FB} = n$ if $t \leq \bar{t}$ and $k_t^{FB} = 0$ if $t > \bar{t}$. Because $\underline{t} \leq \bar{t}$, $k_t^{FB} = k_t^{NL} = k_t^{IL} = n$ for all $t \leq \underline{t}$.

(ii) For $\underline{t} < t < \bar{t}$, $k_t^{NL} = 0$ and $k_t^{FB} = n$. However by (2.10), $k_t^{IL} = n-1$ if $\frac{R}{n} - \frac{cn+g}{np} < V_{t-1,n}^{IL} \leq \frac{R}{n} - \frac{g}{np}$ and $k_t^{IL} = 0$ if $V_{t-1,n}^{IL} > \frac{R}{n} - \frac{g}{np}$. As a result, $k_t^{FB} > k_t^{IL} \geq k_t^{NL}$ for $\underline{t} < t < \bar{t}$.

(iii) For $t \geq \bar{t}$, $k_t^{FB} = k_t^{NL} = 0$. However by (2.10), $k_t^{IL} = n-1$ if $\frac{R}{n} - \frac{cn+g}{np} < V_{t-1,n}^{IL} \leq \frac{R}{n} - \frac{g}{np}$ and $k_t^{IL} = 0$ if $V_{t-1,n}^{IL} > \frac{R}{n} - \frac{g}{np}$. As a result, $k_t^{IL} \geq k_t^{FB} = k_t^{NL}$ for $t \geq \bar{t}$. □

Proof of Proposition 2.2.

By Proposition 2.3, $k_t^{NL} = k_t^{IL} = n$ for all $t \leq \underline{t}$ and therefore $\sum_{i=1}^n V_{t,i}^{IL} = \sum_{i=1}^n V_{t,i}^{NL}$ for all $t \leq \underline{t}$ by (2.3) and (2.9). In addition, $k_t^{NL} = 0$ for $t > \underline{t}$; therefore $\sum_{i=1}^n V_{t,i}^{NL} = \sum_{i=1}^n V_{\underline{t},i}^{NL}$ for $t > \underline{t}$ by (2.3). We fix $t = \underline{t}+1$ and consider two cases: (a) $k_{\underline{t}+1}^{IL} = 0$ and (b) $k_{\underline{t}+1}^{IL} = n-1$.

(a) By Proposition 2.2, $k_{\underline{t}+1}^{IL} = 0$ implies $k_t^{IL} = 0$ for $t > \underline{t}$; therefore $\sum_{i=1}^n V_{t,i}^{IL} = \sum_{i=1}^n V_{\underline{t},i}^{IL} = \sum_{i=1}^n V_{t,i}^{NL}$ for $t > \underline{t}$.

(b) By assumption, $-c-g+pR \geq 0$ and therefore we have

$$\begin{aligned}\sum_{i=1}^n V_{\underline{t}+1,i}^{IL} &= (n-1)(-c-g+pR) + (1-p(n-1))\sum_{i=1}^n V_{\underline{t},i}^{IL} \\ &\geq \sum_{i=1}^n V_{\underline{t},i}^{IL} = \sum_{i=1}^n V_{\underline{t},i}^{NL} = \sum_{i=1}^n V_{\underline{t}+1,i}^{NL}.\end{aligned}$$

Fix $t > \underline{t}$ and suppose $\sum_{i=1}^n V_{t,i}^{IL} \geq \sum_{i=1}^n V_{t-1,i}^{IL}$. Then using the same argument as when $t = \underline{t}+1$, we obtain $\sum_{i=1}^n V_{t-1,i}^{IL} \geq \sum_{i=1}^n V_{t,i}^{NL}$.□

Lemma B-6. *k_∞^{FB} and k_∞^{IL} are unique.*

Proof. We first show the results for the FB solution. By definition, $V_t^{FB} = V_{t-1}^{FB} = V_\infty^{FB}$ and therefore the FB optimal payoff in period t satisfies the following:

$$\begin{aligned}V_\infty^{FB} &= -k_\infty^{FB} c(k_\infty^{FB}) - g(k_\infty^{FB}) + \delta p(k_\infty^{FB})R + \delta(1-p(k_\infty^{FB}))V_\infty^{FB} \\ \text{Hence, } V_\infty^{FB} &= \frac{-k_\infty^{FB} c(k_\infty^{FB}) - g(k_\infty^{FB}) + \delta p(k_\infty^{FB})R}{1-\delta+\delta p(k_\infty^{FB})}.\end{aligned}$$

We define $V(i) = \frac{\delta p(i)R - G(i)}{1-\delta+\delta p(i)}$, in which $G(i) = ic(i) + g(i)$, and show that it has a unique optimum. For the sake of exposition, we assume that $p(i)$, $g(i)$, and $c(i)$ are twice-differentiable. We have

$$V'(i) = \frac{(\delta p'(i)R - G'(i))(1-\delta) - G'(i)\delta p(i) + \delta G(i)p'(i)}{(1-\delta+\delta p(i))^2}$$

Let us define $f(i) = (\delta p'(i)R - G'(i))(1-\delta) - G'(i)\delta p(i) + \delta G(i)p'(i)$. Let k be a solution of $f(i) = 0$. We next show that $f(i)$ is decreasing in i.

$$f'(i) = (\delta p''(i)R - G''(i))(1-\delta) - \delta G''(i)p(i) + \delta p''(i)G(i) \leq 0,$$

given that $p(i)$ is concave and $G(i)$ is convex. Hence, $V(i)$ is pseudo-concave and therefore k_∞^{FB} is unique and is obtained by

$$k_\infty^{FB} = \min\left\{i \in Z^+_{i\leq n} \mid \frac{\Delta[ic(i)h(i)] + \Delta[g(i)h(i)]}{\delta\Delta[p(i)h(i)]} \geq R\right\}, \tag{B-8}$$

in which $h(i) = 1/(1-\delta+\delta p(i))$.

