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On Pairwise Stability and Anti-Competitiveness of Cross-Holdings in Oligopoly*

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Abstract

This paper considers a model of endogenous bilateral cross-holdings. A notion of pairwise stability is applied to analyze firms' incentives for cross-holdings. Under certain conditions and Cournot competition on the output market, it is shown that monopoly is the only outcome of pairwise stable cross-holdings when there are two firms; a wide range of outcomes is possible when there are three firms, including as special cases the triopoly and the duopoly Cournot equilibria without any cross-holding; and the Cournot equilibrium is an outcome of pairwise stable cross-holdings when there are four or more firms. Competitive implications of the results are also briefly discussed.

Keywords: Antitrust, cross-holding, Cournot equilibrium, network, oligopoly, pairwise stability. (*JEL* C72, L1, L41)

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1 Introduction

Cross-holding refers to a situation where a firm holds *passive* ownership in another firm. There are many cases in which firms hold equity shares in their rivals' profits but *not* in decision making. For example, Northwest Airlines acquired a 14% equity position in Continental Airlines in 1998. Microsoft acquired about 7% of the nonvoting stock of Apple in 1997.¹ The Japanese and the U.S. Automobile industries are two of many that feature complex webs of cross-holdings (see Alley, 1997).²

Due to the increasing popularity of cross-holdings, a substantial literature concerning their competitive implications has been emerged. A well-known result in this literature states that with Cournot competition on the output market, the output equilibrium will become less competitive, with aggregate output level falling toward the monopoly level as cross-holdings are increased. As shown in Reynolds and Snapp (1986), after a firm has entered into a long equity position in a rival firm, it is induced to take into consideration the effect of its own output decision on the rival's profit. This consideration makes the firm compete less aggressively, because in so doing the firm can increase the profit to the rival and hence its stake in the rival firm's profit. Thus, if both firms hold passive ownership in each other, they are both induced to produce less which leads to greater profit for both.³

¹“Microsoft Investments Draw Federal Scrutiny,” *Pittsburg Post-Gazette*, August 10, 1997, B-11; “Corel Again Buys a ‘Victim’ of Microsoft Juggernaut,” *The Ottawa Citizen*, February 8, 200, C1. “Antitrust Investigations examine Microsoft, Apple Deal,” *Journal Record The (Oklahoma City)*, by Bob Drummond Bloomberg News, August 20, 1998.

²Gulati and Singh (1998) presented a typology of alliance structure that defines three distinct types of alliance governance structures. One of them is termed *minority investment*. This type of alliance governance structures includes partnerships of which the firms work together without creating a new identity. Instead, one or a collection of partners takes a minority equity position in the others. For example, in the formation of alliances pairing a small entrepreneurial firm with a larger firm, the smaller firm commonly sells a stake of its own equity to its partner.

³In a repeated model, cross-holdings facilitate collusion if they expand the range of discount factors for which tacit collusion can be sustained. Malueg (1992) considered the collusive effect of given cross-holdings in a repeated Cournot duopoly. Since cross-holdings weakens output market competition following a breakdown of a collusive scheme, Malueg showed that cross-holdings have an ambiguous effect on collusion. In contrast, Gilo, Moshe, and Spiegel (2006) considered the collusive effect in a repeated Bertrand oligopoly. Their paper established conditions under which an increase in cross-holdings by one firm in another facilitate

The preceding comparative static analysis lacks the consideration of firms' incentives to increase cross-holdings. As we illustrate in this paper, two firms may sometimes be better off reducing cross-holdings. The reason is because their less competitive behavior may induce the other firms to compete more aggressively. As a result, similar to reasons for the *merger paradox* in Salant, Switzer, and Reynolds (1983), they may be worse off in the end. A complete analysis of the anti-competitiveness of cross-holdings would therefore require cross-holdings be incentive compatible for the firms.

In this paper, we consider a model of endogenous bilateral cross-holdings and apply a network approach to analyze firms' incentives to participate in cross-holdings. Intuitively, when two firms can increase total profit by a bilateral change of cross-holdings involving their own equities between each other, they can always agree on mutually beneficial prices for making the changes (see, for example, Farrell and Shapiro, 1990, p. 287). Thus, a collection of cross-holdings is "pairwise stable" if no two firms can increase total profit by a bilateral change of cross-holdings involving their own equities between each other. This notion of pairwise stability is adapted from Jackson and Wolinsky (1996).⁴

We assume that firms determine cross-holdings before they choose outputs. Price for cross-holdings are either fixed or simultaneously determined with cross-holdings by the firms. Furthermore, the determination of cross-holdings and their prices cannot be contingent upon subsequent output choices. A justification for this lack of contingency is that cross-holdings are passive, so the firms cannot directly influence each other's output choices by conditioning cross-holdings or their prices upon them. Since payments for cross-holdings between any two firms sum to zero, pairwise stability does not depend on them. Finally, given cross-holdings, we assume that firms compete on the output market in Cournot fashion.

We focus on characterizing pairwise stable cross-holdings and their anti-competitive collusion.

