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Contract and Game Theory: Basic Concepts for Settings with Finite Horizons

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Abstract

This paper reports the analysis of a general model of contract in multi-period settings with both external and self-enforcement. In the model, players alternately engage in contract negotiation and take individual actions. A notion of *contractual equilibrium*, which combines a bargaining solution and individual incentive constraints, is proposed and analyzed. The modeling framework helps identify the relation between the manner in which players negotiate and the outcome of the long-term contractual relationship. In particular, the model shows the importance of accounting for the self-enforced component of contract in the negotiation process. Examples and guidance for applications are provided. *JEL Classification: C70, D74, K10.*

Many economic relationships are contractual, in that the parties negotiate and agree on matters of mutual interest and intend for their agreement to be enforced. Enforcement comes from two sources: external agents who are not parties to the contract at hand (external enforcement) and the contracting parties themselves (self-enforcement). Most contracts involve some externally enforced component and some self-enforced component. When parties negotiate an agreement, they are coordinating generally on both of these dimensions.

In this paper, I further a game-theoretic framework for analyzing contractual settings with both externally enforced and self-enforced components. I model a long-term contractual relationship in which, alternately over time, the parties engage in contracting (or recontracting) and they take individual productive actions. I discuss various alternatives in the analysis of contract negotiation and I define a notion of *contractual equilibrium* that combines a bargaining solution with incentive conditions on the individual actions. Existence results, examples, and notes for some applications are also provided.

Period 1 ── Contract formation, with externally
enforced component $m^{\text{hh}}, m^{\text{hl}}, m^{\text{lh}}, m^{\text{ll}}$.

── Worker selects investment I.

Period 2 ── Contract revision/renegotiation, where
$m^{\text{hh}}, m^{\text{hl}}, m^{\text{lh}}$, and m^{ll} can be changed.

── Players simultaneously choose levels
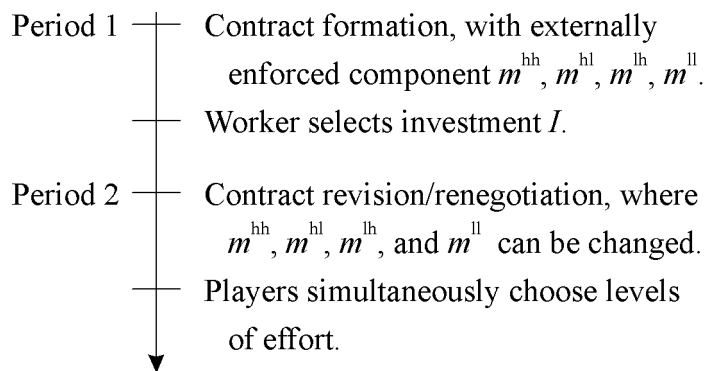of effort.

Figure 1: Time line for the MW example.

The framework developed herein is designed to clarify the relation between the self-enforced and externally enforced components of contract, to demonstrate the advantages of explicitly modeling the self-enforced component in applied research, and to facilitate this type of modeling. The framework emphasizes the roles of history, social convention, and bargaining power in determining the outcome of contract negotiation and renegotiation. The framework also helps one to compare the implications of various theories of negotiation.

MW Example

Here is an example that illustrates some of the issues in modeling both the self- and externally enforced components of contract. Suppose that a manager (player 1) and a worker (player 2) interact over two periods of time, as shown in Figure 1. At the beginning of the first period, the parties engage in contract negotiation. The contract includes vectors $m^{\text{hh}}, m^{\text{hl}}, m^{\text{lh}}, m^{\text{ll}} \in \mathbf{R}^2$, which specify externally enforced monetary transfers between the manager and the worker contingent on actions they take in the second period. Each vector is of the form $m = (m_1, m_2)$, where m_1 is the monetary transfer to player 1 and m_2 is the transfer to player 2. Transfers can be negative and I assume that $m_1 + m_2 \leq 0$, meaning that the parties cannot create money. I also assume that the transfers are "relatively balanced" in that $m_1^{\text{hh}} + m_2^{\text{hh}} = m_1^{\text{hl}} + m_2^{\text{hl}} = m_1^{\text{lh}} + m_2^{\text{lh}} = m_1^{\text{ll}} + m_2^{\text{ll}}$.[1] Later in the first period, the worker selects an investment $I \in \{0, 1, 2, 3, 4\}$; the investment yields an immediate benefit $8I$ to the manager and an immediate cost I^2 to the worker. The manager observes I, but external enforcer cannot observe I.

At the beginning of the second period there is another round of contracting, at which point the specification of transfers may be changed. Then the parties simultaneously select individual effort levels—"high" or "low"—and they receive monetary gains according to

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[1]I elaborate on this assumption in Section 4, where the example is described more formally.

2

the following table:

M \ W	high	low
high	$17 + m_1^{hh}, \ -1 + m_2^{hh}$	$9 + m_1^{hl}, \ m_2^{hl}$
low	$10 + m_1^{lh}, \ -1 + m_2^{lh}$	$m_1^{ll}, \ m_2^{ll}$

The table represents that the parties' efforts generate revenue, which is 18 if both exert high effort, 10 if exactly one of them exerts high effort, and zero otherwise. Each party's cost of high effort is 1. The manager obtains the revenue, minus his effort cost, plus the transfer m_1 (which can be negative). The worker receives the monetary transfer m_2 minus his effort cost. That there is a different vector m for each cell of the table means that the external enforcer can verify the effort choices of both players.

Assume that each player's payoff for the entire relationship is the sum of first- and second-period monetary gains. Thus, an efficient specification of productive decisions is one that maximizes the players' joint value (the sum of their payoffs). Clearly, efficiency requires the worker to choose $I = 4$ in the first period (it maximizes $8I - I^2$) and requires the players to select (high, high) in the second period. Note, however, that the parties would not have the individual incentives to select (high, high) in the second period unless $m_1^{hh} \geq m_1^{lh} - 7$ and $m_2^{hh} \geq m_2^{hl} + 1$; that is, (high, high) must be a Nash equilibrium of the matrix shown above. Furthermore, the worker has the strict incentive to choose $I = 0$ if the first period is viewed in isolation. Whether efficiency, or even $I > 0$, can be achieved depends on the details of the contracting process.

Before thinking about contractual alternatives, consider the parties' incentives regarding their effort choices in the second period. Any of the four effort profiles can be made a Nash equilibrium in dominant strategies with the appropriate selection of externally enforced transfers. This is true even if one restricts attention to balanced transfers (where no money is discarded). For example, transfers satisfying

$$m_2^{hl} > m_2^{hh} - 1, \quad m_2^{ll} > m_2^{lh} - 1, \quad m_2^{hh} > 7 + m_2^{lh}, \quad \text{and} \quad m_2^{hl} > 9 + m_2^{ll},$$

with

$$m_1^{hh} = -m_2^{hh}, \quad m_1^{hl} = -m_2^{hl}, \quad m_1^{lh} = -m_2^{lh}, \quad \text{and} \quad m_1^{ll} = -m_2^{ll},$$

make "low" a dominant action for both players. Numbers that work are $m_2^{hh} = m_2^{ll} = 0$, $m_2^{hl} = 10$, and $m_2^{lh} = -8$. Transfers satisfying the reverse strict inequalities make "high" a dominant strategy for both players. There are also transfers that make "low" dominant for one player and "high" dominant for the other. All of these are examples of *forcing* transfers, in that they force (through the players' incentives) a particular effort profile to be played.

I next sketch three contractual alternatives. While these are presented informally and do not cover the full range of possibilities, they illustrate the contractual components and issues that are formally addressed in this paper.

- **Alternative 1:**

  The players make the following agreement at the beginning of the first period. They choose a specification of transfers that satisfies $m^{ll} = (0,0)$ and that forces (low, low) to be played in the second period. With these transfers, the players would each obtain zero in the second period. The players also agree that if the worker invests $I = 4$ then, at the beginning of period 2, they will recontract to specify transfers that force (high, high) to be played and $m^{hh} = (-17, 17)$. For any other investment I, the players expect not to change the specification of transfers.

  If this contractual agreement holds, then it is rational for the worker to select $I = 4$ in the first period. If he makes this choice, he gets -16 in the first period and, anticipating recontracting to force (high, high), he gets 16 in the second period. If he chooses any other effort level in the first period, the worker obtains no more than zero. An efficient outcome results.

- **Alternative 2:**

  In first-period negotiation, the players choose a specification of transfers that forces (low, low) with $m_2^{ll} = 0$, as in Alternative 1. In the second period, the players renegotiate to pick transfers that force (high, high) with $m^{hh} = (-9, 9)$, and they do so regardless of the worker's first-period investment level. In other words, the outcome of second-period renegotiation is the ex-post efficient point in which the players obtain equal shares of the bargaining surplus relative to their initial contract. Anticipating that his second-period gains do not depend on his investment choice, the worker optimally selects $I = 0$.

- **Alternative 3:**

  The players' initial contract specifies $m^{hh} = m^{hl} = (-9, 9)$, $m^{lh} = (-1, 1)$, and $m^{ll} = (0, 0)$, which yields the following matrix of second-period payoffs:

	W high	W low
M high	8, 8	0, 9
M low	9, 0	0, 0

  Note that, for this matrix, (high, low), (low, high), and (low, low) are the Nash equilibria and that these are ex post inefficient.[2] The players agree that, in the event that they fail to renegotiate in the second period, the equilibrium on which they coordinate will be conditioned on the worker's investment choice. Specifically, if the worker

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[2] The determination of efficiency includes consideration of transfers; the profile (high, low) is inefficient because, with the appropriate transfers, both players can be made better off if they select (high, high). That the Nash equilibria are weak is of little consequence for the point of the example; for simplicity of exposition, this example was constructed to avoid having to examine mixed strategy equilibria.

chose $I \geq 3$ then they coordinate on equilibrium (high, low), whereas if $I < 3$ then they coordinate on equilibrium (low, high). That is, the disagreement point for second-period negotiation depends on the worker's action in the first period.

Assume that the players do renegotiate in the second period so that the efficient profile (high, high) is achieved. The joint surplus of renegotiation is $16 - 9 = 7$. Suppose that player 1 has all of the bargaining power and thus negotiates a transfer that gives him the entire surplus. The worker therefore gets 9 if (high, low) is the disagreement point, whereas he gets 0 if (low, high) is the disagreement point. Clearly, with this contractual arrangement, the worker optimally selects $I = 3$ in the first period.

External enforcement is represented by the way that the contract directly affects the players' payoffs (through vectors $m^{hh}, m^{hl}, m^{lh}, m^{ll}$ in the example). Self-enforcement relates to the way in which the players coordinate their individual actions. For instance, the players agree on the effort levels that they will choose in the second period; self-enforcement requires that these effort levels constitute a Nash equilibrium.

Obviously, the self- and externally enforced components of contract interact, because, at minimum, the externally enforced transfers influence which second-period action profiles are Nash equilibria. In Alternatives 1 and 2, the specification of transfers forces a unique action profile in the second period. In Alternative 3, the initial specification of transfers creates a situation in which there are three Nash equilibria. The self-enforced component of contract specifies how the players will select among these equilibria.

Alternatives 1-3 represent different assumptions about second-period contracting. Alternative 1 is viable only if renegotiation accommodates an ex post inefficient outcome conditional on the worker failing to invest. Alternative 1 also assumes that the worker has a great deal of bargaining power and can thus extract the entire renegotiation surplus. Alternative 2 assumes that ex post efficiency is always reached in the renegotiation process. In Alternative 3, the second-period disagreement point is made a function of first-period play, via the self-enforced component of contract. Note that this manner of conditioning could not be achieved through the externally enforced part, because the external enforcer does not observe the worker's investment (whereas the manager and worker, who coordinate their effort choices, do). Thus, the example shows that, to determine what the players can achieve in a contractual relationship, it is important to model the negotiation process and to differentiate between self-enforced and externally enforced components of contract.

Themes

As the MW example illustrates, self-enforced and externally enforced components of contract differ in important respects. Because the contracting parties may have different information than do external agents, self-enforcement and external enforcement may be conditioned on different sets of events. Furthermore, an "agreement" on the terms of the self-enforced component entails subtleties not present with external enforcement. In particular, self-enforcement relies on the contracting parties to coordinate their behavior in the

future; external enforcement, in contrast, does not require the same kind of coordination because an external agent performs it.[3]

A key issue explored herein is how "actively" the parties engage in contracting. In very active contracting, contract selection depends mainly on the current technology of the relationship (including the bargaining protocol and the future productive alternatives) rather than on historical distinctions that are not payoff relevant from the current time. In less active contracting, social convention and history (including prior agreements) play a greater role. Bargaining theories that represent more active contracting are refinements (in terms of set inclusion) of those that represent less active contracting. My analysis relates activeness of contracting to the extent one assumes that verbal statements made by the players influence how they coordinate in the future. Stronger assumptions about the relation between language and expectations imply more active contracting.

The activeness of contracting is particularly salient for the self-enforced component of contract, because, on this dimension, a verbal statement of agreement does not force the parties to subsequently act accordingly (as external enforcement can do) and there are various possibilities for the link between the parties' verbal agreement and the way they actually coordinate. For example, the parties may say to each other that, at some later time, they will coordinate on the individual-action profile a, whereas they both anticipate actually coordinating on a'. Language is more meaningful, and contracting more active, when statements of agreement—at least those that are consistent with individual incentives—induce the parties to comply.

By differentiating between the two components of contract, this paper furthers my general research agenda, part of which is to (i) discover how the technological details of contractual settings influence outcomes, (ii) demonstrate the importance of carefully modeling these details, and (iii) provide a flexible framework that facilitates the analysis of various applications. By "technological details," I mean the nature of productive actions, the actions available to external enforcers, the manner in which agents communicate and negotiate with one another, and the exact timing of these various elements in a given contractual relationship. In other papers, I focus on the nature of productive actions (Watson 2005a,b), the mechanics of evidence production (Bull and Watson 2004,2006), and contract writing and renegotiation costs (Schwartz and Watson 2004, Brennan and Watson 2002). These papers and the present one show that the technological details can matter significantly.

Relation to the Literature

This paper builds on the large and varied contract-theory literature. In many of the literature's standard models (such as the basic principal-agent problem with moral hazard), the self-enforced aspects of contracting are trivial. However, there are also quite a few studies of contractual relations in which self-enforcement plays a more prominent role. Finitely repeated games constitute an abstract benchmark setting in which there is no external enforcement. For this setting, Benoit and Krishna (1993) provide an analysis of

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[3]There may be ambiguities in the language parties use to communicate with an external enforcer, which could introduce a coordination problem, but herein I ignore this issue.

renegotiation. The basic idea is that, at the beginning of the repeated game, the players agree on a subgame-perfect Nash equilibrium for the entire game, but they jointly reevaluate the selection in each subgame. Benoit and Krishna assume that, in each period, the parties select an equilibrium on the Pareto frontier of the set of feasible equilibria, where feasibility incorporates the implications of negotiation in future periods.

