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STABILITY ANALYSIS OF AXISYMMETRIC SHELLS

By
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EDWARD L. WILSON

REPORT TO
U. S. NAVAL
CIVIL ENGINEERING LABORATORY

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STRUCTURAL ENGINEERING LABORATORY
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STABILITY ANALYSIS

OF

AXISYMMETRIC SHELLS

by

Te-ming Hsueh

Edward Wilson

Report to the

U.S. Naval Civil Engineering Laboratory
Port Hueneme, California
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ABSTRACT

The finite element method is applied to the stability analysis of axisymmetric thin shells subjected to axisymmetric load conditions. The conical shell element is used in the formulation. Several examples are presented to illustrate the application of the method. The use of the program and a listing of the FORTRAN IV program are given in the Appendices.

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NOTATION

R,θ,Z	-	global cylindrical coordinates
S,θ,ξ	-	local coordinates
$\overline{U},\overline{V},\overline{W}$	-	displacement components in global coordinates
U,V,W	-	displacement components in local coordinates
$\{P^o\}$	-	externally applied load vector
p	-	hydrostatic pressure
L	-	element length
ϕ	-	angle defining slope of element
π	-	potential energy of the structural system
U	-	strain energy
V	-	potential energy of externally applied loads
$\{\tau_e\}$	-	stress vector in elasticity
$\{\varepsilon_e\}$	-	strain vector in elasticity
$[C]$	-	elasticity matrix
$\{N\}$	-	stress resultants for conical shell element
$\{\varepsilon\}$	-	strain vector for conical shell element
$[C^*]$	-	elasticity matrix for conical shell element
$[L]\ [L^{-1}]$	-	transformation matrix between local and generalized coordinate systems
$[\lambda]$	-	transformation matrix between global and local coordinate systems
$[T]$	-	transformation matrix between global and generalized coordinate systems
$[D^M]$	-	strain-displacement relations for in-plane strains
$[D^F]$	-	strain-displacement relations for flexure strains

$[S^M]$ - interpolation function matrix for in-plane strains

$[S^F]$ - interpolation function matrix for flexure strains

$[\phi^M],[\phi^F]$ - quadratic displacement coefficients which lead to the elastic stiffness matrix

$[I],[I^M],[I^F]$ - generalized quadratic displacement coefficients which lead to the elastic stiffness matrix

$[\phi_1][\phi_2],[\phi_3]$ - quadratic coefficients which lead to the geometric stiffness matrix

$[I_g][I_s],[I_\theta]$ - generalized quadratic displacement coefficients which lead to the geometric stiffness matrix

$[k_e]$ - element elastic stiffness matrix

$[k_g]$ - element geometric stiffness matrix

$[K_e]$ - structural elastic stiffness matrix

$[K_g]$ - structural geometric stiffness matrix

λ - eigenvalue

$\{\bar{U}\}$ - displacement vector or bucking mode shape

I. INTRODUCTION

The stability of thin shells is of considerable interest in engineering practice. A large amount of theoretical literature has been devoted to this topic. As a result, governing field equations have been available for many years. Certain cases of idealized geometry and restricted load conditions have been solved in closed form. However, most problems involving practical shell structures can only be solved numerically with the aid of digital computers. In this investigation, attention is concentrated on the stability of axisymmetric thin shells of arbitrary shape subjected to axisymmetric loads. The finite element method is used as the basic numerical procedure.

In this report, the shell is modeled as an assembly of discrete truncated conical shell elements. The material properties of the individual elements are assumed to be homogeneous, orthotropic and linearly elastic. Thickness and material properties can vary for different elements. Based on proper kinematic assumptions, the elastic and geometric stiffness matrices are generated. For small deformations the resulting system of linear equations is solved by a Gauss elimination procedure. An eigenvalue problem is generated from the instability analysis. Since only the lowest positive eigenvalue and its associated eigenvector are needed, a special eigenvalue subroutine program was developed for this purpose. The stiffness matrices are in symmetric and banded form, therefore, these properties are recognized in order to minimize computer time and storage.

Several classical examples with known solutions are used to verify the computer program. Finally, detailed results of an stability analysis of a toroid shell are presented.

II. METHOD OF ANALYSIS

In this report, the axisymmetric shell is idealized by a finite number of conical shell elements interconnected at thier nodal circles. The potential energy expression of the structure is computed by summing the contributions of all elements. By means of a variational principle, conditions of stable equilibrium and bucking are defined as the vanishing of the first and second variations of the potential energy.

A. Basic Concepts

1. Potential Energy

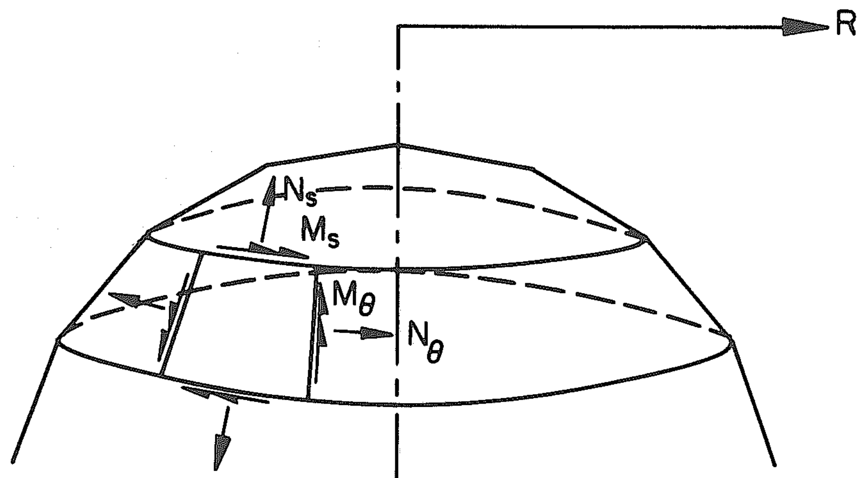
The geometry of an axisymmetric shell is described by cylindrical coordinates (R,θ,Z) as shown in Figure 1. In referring to a particular element between nodal circles I (R_i, Z_i) and J (R_j, Z_j) it is convenient to introduce a local coordinate system (S,θ,ξ). Assuming $\tau_{\xi\xi} = 0$, the strains {ε} and stress resultants {N} are given as

$$< \varepsilon > = < \varepsilon_s \quad \varepsilon_\theta \quad \varepsilon_{s\theta} \quad X_s \quad X_\theta \quad X_{s\theta} >$$

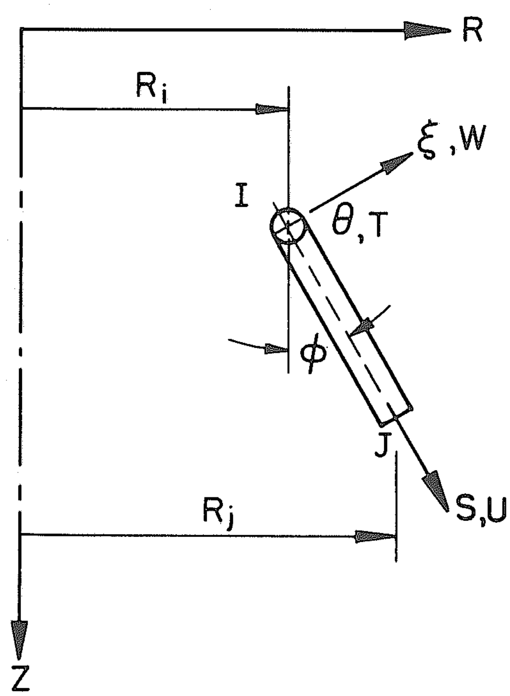
$$< N > = < N_s \quad N_\theta \quad N_{s\theta} \quad M_s \quad M_\theta \quad M_{s\theta} >$$

For a conical shell element under externally applied loads {P} at the nodal circles the potential energy may be written as

$$\pi = U + V$$



STRESS RESULTANTS IN
TYPICAL ELEMENT



ELEMENT LOCAL COORDINATES

FIG. 1 FINITE ELEMENT IDEALIZATION OF AXISYMMETRIC THIN SHELL

Where the internal strain energy is

$$U = \iint_A \frac{1}{2} < N > \{\varepsilon\} \, dA$$

and the potential energy of the externally applied loads is

$$V = - \int_c < P^\circ > \{u\} \, dc$$

2. <u>Constitutive Law</u>

For orthotropic linear elastic solids, the stress-strain relationship may be written in the following matrix form

$$\{\tau_e\} = [C] \{\varepsilon_e\}$$

Where

$$< \tau_e > = < \tau_s \quad \tau_\theta \quad \tau_\xi \quad \tau_{s\theta} >$$

$$< \varepsilon_e > = < \varepsilon_s \quad \varepsilon_\theta \quad \varepsilon_\xi \quad 2\varepsilon_{s\theta} >$$

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{12} & C_{22} & C_{23} & 0 \\ C_{13} & C_{23} & C_{33} & 0 \\ 0 & 0 & 0 & C_{44} \end{bmatrix}$$

After proper integration a constitutive law for a conical elastic thin shell is developed which relates stress resultants to in-plane strains and curvatures.

$$\{N\} = [C^*]\{\varepsilon\}$$

Where

$$[C^*] = \begin{bmatrix} C^*_{11} & C^*_{12} & 0 & 0 & 0 & 0 \\ C^*_{12} & C^*_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & C^*_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{44} \end{bmatrix}$$

Where

$$C^*_{11} = t\,(C_{11} - \frac{C_{13}^2}{C_{33}}), \qquad D_{11} = \frac{t^2}{12}\, C^*_{11}$$

$$C^*_{12} = t\,(C_{12} - \frac{C_{13}\, C_{23}}{C_{33}}), \qquad D_{12} = \frac{t^2}{12}\, C^*_{12}$$

$$C^*_{22} = t\,(C_{22} - \frac{C_{23}^2}{C_{33}}), \qquad D_{22} = \frac{t^2}{12}\, C^*_{22}$$

$$C^*_{44} = t\, C_{44}\,, \qquad D_{44} = \frac{t^2}{12}\, C^*_{44}$$

3. Kinematic Assumptions

On the basis of the Kirchhoff-Love hypothesis, the deformation within a conical thin shell can be uniquely defined by a displacement field $\{U\,(s,\theta)\}$ of the middle surface. This field is represented by a finite Fourier series expansion, which consists of a symmetric and asymmetric expansion about a plane $\theta = 0$. The symmetric part is given as:

$$\begin{Bmatrix} U\ (s,\theta) \\ V\ (s,\theta) \\ W\ (s,\theta) \end{Bmatrix} = \sum_{n=0}^{N} \begin{Bmatrix} U_n\ (s)\ \cos n\theta \\ V_n\ (s)\ \sin n\theta \\ W_n\ (s)\ \cos n\theta \end{Bmatrix}$$

In the corresponding asymmetric part $\sin n\theta$ and $\cos n\theta$ are replaced by $\cos n\theta$ and $\sin n\theta$ respectively.

In the meridional direction the displacements are approximated by interpolation polynomials of the form

$$U_n\ (s) = \alpha_{1n} + \alpha_{2n}\ s$$

$$V_n\ (s) = \alpha_{3n} + \alpha_{4n}\ s$$

$$W_n\ (s) = \alpha_{5n} + \alpha_{6n}\ s + \alpha_{7n}\ s^2 + \alpha_{8n}\ s^3$$

Therefore, it is possible to approximate the displacement field within a conical shell element in terms of the generalized displacements at the two nodal circles. The transformation relationships are

$$\{U_n^o\} = [L]\ \{\alpha_n\}$$

and

$$\{\alpha_n\} = [L^{-1}]\ \{U_n^o\}$$

where

$$< U_n^o > = < U_n^i\ \ W_n^i\ \ V_n^i\ \ \frac{\partial W_n^i}{\partial s}\ \ U_n^j\ \ W_n^j\ \ V_n^j\ \ \frac{\partial W_n^j}{\partial s} >$$

$$< \alpha_n > = < \alpha_{1n}\ \ \alpha_{2n}\ \ \alpha_{3n}\ \ \alpha_{4n}\ \ \alpha_{5n}\ \ \alpha_{6n}\ \ \alpha_{7n}\ \ \alpha_{8n} >$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & L & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L & L^2 & L^3 \\ 0 & 0 & 1 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2L & 3L^2 \end{bmatrix}$$

$$[L^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{L} & 0 & 0 & 0 & \frac{1}{L} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{L} & 0 & 0 & 0 & \frac{1}{L} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{L^2} & 0 & -\frac{2}{L} & 0 & \frac{3}{L^2} & 0 & -\frac{1}{L} \\ 0 & \frac{2}{L^3} & 0 & \frac{1}{L^2} & 0 & -\frac{2}{L^3} & 0 & \frac{1}{L^2} \end{bmatrix}$$

The transformation for the generalized displacements in the conical shell coordinate system and in the global cylindrical coordinate system is

$$\{U_n^o\} = [\lambda] \{\overline{U}_n\}$$

Where

$$<\bar{U}_n> \; = \; <\bar{U}_n^i \quad \bar{W}_n^i \quad \bar{V}_n^i \quad \beta_n^o \quad \bar{U}_n^j \quad \bar{W}_n^j \quad \bar{V}_n^j \quad \beta_n^j>$$

$$[\lambda] = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi & \sin\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$\{\alpha_n\} = [L^{-1}]\,[\lambda]\,\{U_n\}$$

or

$$\{\alpha_n\} = [T]\,\{U_n\}$$

If $a = R_j - R_i$, $b = Z_j - Z_i$

$$[T] = \begin{bmatrix} \frac{b}{L} & \frac{a}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{b}{L^2} & -\frac{a}{L^2} & 0 & 0 & \frac{b}{L^2} & \frac{a}{L^2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{L} & 0 & 0 & 0 & \frac{1}{L} & 0 \\ -\frac{a}{L} & \frac{b}{L} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{3a}{L^3} & -\frac{3b}{L^3} & 0 & -\frac{2}{L} & -\frac{3a}{L^3} & \frac{3b}{L^3} & 0 & -\frac{1}{L} \\ -\frac{2a}{L^4} & \frac{2b}{L^4} & 0 & \frac{1}{L^2} & \frac{2a}{L^4} & -\frac{2b}{L^4} & 0 & \frac{1}{L^2} \end{bmatrix}$$

4. Strain-Displacement Relations

For a thin conical shell having moderate strains, the strain-displacement equations are of the following form:

$$\varepsilon_s = \frac{\partial U}{\partial S} + \frac{1}{2}(\psi_s{}^2 + \psi^{*2})$$

$$\varepsilon_\theta = \frac{1}{R}(U\sin\phi + \frac{\partial V}{\partial \theta} + W\cos\phi) + \frac{1}{2}(\psi_\theta{}^2 + \psi^{*2})$$

$$\varepsilon_{s\theta} = \frac{1}{2R}(R\frac{\partial V}{\partial S} + \frac{\partial U}{\partial \theta} - V\sin\phi) + \frac{1}{2}\psi_s\ \psi_\theta$$

$$X_s = \frac{\partial \psi_s}{\partial S}$$

$$X_\theta = \frac{1}{R}(\frac{\partial \psi_\theta}{\partial \theta} + \psi_s \sin\phi)$$

$$X_{s\theta} = \frac{1}{2}(\frac{\partial \psi_\theta}{\partial S} - \frac{\psi_\theta}{R}\sin\phi)$$

Where

$$\psi_s = -\frac{\partial W}{\partial S}$$

$$\psi_\theta = -\frac{1}{R}(\frac{\partial W}{\partial \theta} - V\cos\phi)$$

$$\psi^* = \frac{1}{2R}(R\frac{\partial V}{\partial S} + V\sin\phi - \frac{\partial U}{\partial \theta}$$

Based on these approximations it is possible to represent the potential energy of the shell in terms of the generalized displacements at the nodal circles. With proper manipulations, the structural problems, both for equilibrium and buckling, can be formulated.

B. Stress Analysis

In a state of stable equilibrium, the stress distribution in a shell is insensitive with respect to geometric changing due to elastic deformations. This is particularly ture for the membrane stress which has significant influence on stability analysis. Therefore, a classical linear analysis may be used to estimate the stress distribution prior to buckling.

1. Strain Energy

In an axisymmetric deformation, the tangential displacement component is equal to zero; therefore, the linearized strain-displacement equations are simplified to

$$\{\varepsilon^o\} = [D^o]\{U^o\}.$$

Where

$$< \varepsilon^o > = < \varepsilon_s \quad \varepsilon_\theta \quad X_s \quad X_\theta >$$

$$< U^o > = < \frac{\partial U_o}{\partial S} \quad \frac{U_o}{R} \quad \frac{W_o}{R} \quad \frac{\partial^2 W_o}{\partial S^2} \quad \frac{1}{R}\frac{\partial W_o}{\partial S} >$$

$$[D^o] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\sin\phi \end{bmatrix}$$

The kinematic assumption becomes

$$\{U^o\} = [S^o] \{\alpha_o\}$$

Where

$$[S^o] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{R} & \frac{S}{R} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R} & \frac{S}{R} & \frac{S^2}{R} & \frac{S^3}{R} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6S \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R} & \frac{2S}{R} & \frac{3S^2}{R} \end{bmatrix}$$

Substituting the above relationships into the strain energy expression, gives.

$$U^o = \iint_A \frac{1}{2} < N^o > \{\varepsilon^o\} \; dA$$

$$= \iint_A \frac{1}{2} < \varepsilon^o > [C^{*o}] \{\varepsilon^o\} \; dA$$

$$= \frac{1}{2} < \alpha_o > \left[\iint_A [S^o]^T \; [D^o]^T \; [C^{*o}] \; [D^o] \; [S^o] \; dA \right] \{\alpha_o\}$$

$$= \frac{1}{2} < \alpha_o > \left[\iint_A [S^o]^T \; [\phi^o] \; [S^o] \; dA \right] \{\alpha_o\}$$

Where

$$[\phi^o] = [D^o]^T \; [C^{*o}] \; [D^o]$$

or

$$[\phi^o] = \begin{bmatrix} C^*_{11} & C^*_{12}S\phi & C^*_{12}C\phi & 0 & 0 \\ C^*_{12}S\phi & C^*_{22}S^2\phi & C^*_{22}S\phi C\phi & 0 & 0 \\ C^*_{12}C\phi & C^*_{22}S\phi C\phi & C^*_{22}C^2\phi & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12}S\phi \\ 0 & 0 & 0 & D_{12}S\phi & D_{22}S^2\phi \end{bmatrix}$$

where $s\phi = \sin\phi$, $c\phi = \cos\phi$.

The integration

$$[I^o] = \left[\iint_A [S^o]^T \; [\phi^o] \; [S^o] \; dA\right]$$

is carried out numerically within the computer program using a ten-point Gauss quadrature integration procedure.

