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# Trading off generations: infinitely lived agent versus OLG

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# Trading Off Generations: Infinitely Lived Agent Versus OLG*

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**Abstract:** The prevailing literature discusses intergenerational trade-offs in climate change predominantly in terms of the Ramsey equation relying on the infinitely lived agent model. We discuss these trade-offs in a continuous time OLG framework and relate our results to the infinitely lived agent setting. We identify three shortcomings of the latter: First, underlying normative assumptions about social preferences cannot be deduced unambiguously. Second, the distribution among generations living at the same time cannot be captured. Third, the optimal solution may not be implementable in overlapping generations market economies.

**Keywords:** climate change, discounting, infinitely lived agents, intergenerational equity, overlapping generations, time preference

**JEL-Classification:** D63, H23, Q54

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## 1 Introduction

How much should society invest into avoiding or at least extenuating anthropogenic climate change? This question is at the heart of the literature on integrated assessment models, which augments economic growth models with a climate module to deliver the quantitative input for policy design. Any climate policy involves major intergenerational transfers. Therefore, a sound analysis of the structural and – implicit or explicit – normative assumptions is crucial. The current debate discusses the intergenerational trade-off between today's mitigation costs and future generation's well-being in terms of the Ramsey equation.[1] The Ramsey equation, like most integrated assessment models, relies on the Ramsey-Cass-Koopmans growth model and, thus, assumes an infinitely lived representative agent (ILA). In this paper, we explore how an explicit model of overlapping generations (OLG) relates to the Ramsey equation. For this purpose, we develop a new continuous time overlapping generations growth model. We compare the preference specifications of the ILA-based Ramsey equation and its OLG counterpart resulting in the same (observed) aggregate market outcomes. We uncover normative assumptions of calibration-based approaches to climate change assessment and explore equity and consistency concerns in normative approaches that refuse intergenerational discounting.

The Stern (2007) review on the economics of climate change, carried out by the former World Bank Chief Economist on behalf of the British government, has drawn significant attention in the political arena. It implies an optimal carbon tax that differs by an order of magnitude from the optimal tax derived by Nordhaus (2008) in his widely known integrated assessment model DICE. Nordhaus (2007) shows that this difference is almost fully explained by the different assumptions on social discounting as summarized in the Ramsey equation. Nordhaus himself favors a *positive* approach to social discounting using a calibration-based procedure that attempts to avoid explicit normative assumptions. In contrast, Stern (2007) advocates a *normative* approach emphasizing that only ethical considerations are valid to address the intergenerational trade-off. The debate over the right discount rate almost exclusively relies on the Ramsey equation. However, the underlying ILA assumption inevitably neglects the crucial distinction: How do we discount the welfare of future generations as opposed to our own future welfare?

The Ramsey equation characterizes how an ILA trades off consumption possibilities accruing at different points in time. In the climate change discussion, the ILA framework is usually interpreted as a utilitarian social welfare function, where each point in time is associated with the utility of a different generation. Barro (1974) shows that appropriate assumptions on altruism and operational bequests imply that finitely lived overlapping generations can be aggregated into a representative ILA. However, recent empirical studies indicate that the altruistic bequest motive is rather weak.[2] As a consequence, the dominant share of savings is

[1] In line with the environmental economic literature we call the the Euler equation of the Ramsey-Kass-Koopmans growth model "Ramsey equation".

[2] See, e.g., Hurd (1987, 1989), Kopczuk and Lupton (2007), Laitner and Juster (1996), Laitner and Ohlsson

driven by individual life-cycle planning rather then by altruistic transfers for future generations. Therefore, a calibration of the Ramsey equation to observed interest rates will necessarily reflect preference parameters that deal with individuals' life-cycle planning over their finite lifetime. In contrast, integrated assessment models evaluate climate change and climate policy impacts on a scale of several centuries. Hence, the disentangling of the life plans of finitely-lived individuals from the long-run plans of a social planner is crucial to analyze the positive and normative assumptions underlying the climate change assessment.

For our analysis, we develop a novel continuous time OLG model around two desiderata. First, in order to relate as closely as possible to the standard Ramsey equation, we choose a model in continuous time where agent's live a finite deterministic life span. In contrast to the models based on Yaari (1965) and Blanchard (1985), where agents have an infinite lifetime and a constant probability of death, our model explicitly captures life-cycle investment. Second, we incorporate economic growth via exogenous technological change in order to make reasonable statements about intergenerational distribution. This feature is also a crucial distinction from the most closely related model in the literature by d'Albis (2007) who examines the influence of demographic structure on capital accumulation.

We first compare our decentralized OLG economy to an ILA economy. We explain why and how the preference parameters of the individual households in the decentralized OLG economy differ from those in the observationally equivalent (i.e., leading to the same aggregate outcomes) ILA economy. We draw attention to the resulting implicit normative assumptions of the "positive" approach. We then introduce a social planner who maximizes the discounted life time utilities of the OLG as, for example, in Calvo and Obstfeld (1988), Burton (1993) and Marini and Scaramozzino (1995). We show that this utilitarian OLG economy is observationally equivalent to an appropriately chosen ILA economy. However, the distribution of consumption between old and young at any given point in time differs substantially from that of the decentralized economy if the rate of time preference (or generational discount rate) of the social planner is lower than that of the individual households. We draw attention to a resulting normative conflict between intergenerational equity and distributional equity among generations alive that arises in the utilitarian social planner framework with a dedicated intergenerational discount rate. Finally, we find that a social planner who is limited to tax labor and capital income cannot achieve the first-best social optimum without age-discriminatory tax schedules.

Related to our analysis, Aiyagari (1985) showed that under certain conditions an overlapping generations model with two-period-lived agents exhibits the same paths of aggregate capital and consumption as the discounted dynamic programming model with infinitely lived agents in discrete time. We complement these results by explicitly deriving the relation between the preference parameters of the OLG model and the observationally equivalent ILA framework

(2001), Wilhelm (1996). These papers suggest either that the bequest motive is statistically insignificant, economically irrelevant, or, if there is a considerable bequest motive, that it is not of the altruistic type (in the sense of Barro 1974 and Becker 1974) but originates from other sources such as the "joy of giving". In all these cases an OLG economy does not reduce to an ILA economy.

in continuous time. The equivalence between the social planner solution in a continuous time OLG setting and an ILA model was already observed by Calvo and Obstfeld (1988). While they focus on time inconsistencies in fiscal policy, our focus is on intergenerational trade-offs.

Several environmental economic applications employ numerical simulations of integrated assessment models to compare interest rates and climate policy between ILA models and OLG frameworks in which agents live for two or three periods. Gerlagh and van der Zwaan (2000) point at differences between the models as a consequence of aging and distributional policies. Howarth (1998) compares the simulation results of a decentralized OLG, a constrained, and an unconstrained utilitarian OLG to the results obtained by Nordhaus (1994) using the ILA model DICE. While the decentralized OLG yields similar results as DICE, he finds substantial differences for the utilitarian OLGs. Calibrating time preference, Howarth (2000) shows that the unconstrained utilitarian OLG model and the ILA model can produce similar outcomes. Stephan et al. (1997) provide a simulation yielding equivalence between a decentralized OLG with bounded rationality and an ILA economy with limited foresight. In contrast, our model elaborates the analytical conditions under which the continuous time ILA and OLG frameworks are observationally equivalent. Burton (1993) and Marini and Scaramozzino (1995) analyze the relationship between individual welfare maximization and the optimal outcome of a benevolent social planner in an overlapping generations model with resources or environmental pollution. With this literature, our paper shares the insight that OLG models provide crucial insights about intergenerational trade-offs that cannot be captured in infinitely lived agent models. The next section explains the positive and normative approaches to social discounting and lays out the further structure of the paper.

## 2 Nordhaus, Stern and the Relation between ILA and OLG Models

Our major presumption is that the world looks more like an overlapping generations model than an infinitely-lived agent framework. Accordingly, we interpret the real world (without policy intervention) as a decentralized OLG economy. We develop a decentralized OLG model and an OLG model where a social planner optimizes the allocations of consumption and capital (either with or without a full set of instruments). We analyze the relation between preference inputs and market outcomes and compare it to a standard Ramsey-Cass-Koopmans model with an infinitely lived decision maker. Figure 1 illustrates the relations that we analyze in this paper.

The majority of economists in the climate change debate takes an observation-based approach to social discounting. This view is exemplarily laid out in Nordhaus' (2007) critical review of the Stern (2007) review of climate change. Individual preferences towards climate change mitigation cannot be observed directly in market transactions because of the public good characteristic of greenhouse gas abatement. However, we observe everyday investment decisions on capital markets that carry information on intertemporal preferences. In particular, we observe the market interest rate and the steady state growth rate of the economy. The positive approach

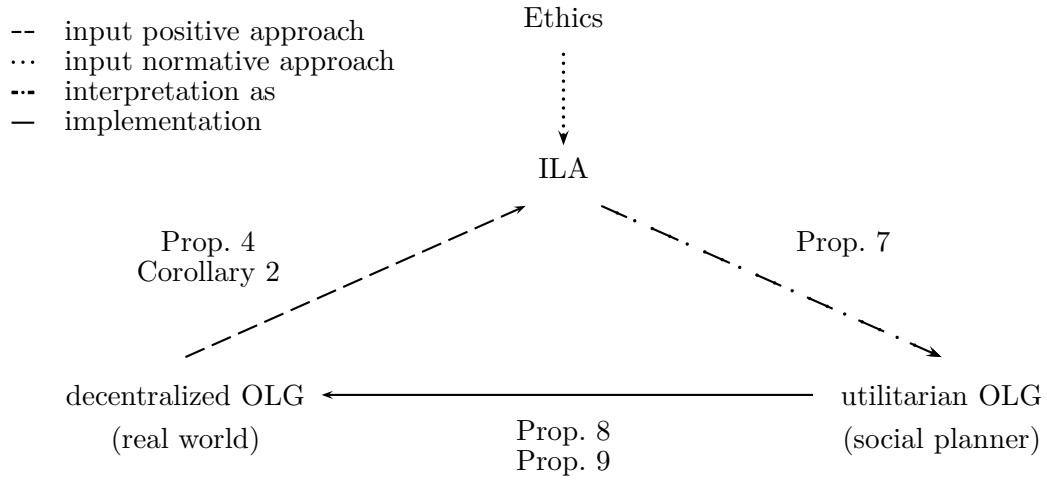


**Figure 1:** Relation between ILA and OLG models. Clockwise interpretation starts from the decentralized OLG economy that we consider the more accurate description of the real world: The integrated assessment literature and the social discounting debate abstract from the real world OLG to an ILA model. After filling in preference information in different ways, the ILA model is interpreted as a social planner model for evaluating climate policy. In the OLG world, the social planner has to implement the policy in the presence of households that optimize their own life-cycle consumption.

translates this information into (pairs of) time preference and a measure for the intertemporal elasticity of substitution. Then, this ILA is interpreted as a utilitarian social planner who confronts the climate problem in an integrated assessment model. In Figure 1, the dashed arrows indicate the positive approach. The ILA model is calibrated according to preferences revealed in the decentralized OLG economy and then interpreted as a social planner weighting the different generations' lifetime utility. The latter is represented by our utilitarian OLG model.