⁴Jackson and Wolinsky (1996) consider networks on the set of players of a game with discrete links, in the sense that the link between two players takes either value 1 (the link is present) or value 0 (the link is absent). Considering a collection of cross-holdings as a (directed) network, the values of the links are continuous. For example, the value of 5% equity shares held by firm i in firm j for the link from i to j is different from the value of 10% equity shares held by i in firm j for the same link. See Jackson (2008) for developments and applications of network models.

implications. To this end, given the non-contingencies as mentioned in the preceding paragraph, we leave the determination of the prices for cross-holdings not explicitly modeled, but require instead that they together with cross-holdings be “individually rational”. Individual rationality in this paper means that no firm is worse off from participating in cross-holdings than not participating at all.

Our analysis of pairwise stable and individually rational cross-holdings and their prices is carried out in two steps. First, we completely characterize pairwise stable cross-holdings without constraining the firms to produce non-negative outputs or to retain some minimum ownership. In this case, given cross-holdings, the subsequent Cournot equilibrium outputs of the firms can be more conveniently characterized by the first-order conditions. Second, we show that no firm can increase profit with one or both of the constraints. This result implies that unconstrained pairwise stable cross-holdings are constrained pairwise stable, provided neither they nor the subsequent Cournot equilibrium outputs violate the constraints. In addition, the individual rationality of cross-holdings and their prices does not depend on the constraints.

With linear demand and identical linear cost functions, we show that a collection of cross-holdings is unconstrained pairwise stable if and only if each firm holds the same amount of the other firm’s equity as what it retains when there are two firms; with three firms, one firm retains all its own equity and holds an equal amount of equity in each of the other two firms, while these two firms do not participate in cross-holdings; with four or more firms, one firm retains all its equity and holds all of the other firms’ equities. With two firms, holding the same amount of the other firm’s equity as what a firm retains means that both firms are induced to maximize total profit. Hence, monopoly is the only outcome. When there are four or more firms, the usual Cournot equilibrium is the only other outcome besides the monopoly outcome. With three firms, however, the star cross-holding structures result in a great variety of outcomes. These outcomes include as special cases the usual triopoly Cournot equilibrium, duopoly Cournot equilibrium, and monopoly. It follows that output market is more collusive with the presence of cross-holdings than without it.

With the non-negative output and/or some minimum retained ownership constraints,

we show that our characterization of unconstrained pairwise stable cross-holdings imply that monopoly is the only outcome of constrained pairwise stable cross-holdings when there are two firms; a significant range of pairwise stable cross-holdings is consistent with the constraints when there are three firms, including those that result in triopoly and duopoly equilibria; with four and more firms, the Cournot equilibrium remains to be an outcome of constrained pairwise stable cross-holdings. Thus, the collusive effect of cross-holdings is similar as in the unconstrained case.

Reitman (1994) considered a model of endogenous bilateral cross-holdings with quantity competition (Cournot or conjectural) on the output market. In that paper, a collection of cross-holdings is in equilibrium if no firm can become more profitable by *completely* withdrawing cross-holdings with one or more other firms. Under this equilibrium notion, deviations are restricted to be discrete and simultaneous deviations from existing cross-holdings with multiple other firms are allowed for an individual firm. It follows that this equilibrium notion is different from pairwise stability. Indeed, as shown in Reitman (1994, p. 318), with three firms, Cournot triopolistic firms do not want to cross-hold in rivals in equilibrium. In comparison, we show that a wide range of cross-holdings are pairwise stable with or without constraints.⁵

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the results and section 4 concludes.

2 Model

We consider a market with n firms producing a homogeneous good. Market inverse demand is $P(X)$, where X is the total output. Firm i 's technology exhibits constant returns to scale, which results in constant marginal cost $c_i > 0$ and zero fixed cost. Denote by $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in})$ the vector of equity shares α_{ij} of firm i in firm j and by $\tau_i = (\tau_{i1}, \tau_{i2}, \dots, \tau_{in})$ with $\tau_{ii} = 0$ the vector of prices τ_{ij} from firm i to firm j for cross-

⁵The reader is referred to Flath (1991) for a model of endogenous cross-holdings based on purchases of equity shares by individual firms on capital markets.

holdings. Let α denote the collection with α_i for all i and by τ the collection with τ_i for all i . Let $\alpha^\circ = (\alpha_{ij}^\circ)$ denote the collection with

$$\alpha_{ij}^\circ = 0, \quad \forall i \neq j.$$

Given α and τ , firm i 's total profit at quantity profile $x = (x_1, x_2, \dots, x_n)$ is given by

$$\pi_i(x, \alpha) + \kappa_i(\alpha, \tau) \tag{1}$$

where

$$\pi_i(x, \alpha) = \sum_{k=1}^n \alpha_{ik} [P(X) - c_k] x_k \tag{2}$$

is firm i 's total profit share from production due to cross-holdings and

$$\kappa_i(\alpha, \tau) = \sum_{j=1}^n [\tau_{ji} \alpha_{ji} - \tau_{ij} \alpha_{ij}] \tag{3}$$

is its total net payment for cross-holdings. By (2), firm i 's production decision is guided by the maximization of profits net of those distributed to competitors.⁶

Remark 1 *The prices for cross-holdings and hence the total net payment for each firm are independent of the firms' subsequent output choices. This is due to cross-holdings being passive.*

Due to the timing difference between the determination of cross-holdings and of outputs, we solve the decision problems by *backward induction*. Notice that from (1)-(3) it follows that the subsequent Cournot equilibrium outputs are independent of prices for cross-holdings. Notice also when $\alpha_{ii} = 0$, it means that all equity shares within firm i are held by other firms. That is, in this case, firm i is bought out by the other firms. Since technologies

⁶See Reynolds and Snapp (1986, p. 144) for a discussion about this formulation.

are identical and of constant returns to scale, there is no need for the buying firms to use its technology. We assume $x_i^*(\alpha) = 0$ if $\alpha_{ii} = 0$.