Utilizing the generalized coordinate transformation relationships, the strain energy expression is given as a quadratic function in terms of the generalized displacements of the nodal circles.

$$U_e^o = \frac{1}{2} < \bar{U}_o > [T]^T \; [I^o] \; [T] \; \{\bar{U}_o\}.$$

Or

$$U_e^o = U_e^o \; (\bar{U}_o^i, \; \bar{U}_o^j \;)$$

2. Direct Stiffness Formulation

With the same assumptions on the displacement pattern, the potential energy of the externally applied loads can be expressed as a linear function of the generalized displacement at nodal circles,

i.e. $$V_e^o = V_e^o (\bar{U}_o^i)$$

Having $U_e^o (\bar{U}_o^i, \bar{U}_o^j)$ and $V_e^o (\bar{U}_o^i)$ for each element the total potential energy of the structural system equals the summation of the contributions of all elements.

$$\pi = U^o (\bar{U}_o^i, \bar{U}_o^j) + V^o (\bar{U}_o^i)$$

Where

$$U^o = \sum_{e=1}^{M} U_e^o (\bar{U}_o^i, \bar{U}_o^j)$$

$$V^o = \sum_{e=1}^{M} V_e^o (\bar{U}_o^i)$$

For equilibrium of the structural system the necessary condition is for the first variation of π to vanish

$$\frac{\partial \pi}{\partial \bar{U}^i} = 0 \qquad i = o,1,\ldots\ldots n$$

This generates a system of linear equations

$$[K] \{\bar{U}_o\} = \{P\}$$

Within the computer program, the structural stiffness matrix [K] is assembled by adding the stiffness coefficients of the submatrices $[k^e]$ into the proper locations in [K]. Symbolically, this process is shown as

$$[K] = \sum_{e=1}^{M} [k^e]$$

where

$$[k^e] = [T_e]^T [I_e^o] [T_e]$$

In this report the nodal point load vector {P} is set equal to the original externally applied nodal load vector plus the contributions due to hydrostatic pressure the latter is computed by means of the tributary area concept,

$$\{P\} = \{P^o\} + \sum_{e=1}^{M} \{P^e\}$$

Where

$$\{P^e\} = 2 \pi L p \begin{Bmatrix} \left(\frac{R_i}{2} + \frac{L \sin\phi}{6}\right) \sin\phi \\ \left(\frac{R_i}{2} + \frac{L \cos\phi}{6}\right) \cos\phi \\ 0 \\ 0 \\ \left(\frac{R_i}{2} + \frac{L \sin\phi}{3}\right) \sin\phi \\ \left(\frac{R_i}{2} + \frac{L \cos\phi}{3}\right) \cos\phi \\ 0 \\ 0 \end{Bmatrix}$$

Experience has shown that lumping of nodal loads is quite satisfactory as compared to the consistant load approach.

To allow for prescribed displacement boundary conditions, the load vecter and stiffness matrix are appropriately adjusted. In the stiffness matrix, the elements of corresponding rows and columns are set to zeros and the diagonal terms are replaced by unity. Then, the equilibrium equations are solved for the generalized displacements by the Gauss elimination method.

3. Stress Computation

Having the generalized displacements at the nodal circles, the element stress resultants can be computed by proper matrix operations. In this report, the stresses are evaluated at the midpoint of each element, where the true slope of the shell is most nearly represented by the straight line approximation. For a particular element, this procedure yields

$$\{N\} = [C^{*o}] [D^o] [S^o] [T] \{\bar{U}^o\}$$

where in $[S^o]$, the coordinate quatity s is replaced by 0.5L.

C. Stability Analysis

1. Potential Energy

Consider a shell which is in equilibrium under a system of conservative axisymmetric loads. Let $\{\tilde{\bar{U}}^o\}$ be the vector defining this displacement field. The investigation of stability of the structural system can be done by introducing an arbitrary infinitesimal disturbance $\{\bar{U}^a\}$ and finding out if the second variation of the potential energy of the system vanishes. Then the total displacements are

$$\{\bar{U}\} = \{\bar{U}^o\} + \{\bar{U}^a\}$$

and the total strains are assumed as

$$\{\varepsilon\} = \{\varepsilon^o\} + \{\varepsilon^a\}$$

The strain energy is

$$U = \frac{1}{2} \iint_A \{\varepsilon\}^T [C^*] \{\varepsilon\} \, dA$$

$$= U_o + U_1 + U_2$$

where

$$U_o = \frac{1}{2} \iint_A \{\varepsilon^o\}^T [C^*] \{\varepsilon^o\} \, dA$$

$$U_1 = \iint_A \{\varepsilon^o\}^T [C^*] \{\varepsilon^a\} \, dA$$

$$U_2 = \frac{1}{2} \iint \{\varepsilon^a\}^T [C^*] \{\varepsilon^a\} \, dA.$$

By using the strain-displacement relationships, U_1 and U_2 can be represented as.

$$U_1 = U_1^1 + U_1^2 + U_1^3$$

$$U_2 = U_2^2 + U_2^3$$

Where the term with superscript 1 is linear in the displacement components and their derivatives. Terms with superscript 2 and superscript 3 are quadratic and cubic (or higher) in the displacement components and their derivatives.

The potential energy of the externally applied load is

$$V = -\int_c < p > \{\bar{U}\} \, dc$$

$$= V_o + V_1$$

where

$$V_o = -\int_c < p > \{\bar{U}^o\} \, dc \quad ; \quad V_1 = -\int_c < p > \{\bar{U}^a\} \, dc$$

The potential energy of the structural system is

$$\pi = U + V$$

$$= \pi_o + \pi_1 + \pi_2 + \pi_3$$

where

$$\pi_o = U_o + V_o$$

$$\pi_1 = U_1^1 + V_1$$

$$\pi_2 = U_1^2 + U_2^2$$

$$\pi_3 = U_1^3 + U_2^3$$

This potential energy function π is in terms of generalized displacements at the nodal circles after introducing the kinematic assumption. Taking the variation with respect to the disturbance $\{\bar{U}^a\}$ leads to

$$\delta \pi = \delta \pi_o + \delta \pi_1 + \delta \pi_2 + \delta \pi_3$$

π_o can be treated as constant at this stage and its variation is zero. The variation of π_1 leads to the system of equations of equilibrium, which have been satisfied previously. π_3 is a higher order infinitesimal, and its contribution to the variation can be neglected. Therefore, the behavior of the second variation $\delta \pi_2$ is crucial in the stability analysis.

It is convenient to write π_2 in the following form.

$$\pi_2 = U^{(2)} = < \bar{U}^a > [K] \{\bar{U}^a\}.$$

where

$$K_{ij} = \frac{U^{(2)}}{\partial \bar{U}_i^a \, \partial \bar{U}_j^a}$$

If the stiffness matrix [K] is positive definite, the structure is in a stable state. When the stiffness matrix [K] is positive-semidefinite, the state is defined to be unstable. When the stiffness matrix [K] is indefinite or negative-definite, it corresponds to postbuckling. At this state, the structure may or may not fail physically. Further investigation for this state is beyond the scope of this report.

In order to predict the buckling load for a given load pattern from a reference load level, an eigenvalue problem is established as follows. A nondimensional parameter λ which is associated with $U_1^{(2)}$ is introduced.

Let

$$\pi_2 = U_2^{(2)} + \lambda \, U_1^{(2)}$$

The vanishing of the second variation of π leads to

$$\delta \pi_2 = < \delta \bar{U}^a > ([K_e] + \lambda [K_g]) \{\bar{U}^a\} = 0$$

This will be satisfied if

$$| [K_e] + \lambda [K_g] | = 0$$

Because of the orthogonality properties of the Fourier Series, the stiffness matrix is uncoupled into a number of blocks, each of which corresponds to a harmonic. Therefore, an separate eigenvalue problem is generated for each harmonic. Among all of the harmonics, the lowest eigenvalue corresponds to the buckling load; therefore, it is necessary to evaluate the lowest positive eigenvalue for each harmonic.

2. <u>Element Elastic Stiffness</u>

In a conical shell element, the part of the strain energy which yields the elastic stiffness for a specified harmonic ($n \geq 1$) can be written as.

$$U_2^2 = \frac{1}{2} \int_0^L \pi R < \varepsilon_n > \{N_n\} \, ds.$$

The integration with respect to circumferential angle θ has already been carried out in this expression.

For convenience, the strains and stresses are divided into membrane and flexure components.

$$< \varepsilon_n > = \ll \bar{\varepsilon}_n > < X_n \gg$$

where $< \bar{\varepsilon}_n > = < \varepsilon_{sn} \; \varepsilon_{\theta n} \; 2\, \varepsilon_{s\theta n} >$

$$< X_n > = < X_{sn} \; X_{\theta n} \; 2\, X_{s\theta n} >$$

$$< N_n > = \ll \bar{N}_n > < M_n \gg$$

Where

$$< \bar{N}_n > \; = \; < N_{sn} \quad N_{\theta n} \quad N_{s\theta n} >$$

$$< M_n > \; = \; < M_{sn} \quad M_{\theta n} \quad M_{s\theta n} >$$

The linearized strain-displacement relationships can be written as:

$$\{ \bar{\varepsilon}_n \} = [D^M] \{U_n^M\}$$

where

$$[D^M] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\phi & n & \cos\phi \\ 0 & 1 & -n & -\sin\phi & 0 \end{bmatrix}$$

$$< U_n^M > = < \frac{\partial U_n}{\partial S} \quad \frac{\partial V_n}{\partial S} \quad \frac{U_n}{R} \quad \frac{V_n}{R} \quad \frac{W_n}{R} >$$

and

$$\{X_n\} = [D^F] \{U_n^F\}$$

where

$$[D^F] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\phi & n\cos\phi & N^2 \\ 0 & \cos\phi & n & -\sin\phi\cos\phi & -n\sin\phi \end{bmatrix}$$

$$< U_n^F > = < \frac{\partial^2 W_n}{\partial S^2} \quad \frac{1}{R}\frac{\partial V_n}{\partial S} \quad \frac{1}{R}\frac{\partial W_n}{\partial S} \quad \frac{V_n}{R^2} \quad \frac{W_n}{R^2} >$$

The interpolation relationships are

$$\{U_n^M\} = [S^M]\{\alpha_n\}$$

where

$$[S^M] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{R} & \frac{S}{R} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R} & \frac{S}{R} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R} & \frac{S}{R} & \frac{S^2}{R} & \frac{S^3}{R} \end{bmatrix}$$

and

$$\{U_n^F\} = [S^F]\{\alpha_n\}$$

where

$$[S^F] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6S \\ 0 & 0 & 0 & \frac{1}{R} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R} & \frac{2S}{R} & \frac{3S^2}{R} \\ 0 & 0 & \frac{1}{R^2} & \frac{S}{R^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R^2} & \frac{S}{R^2} & \frac{S^2}{R^2} & \frac{S^3}{R^2} \end{bmatrix}$$

The strain energy quantity U_2 may be given as

$$U_2^2 = U_2^M + U_2^F$$

The membrane part U_2^M is

$$U_2^M = \frac{1}{2} \int_0^L < \bar{\varepsilon}_n > \{\bar{N}_n\} \pi R \, ds$$

$$= \frac{1}{2} \int_0^L < U_n^M > [D^M]^T [C^M] [D^M] \{U_n^M\} \pi R \, ds$$

$$= \frac{1}{2} < \alpha_n > \int_0^L [S^M]^T [\phi^M] [S^M] \pi R \, ds \quad \{\alpha_n\}$$

$$= \frac{1}{2} < \alpha_n > [I^M] \{\alpha_n\}$$

where

$$[\phi^M] = [D^M]^T [C^M] [D^M]$$

or

$$[\phi^M] = \begin{bmatrix} C_{11}^* & & & & \text{SYMM.} \\ 0 & C_{44}^* & & & \\ C_{12}^* s\phi & -nC_{44}^* & (C_{22}^* s^2\phi + n^2 C_{44}^*) & & \\ nC_{12}^* & -C_{44}^* s\phi & (nC_{22}^* s\phi + nC_{44}^* s\phi) & (n^2 C_{22}^* + C_{44}^* s^2\phi) & \\ C_{12}^* c\phi & 0 & C_{22}^* s\phi c\phi & nC_{22}^* c\phi & C_{22}^* c^2\phi \end{bmatrix}$$

and $$[I^M] = \int_0^L [S^M]^T [\phi^M] [S^M] \pi R \, ds.$$

The flexure part U_2^F is

$$U_2^F = \frac{1}{2} \int_0^L < X_n > \{M_n\} \pi R \, ds$$

$$= \frac{1}{2} \int_0^L < U_n^F > [D^F]^T [C^F] [D^F] \{U_n^F\} \pi R \, ds$$

$$= \frac{1}{2} < \alpha_n > \int_0^L [S^F]^T [\phi^F] [S^F] \pi R \, ds \, \{\alpha_n\}$$

$$= \frac{1}{2} < \alpha_n > [I^F] \{\alpha_n\}$$

where $[\phi^F] = [D^F] [C^F] [D^F]$ which is shown in expanded form on the next page.

And $$[I^F] = \int_0^L [S^F]^T [\phi^F] [S^F] \pi R \, ds$$

then $$U_2^2 = \frac{1}{2} < \alpha_n > [I] \{\alpha_n\}$$

where $$[I] = [I^M] + [I^F]$$

The integrations are evaluated numerically within the computer program using a ten-point Gauss quadrature integration procedure.

The coordinate transformation relationship leads to

$$U_2^2 = \frac{1}{2} < \bar{U}_n > [T]^T [I] [T] \{\bar{U}_n\}$$

$$= \frac{1}{2} < \bar{U}_n > [k_e] \{U_n\}$$

where the element elastic stiffness matrix is given as

$$[k_e] = [T]^T [I] [T]$$

For n = o, the case degenerates to the axisymmetric problem.

or

$$[\phi^F] = \begin{bmatrix} D_{11} & & & & \text{SYMM.} \\ 0 & D_{44}c^2\phi & & & \\ D_{12}s\phi & nD_{44}c\phi & (D_{44}s^2\phi + n^2D_{44}) & & \\ -nD_{12}c\phi & -D_{44}s\phi c^2\phi & (-nD_{22}s\phi c\phi - D_{44}s\phi c\phi) & (n^2D_{22}c^2\phi + D_{44}s^2\phi c^2\phi) & \\ -n^2D_{12} & -nD_{44}s\phi c\phi & (-n^2D_{22}s\phi - n^2D_{44}s\phi) & (n^3D_{22}c\phi + nD_{44}s^2\phi c\phi) & (n^4D_{22} + n^2D_{44}s^2\phi) \end{bmatrix}$$

3. Element Geometric Stiffness

Rewrite $U_1^{(2)}$ for $n \geq 1$ as

$$U_1^{(2)} = \iint_A \{\varepsilon^o\}^T [C^*] \left\{\varepsilon_n^{a(2)}\right\} dA.$$

$$= \iint_A \{N^o\}^T \left\{\varepsilon_n^{a(2)}\right\} dA.$$

where $< N^o > \quad = < N_s^o \quad N_\theta^o \quad 0 \quad M_s^o \quad M_\theta^o \quad 0 >$

$$< \varepsilon_n^{a(2)} > = < \frac{1}{2} (\psi_{sn}^2 + \psi_n^{*2}) \; \frac{1}{2} (\psi_{\theta n}^2 + \psi_n^{*2}) \; \frac{1}{2} \psi_{sn}\psi_{\theta n} \; 0 \; 0 \; 0 >$$

This part of the strain energy involves initial stresses and is quadratic in the displacements and their derivatives therefore, it must yield the geometric stiffness. After integration with respect to the circumferential angle θ, $U_1^{(2)}$ is in the form

$$U_1^{(2)} = \frac{1}{2} \int_0^L [N_s^o (\psi_{sn}^2 + \psi_n^{*2}) + N_\theta^2 (\psi_{\theta n}^2 + \psi_n^{*2})] \pi R \, ds$$

$$= \frac{1}{2} (U_s^2 + U_\theta^2)$$

where $$U_s^2 = \int_0^L N_s^o (\psi_{sn}^2 + \psi_n^{*2}) \pi R \, ds$$

$$U_\theta^2 = \int_0^L N_\theta^2 (\psi_{\theta n}^2 + \psi_n^{*2}) \pi R \, ds$$

here ψ_{sn}, $\psi_{\theta n}$ and ψ_n^* are meridional dependent only.

The kinematic assumption leads to

$$\psi_{5n} = \alpha_{6n} + 2\alpha_{7n}S + 3\alpha_{8n}S^2$$

$$\psi_{\theta n} = \frac{1}{R}(\cos\phi\alpha_{3n} + \cos\phi\alpha_{4n}S + n\alpha_{5n} + n\alpha_{6n}S + n\alpha_{7n}S^2 + n\alpha_{8n}S^3)$$

$$\psi_n^* = \frac{1}{2R}(n\alpha_{1n} + n\alpha_{2n}S + \sin\phi\alpha_{3n} + \sin\phi\alpha_{4n}S + \alpha_{4n}R)$$

or using matrix notation

$$\psi_{3n}^2 = < \alpha_n^{6-8} > [\phi_1] \{\alpha_n^{6-8}\}$$

where

$$< \alpha_n^{6-8} > = < \alpha_{6n} \quad \alpha_{7n} \quad \alpha_{8n} >$$

$$[\phi_1] = \begin{bmatrix} 1 & & \text{SYMM.} \\ 2S & 4S^2 & \\ 3S^2 & 65^3 & 9S^4 \end{bmatrix}$$

$$\psi_{\theta n}^2 = < \alpha_n^{3-8} > [\phi_2] \{\alpha_n^{3-8}\}$$

where

$$< \alpha_n^{3-8} > = < \alpha_{3n} \quad \alpha_{4n} \quad \alpha_{5n} \quad \alpha_{6n} \quad \alpha_{7n} \quad \alpha_{8n} >$$

$$[\phi_2] = \frac{1}{R^2} \begin{bmatrix} \cos^2\phi & & & & & \text{SYMM.} \\ S\cos^2\phi & S^2\cos^2\phi & & & & \\ n\cos\phi & nS\cos\phi & n^2 & & & \\ nS\cos\phi & nS^2\cos\phi & n^2S & n^2S^2 & & \\ nS^2\cos\phi & nS^3\cos\phi & n^2S^2 & n^2S^3 & n^2S^4 & \\ nS^3\cos\phi & nS^4\cos\phi & n^2S^3 & n^2S^4 & n^2S^5 & n^2S^6 \end{bmatrix}$$

$$\psi_n^{*2} = < \alpha_n^{1-4} > [\phi_3] \{\alpha_n^{1-4}\}$$

where $< \alpha_n^{1-4} > = < \alpha_{1n} \quad \alpha_{2n} \quad \alpha_{3n} \quad \alpha_{4n} >$

$$[\phi_3] = \frac{1}{4R^2} \begin{bmatrix} n^2 & & & \text{SYMM.} \\ n^2S & n^2S^2 & & \\ n\sin\phi & nS\sin\phi & \sin^2\phi & \\ (nR + nS\sin\phi) & (nRS + nS^2\sin\phi) & (R\sin\phi + S\sin^2\phi) & (R + S\sin\phi)^2 \end{bmatrix}$$

To evaluate U_s^2 and U_θ^2, the integrations are carried out by the mid-point rule. This yields

$$U_s^2 = < \alpha_n > [I_s] \{\alpha_n\}$$

$$U_\theta^2 = < \alpha_n > [I_\theta] \{\alpha_n\}$$

and $$U_1^{(2)} = \frac{1}{2} < \alpha_n > [I_g] \{\alpha_n\}$$

where $$[I_g] = [I_s] + [I_\theta]$$

The $[I_s]$ Matrix :

$$
\pi N_s \begin{bmatrix}
\frac{n^2 L}{4R} & \frac{n^2 L^2}{8R} & \frac{nL\sin\phi}{4R} & \frac{(2R + L\sin\phi)\, nL}{8R} & 0 & 0 & 0 & 0 \\
 & \frac{n^2 L^3}{12R} & \frac{nL^2\sin\phi}{8R} & \frac{(3R + 2L\sin\phi)\, nL^2}{24R} & 0 & 0 & 0 & 0 \\
 & & \frac{L\sin^2\phi}{4R} & \frac{(2R + L\sin\phi)\, L\sin\phi}{8R} & 0 & 0 & 0 & 0 \\
 & & & \frac{(3R^2 + 3LR\sin\phi + L^2\sin^2\phi)\, L}{12R} & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 & & & & & LR & L^2R & L^3R \\
 & & & & & & \frac{4L^3R}{3} & \frac{3L^4R}{2} \\
 & & \text{SYMM.} & & & & & \frac{9L^5R}{5}
\end{bmatrix}
$$

The $[I_\theta]$ Matrix:

$$
\pi N_\theta
\begin{bmatrix}
\frac{n^2L}{4R} & \frac{n^2L^2}{8R} & \frac{nL\sin\phi}{4R} & \frac{(2R + L\sin\phi)nL}{8R} & 0 & 0 & 0 & 0 \\
 & \frac{n^2L^3}{12R} & \frac{nL^2\sin\phi}{8R} & \frac{(3R + 2L\sin\phi)nL^2}{24R} & 0 & 0 & 0 & 0 \\
 & & \frac{(1+ 3\cos^2\phi)L}{4R} & \frac{(L + 2R\sin\phi + 3L\cos^2\phi)L}{8R} & \frac{nL\cos\phi}{R} & \frac{nL^2\cos\phi}{2R} & \frac{nL^3\cos\phi}{3R} & \frac{nL^4\cos\phi}{4R} \\
 & & & \frac{(3R^2 + L^2 + 3LR\sin\phi + 3L^2\cos^2\phi)L}{12R} & \frac{nL^2\cos\phi}{2R} & \frac{nL^3\cos\phi}{3R} & \frac{nL^4\cos\phi}{4R} & \frac{nL^5\cos\phi}{5R} \\
 & & & & \frac{n^2L}{R} & \frac{n^2L^2}{2R} & \frac{n^2L^3}{3R} & \frac{n^2L^4}{4R} \\
 & & & & & \frac{n^2L^3}{3R} & \frac{n^2L^4}{4R} & \frac{n^2L^5}{5R} \\
 & & & & & & \frac{n^2L^5}{5R} & \frac{n^2L^6}{6R} \\
 & \text{SYMM.} & & & & & & \frac{n^2L^7}{7R}
\end{bmatrix}
$$

The coordinate transformation relationship leads to

$$U_1^{(2)} = \frac{1}{2} < \bar{U}_n > [T]^T [I_g] [T] \{\bar{U}_n\}$$

$$= \frac{1}{2} < \bar{U}_n > [k_g] \{\bar{U}_n\}$$

where the element geometric stiffness matrix is given as

$$[k_g] = [T]^T [I_g] [T]$$

For n = o, $[I_g^o]$ is simplified as following:

$$[I_g^o] = \pi N_s \begin{bmatrix} 0 & & & & & & & \text{SYMM.} \\ 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 2LR & & \\ 0 & 0 & 0 & 0 & 0 & 2L^2R & \frac{8RL^3}{3} & \\ 0 & 0 & 0 & 0 & 0 & 2L^3R & 3RL^4 & \frac{18RL^5}{5} \end{bmatrix}$$

and all other matrix manipulation are the same as previeously indicated.