The normative approach to social discounting aims at treating all generations alike and, therefore, argues that a positive rate of time preference is non-ethical. This view is supported by a number of authors including Ramsey (1928), Pigou (1932), Harrod (1948), Koopmans (1965), Solow (1974), Broome and Schmalensee (1992) and Cline (1992). The Stern (2007) review of climate change effectively uses a zero rate of time preference, but adopts the parameter value $\rho^R = 0.1\%$ in order to capture a small but positive probability that society becomes extinct.[3] Figure 1 represents the normative approach by dotted arrows. It uses ethical arguments to specify the time preference rate of the ILA. As in the positive approach, this time preference rate is

[3] Strictly speaking this is not time preference, but Yaari (1965) shows the equivalence of discounting because of a probability of death/extinction and a corresponding rate of time preference. Our superscript $R$ labels inputs to the Ramsey equation.

interpreted as the weight a social planner attaches to lifetime utilities of different generations.

Finally, the social optimum or preferred policy has to be implemented. As indicated by the solid arrow in Figure 1, in our explicit OLG setting the social planner has to implement the policy in the presence of households that optimize their own life-cycle consumption. In this setting, equity trade-offs become more complex, as the implementation of the first-best solution is limited.

The remainder of the paper is structured as follows. In Section 3, we develop the decentralized, continuous time OLG model and establish conditions for existence and uniqueness of a steady state. Section 4 recalls the ILA Ramsey-Cass-Koopmans economy. Section 5 analyzes the relation between the preference parameters of OLG households and the ILA for observationally equivalent economies. In Section 6, we introduce a social planner into the OLG model and examine the relationship between this utilitarian OLG economy, the ILA model, and the decentralized OLG economy. We consider the case where the utilitarian social planner can fully control the economy and the situation where he is limited to non-age discriminatory taxes on labor and capital income. We apply our formal investigation to the recent debate on climate change mitigation in Section 7. Section 8 concludes.

## 3 An OLG Growth Model in Continuous Time

We introduce an OLG exogenous growth model in continuous time and analyze the long-run individual and aggregate dynamics of a decentralized economy in market equilibrium.

### 3.1 Households

Consider a continuum of households, each living the finite time span $T$. All households exhibit the same intertemporal preferences irrespective of their time of birth $s \in (-\infty, \infty)$. We assume that if households are altruistic, their altruistic preferences are not sufficiently strong for an operative bequest motive. This allows us to abstract from altruism in individual preferences. As a consequence, all households maximize their own welfare $U$, which is the discounted stream of instantaneous utility derived from consumption during their lifetime

$$U(s) \equiv \int_s^{s+T} \frac{c(t,s)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \exp\left[-\rho^H(t-s)\right] dt \ , \tag{1}$$

where $c(t,s)$ is the consumption at calender time $t$ of households born at time $s$, $\sigma^H$ is the constant intertemporal elasticity of substitution and $\rho^H$ denotes the constant rate of (pure) time preference of the households. Each household is endowed with one unit of labor at any time alive, which is supplied inelastically to the labor market at wage $w(t)$. In addition, households

may save and borrow assets $b(t,s)$ at the interest $r(t)$. The household's budget constraint is[4]

$$\dot{b}(t,s) = r(t)b(t,s) + w(t) - c(t,s) \ , \qquad t \in [s, s+T] \ . \tag{2}$$

Households are born without assets and are not allowed to be indebted at time of death. Thus, the following boundary conditions apply for all generations $s$

$$b(s,s) = 0 \ , \qquad b(s+T,s) \geq 0 \ . \tag{3}$$

Because of the non-operative bequest motive, intertemporal welfare $U$ of a household born at time $s$ always increases in consumption at time $s+T$. Thus, in the household optimum, the second boundary condition in equation (3) holds with equality.

Maximizing equation (1) for any given $s$ subject to conditions (2) and (3) yields the well known Euler equation

$$\dot{c}(t,s) = \sigma^H \big[r(t) - \rho^H\big] c(t,s) \ , \qquad t \in [s, s+T] \ . \tag{4}$$

The behavior of a household born at time $s$ is characterized by the system of differential equations (2) and (4) and the boundary conditions for the asset stock (3).

At any time $t \in (-\infty, \infty)$ the size of the population $N(t)$ increases at the constant rate $\nu \geq 0$. Normalizing the population at time $t = 0$ to unity implies the birth rate $\gamma$[5]

$$N(t) \equiv \exp[\nu t] \qquad \Rightarrow \qquad \gamma = \frac{\nu \exp[\nu T]}{\exp[\nu T] - 1} \ . \tag{5}$$

### 3.2 Firms

Consider a continuum of identical competitive firms $i \in [0,1]$. All firms produce a homogeneous consumption good under conditions of perfect competition from capital $k(t,i)$ and *effective labor* $A(t)l(t,i)$. $A(t)$ characterizes the technological level of the economy and grows exogenously at a constant rate $\xi$. Normalizing technological progress at $t = 0$ to unity implies

$$A(t) \equiv \exp[\xi t] \ . \tag{6}$$

All firms have access to the same production technology $F(k(t,i), A(t)l(t,i))$, which exhibits constant returns to scale and positive but strictly decreasing marginal productivity with respect to both inputs capital and effective labor. Furthermore, $F$ satisfies the Inada conditions.

[4] Throughout the paper, partial derivatives are denoted by subscripts (e.g., $F_k(k,l) = \partial F(k,l)/\partial k$), derivatives with respect to calendar time $t$ are denoted by dots and derivatives of functions depending on one variable only are denoted by primes.

[5] The equation is derived by solving $\int_{t-T}^{t} \gamma \exp[\nu s]\, ds = N(t)$, where $\gamma \exp[\nu s]$ denotes the cohort size of the generation born at time $s$. Observe that $\gamma \to 1/T$ for $\nu \to 0$ and $\gamma \to \nu$ for $T \to \infty$. Anticipating definition (12), we can also write $\gamma = 1/Q_T(\nu)$.

Constant returns to scale of the production function and symmetry of the firms allow us to work with a representative firm whose decision variables are interpreted as aggregate variables. With minor abuse of notation, we introduce aggregate capital per effective labor, $k(t)$, and aggregate capital per capita, $\bar{k}(t)$,

$$k(t) \equiv \frac{\int_0^1 k(t,i)\,di}{A(t)\int_0^1 l(t,i)\,di}\ , \qquad \bar{k}(t) \equiv \frac{\int_0^1 k(t,i)\,di}{N(t)}\ . \tag{7}$$

In addition, we define the intensive form production function $f\big(k(t)\big) \equiv F\big(k(t),1\big)$.

Profit maximization of the representative firm yields for the wage $w(t)$ and the interest rate $r(t)$

$$w(t) = A(t)\left[f\big(k(t)\big) - f'\big(k(t)\big)k(t)\right]\ , \tag{8a}$$

$$r(t) = f'\big(k(t)\big)\ . \tag{8b}$$

### 3.3 Market Equilibrium and Aggregate Dynamics

In order to investigate the aggregate dynamics of the economy, we introduce aggregate household variables per effective labor by integrating over all living individuals and dividing by the product of technological level and the labor force of the economy. Analogously to equation (7) we define under slight abuse of notation per effective labor household variables, $x(t)$, and aggregate household variables per capita, $\bar{x}(t)$,

$$x(t) \equiv \frac{\int_{t-T}^{t} x(t,s)\gamma\exp[\nu s]\,ds}{A(t)\int_0^1 l(t,i)\,di}\ , \qquad \bar{x}(t) \equiv \frac{\int_{t-T}^{t} x(t,s)\gamma\exp[\nu s]\,ds}{N(t)}\ , \tag{9}$$

where $x(t,s)$ stands for the individual household variables consumption $c(t,s)$ and assets $b(t,s)$.

The economy consists of three markets: the labor market, the capital market and the consumption good market. We assume the economy to be in market equilibrium at all times $t$. In consequence, labor demand equals the population size, i.e., $\int_0^1 l(t,i)\,di = N(t)$, and capital in terms of effective labor equals aggregate assets in terms of effective labor, i.e., $k(t) = b(t)$. Then, the aggregate dynamics imply[6]

$$\frac{\dot{c}(t)}{c(t)} = \sigma^H\big[r(t) - \rho^H\big] - (\nu+\xi) - \frac{\Delta c(t)}{c(t)}\ , \tag{10a}$$

$$\dot{k}(t) = f\big(k(t)\big) - (\nu+\xi)k(t) - c(t)\ , \tag{10b}$$

[6] Note that $\dot{x}(t) = -(\nu+\xi)x(t) + \exp[-(\nu+\xi)t]\int_{t-T}^{t}\dot{x}(t,s)\gamma\exp[\nu s]\,ds + \gamma\left[x(t,t) - \frac{x(t,t-T)}{\exp[(\nu+\xi)T]}\right]\exp[-\xi t]$.

where the term[7]

$$\Delta c(t) \equiv \frac{\gamma \exp[\nu(t-T)]c(t,t-T) - \gamma \exp[\nu t]c(t,t)}{\exp[\nu t]\exp[\xi t]} \ . \tag{10c}$$

captures the difference in aggregate consumption per effective labor between the generation born and the generation dying at time $t$.