Solve firms' equilibrium quantities for a given collection α and denote the solution by $x_i^*(\alpha)$ for $i = 1, 2, \dots, n$. Set

$$\pi_i^*(\alpha) = \pi_i(x^*(\alpha), \alpha), \quad (4)$$

where $x^*(\alpha) = (x_1^*(\alpha), \dots, x_n^*(\alpha))$.

Cross-holdings are voluntary. Hence, it is natural to require that no firm be made worse off from engaging in cross-holdings than not engaging in any cross-holding.

Definition 1 *We say that a collection α of cross-holdings is individually rational if there exists a collection τ of prices such that for all i ,*

$$\pi_i^*(\alpha) + \kappa_i(\alpha, \tau) \geq \min_{\alpha': \alpha'_{ij} = \alpha'_{ji} = 0, j \neq i} \pi_i^*(\alpha'). \quad (5)$$

The right-hand-side of (5) is the profit level firm i can guarantee itself by not participating in any cross-holding. Such a payoff level is known as the *security level* of firm i (see Binmore, 1992, pp. 224 -225). We show in the next section that a firm's security level coincides with its profit in the Cournot equilibrium without any cross-holding.

A collection of cross-holdings with zero payments may violate the individual rationality. For example, let $n = 3$, $p(X) = 1 - X$ be the market inverse demand, and $c_i = 0$ for all i . Consider $0 < \delta < 1/2$ and consider

$$\alpha = \begin{pmatrix} 1 & \delta & \delta \\ 0 & 1 - \delta & 0 \\ 0 & 0 & 1 - \delta \end{pmatrix}. \quad (6)$$

From the first-order condition for the subsequent Cournot equilibrium it follows

$$x_1^*(\alpha) = \frac{1 - 2\delta}{4 - 2\delta} \text{ and } x_2^*(\alpha) = x_3^*(\alpha) = \frac{1}{4 - 2\delta}.$$

Thus, given α in (6), we have

$$\pi_1^*(\alpha) = \frac{1}{(4-2\delta)^2} \text{ and } \pi_2^*(\alpha) = \pi_3^*(\alpha) = \frac{1-\delta}{(4-2\delta)^2}.$$

Should the payment for cross-holdings be zero, both firm 2 and firm 3's profit would be below their Cournot equilibrium profit levels without any cross-holding. It follows that for α in (6) to be individually rational, we would need prices τ_{12} and τ_{13} to satisfy

$$\frac{\delta}{16(2-\delta)^2} \leq \tau_{12}, \tau_{13} \leq \frac{4-\delta}{32(2-\delta)^2}.$$

3 Pairwise Stability and Individual Rationality

For the rest of the paper, we assume $c_i = c$ for all i and $P(X) = a - X$, where $0 \leq c < a$. With symmetric constant marginal cost, it does not lose any generality to have the slope of the demand equal to -1; otherwise, we can always normalize the unit for measuring the outputs to make the slope equal to -1 without affecting the analysis.

3.1 Unconstrained Pairwise Stable Cross-Holdings

Notice that the payments for cross-holdings in each other's equities between any two firms sum up to zero. For this reason, pairwise stability based on total profit is independent of the prices for cross-holdings.

Definition 2 *We say that a collection α of cross-holdings is pairwise stable if for any $i \neq j$, $(\alpha_{ii}, \alpha_{ij}, \alpha_{ji}, \alpha_{jj})$ solves*

$$\begin{aligned} & \max \pi_i^*(\alpha') + \pi_j^*(\alpha') \\ & \text{subject to:} \\ & \alpha'_{ij} + \alpha'_{jj} = \alpha_{ij} + \alpha_{jj}, \\ & \alpha'_{ii} + \alpha'_{ji} = \alpha_{ii} + \alpha_{ji} \\ & \alpha'_{kl} = \alpha_{kl}, kl \neq ii, ij, ji, jj. \end{aligned} \tag{7}$$

As mentioned earlier, Definition 1 is based on the intuitive assumption that whenever total profit increases by a bilateral change of equity holdings between two firms, they can always agree on a mutually beneficial price for making the changes. The definition is adapted from Jackson and Wolinsky (1996). In that paper, when a link between i and j does not exist (which in our setting means that neither firm holds any passive ownership in the other), pairwise stability also requires that, taking the existing collection of links as given, j be necessarily worse off with the presence of the link should i become better off. This requirement is automatic here because we require the link to maximize joint profit.