4. <u>The Eigenvalue Problem</u>

After the element elastic stiffness matrix has been computed for each element, the structural elastic stiffness matrix is obtained by the direct stiffness method, Symbolically.

$$[K_e] = \sum_{\ell=1}^{M} [k_e^{\ell}]$$

Similarly, the structural geometric stiffness matrix is assembled

$$[K_g] = \sum_{\ell=1}^{M} [k_g^{\ell}]$$

The eigenvalue problem is in the form

$$\left([K_e] + \lambda [K_g]\right)\{\bar{U}\} = \{0\}.$$

To allow for prescribed displacement boundary conditions the corresponding rows and columns are eliminated from the stiffness matrices. The eigenvalue problem reduces to

$$\left([K_e]_R + \lambda [K_g]_R\right)\{\bar{U}\}_R = \{0\}$$

The lowest positive eigenvalue and its associated buckling mode shape are solved by a Gauss elimination technique. Theoretically, the true eigenvalue for instability analysis is unity, this loading condition can be reached by a trial and error method or an extrapolation method.

III. PROGRAM VERIFICATION

Several typical examples were made to verify the method of analysis. The results are presented here, and compared with known analytical solutions.

A. Euler Column

A flat circular plate with a concentric circular hole is taken as the first example. The geometry and material properties are given as follows:

Inside radius:	R_i	=	10000"
Outside radius:	R_o	=	10100"
Thickness:	t	=	1"
Young's modulus:	E	=	$30{,}000^{ksi}$
Poisson's ratio:	ν	=	0

With simply supported boundary conditions, the structure reduces to the classical Euler column of one inch square cross-section. Its critical buckling load is well known as:

$$Pcr = \frac{\pi^2 EI}{L^2} = 2.47^k$$

The structure is modeled into an assembly of four equal elements. Using Pcr as the externally applied compressive load, the eigenvalues of the first five harmonics are found to be

$$\lambda_0 = 1.005$$

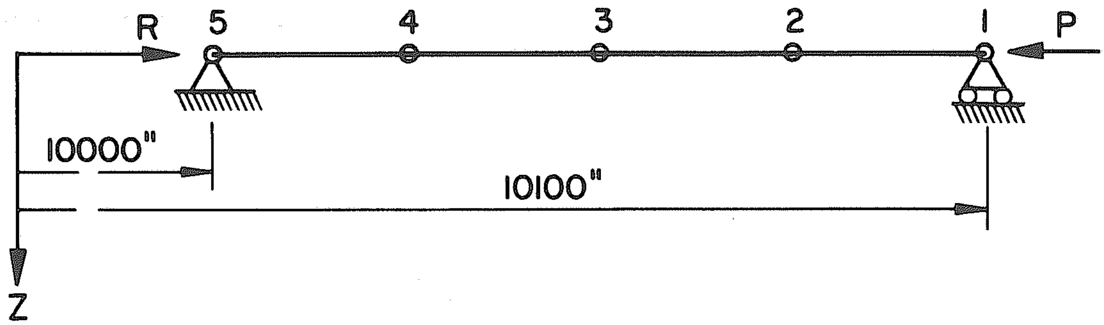
$$\lambda_1 = 1.007$$

$$\lambda_2 = 1.007$$

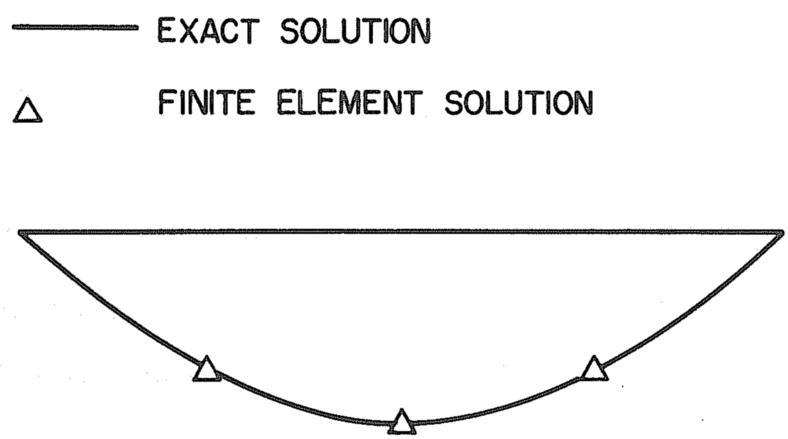
$$\lambda_3 = 1.007$$

$$\lambda_4 = 1.007$$

Insensitivity of the eigenvalues with respect to different harmonic modes is expected. Because of the large radius and zero Poisson's ratio, the lateral coupling effect in higher harmonics is negligible. In Figure 2 the structure and its buckling mode are shown.



FOUR ELEMENT IDEALIZATION OF UNIFORM COLUMN



BUCKLING MODE SHAPE

FIG.2 EULER COLUMN

B. Cylindrical Shell

A uniform circular cylindrical shell, shown in Figure 3, has the following properties:

Radius:	R	=	4"
Length:	L	=	7"
Thickness:	t	=	0.005"
Young's modulus:	E	=	$10{,}000{,}000^{psi}$
Poisson's ratio:	ν	=	0.3

The shell is uniformly compressed in the axial direction. The classical critical stress is

$$\sigma_{cr} = \frac{Et}{R\sqrt{3\,(1-\nu^2)}} = 7570^{psi}$$

It is found that for different boundary conditions, the buckling load might be as low as fifty percent of the value predicted by the classical theory. Therefore, two types of boundary conditions are used:

Case 1: The edge condition is set so that:
Axial displacement is zero,
Radial displacement is zero,
Tangential displacement is zero
and moment is zero.
The analytical solution for critical stress is 7578^{psi}.
It corresponds to harmonic 14.

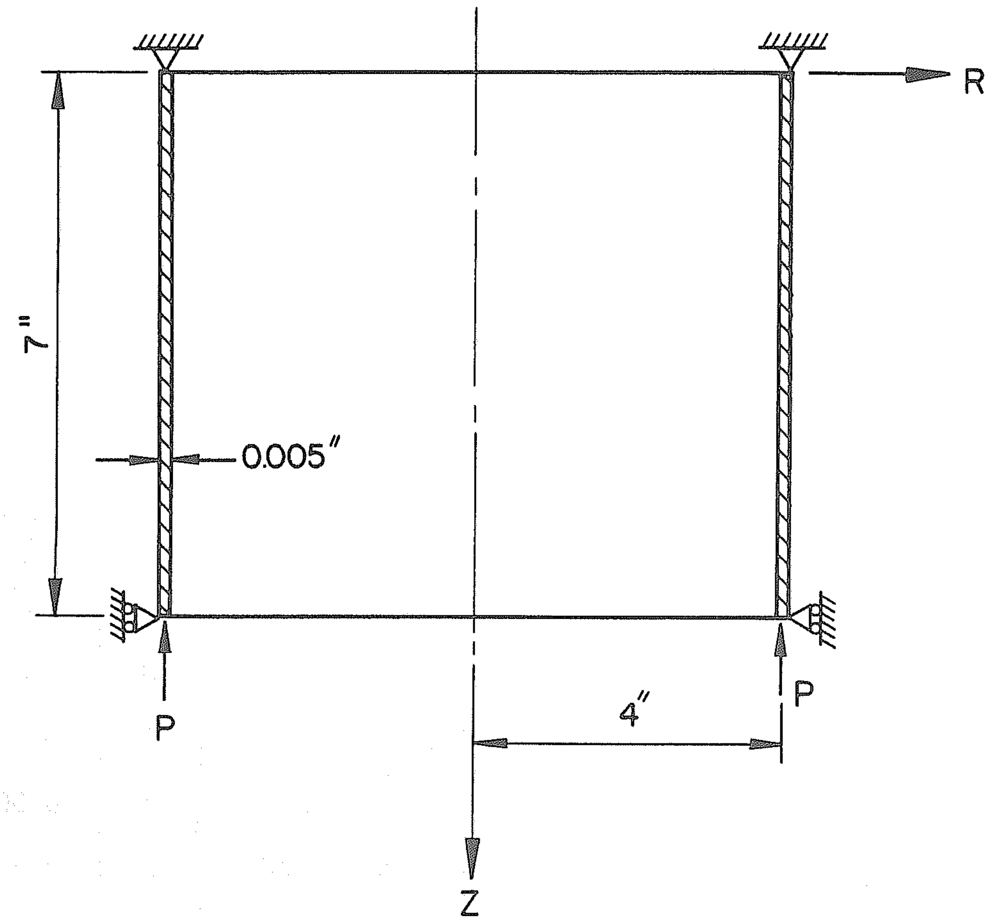


FIG. 3 CYLINDRICAL SHELL

Case 2: The edge condition is set so that:

Axial displacement is zero,

Radial displacement is zero,

Tangential shear stress is zero

and moment is zero.

The analytical solution for critical stress is 3823^{psi}.

It corresponds to harmonic 3.

The stress is analysed at a stress level of 7570^{psi}, and 40 elements are used. The eigenvalues were computed as

CASE 1				CASE 2	
n	λ	n	λ	n	λ
0	1.034	20	0.946	0	1.032
2	1.034	22	0.929	1	0.503
4	1.032	24	0.916	2	0.503
6	1.030	26	0.908	3	0.503
8	1.026	28	0.909	4	0.503
10	1.020	30	0.922	5	0.504
12	1.011	32	0.946	6	0.507
14	0.996	34	0.979	7	0.511
16	0.979	36	1.024	8	0.515
18	0.964	38	1.077	9	0.523

In case 1, during the stress analysis, the ends are allowed to expand freely in the radial direction. But in case 2, the boundary conditions are the same for both the stress and stability analyses.

C. Spherical Shell

In this example, the stability of a spherical shell subjected to a uniform external pressure is studied. The shell shown in Figure 4 has the following properties:

Radius: $R = 40''$

Thickness: $t = 0.1''$

Young's modulus: $E = 30{,}000{,}000^{psi}$

Poisson's ratio: $\nu = 0.3$

The critical pressure is given as

$$P_{cr} = \frac{2Et^2}{R^2\sqrt{3(1-\nu^2)}} = 227^{psi}.$$

Taking advantage of symmetry, half of the shell is used for the analysis. For different element meshes, the eigenvalues are summarized below.

HARMONICS (n)	EIGENVALUE (λ)				
	20 ELS	40 ELS	60 ELS	80 ELS	100 ELS
0	0.8979	0.9722	0.9426	0.9976	0.9995
5	1.0398	0.9561	0.9338	0.9248	0.9209
10	1.0405	0.9404	0.9170	0.9077	0.9038
15	1.0381	0.9304	0.9053	0.8960	0.8921
20	1.0342	0.9209	0.8953	0.8857	0.8818
25	1.0405	0.9155	0.8857	0.8765	0.8726
30	1.0654	0.9248	0.8936	0.8818	0.8779

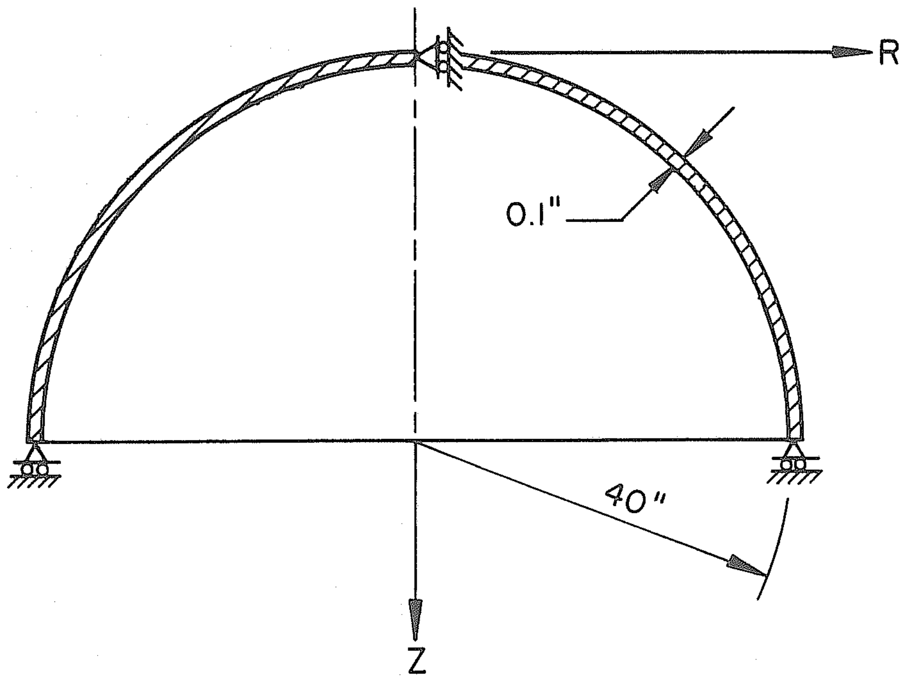


FIG. 4 SPHERICAL SHELL

IV. RESULTS OF STABILITY ANALYSIS OF THE TOROID SHELL

An attempt is made to predict the critical buckling load of a toriod shell subjected to in an ocean enviroment. The geometry of the structure is shown in Figure 5. For practical application, the shell may be reinforced with stiffeners arranged orthotropically. This kind of analysis can be done by calculating the equivalent coefficients of the elasticity matrix corresponding to given structural details. For simplicity, a shell without stiffeners is used as an example here. The material is a 1¼ inches thick steel plate having a Young's modulus of 29,600 ksi. and a Poisson's ratio of 0.3. The coefficients of the elasticity matrix are given as follows:

$$C_{ss} = 39846.153 \text{ ksi}$$

$$C_{st} = 17076.923 \text{ ksi}$$

$$C_{sw} = 17076.923 \text{ ksi}$$

$$C_{tt} = 39846.153 \text{ ksi}$$

$$C_{tw} = 17076.923 \text{ ksi}$$

$$C_{ww} = 39846.153 \text{ ksi}$$

$$G_{st} = 11384.615 \text{ ksi}$$

In the stress analysis, half of the structure is modeled into a finite element assembly shown in Figure 6. In order to match the buckling mode shapes which are both symmetric and antisymmetric about the horizontal section A-A, Two kinds of boundary condition are used in the stability analysis.

