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# Box integrals

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**Abstract.** By a “box integral” we mean here an expectation  $\langle |\vec{r} - \vec{q}|^s \rangle$  where  $\vec{r}$  runs over the unit  $n$ -cube, with  $\vec{q}$  and  $s$  fixed, explicitly:

$$\int_0^1 \cdots \int_0^1 ((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2)^{s/2} dr_1 \cdots dr_n.$$

The study of box integrals leads one naturally into several disparate fields of analysis. While previous studies have focused upon symbolic evaluation and asymptotic analysis of special cases (notably  $s = 1$ ), we work herein more generally—in interdisciplinary fashion—developing results such as: (1) analytic continuation (in complex  $s$ ), (2) relevant combinatorial identities, (3) rapidly converging series, (4) statistical inferences, (5) connections to mathematical physics, and (6) extreme-precision quadrature techniques appropriate for these integrals. These intuitions and results open up avenues of experimental mathematics, with a view to new conjectures and theorems on integrals of this type.

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# 1 Box integrals as expectations

Define a box integral for dimension  $n$  and parameters  $\vec{q}, s$  as the expectation of distance, from a fixed point  $\vec{q}$ , of a point  $\vec{r}$  chosen in equidistributed random fashion over the unit  $n$ -cube:

$$\begin{aligned} X_n(s, \vec{q}) &:= \langle |\vec{r} - \vec{q}|^s \rangle_{\vec{r} \in [0,1]^n} \\ &= \int_{\vec{r} \in [0,1]^n} |\vec{r} - \vec{q}|^s \mathcal{D}\vec{r}, \\ &= \int_0^1 \cdots \int_0^1 ((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2)^{s/2} dr_1 \cdots dr_n, \end{aligned} \quad (1)$$

where here and elsewhere  $\mathcal{D}\vec{r} := dr_1 \cdots dr_n$  is the  $n$ -space volume element. We also shall denote simply by  $r$  the magnitude  $|\vec{r}|$ .

There are two classically important instances/functionals of the  $X$ -integrals, namely  $B_n$  and  $\Delta_n$  defined:

$$\begin{aligned} B_n(s) &:= X_n(s, \vec{0}) = \int_{\vec{r} \in [0,1]^n} r^s \mathcal{D}\vec{r} \\ &= \int_0^1 \cdots \int_0^1 (r_1^2 + \cdots + r_n^2)^{s/2} dr_1 \cdots dr_n, \end{aligned} \quad (2)$$

$$\begin{aligned} \Delta_n(s) &:= \langle X_n(s, \vec{q}) \rangle_{\vec{q} \in [0,1]^n} = \int_{\vec{r}, \vec{q} \in [0,1]^n} |\vec{r} - \vec{q}|^s \mathcal{D}\vec{r} \mathcal{D}\vec{q} \\ &= \int_0^1 \cdots \int_0^1 ((r_1 - q_1)^2 + \cdots + (r_n - q_n)^2)^{s/2} dr_1 \cdots dr_n dq_1 \cdots dq_n. \end{aligned} \quad (3)$$

Note that

1.  $B_n(1)$  is the expected distance of a random point from any *vertex* of the  $n$ -cube
2.  $\Delta_n(1)$  is the expected distance *between* two random points of said cube
3.  $X_n(1, (1/2, 1/2, \dots, 1/2))$  is the expected distance of a random point from the *center* of said cube

and these are the most oft-discussed entities in the literature. There are many others such as the expected distance between points on distinct sides of a cube or hypercube investigated in [7, §1.7] or [5]. We remark that  $B_3(1)$  is also known as the *Robbins constant*, after [9]. Note that the third entity here is not genuinely different, because for general  $s$  one has the expected distance from center as

$$X_n(s, (1/2, 1/2, \dots, 1/2)) = \frac{1}{2^s} B_n(s), \quad (4)$$

as can be shown quickly from relations (1) by setting  $\vec{q} = (1/2, 1/2, \dots, 1/2)$ , changing to  $p_i = (r_i - q_i)/2$ , and observing how the integral has scaled. This is one of the various relations we shall develop that hold for all complex  $s$ ; in particular, we shall address analytic continuation. It will turn out, interestingly, that  $B_n(s)$  is always analytic except for a simple pole at  $s = -n$ .

There have been interesting modern treatments of the  $B_n$  and related integrals, as in [5], [7, p.208], [11], [10]. However, a pivotal, original treatment is the 1976 work of Anderssen et al, [1] who gave a large- $n$  asymptotic series

$$B_n(1) \sim \sqrt{\frac{n}{3}} \left(1 + \frac{n}{10} + \dots\right), \quad (5)$$

together with a convergent series development for  $B_n(1)$  we cite later (and extend to general  $s$ ), and a collection of bounds, derived via statistical theory, such as

$$\sqrt{\frac{n}{4}} \leq B_n(1) \leq \sqrt{\frac{n}{3}}.$$

This asymptotic is especially interesting when one realizes that the positive unit  $n$ -ball sector (the intersection of the  $n$ -ball with the cube  $[0, 1]^n$ ) has volume decaying superexponentially fast with  $n$ . Intuitively speaking, this discrepancy is due to the fact of “so many corners” of the  $n$ -cube, where integrable matter resides. We shall argue using statistical intuition that for general  $s$ ,

$$B_n(s) \sim \left(\frac{n}{3}\right)^{s/2}. \quad (6)$$

A word here is relevant as to the importance of box integrals in other fields of research. It should be noted first that the Anderssen et al. work [1] was motivated by global-optimization study, which explains why the adroit use of statistical principles is apparent in that effort. Secondly, there are problems of lattice theory—such as derivation of what are called “jellium” potentials, that involve  $B_n(s)$  for negative  $s$ . It is easy to imagine how potential theory for a periodic crystal can involve box integrals. We define and discuss later an  $n$ -dimensional jellium potential  $J_n$  as an expectation  $\langle V_n \rangle$  where  $V_n$  is a potential relevant to the  $n$ -dimensional Laplace equation.

As we explain herein, it turns out that both  $B_n(s), \Delta_n(s)$  even for large  $n$  can be numerically evaluated to extreme precision, in much the same way that Bailey et al. [4] resolved the Ising-class integrals  $C_n$  for dimension  $n \sim 1000$  to hundreds of decimals. In that previous work, a modified-Bessel kernel was employed in a 1-dimensional representation suitable for numerical quadrature. In our present case, an error-function kernel is appropriate. These high-precision quadratures have motivated some conjectures and subsequent proofs of same.