Given any collection α , the subsequent Cournot equilibrium, $x^*(\alpha)$, on the output market satisfies the first-order condition: for all i with $\alpha_{ii} > 0$,

$$P'(X)X_i + \alpha_{ii}[P(X) - c] = 0, \quad (8)$$

where $X_i = \sum_{k=1}^n \alpha_{ik}x_k$ is firm i 's total share of outputs in all n firms. Equation (8) implies

$$X_h^*(\alpha) = \sum_{k=1}^n \alpha_{hk}x_k^*(\alpha) = \frac{(a-c)\alpha_{hh}}{1 + \sum_{k=1}^n \alpha_{kk}}, \quad (9)$$

and

$$P(X^*(\alpha)) = \frac{a + c \sum_{k=1}^n \alpha_{kk}}{1 + \sum_{k=1}^n \alpha_{kk}}. \quad (10)$$

Thus, by (2), (4), (9), and (10),

$$\pi_h^*(\alpha) = [P(X^*(\alpha)) - c]X_h^*(\alpha) = \frac{\alpha_{hh}(a-c)^2}{(1 + \sum_{k=1}^n \alpha_{kk})^2}. \quad (11)$$

It is interesting to observe that given a collection α , the profit share $\pi_h^*(\alpha)$ of firm h only depends on retained ownerships α_{kk} for all k . This is because cross-holdings do not have decision rights.

We now show that the individual rationality in Definition 1 implies that sum of total net payment and profit share from cross-holdings result in a sum for each firm no less than

what it gets in the Cournot equilibrium with no cross-holdings.

Theorem 1 *For all i ,*

$$\min_{\alpha': \alpha'_{ij}=\alpha'_{ji}=0, j \neq i} \pi_i^*(\alpha') = \pi_i^*(\alpha^\circ).$$

Proof. By (11), given any collection α' , $\alpha'_{ii} = 1$ implies

$$\pi_i^*(\alpha') = \frac{(a - c)^2}{(1 + \sum_{k=1}^n \alpha'_{kk})^2}.$$

It follows that firm i 's profit is a decreasing function of other firms' retained equity shares. Hence, the minimum profit of firm i is achieved at $\alpha'_{jj} = 1$ for all $j \neq i$. This together with $\alpha'_{ii} = 1$ implies that the minimum profit from retaining all equity is firm i 's Cournot equilibrium profit without any cross-holding. ■

By (11), a simpler but equivalent condition to (7) for pairwise stability is possible. Under this simplified condition, we only need to consider bilateral changes in cross-holdings in one firm's equity only.

Lemma 1 *A collection α is pairwise stable if and only if for any $i \neq j$, $(\alpha_{ij}, \alpha_{jj})$ solves*

$$\begin{aligned} & \max \pi_i^*(\alpha') + \pi_j^*(\alpha') \\ & \text{subject to:} \\ & \alpha'_{ij} + \alpha'_{jj} = \alpha_{ij} + \alpha_{jj}, \\ & \alpha'_{kl} = \alpha_{kl}, kl \neq ij, jj. \end{aligned} \tag{12}$$

Proof. By (11), for any $i \neq j$,

$$\pi_i^*(\alpha) + \pi_j^*(\alpha) = \frac{(\alpha_{ii} + \alpha_{jj})(a - c)^2}{(1 + \sum_{k=1}^n \alpha_{kk})^2}.$$

It follows that the total profit of firms i and j is a concave function of the sum of their retained shares. Consequently, if bilateral cross-holding changes satisfying (7) can increase

the total profit of firms i and j , then so can bilateral cross-holding changes satisfying (12). The converse is clearly also true. ■

Combining (11) with Lemma 1, our next result establishes a simple criterion for pairwise stability.

Theorem 2 *A collection α of cross-holdings is pairwise stable if and only if for all $i \neq j$ and for all bilateral changes $(d\alpha_{ij}, d\alpha_{jj})$ in the holdings of firm j 's equity by firms i and j such that $d\alpha_{ij} + d\alpha_{jj} = 0$,*

$$\left[1 + \sum_{k \neq i, j}^n \alpha_{kk} - (\alpha_{ii} + \alpha_{jj}) \right] d\alpha_{jj} \leq 0 \quad (13)$$

Proof. By (11), $\pi_i^*(\alpha) + \pi_j^*(\alpha)$ is concave and differentiable in α_{jj} . It follows that given α , for $(\alpha_{ij}, \alpha_{jj})$ to solve (12), it is necessary and sufficient

$$d[\pi_i^*(\alpha) + \pi_j^*(\alpha)] \leq 0.$$

Condition (13) follows easily from the above equation together with (11). ■

We now proceed to apply Theorem 2 to completely characterize pairwise stable cross-holdings.

Theorem 3 *Assume $n = 2$. Then, α is pairwise stable if and only if*

$$\alpha_{11} + \alpha_{22} = 1.$$

Proof. The sufficiency of the condition follows directly from (13). Thus, we only need to show the necessity of the condition.

To this end, we first show for $i \neq j$, $\alpha_{jj} = 0$ implies $\alpha_{ii} = 1$, in which case the condition is necessary. The reason is as follows. When $\alpha_{jj} = 0$, a pair $(d\alpha_{ij}, d\alpha_{jj})$ of changes with

$d\alpha_{ij} + d\alpha_{jj} = 0$ and $d\alpha_{jj} > 0$ is feasible. Thus, by (13), $\alpha_{ii} + \alpha_{jj} \geq 1$ must hold which together with $\alpha_{jj} = 0$ and $\alpha_{ii} \leq 1$ implies $\alpha_{ii} = 1$.

We now show $\alpha_{11} + \alpha_{22} \neq 2$. Otherwise, we would have $\alpha_{11} = \alpha_{22} = 1$. In this case, a pair $(d\alpha_{12}, d\alpha_{22})$ of changes with $d\alpha_{12} + d\alpha_{22} = 0$ and $d\alpha_{22} < 0$ is feasible. Thus, by (13), $\alpha_{11} + \alpha_{22} \leq 1$ must hold which results in the desired contradiction.

