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STRUCTURALLY-INDUCED VOLATILITY CLUSTERING

BY

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ABSTRACT: Many standard structural models in economics have the property that they induce persistent, partially predictable heteroskedasticity (“volatility clustering”) in their key dependent variables, even when their underlying stochastic shock variables are all serially independent and homoskedastic, and their structural parameters are all time-invariant. This paper presents examples of this phenomenon, and examines the nature of such induced volatility clustering.

Keywords: Volatility Clustering, Induced Volatility Clustering, Stochastic Volatility, ARCH

JEL Codes: C53, G00, G12

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1. INTRODUCTION

The goal of this paper is to explore ways in which the phenomenon of volatility drift or *volatility clustering*, of the general type modeled by ARCH and other stochastic volatility specifications,¹ can arise even in systems whose nonstochastic structure is time-invariant and whose stochastic shocks are all zero-mean i.i.d. and consequently homoskedastic. In other words, we study when volatility clustering can be represented as being generated by the *structural* (i.e., deterministic) properties of an economic system, rather than entering exogenously through its shocks.

Volatility clustering can arise from a time-invariant structure and zero-mean i.i.d. shocks² $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$ and $\{\tilde{\eta}_t\}_{t=1,2,\dots}$ in either of two ways. The first involves a multiplicative interaction

$$(1) \quad Y_t = Z_t \cdot \tilde{\varepsilon}_t$$

where Z_t is any stationary or nonstationary *drift variable* with homoskedastic innovation term $\tilde{\eta}_t$, such as the random walk, autoregressive or moving average processes (for $\rho, \gamma \in (0,1)$):

$$(2) \quad Z_t = \sum_{\tau=1}^t \tilde{\eta}_\tau \quad Z_t = \rho \cdot Z_{t-1} + \tilde{\eta}_t \quad Z_t = \tilde{\eta}_t + \gamma \cdot \tilde{\eta}_{t-1}$$

or some combination of these. Past values of the dependent variable Y_t and the drift variable Z_t are assumed to be directly observable and thus part of the information set I_t . Past values of $\tilde{\varepsilon}_t$ and $\tilde{\eta}_t$ are not directly observable, though in some cases they could be estimated or algebraically inferred from the data. We refer to any such process for Y_t as a *drifting coefficient process*. For such a process, the mean and variance of Y_t conditional on the information set I_t are given by

$$(3) \quad E[Y_t | I_t] = E[Z_t \cdot \tilde{\varepsilon}_t | I_t] = E[Z_t | I_t] \cdot E[\tilde{\varepsilon}_t | I_t] = 0$$

$$(4) \quad \text{var}[Y_t | I_t] = E[Z_t^2 \cdot \tilde{\varepsilon}_t^2 | I_t] = E[Z_t^2 | I_t] \cdot E[\tilde{\varepsilon}_t^2 | I_t] = [(E[Z_t | I_t])^2 + \sigma_{\tilde{\eta}}^2] \cdot \sigma_{\tilde{\varepsilon}}^2$$

and the latter is seen to drift over time with drift in the value of $E[Z_t | I_t]$. For the three drift processes listed in (2), $E[Z_t | I_t]$ takes the respective forms Z_{t-1} , $\rho \cdot Z_{t-1}$ and $-\sum_{\tau=1}^{\infty} (-\gamma)^\tau \cdot Z_{t-\tau}$.

A second way in which volatility clustering can arise from a time-invariant structure and a *single* source of zero-mean i.i.d. shocks $\{\tilde{\eta}_t\}_{t=1,2,\dots}$ is when a conditionally homoskedastic drift variable Z_t of a form such as (2) has a nonlinear influence on the dependent variable Y_t , either through an explicit or implicit structural relationship of the form

$$(5) \quad Y_t = g(Z_t) \quad \text{or} \quad h(Y_t) = Z_t$$

We refer to this as a *drifting input* (or *drifting implicit input*) process. For small values of $\sigma_{\tilde{\eta}}^2$ the conditional mean and variance of Y_t for the relationship $Y_t = g(Z_t)$ can be approximated by

$$(6) \quad E[Y_t | I_t] \approx g(E[Z_t | I_t])$$

$$(7) \quad \text{var}[Y_t | I_t] \approx g'(E[Z_t | I_t])^2 \cdot \sigma_{\tilde{\eta}}^2$$

When $g''(\cdot) \neq 0$ the volatility of Y_t seen to drift with $E[Z_t | I_t]$, even for time-invariant $\sigma_{\tilde{\eta}}^2$. The conditional mean and variance of Y_t for the relationship $h(Y_t) = Z_t$ are approximated by $E[Y_t | I_t] \approx h^{-1}(E[Z_t | I_t])$ and $\text{var}[Y_t | I_t] \approx h'(h^{-1}(E[Z_t | I_t]))^2 \cdot \sigma_{\tilde{\eta}}^2$, which similarly yields volatility clustering.