### 3.4 Steady State

Our analysis will concentrate on the long-run steady state growth path of the economy, in which both consumption per effective labor and capital per effective labor are constant over time, i.e., $c(t) = c^\star$, $k(t) = k^\star$. From equations (8) follows that in the steady state the interest rate $r(t) = r^\star \equiv f'(k^\star)$ is constant and the wage $w(t)$ grows at the rate of technological progress $\xi$. The wage relative to the technology level is constant in the steady state

$$w^\star \equiv \left.\frac{w(t)}{\exp[\xi t]}\right|_{k=k^\star} = \left[f(k^\star) - f'(k^\star)k^\star\right] \ . \tag{11}$$

For $T \in \mathbb{R}_{++}$ we define the function $Q_T : \mathbb{R} \to \mathbb{R}_+$ as

$$Q_T(r) \equiv \frac{1 - \exp[-rT]}{r} \ , \quad \forall\, r \neq 0 \ , \tag{12}$$

and $Q_T(0) \equiv T$. $Q_T(r)$ can be interpreted as the present value of an annuity received over $T$ years, at the discount rate $r$. Properties of the function $Q_T$ are summarized in Lemma 1 in appendix A.10. Expressing steady state consumption and wealth of individual households relative to the technology level returns functions that only depend on the household's *age* $a \equiv t - s$:

$$c^\star(a) \equiv \left.\frac{c(t,s)}{\exp[\xi t]}\right|_{k=k^\star} = w^\star \frac{Q_T(r^\star - \xi)}{Q_T\big(r^\star - \sigma^H(r^\star - \rho^H)\big)} \exp\left[\big(\sigma^H(r^\star - \rho^H) - \xi\big)a\right] \ , \tag{13a}$$

$$\begin{aligned} b^\star(a) \equiv \left.\frac{b(t,s)}{\exp[\xi t]}\right|_{k=k^\star} &= w^\star Q_a\big(r^\star - \sigma^H(r^\star - \rho^H)\big) \ \exp[(r^\star - \xi)a] \\ &\times \left[\frac{Q_a(r^\star - \xi)}{Q_a\big(r^\star - \sigma^H(r^\star - \rho^H)\big)} - \frac{Q_T(r^\star - \xi)}{Q_T\big(r^\star - \sigma^H(r^\star - \rho^H)\big)}\right] \ . \end{aligned} \tag{13b}$$

Figure 2 illustrates these steady state paths for individual consumption and assets in terms of the technological level of the economy.[8] The individual consumption path grows exponentially over the lifetime of each generation. Individual household assets follow an inverted U-shape, i.e., households are born with no assets, accumulate assets in their youth and consume their

[7] Note that $\Delta c(t)$ includes via $c(t, t-T)$ and $c(t,t)$ all values of $k(s)$ for $s \in [t-T, t+T]$. Thus, (10) defines a system of integro-differential equations. In the steady state, however, $\Delta c(t)/c(t) = \sigma^H\big[r^\star - \rho^H\big] - (\nu + \xi)$, where $r^\star$ denotes the steady state interest rate.

[8] The calculations use the following model specifications: $f(k) = k^\alpha$, $\alpha = 0.3$, $\rho = 3\%$, $\sigma = 1$, $\xi = 1.5\%$, $\nu = 0$, $T = 50$.

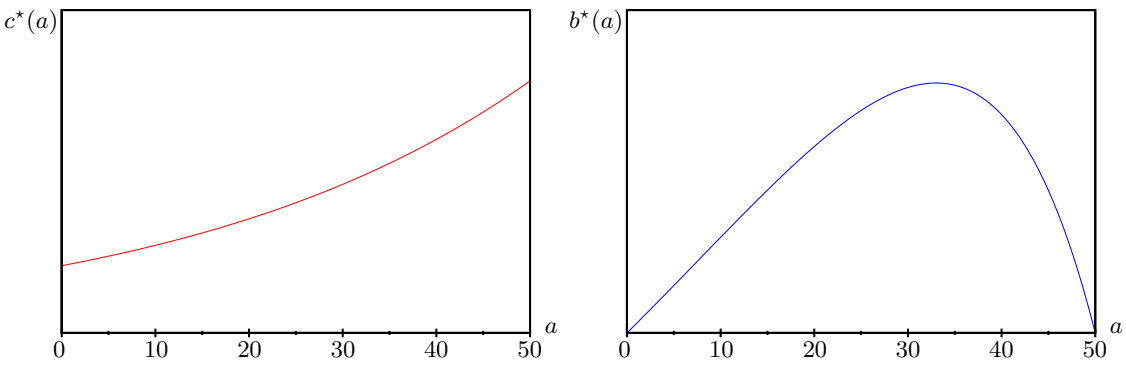


**Figure 2:** Steady state paths of consumption (left) and asset (right) for individual households over age.

wealth towards their death.

Applying the aggregation rule (9), we obtain for the aggregate values per effective labor

$$c^\star = w^\star \frac{Q_T(r^\star - \xi)}{Q_T(\nu)} \frac{Q_T(\nu + \xi - \sigma^H(r^\star - \rho^H))}{Q_T(r^\star - \sigma^H(r^\star - \rho^H))} \ , \tag{14a}$$

$$\begin{aligned} b^\star = {} & \frac{w^\star}{r^\star - \xi} \left[ \frac{Q_T(\xi + \nu - r^\star)}{Q_T(\nu)} - 1 \right] - \frac{w^\star}{r^\star - \sigma^H(r^\star - \rho^H)} \\ & \times \frac{Q_T(r^\star - \xi)}{Q_T(\nu)} \frac{Q_T(\xi + \nu - r^\star) - Q_T(\xi + \nu - \sigma^H(r^\star - \rho^H))}{Q_T(r^\star - \sigma^H(r^\star - \rho^H))} \ . \end{aligned} \tag{14b}$$

The following proposition guarantees the existence of a non-trivial steady state for a large class of production functions, in particular, CES-production functions.

**Proposition 1 (Existence of the steady state)**
*There exists a $k^\star > 0$ solving equations (8) and (14) with $b^\star = k^\star$ if*

$$\lim_{k \to 0} -k f''(k) > 0 \ . \tag{15}$$

The proof is given in the appendix.

In the proof of Proposition 1 we show that steady states may be equal to or larger than the golden rule capital stock $k^{gr}$, which is implicitly defined by $r^{gr} \equiv \nu + \xi = f'(k^{gr})$. As our aim is to compare the decentralized OLG with an ILA economy, we are particularly interested in steady states with $k^\star < k^{gr}$.[9]

**Definition 1 (Decentralized OLG economy)**

*(i) The set $\Gamma \equiv \{f, \xi, \nu, \sigma^H, \rho^H, T\}$ defines a decentralized OLG economy.*

*(ii) $\Gamma^\star \in \{\Gamma \,|\, \exists\, k^\star$ with $0 < k^\star < k^{gr}\}$ defines a decentralized OLG economy with a dynamically*

[9] In the ILA economy only steady states $k^\star < k^{gr}$ may occur.

*efficient capital stock $k^\star < k^{gr}$. For an economy $\Gamma^\star$ we refer by $k^\star$ and $r^\star$ to a steady state satisfying this condition.*

The following proposition shows the existence of dynamically efficient economies $\Gamma^\star$. Analogously to d'Albis (2007), we introduce the share of capital in output, $s(k)$, and the elasticity of substitution between capital and labor, $\epsilon(k)$,

$$s(k) \equiv \frac{kf'(k)}{f(k)} \ , \qquad \epsilon(k) \equiv -\frac{f(k) - f'(k)k}{k^2 f''(k)} \ . \tag{16}$$

**Proposition 2 (Existence and uniqueness of dynamically efficient steady states)**
*Given that condition* (15) *holds, there exists a steady state with $k^\star < k^{gr}$ if*

$$\frac{k^{gr}}{f(k^{gr}) - r^{gr}k^{gr}} > \frac{Q_T'(\nu)}{Q_T(\nu)} - \frac{Q_T'((\nu+\xi)(1-\sigma^H) + \sigma^H\rho^H)}{Q_T((\nu+\xi)(1-\sigma^H) + \sigma^H\rho^H)} \ . \tag{17}$$

*There exists exactly one $k^\star < k^{gr}$ if*

$$s(k) \leq \epsilon(k) \qquad \textit{and} \qquad \frac{d}{dk}\left(\frac{s(k)}{\epsilon(k)}\right) \geq 0 \ , \tag{18a}$$

*and, in case that $\sigma^H > 1$,*

$$\rho^H < \frac{\sigma^H - 1}{\sigma^H}(\nu + \xi) \ . \tag{18b}$$

The proof is given in the appendix.

Although we cannot solve the implicit equation $k^\star = b^\star$ analytically and, therefore, cannot calculate the steady state interest rate $r^\star$, the following proposition determines a lower bound of the steady state interest rates in a dynamically efficient OLG economy.

**Proposition 3 (Lower bound of steady state interest rate)**
*For any economy $\Gamma^*$ (which implies $r^\star > \nu + \xi$) holds*

$$r^\star > \rho^H + \frac{\xi}{\sigma^H} \ .$$

The proof is given in the appendix.

The lower bound of the steady state interest rate in the decentralized OLG economy will play an important role for the comparison with the ILA economy.

## 4 Infinitely Lived Agent Economy and Observational Equivalence

As intergenerational trade-offs are mostly discussed in ILA frameworks rather than in OLG models, we investigate how the macroeconomic observables of an OLG and ILA economy relate

to each other. Therefore, we first introduce the ILA model and then define observational equivalence between two economies. Whenever we compare two different model structures in this paper we assume that population growth and the production side of the economy are identical.

Variables of the ILA model that are not exogenously fixed to its corresponding counterparts in the OLG model are indexed by a superscript $^R$. The ILA model abstracts from individual generations' life cycles only considering *aggregate* consumption and asset holdings. In the ILA model optimal consumption and asset paths per capita are derived by maximizing the discounted stream of instantaneous utility of consumption per capita weighted by population size

$$U^R \equiv \int_0^\infty N(t) \frac{\bar{c}^R(t)^{1-\frac{1}{\sigma^R}}}{1-\frac{1}{\sigma^R}} \exp\big[-\rho^R t\big]\, dt \ , \tag{19}$$

subject to the budget constraint

$$\dot{b}^R(t) = \big[r^R(t) - \xi - \nu\big] b^R(t) + \frac{w^R(t)}{A(t)} - c^R(t) \ , \tag{20}$$

and the transversality condition

$$\lim_{t\to\infty} b(t) \exp\left[-\int_0^t r^R(t')\, dt' + (\xi+\nu)t\right] = 0 \ . \tag{21}$$

Maximizing (19) subject to (20) and (21) yields the well known Euler equation of the ILA model

$$\frac{\dot{c}^R(t)}{c^R(t)} = \sigma^R\big[r^R(t) - \rho^R\big] - \xi \ . \tag{22a}$$

Making use of equation (22a) we know that in a steady state the transversality condition translates into

$$\rho^R > \left(1 - \frac{1}{\sigma^R}\right) \xi + \nu \ . \tag{22b}$$

In the following we assume that the transversality condition is met. Note that it is the strict version for the Ramsey agent of the dynamic efficiency condition for the household in the OLG economy.