The preceding reasoning shows that for the remaining proof of the necessity, we only need to consider cases where $\alpha_{11}\alpha_{22} > 0$ and $0 < \alpha_{11} + \alpha_{22} < 2$. These are the cases in which $\alpha_{jj} < 1$ for at least one j . Thus, both $d\alpha_{jj} > 0$ and $d\alpha_{jj} < 0$ are possible infinitesimal changes in equity shares firm j retains. It follows from (13) that both $\alpha_{ii} + \alpha_{jj} \geq 1$ and $\alpha_{ii} + \alpha_{jj} \leq 1$ must be simultaneously satisfied. ■

Observe that $\alpha_{11} + \alpha_{22} = 1$ implies $\alpha_{ii} = \alpha_{ij}$ for all $i \neq j$. It follows from Theorem 3 that with 2 firms, all pairwise stable cross-holdings perfectly coordinate firms' production decisions so as to result in the monopoly outcome. This conclusion does not require that firm 1 and firm 2 completely merge.

Our next lemma shows that when there are three or more firms, a firm cannot hold another firm's equity shares if some other firm holds its equity.

Lemma 2 *Assume $n \geq 3$. If α is pairwise stable, then $\alpha_{ij} = 0$ for all $j \neq i$ whenever $\alpha_{hi} \neq 0$ for some $h \neq i$.*

Proof. Let α be pairwise stable. We prove the lemma in two steps.

Step 1: $\alpha_{hi}\alpha_{ij} = 0$ for any three different firms h, i , and j .

Suppose on the contrary there exist three distinct firms h, i , and j for which $\alpha_{hi}\alpha_{ij} \neq 0$. Then, $\alpha_{hi} > 0$ implies that $d\alpha_{ii} > 0$ can be arranged between firm h and firm i . Thus, by (13),

$$1 + \sum_{k \neq h, i} \alpha_{kk} \leq \alpha_{hh} + \alpha_{ii}. \quad (14)$$

Similarly, since $\alpha_{ij} > 0$, (12) implies

$$1 + \sum_{k \neq i, j} \alpha_{kk} \leq \alpha_{ii} + \alpha_{jj}. \quad (15)$$

Putting (14) and (15) together, we have $\alpha_{ii} \geq 1$ which contradicts $\alpha_{hi} > 0$.

Step 2: $\alpha_{ij}\alpha_{ji} = 0$.

Suppose on the contrary there exists two distinct firms i and j such that $\alpha_{ij}\alpha_{ji} \neq 0$. Then, both $d\alpha_{ii} > 0$ and $d\alpha_{jj} > 0$ can be arranged between firms i and j . It follows from (13)

$$1 + \sum_{h \neq i, j} \alpha_{hh} = \alpha_{ii} + \alpha_{jj}. \quad (16)$$

Since $\alpha_{ii} < 1$ and $\alpha_{jj} < 1$, (16) implies $\alpha_{hh} < 1$ for all firms $h \neq i, j$. Fix firm $k \neq i, j$. Then, there exists a firm $l \neq k$ such that $\alpha_{lk} > 0$. Since $\alpha_{ij}\alpha_{ji} \neq 0$, it follows from Step 1 that $l \neq i, j$ which is impossible when $n = 3$. Suppose $n > 3$. Then, $\alpha_{ll} < 1$ implies that there exists a firm $m \neq i, j$ such that $\alpha_{ml} > 0$. From Step 1, it must be $m = k$. With $\alpha_{kl}\alpha_{lk} \neq 0$, a similar reason as before establishes

$$1 + \sum_{h \neq k, l} \alpha_{hh} = \alpha_{kk} + \alpha_{ll}. \quad (17)$$

Putting (16) and (17) together, we have $2 \leq 0$ which yields the desired contradiction. ■

When there are three firms, next theorem shows that a wide range of pairwise stable cross-holding matrices is possible. Each of these cross-holding matrices has one firm cross hold equal shares in the other two firms, while neither of the latter two cross holds other's shares.

Theorem 4 *Assume $n = 3$. Then, α is pairwise stable if and only if $\alpha_{ii} = 1$, $\alpha_{ij} = \alpha_{ik}$, and $\alpha_{jk} = \alpha_{kj} = 0$.*

Proof. The sufficiency follows easily from (13). Let α be given. If $\alpha_{hh} < 1$ for all h . Then, each firm's equity shares are cross held by another firm. Thus, by Lemma 2, no firm cross holds any other firm's equity shares. This is clearly a contradiction. Now suppose $\alpha_{ii} = 1$. If $\alpha_{jj} = \alpha_{kk} = 1$, then the conditions in the lemma are satisfied. Without loss of generality, assume $\alpha_{ij} > 0$ and hence $\alpha_{jj} < 1$. Then, $d\alpha_{jj} > 0$ is feasible for firms i and j . Consequently, by (13),

$$1 + \alpha_{kk} \leq \alpha_{ii} + \alpha_{jj} \iff \alpha_{kk} \leq \alpha_{jj}, \quad (18)$$

which implies $\alpha_{kk} < 1$. Since $\alpha_{ij} > 0$, by Lemma 2, $\alpha_{jk} = 0$. We thus conclude $\alpha_{ik} > 0$. Consequently, $d\alpha_{kk} > 0$ is feasible for firms i and k . By (13),

$$1 + \alpha_{jj} \leq \alpha_{ii} + \alpha_{kk} \iff \alpha_{jj} \leq \alpha_{kk}. \quad (19)$$