The proof for the IL structure follows the same, except that for the IL structure $G^{IL}(i) = g(i)$ and $V^{IL}(i) = V(i)/n$. The IL team size is obtained by

$$k_\infty^{IL} = \min\left\{i \in Z^+_{i\leq n} \mid \frac{\Delta[g(i)h(i)]}{\delta\Delta[p(i)h(i)]} \geq R\right\}. \tag{B-9}$$

in which $h(i) = 1/(1-\delta+\delta p(i))$.□ ∎

Proof of Lemma 2.4. The proof uses Lemma B-6. With the similar approach as in Lemma B-6, we obtain $V_{\infty,i}^{NL} = [-c(k_\infty^{NL}) - g(k_\infty^{NL})/n + \delta p(k_\infty^{NL})R/n]/[1-\delta+\delta p(k)]$ for $i \leq k_\infty^{NL}$ and $V_{\infty,i}^{NL} = [-g(k_\infty^{NL})/n + \delta p(k_\infty^{NL})R/n]/[1-\delta+\delta p(k)]$ for $i > k_\infty^{NL}$. As a result considering $G(i) = -nc(i) - g(i)$, k_∞^{NL} can be obtained by the following when it converges:

$$k_\infty^{NL} = \min\left\{ i \in Z_{i\leq n}^+ \mid \frac{nc(i+1)h(i+1) + \Delta[g(i)h(i)]}{\delta\Delta[p(i)h(i)]} \geq R \right\}, \tag{B-10}$$

in which $h(i) = 1/(1-\delta+\delta p(i))$.

(i) The proof follows similar to the proof of Lemma 2.1 part (a) with $\overline{R}_\infty = \min\{R \mid R \geq \frac{nc(n)h(n)+\delta\Delta[g(n-1)h(n-1)]}{\delta\Delta[p(n-1)h(n-1)]}\}$.

(ii) The proof follows similar to the proof of Lemma 2.1 part (b) with $\underline{R}_\infty = \min\{R \mid R \geq \frac{\Delta[(n-1)c(n-1)h(n-1)]+\delta\Delta[g(n-1)h(n-1)]}{\delta\Delta[p(n-1)h(n-1)]}\}$.

(iii) For $R \leq \underline{R}_\infty$, $k_\infty^{FB} < n$, $k_\infty^{NL} < n$, and $k_\infty^{IL} < n$. Comparing the stationary team sizes in (B-8), (B-10), and (B-9), one can see that $k_\infty^{IL} \geq k_\infty^{FB}$, because the left hand side of the inequality in (B-9) is smaller than in (B-8). Similarly, $k_\infty^{FB} \geq k_\infty^{NL}$, because the left hand side of the inequalities in (B-8) is smaller than in (B-10).□

Proof of Proposition 2.3. The proof uses Lemma 2.4.

(i) When $R > \overline{R}_\infty$, $k_\infty^{FB} = k_\infty^{IL} = k_\infty^{NL}$ by Lemma 2.4. Therefore $\sum_i V_{\infty,i}^{IL} = \sum_i V_{\infty,i}^{NL} = V_\infty^{FB}$.

(ii) When $\underline{R}_\infty < R \leq \overline{R}_\infty$, $k_\infty^{FB} > k_\infty^{IL} \geq k_\infty^{NL}$ by Lemma 2.1. Therefore $\sum_i V_{\infty,i}^{IL} \geq \sum_i V_{\infty,i}^{NL}$.

(iii)When $R \leq \underline{R}_\infty$, $k_\infty^{IL} \geq k_\infty^{FB} \geq k_\infty^{NL}$ by Lemma 2.4. According to (B-10) and (B-9), k_∞^{NL} and k_∞^{IL} are increasing in R. Similarly, k_∞^{FB} and $V_\infty^{FB}(k)$ are increasing in R. By Lemma 2.4, $k_\infty^{IL} \geq k_\infty^{FB} \geq k_\infty^{NL}$. Therefore with the similar approach as in Proposition 2.1 part (c), one can show that there exists a unique threshold $\hat{R}_\infty$ above which the NL structure generates a higher total surplus than the IL structure.□

4.3 Appendix C : Proofs for Chapter 3

Lemma C-1. *$V_t(x,m,n)$ is decreasing in t.*

Proof. The proof proceeds by induction. In period $T-1$, we have that $V_{T-1}(x,m,n) \geq R(x) = V_T(x,m,n)$. Suppose next that $V_{t+1}(x,m,n) \geq V_{t+2}(x,m,n)$ for all states (x,m,n). Then,

$$\begin{aligned} V_t(x,m,n) &= Max\{-c_s + \delta\left[p_s V_{t+1}(x+1,m,n) + (1-p_s)\, V_{t+1}(x,m,n)\right], R(x), \\ &\quad -2c_d + \delta[f(m,n)V_{t+1}(x+1,m,n+1) + (1-f(m,n))\, V_{t+1}(x,m+1,n)]\} \\ &\geq Max\{-c_s + \delta[p_s V_{t+2}(x+1,m,n) + (1-p_s)\, V_{t+2}(x,m,n)], R(x), \\ &\quad -2c_d + \delta[f(m,n)V_{t+2}(x+1,m,n+1) + (1-f(m,n))\, V_{t+2}(x,m+1,n)]\} \\ &= V_{t+1}(x,m,n). \end{aligned}$$

This completes the induction step.□ ∎

Lemma C-2. *If $\mathcal{E}_t(x,m,n) = Finish$, then $\mathcal{E}_{t+1}(x,m,n) = Finish$.*