Many papers look at the interaction of self-enforced and externally enforced components of contract in applied settings. Prominent entries include Telser (1980), Bull (1987), MacLeod and Malcomson (1989), Schmidt and Schnitzer (1995), Roth (1996), Bernheim and Whinston (1998), Pearce and Stacchetti (1998), Ramey and Watson (2001, 2002), Baker, Gibbons, and Murphy (2002), and Levin (2003). In each of these papers, a contract specifies an externally enforced element that affects the parties' payoffs in each stage of their game. The parties interact over time and may sustain cooperative behavior by playing repeated-game-style equilibria. Some of these papers address the issue of negotiation (and renegotiation) over the self-enforced component of contract; this is typically done with assumptions on equilibrium selection, such as Pareto perfection, rather than by explicitly modeling how contracts are formed. Of these papers, the most relevant to the work reported here is Bernheim and Whinston (1998). These authors show that, even when external enforcement can be structured to control individual actions, parties may prefer relying on self-enforcement because it can generally be conditioned on more of the history than can externally enforced elements. This is exactly the feature exhibited in the comparison of Alternatives 2 and 3 in the example discussed above.

The distinction of the modeling exercise herein, relative to the existing literature, is the depth of modeling how parties determine the self-enforced aspects of contract. That is, I examine the negotiation process in more explicit terms than is done in the related literature. Here, notions of bargaining power and disagreement points are addressed in the context of self-enforcement. The detailed examination of negotiation yields a foundation for "Pareto perfection" as well as the concepts used in some of my other work (Ramey and Watson 2002, Klimenko, Ramey, and Watson 2004, Watson 2005a,b). Furthermore, it helps one understand the implications of various assumptions about the negotiation process, which offer a guide for future applied work.

Part of the framework developed here was previously sketched in Watson (2002), which contains some of the basic definitions and concepts without the formalities. My analysis of negotiation over self-enforced elements of contract draws heavily from—and extends in a straightforward way—the literature on cheap talk, most notably Farrell (1987), Rabin (1994), Arvan, Cabral, and Santos (1999), and Santos (2000).

Outline of the Paper

The following section begins the theoretical exercise by describing single contracting problems that include self-enforced and externally enforced components. Section 2 addresses how contract negotiation can be modeled. This section first describes bargaining protocols to be analyzed using standard non-cooperative theory. Assumptions on the meaning of language are described. Then, non-cooperative bargaining theories are translated

into a convenient cooperative-theory form and the notion of activeness is discussed. Three benchmark bargaining solutions are defined.

In Section 3, I develop a framework for analyzing long-term contractual relationships in which the parties contract and take individual actions in successive periods of time. I analyze contractual relationships in terms of the sets of continuation values from each period in various contingencies. The notion of contractual equilibrium is defined and partly characterized. Section 4 contains the formal analysis of the MW example and a repeated-game example. Section 5 concludes the body of the paper with some notes for applications.

The appendices present technical details and existence results. Appendix A contains the details of an example discussed in Section 2. Appendix B defines three classes of contractual relationships in which elements of the technology are finite; existence of contractual equilibrium is proved for these classes of relationships. Appendix C contains the proofs of the lemma and theorem that appear in the body of the paper.

1 Contractual Elements in an Ongoing Relationship

Suppose that there are two players in an ongoing relationship and that at a particular time—call it a *negotiation phase*—they have the opportunity to establish a contract or revisit a previously established contract. At this point, the players have a history (describing how they interacted previously) and they look forward to a continuation (describing what is to come). At the negotiation phase, the players form a contract that has two components: an externally enforced part and a self-enforced part.

The externally enforced component of their agreement is a tangible joint action x, which is an element of some set X. For example, x can represent details of a document that the players are registering with the court, who later will intervene in the relationship; x could be an immediate monetary transfer; or x might represent a choice of production technology. There is a special element of X, called the *default joint action* and denoted $\underline{x}$, that is compelled if the players fail to reach an agreement.

The self-enforced component of the players' joint decision is an agreement between them on how to coordinate their future behavior. For example, if the players anticipate interacting in a subgame later in their relationship, then, at the present negotiation phase, they can agree on how each of them will behave in the subgame. The feasible ways in which the players can coordinate on future behavior are exactly those that can be self-enforced, as identified by incentive conditions. Coordination on future behavior yields a continuation value following the negotiation phase.

To describe the set of feasible contracts, it is useful to start by representing the players' alternatives in terms of feasible payoff vectors. Let $Y^x \subset \mathbf{R}^2$ be the set of payoff vectors (continuation values) on which the players can coordinate, conditional on selecting tangible joint action x. The set Y^x is assumed to embody a solution concept for future play.

Defining $\mathcal{Y} \equiv \{Y^x\}_{x \in X}$, the contracting problem is represented as $(X, \underline{x}, \mathcal{Y})$. Define

$$Y \equiv \bigcup_{x \in X} Y^x \text{ and } \underline{Y} \equiv Y^{\underline{x}}.$$

In payoff terms, the contracting problem can be thought of as a joint selection of a vector in Y, with disagreement leading to a vector in $\underline{Y}$.

Some of the analysis uses the following assumption.

Assumption 1: *The set $\underline{Y}$ is compact. Furthermore, the set Y is closed and satisfies $Y \subset \{y \in \mathbf{R}^2 \mid \beta_1 y_1 + \beta_2 y_2 \le 1\}$ for some numbers $\beta_1, \beta_2 > 0$.*

The second part of this assumption is that Y is bounded above (separated) by a negatively-sloped line in $\mathbf{R}^2$; this assures that arbitrarily large utility vectors are not available to the players.

BoS Example

Here is a simple example that illustrates the determination of Y^x. There are two players who interact over two phases of time. The first is a negotiation phase, where the players negotiate and jointly select tangible action $x = (l, m)$. The component l is a "level of interaction" that can be either 1 or 2, whereas $m = (m_1, m_2)$ is a vector of monetary transfers from the set

$$\mathbf{R}^2_- \equiv \{m \in \mathbf{R}^2 \mid m_1 + m_2 \le 0\}.$$

Note that the players can transfer money between them and can throw away money. We have $X = \{1, 2\} \times \mathbf{R}^2_-$. In the second phase, the players simultaneously take individual actions, with each player choosing between "left" and "right." The players then receive payoffs given by the following battle-of-the-sexes matrix:

1 \ 2	left	right
left	$6l + m_1, 3l + m_2$	m_1, m_2
right	m_1, m_2	$3l + m_1, 6l + m_2$

Assume that the default joint action is $(1, (0, 0))$, where no money is transferred and $l = 1$.

Self-enforcement of behavior in the second phase is modeled using Nash equilibrium. There are three Nash equilibria of the battle-of-the-sexes game. For the matrix shown above, the two pure-strategy equilibria yield payoff vectors $(6l + m_1, 3l + m_2)$ and $(3l + m_1, 6l + m_2)$, whereas the mixed-strategy equilibrium yields $(2l + m_1, 2l + m_2)$. Thus, in this example, we have for any $m \in \mathbf{R}^2_-$:

$$Y^{(1,m)} = \{(6 + m_1, 3 + m_2), (3 + m_1, 6 + m_2), (2 + m_1, 2 + m_2)\}$$

and

$$Y^{(2,m)} = \{(12 + m_1, 6 + m_2), (6 + m_1, 12 + m_2), (4 + m_1, 4 + m_2)\}.$$
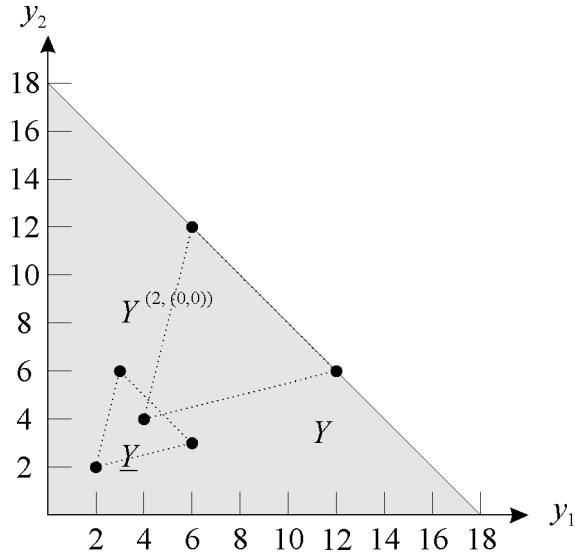
9

Figure 2: Y for the BoS example.

The set Y is given by the shaded region in Figure 2 (which shows the positive quadrant). The figure also shows the three points composing $Y^{(1,(0,0))}$ and the three points composing $Y^{(2,(0,0))}$.

More on the MW Example

For another illustration of how to construct Y^x, consider the manager-worker example from the Introduction. Imagine that the parties are negotiating at the beginning of the second period, in the contingency in which they had selected $m^{hh} = m^{hl} = (-9, 9)$, $m^{lh} = (-1, 1)$, and $m^{ll} = (0, 0)$ in the first period (recall the description of Alternative 3 in the Introduction).

The parties' tangible joint action at the beginning of the second period is a new selection of transfers to supercede those chosen in the first period. Call the second period selections $\hat{m}^{hh}$, $\hat{m}^{hl}$, $\hat{m}^{lh}$, and $\hat{m}^{ll}$. The default action specifies $\hat{m}^{hh} = \hat{m}^{hl} = (-9, 9)$, $\hat{m}^{lh} = (-1, 1)$, and $\hat{m}^{ll} = (0, 0)$. Recall that the matrix induced my these default transfers is:

	W	
M	high	low
high	8, 8	0, 9
low	9, 0	0, 0

Thus, noting the three Nash equilibria of this matrix, we have

$$\underline{Y} = \{(9, 0), (0, 0), (0, 9)\}.$$

As for the set Y, note that the greatest joint payoff attainable in the second period is 16. Further, for any vector $y = (y_1, y_2)$ with $y_1 + y_2 \leq 16$, there is a specification of transfers
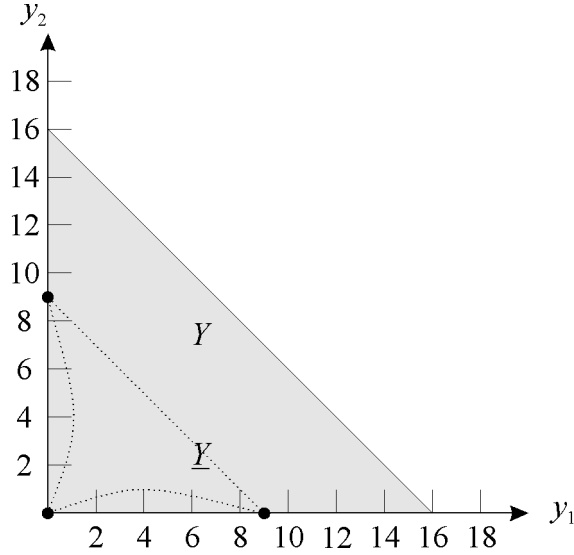
Figure 3: Y for the MW example in a second-period contingency.

such that (high, high) is a Nash equilibrium, $17 + \hat{m}_1^{\text{hh}} = y_1$, and $-1 + \hat{m}_2^{\text{hh}} = y_2$. This implies that

$$Y = \{y \in \mathbf{R}^2 \mid y_1 + y_2 \leq 16\}.$$

The sets $\underline{Y}$ and Y are pictured in Figure 3.

2 Modeling Contract Negotiation

Interaction in the negotiation phase leads to some contractual agreement between the players, which we can represent by a joint action x and a continuation value $y \in Y^x$. The negotiation process can be modeled using a non-cooperative specification that explicitly accounts for the details of the negotiation process. For example, we could assume that negotiation follows a specific offer-counteroffer or simultaneous-demand protocol. This approach basically "inserts" into the time line of the players' relationship a standard non-cooperative bargaining game.

For an illustration, consider the ultimatum-offer bargaining protocol whereby player 1 makes an offer to player 2 and then player 2 decides whether to accept or reject it, ending the negotiation phase. Because the players are negotiating over both the tangible action x and the intangible coordination of their future behavior, player 1's "offer" comprises both of these elements. Because the sets Y^x represent (in payoff terms) the feasible ways in which the players can coordinate future behavior, we can describe an offer by a tuple (x, y). Translated into plain English, (x, y) means "I suggest that we take tangible action x now and then coordinate our future behavior to get payoff vector y."

If an offer is accepted by player 2, then x is externally enforced. Note, however, that "y" is simply cheap talk. When player 1 suggests coordinating to obtain payoff vector

11

y and player 2 says "I accept," there is no externally enforced commitment to play accordingly. In fact, the players could then coordinate to achieve some other payoff vector. Furthermore, there is no requirement that (x, y) satisfy $y \in Y^x$; a player can say whatever he wants to say.

Because *y* is cheap talk, there are always equilibria of the negotiation phase in which this intangible component of the offer is ignored; in other words, the players' future behavior is not conditioned on what they say to each other in the negotiation phase (other than through the externally enforced *x*). In such equilibria, there is a sense in which the players are not actively negotiating over how to coordinate their future behavior. Rather, some social convention or other institution is doing so. To model more active negotiation, we can proceed by combining standard equilibrium conditions with additional assumptions on the meaning of language. These additional requirements can be expressed as constraints on future behavior conditional on statements that the players make in the negotiation phase. Farrell (1987), Rabin (1994), Arvan, Cabral, and Santos (1999), and Santos (2000) conduct such an exercise for settings without external enforcement.[4] This section builds on their work and offers the straightforward extension to settings with external enforcement.

Bargaining Protocols

To formalize these ideas, I begin with some general definitions. I assume that the negotiation phase is modeled using standard noncooperative-theory tools.

Definition 1: *A **bargaining protocol** Γ consists of (i) a game tree that has all of the standard elements except payoffs, (ii) a mapping ψ from the set of terminal nodes N of the game tree to the set X, (iii) a partition $\{N^A, N^D\}$ of N, and (iv) a function $v : N^A \rightarrow \mathbf{R}^2$.*

The mapping ψ identifies, for each path through the tree, a tangible action from X that the players have selected. Reaching terminal node *n* implies that $\psi(n)$ will then be externally enforced. Although no payoff vector is specified for terminal nodes, it is understood that if players reach terminal node *n* in the negotiation phase then their behavior in the continuation leads to some payoff in $Y^{\psi(n)}$. Note that a well-defined extensive-form game can be delineated by assigning a specific payoff vector to each terminal node of Γ.

The additional structure provided by items (iii) and (iv) in the definition is helpful in distinguishing between "agreement" and "disagreement." The set N^A comprises the terminal nodes at which the players have reached an agreement, whereas N^D comprises the terminal nodes at which the players have not reached an agreement. The meaning of "disagreement" requires that $\psi(n) = \underline{x}$ for all $n \in N^D$; that is, the default action is taken when the players fail to reach an agreement. I assume that N^D is nonempty. The function v associates with each terminal node in N^A the continuation payoff vector that was verbally stated in the players' agreement.