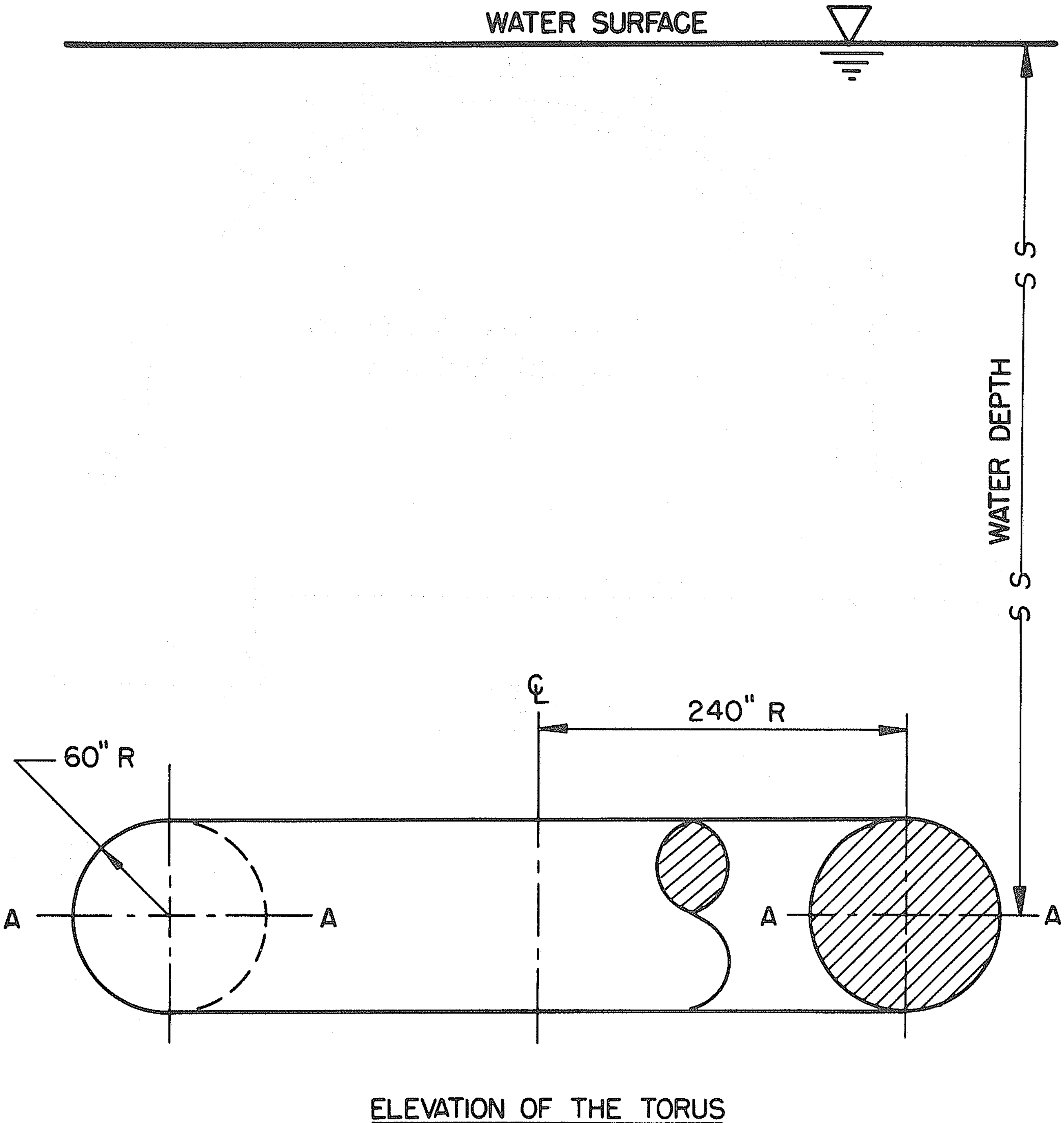


FIG. 5 TORUS SUBJECTED TO HYDROSTATIC PRESSURE

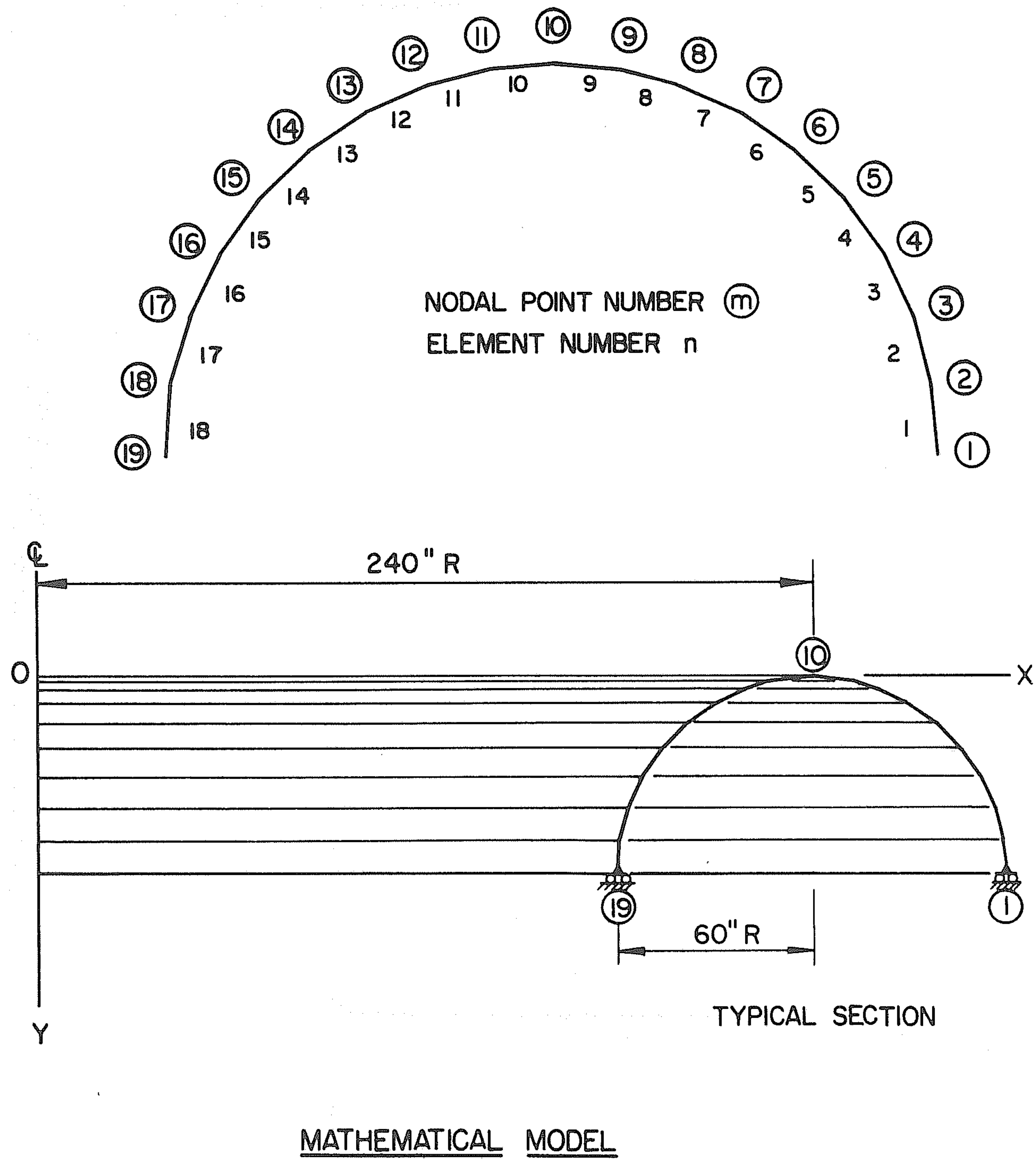


FIG. 6 FINITE ELEMENT REPRESENTATION OF TORUS

Case A: The buckling mode shape is symmetric about section A-A.

The edge conditions are prescribed so that:

Axial displacement is zero

Radial force is zero

Tangential force is zero

and rotation is zero.

Case B: The buckling mode shape is antisymmetric about section A-A.

The edge conditions are prescribed so that:

Axial force is zero

Radial displacement is zero

Tangential displacement is zero

and moment is zero.

With a load condition corresponding to a water depth of 2000 feet, the stability analysis is carried out for the first five harmonics. The eigenvalues are given as:

HARMONICS	EIGENVALUES (λ)	
(n)	CASE A	CASE B
0	1.2920	—
1	—	—
2	0.3525	0.3291
3	0.3877	0.3857
4	0.4404	0.4404

It can be seen that if the shell were buckling as a whole it would buckle at the water depth of 2584 feet. However, the shell will buckle at a water depth of 658 feet which corresponds to harmonic 2 of case B. Rigid body movement is experienced for harmonic 1 in case A and harmonics 0, 1 in case B. Therefore, the eigenvalues are zero for these conditions. The linear displacement shape is shown in Figure 7. Various buckling mode shapes are given in Figures 8-14. It is hoped that this information would give aid in modifying the structure during the course of an engineering design iteration.