¹ E.g., Engle, Bollerslev and Nelson (1994), Engle (1995), Ghysels, Harvey and Renault (1996) and Shephard (1996).

² Throughout this paper, we will use tildes to denote zero-mean i.i.d. random variables.

A structural model can simultaneously exhibit volatility clustering due to *both* drifting coefficient and drifting input effects, such as the form

$$(8) \quad Y_t = f(Z_t, \tilde{\varepsilon}_t)$$

which can be taken either as a direct structural equation for Y_t , or else as it's reduced form equation from an underlying structural model. For small values of $\sigma_{\tilde{\eta}}^2$ and $\sigma_{\tilde{\varepsilon}}^2$, the conditional mean and variance of Y_t can be approximated by

$$(9) \quad E[Y_t|I_t] \approx f(E[Z_t|I_t], E[\tilde{\varepsilon}_t|I_t]) = f(E[Z_t|I_t], 0)$$

$$(10) \quad \text{var}[Y_t|I_t] \approx f_Z(E[Z_t|I_t], 0)^2 \cdot \sigma_{\tilde{\eta}}^2 + f_{\varepsilon}(E[Z_t|I_t], 0)^2 \cdot \sigma_{\tilde{\varepsilon}}^2$$

When $f_{ZZ}(E[Z_t|I_t], 0) \neq 0$, the conditional variance of Y_t is again seen to drift with $E[Z_t|I_t]$, even for time-invariant $\sigma_{\tilde{\eta}}^2$, which is another example of the drifting input process (5). When $f_{Z\varepsilon}(E[Z_t|I_t], 0) \neq 0$, the conditional variance of Y_t is again seen to drift with $E[Z_t|I_t]$, even for time-invariant $\sigma_{\tilde{\varepsilon}}^2$, which is a generalized version of the drifting coefficient process (1), with $\tilde{\varepsilon}_t$'s original drifting coefficient of Z_t replaced by its drifting partial derivative $f_{\varepsilon}(E[Z_t|I_t], 0)$.

The following three sections present some simple economic models with time-invariant structures and zero-mean i.i.d. shocks, which nevertheless imply volatility clustering. These examples involve volatility clustering in the price of a standard commodity, in the period-by-period returns of a financial asset, and the joint clustering properties between the price and quantity of a commodity. The paper concludes with some examples of attributing volatility clustering to structural forms, as opposed to leaving it as an exogenous feature of the model's shocks.

2. INDUCED VOLATILITY CLUSTERING: PRICE OF A STANDARD COMMODITY

The most basic structural model in economics is the supply-demand model for a standard (flow) commodity. As a simple example, consider a commodity with a deterministic market supply function $Q^S(P_t)$, and market demand function $Q^D(P_t, Z_t) + \tilde{\varepsilon}_t$ in terms of the commodity price P_t , an economic input Z_t (such as income or the price of another good) that follows a conditionally homoskedastic drift processes as in (2), and independent zero-mean i.i.d. demand shocks $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$. The equilibrium price P_t^e in period t is determined by the market clearing condition

$$(11) \quad Q^D(P_t^e, Z_t) + \tilde{\varepsilon}_t = Q^S(P_t^e)$$

For small departures of Z_t from its conditional mean $E[Z_t|I_t]$ and small values of $\tilde{\varepsilon}_t$ about its conditional mean of 0, we have

$$(12) \quad dP_t^e = \frac{Q_Z^D(E[P_t^e|I_t], E[Z_t|I_t]) \cdot dZ_t + d\tilde{\varepsilon}_t}{Q_P^S(E[P_t^e|I_t]) - Q_P^D(E[P_t^e|I_t], E[Z_t|I_t])}$$

For small values of $\sigma_{\tilde{\eta}}^2$ and $\sigma_{\tilde{\varepsilon}}^2$ the conditional variance of P_t^e can accordingly be approximated by

$$(13) \quad \text{var}[P_t^e|I_t] \approx \frac{Q_Z^D(E[P_t^e|I_t], E[Z_t|I_t])^2 \cdot \sigma_{\tilde{\eta}}^2 + \sigma_{\tilde{\varepsilon}}^2}{[Q_P^S(E[P_t^e|I_t]) - Q_P^D(E[P_t^e|I_t], E[Z_t|I_t])]^2}$$

It follows that P_t^e will exhibit volatility clustering as a result of the drifting value of $E[Z_t|I_t]$ whenever either $Q_{ZZ}^D(\cdot, \cdot)$ or $Q_{PZ}^D(\cdot, \cdot)$ is nonzero, and exhibit volatility clustering as a result of its own drifting conditional mean $E[P_t^e|I_t]$ whenever either $Q_{PP}^S(\cdot)$, $Q_{PP}^D(\cdot, \cdot)$ or $Q_{PZ}^D(\cdot, \cdot)$ is nonzero.