Proof. The proof uses Lemma C-1. Given that $\mathcal{E}_t(x,m,n) = Finish$, we have $R(x) \leq V_{t+1}(x,m,n) \leq V_t(x,m,n) = R(x)$ and therefore $V_{t+1}(x,m,n) = R(x)$.□ ∎

Proof of Proposition 3.1. We have $\mathcal{E}_1(x,m,n) = Finish$ if and only if $R(x+1) - R(x) \leq \min\{\theta_{fs}(x), \theta_{fd}(x,m,n)\}$. Given that $R(x)$ is concave and $\theta_{fs}(x)$ and $\theta_{fd}(x,m,n)$ are increasing in x. We have $\mathcal{E}_1(x,m,n) = Finish$ implies $\mathcal{E}_1(x+1,m,n) = Finish$. Therefore, we should have $V_2(x+1,m,n) - V_2(x,m,n) = R(x+1) - R(x) \leq \min\{\theta_{fs}(x), \theta_{fd}(x,m,n)\}$. Similarly, $V_1(x+1,m,n) - V_1(x,m,n) = R(x+1) - R(x) \leq \min\{\theta_{fs}(x), \theta_{fd}(x,m,n)\}$ that implies $\mathcal{E}_0(x,m,n) \neq Solo$.□

Lemma C-3. *$V_t(m)$ is decreasing in m.*

Proof. We prove the lemma by induction on t. By definition, we have that $f(m) > f(m+1)$. When $t = T-1$, $V_{T-1}(m) = \max\{-2c_d + \delta f(m)R, -c_s + \delta p_s R\} \geq \max\{-2c_d + \delta f(m+1)R, -c_s + \delta p_s R\} = V_{T-1}(m+1)$. Fix $t < T-1$ and suppose that $V_{t+1}(m+1) \leq V_{t+1}(m)$ for all m. Then, $V_t(m+1 \mid Solo) = -c_s + \delta p_s R + \delta(1-p_s)V_{t+1}(m+1) \leq -c_s + \delta p_s R + \delta(1-p_s)V_{t+1}(m) = V_t(m \mid Solo)$. We next need to show that $V_t(m+1 \mid Duo) \leq V_t(m \mid Duo)$. We have

$$\begin{aligned} f(m)R + (1-f(m))V_{t+1}(m+1) &\geq f(m)R + (1-f(m))V_{t+1}(m+2) \\ &> f(m+1)R + (1-f(m+1))V_{t+1}(m+2), \end{aligned}$$

in which the second inequality is true by the fact that $R \geq V_t(m)$ for all t and m.

As a result, $V_t(m+1 \mid Duo) = -2c_d + \delta\left[f(m+1)R + (1-f(m+1))V_{t+1}(m+2)\right] \leq -2c_d + \delta\left[f(m)R + (1-f(m))V_{t+1}(m+1)\right] = V_t(m \mid Duo)$. This completes the induction step.□ ∎

Proof of Proposition 3.2. The proof uses Lemma C-3. We have that $V_{T-1}(m) = Solo$ for all $m \geq m_s$ and therefore $f(m) \leq \frac{2c_d - c_s + \delta p_s R}{\delta R}$ for all $m \geq m_s$. We show that $V_t(m \mid Duo) < V_t(m \mid Solo)$ for all t and $m \geq m_s$ as follows:

$$\begin{aligned} V_t(m \mid Duo) &= -2c_d + \delta f(m)R + \delta(1-f(m))V_{t+1}(m+1) \\ &\leq -2c_d + \delta\frac{2c_d - c_s + \delta p_s R}{\delta R}\left[R - V_{t+1}(m+1)\right] + \delta V_{t+1}(m+1) \\ &< -c_s + \delta p_s R + \delta(1-p_s)V_{t+1}(m) = V_t(m \mid Solo), \end{aligned}$$

in which the first inequality holds because $R - V_{t+1}(m+1) \geq 0$ and $f(m) \leq \frac{2c_d - c_s + \delta p_s R}{\delta R}$.

And, the last inequality holds by Lemma C-3.□

Lemma C-4. *For $t \leq t_s(m)$, if $\mathcal{E}_{t+1}(m) = \mathcal{E}_{t+1}(m+1) = Solo$, then $\mathcal{E}_t(m) = Solo$.*

Proof. The proof proceeds by induction. We show that when $\mathcal{E}_{t+1}(m) = \mathcal{E}_{t+1}(m+1) = Solo$, $H_t(m) \geq H_{t+1}(m)$. We first show the result in state $m_s - 1$. We have that $\mathcal{E}_{t_s(m_s-1)+1}(m_s - 1) = Duo$ and $\mathcal{E}_{t_s(m_s-1)}(m_s - 1) = Solo$ which imply $H_{t_s(m_s-1)}(m_s - 1) \geq -2c_d + c_s - \delta p_s R \geq H_{t_s(m_s-1)+1}(m_s - 1)$.

Next we consider any time $t < t_s(m_s - 1)$. By Proposition 3.2, we have that $\mathcal{E}_t(m_s) = Solo$. Suppose by induction that $\mathcal{E}_{t+1}(m_s - 1) = Solo$ and $H_{t+1}(m_s - 1) \geq H_{t+2}(m_s - 1)$. We show that $H_t(m_s - 1) \geq H_{t+1}(m_s - 1)$ under two cases (i) $\mathcal{E}_{t+2}(m_s - 1) = Solo$ and (ii) $\mathcal{E}_{t+2}(m_s - 1) = Duo$.