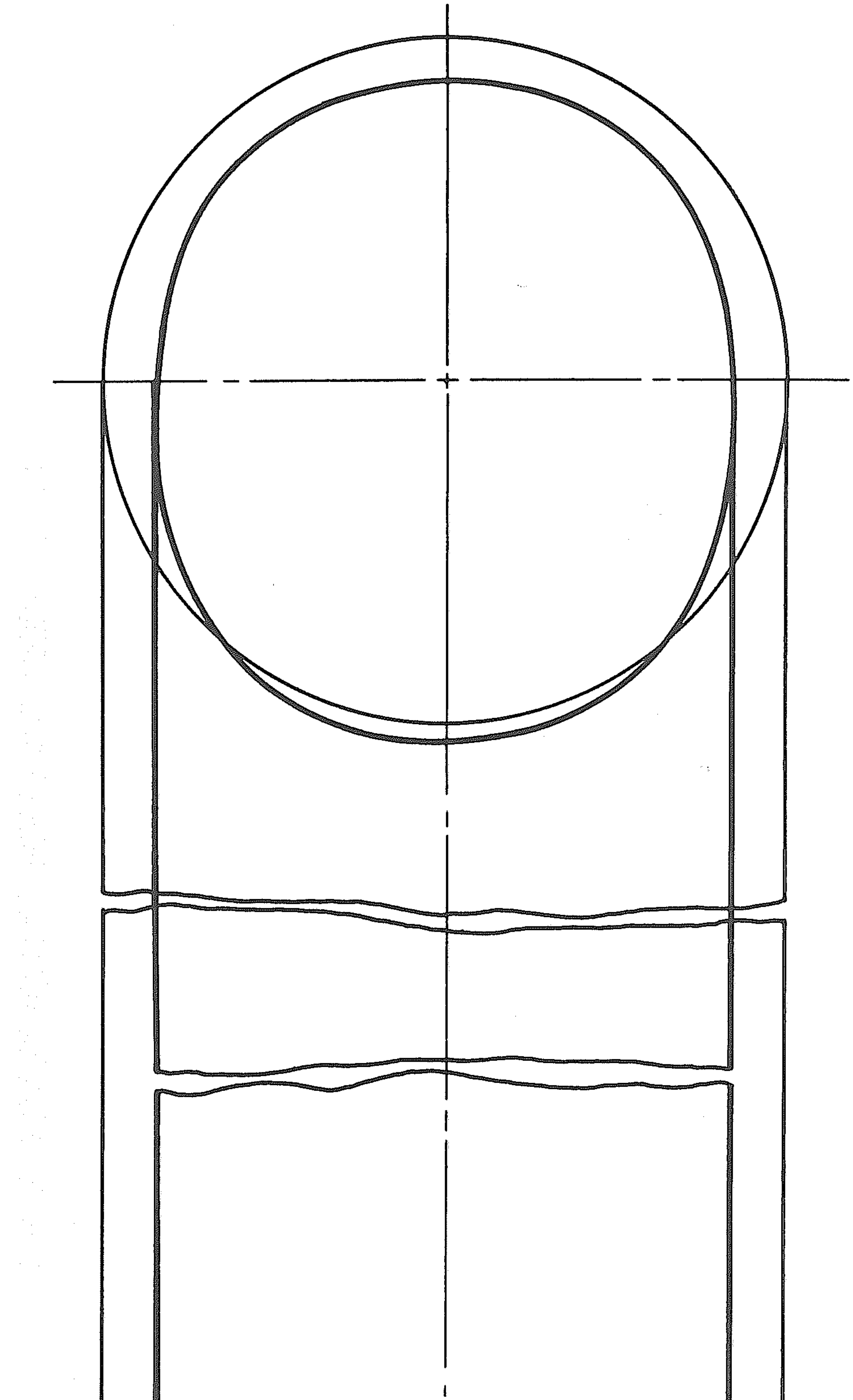


FIG. 7 LINEAR DISPLACEMENT SHAPE

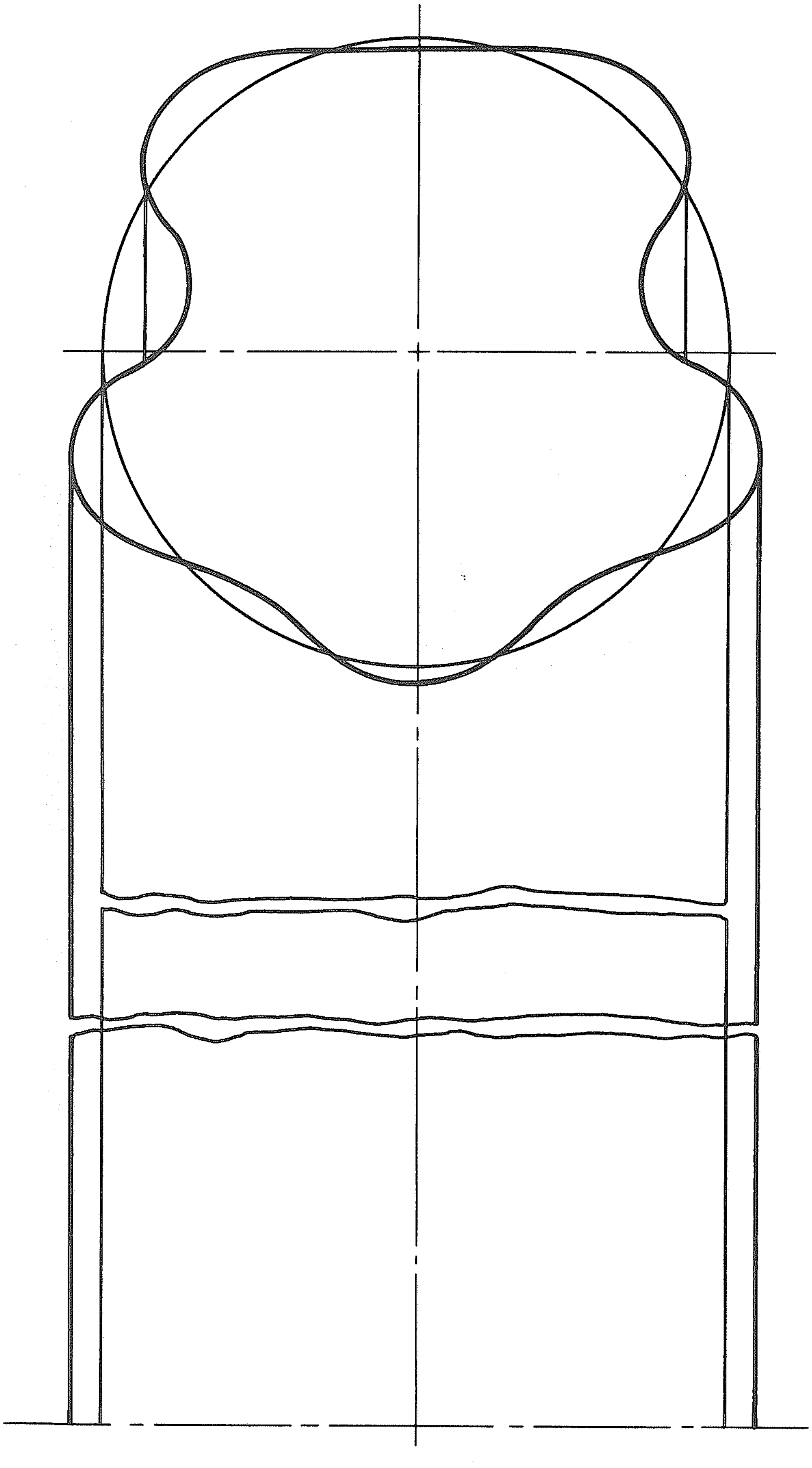


FIG. 8 FUNDAMENTAL BUCKLING MODE SHAPE

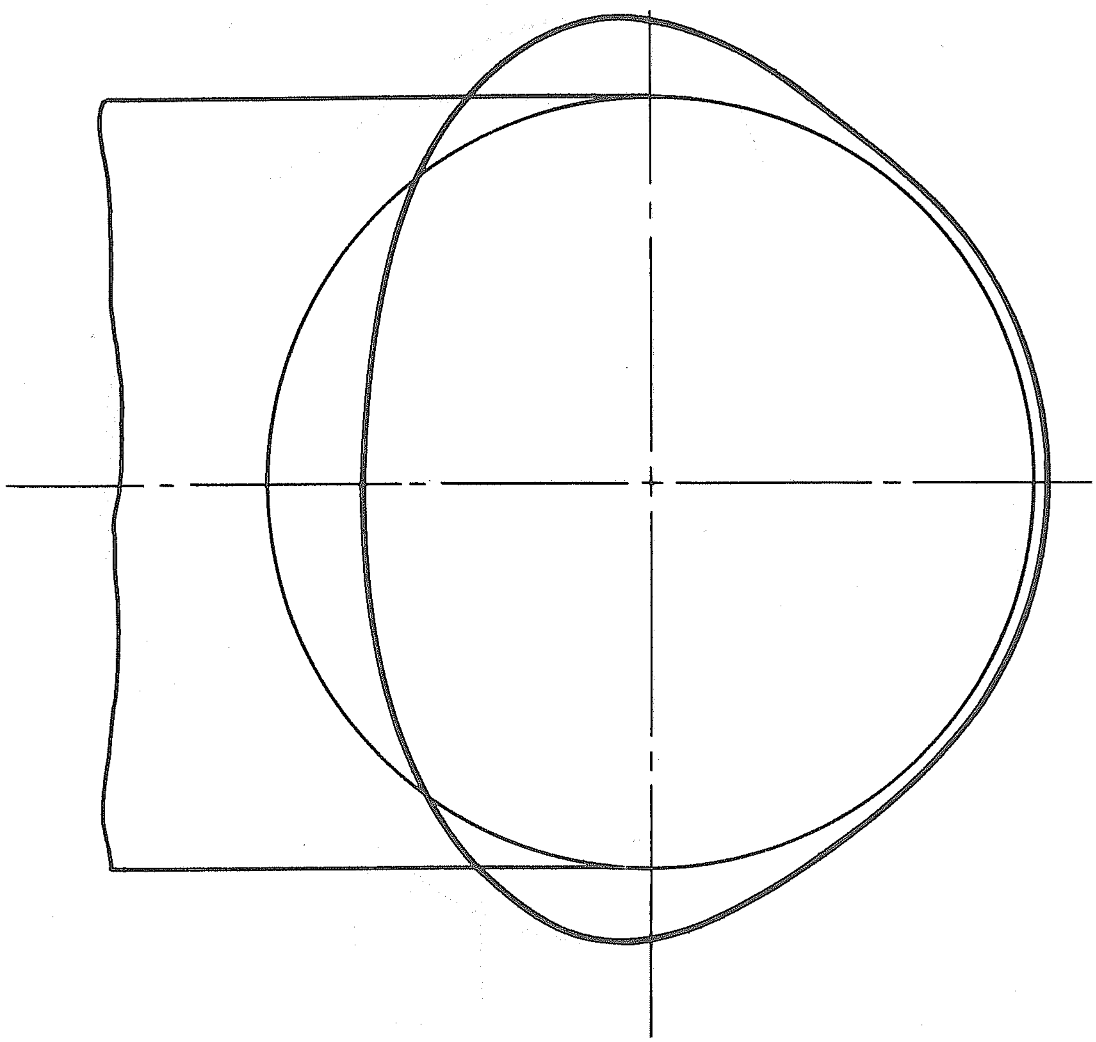


FIG. 9 BUCKLING MODE SHAPE
HARMONIC 2, CASE A

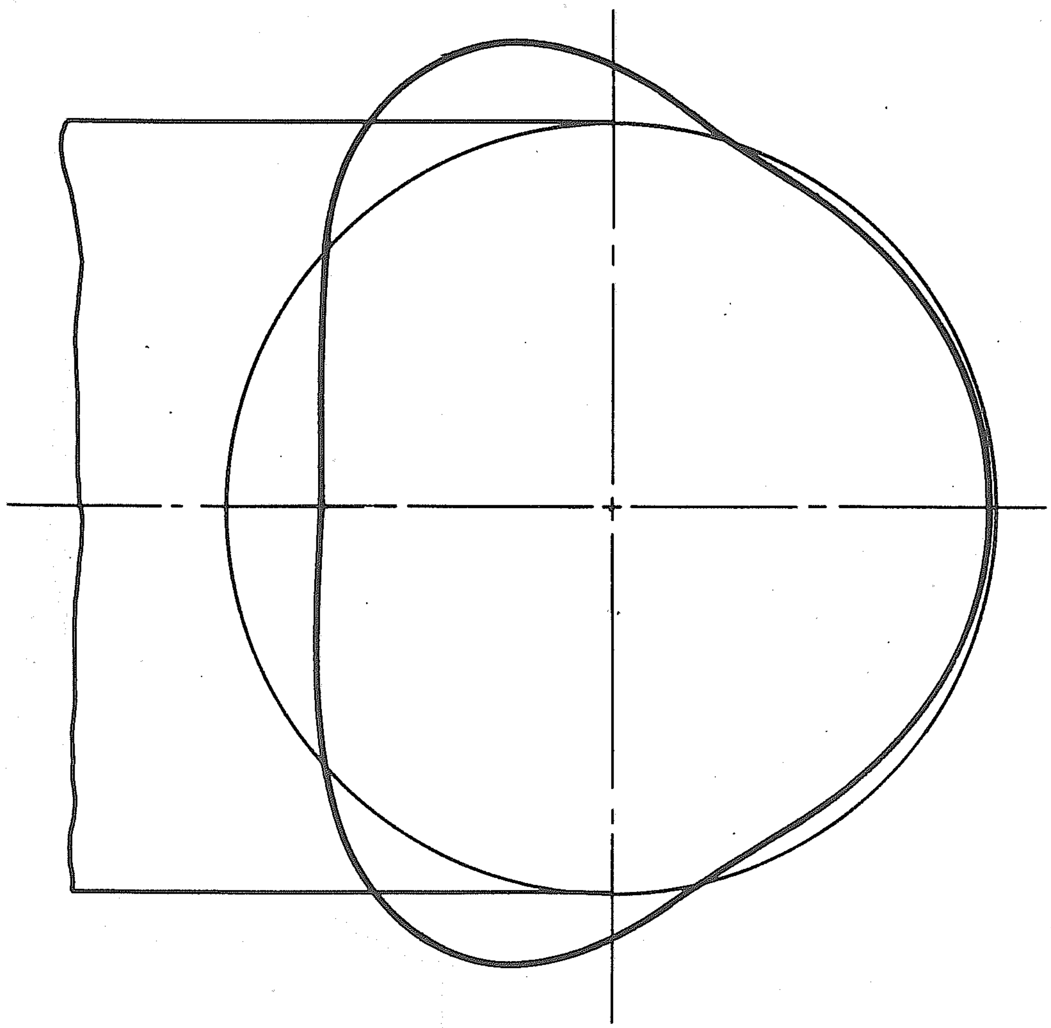


FIG. 10 BUCKLING MODE SHAPE
HARMONIC 3, CASE A

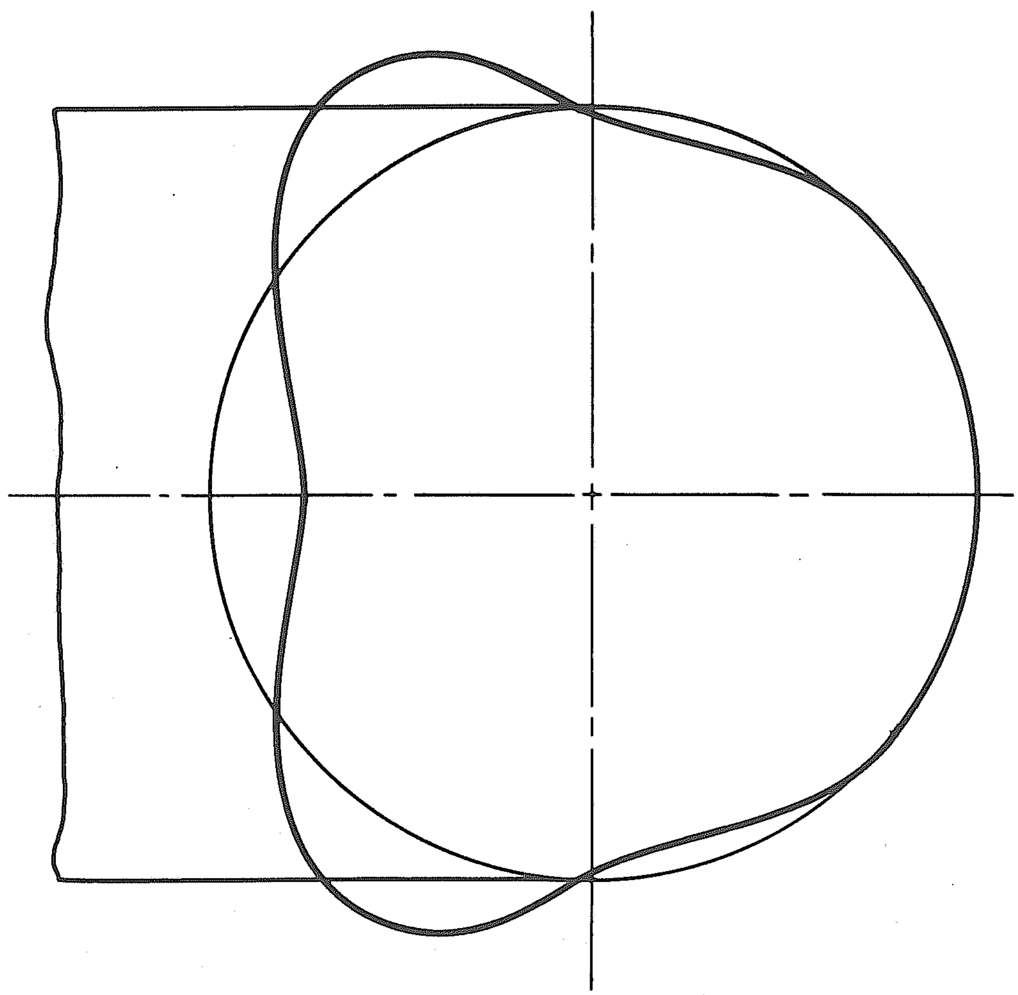


FIG. 11 BUCKLING MODE SHAPE
HARMONIC 4, CASE A

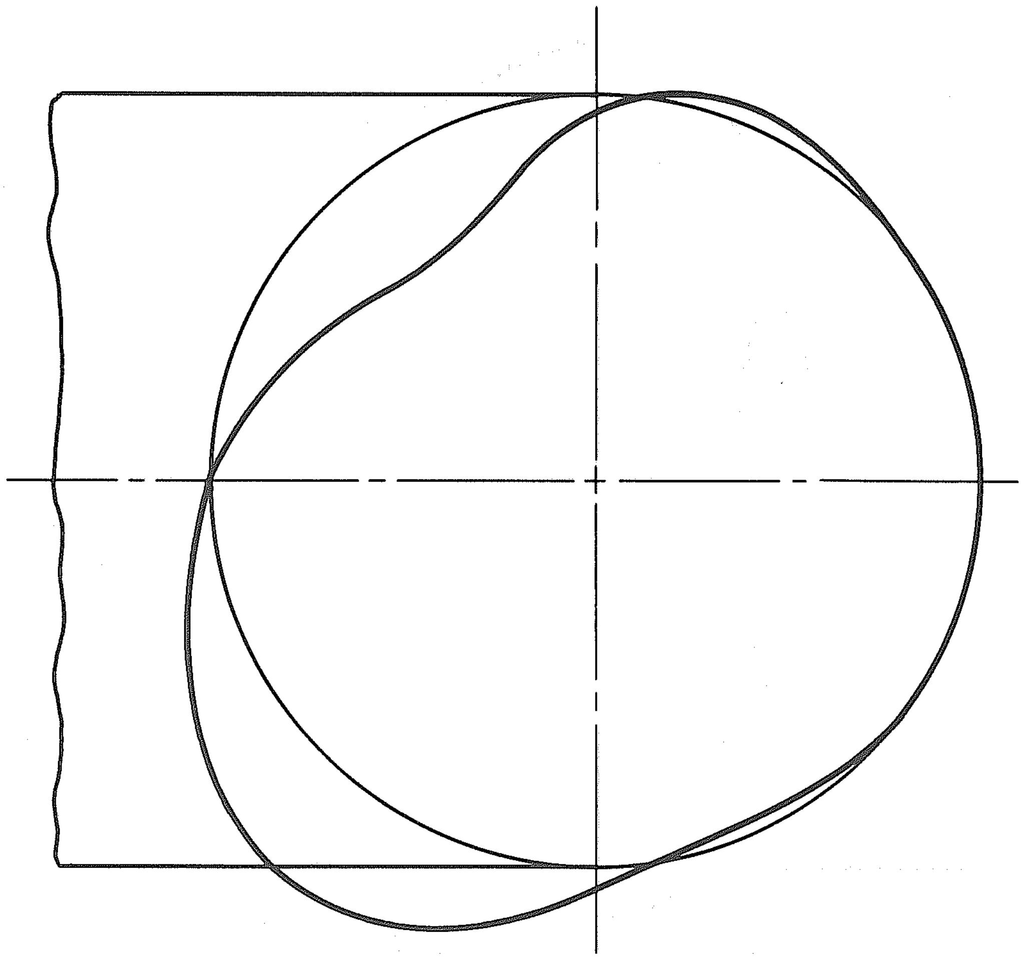


FIG. 12 BUCKLING MODE SHAPE
HARMONIC 2, CASE B

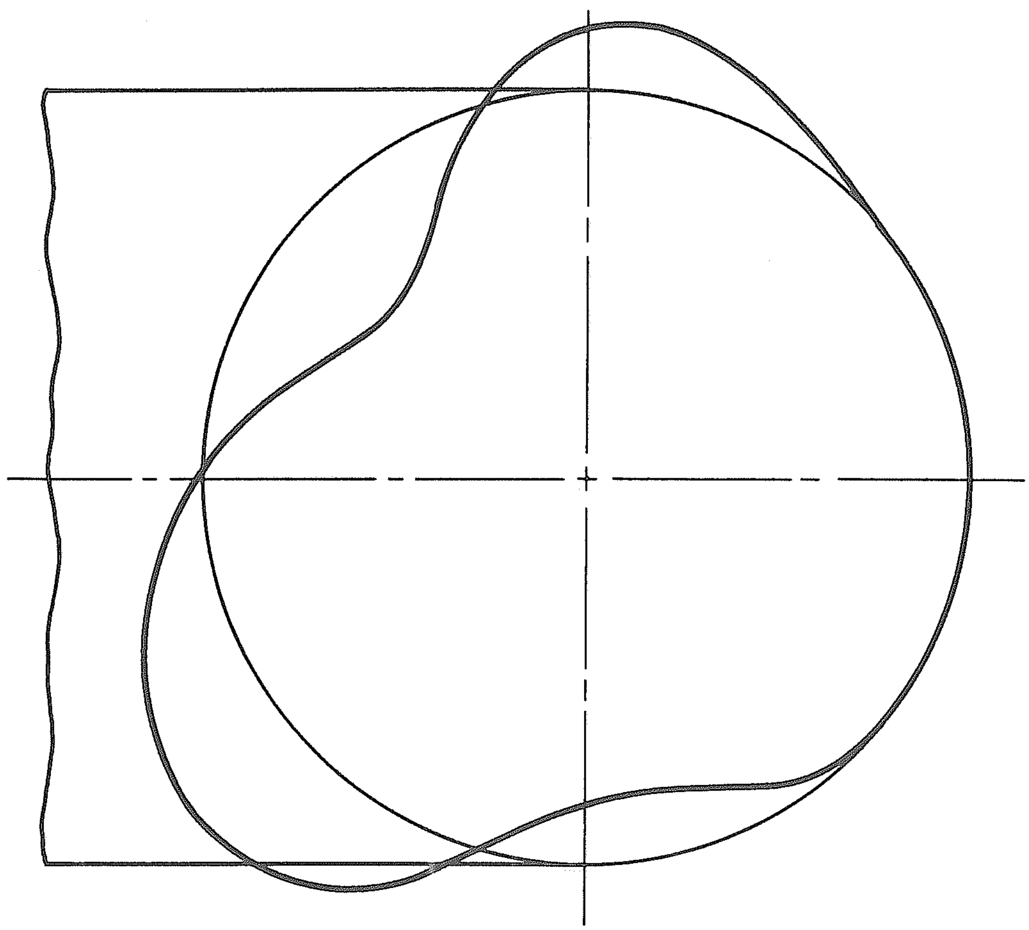


FIG. 13 BUCKLING MODE SHAPE
HARMONIC 3 , CASE B

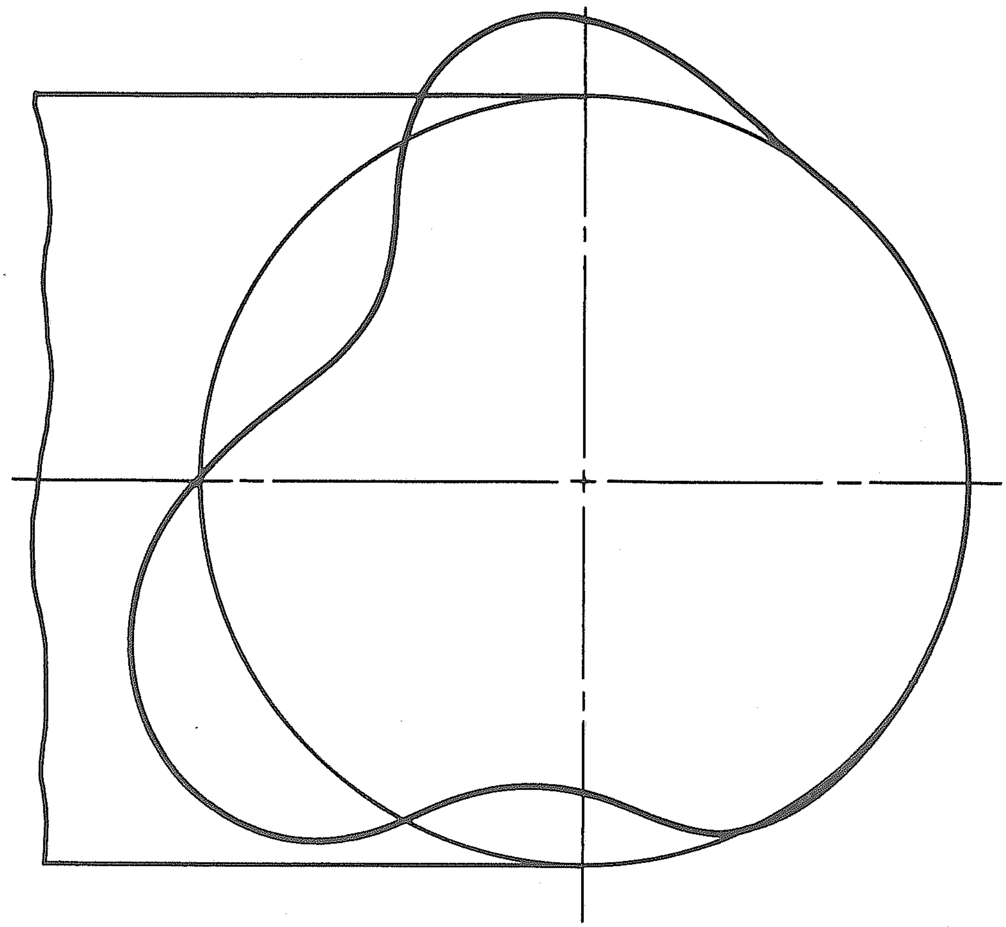


FIG. 14 BUCKLING MODE SHAPE
HARMONIC 4, CASE B

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APPENDIX A

SOLUTION OF LINEAR EQUATIONS

APPENDIX A.

SOLUTION OF LINEAR EQUATIONS

The equilibrium equations for a structural system can be written in the following form:

$$A_{11}X_1 + A_{12}X_2 + A_{13}X_3 \; \cdots\cdots + A_{1N}X_N = B_1 \qquad \text{(A-1a)}$$

$$A_{21}X_1 + A_{22}X_2 + A_{23}X_3 \; \cdots\cdots + A_{2N}X_N = B_2 \qquad \text{(A-1b)}$$

$$A_{31}X_1 + A_{32}X_2 + A_{33}X_3 \; \cdots\cdots + A_{3N}X_N = B_3 \qquad \text{(A-1c)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{N1}X_1 + A_{N2}X_2 + A_{N3}X_3 \; \cdots\cdots + A_{NN}X_N = B_N \qquad (-)$$

or, symbolically,

$$[A][X] = [B] \qquad \text{(A-1)}$$

where

[A] = the stiffness matrix

[X] = the unknown displacements

[B] = the applied loads

A.1 Gaussian Elimination

The first step in the solution of the above set of equations is to solve Eq. (A-1a) for X_1 :

$$X_1 = B_1/A_{11} - (A_{12}/A_{11})X_2 - (A_{13}/A_{11})X_3 \cdots (A_{1N}/A_{11})X_N \qquad \text{(A-2)}$$

If Eq. (A-2) is substituted into Eqs. (A-1b, c,..., N), a modified set of N-1 equations is obtained:

$$A^1_{22}X_2 + A^1_{23}X_3 \;\ldots\ldots + A^1_{2N}X_N = B^1_2 \qquad \text{(A-3a)}$$

$$A^1_{32}X_2 + A^1_{33}X_3 \;\ldots\ldots + A^1_{3N}X_N = B^1_3 \qquad \text{(A-3b)}$$

$$A^1_{N2}X_2 + A^1_{N3}X_3 \;\ldots\ldots + A^1_{NN}X_N = B^1_N \qquad \text{(--)}$$

where

$$A^1_{ij} = A_{ij} - A_{i1}A_{1j}/A_{11} \quad i, j = 2,\ldots, N \qquad \text{(A-4a)}$$

$$B^1_i = B_i - A_{i1}B_1/A_{11} \quad i = 2,\ldots, N \qquad \text{(A-4b)}$$

A similar procedure is used to eliminate X_2 from Eq. (A-3), etc. A general algorithm for the elimination of X_n can be written as

$$x_n = (B^{n-1}_n/A^{n-1}_{nn}) - \Sigma\, (A^{n-1}_{nj}/A^{n-1}_{nn})\, X_j \quad j = n + 1,\ldots, N \qquad \text{(A-5)}$$

$$A^n_{ij} = A^{n-1}_{ij} - A^{n-1}_{in}(A^{n-1}_{nj}/A^{n-1}_{nn}) \quad i, j = n + 1,\ldots, N \qquad \text{(A-6)}$$

$$B^n_i = B^{n-1}_i - A^{n-1}_{in}(B^{n-1}_n/A^{n-1}_{nn}) \quad i = n + 1,\ldots, N \qquad \text{(A-7)}$$

Equations A-5, A-6 and A-7 can be rewritten in compact form:

$$X_n = D_n - \Sigma\, H_{nj}X_j \quad j = n + 1,\ldots, N \qquad \text{(A-8)}$$

$$A^n_{ij} = A^{n-1}_{ij} - A^{n-1}_{in}\, H_{nj} \quad i, j = n + 1,\ldots, N \qquad \text{(A-9)}$$

$$B^n_i = B^{n-1}_i - A^{n-1}_{in}\, D_n \quad i = n + 1,\ldots, N \qquad \text{(A-10)}$$

where

$$D_n = B^{n-1}_n/A^{n-1}_{nn}$$

$$H_{nj} = A^{n-1}_{nj}/A^{n-1}_{nn}$$

After the above procedure is applied N-1 times, the original set of equations is reduced to the single equation

$$A_{NN}^{N-1} X_N = B_N^{N-1}$$

which is solved directly for X_N:

$$X_N = B_N^{N-1} / A_{NN}^{N-1}$$

In terms of the previous notation, this is

$$X_N = D_N \qquad \text{(A-11)}$$

The remaining unknowns are determined in reverse order by the repeated application of Eq. (A-8).

A.2 Simplification for Band Matrices

For many structural systems, the stiffness matrix occurs in a "band" form which results in the concentration of the elements of the stiffness matrix along the main diagonal. Therefore, the following simplifications in the general algorithm (Eqs. [A-8], [A-9], and [A-10] are possible:

$$x_n = D_n - \Sigma H_{nj} X_j \qquad j = n + 1, \ldots, n + M - 1 \qquad \text{(A-12)}$$

$$A_{ij}^{n} = A_{ij}^{n-1} - A_{in}^{n-1} H_{nj} \quad i,\ j = n + 1, \ldots, n + M - 1 \qquad \text{(A-13)}$$

$$B_i^n = B_i^{n-1} - A_{in}^{n-1} D_n \qquad i = n + 1, \ldots, n + M - 1 \qquad \text{(A-14)}$$

where M is the band width of the matrix.

The number of numerical operations can further be reduced by recognizing that the reduced matrix at any stage of the procedure is symmetric. Accordingly, since

$$A_{ji}^{n} = A_{ij}^{n}$$

Eq. (A-13) can be replaced by

$$A_{ij}^{n} = A_{ij}^{n-1} - A_{in}^{n-1} H_{nj} \qquad \begin{matrix} i = n + 1, \ldots, N + M - 1 \\ j = i, \ldots, n + M - 1 \end{matrix} \tag{A-15}$$

The number of numerical operations required for the solution of a band matrix is proportional to NM^2 as compared to N^3 which is required for the solution of a full matrix. Also, the computer storage required by the band matrix procedure is NM as compared to N^2 required by a set of N arbitrary equations. This is the technique used within the computer program presented in this report.

APPENDIX B.

SOLUTION OF EIGENVALUE PROBLEM OF BUCKLING TYPE

APPENDIX B.