These two types of effects can each be graphically illustrated. In each case, the horizontal excess demand shocks $\tilde{\varepsilon}_t$ from (11) are homoskedastic. Figure 1 depicts a pure drifting coefficient effect, in which $Q_{PP}^S(\cdot)$ and $Q_{PP}^D(\cdot, \cdot)$ are both zero so that the market excess demand curve is linear in P , but $Q_{PZ}^D(\cdot, \cdot)$ is non-zero so that drift in Z_t leads to drift in the *slope* of the excess demand curve, and hence volatility clustering in P_t^e . Figure 2 depicts a pure drifting implicit input effect, in which $Q_{PZ}^D(\cdot, \cdot)$ is zero so that drift in Z_t leads to a pure horizontal translation of the excess demand curve, but $Q_{PP}^S(\cdot)$ and/or $Q_{PP}^D(\cdot, \cdot)$ are nonzero so that the excess demand curve is nonlinear in P , which also leads to volatility clustering in P_t^e .

Figure 1
Induced volatility clustering in P_t^e for linear excess demand curve
with drifting slope and homoskedastic horizontal shocks

Figure 2
Induced volatility clustering in P_t^e for nonlinear excess demand curve
with horizontal drift and homoskedastic horizontal shocks

3. INDUCED VOLATILITY CLUSTERING: RETURNS ON A FINANCIAL ASSET

Consider an asset whose cash value in its terminal period T will be given by a nonlinear function $\pi(\varepsilon_1 + \dots + \varepsilon_T)$ of the accumulation of a sequence of zero-mean i.i.d. “news” variables $\{\tilde{\varepsilon}_t\}_{t=1,\dots,T}$, which are realized and observed one period at a time. Define $Z_t = \sum_{\tau=1}^{t-1} \varepsilon_\tau$ as the news available at the start of period t , that is, before the realization of $\tilde{\varepsilon}_t$. Assume that the discount rate is zero, and that the price of the asset at the end of period t (that is, after the realization of $\tilde{\varepsilon}_t$) is given by the expectation of its terminal value, conditional on the news to date:

$$(14) \quad P_t(Z_t + \varepsilon_t) = E_{\tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_T} [\pi(Z_t + \varepsilon_t + \tilde{\varepsilon}_{t+1} + \dots + \tilde{\varepsilon}_T)]$$

Since this also implies

$$(15) \quad P_{t-1}(Z_t) = E_{\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T} [\pi(Z_t + \tilde{\varepsilon}_t + \tilde{\varepsilon}_{t+1} + \dots + \tilde{\varepsilon}_T)]$$

the asset’s one-period gross rate of return viewed from the start of period t is given by the random variable

$$(16) \quad R_t(Z_t, \tilde{\varepsilon}_t) = \frac{P_t(Z_t + \tilde{\varepsilon}_t)}{P_{t-1}(Z_t)} = \frac{E_{\tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_T} [\pi(Z_t + \tilde{\varepsilon}_t + \tilde{\varepsilon}_{t+1} + \dots + \tilde{\varepsilon}_T)]}{E_{\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T} [\pi(Z_t + \tilde{\varepsilon}_t + \tilde{\varepsilon}_{t+1} + \dots + \tilde{\varepsilon}_T)]}$$

Note that while $\tilde{\varepsilon}_t$ appears as an actual random variable in the left and middle terms of (16) as well as in the *numerator* of the right term, it is expected out in the *denominator* of the right term. By the Law of Iterated Expectations and the fact that Z_t is a sufficient statistic for the information set I_t at the start of period t , the conditional mean of the gross return is given by

$$(17) \quad E[R_t(Z_t, \tilde{\varepsilon}_t) | I_t] = E_{\tilde{\varepsilon}_t} [R_t(Z_t, \tilde{\varepsilon}_t)] = \frac{E_{\tilde{\varepsilon}_t, \tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_T} [\pi(Z_t + \tilde{\varepsilon}_t + \tilde{\varepsilon}_{t+1} + \dots + \tilde{\varepsilon}_T)]}{E_{\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T} [\pi(Z_t + \tilde{\varepsilon}_t + \tilde{\varepsilon}_{t+1} + \dots + \tilde{\varepsilon}_T)]} = 1$$

which comes as no surprise, given our assumptions of expectation-based pricing and zero discounting. Expanding about $\tilde{\varepsilon}_t = 0$, the conditional variance of the return is approximated by

$$(18) \quad \text{var}[R_t(Z_t, \tilde{\varepsilon}_t) | I_t] = \text{var}[R_t(Z_t, \tilde{\varepsilon}_t)] \approx \frac{[E_{\tilde{\varepsilon}_{t+1}, \dots, \tilde{\varepsilon}_T} [\pi'(Z_t + \tilde{\varepsilon}_{t+1} + \dots + \tilde{\varepsilon}_T)]]^2 \cdot \sigma_{\tilde{\varepsilon}}^2}{[E_{\tilde{\varepsilon}_t, \dots, \tilde{\varepsilon}_T} [\pi(Z_t + \tilde{\varepsilon}_t + \tilde{\varepsilon}_{t+1} + \dots + \tilde{\varepsilon}_T)]]^2}$$

In spite of the homoskedasticity and serial independence of the news variables $\{\tilde{\varepsilon}_t\}_{t=1,\dots,T}$, the conditional variance of the gross return (and the conditional variance of P_t itself) is seen to drift with the drift in Z_t , through both the numerator and denominator of (18).