Putting (18) and (19) together, we have $\alpha_{jj} = \alpha_{kk}$ which in turn implies $\alpha_{ij} = \alpha_{ik}$. ■

While monopoly is the only outcome resulting from pairwise stable cross-holdings when there are two firms, Theorem 4 implies that a wide range of outcomes are possible when there are three firms. For example, α° is pairwise stable which results in the triopoly Cournot equilibrium with no cross-holdings. On the other hand, α such that $\alpha_{ii} = \alpha_{ij} = \alpha_{ik} = 1$ is also pairwise stable, which results in monopoly. Finally, let α be as in (6). By the first-order condition (8),

$$x_1^*(\alpha) = \frac{(1 - 2\delta)(a - c)}{2(2 - \delta)}.$$

It follows that $x_1^*(\alpha) = 0$ when $\delta = 1/2$. Thus, with $\delta = 1/2$, α results in the duopoly Cournot equilibrium with firms 2 and 3 as the duopolistic firms which do not participate in any cross-holdings.

When two firms engage cross-holdings, they compete less competitively. Their less competitive behavior would make the other firms the primary beneficiaries. For example, with $n = 3$, $\pi_i^*(\alpha^\circ) = (a - c)^2/16$ for all i for all i . Now consider a bilateral change in cross-

holdings between firms 2 and 3 that changes α° to

$$\alpha' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \alpha_{32} & \alpha_{23} \\ 0 & \alpha_{32} & 1 - \alpha_{23} \end{pmatrix}$$

With α' replacing α° , the subsequent Cournot equilibrium becomes

$$\begin{aligned} x_1^*(\alpha') &= \frac{1}{4 - \alpha_{23} - \alpha_{32}}(a - c), \\ x_2^*(\alpha') &= \frac{1 - \alpha_{23}}{4 - \alpha_{23} - \alpha_{32}}(a - c), \\ x_3^*(\alpha') &= \frac{1 - \alpha_{32}}{4 - \alpha_{23} - \alpha_{32}}(a - c). \end{aligned}$$

Observe that due to the cross-holdings, firm 2's and firm 3's outputs decrease, but firm 1's increases. Observe also that the total output decreases as a result of cross-holdings between firms 2 and 3. Hence, firms 2 and 3 are jointly worse off leaving firm 1 the only beneficiary of their cross-holdings. Indeed,

$$\pi_2^*(\alpha') + \pi_3^*(\alpha') = \frac{(2 - \alpha_{23} - \alpha_{32})(a - c)^2}{(4 - \alpha_{23} - \alpha_{32})^2} < \frac{(a - c)^2}{8} = \pi_2^*(\alpha^\circ) + \pi_3^*(\alpha^\circ).$$

Theorem 5 *Assume $n \geq 4$. Then, α is pairwise stable if and only if (a) $\alpha_{ii} = 1$ for all i or (b) for some firm i , $\alpha_{ik} = 1$ for $k = 1, 2, \dots, n$.*

Proof. Notice first that α satisfies (13) whenever α satisfies either (a) or (b). This shows that the conditions are sufficient. Suppose $\alpha_{ii} \neq 1$ for at least one i ; otherwise α satisfies condition (a) in which case the proof of the necessity is completed. Without loss of generality, assume

$$\alpha_{11} < 1. \tag{20}$$

We complete the rest of the proof in four steps.

Step 1: For some firm \bar{i} , $\alpha_{\bar{i}\bar{i}} = 1$.

Since $\alpha_{11} < 0$, there must be another firm which cross holds equity shares in firm 1. Denote this firm by \bar{i} . By Lemma 2, $\alpha_{\bar{i}1} > 0$ implies $\alpha_{\bar{i}\bar{i}} = 1$.

Step 2: $\alpha_{jj} < 1$ for all $j \neq \bar{i}$.

Since $\alpha_{\bar{i}1} > 0$, (20) implies that $d\alpha_{11} > 0$ can be arranged between firms 1 and firm \bar{i} . Thus, by (13),

$$1 + \sum_{k \neq \bar{i}, 1} \alpha_{kk} \leq \alpha_{\bar{i}\bar{i}} + \alpha_{11}.$$

If $\alpha_{jj} = 1$, then the preceding equality would imply $\alpha_{11} = 1$, which contradicts (20).

Step 3: $\alpha_{\bar{i}j} > 0$, $\alpha_{ij} = 0$ for all i and j such that $i \neq \bar{i}, j$.

Suppose $\alpha_{ij} > 0$ for some $i \neq \bar{i}, j$. Then, $d\alpha_{jj} > 0$ can be arranged between firm i and firm j . Thus, by (13),

$$1 + \sum_{k \neq i, j} \alpha_{kk} \leq \alpha_{ii} + \alpha_{jj}.$$

Since $\alpha_{\bar{i}\bar{i}} = 1$, the preceding inequality would imply $\alpha_{jj} \geq 1$, which results in the desired contradiction. This shows $\alpha_{ij} = 0$ for all i and j such that $i \neq \bar{i}, j$. This together with $\alpha_{jj} < 1$ as shown in Step 2 implies $\alpha_{\bar{i}j} > 0$.