Consider, as an example, the ultimatum-offer protocol in which player 1 makes an offer and player 2 accepts or rejects it. For this game tree, each terminal node represents a path

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[4]See also Farrell and Rabin (1996).

consisting of player 1's offer (x, y) and player 2's response ("accept" or "reject"). We have $\psi(x, y; \text{accept}) = x$ and $\psi(x, y; \text{reject}) = \underline{x}$ for every $x \in X$ and $y \in \mathbf{R}^2$. That is, if player 2 accepts (x, y) then external enforcement of x is triggered; otherwise, the default action $\underline{x}$ occurs. Furthermore,

$$N^A = \{(x, y; \text{accept}) \mid x \in X, y \in \mathbf{R}^2\}$$

and $v(x, y; \text{accept}) = y$. That is, if player 1's offer of (x, y) is accepted by player 2, then the players have identified the continuation payoff y. Remember, though, that y is not necessarily a feasible payoff vector in the continuation.

Equilibrium Conditions

An equilibrium in the negotiation phase is a specification of behavior that is rational given some feasible selection of future behavior.

Definition 2: *Take a bargaining protocol Γ and let s be a strategy profile (mixed or pure) for this extensive form. Call s an* **equilibrium of the negotiation phase** *if there is a mapping $\varphi : N \to \mathbf{R}^2$ such that (i) $\varphi(n) \in Y^{\psi(n)}$ for every $n \in N$, and (ii) s is a subgame-perfect equilibrium in the extensive form game defined by Γ with the payoff at each node $n \in N$ given by $\varphi(n)$.[5] Call y^* an* **equilibrium negotiation value** *if it is the payoff vector for some equilibrium of the negotiation phase.*

Implied by the domain of φ is that the behavior in the negotiation phase becomes common knowledge before the players begin interacting in the continuation. Clearly, studying equilibrium in the negotiation phase is equivalent to examining a subgame-perfect equilibrium in the entire game (negotiation phase plus continuation); here, I am simply representing post-negotiation interaction by its continuation value.

Assumptions about the meaning of language can be viewed as constraints on the function φ in the definition of equilibrium of the negotiation phase. It may be natural to assume that if a player accepts an offer (x, y) and if $y \in Y^x$ (that is, y is consistent with x), then the continuation value y becomes focal and the players coordinate their future behavior to achieve it. This is expressed by

Assumption 2 (Agreement Condition): $\varphi(n) = v(n)$ *for all $n \in N^A$ satisfying $\psi(n) \in Y^{\psi(n)}$.*

We could also make an assumption about the continuation value realized when the players fail to agree. One such condition is that the players share responsibility for disagreement in the sense that the continuation value does not depend on how the players arrived at the impasse.

Assumption 3 (Disagreement Condition): *There is a vector $\underline{y} \in Y$ such that $\varphi(n) = \underline{y}$ for all $n \in N^D$.*

---

[5] One could use something other than subgame perfection for condition (ii), but subgame perfection is suitable for my purposes herein.

These assumptions are added to the conditions on φ in Definition 2. To refer to the resulting equilibrium definitions, I use the language "equilibrium negotiation value under Assumption's [2, 3, or both]."

Even with Assumptions 2 and 3, the disagreement value $\underline{y}$ is an arbitrary vector that can depend on the players' history of interaction. One can imagine that $\underline{y}$ is selected by a previous agreement between the players or by some social norm—either way, taken as given by the players in the current negotiation phase. On the other hand, one can imagine a more active form of negotiation in which communication in the current negotiation phase determines how the players coordinate their future behavior when they disagree.

Here is a story for how $\underline{y}$ may be decided in the current negotiation phase. In the event that bargaining ends at an impasse, there is one more round of communication in which "Nature" selects one of the players to make a final declaration about how they should coordinate their future behavior. Assume that Nature chooses player 1 with probability π_1 and player 2 with probability π_2. Also suppose, along the lines of Assumption 2, that the declaration becomes focal, so that the players actually coordinate appropriately. Then, when player i gets to make the final declaration, he will choose the continuation value $\hat{y}^i \in \underline{Y}$ satisfying $\hat{y}_i^i = \max \underline{Y}_i$. For simplicity, assume that

$$\hat{y}_j^i = \max\{y_j \mid (\hat{y}_i^i, y_j) \in \underline{Y}\}$$

(player i is generous where it does not cost him). The disagreement value is thus given by:

Assumption 4 (Active Disagreement): *The disagreement value is the constant vector* $\underline{y} \equiv \pi_1 \hat{y}^1 + \pi_2 \hat{y}^2$.

The vectors $\hat{y}^1$ and $\hat{y}^2$ exist by Assumption 1.

Example: BoS with the Ultimatum-Offer Protocol

To illustrate the implications of Assumptions 2-4, I consider as an example (i) the BoS contracting problem described in Section 1 and (ii) the ultimatum-offer protocol described at the end of the previous subsection. Recall that Figure 2 shows sets Y and $\underline{Y}$ for the BoS contracting problem. What follows are brief descriptions of the sets of equilibrium negotiation values with and without these assumptions. See Appendix A for more details.

The set of equilibrium negotiation values (without Assumptions 2-4) is given by the shaded region in the left picture shown in Figure 4. The points with $y_1 < 3$ are not equilibrium values because, by making an offer of $x = (2, \tilde{m})$ with $\tilde{m}_1 \geq -1$ and $\tilde{m}_2 > -2$, player 1 guarantees himself a payoff of at least 3. This is because (i) if player 2 accepts such an offer, then, regardless of which continuation value they coordinate on, player 1 gets at least $\min Y^{(2,\tilde{m})} = 4 + \tilde{m}_1 \geq 3$, and (ii) player 2 could rationally reject such an offer only if he anticipates getting at least $4 + \tilde{m}_2$ himself, but the two such points in $\underline{Y}$ yield at least 3 to player 1.

The set of equilibrium negotiation values is reduced under Assumption 2, as shown in the right picture of Figure 4. Here, player 1 can guarantee himself a value of $12 - \varepsilon$ by
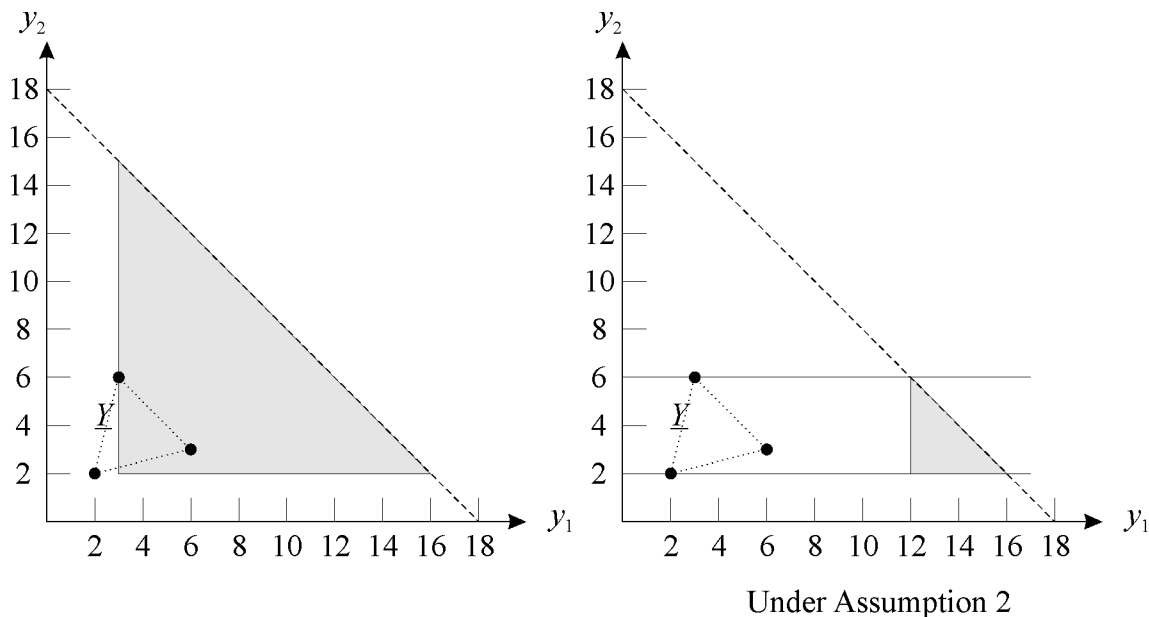
Figure 4: Equilibrium negotiation values in the BoS example.

offering $x = (2, (-\varepsilon, \varepsilon))$ and $y = (12 - \varepsilon, 6 + \varepsilon)$ for any small $\varepsilon > 0$; such an offer must be accepted by player 2 in equilibrium. Points in the shaded region can be supported with appropriately chosen disagreement values.

The set of equilibrium negotiation values is further reduced under Assumptions 2 and 3, as shown in Figure 5. With these assumptions, we have the standard outcome of the ultimatum-offer-bargaining game as though all aspects of the continuation value were externally enforced and the disagreement value were some fixed element of $\underline{Y}$. In the left picture of Figure 5, the set of equilibrium negotiation values (for various $\underline{y} \in \underline{Y}$) comprises the three points indicated on the efficient boundary of Y. The picture on the right shows the single equilibrium negotiation value that results when Assumption 4 is added; the case pictured has $\pi_2 \in (1/2, 1)$.

Cooperative-Theory Representation

Whatever are the assumed bargaining protocol and equilibrium conditions, their implications can be given a cooperative-theory representation in terms of a *bargaining solution* S that maps contracting problems into sets of equilibrium negotiation values. That is, for a contracting problem $(X, \underline{x}, \mathcal{Y})$, $S(X, \underline{x}, \mathcal{Y})$ is the set of equilibrium negotiation values for the given bargaining protocol and equilibrium conditions. It can be helpful to specify a bargaining solution S directly (with the understanding of what it represents) because this allows the researcher to focus on other aspect of the strategic situation without getting bogged down in the analysis of negotiation. One can then compare the effects of various assumptions about negotiation by altering the function S.
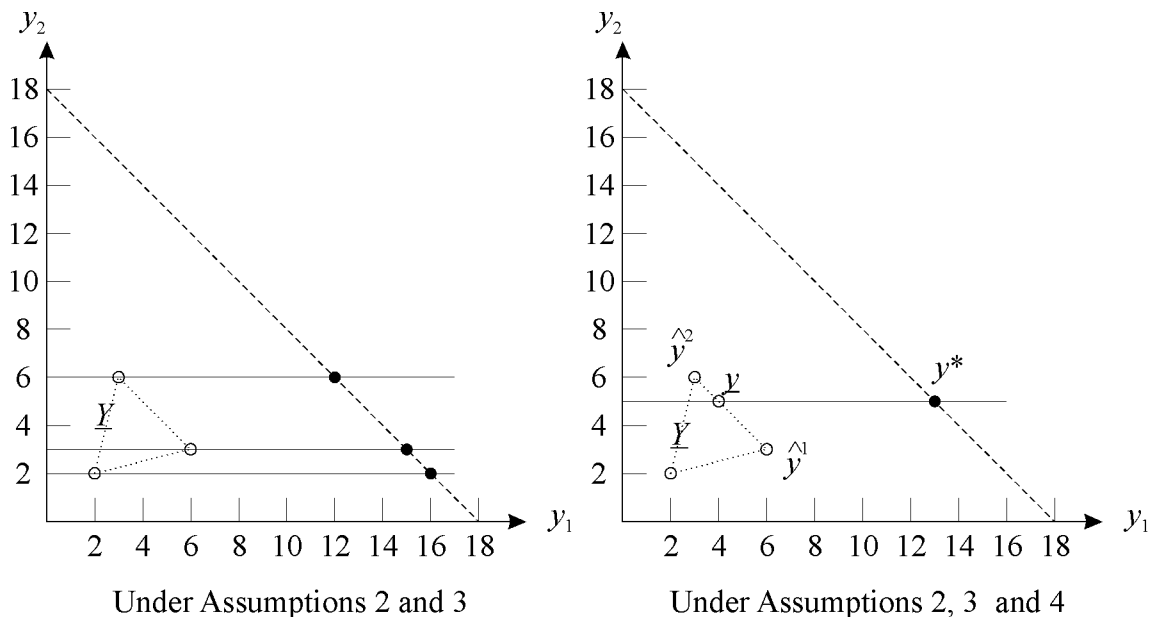
15

Figure 5: More equilibrium negotiation values in the BoS example.

On the technical side, note that $S(X, \underline{x}, \mathcal{Y})$ is a subset of the convex hull of Y. For some applications, it will be the case that $S(X, \underline{x}, \mathcal{Y})$ is contained in Y. Points in the convex hull of Y that are not in Y are possible in settings in which the bargaining protocol involve moves of Nature or where players randomize in equilibrium. One can also use the definition of S to model situations in which the players select lotteries over joint actions and continuation values. In these cases, an equilibrium negotiation value may be a nontrivial convex combination of points in Y.

Activeness of Contracting

When a bargaining solution contains more than one element for a given contracting problem, there is a role for social convention and history to play in the selection of the outcome. The selection may also be determined by a previous contract to which the players agreed. If two bargaining solutions, S' and S, have the relation $S' \subset S$, then I say that S' represents *more active contracting* than does S. In this case, S imposes fewer constraints on the outcome of the negotiation process and it allows a greater role for social convention, history, and prior agreements to play in the selection of the outcome. For example, if the players would arrive at a particular contracting problem $(X, \underline{x}, \mathcal{Y})$ in two or more different contingencies (from two different histories), then S affords wider scope for differentiating between the histories than does S'.

Benchmark Solutions

Various bargaining protocols and equilibrium conditions can be compared on the bases of (i) the extent to which they support outcomes that are inefficient from the negotiation

phase, and (ii) the extent to which they represent the players' active exercise of bargaining power. In this subsection, I describe three benchmark solutions that collectively represent a range of possibilities on these two dimensions.

The first benchmark solution is defined by the set of equilibrium negotiation values for the Nash-demand protocol without Assumptions 2-4. In the Nash-demand protocol, the players simultaneously make demands (x^1, y^1) and (x^2, y^2). If the demands are equal, so $x^1 = x^2$ and $y^1 = y^2$, then x^1 is compelled by the external enforcer; otherwise, the default action $\underline{x}$ is compelled. It is easy to verify that the equilibrium negotiation values in this case are given by

$$S^{\text{ND}}(X, \underline{x}, \mathcal{Y}) \equiv \{y^* \in Y \mid y_1^* \geq \min \underline{Y}_1 \text{ and } y_2^* \geq \min \underline{Y}_2\}.$$

The superscript "ND" here denotes "Nash demand."