Although (16) is similar to the general specification (8) in that it also depends on both a homoskedastic drift variable Z_t and an i.i.d. shock $\tilde{\varepsilon}_t$, it differs from (8) in two respects. The first difference is that the drift variable Z_t in $R_t(Z_t, \tilde{\varepsilon}_t)$ is not the accumulation of *separate variables* $\{\tilde{\eta}_t\}$, but rather, the accumulation of past values of $\tilde{\varepsilon}_t$ itself. But since $\tilde{\varepsilon}_t$ is independent of its past values this difference is not essential, and like (8), the specification (16) exhibits volatility clustering from both a drifting input effect and a drifting coefficient effect: R_t is seen to exhibit volatility clustering from a drifting input effect through its numerator $P_t(Z_t + \tilde{\varepsilon}_t)$, and since this numerator is divided by the drifting predetermined variable $P_{t-1}(Z_t)$, R_t also exhibits induced volatility due to the drifting coefficient $1/P_{t-1}(Z_t)$. These two effects correspond to Z_t ’s influence on the conditional variance formula through the numerator and denominator of (18) respectively.

A second difference between (16) and (8), which may seem to contradict our goal of obtaining volatility clustering from i.i.d. shocks and time-invariant structures, is that (16) involves a *time-dependent* formula $R_t(\cdot, \cdot)$, so it is no surprise that it would generate a time-dependent unconditional variance path $\text{var}[R_1|I_0], \dots, \text{var}[R_T|I_0]$. Two remarks are in order: First, the formulas $R_1(\cdot, \cdot), \dots, R_T(\cdot, \cdot)$ are *not* structural, but rather, are themselves derived from the one true structural formula of the model, namely the terminal cash value function $\pi(\cdot)$, via the economic pricing formula (14). Second, it is important to note that even if a model *does* imply a time-dependent unconditional variance path, it may or may not exhibit volatility clustering. Volatility clustering is *not* defined as a *time-dependent unconditional variance path* $\text{var}[R_t|I_0]$, but rather, as *serially correlated departures in the conditional variance* $\text{var}[R_t|I_t]$ *from its predicted path* $E[\text{var}[R_t|I_t]|I_0]$.³ In other words, it is the persistent drift in the conditional variance $\text{var}[R_t|I_t]$ *about* its (constant or nonconstant) predicted path $E[\text{var}[R_t|I_t]|I_0]$ that constitutes volatility clustering in R_t , and which is implied by the dependence of $\text{var}[R_t|I_t]$ upon the drifting variable Z_t in (18).

By assuming an infinite horizon, we can construct an asset pricing model that exhibits both induced volatility clustering and time-invariant $P(\cdot)$ and $R(\cdot, \cdot)$ functions: Consider an orchard with overlapping cohorts of trees, where each tree yields fruit for $L+1$ periods, and the productivity (net with respect to some average) of trees planted in period t is $\tilde{\varepsilon}_t$. Because of scale effects in processing and marketing, total profits in period t is $\pi(\varepsilon_{t-L} + \dots + \varepsilon_t)$, so the market value of the firm at the end of period t is given by the discounted conditional expectation

$$\begin{aligned}
 P(\varepsilon_{t-L}, \dots, \varepsilon_t) &= \pi(\varepsilon_{t-L} + \dots + \varepsilon_t) \\
 &+ \delta \cdot E_{\tilde{\varepsilon}_{t+1}} [\pi(\varepsilon_{t-L+1} + \dots + \varepsilon_t + \tilde{\varepsilon}_{t+1})] \\
 &+ \delta^2 \cdot E_{\tilde{\varepsilon}_{t+1}, \tilde{\varepsilon}_{t+2}} [\pi(\varepsilon_{t-L+2} + \dots + \varepsilon_t + \tilde{\varepsilon}_{t+1} + \tilde{\varepsilon}_{t+2})] \\
 &\vdots \\
 &+ \delta^L \cdot E_{\tilde{\varepsilon}_{t+1}, \tilde{\varepsilon}_{t+2}, \dots, \tilde{\varepsilon}_{t+L}} [\pi(\varepsilon_t + \tilde{\varepsilon}_{t+1} + \tilde{\varepsilon}_{t+2} \dots + \tilde{\varepsilon}_{t+L})] \\
 &+ \sum_{\tau=L+1}^{\infty} \delta^\tau \cdot E_{\tilde{\varepsilon}_{t+\tau-L}, \dots, \tilde{\varepsilon}_{t+\tau}} [\pi(\tilde{\varepsilon}_{t+\tau-L} + \tilde{\varepsilon}_{t+\tau-L+1} + \dots + \tilde{\varepsilon}_{t+\tau})]
 \end{aligned}
 \tag{19}$$