(i) Suppose $\mathcal{E}_{t+2}(m_s - 1) = Solo$. We have

$$\begin{aligned}
H_t(m_s - 1) &= \delta(f(m_s - 1) - p_s)\,[-c_s + \delta p_s R] - \delta f(m_s - 1)R \\
&+ \delta(1 - p_s)\,[H_{t+1}(m_s - 1) + \delta f(m_s - 1)R] \\
&\geq \delta(f(m_s - 1) - p_s)\,[-c_s + \delta p_s R] - \delta f(m_s - 1)R \\
&+ \delta(1 - p_s)\,[H_{t+2}(m_s - 1) + \delta f(m_s - 1)R] = H_{t+1}(m_s - 1)
\end{aligned}$$

(ii) Suppose $\mathcal{E}_{t+2}(m_s - 1) = Duo$. We have $V_{t+1}(m_s - 1 \mid Solo) > V_{t+1}(m_s - 1 \mid Duo)$ because $\mathcal{E}_{t+1}(m_s - 1) = Solo$. Hence, $H_t(m_s - 1) \geq \delta(1 - p_s)V_{t+1}(m_s - 1 \mid Duo) - \delta(1 - f(m_s - 1))V_{t+1}(m_s) - \delta f(m_s - 1)R = \delta(1 - p_s)[-2c_d + \delta f(m_s - 1)R] - \delta(1 - f(m_s - 1))[-c_s + \delta p_s R] - \delta f(m_s - 1)R = H_{t+1}(m_s - 1)$.

Therefore, $\mathcal{E}_t(m_s - 1) = Solo$ for $t < t_s(m_s - 1)$.

Next, we extend the proof for any m and $t \le t_s(m)$. Similar to state $m_s - 1$, for any state m and period $t_s(m)$ we have that $\mathcal{E}_{t_s(m)+1}(m) = Duo$ and $\mathcal{E}_{t_s(m)}(m) = Solo$ which imply $H_{t_s(m)}(m) \ge -2c_d + c_s - \delta p_s R + 2c_d \ge H_{t_s(m)+1}(m)$ for all m.

Suppose $H_{t+1}(m) \ge H_{t+2}(m)$. We show that $H_t(m) \ge H_{t+1}(m)$. We may have $\mathcal{E}_{t+2}(m+1) = Solo$ or $\mathcal{E}_{t+2}(m+1) = Duo$. However, in both cases we have that $V_{t+2}(m+1 \mid Solo) \le V_{t+2}(m+1)$. Similar to $m_s - 1$, we show the result under two cases: (i) $\mathcal{E}_{t+2}(m) = Solo$ and (ii) $\mathcal{E}_{t+2}(m) = Duo$.

(i) Suppose $\mathcal{E}_{t+2}(m) = Solo$. We have $H_t(m) \ge \delta(f(m) - p_s)[-c_s + \delta p_s R] + \delta(1 - p_s)[H_{t+1}(m) + \delta f(m)R] - \delta f(m)R \ge H_{t+1}(m)$, in which the last inequality holds because $V_{t+2}(m+1 \mid Solo) \le V_{t+2}(m+1)$.

(ii) Suppose $\mathcal{E}_{t+2}(m) = Duo$. We have $H_t(m) \ge \delta(1-p_s)V_{t+1}(m \mid Duo) - \delta(1 - f(m))V_{t+1}(m+1) - \delta f(m)R = \delta(1-p_s)V_{t+2}(m) - \delta(1-f(m))V_{t+2}(m+1 \mid Solo) - \delta f(m)R \ge H_{t+1}(m)$, in which the last inequality holds because $V_{t+2}(m + 1 \mid Solo) \le V_{t+2}(m+1)$.$\square$

∎

Lemma C-5. *Suppose $\hat{m} = 0$. For $m < m_s(t+1)$, if $\mathcal{E}_t(m+1) = Duo$, then $\mathcal{E}_t(m) = Duo$.*

Proof. Similar to (3.6), $\mathcal{E}_t(m) = Duo$ if and only if $H_t(m) \le -2c_d + c_s - \delta p_s R$. We show the results by induction on time and state. We show that if $\mathcal{E}_t(m+1) = \mathcal{E}_{t+1}(m+1) = \mathcal{E}_{t+1}(m) = Duo$, then $H_t(m) \le H_t(m+1)$ and therefore $\mathcal{E}_t(m) = Duo$. To initialize the induction, we show that the result holds in period $T-1$ and also in state $m_s(t) - 1$. By definition, $V_T(m) = 0$ for all m and since $f(m)$ is decreasing in m, we have that $H_{T-1}(m) = -\delta f(m)R \le -\delta f(m+1)R = H_{T-1}(m+1)$. In state

$m_s(t)-1$, by definition, we have that $\mathcal{E}_t(m_s(t)-1) = Duo$ and $\mathcal{E}_t(m_s(t)) = Solo$ and therefore $H_t(m_s(t)-1) \leq -2c_d + c_s - \delta p_s R \leq H_t(m_s(t))$ for all t.

Next, suppose $H_t(m+1) \leq H_t(m+2)$. To complete the induction proof, we show the result under two possible cases: (i) $\mathcal{E}_{t+1}(m+2) = Duo$ and (ii) $\mathcal{E}_{t+1}(m+2) = Solo$.