Assuming markets to be in equilibrium at all times (i.e., $\int_0^1 l(t,i)\, di = N(t)$ and $k^R(t) = b^R(t)$), the dynamics of the capital stock per effective labor in the ILA economy reads

$$\dot{k}^R(t) = f\big(k^R(t)\big) - (\nu+\xi) k^R(t) - c^R(t) \ , \tag{22c}$$

which is formally equivalent to the corresponding equation (10b) of the OLG economy. To compare the different models we use the following definition:

**Definition 2 (Observational equivalence)**

*(i) Two economies $A$ and $B$ are* observationally equivalent *if coincidence in their current observable macroeconomic variables leads to coincidence of their future observable macroeconomic variables. Formally, if for any $c^A(0) = c^B(0)$ and $k^A(0) = k^B(0)$ it holds that $c^A(t) = c^B(t)$ and $k^A(t) = k^B(t)$ for all $t \geq 0$.*

*(ii) Two economies $A$ and $B$ are* observationally equivalent in steady state *if there exist $c^\star$ and $k^\star$ such that both economies are in a steady state.*

Note that observational equivalence in the steady state (ii) is weaker than general observational equivalence (i).

## 5 Decentralized OLG Versus Infinitely Lived Agent Economy

Now, we investigate under what conditions a decentralized OLG economy, as outlined in Section 3, is observationally equivalent to an ILA economy, as defined in Section 4. The following proposition states the necessary and sufficient condition:

**Proposition 4 (Decentralized OLG versus ILA economy)**

*(i) A decentralized OLG economy $\Gamma^\star$ and an ILA economy are observationally equivalent if and only if for all $t \geq 0$ the following condition holds:*

$$\rho^R = \frac{\sigma^H}{\sigma^R}\,\rho^H + \left(1 - \frac{\sigma^H}{\sigma^R}\right) r(t) + \frac{1}{\sigma^R}\left[\frac{\Delta c(t)}{c(t)} + \nu\right] \ . \tag{23}$$

*(ii) For any decentralized OLG economy $\Gamma^\star$ there exists an ILA economy that is observationally equivalent in the steady state.*

The proof is given in the Appendix.

Proposition 4 states that any decentralized OLG economy $\Gamma^\star$ is – at least in the steady state – observationally equivalent to an ILA economy for an appropriate choice of $(\sigma^R, \rho^R)$. Note that $(\sigma^R, \rho^R)$ is, in general, not uniquely determined by (23).

If we assume that the intertemporal propensity to smooth consumption between two periods is the same for the households in the OLG and the ILA economy, i.e., $\sigma^H = \sigma^R$, we obtain the following corollary.

**Corollary 1 (Identical intertemporal elasticity of substitution)**
*For $\sigma^R = \sigma^H$ condition* (23) *reduces to*

$$\rho^R = \rho^H + \frac{1}{\sigma^R}\left[\frac{\Delta c(t)}{c(t)} + \nu\right] \ . \tag{24}$$

To understand why the pure rates of time preference in the ILA economy differs from the observationally equivalent OLG economy, we analyze the two terms in brackets on the right-hand side of equation (24). The first term in brackets captures the difference in consumption between the cohort dying and the cohort just born relative to aggregate consumption. The term is a consequence of the fact that every individual in the OLG model plans his own life cycle, saving while young and spending while old. If there is no population growth, i.e., $\nu = 0$ ($\gamma = 1/T$), individual consumption growth is higher than aggregate consumption growth and the term is always positive. More generally the following proposition states that the first term is positive if and only if individual consumption grows faster than aggregate consumption.[10]

**Proposition 5 (Sign of $\Delta c(t)/c(t)$)**
*For any decentralized OLG economy* $\Gamma^\star$ $\Delta c(t)/c(t) > 0$ *holds if and only if*

$$\frac{\dot{c}(t,s)}{c(t,s)} > \frac{\dot{\bar{c}}(t)}{\bar{c}(t)} + \nu \quad \textit{for all } s \in [t-T, t] \ . \tag{25}$$

**Proof:** The equivalence between $\Delta c(t)/c(t) > 0$ and (25) is obtained by substituting the individual household's Euler equation (4) into the aggregate Euler equation (10a), recalling that $\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\bar{c}}(t)}{\bar{c}(t)} - \xi$ according to (9), and solving for $\Delta c(t)/c(t)$. □

Note that the right hand side of inequality (25) represents the growth rate of aggregate consumption.

The second term in brackets of equation (24) reflects that instantaneous utility in the ILA model is weighted by population size. Hence, for a growing population future consumption receives an increasing weight in the objective function. A corresponding weighting does not occur in the decentralized OLG economy, where all households only maximize own lifetime utility. As a consequence, the time preference rate of an observationally equivalent ILA must be higher to compensate for the greater weights on future consumption.

It follows immediately from Proposition 5 that for $\sigma^R = \sigma^H$ both effects in equation (24) together yield $\rho^R > \rho^H$ whenever $\dot{c}(t,s)/c(t,s) > \dot{\bar{c}}(t)/\bar{c}(t)$, i.e., individual consumption growth dominates growth per capita. The following corollary shows that the latter condition always holds in the steady state and extends the analysis to the general case in which $\sigma^H \neq \sigma^R$.

**Corollary 2 (Comparing time preference rates)**
*Suppose a decentralized OLG economy* $\Gamma^\star$ *is observationally equivalent in the steady state to an ILA economy. Then the following statements hold:*

*(i)* $\sigma^R = \sigma^H \quad \Rightarrow \quad \rho^R > \rho^H$ .

*(ii) In general,*

$$\rho^R > \rho^H \quad \Leftrightarrow \quad \sigma^R > \sigma^H \left[1 + \frac{1}{\xi}\left(\frac{\Delta c(t)}{c(t)} + \nu\right)\right]^{-1} \ . \tag{26}$$

[10] Equation (25) holds for all $s \in [t-T, t]$ if and only if it holds for some $s$.

The proof is given in the Appendix.

Equipping an ILA with a lower intertemporal substitutability than the household in the decentralized OLG economy would ceteris paribus increase the steady state interest rate in the ILA economy (as opposed to the situation whith coinciding elasticities). In order to match the same observed interest rate as before, the ILA's rate of time preference has to be lower. Thus, the time preference relation can flip around if picking the intertemporal elasticity of substitution of the ILA sufficiently below that of the household in the decentralized OLG economy (note that $[\cdot]^{-1} < 1$).

## 6 Utilitarian OLG Versus Infinitely Lived Agent Economy

Consider an OLG economy, which is governed by a social planner maximizing a social welfare function. In this section, we investigate the conditions under which this economy is observationally equivalent to an ILA economy. We assume a utilitarian social welfare function in which the social planner trades off the weighted lifetime utility of different generations. The weight consists of two components. First, the lifetime utility of the generation born at time $s$ is weighted by cohort size. Second, the social planner exhibits a social rate of time preference $\rho^S > 0$ at which he discounts the expected lifetime utility at birth for generations born in the future.[11]

Assuming that the social planner maximizes social welfare from $t = 0$ onward, the social welfare function consists of two parts: (i) the weighted integral of the *remaining* lifetime utility of all generations alive at time $t = 0$, and (ii) the weighted integral of all future generations

$$\begin{aligned} W \equiv & \int_{-T}^{0} \left\{ \int_{0}^{s+T} \frac{c(t,s)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \exp\left[ -\rho^H (t-s)\right] dt \right\} \gamma \exp[\nu s] \exp[-\rho^S s] ds \\ & + \int_{0}^{\infty} \left\{ \int_{s}^{s+T} \frac{c(t,s)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \exp\left[ -\rho^H (t-s)\right] dt \right\} \gamma \exp[\nu s] \exp[-\rho^S s] ds \ . \end{aligned} \tag{27a}$$

The term in the first curly braces is the (remaining) lifetime utility $U(s)$ of a household born at time $s$, as given by equation (1), the functional form of which is a given primitive for the social planner. The term $\gamma \exp[\nu s]$ denotes the cohort size of the generation born at time $s$.

[11] We examine the discounted utilitarian social welfare function of, e.g., Calvo and Obstfeld (1988), Burton (1993) and Marini and Scaramozzino (1995), as it represents the de facto standard in the economic literature. For a general criticism of discounted utilitarianism, as also employed in the climate change debate by Nordhaus (2007) and Stern (2007), see, e.g., Sen and Williams (1982) and Asheim and Mitra (2010). Calvo and Obstfeld (1988) show that social welfare functions which do not treat all present and future generations symmetrically, i.e., discount lifetime utility to the same point of reference (here the date of birth), may lead to time-inconsistent optimal plans.

Changing the order of integration and replacing $t-s$ by age $a$, we obtain

$$W = \int_0^\infty \left\{ \int_0^T \frac{c(t,t-a)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \gamma \exp\left[(\rho^S - \rho^H - \nu)a\right] da \right\} \exp\left[(\nu - \rho^S)t\right] dt \ . \tag{27b}$$

In the following, we consider two different scenarios. In the *unconstrained* utilitarian OLG economy, a social planner maximizes the social welfare function (27b) directly controlling investment and household consumption. Thus, the social planner is in command of a centralized economy. In contrast, in the *constrained* utilitarian OLG economy the social planner relies on a market economy, in which the households optimally control their savings and consumption maximizing their individual lifetime utility (1). In this second scenario, the social planner is constrained to influencing prices by a tax/subsidy regime in order to maximize the social welfare function (27b).