Since the zero-mean productivity shocks $\tilde{\varepsilon}_t$ are i.i.d., the function $P(\cdot, \dots, \cdot)$ is time-invariant, and the one-period gross rate of return viewed from the start of period t , namely

$$R(\varepsilon_{t-L-1}, \dots, \varepsilon_{t-1}, \tilde{\varepsilon}_t) = \frac{P(\varepsilon_{t-L}, \dots, \varepsilon_{t-1}, \tilde{\varepsilon}_t)}{P(\varepsilon_{t-L-1}, \dots, \varepsilon_{t-1})}
 \tag{20}$$

is similarly time-invariant. Since nonlinearity of $\pi(\cdot)$ implies that $\partial P(\varepsilon_{t-L}, \dots, \varepsilon_{t-1}, \varepsilon_t) / \partial \varepsilon_t|_{\varepsilon_t=0}$ will drift with each of the $L+1$ moving sums $\varepsilon_{t-L} + \dots + \varepsilon_{t-1}$, $\varepsilon_{t-L+1} + \dots + \varepsilon_{t-1}$, $\dots$, $\varepsilon_{t-2} + \varepsilon_{t-1}$, ε_{t-1} (which determine the predictable component of profits in the current and each of the next L periods), and $1/P(\varepsilon_{t-L-1}, \dots, \varepsilon_{t-1})$ will drift with the moving sums $\varepsilon_{t-L-1} + \dots + \varepsilon_{t-1}$, $\varepsilon_{t-L} + \dots + \varepsilon_{t-1}$, $\dots$, $\varepsilon_{t-2} + \varepsilon_{t-1}$, ε_{t-1} , we again have that R_t exhibits volatility clustering due to both drifting input effects in its numerator and a drifting coefficient effect from its denominator.

³ Thus, even though a pure random walk $Z_t = \sum_{\tau=1}^t \tilde{\eta}_\tau$ has a time-dependent unconditional variance path $\text{var}[Z_t|I_0] \equiv t \cdot \sigma_{\tilde{\eta}}^2$, it exhibits no volatility clustering, since its conditional variance $\text{var}[Z_t|I_t]$ never departs from its predicted value $E[\text{var}[Z_t|I_t]|I_0] = \sigma_{\tilde{\eta}}^2$.

4. INDUCED COVARIANCE CLUSTERING AND JOINT VOLATILITY CLUSTERING

Give a pair of variables $\{Y_{1t}\}_{t=1,2,\dots}$ and $\{Y_{2t}\}_{t=1,2,\dots}$ we can ask the following two questions about their joint behavior:⁴

1. Do Y_{1t} and Y_{2t} exhibit *covariance clustering* – that is, does their conditional covariance $\text{cov}[Y_{1t}, Y_{2t} | I_t]$ exhibit drift in the sense of serially correlated departures about its predicted path $\text{cov}[Y_{1t}, Y_{2t} | I_0]$?
2. Do Y_{1t} and Y_{2t} exhibit *joint volatility clustering* – that is, do their conditional variances $\text{var}[Y_{1t} | I_t]$ and $\text{var}[Y_{2t} | I_t]$ exhibit contemporaneously correlated departures from their predicted paths $\text{var}[Y_{1t} | I_0]$ and $\text{var}[Y_{2t} | I_0]$?

To see that either of these phenomena can occur without the other, let $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$, $\{\tilde{\eta}_t\}_{t=1,2,\dots}$ and $\{\tilde{\omega}_t\}_{t=1,2,\dots}$ be independent standard normal shock variables, and Z_t be a drift variable with values in $[0,1]$ and which is in the information set I_t . Since the pair of variables $Y'_{1t} = \sqrt{Z_t} \cdot \tilde{\varepsilon}_t + \sqrt{1-Z_t} \cdot \tilde{\eta}_t$ and $Y'_{2t} = \sqrt{Z_t} \cdot \tilde{\varepsilon}_t + \sqrt{1-Z_t} \cdot \tilde{\omega}_t$ satisfy $E[Y'_{1t} | I_t] = E[Y'_{2t} | I_t] = 0$, $\text{cov}[Y'_{1t}, Y'_{2t} | I_t] = Z_t$ and $\text{var}[Y'_{1t} | I_t] = \text{var}[Y'_{2t} | I_t] = 1$, they exhibit covariance clustering, but neither joint nor individual volatility clustering. Conversely, since the variables $Y''_{1t} = Z_t \cdot \tilde{\varepsilon}_t$ and $Y''_{2t} = Z_t \cdot \tilde{\eta}_t$ satisfy $E[Y''_{1t} | I_t] = E[Y''_{2t} | I_t] = 0$, $\text{var}[Y''_{1t} | I_t] = \text{var}[Y''_{2t} | I_t] = Z_t^2$ and $\text{cov}[Y''_{1t}, Y''_{2t} | I_t] = 0$, they exhibit both individual and joint volatility clustering, but not covariance clustering.