(i) We have that $\mathcal{E}_{t+1}(m) = \mathcal{E}_{t+1}(m+1) = \mathcal{E}_{t+1}(m+2) = Duo$ and therefore

$$\begin{aligned} H_t(m) &= \delta(1-f(m))H_{t+1}(m+1) + \delta(1-p_s)\left[-2c_d + \delta f(m)R\right] \\ &- \delta(1-f(m))\left[-2c_d\right] - \delta f(m)R \\ &\leq \delta(1-f(m))\left(H_{t+1}(m+2) + \delta f(m+2)R\right) + 2c_d\delta p_s - \delta^2 f(m+2)R \\ &+ \delta f(m)\left[\delta R(1-p_s) - 2c_d + \delta f(m+2)R - R\right] \end{aligned}$$

Because $f(m)$ is decreasing in m and that $H_{t+1}(m+2) + \delta f(m+2)R = \delta(1-p_s)V_{t+1}(m) - \delta(1-f(m))V_{t+1}(m+1) \geq 0$, we have $\delta(1-f(m))[H_{t+1}(m+2) + \delta f(m+2)R] \leq \delta(1-f(m+1))[H_{t+1}(m+2) + \delta f(m+2)R]$. In addition, since $\hat{m} = 0$, $\delta R(1-p_s) - 2c_d + \delta f(m+2)R - R \leq 0$ and therefore $\delta f(m)[\delta R(1-p_s) - 2c_d + \delta f(m+2)R - R] < f(m+1)[\delta R(1-p_s) - 2c_d + \delta f(m+2)R - R]$. As a result, $H_t(m) \leq H_t(m+1)$.

(ii) Suppose $\mathcal{E}_{t+1}(m+2) = Solo$. Similar to (i), we obtain

$$\begin{aligned} H_t(m) &\leq \delta(1-p_s)\left(-2c_d + \delta f(m)R + \delta(1-f(m))V_{t+2}(m+1)\right) \\ &- \delta(1-f(m))\left(-c_s + \delta p_s R + \delta(1-p_s)V_{t+2}(m+1)\right) - \delta f(m)R \\ &\leq \delta(1-p_s)(-2c_d) + \delta f(m+1)\left[\delta R - c_s - R\right] + \delta c_s - \delta^2 p_s R = H_t(m+1), \end{aligned}$$

in which the first inequality holds because $V_{t+1}(m+1 \mid Duo) > V_{t+1}(m+1 \mid Solo) \geq 0$ and the second inequality holds because that $f(m)$ is decreasing in m and $\delta R - c_s - R < 0.\square$

∎

Proof of Proposition 3.3. The proof uses Lemmas C-4 and C-5.

(i) We show in Lemma C-4 that $\mathcal{E}_{t+1}(m) = \mathcal{E}_{t+1}(m+1) = Solo$ implies $\mathcal{E}_t(m) = Solo$ for $t < t_s(m)$. In addition, we show in Lemma C-5 that $\mathcal{E}_t(m) = Duo$ for $m \leq m_s(t)$. These two results imply that $t_s(m)$ is increasing in m and $m_s(t)$ is increasing in t. We next show that for $m > m_d$, $m_s(t)$ is strictly increasing in t. Suppose $\mathcal{E}_{t+1}(m+1) = Solo$ and $\mathcal{E}_{t+1}(m) = Duo$. Then by (3.10), $H_t(m) = \delta(1-f(m))[-2c_d+\delta f(m)R] - \delta(1-f(m))[-c_s+\delta p_s R] - \delta f(m)R \geq -2c_d + c_s - \delta p_s R$. In particular, rewriting the inequality in (3.10), we obtain $\delta(1-p_s)[-2c_d+\delta f(m)R] - \delta(1-f(m))[-c_s+\delta p_s R] - \delta f(m)R \geq 2c_d + c_s - \delta p_s R$ for all $m > m_d$.

(ii) By definition of $t_s(m)$, we have that $\mathcal{E}_t(m_d+1) = Duo$ for all $t > t_s(m_d+1)$. In addition by Lemma C-4, $\mathcal{E}_t(m_d+1) = Solo$ for all $t \leq t_s(m_d+1)$. Therefore by Lemma C-5, $\mathcal{E}_t(m) = Duo$ for $m \leq m_d$ and $t > t_s(m_d+1)$. We next show $\mathcal{E}_t(m) = Duo$ for $m \leq m_d$ and $t \leq t_s(m_d+1)$.

We first show that when $\mathcal{E}_{t+1}(m) = \mathcal{E}_{t+1}(m+1) = Duo$, then $\mathcal{E}_t(m) = Duo$ for $m \leq m_d$. Since $\mathcal{E}_{t+1}(m+1) = Duo$, we have that $V_{t+1}(m+1) > V_{t+1}(m+1 \mid Solo)$. Hence by (3.10), $H_t(m) \leq \delta(1-p_s)[-2c_d+\delta f(m)R+\delta(1-f(m))V_{t+2}(m+1)] - \delta(1-f(m))[-c_s+\delta p_s R+\delta(1-p_s)V_{t+2}(m+1)] - \delta f(m)R \leq -2c_d + c_s - \delta p_s R$.

We next consider $\mathcal{E}_{t_s(m_d+1)}(m_d) = Duo$ and $\mathcal{E}_{t_s(m_d+1)}(m_d+1) = Solo$. By (3.10), we obtain $H_{t_s(m_d+1)-1}(m_d) = \delta(1-f(m_d))[-2c_d+\delta f(m_d)R] - \delta(1-f(m_d))[-c_s+\delta p_s R] - \delta f(m_d)R \leq -2c_d + c_s - \delta p_s R$.

By Lemma C-4, $\mathcal{E}_t(m_d+1) = Solo$ for all $t \leq t_s(m_d)$. Therefore similar to

$H_{t_s(m_d+1)-1}(m_d)$, one can show that $H_t(m_d) = H_{t_s(m_d+1)}(m_d) \leq -2c_d + c_s - \delta p_s R$ for all $t < t_s(m_d + 1)$. As a result, $\mathcal{E}_t(m_d) = Duo$ for all $t < t_s(m_d + 1)$. As a result, $\mathcal{E}_t(m) = Duo$ for all $m \leq m_d$ and $t < T$.$\square$

Bibliography

[AD72] A. A. Alchian and H. Demsetz. “Production, Information Costs, and Economic Organization.” *The American Economics Review*, **62**:777–795, 1972.