The second and third benchmark solutions refer to the set of equilibrium negotiation values for a standard K-round, alternating-offer bargaining protocol under Assumptions 2 and 3. In any odd-numbered round, player 1 makes an offer to player 2, who then decides whether to accept or reject it. If player 2 accepts, then the negotiation phase ends. If player 2 rejects player 1's offer, then a random draw determines whether the negotiation phase ends or continues into the next round; the latter occurs with probability $e^{-\Delta \pi_1}$. In an even-numbered round, the players' roles are reversed and $e^{-\Delta \pi_2}$ is the probability that the following round is reached in the event that player 1 rejects player 2's offer. I assume that the values π_1 and π_2 are nonnegative and sum to one, and I write $\pi = (\pi_1, \pi_2)$. The negotiation phase ends for sure after round K. Let $\Gamma^{\text{A}}(K, \pi, \Delta)$ denote this bargaining protocol.[6]

It is easy to verify that, under Assumption 1, there is an equilibrium of $\Gamma^{\text{A}}(K, \pi, \Delta)$ that satisfies Assumptions 2 and 3 and that has agreement in the first round.[7] For any disagreement value $\underline{y} \in \underline{Y}$, let $Q(\underline{y}, \Delta)$ be defined as the limit superior of the set of implied equilibrium negotiation values as $K \to \infty$. Define the limit in terms of the Hausdorff metric, so that $y^* \in Q(\underline{y}, \Delta)$ if and only if there is a sequence of vectors $\{y^K\}_{K=1}^{\infty}$ such that (i) y^K is an equilibrium negotiation value for $\Gamma^{\text{A}}(K, \pi, \Delta)$ under Assumptions 2 and 3, and (ii) $\{y^K\}$ has a subsequence that converges to y^*.

Consider the limit superior of $Q(\underline{y}, \Delta)$ as the "time between offers" Δ converges to zero:

$$\overline{Q}(\underline{y}) \equiv \lim_{\Delta \to 0} Q(\underline{y}, \Delta).$$

$\overline{Q}(\underline{y}) \subset Y$ is implied by Assumption 1 and by the construction of $\overline{Q}$. I define the second benchmark bargaining solution to be the set that results by looking at all feasible disagreement values:

$$S^{\text{AA}}(X, \underline{x}, \mathcal{Y}) \equiv \bigcup_{\underline{y} \in \underline{Y}} \overline{Q}(\underline{y}).$$

---

[6] One could have $K = \infty$, but the case of a finite K has perhaps a more straightforward interpretation whereby the continuation of the relationship must start at ΔK units of time from when negotiation begins.

[7] For some nonconvex examples of Y, one can find mixed equilibria and/or equilibria in which delay occurs. I ignore these.

For the third benchmark solution, I add Assumption 4 to focus on the "active disagreement" vector:

$$S^{AD}(X, \underline{x}, \mathcal{Y}) \equiv \overline{Q}(\pi_1 \hat{y}^1 + \pi_2 \hat{y}^2).$$

By taking advantage of Binmore, Rubinstein, and Wolinsky's (1986) analysis, we can show that, assuming Y is convex, $\overline{Q}(\underline{y})$ is the generalized Nash bargaining solution evaluated on bargaining set Y, with disagreement value $\underline{y}$, and with the players' bargaining weights given by π.

Lemma 1: *If Y is convex then* $\overline{Q}(\underline{y}) = \underset{\substack{y \in \mathrm{SPB}(Y) \\ y \geq \underline{y}}}{\arg \max} \; (y_1 - \underline{y}_1)^{\pi_1}(y_2 - \underline{y}_2)^{\pi_2}.$

Here, $\mathrm{SPB}(Y)$ denotes the strong Pareto boundary of the set Y. Note, therefore, that if Y is convex then bargaining solution S^{AA} is a subset of, and bargaining solution S^{AD} is a point on, the strong Pareto boundary of Y.

3   Contractual Relationships

In this section, I develop a framework for examining multi-period contractual relationships. Each period consists of two phases of time: the *negotiation phase* and the *individual-action phase*. In the negotiation phase, the players make a joint contracting decision. Interaction in the negotiation phase is modeled as described in the preceding sections. In the individual-action phase, the players make independent decisions that are modeled as non-cooperative interaction. I constrain attention to finite-horizon settings in which, in each negotiation phase, all payoff-relevant information about future interaction is commonly known.

Formal Description of the Game with Joint Actions

Suppose that two players interact over a finite number of discrete periods. At the beginning of each period, a state variable z represents the payoff-relevant aspects of the players' history as well as any events on which the players are assumed to condition their behavior. With common knowledge of z, the players first select a joint action x. Then, simultaneously and independently, the players choose individual actions a_1 and a_2 (for players 1 and 2, respectively) and an exogenous random draw a_0 is realized. Write $a = (a_0, a_1, a_2)$ as the individual-action profile. At the end of the period, the players receive payoffs given by a vector $u(z, x, a)$. The state is then updated and interaction continues in the next period or the game terminates.

To be more precise, fundamentals of the game include a set of states Z, a set of joint actions $\mathcal{X}$, and a set of individual-action profiles $\mathcal{A}$. There is a correspondence $X : Z \rightrightarrows \mathcal{X}$ and a function $\underline{x} : Z \to \mathcal{X}$ such that $X(z)$ gives the set of joint actions available to the players in a period that begins in state z, and $\underline{x}(z) \in X(z)$ denotes the default joint action in this state. Further, $A_i(z, x)$ denotes the set of feasible individual actions for player i in a

period in which z is the state and x is the joint action that the players selected. The feasible individual-action profiles are given by

$$A(z, x) = A_0(z, x) \times A_1(z, x) \times A_2(z, x),$$

where $A(z, x) \subset \mathcal{A}$. Assume that Nature selects a_0 according to some probability distribution $\alpha_0(z, x)$ in each period. Note that Nature's actions and probability distribution may be influenced by the state and joint action of the current period.

There is a transition function $f : Z \times \mathcal{X} \times \mathcal{A} \to Z$ that defines the state in the following period as a function of the current period's state and actions.[8] Specifically, if x and a are the actions taken in a period that begins in state z, then $z' = f(z, x, a)$ is the state in the next period. There is an *initial state* of the relationship $z^{\text{init}} \in Z$ in effect at the beginning of the first period. There is also a nonempty set of *terminal states* $Z^{\text{term}} \subset Z$ that mark the end of the game. A *feasible path* of play is a sequence $\{(z^t, x^t, a^t)\}_{t=1}^T$ such that

(i) $z^1 = z^{\text{init}}$;

(ii) $x^t \in X(z^t)$, $a^t \in A(z^t, x^t)$, and $z^{t+1} = f(z^t, x^t, a^t)$
for all $t = 1, 2, \ldots T - 1$; and

(iii) $f(z^T, x^T, a^T) \in Z^{\text{term}}$.

Assumption 5: *There is a positive integer τ such that every feasible path of play has $T \leq \tau$ and at least one feasible path has $T = \tau$.*

This assumption is maintained hereinafter. It implies that Z can be partitioned into sets $Z^1, Z^2, \ldots, Z^{\tau+1}$ with the following properties. First, $Z^1 = \{z^{\text{init}}\}$ and $Z^{\tau+1} = Z^{\text{term}}$. Second, for each $t \in \{1, 2, \ldots, \tau\}$, each $z \in Z^t$, and actions $x \in X(z)$ and $a \in A(z, x)$, we have

$$f(z, x, a) \in \mathcal{Z}^{t+1} \equiv \bigcup_{k=t+1}^{\tau+1} Z^k.$$

Partitioning Z in this way facilitates backward-induction analysis.

Finally, there is a payoff function $u : Z \times \mathcal{X} \times \mathcal{A} \to \mathbf{R}^2$. For any given path of play $\{z^t, x^t, a^t\}_{t=1}^T$, the payoff vector for the entire game is the sum of per-period payoffs:

$$\sum_{t=1}^T u(z^t, x^t, a^t).$$

One can incorporate discounting by defining u and the state system appropriately.

---

[8]By making z the full history of the relationship, we have a setting of almost perfect information. From this, assumptions about on what the players condition their behavior are achieved by restricting Z. Imperfect information can be accommodated by specifying that the state does not fully record all of the players' past actions; in this case, one's attention is limited to settings in which the events that are not common knowledge are also not payoff relevant for future interaction, and then we are assuming public equilibrium.

Contractual Equilibrium

To analyze the contractual relationships described in the previous subsection, I combine a bargaining solution S with individual incentive conditions. The former models how players select among joint actions in the negotiation phase, whereas the latter identify Nash equilibria for the individual-action phase in each period. The analysis is most conveniently conducted using a recursive formulation in which we focus on within-period payoffs and continuation values.

In the recursive formulation, we posit a value correspondence $V : Z \rightrightarrows \mathbf{R}^2$ (a mapping from states to subsets of $\mathbf{R}^2$) with the interpretation that, for each $z \in Z$, $V(z)$ is a set of continuation-value vectors for the players at the start of a period in state z. For V to be consistent with the bargaining solution S, it must be that $V(z) \equiv S(X(z), \underline{x}(z), \mathcal{Y}(z))$, where $(X(z), \underline{x}(z), \mathcal{Y}(z))$ is the contracting problem that the players face at the beginning of the period in state z.

In turn, $\mathcal{Y}(z) = \{Y^x(z)\}_{x \in X(z)}$ should correctly describe the feasible continuation values over the joint actions that are available to the players. For a specific $x \in X(z)$, every feasible continuation value is supported by some (possibly mixed) action profile (α_1^*, α_2^*) that is a Nash equilibrium in the individual-action phase of the current period. The Nash equilibrium condition is

$$
\begin{aligned}
y_i &= \mathrm{E}\left[u_i(z, x, (a_0, a_i, a_j)) + v_i(f(z, x, (a_0, a_i, a_j))) \mid \alpha_0(z, x), \alpha_i^*, \alpha_j^*\right] \\
&\geq \mathrm{E}\left[u_i(z, x, (a_0, a_i', a_j)) + v_i(f(z, x, (a_0, a_i', a_j))) \mid \alpha_0(z, x), \alpha_j^*\right]
\end{aligned}
\tag{1}
$$

for all $a_i' \in A_i(z, x)$ and for $i = 1, 2$. In this expression, y_i is player i's continuation payoff from the beginning of the current period and E denotes the expectation taken over Nature's probability distribution $\alpha_0(z, x)$ and whatever randomizing the players do. The function v selects continuation-value vectors for the start of the next period. These should be consistent with the behavioral theory, meaning that $v(z') \in V(z')$. Thus, for every $x \in X(z)$, we have

$$
\begin{aligned}
Y^x(z) \equiv \{ y \in \mathbf{R}^2 \mid \text{ there exists } v : \mathcal{Z}^{t+1} &\to \mathbf{R}^2 \text{ and } (\alpha_1^*, \alpha_2^*) \in \triangle A_{-0}(z, x) \text{ such that} \\
\text{Condition (1) holds for } i = 1, 2, \text{ and } &v(z') \in V(z') \text{ for all } z' \in \mathcal{Z}^{t+1} \}.
\end{aligned}
\tag{2}
$$

In this expression, $\triangle A_{-0}(z, x)$ denotes the set of uncorrelated probability distributions over $A_1(z, x) \times A_2(z, x)$.

These elements compose the notion of rational behavior.

Definition 3: *Let S be a given bargaining solution. A value correspondence $V : Z \rightrightarrows \mathbf{R}^2$ is said to* **represent contractual equilibrium** *if*

(i) $V(z) \equiv \{(0,0)\}$ for every $z \in Z^{\text{term}}$ and

(ii) for every $z \in Z \setminus Z^{\text{term}}$, we have $V(z) = S(X(z), \underline{x}(z), \mathcal{Y}(z)) \neq \emptyset$, where $\mathcal{Y}(z)$ is defined by Equation (2).

If such a correspondence exists, say that **contractual equilibrium exists**.

Theorem 1: *Take as given a contractual relationship and a bargaining solution S. If Assumption 5 is satisfied and contractual equilibrium exists, then there is a unique value correspondence that represents contractual equilibrium.*

This result is a straightforward consequence of the backward-induction construction. The result is proved in Appendix C; existence results for three classes of contractual relationships are presented in Appendix B.

4   Analysis of Examples

This section contains analysis of the MW example and an example of a repeated game with contracting. These examples illustrate the procedure for characterizing contractual equilibrium and they demonstrate how the activeness of contracting influences the outcome.

MW Example

In this subsection, I formally elaborate the MW example from the Introduction. A state is denoted $z = (h, \mu)$, where h is the history of individual actions to a particular period and μ is a function that tells the external enforcer what monetary transfer to make contingent on the history; that is, following history h, the external enforcer compels transfer $\mu(h) = (m_1, m_2)$, where m_i is the amount given to player i. Let $H \equiv H^1 \cup H^2 \cup H^3$, where

$H^1 = \{h^{\text{init}}\}$ is the set containing the initial (null) history of actions h^{init},

$H^2 = \{\phi\} \times \{0, 1, 2, 3, 4\}$ is the set of individual-action profiles in the first period (investment levels for the worker and a null, trivial action "ϕ" for the manager), and

$H^3 = \{\phi\} \times \{0, 1, 2, 3, 4\} \times \{\text{high}, \text{low}\}^2$ is the histories of individual actions through the end of the second period.

Note that H^3 includes the worker's first-period investment choice and both players' second-period effort choices.

The transfer function μ maps H to $\mathbf{R}^2$ with the following constraints. First, recall that, in the story, the external enforcer observes second-period effort choices but not the worker's first-period investment. This is represented by assuming that the transfer function is measurable with respect to the partition

$$\mathcal{H} \equiv \big\{ H^1, H^2, H^2 \times \{(\text{high}, \text{high})\}, H^2 \times \{(\text{high}, \text{low})\},$$
$$H^2 \times \{(\text{low}, \text{high})\}, H^2 \times \{(\text{low}, \text{low})\} \big\}$$

of H. The story also indicates that the transfer function must be relatively balanced. The interpretation I offer for this is that the transfer is actually the sum of (i) an immediate, voluntary transfer that the players make as they reach agreement in the negotiation phase,

and (ii) a contingent transfer compelled by the external enforcer after individual actions are taken in a given period. Furthermore, the external enforcer only compels balanced transfers that sum to zero. I make the additional technical assumption that transfers are contained in some set $M = \{m \in \mathbf{R}^2 \mid m_1, m_2 \geq -\beta,\ m_1 + m_2 \leq 0\}$, where β is arbitrarily large. To represent this idea, define $\mathcal{X}$ to be the subset of transfer functions of the form $\mu : H \to M$ that are measurable with respect to $\mathcal{H}$ and have the following property. For $t = 2, 3$ and $h, h' \in H^t$, $\mu_1(h) + \mu_2(h) = \mu_1(h') + \mu_2(h')$.