As with volatility clustering, both covariance clustering and joint volatility clustering can arise in structural systems that are time-invariant and whose shocks are all zero-mean i.i.d. In the supply-demand example of Section 2, since the market clearing quantity Q_t^e is a deterministic function $Q^S(\cdot)$ of market clearing price P_t^e , equation (12) implies that for small departures of Z_t from its conditional mean $E[Z_t | I_t]$ and small values of $\tilde{\varepsilon}_t$ about its conditional mean of 0, we have

$$(21) \quad dQ_t^e = Q_P^S(E[P_t^e | I_t]) \cdot \frac{Q_Z^D(E[P_t^e | I_t], E[Z_t | I_t]) \cdot dZ_t + d\tilde{\varepsilon}_t}{Q_P^S(E[P_t^e | I_t]) - Q_P^D(E[P_t^e | I_t], E[Z_t | I_t])}$$

For small values of $\sigma_{\tilde{\eta}}^2$ and $\sigma_{\tilde{\varepsilon}}^2$ the conditional variance of Q_t^e and conditional covariance of P_t^e and Q_t^e can be approximated by

$$(22) \quad \text{var}[Q_t^e | I_t] \approx Q_P^S(E[P_t^e | I_t])^2 \cdot \frac{Q_Z^D(E[P_t^e | I_t], E[Z_t | I_t])^2 \cdot \sigma_{\tilde{\eta}}^2 + \sigma_{\tilde{\varepsilon}}^2}{[Q_P^S(E[P_t^e | I_t]) - Q_P^D(E[P_t^e | I_t], E[Z_t | I_t])]^2}$$

$$(23) \quad \text{cov}[P_t^e, Q_t^e | I_t] \approx Q_P^S(E[P_t^e | I_t]) \cdot \frac{Q_Z^D(E[P_t^e | I_t], E[Z_t | I_t])^2 \cdot \sigma_{\tilde{\eta}}^2 + \sigma_{\tilde{\varepsilon}}^2}{[Q_P^S(E[P_t^e | I_t]) - Q_P^D(E[P_t^e | I_t], E[Z_t | I_t])]^2}$$

Since drift in $E[Z_t | I_t]$ affects $\text{var}[P_t^e | I_t]$, $\text{var}[Q_t^e | I_t]$ and $\text{cov}[P_t^e, Q_t^e | I_t]$, P_t^e and Q_t^e will generally exhibit both covariance clustering and joint volatility clustering. Since the partial derivative $Q_P^D(\cdot, \cdot)$ is negative and $Q_P^S(\cdot)$ is positive, the drifting conditional covariance $\text{cov}[P_t^e, Q_t^e | I_t]$ is always positive.⁵ However, whether the conditional variances $\text{var}[P_t^e | I_t]$ and $\text{var}[Q_t^e | I_t]$ tend to drift in the same direction as each other or in opposite directions will depend on the signs and magnitudes of

⁴ We use the following pair of terms in place of the term “covolatility clustering” on the grounds that the latter is ambiguous: If “covolatility clustering” is taken to mean “clustering of covolatility,” then it corresponds to what we term “covariance clustering.” On the other hand, if “covolatility clustering” is taken to mean “correlation of volatility clustering,” then it corresponds to what we term “joint volatility clustering.”

⁵ This is natural since we assume a drifting stochastic demand curve and a deterministic supply curve.

the second-order derivatives of the demand and supply functions. The following figures illustrate this in the simple case of a demand function $Q^D(P_t) + Z_t + \tilde{\varepsilon}_t$. In Figure 3, where demand is convex in price and supply is linear, $\text{var}[P_t^e|I_t]$ and $\text{var}[Q_t^e|I_t]$ both unambiguously rise/fall as the demand curve drifts to the right/left. In Figure 4, where demand is convex in price and supply is concave, $\text{var}[P_t^e|I_t]$ unambiguously rises/falls as demand drifts to the right/left, but the behavior of $\text{var}[Q_t^e|I_t]$ will depend on the exact magnitudes of $Q_{PP}^S(\cdot)$ and $Q_{PP}^D(\cdot, \cdot)$, via (22).