### 6.1 Unconstrained Utilitarian OLG Economy

We determine the unconstrained social planner's optimal allocation by maximizing (27b) subject to the budget constraint (10b) and the transversality condition

$$\lim_{t\to\infty} k(t) \exp\left[ - \int_0^t f'(k(t'))\, dt' + (\xi + \nu)t \right] = 0 \ . \tag{28}$$

Following the approach of Calvo and Obstfeld (1988), we interpret the unconstrained social planner's optimization problem as two nested optimization problems. The first problem is obtained by defining

$$V\left(\bar{c}(t)\right) \equiv \max_{\{c(t,t-a)\}_{a=0}^T} \int_0^T \frac{c(t,t-a)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \gamma \exp\left[(\rho^S - \rho^H - \nu)a\right] da \ , \tag{29}$$

subject to

$$\int_0^T c(t,t-a)\gamma \exp[-\nu a] da \ \leq \ \bar{c}(t) \ . \tag{30}$$

The solution to this maximization problem is the social planner's optimal distribution of consumption between all generations alive at time $t$.

**Proposition 6 (Optimal consumption distribution for given time $t$)**
*The optimal solution of the maximization problem* (29) *subject to condition* (30) *is*

$$c(t,t-a) = \bar{c}(t) \frac{Q_T(\nu)}{Q_T(\nu + \sigma^H(\rho^H - \rho^S))} \exp\left[ - \sigma^H (\rho^H - \rho^S) a \right] \ . \tag{31}$$

*As a consequence, all households receive the same amount of consumption at time t irrespective of age for $\rho^H = \rho^S$, and receive less consumption the older (younger) they are at a given time*

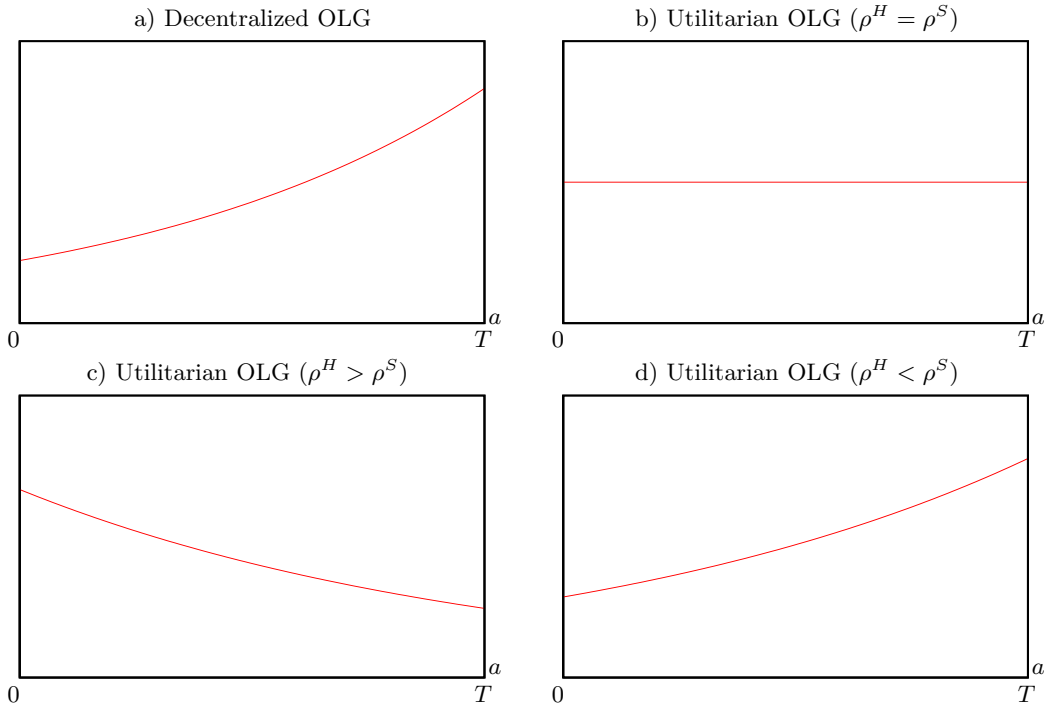


**Figure 3:** Distribution of consumption across all generations alive at given time $t$ dependent on age $a$ for the decentralized OLG and three different utilitarian OLGs.

*$t$ for $\rho^H > \rho^S$ ($\rho^H < \rho^S$).*

The proof is given in the appendix.

Proposition 6 states that the difference between the households' rate of time preference $\rho^H$ and the social rate of time preference $\rho^S$ determines the social planner's optimal distribution of consumption across households of different age at some given time $t$. In particular, if $\rho^H > \rho^S$ the consumption profile with respect to age is qualitatively opposite to that of the decentralized solution at any time $t$, as following from the Euler equation (4) and illustrated in Figure 3.[12] That is, in the social planner's solution households receive less consumption the older they are, whereas they would consume more the older they are in the decentralized OLG economy. The intuition for this result is as follows. The social planner weighs the lifetime utility of every individual discounted to the time of birth. Thus, the instantaneous utility at time $t$ of those who are younger (born later) is discounted for a relatively longer time at the social planner's time preference (before birth) and for a relatively shorter time by the individual's time preference (after birth) than is the case for the instantaneous utility at time $t$ of those who are older (born earlier). For $\rho^H > \rho^S$ the social planner's time preference is smaller and, thus, the young

[12] We do not take up a stance on the relationship between the individual and the social rate of time preference, but merely hint at the resulting consequences. This is in line with Burton (1993) and Marini and Scaramozzino (1995), who argue that they represent profoundly different concepts and, thus, may differ. In fact, $\rho^H$ trades off consumption today versus consumption tomorrow *within* each generation, while $\rho^S$ trades off lifetime utilities *across* generations. If they are supposed to differ, then it is usually assumed that $\rho^H > \rho^S$ (see also Heinzel and Winkler 2011).

generation's utility at time $t$ receives higher weight.

Proposition 6 shows that the standard approach of weighted intergenerational utilitarianism poses a trade-off between *intertemporal* generational equity and *intratemporal* generational equity to the social planner whenever households exhibit a positive rate of time preference. Lifetime utilities of today's and future generations would receive equal weight if and only if the social rate of time preference were zero. Approaching this by a close to zero social time preference rate, $\rho^H > \rho^S \approx 0$ implies that at each point in time the young enjoy higher consumption than the old.[13] In contrast, an equal distribution of consumption among the generations alive is obtained if and only if social time preference matches individual time preference. However, a positive social rate of time preference comes at the expense of an unequal treatment of lifetime utilities of different generations. This trade-off practically vanishes only if the individuals' and the social planner's rates of time preference are both very close to zero. Such an equality trade-off can only be captured in an OLG model which explicitly considers the life cycles of different generations.

We now turn to the second part of the maximization problem, which optimizes $\bar{c}(t)$ over time. It is obtained by replacing the term in curly brackets in equation (27b) by the left hand side of equation (29) resulting in

$$\max_{\{\bar{c}(t)\}_{t=0}^{\infty}} \int_0^{\infty} V\big(\bar{c}(t)\big) \exp[\nu t] \exp\big[-\rho^S t\big]\, dt \;, \tag{32}$$

subject to the budget constraint (10b). Observe that problem (32) is formally equivalent to an ILA economy with the instantaneous utility function $V\big(\bar{c}(t)\big)$ and the time preference rate $\rho^S$.[14] We obtain $V\big(\bar{c}(t)\big)$ by inserting the optimal consumption profile (31) into equation (29) and carrying out the integration

$$V\big(\bar{c}(t)\big) = \left[\frac{Q_T\big(\nu + \sigma^H(\rho^H - \rho^S)\big)}{Q_T(\nu)}\right]^{\frac{1}{\sigma^H}} \frac{\bar{c}(t)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}} \,. \tag{33}$$

The social planner's maximization problem (32) is invariant under affine transformations of the objective function (33), in particular, under a multiplication with the inverse of the term in square brackets. Thus, problem (32) is identical to the optimization problem in the ILA economy when setting the intertemporal elasticity of substitution $\sigma^R = \sigma^H$ and the time preference rate $\rho^R = \rho^S$.

**Proposition 7 (Unconstrained utilitarian OLG and ILA economy)**
*For an unconstrained utilitarian OLG economy, i.e., a social planner maximizing the social welfare function* (27b) *subject to the budget constraint* (10b) *and the transversality condition* (28)*, the following statements hold:*

[13] Note that for $\rho^S = 0$ the maximization problem of the unconstrained social planner is not well defined.
[14] Such an equivalence was already observed by Calvo and Obstfeld (1988).

*(i) An unconstrained utilitarian OLG economy is observationally equivalent to the ILA economy if and only if* $\sigma^R = \sigma^H$ *and* $\rho^R = \rho^S$.

*(ii) An unconstrained utilitarian OLG economy is observationally equivalent in the steady state to an ILA economy if and only if*

$$\rho^R = \rho^S + \xi \, \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H} \ . \tag{34}$$

The proof is given in the appendix.

Proposition 7 states that, maximizing the utilitarian social welfare function (27b) yields the same aggregate consumption and capital paths as maximizing the welfare (19) in the ILA model with $\sigma^R = \sigma^H$ and $\rho^R = \rho^S$. This result, however, does not imply that the unconstrained social planner problem can, in general, be replaced by an ILA model.

First, to derive the equivalence result, we have assumed a social planner who does *not* exhibit any preferences for smoothing lifetime utility across generations. The parameter $\sigma^H$ in equation (33) stems from the individuals' preferences to smooth consumption within the lifetime of each generation. It is therefore a given primitive to the social planner. Thus, the only normative parameter the social planner may choose is the social time preference rate $\rho^S$. It remains an open question for future research whether a different welfare functional for the unconstrained utilitarian social planner exists that permits a normative choice of $\sigma^S$ for the social planner and still delivers observational equivalence to an ILA model with $\rho^S = \rho^R$.

Second, in the ILA setting, the first-best solution can easily be decentralized, for example, via taxes that ensure the optimal path of the aggregate capital stock. However, this may not be the case for the unconstrained social planner's problem as the latter is also concerned about the intratemporal allocation of consumption across all generations alive at a certain point in time. Before, we investigate the decentralization of the social optimum in the next section, we compare the outcome of the OLG economy managed by the unconstrained social planner to that of a decentralized OLG economy. In all comparisons between a utilitarian and a decentralized OLG economy, we assume identical preferences of the individual households in both economies.