The initial state is defined as $z^{\text{init}} = (h^{\text{init}}, \mu^0)$, where μ^0 denotes the constant transfer function that always specifies zero transfers. At the beginning of any period in state $z = (h, \underline{\mu})$, the set of feasible joint actions is $X(z) \equiv \mathcal{X}$ and the default joint action is $\underline{\mu}$. Regarding individual actions, there is no random draw. The set of feasible individual-action profiles in the first period is given by $A(z^{\text{init}}, \mu) = \{\phi\} \times \{1, 2, 3, 4\}$. The set of individual-action profiles in the second-period is $A((h, \underline{\mu}), \mu) = \{\text{low}, \text{high}\}^2$ for all $h \in H^2$. The transition of the state works in the obvious way. From state $z = (h, \underline{\mu})$, where $h \in H^1 \cup H^2$, if μ is the joint action (transfer function selected) and a is the individual action profile in the current period, then the state in the following period is $z' = (h', \mu)$ where $h' = (h; a)$ is the history formed by appending a to h. Every state of the form $z = (h, \mu)$ for $h \in H^3$ is a terminal state.

On payoffs, in the first period for any $a = (\phi, I)$ and $\mu \in M$, we have $u(z^{\text{init}}, \mu, a) = (8I, -I^2) + \mu(a)$. Regarding the second period, take any $z = (h, \underline{\mu})$ where $h \in H^2$ and $a \in \{\text{low}, \text{high}\}^2$, the vector $u(z, \mu, a)$ is given by $\mu(h; a)$ plus the vector in the cell of the matrix

| W | | |
M	high	low
high	17, -1	9, 0
low	10, -1	0, 0

that is relevant for action profile a.

Minor modifications represented by this formal description of the game, relative to the description in the Introduction, are (i) that the monetary transfers are restricted to be in some compact set and (ii) that an externally enforced transfer can be specified for the first period in addition to those specified for the second period. Item (i) allows an existence result from Appendix B to apply (in particular, Theorem 3); item (ii) was left out of the description in the Introduction because it was not central to the discussion.

For convenience, given any transfer function μ, I shall let m^{ll} be the vector specified for each $h \in H^3$ in which (low, low) is the individual-action played in the second period, I let m^{hl} be the vector specified for each $h \in H^3$ in which (high, low) is the individual-action played in the second period, and so on. This is in accord with the notation used in the presentation of the example in the Introduction.

I next characterize contractual equilibrium and find the maximum joint value that the players can attain, for each of the benchmark bargaining solutions S^{ND}, S^{AA}, and S^{AD} defined in Section 2. By Theorem 3 in Appendix B, contractual equilibrium exists in the cases of S^{ND} and S^{AA}; existence will be obvious for the case of S^{AD}. Write V^{ND}, V^{AA}, and V^{AD} for the value correspondences that represent contractual equilibrium for these three cases. The following analysis establishes that

$$\max\{v_1 + v_2 \mid v \in V^{\text{ND}}(z^{\text{init}})\} = 32,$$

$$v_1 + v_2 = 31 \text{ for all } v \in V^{\text{AA}}(z^{\text{init}}),$$

and

$$v_1 + v_2 = 16 \text{ for all } v \in V^{\text{AD}}(z^{\text{init}}).$$

Thus, in this example, more active contracting implies a lower attainable joint value.

First consider S^{ND}. Let $\tilde{\mu}$ denote a transfer function that forces (low, low) to be played in the second period and has $\tilde{m}^{\text{ll}} = (0,0)$. This is the transfer function from Alternative 1 in the Introduction. Take any $h \in H^2$ and let $z = (h, \tilde{\mu})$ be any state at the start of the second period where $\tilde{\mu}$ is the current transfer function. Note that $\underline{Y}(z) = (0,0)$ because, unless the transfer function is renegotiated, the players will have the incentive to play (low, low) at the end of the period. Also note that the players can renegotiate to pick a transfer function that forces (high, high) and arbitrarily divide or throw away the value. This means that

$$B \equiv \{y \in \mathbf{R}^2 \mid y_1, y_2 \geq 0, \ y_1 + y_2 \leq 16\} \subset Y(z).$$

By the definition of S^{ND}, we therefore have

$$V^{\text{ND}}(z) = S^{\text{ND}}(X(z), \underline{x}(z), \mathcal{Y}(z)) = B.$$

In the first period, the players can select $\tilde{\mu}$ and agree on a second-period continuation value of $(0, 16)$ if the worker chooses $I = 4$ and a value of $(0, 0)$ if the worker chooses any $I \neq 4$. That is, the players agree to coordinate on these history-dependent continuation values in the negotiation phase at the start of the second period.

When the players anticipate coordinating in this way, player 2 has the incentive to choose $I = 4$ in the first period and therefore (if no transfer is specified for period 1) the payoff from the beginning of the game is $(32, -16) + (0, 16) = (32, 0)$. All or part of the joint value of 32 can be transferred to player 2 by way of an additional constant transfer (in period 1, for instance). The bottom line is that

$$\{y \in \mathbf{R}^2 \mid y_1, y_2 \geq 0, \ y_1 + y_2 \leq 32\} \subset Y(z^{\text{init}}).$$

In addition, it is not difficult to check that

$$\underline{Y}(z^{\text{init}}) = \{y \in \mathbf{R}^2 \mid y_1 \geq 9, \ y_2 \geq 0, \ y_1 + y_2 \leq 16\}.$$

By the definition of S^{ND}, we conclude that

$$V^{\text{ND}}(z^{\text{init}}) = \{y \in \mathbf{R}^2 \mid y_1 \geq 9, \ y_2 \geq 0, \ y_1 + y_2 \leq 32\}.$$

Next consider S^{AA} with bargaining weights given by $\pi = (1, 0)$, as described in the Introduction. Let $\tilde{\mu}$ denote the transfer function associated with Alternative 3. That is, $\tilde{m}^{\text{hh}} = \tilde{m}^{\text{hl}} = (-9, 9)$, $\tilde{m}^{\text{lh}} = (-1, 1)$, and $\tilde{m}^{\text{ll}} = (0, 0)$. Take any $h \in H^2$ and let $z = (h, \tilde{\mu})$ be any state at the start of the second period where $\tilde{\mu}$ is the current transfer function. Noting that

$$\underline{Y}(z) = \{(0, 0), (0, 9), (9, 0)\}$$

and $B \subset Y(z)$ (as with the previous case), and using the definition of S^{AA}, we have

$$V^{\text{AA}}(z) = \{(16, 0), (7, 9)\}.$$

In the first period, the players can select $\tilde{\mu}$ and agree on a second-period continuation value of $(7, 9)$ if the worker chooses $I = 3$ and a value of $(16, 0)$ if the worker chooses any $I \neq 3$. That is, the players agree to coordinate on these history-dependent continuation values in the negotiation phase at the start of the second period. Importantly, it is the *disagreement point* for their second-period negotiation that achieves history dependence.

When the players anticipate coordinating in this way, player 2 has the incentive to choose $I = 3$ in the first period and therefore (if no transfer is specified for period 1) the payoff from the beginning of the game is $(24, -9) + (7, 9) = (31, 0)$. One can easily verify that no contract achieves a higher joint value than 31. To do so would require a specification of second-period transfers such that (i) there are two equilibria in the individual-action phase, and (ii) player 2's payoffs in these two equilibria differ by strictly more than 9. It is not difficult to calculate that, because the transfer function is constrained to be relatively balanced (so $m_1^{\text{hh}} + m_2^{\text{hh}} = m_1^{\text{hl}} + m_2^{\text{hl}} = m_1^{\text{lh}} + m_2^{\text{lh}} = m_1^{\text{ll}} + m_2^{\text{ll}}$), the maximum difference in player 2's payoffs in multiple equilibria of the individual-action phase is 9.

Continuing with the case of S^{AA}, one can easily calculate that $\underline{Y}(z^{\text{init}}) = \{(16, 0)\}$. Combining this with the analysis of the preceding paragraph, we conclude that $V^{\text{AA}}(z^{\text{init}}) = \{(31, 0)\}$.

Finally, consider S^{AD}, with bargaining weights given by $\pi = (1, 0)$. In this case, $V^{\text{AD}}(z)$ is a singleton for every $z \in H^2$. Furthermore, on H^2, V^{AD} is invariant in the "h" part of the state. Thus, whatever contract is formed in the first period, player 2's continuation value from the beginning of the second period does not depend on his investment choice. He optimally selects $I = 0$. The players can do no better than agree to the contract of Alternative 2 in the Introduction. Player 1's bargaining power gets him the entire joint value, so $V^{\text{AD}}(z^{\text{init}}) = \{(16, 0)\}$.

A Repeated Game with Contracting

The next example is a repeated game with no transfers and no external enforcement. Contracting thus relates only to the self-enforced activity. The example demonstrates that

1 \ 2	a_2^1	a_2^2	a_2^3	a_2^4	a_2^5
a_1^1	1, 3	0, 3	$-1, 3$	$-1, 0$	$-1, -1$
a_1^2	1, 0	$3-\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}$	$0, 1-\frac{\varepsilon}{2}$	0, 0	$0, -1$
a_1^3	$1, -1$	$3-\frac{\varepsilon}{2}, 0$	3, 1	$4+\frac{\varepsilon}{2}, -4$	$14, -5$
a_1^4	$0, -1$	0, 0	$-4, 6+\frac{\varepsilon}{2}$	4, 6	0, 0
a_1^5	$-1, -1$	$-1, 0$	$-5, 14$	0, 0	12, 12

Figure 6: Example 1 stage game.

activeness of contracting is, in general, not monotonically related to the maximally attainable joint values.

Consider a three-period relationship with no joint actions and where the individual actions and payoffs in each period are given by the stage-game matrix in Figure 6. Assume that ε is a small positive number (less than 1 suffices). Define the state to be the history of individual actions. This is a finitely repeated game with contracting. Formally:

$$A_1 = \{a_1^1, a_1^2, a_1^3, a_1^4, a_1^5\}$$
$$A_2 = \{a_2^1, a_2^2, a_2^3, a_2^4, a_2^5\}$$
$$X = \{\underline{x}\}$$
$$Z = \{z^{\text{init}}\} \cup A \cup A^2 \cup \{z^{\text{term}}\}.$$

The transition function is defined by $f(z^{\text{init}}, \underline{x}, a) = a$, $f(z, \underline{x}, a) = (z; a)$ for each $z \in A$, and $f(z, \underline{x}, a) = z^{\text{term}}$ for each $z \in A^2$. Here, "$(z; a)$" denotes the history formed by appending a to z. Also, $X(z) = X$ and $A(z, \underline{x}) = A$ for all $z \in Z$.

Consider the bargaining solution S^δ defined by:

$$S^\delta(X, \underline{x}, \mathcal{Y}) \equiv \{y^* \in Y \mid \text{ there exists } y \in \text{SPB}(Y) \text{ and } \underline{y} \in \underline{Y} \text{ such that}$$
$$y^* \geq \underline{y} \text{ and } |y_1 - y_1^*| + |y_2 - y_2^*| \leq \delta\}, \qquad (3)$$

That is, S^δ is the set of vectors that weakly dominate a disagreement value and are within δ of the strong Pareto boundary of Y. If $\delta = 0$ then this is simply the strong Pareto boundary weakly above the disagreement set and, in the context of the repeated game, will yield "strong Pareto perfection." If $\delta > 0$ then inefficient points are included. The bargaining solution S^δ is regular and exhibits an ordering by inclusion in the sense that $S^\delta \subset S^{\delta'}$ whenever $\delta \leq \delta'$.

By Theorem 2 in Appendix B, contractual equilibrium exists. Note that, because the set of joint actions is trivial, $\underline{Y}(z) = Y(z)$ for every z. Note also that, in this relationship,
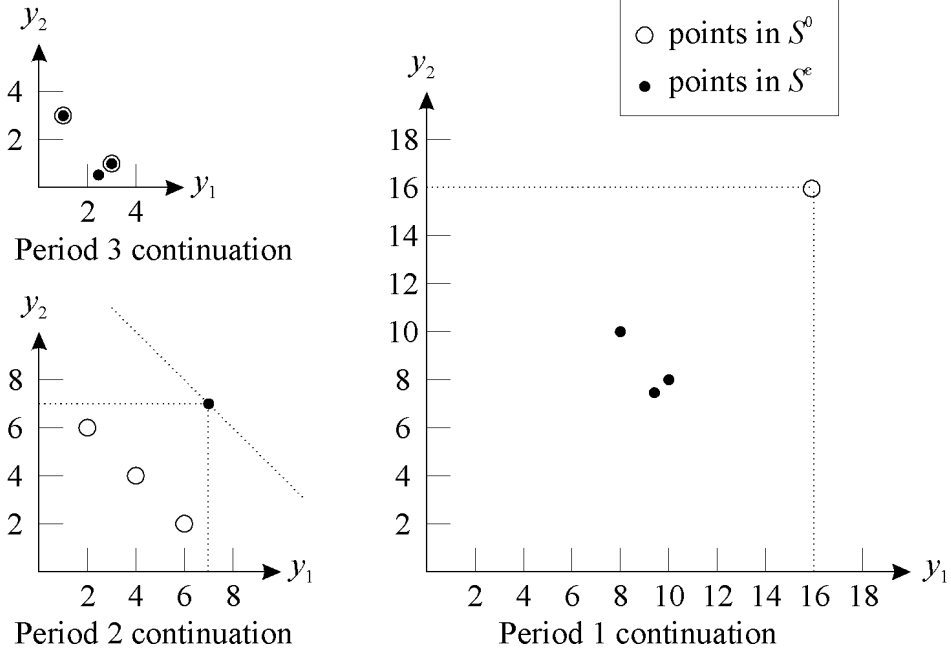
Figure 7: Example 1 equilibrium negotiation values.

the actions in a period have no direct effect on the set of feasible actions or the payoffs in future periods. Therefore, for any two histories z and z' of the same length (for example, $z, z' \in A^2$), we have $V(z) = V(z')$. I next calculate the value correspondences V^0 and V^ε for the cases $\delta = 0$ and $\delta = \varepsilon$, respectively.

Start with period 3 and consider any $z \in A^2$. In this period (after which the game ends), the players must be playing a Nash equilibrium of the stage game. There are three Nash equilibria: (a_1^1, a_2^1), (a_1^2, a_2^2), and (a_1^3, a_2^3). Thus, we have

$$Y(z) = \left\{ (1, 3), \left(3 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2} \right), (3, 1) \right\},$$

for each $z \in A^2$. The first and third are on the strong Pareto boundary of the equilibrium set; their payoff vectors compose the set $S^0(X(z), \underline{x}, \mathcal{Y}(z))$, with each one achieved by using itself as the disagreement value. As for the S^ε bargaining solution, we have $S^\varepsilon(X(z), \underline{x}, \mathcal{Y}(z)) = Y(z)$; the vector $\left(3 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2} \right)$ is included because it is within ε of the strong Pareto boundary with itself as the disagreement point. We have

$$V^0(z) = \{ (1, 3), (3, 1) \}$$

and

$$V^\varepsilon(z) = \left\{ (1, 3), \left(3 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2} \right), (3, 1) \right\},$$

as illustrated in Figure 7.