## 2 Dimensional reduction via vector-field calculus

It turns out that a box integral  $X_n(s, \vec{q})$  can be reduced to a suitable integral over the *faces* of a displaced  $n$ -cube, in some instances reducible yet further to edges, and so on. Let us write

$$X_n(s, \vec{q}) = \int_{\vec{r} \in \mathcal{C}} r^s \mathcal{D}\vec{r},$$

where  $r := |\vec{r}|$  and the integration is over a translated cube

$$\mathcal{C} := [0, 1]^n - \vec{q}.$$

We may then invoke an elegant procedure from mathematical physics; namely, we attempt to write the (radially symmetric) integrand  $r^s$  as the Laplacian of a scalar field. That is, we seek a function  $\Phi$  of position, such that

$$\nabla^2 \Phi(\vec{r}) = r^s.$$

A radially symmetric solution will satisfy the radial part of the Laplacian relation, as

$$\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \right) = r^s,$$

whence there is a solution satisfying

$$\frac{\partial \Phi}{\partial r} = \frac{1}{n+s} r^{s+1}.$$

The point of these machinations is that we may now utilize the divergence theorem for vector fields, in the form<sup>1</sup>

$$\int_R \nabla \cdot \vec{\mathcal{F}} \mathcal{D}\vec{r} = \int_{\partial R} \vec{\mathcal{F}} \cdot \mathcal{D}\vec{a},$$

where  $\vec{\mathcal{F}}(\vec{r})$  is a vector field, the left-hand integral is over the interior of a region  $R$ , the right integral is over the boundary, with  $\mathcal{D}\vec{a}$  denoting an area element with vector direction always normal to the surface.

The next step is to consider the vector field defined  $\vec{\mathcal{F}} := \nabla \Phi$ . Using the above observations, we conclude

$$X_n(s, \vec{q}) = \frac{1}{n+s} \int_{\vec{r} \in \partial \mathcal{C}} r^s \vec{r} \cdot \hat{\mu} da \quad (7)$$

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<sup>1</sup>Known classically as the Gauss theorem for vector fields, this integral relation is ubiquitous in electrostatics and hydrodynamics.

where  $da$  is the surface element with normal unit vector  $\hat{\mu}$ . Note that we have hereby reduced the box integral to an integral over the *faces* of a certain, displaced unit cube. For the box integrals  $B_n(s)$ , so we have offset  $\vec{q} = (1/2, \dots, 1/2)$ , we realize there are  $2n$  symmetrically situated faces, and our results boils down to the dimensional-reduction relation

$$B_n(s) = \frac{n}{n+s} \int_{\vec{r} \in [0,1]^{n-1}} (r^2 + 1)^{s/2} \mathcal{D}\vec{r}. \quad (8)$$

So for example the 2-dimensional case reduces to a 1-dimensional integral and a final hypergeometric evaluation.

$$\begin{aligned} B_2(s) &:= \int_{\vec{r} \in [0,1]^2} r^s \mathcal{D}\vec{r} = \frac{2}{2+s} \int_0^1 (x^2 + 1)^{s/2} dx \\ &= \frac{2}{2+s} {}_2F_1\left(\frac{1}{2}, -\frac{s}{2}; \frac{3}{2}; -1\right). \end{aligned} \quad (9)$$

This hypergeometric entity is rational when  $s$  is a nonnegative even integer, and evidently is always a surd plus the log of a surd for  $s$  a nonnegative odd integer (see Section 7 for some closed forms).

For the 3-dimensional case, we are able to reduce one further dimension by employing, after the first reduction step from (8), a 2-dimensional solution to

$$\nabla^2 \Phi = (r^2 + 1)^{s/2},$$

which solution has the property

$$r \frac{\partial \Phi}{\partial r} = \frac{(r^2 + 1)^{s/2} - 1}{s + 2},$$

to get a 1-dimensional representation, like so:

$$\begin{aligned} B_3(s) &= \frac{3}{3+s} \int_{\vec{r} \in [0,1]^2} (r^2 + 1)^{s/2} \mathcal{D}\vec{r} \\ &= \frac{3}{3+s} \frac{2}{2+s} \int_0^1 \frac{(y^2 + 2)^{s/2} - 1}{y^2 + 1} dy \\ &= \frac{6}{(3+s)(2+s)} \left( -\frac{\pi}{4} + \int_0^{\pi/4} (1 + \sec^2 t)^{s/2+1} dt \right). \end{aligned} \quad (10)$$

As with the cases  $B_2(s)$ , these  $B_3(s)$  do enjoy some closed forms, as in Section 7.

### 3 Error-function formalism and combinatorics

We have seen that an  $n$ -dimensional box integral can be reduced by at least one dimension. It turns out that, for numerical quadrature applications, one may achieve a *one*-dimensional integral representation of either  $B_n$  or  $\Delta_n$ . The procedure runs as follows.<sup>2</sup> We start with a certain representation of complex powers:

$$z^\rho = \frac{\rho}{\Gamma(1-\rho)} \int_0^\infty t^{-\rho-1} (1 - e^{-tz}) dt, \quad (11)$$

valid for  $\Re(z) > 0$  and  $\Re(\rho) \in (0, 2)$ . We next define two key functions

$$b(u) := \int_0^1 e^{-u^2 x^2} dx = \frac{\sqrt{\pi} \operatorname{erf}(u)}{2u}, \quad (12)$$

$$d(u) := \int_0^1 \int_0^1 e^{-u^2(x-y)^2} dx dy = \frac{-1 + e^{-u^2} + \sqrt{\pi} u \operatorname{erf}(u)}{u^2}. \quad (13)$$

Now, the defining integrals (2) and (3), and the representation (11), lead to 1-dimensional integrals for each of  $B_n, \Delta_n$ , like so:

$$B_n(s) = \frac{s}{\Gamma(1-s/2)} \int_0^\infty \frac{du}{u^{s+1}} (1 - b(u)^n), \quad (14)$$

$$\Delta_n(s) = \frac{s}{\Gamma(1-s/2)} \int_0^\infty \frac{du}{u^{s+1}} (1 - d(u)^n), \quad (15)$$

both of which being convergent integrals for  $\Re(s) \in (0, 2)$ . Incidentally, these integrals prove immediately that both  $B_n, \Delta_n$  for any fixed real  $s$  are monotonic increasing in  $n$ .

We discuss the issue of numerical quadrature of these error-function representations later. For the moment, we give relevant series developments, as these, relevant to computations on  $B_N$ :

$$b(u) = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{k!(2k+1)} = e^{-u^2} \sum_{k=0}^{\infty} \frac{2^k u^{2k}}{(2k+1)!!}, \quad (16)$$

and these for  $\Delta_n$  work:

$$\begin{aligned} d(u) &= \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{(k+1)!(2k+1)} \\ &= e^{-u^2} \sum_{k=0}^{\infty} u^{2k} \left( \frac{2^{k+1}}{(2k+1)!!} - \frac{1}{(k+1)!} \right). \end{aligned} \quad (17)$$

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<sup>2</sup>The present authors developed this technique for  $B_n$ , with a view to extreme-precision quadrature and subsequent experimental mathematics. We found later that M. Trott had previously applied a similar approach for the  $\Delta_n$  [11]. In a sense, the present treatment is an attempt at unification of the ideas, for more general box integrals.