**Proposition 8 (Unconstrained utilitarian OLG and decentralized OLG)**

*(i) For any economy* $\Gamma^*$ *there exists an unconstrained utilitarian OLG that is observationally equivalent in the steady state. In such a steady state* $\rho^S > \rho^H$.

*(ii) In the steady state, an economy* $\Gamma^*$ *and an unconstrained utilitarian OLG exhibit the same allocation of consumption across the generations alive at each point in time if and only if they are observationally equivalent in the steady state.*

The proof is given in the appendix.

**Remark:** The converse of (i) is not true, as there exists no economy $\Gamma^*$ that would be observationally equivalent to an unconstrained utilitarian OLG with $\rho^S < \rho^H$.

Proposition 8 implies that an unconstrained utilitarian OLG economy exhibits the same aggregate steady state as the decentralized OLG economy if and only if the intratemporal distribution of consumption between all generations alive coincide. For this to hold, the social planner's rate of time preference has to be higher than the individual households' rate of time preference.

### 6.2 Constrained Utilitarian OLG Economy

As seen in Proposition 8, the optimal solution of a social planner maximizing (27b) subject to the budget constraint (10b) and the transversality condition (28) is, in general, not identical to the outcome of a decentralized OLG economy.[15] Thus, the question arises whether and if so how the social optimum is implementable in a decentralized market economy. Calvo and Obstfeld (1988) show that it is possible to implement the social optimum by a transfer scheme discriminating by date of birth $s$ and age $a$. Such a transfer scheme may be difficult to implement because of its administrative burden. In addition, it is questionable whether taxes and subsidies which are conditioned on age per se are politically viable.[16]

As a consequence, we consider a social planner that cannot discriminate transfers by age but may only influence prices via taxes and subsidies. In particular, we assume that the social planner may impose taxes/subsidies on capital and labor income. Let $\tau_r(t)$ and $\tau_w(t)$ denote the tax/subsidy on returns on savings and on labor income, respectively.[17] The individual households of the OLG economy base their optimal consumption and saving decisions on the effective interest rate $r^{\mathrm{e}}\big(t,\tau_r(t)\big)$ and the effective wage $w^{\mathrm{e}}\big(t,\tau_w(t)\big)$ defined by

$$r^{\mathrm{e}}\big(t,\tau_r(t)\big) = r(t) - \tau_r(t)\ , \tag{35a}$$

$$w^{\mathrm{e}}\big(t,\tau_w(t)\big) = w(t)\big[1-\tau_w(t)\big]\ . \tag{35b}$$

Then, the individual budget constraint reads

$$\dot{b}^{\mathrm{e}}(t,s) = r^{\mathrm{e}}\big(t,\tau_r(t)\big)\, b^{\mathrm{e}}(t,s) + w^{\mathrm{e}}\big(t,\tau_w(t)\big) - c^{\mathrm{e}}(t,s)\ . \tag{35c}$$

Given this budget constraint, individual households choose consumption paths which maximize lifetime utility (1). Thus, the optimal consumption path $c^{\mathrm{e}}\big(t,s,\{r(t'),\tau_r(t'),\ \tau_w(t')\}_{t'=s}^{s+T}\big)$ is a function of the paths of the interest rate $r(t)$ and the taxes $\tau_r(t)$ and $\tau_w(t)$.

Note that for a given path of the interest rate and given tax/subsidy schemes $\{r(t),\tau_r(t),\ \tau_w(t)\}_{t=s}^{s+T}$ the individual household's optimal paths of consumption and assets can be characterized as in the decentralized OLG economy by (2) and (4) when using $r^{\mathrm{e}}\big(t,\tau_r(t)\big)$ and

[15] Recall that we assume the individual preference parameters to be identical in both economies.

[16] See also the "Age Discrimination Act of 1975" for the US stating that "...no person in the United States shall, on the basis of age, be excluded from participation in, be denied the benefits of, or be subjected to discrimination under, any program or activity receiving Federal financial assistance." Note that programs like medicare use age as a proxy for the health condition and do not discriminate by age per se.

[17] Following the standard convention, $\tau_i(t)$ is positive if it is a tax and negative if it is a subsidy.

$w^{\mathrm{e}}(t, \tau_w(t))$ instead of $r(t)$ and $w(t)$, respectively. Applying the aggregation rule (9) yields aggregate consumption per effective labor $c^{\mathrm{e}}(t, \{r(t'), \tau_r(t'), \tau_w(t')\}_{t'=t-T}^{t+T})$. To analyze observational equivalence between such a constrained utilitarian OLG economy and an ILA economy, we have to restrict redistribution to mechanisms which do not alter the aggregate budget constraint (10b) of the economy. We consider the following redistribution scheme which yields a balanced government budget at all times

$$\tau_w(t)w(t) = -\tau_r(t)\bar{b}(t) \ . \tag{35d}$$

Under these conditions the social optimum is, in general, not implementable.

**Proposition 9 (Implementation of the social optimum)**
*The optimal solution of a social planner maximizing* (27b) *subject to the budget constraint* (10b) *and the transversality condition* (28) *is not implementable by a tax/subsidy regime satisfying* (35) *unless this solution is identical to the outcome of the unregulated decentralized OLG economy* $\Gamma^{\star}$*.*

The proof is given in the appendix.

Proposition 9 states that a constrained social planner who can only impose a tax/subsidy regime on interest and wages cannot achieve the first-best social optimum. The intuition is that the constrained social planner can achieve the socially optimal aggregate levels of capital and consumption, but cannot implement the socially optimal intratemporal distribution of consumption across generations living at the same time. The only exception occurs if the social optimum happens to be identical to the outcome of the unregulated OLG economy. In this case, there is no need for the social planner to interfere and, thus, it does not matter whether the social planner can freely re-distribute consumption among generations or is constrained to a self-financing tax/subsidy scheme. In all other cases, the constrained social planner will choose a tax path such as to achieve a second-best optimum. In consequence, Proposition 9 questions the validity of the ILA model in deriving distributional policy advice for a democratic government that, most likely, is not able to redistribute by age between the generations alive.

## 7 Stern vs. Nordhaus – A Critical Review of Choosing the Social Rate of Time Preference

A prime example for questions of intergenerational equity is the mitigation of anthropogenic climate change, as most of its costs accrue today while the benefits spread over decades or even centuries. The question of optimal greenhouse gas abatement has been analyzed in integrated assessment models combining an ILA economy with a climate model. Interpreting the ILA's utility function (19) as a utilitarian social welfare function, intergenerational equity concerns are closely related to the choice of intertemporal elasticity of substitution $\sigma^R$ and the rate of time preference $\rho^R$. This is illustrated well by Nordhaus (2007), who compares two runs

of his open source integrated assessment model DICE-2007. The first run uses his preferred specifications $\sigma^R = 0.5$ and $\rho^R = 1.5\%$. The second run employs $\sigma^R = 1$ and $\rho^R = 0.1\%$, which are the parameter values chosen by Stern (2007). These different parameterizations cause a difference in the optimal reduction rate of emissions in the period 2010–2019 of 14% versus 53% and a difference in the optimal carbon tax of 35$ versus 360$ per ton C.

The previous sections derived important differences between the OLG economy and an ILA model that have to be considered when evaluating climate change mitigation polices. In this section, we will relate these results to the positive and to the normative approach to social discounting.

### 7.1 The "positive" approach

Our paper provides the tools to critically re-examine the positive approach explicitly accounting for the finite lifespan of individuals living in an OLG economy. As described and illustrated in Figure 1 in Section 2, there are two steps associated with the positive approach: First the preference parameters of the ILA are set so that the ILA framework is observationally equivalent to the decentralized OLG. Second, the resulting ILA is interpreted as a utilitarian social planner OLG.

Concerning the first step, we showed in Proposition 4 and Corollary 2 that the rate of time preference of the ILA does not reflect the actual time preference rate of the (homogeneous) individuals in the decentralized OLG economy. The ILA model overestimates the rate of time preference for two reasons. First, the ILA model assumes that every individual plans for an infinite future when taking their market decisions. However, the households in the OLG economy only plan for their own lifespan when revealing their preferences on the market. Interpreting these decisions as if being taken with an infinite time horizon overstates the actual pure time preference. Second, the ILA model assumes that the representative consumer accounts for population growth by giving more weight to the welfare of the larger future population. If the households in the OLG economy dismiss this farsighted altruistic reasoning, the ILA approach once more overestimates individual time preference rates.

A numerical illustration shows how the inferred ILA preferences differ from actual household preferences. Assuming the elasticities $\sigma^H = \sigma^R = .5$ as in Nordhaus (2008) latest version of DICE, the ILA model implies a rate of time preference of the representative household (and social planner) of $\rho^R = 1.5\%$, while the individuals of the decentralized OLG economy exhibit a time preference of $\rho^H = -5.3\%$.[18] The surprising finding of a negative rate of time preference questions the plausibility of the above specifications. A simple sensitivity check suggests that increasing the intertemporal elasticity of substitution is most promising for resolving the nega-

[18] The calculation solves equation (14b) or, alternatively, $F(5.5\%) = J(5.5\%)$ in the notation introduced in the proof of Proposition 1. We choose the following exogenous parameters: capital share $\alpha = .3$, rate of technological progress $\xi = 2\%$, rate of population growth $\nu = 0\%$, lifetime $T = 50$ and interest rate $r = 5.5\%$.

tivity puzzle. The more recent asset pricing literature suggests an estimate of the intertemporal elasticity of substitution of $\sigma = 1.5$ which, in combination with a disentangled measure of risk attitude, also explains various asset prizing puzzles.[19] Adopting this estimate we find $\rho^H = 1.9\%$ for the households in the decentralized OLG economy and $\rho^R = 4.2\%$ in the observationally equivalent ILA economy. The wide-spread assumption of logarithmic utility ($\sigma = 1$) chosen in the Stern (2007) review implies that households have the same rate of pure time preference of $\rho^H = 0.1\%$ that the review chose for the social planner based on normative reasoning.