Step 4: $\alpha_{jj} = 0$ for all $j \neq \bar{i}$.

From Step 3, $\alpha_{\bar{i}j} > 0$ for all $j \neq \bar{i}$. It follows that for all $j \neq \bar{i}$, $d\alpha_{jj} > 0$ can be arranged between firm \bar{i} and firm j . Hence,

$$1 + \sum_{k \neq \bar{i}, j} \alpha_{kk} \leq \alpha_{\bar{i}\bar{i}} + \alpha_{jj}.$$

Since $\alpha_{\bar{i}\bar{i}} = 1$, the preceding equation implies

$$\sum_{k \neq \bar{i}} \alpha_{kk} \leq 2\alpha_{jj}.$$

Summing both sides of the above equation over $j \neq \bar{i}$, we get

$$(n - 3) \sum_{k \neq \bar{i}} \alpha_{kk} \leq 0.$$

Since $n > 3$, it must be $\alpha_{jj} = 0$ for all $j \neq \bar{i}$. ■

Theorem 5 shows that with four or more firms, the usual Cournot equilibrium and monopoly are the only two possible outcomes that will result from pairwise stable cross-holdings. Unlike the case with two firms, to have monopoly one firm must hold all the equity of the other firms and supply to the entire market.

3.2 Constrained Pairwise Stable Cross-Holdings

We now apply the characterization result in the preceding subsection to analyze pairwise stability of cross-holdings with the non-negative output and minimum ownership constraints.

Definition 3 *We say that a collection α of cross-holdings is pairwise stable subject to the non-negative output and minimum retained ownership $\underline{\alpha}_{hh}$ for all h constraints, if for any $i \neq j$, $(\alpha_{ii}, \alpha_{ij}, \alpha_{ji}, \alpha_{jj})$ solves*

$$\begin{aligned} & \max \pi_i^*(\alpha') + \pi_j^*(\alpha') \\ & \text{subject to:} \\ & \alpha'_{ij} + \alpha'_{jj} = \alpha_{ij} + \alpha_{jj}, \\ & \alpha'_{ii} + \alpha'_{ji} = \alpha_{ii} + \alpha_{ji} \\ & \alpha'_{kl} = \alpha_{kl}, kl \neq ii, ij, ji, jj \\ & \alpha'_{hh} \geq \underline{\alpha}_{hh}, x_h^*(\alpha') \geq 0, \forall h. \end{aligned} \tag{21}$$

With the constraints present, the first-order condition for interior Cournot equilibrium (8) or the first-order condition (13) for the bilateral optimality of the cross-holdings may not be applicable. Due to the similarity of analysis, we focus on the case with the non-negative output constraint only (i.e., we take $\underline{\alpha}_{hh} = 0$ for all h).

As an example of the possibility for a corner Cournot equilibrium, consider $n = 3$ and α in (6). Without the non-negativity constraint, the first-order condition (8) implies

$$x_1^*(\alpha) = \frac{(1 - 2\delta)(a - c)}{2(2 - \delta)}.$$

It follows that $x_1^*(\alpha) < 0$ when $\delta > 1/2$, in which case the non-negativity constraint is binding on the subsequent Cournot equilibrium.

Let $\pi_i^{*NC}(\alpha)$ denote firm i 's profit from the Cournot equilibrium with collection α and with firms constrained to produce non-negative outputs. Our next theorem shows that no firm can be better off with the non-negativity constraint than without it and the security levels remain unchanged.

Theorem 6 *For all i with $\alpha_{ii} > 0$, $\pi_i^{*NC}(\alpha) \leq \pi_i^*(\alpha)$ and firm i 's security level remains at $\pi_i^*(\alpha^\circ)$.*

Proof. By analogy, it suffices to prove that firm 1 cannot be better off with the constraint. To this end, note first that with the non-negativity constraint, the Kuhn-Tucker condition for firm i with $\alpha_{ii} > 0$ is

$$P'(X)X_i + \alpha_{ii}[P(X) - c] \leq 0 \text{ and } \{P'(X)X_i + \alpha_{ii}[P(X) - c]\}x_i = 0 \quad (22)$$

where $X_i = \sum_j \alpha_{ij}x_j$. Since $p(X) = a - X$, by summing (22) over all i ,

$$P'(X)X + \sum_i \alpha_{ii}[P(X) - c] \leq 0 \Rightarrow X \geq \frac{(a - c) \sum_i \alpha_{ii}}{1 + \sum_i \alpha_{ii}}. \quad (23)$$

Suppose first $x_1 > 0$. Then, (22) reduces to $P'(X)X_1 + \alpha_{11}[P(X) - c] = 0$ or $X_1 = a_{11}(p(X) - c)$. Thus, by (23),

$$\pi_1^{*NC}(\alpha) = [P(X) - c]X_1 = \alpha_{11}[P(X) - c]^2 \leq \frac{a_{11}(a - c)^2}{(1 + \sum_i \alpha_{ii})^2}. \quad (24)$$

Together, (11) and (24) imply $\pi_1^{*NC}(\alpha) \leq \pi_1^*(\alpha)$.