SOLUTION OF EIGENVALUE PROBLEM OF BUCKLING TYPE

For a discrete structural system, the eigenvalue problem of buckling type can be written in the following form:

$$(E_{11}- \lambda G_{11})X_1 + (E_{12}- \lambda G_{12})X_2 + (E_{13}- \lambda G_{13})X_3 + \dots + (E_{1N}- \lambda G_{1N})X_N = 0$$

$$(E_{21}- \lambda G_{21})X_1 + (E_{22}- \lambda G_{22})X_2 + (E_{23}- \lambda G_{23})X_3 + \dots + (E_{2N}- \lambda G_{2N})X_N = 0$$

$$(E_{31}- \lambda G_{31})X_1 + (E_{32}- \lambda G_{32})X_2 + (E_{33}- \lambda G_{33})X_3 + \dots + (E_{3N}- \lambda G_{3N})X_N = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(E_{N1}- \lambda G_{N1})X_1 + (E_{N2}- \lambda G_{N2})X_2 + (E_{N3}- \lambda G_{N3})X_3 + \dots + (E_{NN}- \lambda G_{NN})X_N = 0$$

or, symbolically,

$$[E]\{X\} - \lambda [G]\{X\} = \{0\}$$

where

$[E]$ = the elastic stiffness matrix

$[G]$ = the geometric stiffness matrix

λ = the eigenvalue

$\{X\}$ = the associated eigenvector

In general, there are N eigenvalues and N associated eigenvectors. To obtain the complete solution of this kind of problem is a time consuming process. Fortunately in the stability analysis, only the lowest positive eigenvalue is of practical interest. Therefore a much simplified algorithm for solving the problem may be developed. An inverse iteration scheme had been successfully used to find the lowest eigenvalue and its associated eigenvector. However, for shell buckling problems, there might be negative eigenvalues which may or may not have physical significance. Sometimes, the lowest negative eigenvalue is associated with a reverse load pattern. In that case, a seperate analysis should be carried out if a reverse load condition is likely to happen. Therefore, a negative eigenvalue is not an objective of the analysis. In order to find a more effective means for getting the lowest positive eigenvalue, a direct trial and error numerical scheme was used.

Mathematically if λ_o is an eigenvalue of the system the following determinant

$$|[A]| = |[E] - \lambda_o [G]|$$

vanishes. In this case an element on the main diagonal will be zero in the Gaussian elimination. Furthermore, if λ_o is the lowest positive eigenvalue, in the upper triangular form $[A^{n-1}]$, all elements on the main diagonal are positive, except the last one is equal to zero. That is

$$A_{nn}^{n-1} = 0.$$

To approach λ_o, the trial and error procedure is outlined below:

1. Let λ^U be the upper bound of λ_o and λ^L be the lower bound of λ_o, such that

$$\lambda^U \geq \lambda_o \geq \lambda^L$$

and

$$\lambda^U > o \; ; \; \lambda^L \geq o$$

Assume a trial eigenvalue

$$\lambda^T = \frac{1}{2} (\lambda^U + \lambda^L)$$

2. With λ^T, compute [A], i.e.

$$[A] = [E] - \lambda [G]$$

3. Reduce matrix [A] to upper triangular form by Gauss elimination. During this process, one of three possible situations will actually take place:

a. Matrix [A] is positive definite, i.e.

$$A_{nn}^{n-1} > o \; , \; n = 1, 2, 3, \ldots, N$$

Modify the lower bound

$$\lambda^L = \lambda^T$$

b. Matrix [A] is indefinite or negative definite, i.e.

$$A_{nn}^{n-1} \leq o \quad , \quad n = 1, 2, 3, \ldots, N-1$$

or

$$A_{nn}^{n-1} < o.$$

Modify the upper bound

$$\lambda^U = \lambda^T$$

c. Matrix [A] is positive-semidefinite , i.e.

$$A_{nn}^{n-1} = o,$$

For the first two cases in step 3, the above procedure would be repeated until the difference between the upper bound and the lower bound vanishes sufficiently. In the third case, the trial eigenvalue is the lowest positive eigenvalue.

After the lowest positive eigenvalue has been found, its associated eigenvector is solved by back substitution. First of all, X_n is set to equal unity. The remaining unknowns are determined in reverse order by the repeated application of Eq. (A-8). Then, the eigenvector is normalized with respect to the element of maximum absolute value.

To initialized the trial and error procedure, the lower bound is set to zero and the trial eigenvalue is taken to be unity. If the matrix [A] is positive definite, the second trial eigenvalue is set to two. If the matrix [A] is still positive definite, four is used. Until, after n trials the matrix [A] is no longer positive definite. Therefore, the initial bounds can be established:

$$\lambda^U = 2^{n-1}$$

$$\lambda^L = 2^{n-2} \quad , \text{ if } n > 1$$

$$\lambda^L = 0 \quad , \text{ if } n = 1$$

Both elastic and geometric stiffness matrices are in "band" form; therefore, the computer time and storage may be minimized in the Gauss elimination process.

APPENDIX C.

COMPUTER PROGRAM USAGE

APPENDIX C.

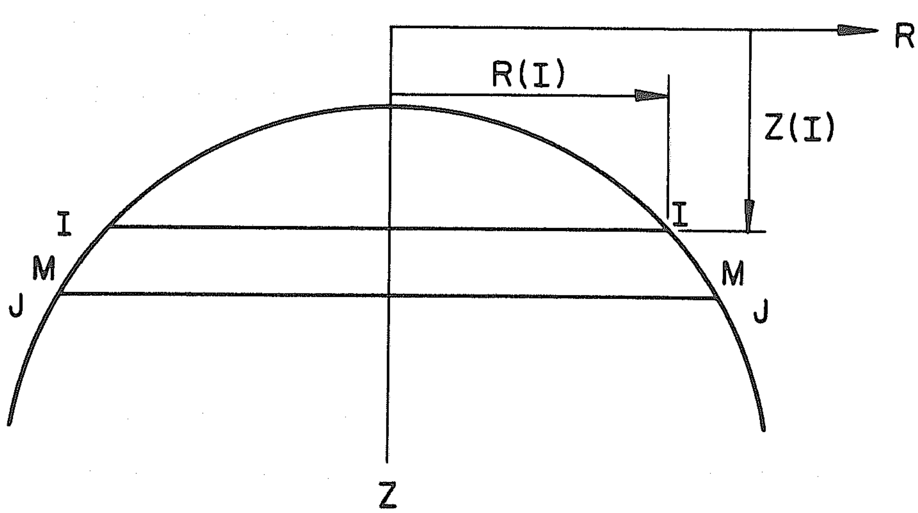
COMPUTER PROGRAM USAGE

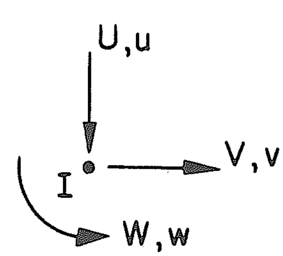
1. IDENTIFICATION

SAAS — Stability Analysis of Axisymmetric Shells

2. PURPOSE

The purpose of this computer program is to provide a stability analysis of axisymmetric thin shells under axisymmetric loading conditions. For each shell to be analysed, an assembly of conical finite elements is used to represent the continuous elastic body. The nodal displacements and element stress resultants are determined by means of conventional linear structural theory. The buckling mode shapes are assumed to be infinitesimal and in forms of Fourier series expansions. For each harmonic, a discrete eigenvalue problem is set up, and the lowest positive eigenvalue and its associated eigenvector are solved. The positive definitions for the input and output data are shown below:





3. INPUT DATA

For each shell to be analysed, a group of punched cards is required in this sequence.

A. START CARD (5H)

The word START must be punched in columns 1 to 5 on a separate card at the beginning of each problem.

B. TITLE CARD (12 A6)

Columns	1 - 72	Alphanumeric data for problem identification.

C. CONTROL CARD (7I5)

Columns	1 - 5	Number of nodal circles
	6 - 10	Number of elements
	11 - 15	Number of material types
	16 - 20	Number of harmonics
	21 - 25	Number of harmonics to be skipped (n). If n is greater than one, the harmonic buckling mode shapes in sequence starting from harmonic zero are analysed with (n-1) harmonics being skipped. When n is zero or one, no skipping takes place.
	26 - 30	Number of nodal boundary codes in stability analysis which are different from those in stress analysis. When a non-zero positive integer is assigned in this field, a same number of boundary code cards must follow the nodal data cards. (maximum = 10)

	31 - 35	Geometric code in stability analysis (0 or 1)
		0: deformed geometry is used in stability analysis.
		1: initial geometry is used in stability analysis.

D. MATERIAL PROPERTY CARDS (I5, 7F10.0)

A card must be supplied for each different material. The material identification numbers must be in sequence starting from one.

Columns	1 - 5	Material identification number
	6 - 15	C_{ss} elasticity element
	16 - 25	$C_{s\theta}$ elasticity element
	26 - 35	$C_{s\xi}$ elasticity element
	36 - 45	$C_{\theta\theta}$ elasticity element
	46 - 55	$C_{\theta\xi}$ elasticity element
	56 - 65	$C_{\xi\xi}$ elasticity element
	66 - 75	$G_{s\theta}$ elasticity element.

E. NODAL CIRCLE CARDS (I5, 2F10.4, I5, 4F10.4, I1, F9.0)

One card for each nodal circle.

Columns	1 - 5	Nodal circle number
	6 - 15	Radial coordinate (R)
	16 - 25	Axial coordinate (Z)

26 - 30	Nodal boundary code - see *below
31 - 40	Axial force or displacement
41 - 50	Radial force or displacement
51 - 60	Tangential force or displacement (to be left blank)
61 - 70	Moment or angular displacement
71	ID - see** below
72 - 80	Curvature - see** below.

*The boundary code is a 4 digit number (consisting of 0 or 1) which specifies if applied "loads" are forces or displacements.

1 specified displacement

0 specified force

e.g. 1001 U(I) is specified displacement
V(I) is specified force
T(I) is specified force
W(I) is specified rotation

**These are optional input data needed if automatic nodal circle data generation is being used. See later section.

F. NODAL BOUNDARY CODE CARDS (2I5)

One card for each nodal circle whose nodal boundary code in stability analysis is different from that in stress analysis.

Columns		
	1 - 5	Nodal circle number.
	6 - 10	Nodal boundary code in stability analysis.

G. ELEMENT CARDS (3I5, F10.4, I5, F10.4)

One card for each element

Columns	1 - 5	Element number
	6 - 10	Number of node I
	11 - 15	Number of node J
	16 - 25	Element thickness
	26 - 30	Number of material type.
	31 - 40	Normal pressure (force per unit area)

H. STOP CARD (4H)

For normal termination of execution the complete data check (not each individual problem) finishes with a card with the work STOP punched in columns 1 - 4.

AUTOMATIC DATA GENERATION.

A. NODAL CIRCLE DATA.

This may be used if a series of nodes is on

a) A straight line i or

b) A circular arc

Automatic data generation is activated if two successive input nodal circle numbers differ by more than one. Nodal circle are required for the first and last nodes in the series. The nodes must have either

a) Zero boundary code - set ID = 0 on the last card in the series; or

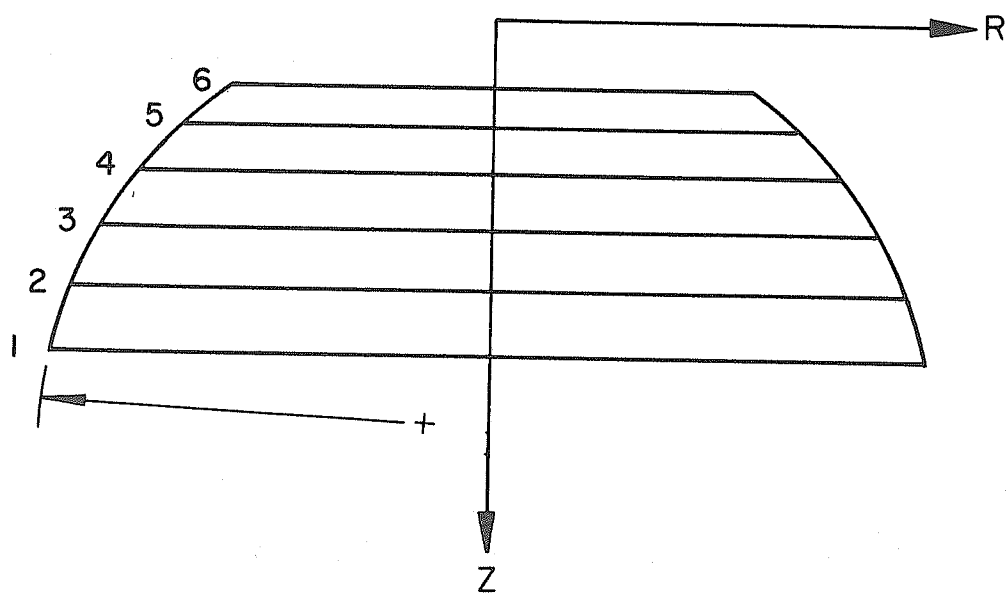
b) Same boundary code as the last - set ID = 1 on the last card in the series.

The last card must also contain the curvature of the line,

a) Straight line - Curvature = 0

b) Circle of radius R - Curvature = 1/R

The sketch below shows positive curvature and appropriate nodal circle numbering sequence.



B. ELEMENT DATA

This may be used if a series of elements has the same

a) Material

b) Thickness

c) Normal pressure

and the nodal circle numbering is sequential from the first to the last. Element cards are required for the first and last elements in the series.

4. OUTPUT INFORMATION

The program prints the following output:

A. INPUT AND GENERATED DATA.

B. THE RESULTS OF THE STRESS ANALYSIS.

a) Nodal circle displacements

b) Stress resultants evaluated at the mid-circle of each element.

C. THE RESULTS OF THE STABILITY ANALYSIS FOR EACH HARMONIC.

a) Eigenvalue for every trial and error cycle.

b) Buckling mode shape.

APPENDIX D.