Next examine incentives in the individual action phase of period 2. Take any history from the first period, $z \in A$. Using continuation values from V^0, the only action profiles

26

that can be supported in period 2 are the three stage-game Nash equilibria. In this case, $Y(z)$ is the set of vectors achieved by adding the stage-game Nash payoff vectors to $(1, 3)$ and $(3, 1)$:

$$Y(z) = \left\{(2, 6), (4, 4), (6, 2), \left(4 - \frac{\varepsilon}{2}, 4 - \frac{\varepsilon}{2}\right), \left(6 - \frac{\varepsilon}{2}, 2 - \frac{\varepsilon}{2}\right)\right\}.$$

The bargaining solution picks the points on the strong Pareto boundary of this set, so

$$V^0(z) = S^0(X(z), \underline{x}, \mathcal{Y}(z)) = \{(2, 6), (4, 4), (6, 2)\}.$$

As for S^ε, note that an additional action profile is supported in period 2 using continuation values from V^ε. Specifically, (a_1^4, a_2^4) satisfies the equilibrium incentive conditions if: (i) play of (a_1^4, a_2^4) leads to the continuation value $(3, 1)$ in period 3, and (ii) any deviation by either player leads to the continuation value $(3 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2})$ in period 3. In this case, $Y(z)$ includes the vector $(7, 7)$. Furthermore, it is easy to check that $(7, 7)$ dominates (by more than ε) all other elements of $Y(z)$. This implies that

$$V^\varepsilon(z) = S^\varepsilon(X(z), \underline{x}, \mathcal{Y}(z)) = \{(7, 7)\},$$

as shown in Figure 7.

Finally, consider the incentives in period 1. Using continuation values from V^0, play of (a_1^5, a_2^5) can be supported. Specifically, this profile leads to the period-2 continuation value $(4, 4)$; any deviation by player 1 leads to the continuation value $(2, 6)$, whereas any deviation by player 2 leads to $(6, 2)$. One can quickly confirm that the resulting payoff vector $(16, 16)$ dominates all other elements of $Y(z^{\text{init}})$, so

$$V^0(z^{\text{init}}) = S^0(X(z^{\text{init}}), \underline{x}, \mathcal{Y}(z^{\text{init}})) = \{(16, 16)\}.$$

Regarding S^ε, because $V^\varepsilon(z)$ is a singleton constant over all $z \in A$, only stage-game Nash equilibria can be supported in the first period in this case. Therefore,

$$V^\varepsilon(z^{\text{init}}) = S^\varepsilon(X(z^{\text{init}}), \underline{x}, \mathcal{Y}(z^{\text{init}})) = \left\{(8, 10), \left(10 - \frac{\varepsilon}{2}, 8 - \frac{\varepsilon}{2}\right), (10, 8)\right\}.$$

These sets are shown in Figure 7.

In this example, $S^0 \subset S^\varepsilon$ yet $V^0(z^{\text{init}})$ *dominates* $V^\varepsilon(z^{\text{init}})$ in that the vector in the former set is strictly greater (for both players) than are the vectors in the latter set. From the beginning of the game, the players fare strictly better if S^0 describes their negotiation behavior than if S^ε does—that is, the contractual outcome is better under more active contracting.

5 Conclusion

The modeling exercise reported here has two objectives. First, it seeks to elaborate on the existing contract-theory literature by clarifying some basic concepts and, in so doing, encouraging research that explicitly considers both the externally enforced and self-enforced

aspects of contract enforcement. The framework I develop brings together elements of the current contract-theory literature (which, although not exclusively, tends to focus on renegotiation of externally enforced contracts), the repeated-game literature (which focuses on renegotiation of self-enforced contracts), and the cheap-talk literature (which focuses on how verbal statements may influence subsequent behavior in equilibria of non-cooperative games).

The second objective of this modeling exercise is to introduce the idea of activeness of contracting and to show that the relation between activeness and the predicted outcomes of a contractual relationship is subtle and interesting. I do not take a position on whether people in real contractual settings are more or less active in their contract negotiation, for this is an empirical issue. Rather, the model helps define activeness and explore its implications, which is a prerequisite for any empirical evaluation.

The modeling framework developed herein lends itself to a wide variety of applications. Repeated-game models are one special case. For example, using the terminology developed here, the analysis of Benoit and Krishna (1993) identifies contractual equilibrium associated with the bargaining solution that selects weakly efficient continuation values. The point of the present modeling exercise is not to reinvent the renegotiation-in-repeated-games wheel, of course, but to expand the theory in terms of generalizing the bargaining theory and including external enforcement. For instance, the framework can be used to analyze more complicated and realistic settings than are commonly studied in the literature (for example, settings with monetary transfers and external enforcement, and where players take actions that directly affect the structure of the game in future periods). Another special case are relationships with unverifiable investment and hold-up (Hart and Moore 1988, Maskin and Moore 1999, and the ensuing body of work). In this category, the framework facilitates the careful analysis of individual trade actions (Watson 2005a) and the examination of settings with more complicated dynamics than has been studied to date (such as Watson 2005b). On the abstract side, the framework developed here may be useful in analyzing problems of "design" with all sorts of real constraints (on, say, liquidity, the timing and alienability of actions, and external enforcement) that are currently not extensively studied.[9]

Additional links to the literature are worth pointing out. Watson (2002) put forth an informal version of the contractual equilibrium concept called *negotiation equilibrium*, whose definition can be made more precise. A negotiation equilibrium is the instance of contractual equilibrium in which the parties negotiate actively over only the externally enforced component of contract. Specifically, a continuation value is selected and fixed for

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[9]There are some obvious directions for further research. The relation between activeness and maximal joint values should be clarified. Also, contracts along the lines of Alternative 3 in the MW example represent a nuanced view of Bernheim and Whinston's (1998) "strategic ambiguity," where players take full advantage of external enforcement to create situations with multiple equilibria and then select among these equilibria as a function of the history. To create the multiplicity of equilibrium, players may want to use "less complete" contracts rather than "more complete" ones (Bernheim and Whinston restrict attention to forcing contracts, where multiplicity is negatively related to a notion of completeness). I plan to address these two theoretical issues in future projects.

each individual joint action, and then the bargaining set is defined as the set of these values, yielding a conventional bargaining problem (with a single disagreement point) to which a bargaining solution is applied. Ramey and Watson (2002) and Klimenko, Ramey, and Watson (2004) examine another variant that the second group calls a *recurrent agreement*; it is built on the theoretical foundation developed here but is based on the idea that, in a repeated game, players revert to a Nash equilibrium of the stage game when they fail to reach an agreement on how to play in the continuation.

A Bargaining Protocol Example: Ultimatum Offer

This appendix contains details of the analysis of the ultimatum-offer protocol, which is sketched (in the context of the BoS contracting problem from Section 1) in Section 2.

To calculate the set of equilibrium negotiation values, some additional notation is useful. For any set $D \subset \mathbf{R}^2$, let D_i denote the projection on dimension i (player i's value). Also, for any number d_2, define $\lambda_1^x(d_2)$ to equal

$$\min\{y_1 \mid \text{there exists } y_2 \geq d_2 \text{ such that } (y_1, y_2) \in Y^x\}$$

if the bracketed set is nonempty and ∞ otherwise. The following lemma characterizes the equilibrium negotiation values for the ultimatum-offer protocol.

Lemma 2: *For the ultimatum-offer bargaining protocol, the set of equilibrium negotiation values is*

$$S(X, \underline{x}, \mathcal{Y}) = \{y^* \in Y \mid y_2^* \geq \min \underline{Y}_2 \text{ and, for every } x \in X, \text{ either}$$
$$y_1^* \geq \lambda_1^x(\min \underline{Y}_2) \text{ or } y_1^* \geq \lambda_1^{\underline{x}}(\min Y_2^x) \text{ or both}\}.$$

This result is easily proved by calculating, for feasible continuation values, the offers that player 2 can rationally accept or reject.

To calculate the set of equilibrium negotiation values under Assumption 2, define $\gamma_1^x(d_2)$ to equal

$$\sup\{y_1 \mid \text{there exists } y_2 > d_2 \text{ such that } (y_1, y_2) \in Y^x\}$$

if the bracketed set is nonempty and $-\infty$ otherwise.

Lemma 3: *For the ultimatum-offer bargaining protocol, the set of equilibrium negotiation values under Assumption 2 is*

$$S(X, \underline{x}, \mathcal{Y}) = \{y^* \in Y \mid y_2^* \geq \min \underline{Y}_2, \ y_1^* \geq \lambda_1^{\underline{x}}(\max \underline{Y}_2),$$
$$\text{and } y_1^* \geq \gamma_1^x(\max \underline{Y}_2) \text{ for every } x \neq \underline{x}\}.$$

Next, I calculate the set of equilibrium negotiation values under Assumptions 2 and 3. In this case, for a given value y, the model of negotiation is equivalent to one in which *both* the tangible action x *and* the value y are externally enforced. Define $\gamma_1(d_2) \equiv \sup_{x \in X} \gamma_1^x(d_2)$.

Lemma 4: *For the ultimatum-offer bargaining protocol, the set of equilibrium negotiation values under Assumptions 2 and 3 is*

$$S(X, \underline{x}, \mathcal{Y}) = \{y^* \in Y \mid y^* \geq \underline{y} \text{ and } y_1^* \geq \gamma_1(\underline{y}_2) \text{ for some } \underline{y} \in \underline{Y}\}.$$

With the addition of Assumption 4, the characterization is refined further.

B Classes of Contractual Relationships and Existence

In this appendix, I delineate some classes of contractual relationships and provide existence results for them. Consider the following classes.

Setting 1: Finite Actions

In this setting, $\mathcal{X}$ and $\mathcal{A}$ are finite sets, which means that Z can also be taken to be finite.

Setting 2: Finite Except for Spot Transfers

In this setting, each joint action can be written as $x = (l, m)$, where $m \in \mathbf{R}^2_-$ is a monetary transfer in the current period. For each $z \in Z$, the feasible joint actions satisfy $X(z) = L(z) \times \mathbf{R}^2_-$ for some set $L(z)$. The default action specifies a transfer of zero. Let $\mathcal{L} \equiv \bigcup_{z \in Z} L(z)$. The sets $\mathcal{A}$ and $\mathcal{L}$ are assumed to be finite and there is a correspondence $\tilde{A} : Z \times \mathcal{L} \rightrightarrows \mathcal{A}$ such that, for each $z \in Z$ and $x = (l, m) \in X(z)$, we have $A(z, x) = \tilde{A}(z, l)$.

Furthermore, there exist functions $\tilde{u} : Z \times \mathcal{L} \times \mathcal{A} \to \mathbf{R}^2$ and $\tilde{f} : Z \times \mathcal{L} \times \mathcal{A} \to Z$, such that, for each $z \in Z$, $x = (l, m) \in X(z)$, and $a \in A(z, x)$, the payoff vector in the current period satisfies $u(z, x, a) = \tilde{u}(z, l, a) + m$ and the transition function satisfies $f(z, x, a) = \tilde{f}(z, l, a)$. Thus, the action spaces and state transition do not depend on transfers made previously, and the transfers affect payoffs additively. The assumptions imply that Z can be taken to be finite.

The BoS example from Section 1 is a one-period contractual relationship that has finite actions except for spot transfers.

Setting 3: Finite Except for Contingent Transfers

I shall first describe the particular structure of this setting and then show how it maps into the general model. In a nutshell, there is an externally enforced productive action p and an externally enforced transfer m in each period, in addition to the individual action a. This *public action* (p, m) takes place at the end of the period and is conditioned on the history as specified by the externally enforced component of the players' contract. Assume $a \in \mathcal{A}$, $p \in \mathcal{P}$, and $m \in M$, for some sets $\mathcal{A}$ and $\mathcal{P}$, and assume that $M \subset \mathbf{R}^2_-$. Payoffs and feasible actions are assumed to be functions of the productive actions $a_{-0} = (a_1, a_2)$ and p taken in each period.

Let H denote the set of histories of productive actions (the a's and p's); a representative t-period history is $(a^1, p^1; a^2, p^2; \ldots; a^t, p^t)$. The initial, null history is defined as h^{init}. There is a set of terminal histories H^{term}. Let $\hat{H} \equiv H \setminus H^{\text{term}}$ denote the non-terminal histories. For $h \in \hat{H}$, the set of feasible individual actions is $\tilde{A}(h) = \tilde{A}_0 \times \tilde{A}_1(h) \times \tilde{A}_2(h)$, where $\tilde{A}_0$ does not depend on the history (this is without loss of generality) and we have $\tilde{A}(h) \subset \mathcal{A}$. Given h and $a \in \tilde{A}(h)$, the set of feasible public actions is $P(h, a) \subset \mathcal{P}$. From history h, if a and p are the individual and public actions selected in the current period,

then the history at the start of the next period is $h' = (h; a, p)$, where $(h; a, p)$ is the history formed by appending a and p to h.

The players' externally enforced contract is a vector of functions $x = (\tilde{\alpha}_0, \rho, \mu)$, with the mappings defined as $\tilde{\alpha}_0 : H \rightarrow \Delta \tilde{A}_0$, $\rho : H \times A \rightarrow \mathcal{P}$, and $\mu : H \times A \rightarrow M$, with the property that $\rho(h, a) \in P(h, a)$. Given history h to the start of the current period, $\tilde{\alpha}_0(h) = \alpha_0$ is the probability distribution for Nature's action a_0 that the contract specifies. For history h and individual action profile a in the current period, the contract specifies (and the external enforcer compels) the productive action $p = \rho(h, a)$ and the monetary transfer $m = \mu(h, a)$. Note that the contract can effectively randomize over productive actions by using the random draw a_0. Thus, the function ρ can be viewed as a mapping from $\hat{H} \times A_{-0}$ to $\Delta \mathcal{P}$. To represent limited verifiability, we can suppose that ρ and μ are measurable with respect to some partition $\mathcal{H}$ of $\hat{H} \times A$.

There is an initial contract x^{init} in place at the start of the first period. At the beginning of each period, the players can renegotiate the contract. Assume, however, that when renegotiating from any history h, the players are restricted to contracts that are consistent with h occurring—that is, the players are renegotiating on the contractual terms for only the current and future periods. To make this formal, consider some t'-period history $h = (a^1, p^1; a^2, p^2; \ldots; a^{t'}, p^{t'})$. Contract x *is consistent with h* if $p^t = \rho(h^t, a^t)$ for $t = 1, 2, \ldots, t'$, where h^t is the t-period truncation (the first t elements) of h. Let $\tilde{X}(h)$ denote the set of contracts that are consistent with history h and that satisfy the measurability requirement relative to $\mathcal{H}$, with the understanding that all contracts are consistent with h^{init}.

Payoffs are linear in money and are defined by a function $\tilde{u} : H \setminus H^{\text{init}} \rightarrow \mathbf{R}^2$. Specifically, in a period that began with history h and saw productive actions a and p and transfer m (that is, $p = \rho(h, a)$ and $m = \mu(h, a)$), the payoff vector is given by $\tilde{u}(h') + m$, where $h' = (h; a, p)$. Assume that the function $\tilde{u}$ depends neither on the current-period random draw a_0 nor on the history of random draws.