We discuss in Section 8 how these series play key roles in numerical quadrature. For the moment, we analyze properties of the 1-dimensional integral representations (14), and (15). Important relations along these lines will be these two, where coefficients  $\beta_{nk}, \delta_{nk}$  are implicitly defined

$$\left(b\left(\sqrt{t/2}\right)e^{t/2}\right)^n =: \sum_{k \geq 0} \beta_{nk} t^k, \quad (18)$$

$$\left(d\left(\sqrt{t/2}\right)e^{t/2}\right)^n =: \sum_{k \geq 0} \delta_{nk} t^k. \quad (19)$$

Inserting these into (14, 15) we obtain two formal series:

$$B_n(s) = n^{s/2} \sum_{k=0}^{\infty} \left(\frac{2}{n}\right)^k (-s/2)_k \beta_{nk}, \quad (20)$$

$$\Delta_n(s) = n^{s/2} \sum_{k=0}^{\infty} \left(\frac{2}{n}\right)^k (-s/2)_k \delta_{nk}, \quad (21)$$

where  $(z)_k$  is the *Pochhammer symbol*<sup>3</sup>. Though developed formally, with regard for convergence issues, it can be shown that each series here converges absolutely whenever  $\Re(s) + n > 0$ .

To sketch the convergence argument, we initially focus on combinatorial relations for the  $\beta_{nk}$  (the analysis for the  $\Delta_n$  series is similar). An elementary observation is in order. First, for  $s = 2m$  with  $m$  a nonnegative integer, the box integral  $B_n(2m)$  can always be written as a finite combinatorial sum of rational components, via simple expansion of the defining integrand. Equivalently, series (20) devolves for  $s = 2m$  an even integer, into a finite sum

$$\begin{aligned} B_n(2m) &:= \int_{\vec{r} \in [0,1]^n} (r_1^2 + \dots + r_n^2)^m \mathcal{D}\vec{r} \\ &= n^m \sum_{k=0}^m (-m)_k \left(\frac{2}{n}\right)^k \beta_{nk}. \end{aligned} \quad (22)$$

This representation of  $B_n$  at even integers will prove quite useful in further analysis. Next, stemming from the implicit definition (18) one can derive various relations, the first of which being a beautiful reciprocal relation with the finite sums  $B_n(2m)$ :

$$\beta_{nk} = \frac{n^k}{2^k k!} \sum_{j=0}^k \binom{k}{j} \left(\frac{-1}{n}\right)^j B_n(2j). \quad (23)$$

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<sup>3</sup>The Pochhammer symbol  $(z)_k := z(z+1)\cdots(z+k-1)$  is extended, for  $z$  not a positive integer, by  $(z)_k := \Gamma(m-z)/\Gamma(1-z)$ , and for all  $z$  we define  $(z)_0 := 1$ .



Other derivable relations are

$$\beta_{nk} = \frac{n^k}{2^k k!} \int_{\vec{r} \in [0,1]^n} \left(1 - \frac{r^2}{n}\right)^k \mathcal{D}\vec{r}, \quad (24)$$

$$\beta_{nk} = \sum_{k_1 + \dots + k_n = k} \frac{1}{(2k_1 + 1)!!} \cdots \frac{1}{(2k_n + 1)!!},$$

$$\beta_{nk} = \sum_{j=0}^k \frac{1}{(2k + 1)!!} \beta_{n-1, k-j}. \quad (25)$$

In checking all of these combinatorial relations, it is convenient to know some “starting cases.” We define  $\beta_{n0} := 1$  if  $n = 0$ , else 0, and note

$$\begin{aligned} \beta_{1k} &= \frac{1}{(2k + 1)!!}, & \beta_{n1} &= \frac{n}{3}, & \beta_{n2} &= \frac{n^2}{18} + \frac{n}{90}, \\ B_n(0) &= 1, & B_n(2) &= \frac{n}{3}, & B_n(4) &= \frac{n^2}{9} + \frac{4n}{45}, \end{aligned}$$

and so on. Now to the convergence issue for the general expansion (20). From relation (24) one can show

$$\beta_{nk} \leq \frac{n^k}{2^k k!} \max(1, (n/k)^{n/2}),$$

and one has for the relevant Pochhammer symbol

$$|(-s/2)_k| = O(k! k^{-1 - \Re(s/2)}).$$

Thus the  $k$ -th summand in (20) is  $O(1/k^{n + \Re(s) + 1})$  and absolute convergence obtains whenever  $n + \Re(s) > 0$ .

Using the above analysis for the general series (20)—and after a similar analysis for (21)—we see that several results accrue. We obtain convenient expansions for the even-argument  $B_n(2m)$ ,  $\Delta_{2m}$ , and an analytic continuation at least for  $n + \Re(s) > 0$ . There are various additional inferences we may pursue, such as asymptotic behavior (see Section 6), but first we shall describe a more powerful analytic continuation—and more rapidly converging general series—for the  $B_n$  in particular.

## 4 Analytic continuations

Remarkably, and perhaps surprisingly, the relation (8) actually leads to a rapidly (linearly) converging general series for  $s$ , and a subsequent analytic continuation to *all* complex  $s$ .

Indeed, using the ideas behind (14) we can infer from the dimensional-reduction formula (8) that

$$B_n(s) = \frac{n}{n+s} \frac{s}{\Gamma(1-s/2)} \int_0^\infty \frac{du}{u^{s+1}} \left(1 - e^{-u^2} b(u)^{n-1}\right), \quad (26)$$

leading, after term-by-term integration as before, to an efficient general expansion

$$B_n(s) = n^{s/2} \frac{n}{n+s} \sum_{k=0}^{\infty} \left(\frac{2}{n}\right)^k (-s/2)_k \beta_{n-1,k}. \quad (27)$$

The rather innocent-looking modifications here over the generally slower series (20) give a much more efficient series. Indeed, since

$$\left(\frac{2}{n}\right)^k (-s/2)_k \beta_{n-1,k} = O\left(\left(1 - 1/n\right)^k k^{-1 - \Re(s/2) - n/2}\right)$$

the sum in the general series (27) is linearly convergent for fixed  $n$ . Thus, (27) *provides an analytic continuation of  $B_n$  to all complex  $s$  except for a simple pole at  $s = -n$ .*

It is not hard to see how analytic continuation works for the box integrals  $B_n$ . Take the trivial scenario of  $n = 1$  dimension. Then, formally,  $B_1(s) = \int_0^1 x^s dx = 1/(s+1)$  and though the integral diverges for  $s = -1$ , the analytic continuation of  $B_1$  is the function  $1/(s+1)$ . The same kind of thinking reveals that in  $n$  dimensions, the integrand  $r^s$  does diverge for  $s = -n$ ; yet, there is an analytic continuation to finite  $B_n$  values at any other  $s$ . An example of a continued value—when the literal integral of  $r^s$  is infinite—is

$$B_4(-5) = -0.96120393268995345712165978002474521286412992715\dots,$$

which could well have a closed form but we do not know it; this approximate value was obtained from the series (27). Note in this regard that our previous, hypergeometric-like reductions (9, 10) for  $B_2(s), B_3(s)$  respectively are *already* in analytic continuation form.