Figure 3
Individual and joint volatility clustering in P_t^e and Q_t^e

Figure 4
Volatility clustering in P_t^e , but not necessarily
volatility clustering in Q_t^e or joint volatility clustering.

5. STRUCTURAL ATTRIBUTION OF OBSERVED VOLATILITY CLUSTERING

If the residuals of a regression equation $Y_t = \alpha + \beta \cdot X_t + u_t$ are found to exhibit serial correlation of the form $u_t = \tilde{\varepsilon}_t + \gamma \cdot \tilde{\varepsilon}_{t-1}$ for some γ and i.i.d. series $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$, it is possible to fully “structurally attribute” this serial correlation by simply expressing the complete model as

$$(24) \quad Y_t = \alpha + \beta \cdot X_t + \tilde{\varepsilon}_t + \gamma \cdot \tilde{\varepsilon}_{t-1}$$

that is, by assuming that the i.i.d. structural variables contained in $\tilde{\varepsilon}_t$ not only affect Y_t but also have structural effect γ upon Y_{t+1} . If the serial correlation in $Y_t = \alpha + \beta \cdot X_t + u_t$ is found to take the alternative form $u_t = \rho \cdot u_{t-1} + \tilde{\varepsilon}_t$ for some $\rho \in [0,1)$ and i.i.d. $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$, then it can be fully structurally attributed by expressing the complete model in either of the equivalent forms

$$(25) \quad Y_t = \alpha + \beta \cdot X_t + \sum_{\tau=0}^{\infty} \rho^\tau \cdot \tilde{\varepsilon}_{t-\tau}$$

$$(25)' \quad Y_t = (1-\rho) \cdot \alpha + \beta \cdot X_t + \rho \cdot Y_{t-1} - \rho \cdot \beta \cdot X_{t-1} + \tilde{\varepsilon}_t$$

where the first form posits that the structural variables contained in $\tilde{\varepsilon}_t$ affect Y_t and also have geometrically diminishing structural effects $\rho, \rho^2, \dots$ on its future values $Y_{t+1}, Y_{t+2}, \dots$, and the second form posits that Y_t and X_t both directly structurally affect Y_{t+1} . The reason that each of the models (24), (25) and (25)' can be said to *fully structurally attribute* the serial correlation is that each is expressed only in terms of zero-mean i.i.d. shocks $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$. Using standard techniques, most ARIMA processes for u_t can be treated in a similar manner.

In this section we examine the possibilities for full structural attribution of observed *volatility clustering* in a variable, or more generally, in the residuals of a regression equation. Specifically, we show how each of three standard specifications of volatility clustering can be fully represented by the interaction of a time-invariant structural model and zero-mean i.i.d. shocks.⁶

For example, if the (zero-mean) residuals in a regression equation $Y_t = \alpha + \beta \cdot X_t + u_t$ are found to exhibit volatility clustering of the simple form $\text{var}(u_t|I_t) = \rho \cdot u_{t-1}^2$, they can be generated by a process of the form

$$(26) \quad u_t = \tilde{\varepsilon}_t \cdot \prod_{\tau=1}^{t-1} |\tilde{\varepsilon}_\tau| = \tilde{\varepsilon}_t \cdot \left| \tilde{\varepsilon}_{t-1} \cdot \prod_{\tau=1}^{t-2} |\tilde{\varepsilon}_\tau| \right| = \tilde{\varepsilon}_t \cdot |u_{t-1}|$$

for zero-mean i.i.d. $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$ with $\text{var}(\tilde{\varepsilon}_t) = \rho$, so that by the substitution $Z_t = \ln(|u_t|)$, this form of volatility clustering can be fully structurally attributed by an i.i.d. shock, time-invariant system of the form

$$(27) \quad \begin{aligned} Y_t &= \alpha + \beta \cdot X_t + \exp(Z_{t-1}) \cdot \tilde{\varepsilon}_t \\ Z_t &= Z_{t-1} + \ln(|\tilde{\varepsilon}_t|) \\ \{\tilde{\varepsilon}_t\}_{t=1,2,\dots} &\text{ zero-mean i.i.d. with } \text{var}(\tilde{\varepsilon}_t) = \rho \end{aligned}$$

where Y_t is seen to exhibit induced volatility clustering via the drifting coefficient $\exp(Z_{t-1})$, for the conditionally homoskedastic random walk drift variable Z_t .

⁶ In addition to the three specifications considered below, equations (4) and (7) can be used to link the range of the conditional volatility $E[Y_t|I_t]$ to the ranges of $E[Z_t|I_t]$ and $g'(\cdot)$ in the simple drifting coefficient and drifting input specifications (1) and (5).