FORTRAN IV PROGRAM LISTING

```
      PROGRAM SAAS (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C *** STABILITY ANALYSIS OF AXISYMMETRIC SHELLS
C *** AXISYMMETRIC LOAD CONDITION
C *** ORTHOTROPIC, LINEARLY ELASTIC MATERIALS
C *** TRUNCATED CONICAL SHELL ELEMENTS
C *** AN EIGENVALUE PROBLEM IS GENERATED AND SOLVED
C
      COMMON HED(12),NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,MBAND,NUMCON,NUMOB
     1 ,IDEF,SK(8,8),CNFV(8),FN,FN2,NE,MS,MOB(10),KODEM(10),A(15000)
      DIMENSION WORD(2)
      DATA WORD /6HSTART ,6HSTOP  /
C
10    READ (5,1000) WORD1
      IF (WORD1.EQ.WORD(1)) GO TO 20
      IF (WORD1.EQ.WORD(2)) STOP
      GO TO 10
C
C *** READ AND PRINT OF CONTROL INFORMATION
C
20    READ  (5,1010) HED,NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,NUMOB,IDEF
      WRITE (6,2010) HED,NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,NUMOB
C
      NEQ=4*NUMNP
      N2=1+NUMNP
      N3=N2+NUMNP
      N4=N3+NUMEL
      N5=N4+NUMEL
      N6=N5+7*NUMMAT
      N7=N6+NEQ
      N8=N7+NUMNP
      N9=N8+NUMEL
      N10=N9+NUMEL
      N11=N10+NUMEL
C
C *** READ, GENERATE AND PRINT OF INPUT DATA
C
      CALL INPUT (A(1),A(N2),A(N3),A(N4),A(N5),A(N6),A(N7),A(N8),A(N9)
     1 ,A(N10))
C
      N12=N11+NUMEL
      N13=N12+NUMEL
      N14=N13+MBAND*NEQ
      N15=N14+MBAND*NEQ
      N16=N15+MBAND*NEQ
C
C *** STABILITY ANALYSIS
C
      CALL SOLVE (A(1),A(N2),A(N3),A(N4),A(N5),A(N6),A(N7),A(N8),A(N9)
     1 ,A(N10),A(N11),A(N12),A(N13),A(N14),A(N15),A(N16),NEQ)
C
      WRITE (6,2020)
      GO TO 10
C
1000  FORMAT (A6)
```

```
1010  FORMAT (12A6/16I5)
2010  FORMAT (1H1,12A6/'
     1 10X,30HNUMBER OF JOINTS-------------=,I5//
     2 10X,30HNUMBER OF ELEMENTS-----------=,I5//
     3 10X,30HNUMBER OF MATERIALS----------=,I5//
     4 10X,30HNUMBER OF HARMONICS----------=,I5//
     5 10X,30HNUMBER OF SKIPPING HARMONICS-=,I5//
     6 10X,30HNUMBER OF MODIFIED KODES-----=,I5)
2020  FORMAT (* THE END OF THE PROBLEM *)
      END
```

```
      SUBROUTINE INPUT (R,Z,PRES,T,E,B,KODE,JI,JJ,MATERL)
C
C *** READ, GENERATE AND PRINT OF INPUT DATA
C
      COMMON HED(12),NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,MBAND,NUMCON,NUMOB
     1 ,IDEF,SK(8,8),CNFV(8),FN,FN2,NE,MS,MOB(10),KODEM(10)
      DIMENSION R(1),Z(1),PRES(1),T(1),E(7,1),B(1),KODE(1),JI(1),JJ(1)
     1 ,MATERL(1)
C
C *** READ AND PRINT OF MATERIAL PROPERTIES
C
      WRITE (6,2000)
      DO 10 N=1,NUMMAT
      READ  (5,1000) N,(E(I,N),I=1,7)
10    WRITE (6,2001) N,(E(I,N),I=1,7)
C
C *** READ AND PRINT OF NODAL POINT DATA
C
C *** AUTOMATIC GENERATION OF NODAL POINT DATA
C *** IF A SERIES OF NODAL POINTS LIE EITHER
C ***   (1) ON A STRAIGHT LINE, OR
C ***   (2) ON A CIRCULAR ARC
C *** NODAL POINT CARDS ARE NEEDED FOR  THE FIRST AND LAST IN SERIES
C *** BOUNDARY KODE FOR INTERMEDIATE NODES MAY BE EITHER
C ***   (1) FREE NODES (KODE=0), SET ID=0, OR
C ***   (2) SAME KODE AND LOADS AS LAST IN SERIES, SET ID=1
C
C *** NODAL POINT DATA GENERATED WHEN TWO SUCCESSIVE
C *** INPUT NODE NUMBERS DIFFER BY MORE THAN ONE
C
      NPREV=0
      N=0
      WRITE (6,2002)
11    IF (N.EQ.NUMNP) GO TO 18
      NPREV=N
      READ (5,1001)N,R(N),Z(N),KODE(N),B(4*N-3),B(4*N-2),B(4*N-1),B(4*N)
     1,ID,CURV
      IF (N-NPREV-1) 11,11,12
12    NX=N
      IF (ID.EQ.0) GO TO 13
      KOD=KODE(N)
      B1=B(4*N-3)
      B2=B(4*N-2)
      B3=B(4*N-1)
      B4=B(4*N)
      GO TO 14
13    KOD=0
      B1=0.
      B2=0.
      B3=0.
      B4=0.
14    NMND=N-NPREV-1
      DO 50 I=1,NMND
      K=NPREV+I
      KODE(K)=KOD
```

```
      B(4*K-3)=B1
      B(4*K-2)=B2
      B(4*K-1)=B3
50    B(4*K)=B4
      DR=R(N)-R(NPREV)
      DZ=Z(N)-Z(NPREV)
      DEL=NMND+1
C
      IF (CURV) 100,200,100
C
100   DL=SQRT(DR*DR+DZ*DZ)
      THET=ASIN(DL*CURV*0.5)
      FI=ATAN(DZ/DR)
      BET=2.0*ATAN(1.0)-FI-THET
      RC=R(NPREV)-COS(BET)/CURV
      ZC=Z(NPREV)+SIN(BET)/CURV
      DEL=2.0*THET/DEL
      DO 150 I=1,NMND
      K=NPREV+I
      BET=BET+DEL
      R(K)=RC+COS(BET)/CURV
150   Z(K)=ZC-SIN(BET)/CURV
      GO TO 11
200   DR=DR/DEL
      DZ=DZ/DEL
      DO 250 I=1,NMND
      K=NPREV+I
      R(K)=R(K-1)+DR
250   Z(K)=Z(K-1)+DZ
      GO TO 11
18    CONTINUE
      DO 19 N=1,NUMNP
19    WRITE(6,2003)N,R(N),Z(N),KODE(N),B(4*N-3),B(4*N-2),B(4*N-1),B(4*N)
C
C *** READ AND PRINT OF CHANGING BOUNDARY KODES
C
      IF (NUMOB.EQ.0) GO TO 26
      WRITE (6,2006)
      DO 25 I=1,NUMOB
      READ  (5,1003) MOB(I),KODEM(I)
25    WRITE (6,2007) I,MOB(I),KODEM(I)
26    CONTINUE
C
C *** READ AND PRINT OF ELEMENT DATA
C
C *** AUTOMATIC GENERATION OF ELEMENT DATA
C *** IF ELEMENTS OF A SERIES HAVE
C *** SEQUENTIAL NUMBERING OF CORRESPONDING NODAL POINTS AND COMMON
C ***   (1) MATERIAL PROPERTIES
C ***   (2) THICKNESS
C ***   (3) DISTRIBUTED LOADING
C *** CARDS REQUIRED FOR FIRST AND LAST ELEMENT IN SERIES
C
      N=0
      MBAND=0
```

```
      WRITE (6,2004)
20    READ  (5,1002) NX,JINX,JJNX,TN,MATL,ELPRES
      IF (NX.GT.(N+1)) GO TO 21
      N=NX
      JIN=JINX
      JJN=JJNX
      GO TO 23
21    KON=NX-N
      INC=(JINX-JIN)/KON
22    JIN=JIN+INC
      JJN=JJN+INC
      N=N+1
23    CONTINUE
      JI(N)=JIN
      JJ(N)=JJN
      T(N)=TN
      MATERL(N)=MATL
      PRES(N)=ELPRES
      WRITE (6,2005) N,JIN,JJN,TN,MATL,ELPRES
C
C *** COMPUTE BANDWIDTH
C
      MB=IABS(JIN-JJN)
      IF (MB.GT.MBAND) MBAND=MB
      IF (N.LT.NX) GO TO 22
      IF (NUMEL-N) 24,24,20
24    CONTINUE
      MBAND=4*MBAND+4
C
      RETURN
1000  FORMAT (I5,7F10.4)
1001  FORMAT (I5,2F10.4,I5,4F10.4,I1,F9.0)
1002  FORMAT (3I5,F10.4, I5, F10.4)
2000  FORMAT (///,1X,19HMATERIAL PROPERTIES/
     1 1X,5H TYPE,8X,4HC-SS,8X,4HC-ST,8X,4HC-SW,8X,4HC-TT,8X,4HC-TW
     2,8X,4HC-WW,8X,4HG-ST//)
2001  FORMAT (1X,I5,7E12.4)
2002  FORMAT (///,1X,9HNODE DATA/
     1 4X,6HNUMBER,10X,8HR CO-ORD,10X,8HZ CO-ORD,10X,4HKODE
     2,10X,6HLOAD Z,10X,6HLOAD R,10X,6HLOAD T,10X,6HLOAD W//)
2003  FORMAT       (5X,I5,2(9X,F10.4),9X,I5,4(6X,F10.4))
2004  FORMAT (///1X,12HELEMENT DATA/
     11X,90H          NUMBER           NODE I           NODE J        THICKNESS
     1        MATERIAL        PRESSURE//)
2005  FORMAT (1X,3(10X,I5),5X,F10.4,10X,I5,5X,F10.4)
1003  FORMAT (2I5)
2006  FORMAT (///1X,*BOUNDARY CONDITIONS IN STABILITY ANALYSIS WHICH ARE
     1 DIFFERENT FROM THOSE IN EQUILIBRIUM ANALYSIS*/
     2 1X,45H          NUMBER           NODE            KODE//)
2007  FORMAT (1X,3(10X,I5))
      END
```

```
      SUBROUTINE SOLVE (R,Z,PRES,T,E,B,KODE,JI,JJ,MATERL,SNM,SNT,SKE,SKG
     1 ,A,NPP,NEQ)
C
C *** GENERATE AND SOLVE EIGENVALUE PROBLEM OF BUCKLING TYPE
C
      COMMON HED(12),NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,MBAND,NUMCON,NUMOB
     1 ,IDEF,SK(8,8),CNFV(8),FN,FN2,NE,MS,MOB(10),KODEM(10)
      DIMENSION R(1),Z(1),PRES(1),T(1),E(7,1),B(1),SNM(1),SNT(1),NPP(1)
     1 ,SKE(NEQ,1),SKG(NEQ,1),A(NEQ,1),KODE(1),JI(1),JJ(1),MATERL(1)
C
C *** STRESS ANALYSIS
C
      DO 22 I=1,NEQ
      DO 22 J=1,MBAND
22    SKE(I,J)=0.
      FN=0.
      FN2=0.
      DO 11 NE=1,NUMEL
C
C *** FORM NODAL LOAD VECTOR
C
      JIN=JI(NE)
      JJN=JJ(NE)
      DR=R(JJN)-R(JIN)
      DZ=Z(JJN)-Z(JIN)
      DL=SQRT(DR*DR+DZ*DZ)
      SN=DR/DL
      CS=DZ/DL
      RI=R(JIN)
      RJ=R(JJN)
      CI=RI+DR/3.
      CJ=CI+DR/3.
      CON=3.141592654*DL*PRES(NE)
      CNFV(1)=CON*CI*SN
      CNFV(2)=CON*CI*(-CS)
      CNFV(5)=CON*CJ*SN
      CNFV(6)=CON*CJ*(-CS)
      CNFV(3)=0.0
      CNFV(4)=0.0
      CNFV(7)=0.0
      CNFV(8)=0.0
C
C *** FORM ELASTIC STIFFNESS MATRIX
C
      CALL ELASTI (R,Z,T,E,JI,JJ,MATERL)
      CALL MODIFY (SKE,B,JI,JJ,KODE,NEQ,1)
C
      DO 40 I=1,4
40    B(4*JIN-4+I)=B(4*JIN-4+I)+CNFV(I)
      DO 41 I=5,8
41    B(4*JJN-8+I)=B(4*JJN-8+I)+CNFV(I)
11    CONTINUE
C
C *** SOLVE FOR NODAL DISPLACEMENTS
C
```

```
      CALL SYMSOL (SKE,B,NEQ,MBAND,1)
      CALL SYMSOL (SKE,B,NEQ,MBAND,2)
C
C *** PRINT OUT NODAL DISPLACEMENTS
C
      WRITE (6,1000)
      DO 50 N=1,NUMNP
50    WRITE (6,1001) N,B(4*N-3),B(4*N-2),B(4*N)
C
C *** COMPUTE THE STRESS RESULTANTS
C
      CALL ELSTRS (R,Z,T,E,B,SNM,SNT,JI,JJ,MATERL)
C
C *** STABILITY ANALYSIS
C
      IF (IDEF.NE.0) GO TO 23
      DO 21 I=1,NUMNP
      Z(I)=Z(I)+B(4*I-3)
21    R(I)=R(I)+B(4*I-2)
23    CONTINUE
C
      IF (NUMOB.EQ.0) GO TO 20
      DO 10 I=1,NUMOB
      J=MOB(I)
10    KODE(J)=KODEM(I)
20    CONTINUE
C
      DO 38 NH=1,NUMHAR
C
C *** ZERO STIFFNESS MATRICES
C
      DO 42 I=1,NEQ
      DO 42 J=1,MBAND
      SKE(I,J)=0.
42    SKG(I,J)=0.0
C
      NF=NH-1
C
C *** SKIP NUMSKH-1 HARMONICS
C
      IF (NUMSKH.GT.0) NF=NF*NUMSKH
      FN=NF
      FN2=FN*FN
      IF (NH.EQ.2) CALL MODIFY1 (SKE,KODE,NPP,NEQ,NEQ1,1)
      IF (FN.NE.0.) GO TO 100
      CALL MODIFY1 (SKE,KODE,NPP,NEQ,NEQ1,1)
C
C *** FORM ELASTIC AND GEOMETRIC STIFFNESS MATRICES FOR ZERO HARMONIC
C
      DO 101 NE=1,NUMEL
      CALL GEOSTI (R,Z,SNM,SNT,JI,JJ)
      CALL MODIFY (SKG,B,JI,JJ,KODE,NEQ,2)
      CALL ELASTI (R,Z,T,E,JI,JJ,MATERL)
      CALL MODIFY (SKE,B,JI,JJ,KODE,NEQ,2)
101   CONTINUE
```

```
      GO TO 102
100   CONTINUE
C
C *** FORM ELASTIC AND GEOMETRIC STIFFNESS MATRICES FOR HIGHER HARMONICS
C
      DO 37 NE=1,NUMEL
      CALL ELASTI (R,Z,T,E,JI,JJ,MATERL)
      CALL MODIFY (SKE,B,JI,JJ,KODE,NEQ,2)
      CALL GEOSTI (R,Z,SNM,SNT,JI,JJ)
      CALL MODIFY (SKG,B,JI,JJ,KODE,NEQ,2)
37    CONTINUE
C
C *** SOLVE FOR THE SMALLEST POSITIVE EIGENVALUE AND ASSOCIATED EIGENVECTOR
C
102   WRITE(6,1002) NF
      CALL MODIFY1 (SKE,KODE,NPP,NEQ,NEQ1,2)
      CALL MODIFY1 (SKG,KODE,NPP,NEQ,NEQ1,2)
      CALL BUEIG (A,SKE,SKG,B,NEQ,NEQ1,MBAND,1)
      CALL BUEIG (A,SKE,SKG,B,NEQ,NEQ1,MBAND,2)
C
C *** PRINT OUT BUCKLING MODE SHAPE
C
      DO 60 I=1,NUMCON
      NE=NEQ-NPP(I)
      IF (NE.EQ.0) GO TO 63
      DO 61 J=1,NE
61    B(NEQ-J+1)=B(NEQ-J)
63    NE=NPP(I)
      B(NE)=0.
60    CONTINUE
      WRITE (6,3001) NF
      DO 62 N=1,NUMNP
62    WRITE (6,3002) N,B(4*N-3),B(4*N-2),B(4*N-1),B(4*N)
38    CONTINUE
C
48    RETURN
1000  FORMAT    (1H1,18HNODE DISPLACEMENTS,6X,4HNODE,5X,
     110H   DISPL(Z),7X,10H   DISPL(R),7X,11HROTATION(W)//)
1001  FORMAT (23X,I5,3(5X,E12.5))
1002  FORMAT (///6X,*EIGENVALUE OF HARMONIC*,I5/
     1 6X,5HCYCLE,20H              EIGENVALUE//)
3001  FORMAT (///,6X,*BUCKLING MODE SHAPE OF HARMONIC*,I5,/
     1 11X,4HNODE,6X,14HZ-DISPLACEMENT,6X,14HR-DISPLACEMENT,6X,
     2 14HT-DISPLACEMENT,6X,14H       ROTATION//)
3002  FORMAT(10X,I5,4(6X,E14.6))
      END
```

```
      SUBROUTINE ELASTI (R,Z,T,E,JI,JJ,MATERL)
C
C *** FORM ELEMENT ELASTIC STIFFNESS MATRIX
C
      COMMON HED(12),NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,MBAND,NUMCON,NUMOB
     1 ,IDEF,SK(8,8),CNFV(8),FN,FN2,NE,MS,MOB(10),KODEM(10)
      REAL  NF,NF2
      DIMENSION
     1 IG(10), JG(10), KG(10), IH(12), JH(12), KH(12), H(12), XS(10),
     2 XW(10), Y(10), ST(8, 8), B1(5,5), B2(5,5), A(8, 8), XI(10,10)
     3 ,R(1),Z(1),T(1),E(7,1),JI(1),JJ(1),MATERL(1)
      EQUIVALENCE (Y(2),RS), (Y(6),S)
      DATA
     1 IG /1,2,3,3,4,4,5,5,5,5/, JG /1,1,2,3,2,3,2,3,4,5/,
     2 KG /2,4,1,2,3,4,5,6,7,8/, IH /1,1,2,3,3,3,4,4,5,5,5,5/,
     3 JH /1,6,2,2,3,4,7,8,7,8,9,10/, KH /7,8,4,6,7,8,3,4,5,6,7,8/,
     4 H /2.,6.,1.,1.,2.,3.,1.,1.,1.,1.,1.,1./,             Y(1) /1./,
     5 XS /-.97390653,-.86506337,-.67940957,-.43339539,-.14887434,
     6 .14887434,.43339539,.67940957,.86506337,.9739 653/,
     7 XW /.06667134,.14945135,.21908636,.26926672,.29552422,.29552422,
     8 .26926672,.21908636,.14945135,.06667134/
C***********************************************************************
C
C          INITIALIZATION, GEOMETRY, AND MATERIAL PROPERTIFS
C
      NF=FN
      NF2=FN2
      MA=MATERL(NE)
      JIN=JI(NE)
      JJN=JJ(NE)
      DR=R(JJN)-R(JIN)
      DZ=Z(JJN)-Z(JIN)
      TT=T(NE)
      XX=TT*3.141592654
      IF (FN.EQ.0.) XX=XX*2.
      X=1./E(6,MA)
      C11=XX*(E(1,MA)-E(3,MA)*E(3,MA)*X)
      C12=XX*(E(2,MA)-E(3,MA)*E(5,MA)*X)
      C22=XX*(E(4,MA)-E(5,MA)*E(5,MA)*X)
      C44=XX*E(7,MA)
      RI=R(JIN)
      XL=SQRT(DR*DR+DZ*DZ)
      X=0.5*XL
      XB=0.5*DR
      SS=-DZ/XL
      SS=-SS
      CS=DR/XL
C
C          COMPUTE THE VOLUME INTEGRALS
C
      DO 50 I=1,10
      DO 50 J=I,10
   50 XI(I,J)=0.
      DO 100 K=1,10
      XX=1.+XS(K)
```

```
      S=X*XX
      RS=RI+XB*XX
      RS=1./RS
      Y(3)=S*RS
      Y(4)=S*Y(3)
      Y(5)=S*Y(4)
      Y(7)=RS*RS
      Y(8)=S*Y(7)
      Y(9)=S*Y(8)
      Y(10)=S*Y(9)
      XX=X/RS
      DO 100 I=1,10
      XXX=XX*XW(K)*Y(I)
      DO 100 J=I,10
  100 XI(I,J)=XI(I,J)+XXX*Y(J)
      DO 150 I=2,10
      DO 150 J=1,I
  150 XI(I,J)=XI(J,I)
C
C         FORM THE STIFFNESS MATRIX IN GENERALIZED COORDINATES
C
      DO 160 I=1,5
      DO 160 J=1,5
  160 B1(I,J)=0.0
      B1(1,1)=C11
      B1(1,2)=0.
      B1(1,3)=CS*C12
      B1(1,4)=NF*C12
      B1(1,5)=SS*C12
      B1(2,2)=C44
      B1(2,3)=-NF*C44
      B1(2,4)=-CS*C44
      B1(2,5)=0.
      B1(3,3)=CS*CS*C22+NF2*C44
      B1(3,4)=NF*CS*(C22+C44)
      B1(3,5)=SS*CS*C22
      B1(4,4)=NF2*C22+CS*CS*C44
      B1(4,5)=NF*SS*C22
      B1(5,5)=SS*SS*C22
C
      X=TT*TT/12.0
      C11=X*C11
      C12=X*C12
      C22=X*C22
      C44=X*C44
C
      DO 165 I=1,5
      DO 165 J=1,5
  165 B2(I,J)=0.0
      B2(1,1)=C11
      B2(1,2)=0.
      B2(1,3)=CS*C12
      B2(1,4)=-NF*SS*C12