There is another way to obtain an efficient series and subsequent continuation, which is foreshadowed by the statistical work in [1] working exclusively with  $B_n(1)$ . Within our present formalism we can generalize to arbitrary  $s$  by contemplating the expectation

$$\langle r^s \rangle_{\vec{r} \in [0,1]^n} = \frac{n}{n+s} \left\langle \left( (1 + (n-1)/2) + (r^2 - (n-1)/2) \right)^{s/2} \right\rangle_{\vec{r} \in [0,1]^{n-1}},$$

where we have written  $1 + r^2$  in an intentionally cumbersome way in order to invoke the binomial theorem for power  $s/2$ . After manipulation, we obtain a very efficient series

$$B_n(s) = \left(\frac{n+1}{2}\right)^{s/2} \frac{n}{n+s} \sum_{k=0}^{\infty} \binom{s/2}{k} \frac{\alpha_{n-1,k}}{(n+1)^k}, \quad (28)$$

where the new  $\alpha$ -coefficients are defined

$$\alpha_{nk} := 2^k \sum_{j=0}^k \binom{k}{j} \left(\frac{1-n}{2}\right)^{k-j} B_n(2j).$$

Incidentally, in computations involving the series (28) it is useful to know, as a simple consequence of (8), that

$$B_n(2m) = \frac{n}{n+2m} \sum_{k=0}^m \binom{m}{k} B_{n-1}(2k).$$

In this way, numerical evaluation of (28) becomes an exercise in the use of recursion relations.

Again we have a convergent series for all complex  $s$  except for the pole at  $s = -n$ ; indeed (28) appears to be the fastest converging series we have, although (27) has certain practical features, such as the appearance of the  $\beta$ -terms which in turn can be evaluated via fast convolution from (25).

## 5 “Jellium” physics and box integrals

Given an  $n$ -cube of uniformly charged jelly of total charge  $+1$ , what is the electrostatic potential of an electron (having charge  $-1$ ) at the cube center? This question cannot be answered until we settle on suitable potentials in  $n$  dimensions. One possibility—which we hereby adopt—is to take the radial potential at distance  $r$  from the electron-center as  $V_n(r)$ , where

$$V_1(r) := r - 1/2, \tag{29}$$

$$V_2(r) := \log(2r), \tag{30}$$

$$V_n(r) := 2^{n-2} - \left(\frac{1}{r}\right)^{n-2}, \quad n > 2. \tag{31}$$

These potentials a) satisfy the Laplace equation in  $n$  dimensions, and b) vanish on any face-center. (We are free in electrostatic theory to give any potential a constant offset.)

Let us then define the  $n$ -th *jellium potential* as

$$J_n := \langle V_n(r) \rangle_{\vec{r} \in [-1/2, 1/2]^n}.$$

Interestingly, every  $J_n$  except  $J_2$  is essentially—up to offset—a box integral. We can dispense with exact evaluations for  $n = 1, 2$  (see Section 7), and observe that

$$J_n = 2^{n-2}(1 - B_n(2-n)), \quad n > 2. \tag{32}$$

Because the general jellium potential involves negative parameter  $s = 2 - n$  for  $n > 2$ , we are moved to use one of (20, 27, 28) for evaluation. However, reminiscent of relations (14), (15) one may derive an additional error-function representation

$$B_n(s) = \frac{2}{\Gamma(-s/2)} \int_0^\infty \frac{du}{u^{s+1}} b(u)^n, \quad (33)$$

valid for  $\Re(s)$  in the negative interval  $(-n, 0)$ .

## 6 Intuition via the central limit theorem

The *central limit theorem* of classical probability theory tells us that in some appropriate sense, the distribution of the random variable  $\chi := \vec{r} \cdot \vec{r}$  over  $\vec{r} \in [0, 1]^n$  is Gaussian-normal, with mean and variance

$$\begin{aligned} \langle \chi \rangle &= \frac{n}{3}, \\ \langle (\chi - \langle \chi \rangle)^2 \rangle &= \langle (r^2 - n/3)^2 \rangle = \frac{4n}{45}. \end{aligned}$$

Heuristically speaking, then, we should have

$$B_n(s) \sim_n \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} (n/3 + x)^s e^{-x^2/(2v)} dx,$$

where  $v := 4n/45$ . Interestingly, even though error terms in central-limit expansions can be problematic, binomial development of the integrand gives

$$B_n(s) \sim_n \left(\frac{n}{3}\right)^{s/2} \left(1 + \frac{s(s-2)}{10n} + \dots\right).$$

This agrees, at least through the first two parenthetical terms, with the proven asymptotic of Anderssen et al. [1] for their case  $s = 1$ .

Still thinking statistically and heuristically, there is another intriguing way to infer that  $B_n(s) \sim_n (n/3)^{s/2} (1 + c/n)$  for constant  $c$ , which is to rewrite (24) in the form

$$\beta_{nk} = \frac{n^k}{3^k k!} \int_{\vec{r} \in [0,1]^n} \left(1 - \frac{3r^2 - n}{2n}\right)^k \mathcal{D}\vec{r}, \quad (34)$$

where we note that there is now a  $3^k$  in the denominator. Evidently, then, the  $\beta$ -coefficient is seen to depend on moments  $\mu_m := \langle (r^2 - n/3)^m \rangle$ ; additionally we have  $\mu_0 = 1, \mu_1 = 0, \mu_2 = 4n/45$ , and generally  $\mu_m = O(1 + m^2/n)$ . Now these estimates for  $\mu$ , when inserted into the converging series (20), can be seen to give the desired asymptotic. We have not made this argument rigorous; however, relation (34) is promising in this regard.

## 7 Various closed forms

We next state some known closed and nearly closed forms. The nearly-closed forms engage four unresolved integrals which are given numerically to high precision in Appendix 1.

### 1. Box integrals as expectations of distance—or inverse-distance-from-vertex:

$$\begin{aligned}
 B_2(-1) &= \log(3 + 2\sqrt{2}), \\
 B_3(-1) &= -\frac{\pi}{4} - \frac{1}{2} \log 2 + \log(5 + 3\sqrt{3}), \\
 B_1(1) &= \frac{1}{2}, \\
 B_2(1) &= \frac{\sqrt{2}}{3} + \frac{1}{3} \log(\sqrt{2} + 1), \\
 B_3(1) &= \frac{\sqrt{3}}{4} + \frac{1}{2} \log(2 + \sqrt{3}) - \frac{\pi}{24}, \\
 B_4(1) &= \frac{2}{5} + \frac{7}{20} \pi \sqrt{2} - \frac{1}{20} \pi \log(1 + \sqrt{2}) + \log(3) - \frac{7}{5} \sqrt{2} \arctan(\sqrt{2}) + \frac{1}{10} \mathcal{K}_0,
 \end{aligned}$$

where the one unresolved term, namely

$$\mathcal{K}_0 := \int_0^1 \frac{\log(1 + \sqrt{3 + y^2}) - \log(-1 + \sqrt{3 + y^2})}{1 + y^2} dy = 2 \int_0^1 \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{3 + y^2}}\right)}{1 + y^2} dy, \quad (35)$$

is a dilogarithm-like entity that can be evaluated reasonably rapidly, via the 2-dimensional sum

$$\mathcal{K}_0 = \sum_{m,k \geq 0} \frac{2^{k+1}}{2m+1} \frac{I_{m+k}}{3^{m+k+1}} = \frac{2}{3} \sum_{p=0}^{\infty} I_p \left(\frac{2}{3}\right)^p \sum_{n=0}^p \frac{1}{2^n (2n+1)},$$

where  $I_0 := 1/2$  and

$$I_m = \frac{1}{2m+1} \left\{ 2m I_{m-1} + \left(\frac{3}{4}\right)^m \frac{1}{2} \right\}.$$