In the second step, the positive approach interprets the observationally equivalent ILA framework as a social planner economy. Proposition 7 indeed verifies that the ILA model can be a shortcut to a social planner maximizing intertemporal utilitarian welfare in an OLG economy. However, calibrating the ILA framework to the decentralized OLG economy and then interpreting the ILA framework as a social planner economy is not innocuous. Our full-fletched model of the unconstrained utilitarian planner in Section 6.1 reveals an implicit normative assumption hidden in this approach. Proposition 8 states that the condition $\rho^S > \rho^H$ holds whenever we calibrate the unconstrained utilitarian OLG economy to the decentralized unregulated market equilibrium. In consequence, the calibrated ILA economy contains the assumption that the intergenerational time preference rate of the social planner is higher than the individual time preference rate.

This assumption not only stands in sharp contrast to most of the literature on intergenerational ethics, it also calls for an explicit justification, as these preferences differ from those of the individuals in the economy. In a purely positive approach to climate change mitigation the social planner would capture only current observed preferences. In a decentralized OLG economy this would imply to terminate the time horizon of the social planner $T$ periods into the future, use individual households' rates of time preference, and introduce a weight that reflects the current individuals still alive at a given point of time in the future. In fact, climate change mitigation would not be optimal with such an approach, if the benefits of mitigation investments accrue beyond the lifetime $T$ of individual households.

If we acknowledge that climate change is a problem where individuals agree to adopt time horizons that exceed their own lifetime, we can adopt a longer or even infinite planning horizon. Then, however, on what grounds is it justified to increase the social planner's impatience over that of the individuals? The same question arises in the context of increasing impatience in order to match the fact that observed individuals in the decentralized OLG economy do not take account of future population growth. If one considers it adequate to endow the social planner with a welfare function giving more weight to larger (future) populations, then why would one increase the time preference rate to crowd out this effect? We do not provide an answer to these normative questions, but point to the normative content of the positive approach and its

[19] Vissing-Jørgensen and Attanasio (2003) estimate and Bansal and Yaron (2004) and Bansal et al. (2010) calibrate intertemporal substitutability to this value based on approaches employing Epstein and Zin (1991) preferences and Campbell's (1996) log-linearization of the Euler equation.

possible normative inconsistencies.

### 7.2 The normative approach

In a normative approach to social discounting it seems more natural to jump straight to an ILA model. By normatively justified assumptions the social planner exhibits an infinite planning horizon and particular values of the time preference rate and the intertemporal elasticity of substitution. It is obvious, however, that the ILA model cannot capture any distinction or interaction between intergenerational weighting and individual time preference. Nevertheless, Proposition 7 shows that a social planner fully controlling an OLG economy is observationally equivalent to an ILA economy if the parameters $\sigma^R$ and $\rho^R$ are appropriately chosen. In particular, the intertemporal path of aggregate consumption does not depend on the individual rate of time preference $\rho^H$, but only on the social planner's rate of time preference $\rho^S$. In fact, the time preference rate of the social planner coincides with the rate of time preference $\rho^R$ of the observationally equivalent ILA economy. This finding provides some support for Stern's (2007) normative approach to intergenerational equity in the ILA model.

However, the shortcut of setting up an ILA economy exhibits a number of caveats as questions of intergenerational equity are more complex than the ILA model reveals. First, according to Proposition 7, the interpretation of the time preference rate of the ILA economy as the time preference rate of a social planner in an observationally equivalent social planner OLG economy ($\rho^R = \rho^S$) requires that the intertemporal elasticity of substitution in the ILA economy be equal to that of the individual households in the OLG economy, i.e., $\sigma^R = \sigma^H$. This constraint, however, implies that the intertemporal elasticity of substitution is a primitive to the social planner and cannot be chosen to match particular normative considerations.[20]

Second, interpreting the ILA economy as a utilitarian social planner OLG neglects the *intratemporal* allocation of consumption across all generations alive at each point in time. The utilitarian OLG model allows us to explicitly analyze the social planner's optimal intratemporal distribution of consumption. As shown in Proposition 6, it depends on the difference between the social planner's and the individual households' rates of time preference. Usually, it is assumed that the normatively chosen social rate of time preference $\rho^S$ is smaller than the individual rate of time preference $\rho^H$.[21] According to Proposition 6, in this case the oldest generation receives least consumption while the newborns get most among all generations alive (see Figure 3, part c). In contrast, the decentralized OLG economy would distribute relatively more to the old (see Figure 3, part a). As a consequence, the standard discounted utilitarianism implies a trade-off between *intertemporal* and *intratemporal* generational equity whenever

[20] Note that the social welfare function (27b) we considered does not include any preferences for smoothing lifetime utility of different generations over time. Of course, such functional forms are conceivable but it is not clear whether and how such a utilitarian OLG economy translates into an observationally equivalent ILA economy.

[21] This assumption seems particularly reasonable if $\rho^S$ is close to zero. With respect to the Stern review, it implies that the individual households' time preference rates exceed $\rho^S = 0.1\%$.

households exhibit a positive rate of pure time preference. The aim of 'treating all generations alike' is therefore neither implemented easily in the economy nor captured in the utilitarian objective function.

Finally, there is an additional caveat, which applies to both the positive and the normative approach to social discounting. The ILA shortcut to the social planner OLG economy conceals that the first-best solution has to be implemented in a decentralized OLG instead of a Ramsey-Cass-Koopmans economy. In general, the social optimum not only requires re-distribution across time but also across different generations living at the same time. Apart, from the question whether consumption discrimination by age is justified on ethical grounds, it is questionable whether it is implementable. In Proposition 9 we show that, in general, a social planner whose policy instruments are limited to non-age-discriminating taxes and subsidies cannot implement the first-best solution. In fact, the first-best social optimum can only be achieved in the special case that it coincides with the outcome of the decentralized OLG economy without any regulatory intervention. Thus, the ILA economy, interpreted as an unconstrained social planner model, cannot capture this second-best aspect of optimal policies.

## 8 Conclusions

In the climate change debate intergenerational trade-offs are most often discussed within ILA frameworks, which are interpreted as a utilitarian social welfare function. In this paper, we analyzed to what extent these models can represent the relevant intertemporal trade-offs if an altruistic bequest motive is non-operative.

We showed under which conditions an ILA economy is observationally equivalent to (i) a decentralized OLG economy and (ii) an OLG economy in which a social planner maximizes a utilitarian welfare function. We found that preference parameters differ in the decentralized OLG and the observationally equivalent ILA economy. In general, pure time preference of an ILA planner is higher than pure time preference of the households in the observationally equivalent OLG economy. Moreover, in a normative setting, a utilitarian social planner faces a trade-off between intergenerational and intragenerational equity that cannot be captured in the ILA model. Finally, the limited implementability of the first best allocation can only be observed and discussed in the OLG context.

Our results have important implications for the recent debate on climate change mitigation and, more generally, for ILA based integrated assessment and cost benefit analysis that relies on the Ramsey equation. First, the positive approach to specify the social welfare function implicitly assumes that the time preference rate of the social planner exceeds the one of the individual households. Second, the ILA model does not capture the distribution of consumption among generations alive at a given point in time. The utilitarian OLG model implies that a more equal treatment of lifetime utilities between present and future generations can come at

the expense of a more unequal treatment of the generations alive at a given point in time – at least if individuals possess a positive rate of pure time preference. Thus, the utilitarian ILA in the normative approach to social discounting misses an important generational inequality trade-off. Third, the ILA approach overlooks a limitation in the implementability that arises if the intergenerational discount rate of the social planner in a utilitarian OLG economy does not coincide with the time preference rate of individual households. Then, the social optimum involves re-distribution among generations at each point in time, which would have to rely on age-discriminating taxes.

Our analysis employs two central assumptions. First, we assume selfish individual households. Although several empirical studies suggest that altruistic bequest motives are rather weak, extending the model to include different degrees of altruism is an interesting venue for future research. Second, part of our analysis assumes a specific utilitarian social welfare function. Although commonplace in the literature, this assumption drives some of our results, such as the trade-off between intra- and intergenerational equity. In particular, discounted utilitarianism in general has been questioned as an appropriate approach to deal with questions of intergenerational equity (e.g., Asheim and Mitra 2010).

# A Appendix

## A.1 Proof of Proposition 1

To prove the existence of a non-trivial steady state, i.e. $k^\star \neq 0$, we follow closely part (A) of the proof of Proposition 2 in Gan and Lau (2010). We re-write equation (14b) for $r^* \notin \{\xi, \nu+\xi\}$ as[22]

$$b^\star = \frac{w^\star}{r^\star - \nu - \xi}\left\{\frac{Q_T(r^\star-\xi)}{Q_T(\nu)}\frac{Q_T\big(\nu+\xi-\sigma^H(r^\star-\rho^H)\big)}{Q_T\big(r^\star-\sigma^H(r^\star-\rho^H)\big)} - 1\right\} . \tag{A.1}$$

We define the function $J:\mathbb{R}\to\mathbb{R}$ by

$$J(r) \equiv \frac{Q_T(r-\xi)}{Q_T(\nu)}\frac{Q_T\big(\nu+\xi-\sigma^H(r-\rho^H)\big)}{Q_T\big(r-\sigma^H(r-\rho^H)\big)} , \quad \forall\ r\in\mathbb{R} \tag{A.2}$$

for which Lemma 2 in Appendix A.10 summarizes some useful properties. Defining further

$$\phi(k) \equiv \frac{f(k)-f'(k)k}{f'(k)-\nu-\xi}\big[J\big(f'(k)\big)-1\big] , \tag{A.3}$$

the steady state is given by the solution of the equation $k=\phi(k)$, or equivalently

$$\lambda(k) \equiv \frac{J\big(f'(k)\big)-1}{f'(k)-\nu-\xi} - \frac{k}{f(k)-f'(k)k} = 0 . \tag{A.4}$$

Note that $\lambda(k)$ exhibits a removable pole at the golden rule capital stock $k^{gr}$ which is given by $f'(k^{gr})=\nu+\xi\equiv r^{gr}$. By defining

$$\lambda(k^{gr}) \equiv \lim_{k\to k^{gr}}\lambda(k) = J'\big(f'(k^{gr})\big) - \frac{k^{gr}}{f(k^{gr})-f'(k^{gr})k^{gr}} \tag{A.5}$$

where we use l'Hospital's rule (recognizing that $J\big(f'(k^{gr})\big)=1$), we establish that $\lambda(k)$ is a well-defined and continuous function on $k\in\mathbb{R}$. We now show that

$$\lim_{k\to 0}\lambda(k) = +\infty , \qquad \text{and} \qquad \lim_{k\to\infty}\lambda(k) = -\infty , \tag{A.6}$$

which proves the existence of $k^\star\in(0,\infty)$ with $\lambda(k^\star)=0$ or equivalently $\phi(k^\star)=k^\star$.