      B2(1,5)=-NF2*C12
      B2(2,2)=SS*SS*C44
```

```
      B2(2,3)=NF*SS*C44
      B2(2,4)=-SS*SS*CS*C44
      B2(2,5)=-NF*SS*CS*C44
      B2(3,3)=CS*CS*C22+NF2*C44
      B2(3,4)=-NF*SS*CS*(C22+C44)
      B2(3,5)=-NF2*CS*(C22+C44)
      B2(4,4)=SS*SS*(NF2*C22+CS*CS*C44)
      B2(4,5)=NF*SS*(NF2*C22+CS*CS*C44)
      B2(5,5)=NF2*(NF2*C22+CS*CS*C44)
C
      DO 200 I=2,5
      DO 200 J=1,I
      B1(I,J)=B1(J,I)
  200 B2(I,J)=B2(J,I)
      DO 10 I=1,8
      DO 10 J=1,8
10    SK(I,J)=0.0
C
      DO 300 NG=1,10
      I=KG(NG)
      K=JG(NG)
      M=IG(NG)
      DO 300 MG=1,10
      J=KG(MG)
      L=JG(MG)
      N=IG(MG)
  300 SK(I,J)=SK(I,J)+B1(M,N)*XI(K,L)
C
      DO 400 NG=1,12
      I=KH(NG)
      K=JH(NG)
      M=IH(NG)
      DO 400 MG=1,12
      J=KH(MG)
      L=JH(MG)
      N=IH(MG)
  400 SK(I,J)=SK(I,J)+H(NG)*B2(M,N)*H(MG)*XI(K,L)
C
C           FORM DISPLACEMENT TRANSFORMATION MATRIX
C
      XL=1./XL
      DR=DR*XL
      DZ=DZ*XL
      X=XL*XL
C
      DO 450 I=1,8
      DO 450 J=1,8
  450 A(I,J)=0.0
C
      A(1,1)=DZ
      A(1,2)=DR
      A(2,1)=-DZ*XL
      A(2,2)=-DR*XL
      A(2,5)=-A(2,1)
      A(2,6)=-A(2,2)
```

```
      A(3,3)=1.0
      A(4,3)=-XL
      A(4,7)=XL
      A(5,1)=-DR
      A(5,2)=DZ
      A(6,4)=1.0
      A(7,1)=3.0*DR*X
      A(7,2)=-3.0*DZ*X
      A(7,4)=-2.0*XL
      A(7,5)=-A(7,1)
      A(7,6)=-A(7,2)
      A(7,8)=-XL
      A(8,1)=-2.0*DR*X*XL
      A(8,2)=2.0*DZ*X*XL
      A(8,4)=X
      A(8,5)=-A(8,1)
      A(8,6)=-A(8,2)
      A(8,8)=X
C
C          FORM THE STIFFNESS MATRIX IN GLOBAL COORDINATES
C
      DO 500 I=1,8
      DO 500 J=1,8
      ST(I,J)=0.
      DO 500 K=1,8
  500 ST(I,J)=ST(I,J)+SK(I,K)*A(K,J)
      DO 600 I=1,8
      DO 600 J=1,8
      SK(I,J)=0.
      DO 600 K=1,8
  600 SK(I,J)=SK(I,J)+A(K,I)*ST(K,J)
C
      RETURN
      END
```

```
      SUBROUTINE GEOSTI (R,Z,SNM,SNT,JI,JJ)
C
C *** FORM ELEMENT GEOMETRIC STIFFNESS MATRIX
C
      COMMON HED(12),NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,MBAND,NUMCON,NUMOB
     1 ,IDEF,SK(8,8),CNFV(8),FN,FN2,NE,MS,MOB(10),KODEM(10)
      DIMENSION R(1),Z(1),SNM(1),SNT(1),JI(1),JJ(1),TS(8,8),A(8,8)
C
      JIN=JI(NE)
      JJN=JJ(NE)
      DX=R(JJN)-R(JIN)
      DY=Z(JJN)-Z(JIN)
      DL=SQRT(DX*DX+DY*DY)
      XI=R(JIN)
      XJ=R(JJN)
      SN=DX/DL
      CS=DY/DL
      SM=SNM(NE)*DL*6.2831853/(XI+XJ)
      ST=SNT(NE)*DL*6.2831853/(XI+XJ)
      SM=-SM*2.
      ST=-ST*2.
      DO 10 I=1,8
      DO 10 J=1,8
10    SK(I,J)=0.0
      XR=(XI+XJ)*0.5
      IF (FN.EQ.0.0) GO TO 20
C
C *** CONTRIBUTION OF MERIDIONAL STRESS
C
      SK(1,1)=SK(1,1)+SM*FN*FN/8.0
      SK(1,2)=SK(1,2)+SM*FN*FN*DL/16.0
      SK(1,3)=SK(1,3)+SM*FN*SN/8.0
      SK(1,4)=SK(1,4)+SM*FN*(XI+XJ+DL*SN)/16.0
      SK(2,2)=SK(2,2)+SM*FN*FN*DL*DL/24.0
      SK(2,3)=SK(2,3)+SM*FN*DL*SN/16.0
      SK(2,4)=SK(2,4)+SM*FN*DL*(3.0*XR+2.0*DL*SN)/48.0
      SK(3,3)=SK(3,3)+SM*SN*SN/8.0
      SK(3,4)=SK(3,4)+SM*SN*(2.0*XR+DL*SN)/16.0
      SK(4,4)=SK(4,4)+SM*(3.0*XR*XR+3.0*XR*DL*SN+DL*DL*SN*SN)/24.0
      SK(6,6)=SK(6,6)+SM*XR*XR*0.5
      SK(6,7)=SK(6,7)+SM*XR*XR*DL*0.5
      SK(6,8)=SK(6,8)+SM*XR*XR*DL*DL*0.5
      SK(7,7)=SK(7,7)+SM*XR*XR*DL*DL*2.0/3.0
      SK(7,8)=SK(7,8)+SM*XR*XR*DL*DL*DL*0.75
      SK(8,8)=SK(8,8)+SM*XR*XR*DL*DL*DL*DL*0.9
C
C *** CONTRIBUTION OF TANGENTIAL STRESS
C
      SK(1,1)=SK(1,1)+ST*FN*FN/8.0
      SK(1,2)=SK(1,2)+ST*FN*FN*DL/16.0
      SK(1,3)=SK(1,3)+ST*FN*SN/8.0
      SK(1,4)=SK(1,4)+ST*FN*(2.0*XR+DL*SN)/16.0
      SK(2,2)=SK(2,2)+ST*FN*FN*DL*DL/24.0
      SK(2,3)=SK(2,3)+ST*FN*DL*SN/16.0
      SK(2,4)=SK(2,4)+ST*FN*DL*(3.0*XR+2.0*DL*SN)/48.0
```

```
      SK(3,3)=SK(3,3)+ST*(1.0+3.0*CS*CS)/8.0
      SK(3,4)=SK(3,4)+ST*(DL+2.0*XR*SN+3.0*DL*CS*CS)/16.0
      SK(3,5)=SK(3,5)+ST*FN*CS/2.0
      SK(3,6)=SK(3,6)+ST*FN*DL*CS/4.0
      SK(3,7)=SK(3,7)+ST*FN*DL*DL*CS/6.0
      SK(3,8)=SK(3,8)+ST*FN*DL*DL*DL*CS/8.0
      SK(4,4)=SK(4,4)+ST*(3.*XR*XR+DL*DL+3.*XR*DL*SN 3.*DL*DL*CS*CS)/24.
      SK(4,5)=SK(4,5)+ST*FN*DL*CS/4.0
      SK(4,6)=SK(4,6)+ST*FN*DL**2*CS/6.0
      SK(4,7)=SK(4,7)+ST*FN*DL**3*CS/8.0
      SK(4,8)=SK(4,8)+ST*FN*DL**4*CS/10.0
      SK(5,5)=SK(5,5)+ST*FN*FN/2.0
      SK(5,6)=SK(5,6)+ST*FN*FN*DL/4.0
      SK(5,7)=SK(5,7)+ST*FN*FN*DL**2/6.0
      SK(5,8)=SK(5,8)+ST*FN*FN*DL**3/8.0
      SK(6,6)=SK(6,6)+ST*FN*FN*DL**2/6.0
      SK(6,7)=SK(6,7)+ST*FN*FN*DL**3/8.0
      SK(6,8)=SK(6,8)+ST*FN*FN*DL**4/10.0
      SK(7,7)=SK(7,7)+ST*FN*FN*DL**4/10.0
      SK(7,8)=SK(7,8)+ST*FN*FN*DL**5/12.0
      SK(8,8)=SK(8,8)+ST*FN*FN*DL**6/14.0
C
      IF (FN.NE.0.0) GO TO 21
20    SK(6,6)=SM*XR*XR
      SK(6,7)=SK(6,6)*DL
      SK(6,8)=SK(6,7)*DL
      SK(7,7)=SK(6,8)*4.0/3.0
      SK(7,8)=SK(6,8)*DL*1.5
      SK(8,8)=SK(6,8)*DL*DL*1.8
C
21    DO 22 I=2,8
      DO 22 J=1,I
22    SK(I,J)=SK(J,I)
C
C *** TRANSFORMATION MATRIX
C
      DO 11 I=1,8
      DO 11 J=1,8
11    A(I,J)=0.0
C
      XL=1.0/DL
      DR=DX*XL
      DZ=DY*XL
      X=XL*XL
C
      A(1,1)=DZ
      A(1,2)=DR
      A(2,1)=-DZ*XL
      A(2,2)=-DR*XL
      A(2,5)=-A(2,1)
      A(2,6)=-A(2,2)
      A(3,3)=1.0
      A(4,3)=-XL
      A(4,7)=XL
      A(5,1)=-DR
```

```
      A(5,2)=DZ
      A(6,4)=1.0
      A(7,1)=3.0*DR*X
      A(7,2)=-3.0*DZ*X
      A(7,4)=-2.0*XL
      A(7,5)=-A(7,1)
      A(7,6)=-A(7,2)
      A(7,8)=-XL
      A(8,1)=-2.0*DR*X*XL
      A(8,2)=2.0*DZ*X*XL
      A(8,4)=X
      A(8,5)=-A(8,1)
      A(8,6)=-A(8,2)
      A(8,8)=X
C
C *** FORM THE GEOMETRIC STIFFNESS MATRIX IN GLOBAL COORDINATES
C
      DO 12 I=1,8
      DO 12 J=1,8
      TS(I,J)=0.0
      DO 12 K=1,8
12    TS(I,J)=TS(I,J)+SK(I,K)*A(K,J)
      DO 14 I=1,8
      DO 14 J=1,8
      SK(I,J)=0.0
      DO 14 K=1,8
14    SK(I,J)=SK(I,J)+A(K,I)*TS(K,J)
C
      RETURN
      END
```

```
      SUBROUTINE MODIFY (SKE,B,JI,JJ,KODE,NEQ,KKK)
C
C *** ASSEMBLE STRUCTURAL STIFFNESS MATRIX
C
      COMMON HED(12),NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,MBAND,NUMCON,NUMOB
     1 ,IDEF,SK(8,8),CNFV(8),FN,FN2,NE,MS,MOB(10),KODEM(10)
      DIMENSION SKE(NEQ,1),B(1),JI(1),JJ(1),KODE(1),LM(8)
C
      JIN=JI(NE)
      JJN=JJ(NE)
      GO TO (1000,2000) KKK
1000  KK=KODE(JIN)
      KOUNT=1
      GO TO 28
27    KK=KODE(JJN)
      KOUNT=2
28    XK=KK
      IF (KK) 33,33,29
29    CONTINUE
      DO 32 K=1,4
      XK=XK/10.0
      KK=KK/10
      DKK=KK
      IF (FN.EQ.0..AND.K.EQ.2) GO TO 30
      IF (XK-DKK) 32,32,30
30    CONTINUE
      IJ=JIN
      IF (KOUNT.EQ.2) IJ=JJN
      IS=4*KOUNT-K+1
C
C *** MODIFY LOAD VECTOR
C
      IK=4*IJ-K+1
      PARTY=B(IK)
      DO 10 NI=1,4
10    B(4*JIN-4+NI)=B(4*JIN-4+NI)-SK(NI,IS)*PARTY
      DO 11 NI=1,4
11    B(4*JJN-4+NI)=B(4*JJN-4+NI)-SK(NI+4,IS)*PARTY
      B(IK)=PARTY
      CNFV(IS)=0.0
12    CONTINUE
      DO 31 NI=1,8
      SK(NI,IS)=0.0
31    SK(IS,NI)=0.0
      SK(IS,IS)=1.0
32    XK=DKK
33    CONTINUE
      GO TO (27,34),KOUNT
34    CONTINUE
C
C *** ADD ELEMENT ELASTIC STIFFNESS TO STRUCTURAL ELASTIC STIFFNESS
C
2000  LM(4)=4*JIN
      LM(3)=LM(4)-1
      LM(2)=LM(3)-1
```

```
      LM(1)=LM(2)-1
      LM(8)=4*JJN
      LM(7)=LM(8)-1
      LM(6)=LM(7)-1
      LM(5)=LM(6)-1
      DO 36 I=1,8
      IL=LM(I)
      DO 36 J=1,8
      JL=LM(J)-IL+1
      IF (JL) 36,36,35
35    SKE(IL,JL)=SKE(IL,JL)+SK(I,J)
36    CONTINUE
C
      RETURN
      END
```

```
      SUBROUTINE MODIFY1 (SKE,KODE,NPP,NEQ,NEQ1,KKK)
C
C *** CONTRACT THE STIFFNESS MATRIX WITH RESPECT TO THE DEGREES OF FREEDOM
C *** ASSOCIATED WITH THE PRISCRIBED DISPLACEMENT BOUNDARY CONDITIONS
C
      COMMON HED(12),NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,MBAND,NUMCON,NUMOB
     1 ,IDEF,SK(8,8),CNFV(8),FN,FN2,NE,MS,MOB(10),KODEM(10)
      DIMENSION SKE(NEQ,1),KODE(1),NPP(1)
C
      GO TO (1000,2000) KKK
C
C *** COLLECT THE ELIMINATING DEGREES OF FREEDOM
C
1000  NUMCON=0
      DO 100 NP=1,NUMNP
      KK=KODE(NP)
      XK=KK
      IF (KK.NE.0.OR.FN.NE.0.) GO TO 110
      NUMCON=NUMCON+1
      NPP(NUMCON)=4*NP-1
110   CONTINUE
      IF (KK) 100,100,10
10    DO 30 K=1,4
      XK=XK/10.
      KK=KK/10
      DKK=KK
      IF (FN.EQ.0..AND.K.EQ.2) GO TO 20
      IF (XK-DKK) 30,30,20
20    NUMCON=NUMCON+1
      NPP(NUMCON)=4*NP-K+1
30    XK=DKK
100   CONTINUE
      DO 40 I=1,NUMCON
      DO 40 J=I,NUMCON
      IF (NPP(I).LT.NPP(J)) GO TO 40
      IJ=NPP(I)
      NPP(I)=NPP(J)
      NPP(J)=IJ
40    CONTINUE
      NEQ1=NEQ-NUMCON
      GO TO 1500
C
C *** CONTRACTION PROCESS
C
2000  DO 900 NP=1,NUMCON
      KE=NPP(NUMCON-NP+1)
      IF (KE.EQ.1) GO TO 500
      KL=MIN0(MBAND,KE+1)
      ML=MAX0(KE-MBAND+2,1)
      NL=KE-1
      IF (KE.GE.MBAND) SKE(ML-1,MBAND)=0.
      DO 400 I=ML,NL
      DO 300 J=KL,MBAND
300   SKE(I,J-1)=SKE(I,J)
      SKE(I,MBAND)=0.
```

```
400    KL=KL-1
500    ML=KE+1
       IF (KE.EQ.NEQ) GO TO 900
       DO 600 I=ML,NEQ
       DO 600 J=1,MBAND
600    SKE(I-1,J)=SKE(I,J)
       CONTINUE
C
1500   RETURN
       END
```

```
      SUBROUTINE ELSTRS (R,Z,T,E,B,SNM,SNT,JI,JJ,MATERL)
C
C *** ELEMENT STRESS RESULTANTS AT MIDPOINT OF ELEMENT
C
      COMMON HED(12),NUMNP,NUMEL,NUMMAT,NUMHAR,NUMSKH,MBAND,NUMCON,NUMOB
     1 ,IDEF,SK(8,8),CNFV(8),FN,FN2,NE,MS,MOB(10),KODEM(10)
      DIMENSION R(1),Z(1),T(1),E(7,1),B(1),SNM(1),SNT(1),JI(1),JJ(1)
     1 ,MATERL(1),U(6),EPS(8),STRESS(8),D(4,4)
C
      WRITE (6,2000)
      DO 20 NE=1,NUMEL
      IN=JI(NE)
      JN=JJ(NE)
      TT=T(NE)
      MA=MATERL(NE)
      DR=R(JN)-R(IN)
      DZ=Z(JN)-Z(IN)
      DL=SQRT(DR*DR+DZ*DZ)
      SN=DR/DL
      CS=DZ/DL
      X=(R(IN)+R(JN))*.5
      U(1)=B(4*IN-3)
      U(2)=B(4*IN-2)
      U(3)=B(4*IN)
      U(4)=B(4*JN-3)
      U(5)=B(4*JN-2)
      U(6)=B(4*JN)
      DO 10 I=1,4
      DO 10 J=1,4
10    D(I,J)=0.0
      D(1,1)=TT*(E(1,MA)-E(3,MA)*E(3,MA)/E(6,MA))
      D(1,2)=TT*(E(2,MA)-E(3,MA)*E(5,MA)/E(6,MA))
      D(2,2)=TT*(E(4,MA)-E(5,MA)*E(5,MA)/E(6,MA))
      D(2,1)=D(1,2)
      TT=TT*TT/12.
      D(3,3)=D(1,1)*TT
      D(4,4)=D(2,2)*TT
      D(3,4)=D(1,2)*TT
      D(4,3)=D(3,4)
      RL1=U(1)*CS+U(2)*SN
      RL2=-SN*U(1)+CS*U(2)
      RL3=-U(3)
      RL4=CS*U(4)+SN*U(5)
      RL5=-SN*U(4)+CS*U(5)
      RL6=-U(6)
C
C *** EVALUATION OF STRAINS AT MIDPOINT OF ELEMENT
C
      EPS(1)=(1./DL)*(-RL1+RL4)
      EPS(2)=(0.5/X)*(SN*(RL1+RL4)+CS*(RL2+RL5)+(0.25*DL)*CS*(RL3-RL6))
      EPS(3)=(1./DL)*(-RL3+RL6)
      EPS(4)=(SN/X)*((1.5/DL)*(RL2-RL5)+0.25*(RL3+RL6))
C
C *** EVALUATION OF STRESS RESULTANTS
C
```

```
      DO 11 K=1,4
      STRESS(K)=0.
      DO 11 J=1,4
11    STRESS(K)=STRESS(K)+D(K,J)*EPS(J)
      SNM(NE)=STRESS(1)
      SNT(NE)=STRESS(2)
      WRITE (6,2001) NE,(STRESS(K),K=1,4)
20    CONTINUE
C
      RETURN
2000  FORMAT (///,1X,14HSHELL STRESSES/
     1 1X,90H    ELEMENT         MERID. STRESS             HOOP STRESS         ME
     2RID. MOMENT          HOOP MOMENT//)
2001  FORMAT (6X,I5,4(8X,F12.5))
      END
```

```
      SUBROUTINE SYMSOL (A,B,NN,MM,KKK)
      DIMENSION A(NN,1),B(1)
C
      GO TO (1000,2000),KKK
C
C     REDUCE MATRIX
C
 1000 DO 280 N=1,NN
      DO 260 L=2,MM
      C=A(N,L)/A(N,1)
      I = N+L-1
      IF(NN-I) 260,240,240
  240 J=0
      DO 250 K=L,MM
      J=J+1
  250 A(I,J)=A(I,J)-C*A(N,K)
  260 A(N,L)=C
  280 CONTINUE
      GO TO 500
C
C     REDUCE VECTOR
C
 2000 DO 290 N=1,NN
      DO 285 L=2,MM
      I=N+L-1
      IF(NN-I) 290,285,285
  285 B(I)=B(I)-A(N,L)*B(N)
  290 B(N)=B(N)/A(N,1)
C
C     BACK SUBSTITUTION
C
      N=NN
  300 N = N-1
      IF(N) 350,500,350
  350 DO 400 K=2,MM
      L = N+K-1
      IF(NN-L) 400,370,370
  370 B(N) = B(N) - A(N,K) * B(L)
  400 CONTINUE
      GO TO 300
C
  500 RETURN
      END
```

```
      SUBROUTINE BUEIG (A,SEK,SGK,B,NMAX,NN,MM,KK)
      DIMENSION A(NMAX,1),SEK(NMAX,1),SGK(NMAX,1),B(1)
C
      GO TO (1000,2000) KK
C
C *** TRIAL AND ERROR PROCESS FOR FINDING THE SMALLEST POSITIVE EIGENVALUE
C
1000  TOL=0.001
      BL=0.
      BU=1.
      U=1.
      DIF=1.
      N2=1
      NCY=0
1     NCY=NCY+1
      WRITE (6,2001) NCY,U
      IF (ABS(DIF).LE.TOL) GO TO 500
      IF (U.GE.1000.) GO TO 700
      IF (U.LE.0.001) GO TO 800
      IF (NCY.EQ.51) GO TO 900
      DO 100 I=1,NN
      DO 100 J=1,MM
100   A(I,J)=SEK(I,J)-U*SGK(I,J)
      DO 280 N=1,NN
      IF (N.EQ.NN.AND.ABS(A(NN,1)).LT.TOL) GO TO 500
      IF (A(N,1).LE.0.) GO TO 290
      DO 260 L=2,MM
      C=A(N,L)/A(N,1)
      I=N+L-1
      IF (NN-I) 260,240,240
240   J=0
      DO 250 K=L,MM
      J=J+1
250   A(I,J)=A(I,J)-C*A(N,K)
260   A(N,L)=C
280   CONTINUE
      BL=U
      U=2.*U
      IF (N2.EQ.2) U=0.5*(BL+BU)
      DIF=U-BL
      GO TO 1
290   N2=2
      BU=U
      U=0.5*(BL+BU)
      DIF=BU-U
      GO TO 1
700   WRITE (6,2010)
      GO TO 500
800   WRITE (6,2020)
      GO TO 500
      WRITE (6,2030)
      GO TO 500
C
C *** CALCULATE BUCKLING MODE SHAPE BY BACK SUBSTITUTION
C
```

```
2000  B(NN)=1.
      N=NN
300   N=N-1
      IF (N) 350,600,350
350   B(N)=0.
      DO 400 K=2,MM
      L=N+K-1
      IF (NN-L) 400,370,370
370   B(N)=B(N)-A(N,K)*B(L)
400   CONTINUE
      GO TO 300
600   CONTINUE
      BM=B(1)
      DO 610 I=2,NN
      IF (ABS(BM).LT.ABS(B(I))) BM=B(I)
610   CONTINUE
      DO 620 I=1,NN
620   B(I)=B(I)/BM
C
500   RETURN
2001  FORMAT (6X,I5,5X,E15.5)
2010  FORMAT (* TOO STIFF STRUCTURE *)
2020  FORMAT (* TOO FLEXIBLE STRUCTURE *)
2030  FORMAT (* THE PROCESS IS NOT CONVERGENT *)
      END
```