[ADM08] R. Alonso, W. Dessein, and N. Matouschek. “When does coordination require centralization.” *Amer. Econom. Rev.*, **98**(1):145–179, 2008.

[AJW12] S. Adlakha, R. Johari, and G.Y. Weintraub. “Equilibria of Dynamic Games with Many Players: Existence, Approximation, and Market Structure.” Caltech Working Paper, 2012.

[AKK11] U. Apte, U. S. Karmarkar, C. Kieliszewski, and Y.T. Leung. “Evaluation of Methods for Representation of Complex Processes in Information Intensive Services.” *Int. J. Services Operations and Informatics*, **7**:52–78, 2011.

[ANC02] R.D. Austin, R.L. Nolan, and M.J. Cotteleer. “Cisco Systems, Inc.: Implementing ERP.”, 2002. HBS Case 9-699-022.

[AP91] A. R. Admati and M. Perry. “Joint Projects without Commitment.” *Rev. Econ. Stud.*, **58**(2):259–276, 1991.

[BBM10] R. Bapna, A. Barua, D. Mani, and A. Mehra. “Cooperation, Coordination, and Governance in Multisourcing: An Agenda for Analytical and Empirical Research.” *Information Systems Research*, **21**(4):785–795, 2010.

[BC05] M. Bayiz and C. J. Corbett. “Coordination and Incentive Contracts in Project Management under Asymmetric Information.” UCLA Working Paper, 2005.

[BD05] P. Bolton and M. Dewatripont. *Contract Theory*. MIT Press, London, England, 2005.

[BH99] P. Bolton and Ch. Harris. “Strategic Experimentation.” *Econometrica*, **67**(2):349–374, 1999.

[BH11] A. Bonatti and J. Horner. "Collaborating." *Amer. Econ. Rev.*, **101**(2):632–663, 2011.

[BK09] S. R. Bhaskaran and V. Krishnan. "Effort, revenue, and cost sharing mechanisms for collaborative new product development." *Management Sci.*, **55**(7):1152–1169, 2009.

[BL95] S. Bhattacharyya and F. Lafontaine. "Double-sided moral hazard and the nature of share contracts." *RAND J. Econ.*, **26**(4):761–781, 1995.

[BM92] G.S. Becker and K.M. Murphy. "The Division of Labor, Coordination Costs, and Knowledge." *The Quarterly Journal of Economics*, **107**(4), 1992.

[BO99] T. Başar and G. Olsder. *Dynamic Noncooperative Game Theory*. Classics in Applied Mathematics. SIAM, Philadelphia, PA, 2nd edition edition, 1999.

[Bro75] F. Brooks. *The Mythical Man-Month: Essays on Software Engineering*. Addison-Wesley, Boston, MA, 1975.

[CK10] M. Christensen and Th. Knudsen. "Design of Decision-Making Organizations." *Management Sci.*, **56**(1):71–89, 2010.

[CS10] F. Czerniawska and P. Smith. *Buying Professional Services: How to Get Value for Money from Consultants and Other Professional Service Providers*. The Economist Press, Wiley, 2010.

[Des07] W. Dessein. "Why a Group Needs a Leader: Decision-making and Debate in Committees." Columbia Working Paper, 2007.

[DKS13] E. M. Demirezen, S. Kumar, and B. Shetty. "Co-Creation of Value: A Differential Games Approach." Texas A&M Working Paper, 2013.

[DS10] U. Doraszelski and M. Satterthwaite. "Computable Markov-perfect industry dynamics." *RAND J. Econom.*, **41**(1):215–243, 2010.

[DTC02] T. H. Davenport, R.J. Thomas, and S. Cantrell. "The mysterious art and science of knowledgeworker performance." *MIT Sloan Manage*, **44**(1):23–30, 2002.

[EK08] S. Erat and S. Kavadias. "Sequential Testing of Product Designs: Implications for Learning." *Management Sci.*, **54**(5):956968, 2008.

[FT91] D. Fudenberg and J. Tirole. *Game Theory*. The MIT Press, Cambridge, MA, 1991.

[Fuc68] V. R. Fuchs. *The Service Economy.* National Bureau of Economic Research, New York, NY, 1968.

[Gai13] G. (CEO of Xoriant) Gaitonde. Personal Communication, 2013.

[Gar00] L. Garicano. "Hierarchies and organization of knowledge in production." *The Journal of Political Economy*, **108**(5):874–904, 2000.

[GE07] L. Gratton and T. Erickson. "Eight ways to build collaborative teams." *Harvard Business Review*, **85**(11):100–109, 2007.

[Geo12] G. Georgiadis. "Projects and Team Dynamics." UCLA Working Paper, 2012.

[Gol97] E.M. Goldratt. *Critical Chain.* The North River Press, Great Barrington, MA, 1997.

[GP06] F. Gino and G. Pisano. "Teradyne Corporation: The Jaguar Project.", 2006. HBS Case 9-606-042.

[Hac95] J. R. Hackman. *Encyclopedic Dictionary of Organizational Behavior. N. Nicholson (Ed.).* Oxford, Blackwel, UKl, 1995.

[Hac98] J. R. Hackman. *Theory and Research on Small Groups, R. S. Tindale et al., (Eds.).* Springer, New York: Plenum, 1998. Abridged version in Leader to Leader, pages 24-31.

[Hac02] J. R. Hackman. "New rules for team building." *Optimize*, **July**:50–62, 2002.