I make the following extra technical assumptions. First, I assume that A and $\mathcal{P}$ are finite sets, implying that H is finite. Second, I assume that M is compact and contains the zero transfer $(0, 0)$. Third, I assume that there is a positive integer τ such that all feasible histories are no more than τ periods in length; this implies Assumption 5.

Here is how Setting 3 maps into the general model. A state is given by $z = (h, \underline{x})$, where h is the history of productive actions to the current period and $\underline{x}$ is the contract in effect at the beginning of this period. The set of states is $Z \equiv \{(h, \underline{x}) \mid \underline{x} \in \tilde{X}(h)\}$. In the negotiation phase in state $z = (h, \underline{x})$, players renegotiate their externally enforced contract by taking a joint action $x \in \mathcal{X}$, where $\mathcal{X}$ is the set of all contracts of the form $x = (\tilde{\alpha}_0, \rho, \mu)$ as described above. The set of feasible joint actions is $X(z) \equiv \tilde{X}(h)$ and the default joint action is $\underline{x}(z) \equiv \underline{x}$. In the individual-action phase, feasible actions are given by $A(z, x) \equiv \tilde{A}(h)$. The payoff function is given by $u(z, x, a) = \tilde{u}(h; a, \rho(h, a)) + \mu(h, a)$, where ρ and μ are components of x. Finally, for any state z, joint action $x = (\tilde{\alpha}_0, \rho, \mu)$, and individual action profile a, the state transition is given by $f(z, x, a) \equiv ((h; a, \rho(h, a)), x)$.

Note that $\tilde{\alpha}_0$, as a probability distribution over the finite set $\tilde{A}_0$, is a point in the $|\tilde{A}_0|$-dimensional simplex. Also, ρ is a function with a finite domain and codomain, and μ

has finite domain and compact codomain. Thus, $\mathcal{X}$ and $X(z)$ are compact subsets of a Euclidean space. That H is finite further implies that Z is compact.

The MW example described in Section 4 (and less formally in the Introduction) is finite except for contingent transfers. The MW example is especially simple in that there is no public productive action p and no random draw a_0. Also fitting into Setting 3 is the finite-period contracting model of Watson (2005b).[10]

Existence Results

I next provide existence results for the settings just described. To state the first existence result, I use the following definition.

Definition 4: *A bargaining solution S is said to be **regular** if it is non-empty and compact-valued on the set of contracting problems that satisfy Assumption 1.*

The three benchmark solutions described in Section 2 are regular.

Theorem 2: *Take as given a contractual relationship and a regular bargaining solution S. If the contractual relationship satisfies Assumption 5 and either has finite actions (Setting 1) or is finite except for spot transfers (Setting 2), then contractual equilibrium exists.*

The second existence result focuses on settings with externally enforced transfers and some verifiability of actions (Setting 3). Additional assumptions on the bargaining solution are required for this case. In the following definitions, it is understood that $\tilde{\mathcal{Y}} = \{\tilde{Y}^x\}_{x \in X}$, $\hat{\mathcal{Y}} = \{\hat{Y}^x\}_{x \in X}$, $\mathcal{Y}' = \{Y'^x\}_{x \in X}$, and $\mathcal{Y}'' = \{Y''^x\}_{x \in X}$.

Definition 5: *A bargaining solution S is said to be **continuous in the disagreement set** if the following holds for any given X, $\tilde{\mathcal{Y}}$, $\hat{\mathcal{Y}}$, $\overline{x}$, and sequence $\{\underline{x}^k\} \subset X$, such that $\tilde{Y} = \hat{Y}$ and $\tilde{Y}^{\underline{x}^k}$ converges to $\hat{Y}^{\overline{x}}$ in the Hausdorff metric. We have that $S(X, \underline{x}^k, \tilde{\mathcal{Y}})$ converges to $S(X, \overline{x}, \hat{\mathcal{Y}})$ in the Hausdorff metric.*

Definition 6: *A bargaining solution S is said to be **monotone in the disagreement set** if the following holds for any given X, $\underline{x}$, $\mathcal{Y}'$, and $\mathcal{Y}''$ that satisfy Assumption 1. If $Y' = Y''$ and if $Y'^{\underline{x}} \subset Y''^{\underline{x}}$, then $S(X, \underline{x}, \mathcal{Y}') \subset S(X, \underline{x}, \mathcal{Y}'')$.*

The three benchmark solutions are all continuous in the disagreement set; the first two (S^{ND} and S^{AA}, but not S^{AD}) are monotone in the disagreement set.

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[10]In Watson (2005b), $\mathcal{P}$ is allowed to be infinite (but compact) and it is assumed that $M = \mathbf{R}^2$. However, the players' individual actions are assumed to be payoff irrelevant (they are just messages), the payoff function is continuous, and the implementation exercise in the paper does not rely on unbounded transfers. It is not difficult to verify that, for these reasons, the second existence result here (Theorem 3) holds for the model in Watson (2005b). Note also that in that paper the relevant history is defined as the history of messages (the a's) and the current externally enforced contract; it does not include the public actions, but these can be reconstructed from the messages and contract.

Theorem 3: *Take as given a contractual relationship and a regular bargaining solution S. Assume that the contractual relationship satisfies Assumption 5 and is finite except for contingent transfers (Setting 3). If S is continuous and monotone in the disagreement set, then contractual equilibrium exists.*

Proofs of the two existence results follow.

Proof of Theorem 2

Take any $t \in \{1, 2, \ldots, \tau\}$. Presume that V is well-defined on $\mathcal{Z}^{t+1} = \bigcup_{k=t+1}^{\tau+1} Z^k$ and that, for each state z' in this set, $V(z')$ is nonempty and compact (as is true on Z^{term}). I will show that these properties extend to the domain of Z^t. Take any $z \in Z^t$.

Referring to Step 1 of the inductive procedure described in the proof of Theorem 1, I first show that $Y^x(z)$ is nonempty for each $x \in X(z)$. An arbitrary function v defined on $\mathcal{Z}^{t+1}$ implies an induced simultaneous-move game in the individual-action phase (in state z following a joint action x). This game has action spaces $A_1(z, x)$ and $A_2(z, x)$ and it has payoffs given by $\mathrm{E}[u(z, x, a) + v(f(z, x, a)) \mid \alpha_0]$. This is a finite game under the assumption that the contractual relationship is essentially finite; thus, the game has a Nash equilibrium, which implies that $Y^x(z) \neq \emptyset$.

It is further true that $Y^x(z)$ is compact. To see this, note that the graph of the Nash equilibrium correspondence, as a function of the payoff parameters, is closed (see Fudenberg and Tirole 1991, Section 1.3.2 for example). Also recall that the set of feasible v functions is closed and bounded by the presumption that V is compact-valued on $\mathcal{Z}^{t+1}$. These properties together imply that $Y^x(z)$ is closed; the boundedness of the set of v functions implies that $Y^x(z)$ is bounded.

I next show that the contracting problem at state $z \in Z^t$ satisfies Assumption 1. The preceding paragraphs establish the requirements for $\underline{Y}(z)$. To check the properties of $Y(z)$, I examine separately the two settings covered by the theorem. In the setting of finite actions, $Y(z)$ is compact by virtue of each $Y^x(z)$ being compact and $X(z)$ being finite; hence, Assumption 1 is satisfied.

In a relationship that is finite except for spot transfers, $Y(z)$ is not compact, but the arbitrary spot transfers add only $\mathbf{R}_-^2$ to a compact set. To see this, note that the spot transfer in the current period affects neither incentives in this period nor feasible continuation values from the start of the next period (recall Equation (1) and the payoff specification). Thus, for each $l \in L(z)$ and $m \in \mathbf{R}_-^2$, $Y^{(l,m)} = Y^{(l,(0,0))} + m$ holds.[11] This implies that

$$Y(z) = \{y \in \mathbf{R}^2 \mid y_1 + y_2 \leq \sigma\},$$

where

$$\sigma = \max\{y_1 + y_2 \mid y \in Y^{(l,(0,0))}(z), \, l \in L(z)\}.$$

The maximum exists because $L(z)$ is finite. The set $Y(z)$ is therefore closed and separated by a line of negative slope, so it satisfies Assumption 1.

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[11]For $Y, Y' \subset \mathbf{R}^2$ and $y'' \in \mathbf{R}^2$, $Y + Y' \equiv \{y + y' \mid y \in Y, \, y' \in Y'\}$ and $Y + y'' \equiv \{y + y'' \mid y \in Y\}$.

Moving to Step 2 of the inductive procedure, and using the assumption that S is regular, we have that $S(X(z), \underline{x}(z), \mathcal{Y}(z))$ exists and is compact. Thus, V extended to Z^t is nonempty and compact-valued. By induction, contractual equilibrium exists.

Proof of Theorem 3

This proof parallels that of Theorem 2. Note that, in the setting of the theorem, the set H can be partitioned into sets $H^1, H^2, \ldots, H^{\tau+1}$ such that, for any $\underline{x} \in \mathcal{X}$ and $t \in \{1, 3, \ldots, \tau\}$, we have $z = (h, \underline{x}) \in Z^t$ if and only if $h \in H^t$ and $\underline{x} \in \tilde{X}(h)$. Define $\mathcal{H}^t \equiv \bigcup_{t'=t}^{\tau+1} H^{t'}$ and, as in the text, $\mathcal{Z}^t \equiv \bigcup_{t'=t}^{\tau+1} Z^{t'}$.

Take any $t \in \{1, 2, \ldots, \tau\}$. Presume that V is well-defined on $\mathcal{Z}^{t+1}$ and that, for each state z' in this set, $V(z')$ is nonempty and compact (as is true on Z^{term}). Also presume that, on this domain, V is upper hemi-continuous as a function of x on the feasible set $X(z)$; that is, for a fixed $h \in \mathcal{H}^{t+1}$, presume that $V(h, \cdot)$ is upper hemi-continuous. I will show that these properties extend to the domain of Z^t.

Take any $z \in Z^t$. The argument used in the proof of Theorem 2 establishes that $Y^x(z)$ is nonempty and compact, for each $x \in X(z)$. In addition, we have the following fact.

Lemma 5: *Assume the presumptions of Theorem 3. Given $z \in Z^t$, take any sequence $\{y^k, v^k, x^k, \alpha^k\}$ such that for all k, (i) $y^k \in \mathbf{R}^2$, (ii) v^k maps $\mathcal{Z}^{t+1}$ to $\mathbf{R}^2$ and satisfies $v^k(z') \in V(z')$ for all $z' \in \mathcal{Z}^{t+1}$, (iii) $x^k \in X(z)$, and (iv) α^k is an uncorrelated distribution over $A(z, x^k)$ with α_0^k being the distribution over Nature's actions that is specified by the contract x^k. Further suppose that*

$$y^k = \mathrm{E}[u(z, x^k, a) + v^k(f(z, x^k, a)) \mid \alpha^k]$$

holds and the Nash equilibrium Condition (1) is satisfied (using y^k, v^k, x^k, and α^k in place of y, v, x, and α^*) for every k. Then there is a subsequence $\{y^{k_r}, v^{k_r}, x^{k_r}, \alpha^{k_r}\}$ such that x^{k_r} converges to some contract $\overline{x} \in X(z)$ and y^{k_r} converges to a point $\overline{y} \in Y^{\overline{x}}(z)$.

Proof of Lemma 5: Because $\mathcal{A}$ is finite and $X(z)$ is compact, we can assume (by taking a subsequence) that α^k converges to some distribution $\overline{\alpha} \in \Delta\mathcal{A}$ and that x^k converges to some contract $\overline{x} \in X(z)$. The structure of feasible actions implies that $\overline{\alpha} \in \Delta A(z, \overline{x})$. Because $\mathcal{A}$ and $\mathcal{P}$ are finite, we can use the same argument to justify assuming that, for every $h' \in \mathcal{H}^{t+1}$, $v^k(h', x^k)$ converges (to a real number). Define function $\overline{v}: \mathcal{Z}^{t+1} \to \mathbf{R}^2$ by

$$\overline{v}(h', \overline{x}) \equiv \lim_{k \to \infty} v^k(h', x^k)$$

for every $h' \in \mathcal{H}^{t+1}$, and $\overline{v}(h', x)$ is an arbitrary selection from $V(h', x)$ for every $h' \in \mathcal{H}^{t+1}$ and $x \neq \overline{x}$.

We have $v^k(h', x^k) \in V(h', x^k)$ for every k, so upper hemi-continuity of V on the domain $\mathcal{Z}^{t+1}$ implies that $\overline{v}(h', \overline{x}) \in V(h', \overline{x})$. Thus, we have $\overline{v}(z') \in V(z')$ for every $z' \in \mathcal{Z}^{t+1}$. By the convergence of x^k and finiteness of $\mathcal{H}^{t+1}$, we know that, for a fixed a

there is a history $h' \in \mathcal{H}^{t+1}$ such that $f(z, x^k, a) = (h', x^k)$ for all large k. By construction of $\overline{v}$, we thus have that $v^k(f(z, x^k, a))$ converges to $\overline{v}(f(z, \overline{x}, a))$. Also, noting that u is continuous, we know that $u(z, x^k, a)$ converges to $u(z, \overline{x}, a)$. Thus, we obtain

$$\overline{y} = \mathrm{E}[u(z, \overline{x}, a) + \overline{v}(f(z, \overline{x}, a)) \,|\, \overline{\alpha}],$$

Another implication of the convergence is that the Nash equilibrium Condition (1) is satisfied for $\overline{y}$, $\overline{v}$, $\overline{x}$, and $\overline{\alpha}$ in place of y, v, x, and α^*. To see this, first note that $A(z, x)$ is independent of x, meaning that available individual actions do not depend on the current contract. In addition, if the best-response inequality did not hold for some player i and some action a_i', it would imply failure of the same best-response condition for strategy profile α^k in the context of contract x^k and continuation-value function v^k for large enough k, which is a contradiction. We can thus conclude that $\overline{y} \in Y(z)$, proving the lemma. *Q.E.D.*

Returning to the proof of Theorem 3, continue to consider any $z = (h, \underline{x}) \in Z^t$. Refer to Step 1 of the inductive procedure described in the proof of Theorem 1. The argument used in the proof of Theorem 2 establishes that $Y^x(z)$ is nonempty for every $x \in X(z)$. Lemma 5 implies that $Y(z)$ is compact and that $Y^x(z)$ is compact for every $x \in X(z)$. For any particular x, the latter conclusion follows from looking at a sequence $\{y^k, v^k, x^k, \alpha^k\}$ with $x^k = x$ and $y^k \in Y^x(z)$ and getting $\overline{y} \in Y^x(z)$. Thus, for $z = (h, \underline{x}) \in Z^t$, the contracting problem $(X(z), \underline{x}, \mathcal{Y}(z))$ satisfies Assumption 1.

Moving to Step 2 of the inductive procedure, and using the assumption that S is regular, we have that $S(X(z), \underline{x}(z), \mathcal{Y}(z))$ exists and is compact. Thus, V extended to Z^t is nonempty and compact-valued.