Now  $\mathcal{K}_0$  can also be recast from this sum in a form revealing more obviously a linear convergence (essentially, by powers of  $(2/3)$ ):

$$\mathcal{K}_0 = \sum_{n=1}^{\infty} \beta\left(\frac{1}{2}, n\right) \kappa_n \left(\frac{2}{3}\right)^n - \frac{1}{2} \sum_{n=1}^{\infty} {}_2F_1\left(1, n + \frac{1}{2}, n + 1, \frac{3}{4}\right) \kappa_n \left(\frac{1}{2}\right)^n$$

where

$$\kappa_n := \sum_{m=1}^n \frac{1}{(2m-1)2^m}.$$

The expression for  $B_4(1)$  results from two dimension reductions followed by substantial symbolic computation with the remaining two-dimensional integrals, all of which ultimately resolved—via dilogarithms—except for  $\mathcal{K}_0$ .

**2. Average distance**—or inverse-distance between two points:

$$\Delta_2(-1) = \frac{2}{3} - \frac{4}{3}\sqrt{2} + 4\log(1 + \sqrt{2}),$$

$$\Delta_1(1) = \frac{1}{3},$$

$$\Delta_2(1) = \frac{1}{15} \left( 2 + \sqrt{2} + 5\log(1 + \sqrt{2}) \right),$$

$$\Delta_3(1) = \frac{4}{105} + \frac{17}{105}\sqrt{2} - \frac{2}{35}\sqrt{3} + \frac{1}{5}\log(1 + \sqrt{2}) + \frac{2}{5}\log(2 + \sqrt{3}) - \frac{1}{15}\pi,$$

$$\begin{aligned} \Delta_4(1) &= \frac{26}{15}G - \frac{34}{105}\pi\sqrt{2} - \frac{16}{315}\pi + \frac{197}{420}\log(3) + \frac{52}{105}\log(2 + \sqrt{3}) \\ &+ \frac{1}{14}\log(1 + \sqrt{2}) + \frac{8}{105}\sqrt{3} + \frac{73}{630}\sqrt{2} - \frac{23}{135} + \frac{136}{105}\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right) \\ &- \frac{1}{5}\pi\log(1 + \sqrt{2}) + \frac{4}{5}\alpha\log(1 + \sqrt{2}) - \frac{4}{5}\text{Cl}_2(\alpha) - \frac{4}{5}\text{Cl}_2\left(\alpha + \frac{\pi}{2}\right). \end{aligned}$$

$$\begin{aligned} \Delta_5(1) &= \frac{65}{42}G - \frac{380}{6237}\sqrt{5} + \frac{568}{3465}\sqrt{3} - \frac{4}{189}\pi - \frac{449}{3465} - \frac{73}{63}\sqrt{2}\arctan\left(\frac{\sqrt{2}}{4}\right) - \frac{184}{189}\log(2) \\ &+ \frac{64}{189}\log(\sqrt{5} + 1) + \frac{1}{54}\log(1 + \sqrt{2}) + \frac{40}{63}\log(\sqrt{2} + \sqrt{6}) - \frac{5}{28}\pi\log(1 + \sqrt{2}) \\ &+ \frac{52}{63}\pi\log(2) + \frac{295}{252}\log(3) + \frac{4}{315}\pi^2 + \frac{3239}{62370}\sqrt{2} - \frac{8}{21}\sqrt{3}\arctan\left(\frac{1}{\sqrt{15}}\right) \\ &- \frac{52}{63}\pi\log(\sqrt{2} + \sqrt{6}) + \frac{5}{7}\alpha\log(1 + \sqrt{2}) - \frac{5}{7}\text{Cl}_2(\alpha) - \frac{5}{7}\text{Cl}_2\left(\alpha + \frac{\pi}{2}\right) + \frac{52}{63}\mathcal{K}_1, \end{aligned}$$

where the unresolved quantity is the integral is

$$\mathcal{K}_1 := \int_3^4 \frac{\text{arcsec}(x)}{\sqrt{x^2 - 4x + 3}} dx, \quad (36)$$

and where  $\alpha := \arcsin(2/3 - 1/6\sqrt{2})$ ,  $G$  is the *Catalan constant*,  $\text{Cl}_2$  is the order-2 *Clausen function*,  $\Psi_n$  is the order- $n$  *polygamma function*, and  $\text{Li}_n$  is the *polylogarithm function*—see [7] or [8] for details.

The evaluations for  $\Delta_4(1)$  and  $\Delta_5(1)$  come from taking those given in [11] and then (i) carefully eliminating dependent terms (which often entails reexpressing logarithms and polylogarithms) and (ii) using *Kummer's formula* [8, eq. (5.5)] to express the remaining polylogarithms as Clausen functions.

### 3. Jellium potentials vs. dimension:

$$\begin{aligned}
J_1 &= -\frac{1}{4}, \\
J_2 &= -\frac{3}{2} + \frac{\pi}{4} + \frac{1}{2} \log 2, \\
J_3 &= 2 + \frac{1}{2} \pi - 3 \log(2 + \sqrt{3}), \\
J_4 &= 4 + \pi^2 + 8G - \frac{4}{3} \ln(2 + \sqrt{3}) \pi - 4 \text{Cl}_2\left(\frac{1}{6} \pi\right) - 4 \text{Cl}_2\left(\frac{5}{6} \pi\right) - 16 \mathcal{K}_2 \\
J_5 &= 8 - \frac{5}{3} \pi^2 - 20 \ln(2 + \sqrt{3}) \pi + 80 \mathcal{K}_3.
\end{aligned}$$