More generally, if the residuals in $Y_t = \alpha + \beta \cdot X_t + u_t$ are found to exhibit volatility clustering of the standard ARCH form $\text{var}(u_t|I_t) = c + \rho \cdot u_{t-1}^2$, they can be generated by a process of the form⁷

$$\begin{aligned}
u_t &= \tilde{\varepsilon}_t \cdot \sqrt{(c/\rho) + (c/\rho) \cdot \sum_{s=1}^{t-1} \prod_{\tau=s}^{t-1} \tilde{\varepsilon}_\tau^2} \\
&= \tilde{\varepsilon}_t \cdot \sqrt{(c/\rho) + [(c/\rho) \cdot \tilde{\varepsilon}_{t-1}^2 + (c/\rho) \cdot \tilde{\varepsilon}_{t-1}^2 \cdot \sum_{s=1}^{t-2} \prod_{\tau=s}^{t-2} \tilde{\varepsilon}_\tau^2]} \\
&= \tilde{\varepsilon}_t \cdot \sqrt{(c/\rho) + \tilde{\varepsilon}_{t-1}^2 \cdot [(c/\rho) + (c/\rho) \cdot \sum_{s=1}^{t-2} \prod_{\tau=s}^{t-2} \tilde{\varepsilon}_\tau^2]} \\
&= \tilde{\varepsilon}_t \cdot \sqrt{(c/\rho) + u_{t-1}^2}
\end{aligned}
\tag{28}$$

for zero-mean i.i.d. $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$ with $\text{var}(\tilde{\varepsilon}_t) = \rho$, so that by the substitution $Z_t = u_t^2$, this form of volatility clustering can be fully structurally attributed by a i.i.d. shock, time-invariant system of the form

$$\begin{aligned}
Y_t &= \alpha + \beta \cdot X_t + \tilde{\varepsilon}_t \cdot \sqrt{(c/\rho) + Z_{t-1}} \\
Z_t &= \tilde{\varepsilon}_t^2 \cdot [(c/\rho) + Z_{t-1}] \\
\{\tilde{\varepsilon}_t\}_{t=1,2,\dots} &\text{ zero-mean i.i.d. with } \text{var}(\tilde{\varepsilon}_t) = \rho
\end{aligned}
\tag{29}$$

where Y_t exhibits induced volatility clustering via the drifting coefficient $\sqrt{(c/\rho) + Z_{t-1}}$, for a variable Z_t that is subject to both an additive trend c/ρ and homoskedastic multiplicative drift $\tilde{\varepsilon}_t^2$.

Finally, if the residuals in $Y_t = \alpha + \beta \cdot X_t + u_t$ are found to exhibit volatility clustering of the form $\text{var}(u_t|I_t) = |u_{t-1}|^{2\rho}$, they can be generated by a process of the form

$$u_t = \tilde{\varepsilon}_t \cdot \prod_{\tau=1}^{t-1} |\tilde{\varepsilon}_\tau|^{\rho^{(t-\tau)}} = \tilde{\varepsilon}_t \cdot [|\tilde{\varepsilon}_{t-1}| \cdot \prod_{\tau=1}^{t-2} |\tilde{\varepsilon}_\tau|^{\rho^{(t-1-\tau)}}]^\rho = \tilde{\varepsilon}_t \cdot |u_{t-1}|^\rho
\tag{30}$$

for zero-mean i.i.d. $\{\tilde{\varepsilon}_t\}_{t=1,2,\dots}$ with $\text{var}(\tilde{\varepsilon}_t) = 1$, so that by the substitution $Z_t = \ln(|u_t|)$, this form of volatility clustering can be fully structurally attributed by an i.i.d. shock, time-invariant system of the form

$$\begin{aligned}
Y_t &= \alpha + \beta \cdot X_t + \exp(\rho \cdot Z_{t-1}) \cdot \tilde{\varepsilon}_t \\
Z_t &= \rho \cdot Z_{t-1} + \ln(|\tilde{\varepsilon}_t|) \\
\{\tilde{\varepsilon}_t\}_{t=1,2,\dots} &\text{ zero-mean i.i.d. with } \text{var}(\tilde{\varepsilon}_t) = 1
\end{aligned}
\tag{31}$$

where Y_t exhibits induced volatility clustering via the drifting coefficient $\exp(\rho \cdot Z_{t-1})$, for the conditionally homoskedastic autoregressive drift variable Z_t .

Of course, the fact that we can construct structural models to generate specific forms of observed volatility clustering in a variable does not imply that the causal variables and relationships in these models necessarily exist. Rather, we hope that by exploring the volatility clustering implications of standard structural economic models we can obtain additional stochastic predictions from these models, and conversely, by exploring the potential structural generation of different forms of volatility clustering we can strengthen the extent to which observed stochastic properties of a variable can be used to suggest potential causal variables and structural forms.

⁷ The following form can be equivalently written as $u_t = \text{sgn}(\tilde{\varepsilon}_t) \cdot [(c/\rho) \cdot \sum_{s=1}^t \prod_{\tau=s}^t \tilde{\varepsilon}_\tau^2]^{1/2}$.

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