Suppose now $x_1 = 0$. Then, there are two cases. First, $P'(X)X_1 + \alpha_{11}[P(X) - c] = 0$ holds. In this case, the preceding reasoning can be directly applied to complete the proof. Second, $P'(X)X_1 + \alpha_{11}[P(X) - c] < 0$. In this case, firm 1's unconstrained profit-maximizing output is negative. Denote this negative output by y_1 . Replacing $x_1 = 0$ with y_1 in (22) would make the equality hold. Maintaining the non-negativity constraint on firm i 's output for all $i \neq 1$ and letting $X' = y_1 + \sum_{i \neq 1} x_i$ and $X'_1 = \alpha_{11}y_1 + \sum_{i \neq 1} \alpha_{ii}x_i$, a similar application of (22) as in the previous paragraph shows

$$P'(X')X' + \sum_i \alpha_{ii}[P(X') - c] \leq 0 \Rightarrow X' \geq \frac{(a - c) \sum_i \alpha_{ii}}{1 + \sum_i \alpha_{ii}}.$$

This in turn implies

$$\pi_1^{*NC}(\alpha) \leq [P(X') - c]X'_1 = \alpha_{11}[P(X') - c]^2 \leq \frac{a_{11}(a - c)^2}{(1 + \sum_i \alpha_{ii})^2}. \quad (25)$$

Thus, (11) and (25) together imply $\pi_1^{*NC}(\alpha) \leq \pi_1^*(\alpha)$.

Since firms' security levels without the non-negativity constraint are their Cournot equilibrium profits with no cross-holdings, they are therefore achievable with the constraint. This shows that the firms' security levels remain unchanged. ■

Theorem 6 implies that when two firms cannot increase total profit without the non-negativity constraint, neither can they with the constraint. A direct application of this result establishes

Corollary 1 *Suppose that a collection of cross-holdings is unconstrained pairwise stable and individually rational. Suppose further that the firms' subsequent Cournot equilibrium outputs are non-negative under the collection. Then, the collection remains to be pairwise stable and individually rational when the firms are constrained to produce non-negative outputs.*

To illustrate the corollary, let $n = 3$. Without the non-negativity constraint, the α in (6) implies that in the subsequent Cournot equilibrium, $x_i^*(\alpha) > 0$ for all i when $\delta < 1/2$ and $x_1^*(\alpha) = 0$ and $x_2^*(\alpha) = x_3^*(\alpha) = (a - c)/3$ when $\delta = 1/2$. Thus, all firms' subsequent Cournot equilibrium outputs are non-negative whenever $\delta \leq 1/2$, even though negative outputs are permissible. Assume $\delta = 1/2$. In this case,

$$\pi_2^{*NC}(\alpha) = \pi_2^*(\alpha) = (a - c)^2/18 \text{ and } \pi_3^{*NC}(\alpha) = \pi_3^*(\alpha) = (a - c)^2/18. \quad (26)$$

Now suppose that firm 2 holds as passive ownership a fraction $0 < \gamma < 1 - \delta$ of firm 3's equity. This feasible bilateral change between firms 2 and 3 changes α to

$$\alpha' = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \gamma \\ 0 & 0 & \frac{1}{2} - \gamma \end{pmatrix}.$$

With α' , firm 2 will produce less due to its concern with 3's profit. The reduction in 3's output induces firm 1 to produce. Indeed, the subsequent Cournot equilibrium becomes

$$\begin{cases} x_1^*(\alpha') = \frac{\gamma(a-c)}{3-\gamma} \\ x_2^*(\alpha') = \frac{a-c}{3-\gamma} \\ x_3^*(\alpha') = \frac{(a-c)(1-2\gamma)}{3-\gamma} \end{cases} \Rightarrow \pi_2^*(\alpha') + \pi_3^*(\alpha') = \frac{(1-\gamma)(a-c)^2}{(3-\gamma)^2}. \quad (27)$$

Putting (26) and (27) together, the bilateral change between 2 and 3 would result in a smaller total profit. With 2 holding γ fraction of 3's equity, equation (27) implies that in the subsequent Cournot equilibrium, total output for 2 and 3 together will be

$$x_2^*(\alpha') + x_3^*(\alpha') = \frac{2(a-c)(1-\gamma)}{3-\gamma}$$

which is less than their total output $2(a-c)/3$ before. Of course, this reduction will not cause their total profit to decrease if the new equilibrium price is large enough. However, although

firm 1's new equilibrium output is not so large as to increase the total equilibrium output, its equilibrium output is large enough to make the new equilibrium price not profitable for 2 and 3 collectively.⁷

4 Conclusion

In this paper we analyzed pairwise stable and individually rational cross-holdings in oligopoly. Pairwise stability in this context means that no two firms can increase total profit by a bilateral change of equity holdings in their own firms. Thus, at pairwise unstable cross-holdings there will exist mutually beneficial prices for two firms to bilaterally change equity holdings in their own equities. It follows that pairwise stability is a natural equilibrium concept for cross-holdings. We showed that (i) pairwise stability implies monopoly when there are two firms; (ii) a wide range of outcomes is possible when there are three firms, including the triopoly Cournot equilibrium and the duopoly Cournot equilibrium as special cases; and (iii) the usual Cournot equilibrium with no cross-holdings is an outcome of pairwise stable when there are four or more firms. Thus, the output market is more collusive with the presence of cross-holdings than without. Our results provide a reason for why antitrust concerns with cross-holdings should be raised.

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⁷From (27), total new equilibrium output is $(a - c)(2 - \gamma)/(3 - \gamma)$ which is less than $2(a - c)/3$.

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