The unresolved quantities in the above are given by

$$\begin{aligned}
\mathcal{K}_2 &:= \int_0^{\pi/4} \sqrt{1 + \sec^2(a)} \arctan\left(\frac{1}{\sqrt{1 + \sec^2(a)}}\right) da \\
&= \frac{\pi^2}{16} - \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \sum_{k=1}^{m-1} \frac{\binom{-(m-k)}{k}}{2(m-k)+1} \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \\
&= 1 - \frac{\pi}{4} + \frac{\pi^2}{16} - \sum_{N=1}^{\infty} \left(\frac{1}{2}\right)^N \sum_{n=1}^N \binom{N-1}{n-1} \sum_{m=1}^{N-1} \frac{(-1)^{n+m}}{(2m+1)(2n+1)} \\
&\quad + \sum_{N=1}^{\infty} \left(\frac{1}{2}\right)^N \sum_{n=1}^N \binom{N-1}{n-1} \sum_{m=1}^n \frac{(-1)^{n+N-m}}{(2N-2m+1)(2n+1)} \tag{37}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}_3 &:= \int_0^{\pi/4} \int_0^{\pi/4} \sqrt{1 + \sec^2(a) + \sec^2(b)} da db \tag{38} \\
&= -\sqrt{3} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}/(2n-1)}{12^n} \sum_{k=0}^n \binom{n}{k} \left(\frac{\pi}{4} - \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1}\right) \left(\frac{\pi}{4} - \sum_{j=0}^{n-k-1} \frac{(-1)^j}{2j+1}\right).
\end{aligned}$$

The value of  $J_4$  was obtained from (8). The even-more partial expansion of  $J_5$  was likewise obtained from two polar transformations.

Note again that the dimensions  $n = 1, 2$  have special status, as per (29), (30) and (31).

## 8 Extreme-precision quadrature

Using the 1-dimensional integral representations (14), (15) and (33), we were able to generate extreme-precision values of  $B_n := B_n(1)$ ,  $\Delta_n := \Delta_n(1)$  and  $B_n(2 - n)$ , respectively, for a selection of  $n$ . Note that  $J_n$  can be readily and accurately evaluated from  $B_n(2 - n)$  by using (32). These numerical values are given explicitly in Appendix 1, together with values for the unresolved integrals  $\mathcal{K}_n$  for  $n = 0, 1, 2$  and  $3$ , which we computed using (35), (36), (37) and (38), respectively.

These integrals were computed using the tanh-sinh quadrature scheme. Tanh-sinh quadrature is remarkably effective in evaluating integrals to very high precision, even in cases where the integrand function has an infinite derivative or blow-up singularity at one or both endpoints. It is well-suited for highly parallel evaluation [2], and is also amenable to computation of provable bounds on the error [3]. It is based on the transformation  $x = g(t)$ , where  $g(t) = \tanh[\pi/2 \cdot \sinh(t)]$ . In a straightforward implementation of the tanh-sinh scheme, one first calculates a set of *abscissas*  $x_k$  and *weights*  $w_k$

$$\begin{aligned} x_j &:= \tanh[\pi/2 \cdot \sinh(jh)] \\ w_j &:= \frac{\pi/2 \cdot \cosh(jh)}{\cosh^2[\pi/2 \cdot \sinh(jh)]}, \end{aligned}$$

where  $h$  is the interval of integration. Then the integral of the function  $f(t)$  on  $[-1, 1]$  is performed as

$$\int_{-1}^1 f(x) dx = \int_{-\infty}^{\infty} f(g(t))g'(t) dt \approx \sum_{-N}^N w_j f(x_j)$$

where  $N$  is chosen so that the terms  $w_j f(x_j)$  are sufficiently small that they can be ignored for  $j > N$ . Full details of a robust implementation are given in [6].

Computing  $B_n$  using (14) requires one to perform two integrals, one with the integrand function  $f(u) = (1 - (\sqrt{\pi}/(2u) \cdot \operatorname{erf}u)^n)/u^{s+1}$ , from 0 to 1, and a second integral of  $f(1/u)/u^2$ , from 0 to 1. Adding the two together gives the integral from 0 to  $\infty$ . Computing these integrals is complicated by the fact that in tanh-sinh quadrature, the integrand function must be evaluated to high precision very close to the endpoints. In this case, it is not sufficient just to compute erf to high relative precision near zero; because of the subtractions here, one must use a Taylor series expansion for the integrand function when the argument is within say  $10^{-10}$  of zero. Computing these Taylor series coefficients (which we did using *Mathematica*) turned out to be the most expensive part of the entire computation. Once highly accurate integrand functions were available, the quadrature evaluations for  $B_n$  were completed in less than one minute each.

Computing  $\Delta_n$  using (15) also required Taylor series expansions, at least for the first of the two integrals to be performed. Again, obtaining these Taylor series coefficients turned



out to be the most expensive aspect of the computation. Computing the  $B_n(2 - n)$  integrals required no Taylor series expansions and was completely straightforward.

Computing  $\mathcal{K}_0$ ,  $\mathcal{K}_1$  and  $\mathcal{K}_2$  using (35), (36) and (37) was relatively straightforward. However, computing  $\mathcal{K}_3$  using (38) requires 2-dimensional quadrature. We were able to do this by a straightforward extension of the 1-dimensional tanh-sinh scheme to two dimensions. However, because many times more function evaluations are required, the run time was correspondingly longer—four hours, as opposed to a few seconds for the others. We also computed  $\mathcal{K}_3$  using the nested infinite series given just following (38), but this required even more run time. The two numerical values, however, agreed.

## 9 Open problems

- Can the jellium potential  $J_3$  be generalized for different offset vectors (but still in a 3 dimensional setting), to yield via summation the true jellium potential due to an *infinite* cube of charged jelly?

This leads to the intriguing research area of obtaining other Madelung and Wigner lattice sums via box integrals with changing offset. Note the fixed point  $\vec{q}$  in the very definition of  $X_n$  does *not* have to be within the unit cube.

- What is the precise asymptotic behavior of  $\Delta_n(s)$ ?
- The authors of the original treatment [1] pointed out that a series of the type (28) does not seem to be available even for their parameter case of  $s = 1$ . Nor do we know presently how to convert (27) for the  $\Delta_n$  problems. We do have (21) which converges, albeit slowly. So, what is a rapidly converging series for  $\Delta_n$ ?
- How can (7) be used to reduce the dimension—in a convenient way—for some specific  $\Delta_n$ ? We say “convenient” because the many symmetries of the  $B_n$  cases allowed us to make practical use of (8).
- How can (34) be used to establish a precise asymptotic expansion for  $B_n(s)$ ? The original reference [1] perhaps contains sufficient clues.
- Which of the  $\mathcal{K}_i$  integrals can be further or completely resolved?

## 10 Appendix 1: Numerical values

We give here some numerical values, to 400-digit precision, for  $B_n$ ,  $\Delta_n$  and  $B_n(2 - n)$ , using formulas (14), (15) and (33), respectively, and via methodology described in Section 8. Note that  $J_n$  can be readily and accurately calculated from  $B_n(2 - n)$  – see (32). These constants are followed by 400-digit values for  $\mathcal{K}_0$ ,  $\mathcal{K}_1$ ,  $\mathcal{K}_2$ ,  $\mathcal{K}_3$ . Note that closed forms have already been presented in Section 7 for several instances from each class.