[HIL09] W. J. Hopp, S. M. R. Iravani, and F. Liu. "Managing white-collar work: An operations-oriented survey." *Prod. Oper. Management*, **18**(1):1–32, 2009.

[HK12] J. Hutchison-Krupat and S. Kavadias. "Compensation Challenges for Cross-Functional Product Development Teams." Georgia Tech. Working Paper, 2012.

[HL96] B. A. Huberman and C. H. Loch. "Collaboration, motivation, and size of organizations." *J. Organ. Comput. Electronic Commerce.*, **6**(2):109–130, 1996.

[HL01] C. Huchzermeier and H. Loch. "Project management under risk: Using the real options approach to evaluate flexibility in R&D." *Management Sci.*, **47**(1):85–101, 2001.

[Hol82] B. Holmström. "Moral Hazard in Teams." *Bell J. Econ.*, **13**(2):324–340, 1982.

[HP95] A.Y. Ha and E.L. Porteus. "Optimal Timing of Reviews in Concurrent Design for Manufacturability." *Management Sci.*, **41**(9):1431–1447, 1995.

[HWR00] J. R. Hackman, R. Wageman, T. M. Ruddy, and C. R. Ray. *Industrial and Organizational Psychology: Theory and Practice, C. Cooper and E. A. Locke (Eds.).* Oxford, UK: Blackwell, 2000. Chapter 5: Team effectiveness in theory and practice.

[KBB10] C.A. Kieliszewski, J.H. Bailey, and J. Blomberg. "A service practice approach: people, activities and information in highly collaborative knowledge-based service systems." In P. P. Maglio, C. A. Kieliszewski, and J. C. Spohrer, editors, *Handbook of Service Science*. Springer, New York, NY, 2010.

[KK05] C. J. König and M. Kleinmann. "Deadline Rush: A Time Management Phenomenon and Its Mathematical Description." *The Journal of Psychology*, **139**(1):3345, 2005.

[KLT10] H. D. Kwon, S. A. Lippman, and Ch. S. Tang. "Optimal time-based and cost-based coordinated project contracts with unobservable work rates." *Int. J. Prod. Econ.*, **126**(1):247–254, 2010.

[KN13] S. H. Kim and S. Netessine. "Collaborative Cost Reduction and Component Procurement under Information Asymmetry." *Management Sci.*, **59**(1):189–206, 2013.

[KP95] U. Karmarkar and R. Pitbladdo. "Service markets and competition." *J. Oper. Management*, **12**(34):397–411, 1995.

[KR08] N. Klein and S. Rady. "Negatively Correlated Bandits." Center for Economic and Policy Research Discussion Paper 6983, 2008.

[KRC05] G. Keller, S. Rady, and M. Cripps. "Strategic Experimentation with Exponential Bandits." *Econometrica*, **73**(1):39–68, 2005.

[KS93] J.R. Katzenback and D.K. Smith. *The Wisdom of Teams.* Harper Business, New York, 1993.

[KS09] S. Kavadias and S. Sommer. "The Effects of Problem Structure and Team Diversity on Brainstorming Effectiveness." *Management Sci.*, **55**(12):1899–1913, 2009.

[KW93] S.J. Karau and K.D. Williams. "Social loafing: A meta-analytic review and theoretical integration." *Journal of Personality and Social Psychology*, **65**(4):681–706, 1993.

[Lam01] R.A. Lambert. “Contracting theory and accounting.” *J. Accounting Econom*, **32**(1):3–87, 2001.

[LM02] J. J. Laffont and D. Martimort. *The Theory of Incentives: The Principal-Agent Model*. Princeton University Press, Princeton, NJ, 2002.

[LS12] M. LiCalzi and O. Surucu. “The Power of Diversity over Large Solution Spaces.” *Management Sci.*, **58**(7):1408–1421, 2012.

[LT98] Ch. Loch and Ch. Terwiesch. “Communication and Uncertainty in Concurrent Engineering.” *Management Sci.*, **44**(8):1032–1048, 1998.

[LT05] Ch. Loch and Ch. Terwiesch. “Rush and be wrong or wait and be late? A model of information in collaborative processes.” *Prod. Oper. Management*, **14**(3):331–343, 2005.

[LTT01] Ch. Loch, Ch. Terwiesch, and S. Thomke. “Parallel and Sequential Testing of Design Alternatives.” *Management Sci.*, **47**(5):663–678, 2001.

[Lus82] H Luss. “Operations Research and Capacity Expansion Problems: A Survey.” *Oper. Res*, **30**(1):907–947, 1982.

[Mac97] A. Mackenzie. *The Time Trap*. AMACOM, New York, NY, 3nd edition edition, 1997.

[Mih10] J. Mihm. “Incentives in New Product Development Projects and the Role of Target Costing.” *Management Sci.*, **56**(8):1324–1344, 2010.

[MLH03] J. Mihm, C.H. Loch, and A. Huchzermeier. “Problem-Solving Oscillations in Complex Engineering Projects.” *Management Sci.*, **46**(6):733–750, 2003.

[MLK07] H. Maldonado, B. Lee, S. R. Klemmer, and R. D. Pea. “Patterns of Collaboration in Design Courses: Team dynamics affect technology appropriation, artifact creation, and course performance.” In C. Chinn, G. Erkens, and S. Puntambekar, editors, *Computer Supportive Collaborative Learning: Mice, Minds, and Society. Proceedings of the Seventh International Computer Supported Collaborative Learning Conference (CSCL 2007)*, pp. 486–495, Mahwah, NJ, 2007. Lawrence Erlbaum Associates.

[MLW10] J. Mihm, C.H. Loch, D. Willkinson, and B.A. Huberman. “Hierarchical Structure and Search in Complex Organizations.” *Management Sci.*, **56**(5):831–848, 2010.