Finally, I must establish that $V(h, \cdot)$ is upper hemi-continuous for all $h \in H^t$. To this end, fix $h \in H^t$. Note that $X(h, \underline{x}) = X(h, \underline{x}')$ and $Y^x(h, \underline{x}) = Y^x(h, \underline{x}')$, for all $\underline{x}, \underline{x}' \in \tilde{X}(h)$. The second equality holds because the externally enforced contract in force at the beginning of the period ($\underline{x}$ or $\underline{x}'$) does not directly affect the attainable continuation payoffs; only the newly chosen contract x affects incentives and payoffs in the continuation. Thus, since h is fixed for the remainder of this proof, I will suppress the state argument and simply write X, Y^x, and Y. Also note that, for the fixed h, Lemma 5 implies that Y^x is upper hemi-continuous in x. This follows by limiting attention to sequences such that x^k converges.

Next, take any specific sequence $\{x^k\}$ that converges to some $\overline{x}$. I will compare various contracting problems that are associated with the sequence. To formulate them, let $\underline{Y}$ be the limit superior of $Y^{\underline{x}^k}$ as $k \to \infty$:

$$\underline{Y} \equiv \bigcap_{k'=1}^{\infty} \mathrm{closure} \left[\bigcup_{k=k'}^{\infty} Y^{\underline{x}^k} \right].$$

For each k, we have the contracting problem $(X, \underline{x}^k, \mathcal{Y})$, where $\mathcal{Y} = \{Y^x\}_{x \in X}$. Consider also the artificial contracting problem $(X, \underline{x}^k, \tilde{\mathcal{Y}})$, where we define $\tilde{\mathcal{Y}}$ by $\tilde{Y}^x \equiv Y^x \cup \underline{Y}$ for all $x \in X$. In addition, consider the contracting problem $(X, \overline{x}, \mathcal{Y})$ and another artificial

problem $(X, \underline{\overline{x}}, \hat{\mathcal{Y}})$, where $\hat{\mathcal{Y}}$ is identical to $\mathcal{Y}$ except for joint action $\underline{\overline{x}}$ for which we define $\hat{Y}^{\underline{\overline{x}}} \equiv \overline{Y}$.

By upper hemi-continuity of Y^x, we have $\overline{Y} \subset Y^{\underline{\overline{x}}}$. Thus, $\overline{Y} \subset Y$ and $\tilde{Y} = \hat{Y} = Y$. That is, the sets of feasible continuation payoffs (over all x) in the artificial contracting problems $(X, \underline{x}^k, \tilde{\mathcal{Y}})$ and $(X, \underline{x}^k, \hat{\mathcal{Y}})$ are the same as that in the contracting problems $(X, \underline{x}^k, \mathcal{Y})$ and $(X, \underline{\overline{x}}, \mathcal{Y})$. All of these contracting problems differ only in the disagreement sets. Furthermore, one can easily verify that, by construction of $\tilde{Y}^x$ and because Y is compact, $\tilde{Y}^{\underline{x}^k}$ converges to $\overline{Y}$ in the Hausdorff metric.

Using the fact that $Y^{\underline{x}^k} \subset \tilde{Y}^{\underline{x}^k}$ and that S is monotone in the disagreement set, we have

$$V(h, \underline{x}^k) = S(X, \underline{x}^k, \mathcal{Y}) \subset S(X, \underline{x}^k, \tilde{\mathcal{Y}}). \tag{4}$$

Because $\tilde{Y}^{\underline{x}^k}$ converges to $\overline{Y}$ and S is continuous in the disagreement set, we obtain

$$S(X, \underline{x}^k, \tilde{\mathcal{Y}}) \text{ converges to } S(X, \underline{\overline{x}}, \hat{\mathcal{Y}}). \tag{5}$$

Finally, using the monotone property of S again and that $\overline{Y} \subset Y^{\underline{\overline{x}}}$, we have

$$S(X, \underline{\overline{x}}, \hat{\mathcal{Y}}) \subset S(X, \underline{\overline{x}}, \mathcal{Y}) = V(h, \underline{\overline{x}}). \tag{6}$$

Along with the sequence $\{\underline{x}^k\}$, take a convergent sequence $\{y^k\} \subset \mathbf{R}^2$ such that $y^k \in V(h, \underline{x}^k)$ for each k. Relations 4-6 above establish that the limit of $\{y^k\}$ is contained in $V(h, \underline{\overline{x}})$, which proves upper hemi-continuity of $V(h, \cdot)$. Q.E.D.

C Other Proofs

Lemma 1 and Theorem 1 are restated and proved here.

Lemma 1: If Y is convex then $\overline{Q}(\underline{y}) = \underset{\substack{y \in \mathrm{SPB}(Y) \\ y \geq \underline{y}}}{\arg\max} \ (y_1 - \underline{y}_1)^{\pi_1}(y_2 - \underline{y}_2)^{\pi_2}.$

Proof: For $i = 1, 2$, let $g_i(y_j) \equiv \max\{y_i \mid (y_i, y_j) \in Y\}$. Further, for any fixed $\underline{y}$, define

$$G_i(y_i) \equiv g_i(e^{-\Delta\pi_i} g_j(e^{-\Delta\pi_j} y_i + (1 - e^{-\Delta\pi_j})\underline{y}_i) + (1 - e^{-\Delta\pi_i})\underline{y}_j).$$

Consider these functions defined on strong Pareto boundary of the set Y, where (because Y is convex) g_1 and g_2 are continuous and inverses. These functions characterize the equilibrium of the K-round bargaining protocol under Assumptions 2 and 3. Suppose, for example, that player 2 would have the offer in round $k + 1$, where his equilibrium continuation value is y_2. Then player 1's equilibrium offer in round k is a vector that makes player 2 indifferent between accepting and rejecting; this offer gives player 1 the amount $g_1(e^{-\Delta\pi_1} y_2 + (1 - e^{-\Delta\pi_1})\underline{y}_2)$, which is player 1's equilibrium value from round k.

Function G_i relates player i's equilibrium continuation values across two rounds where player i makes the offer. Convexity of Y implies that g_1 and g_2 are convex functions, which further implies that G_i is a contraction. To see this, consider the differentiable case, where we have $G_i'(y_i) = e^{-\Delta\pi_1} e^{-\Delta\pi_2} g_i'(y_i) g_j'(y_j)$. Here y_i and y_j are numbers satisfying $y_i < g_i(y_j)$. Recalling that g_1 and g_2 are inverses, this implies that $g_i'(y_i) g_j'(y_j) \in (0, 1)$. Thus, $G_i' \in (0, 1)$.

That G_i is a contraction implies that player i's equilibrium payoff in the finite-round protocol converges to the fixed point of G_i, which is i's equilibrium payoff in the infinite-round protocol. From here, one can use the asymmetric version of Binmore, Rubinstein, and Wolinsky's (1986) Proposition 5 to complete the proof. *Q.E.D.*

Theorem 1: Take as given a contractual relationship and a bargaining solution S. If Assumption 5 is satisfied and contractual equilibrium exists, then there is a unique value correspondence that represents contractual equilibrium.

Proof: A value correspondence V can be identified by backward induction, using the partition $\{Z^1, Z^2, \ldots, Z^{\tau+1}\}$ of Z that was described earlier in this section. We start with $Z^{\tau+1} = Z^{\text{term}}$ and have, for each z in this set, $V(z) = \{(0, 0)\}$. Then, for any $t \in \{1, 2, \ldots, \tau\}$, presume that we have defined V on $\bigcup_{k=t+1}^{\tau+1} Z^k$. We can extend V to Z^t as follows.

> **Step 1:** For every $z \in Z^t$ and every $x \in X(z)$, define $Y^x(z)$ by Equation (2). In doing so, note that v need only be defined on $\bigcup_{k=t+1}^{\tau+1} Z^k$, because no other states can be reached from z. If every $Y^x(z)$ is nonempty, then this yields a well-defined contracting problem $(X(z), \underline{x}(z), \mathcal{Y}(z))$ at state z, for every $z \in Z^t$.
>
> **Step 2:** Define $V(z) \equiv S(X(z), \underline{x}(z), \mathcal{Y}(z))$, for every $z \in Z^t$.

If, during this procedure, we find that $Y^x(z) = \emptyset$ for some z and x in Step 1, then the correspondence V is not well-defined and contractual equilibrium does not exist. Likewise, if $S(X(z), \underline{x}(z), \mathcal{Y}(z))$ is empty for some z in Step 2, then contractual equilibrium does not exist. Otherwise, contractual equilibrium exists and the induction procedure identifies a unique value correspondence by construction. *Q.E.D.*

References

Arvan, L., L. Cabral, and V. Santos, "Meaningful Cheap Talk Must Improve Equilibrium Payoffs," *Mathematical Social Sciences* 37 (1999): 97-106.

Baker, G., R. Gibbons, and K. J. Murphy, "Relational Contracts and the Theory of the Firm," *Quarterly Journal of Economics* 117 (2002): 39-83.

Benoit, J.-P. and V. Krishna, "Renegotiation in Finitely Repeated Games," *Econometrica* 61 (1993): 303-323.

Bernheim, B. D. and M. Whinston, "Incomplete Contracts and Strategic Ambiguity," *American Economic Review* 88 (1998): 902-932.

Binmore, K., A. Rubinstein, and A. Wolinsky, "The Nash Bargaining Solution in Economic Modelling," *Rand Journal of Economics* 17 (1986): 176-188.

Brennan, J. and J. Watson, "The Renegotiation-Proofness Principle and Costly Renegotiation," UCSD working paper, 2002.

Bull, C., "The Existence of Self-enforcing Implicit Contracts," *Quarterly Journal of Economics* 102 (1987): 147-159.

Bull, J. and J. Watson, "Evidence Disclosure and Verifiability," *Journal of Economic Theory* 118 (Issue 1, September 2004): 1-31.

Bull, J. and J. Watson, "Hard Evidence and Mechanism Design," *Games and Economic Behavior*, in press (2006).

Che, Y.-K. and D. Hausch, "Cooperative Investments and the Value of Contracting," *American Economic Review* 89 (1999): 125-147.

Dewatripont, M. "Renegotiation and Information Revelation over Time: The Case of Optimal Labor Contracts," *Quarterly Journal of Economics* 104 (1989): 589-619.

Edlin, A. and B. Hermalin, "Contract Renegotiation and Options in Agency Problems," *Journal of Law, Economics, and Organization* 16 (2000): 395-423.

Edlin, A. and S. Reichelstein, "Holdups, Standard Breach Remedies, and Optimal Investment," *American Economic Review* 86 (1996): 478-501.

Farrell, J., "Cheap Talk, Communication, and Entry," *Rand Journal of Economics* 18 (1987): 34-39.

Farrell, J. and M. Rabin, "Cheap Talk," *Journal of Economic Perspectives* 10 (1996): 103-118.

Hart, O. and J. Moore, "Incomplete Contracts and Renegotiation," *Econometrica* 56 (1988): 755-785.

Hart, O. and J. Moore, "Foundations of Incomplete Contracts," *Review of Economic Studies* 66 (1999): 115-138.

Hart, O.D. and J. Tirole, "Contract Renegotiation and Coasian Dynamics," *Review of Economic Studies* 55 (1988): 509-540.

Hurwicz, L., "Economic Design, Adjustment Processes, Mechanisms, and Institutions," *Economic Design* 1 (1994): 1-14.

Klimenko, M., G. Ramey, and J. Watson, "Recurrent Trade Agreements and the Value of External Enforcement," UCSD working paper, revised 2004.

Laffont, J.J. and J. Tirole, "Adverse Selection and Renegotiation in Procurement," *Review of Economic Studies* 57 (1990): 597-625.

Levin, J. "Relational Incentive Contracts," *American Economic Review* 93 (2003): 835-847.

MacLeod, W.B. and J.M. Malcomson, "Implicit Contracts, Incentive Compatibility, and Involuntary Unemployment," *Econometrica* 57 (1989): 447-480.

MacLeod, W. B. and J. M. Malcomson, "Investments, Holdup, and the Form of Market Contracts," *American Economic Review* 83 (1993): 811-837.

Maskin, E., "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies* 66 (1999): 23-38.

Maskin, E. and J. Moore, "Implementation and Renegotiation," *Review of Economic Studies* 66 (1999): 39-56.

Moore, J. and R. Repullo, "Subgame Perfect Implementation," *Econometrica* 56 (1988): 1191-1220.

Myerson, R. B., "Optimal Coordination Mechanisms in Generalized Principal-Agent Problems," Journal of Mathematical Economics 10 (1982): 6781.

Myerson, R. B., Game Theory: Analysis of Conflict, Cambridge, MA: Harvard University Press (1991).

Nöldeke, G. and K. Schmidt, "Option Contracts and Renegotiation: A Solution to the Hold-Up Problem," *RAND Journal of Economics* 26 (1995): 163-179.

Nöldeke, G. and K. Schmidt, "Sequential Investments and Options to Own," *RAND Journal of Economics* 29 (1998): 633-653.

Pearce, D. and E. Stacchetti, "The Interaction of Implicit and Explicit Contracts in Repeated Agency," *Games and Economic Behavior* 23 (1998): 75-96.

Rabin, M., "A Model of Pre-Game Communication," *Journal of Economic Theory* 63 (1994): 370-391.

Ramey, G. and J. Watson, "Bilateral Trade and Opportunism in a Matching Market," *Contributions to Theoretical Economics* Vol. 1, No. 1 (2001), Article 3.

Ramey, G. and J. Watson, "Contractual Intermediaries," *Journal of Law, Economics, and Organization* 18 (2002): 362-384.

Roth, D., "A Theory of Partnership Dynamics," *Games and Economic Behavior* 12 (1996): 95-112.

Santos, V., "Alternating-Announcements Cheap Talk," *Journal of Economic Behavior and Organization* 42 (2000): 405-416

Schmidt, K. and M. Schnitzer, "The Interaction of Explicit and Implicit Contracts," *Economics Letters* 48 (1995) : 193-199.

Schwartz, A. and J. Watson, "The Law and Economics of Costly Contracting," *Journal of Law, Economics, and Organization* 20 (April 2004): 2-31.

Segal, I., "Complexity and Renegotiation: A Foundation for Incomplete Contracts," *Review of Economic Studies* 66 (1999): 57-82.

Segal, I. and M. Whinston, "The Mirrlees Approach to Mechanism Design with Renegotiation (with Applications to Hold-Up and Risk-Sharing)," *Econometrica* 70 (2002): 1-45.

Telser, L.G., "A Theory of Self-enforcing Agreements," *Journal of Business* 53 (1980): 27-44.

Watson, J., *Strategy: An Introduction to Game Theory*, New York: W.W. Norton and Company (2002).

Watson, J., "Contract, Mechanism Design, and Technological Detail," UCSD working paper, revised 2005a.

Watson, J., "Contract and Mechanism Design in Settings with Multi-Period Trade," manuscript (2005b).