B\_3:

0.9605919564550529594251079513938063602409769075457239876908985153103876633401  
632890312279356917748245312164166986007359737684845127383162930777389088732003  
706911770347511241028894972718344134849125281638669503920734964634783372014360  
360629072371339141477787290367440166360494296109519496758260098146943820205650  
037755329676853665979934600683189270145391844972059784188539292526013795172127

B\_4:

1.1218996187158609773516151755675427092008079564395458308367924669164035486069  
153490246731455786376449763403760041726801510115007963136975718767737785523598  
132759365746125169205517189790894487626923594342121547934391865401675773455401  
521775713691478521517176414143131711306778574325608937643218519398649169489909  
522948451157118892878523150581712760768255084802396618535520861420213653932593

B\_5:

1.2624066350362518542778971151593382082288086573637362914268783135965228864710  
809323680157231830034604534204448310384862710340195052011729749887345427570065  
082501592539413681262968331488923311762549603401740793121456379069820698694973  
075038600753175819644310767381624580201045649818667252811032715595551399163489  
446756444073252195468470480314321829137342131214283170243933102113352962316806

B\_6:

1.3885740844573478425302540730307888159109450887822070297589331397626378969376  
828857918435778645157478203781995703617965136256197518532843932099952105974084  
999316093873716806095766615617270764496948791352915099606840664925798852391235  
786066874469900136908824049172113121733139469824376583463293684849179984891522  
368344310820087517017632357629765266938986058767300542248274361017926206913592

B\_8:

1.6112735561015292481937545071626850017609920366064321723256663240164844378504  
962946572519522653496438315265416767418629447113251844391541467333878599346677  
066515894137486870981831511946441509631255223604133732543900168933939873826511  
382292244259867993717103648583798726351597191384821216795984755606711335615215  
100056496138785454851984879009456520115101872216250264779767705574683701428346

B\_16:

2.2945271609776747896246757146222629228303704963642261093084540978512791308021  
437762549837382632049769176853471487876775346557608751869986721449421077926436

101255160954280724924269238087876292596192727163208869105360991836289642801466  
401272049627648229562462148709890291831957062252771599892062187844719633706915  
710155172405268837114099536129195015086837245340910787049761085749867136336110

B\_32:

3. 2556283475032900815009371018742820445128021157075055493354242041453691108454  
763793956202278954768649170992784275916789511134789482244391368377323724162418  
196361789538509553562126695610126536752389471590925092067646141659538445470545  
963004709429455213743629772131445753502591752295401706184582312009771147777407  
961622478252152274122141744140730906603946153335290222877695992364260102810263

B\_64:

4. 6115322767003246128305525097112933014472817159774601313045937775091789232449  
747359202696312418750090050344390288044712387410803388799972697196128275343571  
392625970820309617505081657139626489706423605106798986318873735068937645390508  
647994586029201698442886260464652603058580830010371732628827045822619832161887  
750889978483292913691008737527145552055442630959599251143130988503490398277913

B\_128:

6. 5268509205598172200508604171335034877898034509295274494798693329553663801008  
817895150543200275733511980807698314999096208532467803212795917985470058926083  
571385861160359728536434403116615724260549926218208205200402777816591505191330  
029095745468549696863328099670196135890376416301598632967462395378638474819055  
635751998541425875037456733584100038158396406071096705230642986394366702900431

B\_256:

9. 2339893036017979821813299560317564867759184854981270468525451313188138520846  
645147515212921977581551287520595738011766244530522680719248983179885564954351  
678524446709178250314847122442711343388381616728693944871815358430832564799333  
069267074687407255092374804523896145120970615296607468123763911942145394213127  
209730398544886001708341630756123541378173396548339451661490297352140299836668

B\_512:

13. 061391425743765374699689988825238836400628480849388076753408486929842136376  
312123831816040251049846788764704571430912641718909199764443175573104436620534  
740992015621847637895932628830925229110454520981940403611568915233145813780927  
437286664144716873378067302512765155025944025247165529560783697907304746045962  
261115313028133859985746883221884608151788554136622452316355350723098731990395

B\_1024:

18.473403575815224768394892605266856038204337522823578150467252017100071635212  
124293170103398091079082263366569845177540455130598719868405480868501124703919  
306722343865527853613544616020301541142229423057526514493758459077280030869091  
989606908946900162941152217494957040830364963218638677469330051513767345888894  
705197416642970965243018829790657940322084776757106515814924850571987373958832

Delta\_3:

0.6617071822671762351558311332484135817464001357909536048089442294795846461385  
976313066524807681071201517097753107594109724786805816437216874533242072298244  
423276409229206078600086480533266938951526942028215425692085403456100394606163  
834472771107263924054689743459232206969510457176785303874823891119488713091981  
047559429531205455891503267539401643933207902944734734790101329001545166600642

Delta\_4:

0.7776656535862671153379340946178195099628827244171330580234459648650573531592  
654011461516568931681884655034447831370951051591390267441809140438428998517624  
210232396857024987342411951696562691690582806984694082633284285876412443029864  
236559482249802453834733379102480251274813941020682776748739186655210948229736  
258759623285258102287734298695709044584888772916586825699031845023601414880863

Delta\_5:

0.8785309152674802771789167963362247495118969910897683251376162909611903799327  
949972543463494842970110990898249207706783047263811919449405451224631840451809  
715167996974471022016195484264140487939336031273898799813162066312813605600588  
534362709232451407189347366930101386684205359156108266436021195147574508781297  
948585440545276532589325909472353724500198989411976100590886941736128344150490

Delta\_6:

0.9689420830367467728060658946668998649945711532713998177258046433716722812520  
783658332238668906844993303437015022027291282267370210858131364266978886399883  
565536536487074414207107813430509386417558465440695243868933310873069164192705  
224174401309497623500010734590819535081373100362474067523492597730107954858514  
787605971380025005236671922834828496545558514833526001322020728950261679637232

Delta\_8:

1.1281653402412599846612009394532321854563251231539804097940714333981430537331  
000164158380650316945397064397038462788265030487689322652826650585865724660279

586798780716236652033742272779804200228289773702015019662201245169436721483503  
115379382642242930804186256869191401242132987772484464478700954069253424104663  
027590955571621766420091665288614036614629755969645709283880859098711181164994

Delta\_16:

1. 6146680158164524396585398092175603932999328397589143275304908277836544921047  
242112058902770868433426559163698862715874270432620530347314879415808260145815  
701821516180561108002203979182180621324988737049341612243216518884863909531807  
811624528053216201875642128824787218216892462543183805528242146112004504799071  
913063489404914006433166423335632609903796024777305437835420883234020300878316

Delta\_32:

2. 2966071387598634257091215228252877603275252461721809960501397417063958849102  
582680426664392340469815440508294297647158103265149363544074310728653513740596  
926145008546179815629717021943361252852446382451340265804886547908173602840869  
865445656430566957370483134174185065952444501831234088035970952691130820061342  
878702339508768418648293965733914939882526222315829093283975719264922522230702

Delta\_64:

3. 2569978703061582629113130570458302026732231791059539553462576793146704933110  
950379696597426638423150522793798218971061034534906534685159282691952539849076  
638012991744455493369451096222952081156129432426331297785649545487370191368689  
602403236455057021370205833619207381560789026838556247766772858364632190089314  
159705707376991264744467586166967623739728073489003152648164804263768099402476

Delta\_128:

4. 6124668995266662261327146890574713789476754267630572531665825113264923712008  
775751924127308239003707236195631370210588456717190034834428027897921547812234  
262103051475209516174198235912692454375348126019560488485290755004148746839797  
903426336886311580713828814189969498498368024109989437719583960037566311604138  
445589233412009351762157737887854134426028558427911248620077892739397409719239

Delta\_256:

6. 5275001948905222214786015823845510522953886103465739150365673291750121339601  
533574360517181319698297205610548945697713557067034260424212678994736833277734  
997555196634258491928424771895203515110586051483835893144497750311964600371373  
686491293743382444908121654329130136231929294863991927209344165856809401983117  
863547768072347508776719185657627973945516326755471528687654583856938527045317

Delta\_512:

9.234444365232834858908454787294859711673083249566318745723727247418740772571  
760903663327232671211644576829604443233960679766812133386147916951221402709516  
113233961926839299796907636386892743075688603018733523665769503599028670660625  
923168989356968874523683285358483090810651130235500756133359970800947883255058  
430768757550123746853458514617479226692811872559381497794859044914030353340161

B\_3(-1):

1.1900386819897767533219086751420769449911806073574982644089722373037361765531  
137144543198138396234083391611332200868923179412641407719117411588051665327607  
359961756674889008069793157448495583291788379845629923540847555446809479572771  
934623476322161550548542200537852188419384631072870740888098683693858630245580  
883325746704942692861162141888231736531968185754545526237096212805578129604969

B\_4(-2):

1.0717135155754604706708086778443375210172779320832659847301510617601797742918  
025928479805177071789996585886840849426361343769802899675126794784919625780854  
210333837425873688425508214010025545714063045391348224294643168231697825080225  
435282475698015416250011874216927303965529638404182110457986792008233316685244  
513238791005088281028544803215513071529445581445132214591977201405126716834985

B\_5(-3):

8.0012321133838410594859790093424520754091697421787928390283028883874828334915  
527911032446925213431866540122957252759629118145540072237559291499710396102833  
850950668087714103367221644783746150624235057271320941217304847330608442452002  
704421057032394368783248220382940304917501048562247134687225815446166814618720  
718616800490203110901044614877824149877226921950990355789999399140860844039385e-1

B\_6(-4):

5.1922068443349364720879982234783151043232263181242380771895160697611320027476  
050225434029434473410871685977389818688348366546211459575361138470884749121069  
448389643684872060064064510517468237817560349906131906641798818995802239824409  
967335617826962319786561710220145838407740088635635710634629343335546091273601  
980026329270202782401297689824491531047592107626188080572169970364630206892e-1

B\_8(-6):

1.5953780295854604168166455621857823551584293826352304839592865693680317588179  
884953536596814947773932557478441231873099732665411110190303180662932479426213

926593279193799140729367878638269592698582626441041895117682886800522808866564  
808077073991918446486735563591093167578452211242079094750794462239391216496502  
556724535724291487864577332850093536980397940583465609964981365958726338568e-1

B\_16(-14) :

1.0988310942155857566302294019454581398383308836157279977360943219308669801105  
750878353755889372508456049563895222915645920364314687434139463609694218162488  
037319178287784924970395955238553599562273056425130078592165445655669439732608  
855150377505905112524377065156554977938617786974773146509050262639906127983995  
795501125811421439342086214922710867782214870784569566786842653399962525738e-4

B\_32(-30) :

9.7046001415525909361604981009971061122931328238789176398236263636346557572500  
476038357558248166910798376364722034996142541924602216899067452196217240240603  
706332512008366517960190083600368804801758296884027661051538308204071970419842  
349760366073130108621385565950710421011748826921807462892893823325264144048172  
60517622071767843973251017872032579544956420719087223700447388089168620245e-14

B\_64(-62) :

5.2912118345129707451288801969572510798557499233220704679746395572024722841683  
302494313684987388920809741972596854486445926026311421895783770738322034554293  
611647210069108553770408421573187557920294069987892998116901522986601868740666  
746899079921121744124974492649694973248723594478550040674057069230070034238498  
53240962642528517172345031405880358961178660966838577191047552487926167523e-37

B\_128(-126) :

1.6200746170781777504605426516026963940780798365909962583516816452325965643001  
016818318151880638577091006847622812979685586882697357373665882526069393996495  
353366642805968444162543310432651191959824303989649762894340613207805296127040  
453702802640951800184450548004351225095336200327115280696222507602381355449641  
43283392443781560168737237017787474130033326281105208552574665827074590323e-93

B\_256(-254) :

3.5350805720914227767089798687883314710396249627140676295904382269230853726478  
154558011233885009930466983366183538201823581813190254210827269008933585063842  
661534842038515790774375506907990914040258956292913258551548165885667154534979  
643968316374061999900537555017804444703610235647116037989753088917945545749807  
2066740263602708373733644665053339029843514259454436776150289160536401348e-226

B\_512(-510) :

2.0714779154945833548195268340089879693285608528781642062369216657137708286908  
567691686485585979574489157625716683351290916489400753129411680868691113915262  
962429703905115257245781386437981167271382027924697148177162824312538711027203  
484663189986180880756577933597065676826661163134395645548238434651661268054409  
8744309872683443225200186123587219842735439730877208370759699996856999303e-530

K\_0:

0.9815454579166903183254345064067652807638466955171693124852520241562726831097  
573999425144360715791046684112604523265251137555157109158423061788829413678603  
497635541446270319117899059918624337253292459941632757677351380232610157093735  
135802137246172630900866337076465453049725921279000748256456180655955381933657  
683531900278173497487605599543210786979876048268349094015034359315261874663010

K\_1:

1.6619078747381233774065816861630594973488686732512589183415081943423549310930  
452066938483805687234510381894905136878662909045609227412670328732394609024403  
620972625584993222612064376149581805846498737804601038156457864420832794816980  
986297511979120643170000822509758940449434954440813415143898294962059870207692  
949825313946737277789333826151704801556903907967063865651098248564855419302661

K\_2:

1.4933085037789944562935729344558833604589128885764255629365031757311429406189  
228852008436255405245028991566979365055943001890238695703641363742859085508361  
903676483223286809226261534973297686656884274625467933409426721652718721269375  
43018945087630556535315589839744434271664966420886150143520272583986723716894  
141342989881707266783031732661253882074983844213478856535905829697544396518115

K\_3:

1.5990143092929271524112167925260383396726642062569657287773576892900137332067  
577671174042459829254506767901634197792296705483567942209952566800231683117732  
919239465529828717306222684173003079500611954182943247026156122135513246038397  
836463154374084504973401423693647756065999161150202292250219347643066548305023  
425106559607837511279164308097740680098218462439207715368000480687311324768491



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