[MM00] L. Marx and S. Matthews. “Dynamic Voluntary Contribution to a Public Project.” *Rev. Econ. Stud.*, **67**(2):327–358, 2000.

[MR72] J. Marschak and R. Radner. *Economic Theory of Teams*. Yale University Press, New Haven and London, 1972.

[MT01] E. Maskin and J. Tirole. "Markov Perfect Equilibrium I: Observable Actions." *J. Econ. Theory*, **100**(2):191–219, 2001.

[OGK13] F.G. Ozkan, Ch. Gaimon, and S. Kavadias. "Knowledge Management Strategies for Product and Process Design Teams." Georgia Tech Working Paper, 2013.

[Osb02] M.J. Osborne. *An Introduction to Game Theory*. Oxford University Press, Oxford, 2002.

[PS00] E.J. Pinker and R.A Shumsky. "The efficiency-quality trade-off of cross-trained workers." *Manufacturing and Service Operation Management*, **2**(1):32–48, 2000.

[PT06] E. Plambeck and T. Taylor. "Partnership in a dynamic production system with unobservable actions and noncontractible output." *Management Sci.*, **52**(10):1509–1527, 2006.

[RBN96] G. Ruel, N. Bastiaans, and A. Nauta. "Free riding and team performance in project education." *International Journal of Management Education*, **3**(1):26–38, 1996.

[Rep01] A.P. Repenning. "Understanding fire fighting in new product development." *Journal of Product Innovation Management*, **18**(5):285–300, 2001.

[Rin13] M. Ringelmann. "Research on Animate Sources of Power: The Work of Man." *Annales de l'Institut National Agronomique*, **12**(2):1–40, 1913.

[RKC10] G. Roels, U.S. Karmarkar, and S. Carr. "Contracting for collaborative services." *Management Sci.*, **56**(5):849–863, 2010.

[Sch04] K. Schwaber. *Agile Project Management with Scrum*. Microsoft Press, Redmond, Washington, 2004.

[SDA96] W. Stroebe, M. Diehl, and G. Abakoumkin. "Social compensation and the K ohler effect: Toward a theoretical explanation of motivation gains in group productivity." *Understanding Group Behavior*, **2**(1):37–65, 1996.

[Sha53] L. S. Shapley. "Stochastic Games." *Proc. National Acad. Sci.*, **39**(10):1095–1100, 1953.

[She10] T. Sheedy. "The Right Pricing Model For Your Systems Integration And IT Consulting Projects." Technical report, Forrester Research, 2010. February 10.

[SL09] M. Shao and H. Lee. “Cisco Systems, Inc.: Collaborating on New Product Introduction.”, 2009. Stanford GSB Case GS-66.

[SML12] F.J. Sting, J. Mihm, and C.H. Loch. “Collaborative Search.” INSEAD Working Paper, 2012.

[Sob71] M. Sobel. “Non-cooperative stochastic games.” *Ann. Math. Statist.*, **42**(6):1930–1935, 1971.

[SS88] R.K. Sah and J.E. Stiglitz. “Commiyyees, hierarchies and polyarchies.” *The Economic Journal*, **98**(1):451–470, 1988.

[SS91] R.K. Sah and J.E. Stiglitz. “The Quality of Managers in Centralized versus Decentralized Organizations.” *Quarterly Journal of Economics*, **106**(1):289–295, 1991.

[SU11] B.R. Staats and D.M. Upton. “Lean knowledge work.” *Harvard Business Review*, pp. 1–11, 2011. October 2011.

[SV05] L. P. Santiago and P. Vakili. “On the value of flexibility in R&D projects.” *Management Sci.*, **51**(8):1206–1218, 2005.

[TE12] A. Tripathy and S.D. Eppinger. “Structuring Work Distribution for Global Product Development Organizations.” *Prod. Oper. Management*, 2012. Forthcoming.

[Teo97] S.H. Teoh. “Information disclosure and voluntary contributions to public goods.” *RAND J. Econ.*, **28**(3):385–406, 1997.

[Tho98] S. H. Thomke. “Managing experimentation in the design of new products.” *Management Sci.*, **44**(6):743–762, 1998.

[Tho03] S. H. Thomke. *Experimentation Matters- Unlocking the Potential of New Technologies for Innovation*. Harvard Business School Press, Cambridge, MA, 2003.

[Tho13] L. Thompson. *Creative Conspiracy: The New Rules of Breakthrough Collaboration*. Harvard Business School Press, Cambridge, MA, 2013.

[TL04] Ch. Terwiesch and Ch. Loch. “Collaborative prototyping and pricing of custom-designed products.” *Management Sci.*, **50**(2):145–158, 2004.

[Van03] J.A. Van Mieghem. “Capacity management, investment and hedging: Review and recent developments.” *Manufacturing and Service Operations Management*, **5**(1):269–302, 2003.

[Var94] H.R. Varian. "Sequential Contributions to Public Goods." *J. Public Econ.*, **53**(1):165–186, 1994.

[VG13] J.A. VanMieghem and I. Gurvich. "Collaboration In Service Networks: Architectures, Bottlenecks And Throughput." Northwestern University Working Paper, 2013.

[Wei11] P. Weinschenk. "Procrastination in Teams, Contract Design and Discrimination." Max Planck Institute Working paper, 2011.

[WH94] K.S. Weber and L.A Hill. "Rudi Gassner at BMG International.", 1994. HBS Case 494523-VID-ENG.

[WRK13] Y. Wu, K. Ramachandran, and V. Krishnan. "Managing Cost Salience and Procrastination in Projects: Compensation and Team Composition." NUS Working paper, 2013.