For $k\to 0$, $f'(k)$ tends to $\infty$, $f(k)-f'(k)k$ tends to 0 and $J\big(f'(k)\big)$ tends to $\infty$. The latter holds, as $\lim_{r\to\infty}J'(r)/J(r) > 0$ (see part (iii) and (v) of Lemma 2), which implies that

[22] The equivalence of equation (14b) and (A.1) is easily verified by multiplying over the terms in the denominator and expanding the resulting expressions. In addition, the domain of the functions making up the right hand side of equations (14b) and (A.1) can be extended to $r^*\in\{\xi,\nu+\xi\}$ by limit. Both right hand side functions are continuous and coincide for these points. Thus, the two equations are equivalent for all $r^\star$.

$\lim_{r\to\infty} J(r) = +\infty$ and $\lim_{r\to\infty} J'(r) = +\infty$. Applying l'Hospital's rule we obtain

$$\lim_{k\to 0} \lambda(k) = \lim_{k\to 0} J'(f(k)) - \frac{1}{f''(k)k} = +\infty \ , \tag{A.7}$$

as $\lim_{k\to 0} 1/(f''(k)k)$ is finite by virtue of assumption (15).

For $k \to \infty$, $f(k)$ tends to $\infty$ and $f'(k)$ tends to 0. Thus, the first summand of $\lambda(k)$ tends to $[1 - J(0)]/(\nu + \xi)$, which is finite. For the second summand observe that

$$\lim_{k\to\infty} \frac{f(k) - f'(k)k}{k} = \lim_{k\to\infty} \left[\frac{f(k)}{k} - f'(k)\right] = 0 \ . \tag{A.8}$$

As $f(k) - f'(k)k > 0$ for $k > 0$ this implies that $\lim_{k\to\infty} k/[f(k) - f'(k)k] = +\infty$ and, therefore, $\lim_{k\to\infty} \lambda(k) = -\infty$. □

### A.2 Proof of Proposition 2

To prove the proposition, we re-write the steady state condition (A.3) for $k \neq k^{gr}$ as

$$\frac{f(k) - (\nu + \xi)k}{f(k) - f'(k)k} = J\left(f'(k)\right) \ , \tag{A.9}$$

which allows to distinguish between efficient and inefficient steady states. Moreover, we discuss solutions to equation (A.9) in terms of the interest rate $r$ instead of the capital stock $k$. Therefore, we define

$$F(r) \equiv \frac{f\big(k(r)\big) - (\nu + \xi)k(r)}{f\big(k(r)\big) - f'\big(k(r)\big)k(r)} \ , \tag{A.10}$$

where $k(r) = f'^{-1}(r)$, which is well defined due to the strict monotonicity of $f'(k)$. Observe that $k'(r) = 1/f''\big(k(r)\big)$. The derivative of $F$ with respect to $r$ yields:

$$F'(r) = \frac{f'\big(k(r)\big) - (\nu + \xi)}{f''\big(k(r)\big)\left[f\big(k(r)\big) - f'\big(k(r)\big)k(r)\right]} + \frac{k(r)\left[f\big(k(r)\big) - (\nu + \xi)k(r)\right]}{\left[f\big(k(r)\big) - f'\big(k(r)\big)k(r)\right]^2} \ . \tag{A.11}$$

Then, for $r^\star \neq r^{gr}$, a steady state is given by the solution of the equation $F\left(r^\star\right) = J\left(r^\star\right)$.

>From (A.5) we observe that

$$J'\left(r^{gr}\right) = \frac{k^{gr}}{f\left(k^{gr}\right) - r^{gr}k^{gr}} = F'\left(r^{gr}\right) \ , \tag{A.12}$$

has to hold for $r = r^{gr}$ respectively $k = k^{gr}$ to be a steady state. In addition, we find for $r = r^{gr}$ that

$$F\left(r^{gr}\right) = 1 = J\left(r^{gr}\right) \ . \tag{A.13}$$

>From the proof of Proposition 1 follows that, given condition 15 holds, there exists an efficient steady state with $r^\star > r^{gr}$ and $k^\star < k^{gr}$ for $F'\left(r^{gr}\right) > J'\left(r^{gr}\right)$. This can be seen from equation (A.5), which implies $\lambda(k^{gr}) < 0$, and $\lim_{k\to 0}\lambda(k) = \lim_{r\to\infty}\lambda\big(k(r)\big) = +\infty$. The condition $F'\left(r^{gr}\right) > J'\left(r^{gr}\right)$ is equivalent to condition (17).

We now derive sufficient conditions such that there exists only one steady state $k^\star < k^{gr}$. Suppose that condition (15) holds, which guarantees existence of a dynamically efficient steady state. There exists only one steady state interest rate $r^\star$ with $r^\star > r^{gr}$ if and only if

$$\begin{aligned} & F'(r)|_{r=r^\star} < J'(r)|_{r=r^\star} \ , \quad \forall\, r^\star > r^{gr} \\ & \qquad \Leftrightarrow \left.\frac{F'(r)}{F(r)}\right|_{r=r^\star} < \left.\frac{J'(r)}{J(r)}\right|_{r=r^\star} \ , \quad \forall\, r^\star > r^{gr} \ . \end{aligned} \tag{A.14}$$

The second line holds, as $F(r) = J(r)$ for all $r = r^\star$. A sufficient condition for (A.14) to hold is that

$$\frac{d}{dr}\left(\left.\frac{F'(r)}{F(r)}\right|_{r=r^\star}\right) < 0 \qquad \wedge \qquad \frac{d}{dr}\left(\left.\frac{J'(r)}{J(r)}\right|_{r=r^\star}\right) > 0 \ , \quad \forall\, r^\star > r^{gr} \ . \tag{A.15}$$

>From part (ii) and (iv) of Lemma 2 we know that the second condition holds for all $r > r^{gr}$ if, in case that $\sigma > 1$, also condition (18b) holds.

$$\left.\frac{F'(r)}{F(r)}\right|_{r=r^\star} = \left[\frac{r-\nu-\xi}{f''(k(r))\left[f(k(r))-(\nu+\xi)k(r)\right]} + \frac{k(r)}{f(k(r))-rk(r)}\right]\Bigg|_{r=r^\star} \tag{A.16a}$$

$$= \left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{F(r)}\right) + \frac{k(r)}{f(k(r))-rk(r)}\right]\Bigg|_{r=r^\star} \tag{A.16b}$$

$$= \left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{J(r)}\right) + \frac{k(r)}{f(k(r))-rk(r)}\right]\Bigg|_{r=r^\star} \tag{A.16c}$$

$$= \underbrace{\frac{k(r)}{f(k(r))-rk(r)}}_{\equiv g_1(r)}\left[1-\left(1-\frac{1}{J(r)}\right)\underbrace{\frac{f(k(r))-rk(r)}{-k^2(r)f''(k(r))}}_{\equiv g_2(r)}\right]\Bigg|_{r=r^\star} . \tag{A.16d}$$

>From the second to the third line we employed $F(r) = J(r)$ for all $r = r^\star$. We show in the following that $g_1'(r) \leq 0$ and $g_2'(r) \geq 0$ are sufficient for $\frac{d}{dr}\left(\left.\frac{F'(r)}{F(r)}\right|_{r=r^\star}\right) < 0$.

First, observe from equation (A.3) that $J\left(r^\star\right) > 1$ for all $r^\star > r^{gr}$. As $J(r)$ is U-shaped on $r \in (r^{rg}, \infty)$ because of part (ii) and (iv) of Lemma 2 and $J\left(r^{gr}\right) = 1$, this implies that $J'\left(r^\star\right) > 0$ for all $r^\star > r^{gr}$.

Second, we show that $\frac{F'(r)}{F(r)}\Big|_{r=r^\star} > 0$ for all $r^\star > r^{gr}$ if $g_2'(r) \geq 0$. Observe that

$$\lim_{r^\star\to\infty} \frac{F'(r)}{F(r)}\bigg|_{r=r^\star} = \lim_{r\to\infty} \left[\frac{1}{k(r)f''(k(r))}\left(1-\frac{1}{J(r)}\right) + \frac{k(r)}{f(k(r))-rk(r)}\right] \tag{A.17a}$$

$$= \lim_{r\to\infty} \left[\frac{1}{k(r)f''(k(r))} + \frac{k(r)}{f(k(r))-rk(r)}\right] \tag{A.17b}$$

$$= \lim_{r\to\infty} \left[\frac{1}{k(r)f''(k(r))} - \frac{1}{k(r)f''(k(r))}\right] = 0 \ . \tag{A.17c}$$

In addition, we know that $g_1(r) > 0$ for all $r > 0$ and

$$\lim_{r\to\infty} g_1(r) = \lim_{r\to\infty} \frac{1}{k(r)f''(k(r))} > 0 \ . \tag{A.18}$$

The latter implies together with equation (A.17)

$$\lim_{r\to\infty} g_2(r)\left(1-\frac{1}{J(r)}\right) = 1 \ . \tag{A.19}$$

As $g_2(r)\left(1-\frac{1}{J(r)}\right)$ equals zero at $r = r^{gr}$ and is monotonically increasing in $r$ for $g_2'(r) \geq 0 = 0$, this implies that $F'(r)/F(r)|_{r=r^\star} > 0$ for all $r^\star > r^{gr}$. Then, we obtain for $g_1'(r) \leq 0$ and $g_2'(r) \geq 0$

$$\frac{d}{dr}\left(\frac{F'(r)}{F(r)}\bigg|_{r=r^\star}\right) = g_1'(r)\left[1-\left(1-\frac{1}{J(r)}\right)g - 2(r)\right] - g_1(r)g_2(r)\frac{J'(r)}{J^2(r)}$$
$$- g_1(r)g_2'(r)\left(1-\frac{1}{J(r)}\right) < 0 \ . \tag{A.20}$$

The conditions $s(k) \geq \epsilon(k)$ and $\frac{d}{dk}\left(\frac{s(k)}{\epsilon(k)}\right)$ are sufficient for $g_1'(r) \leq 0$ and $g_2'(r) \geq 0$. □

### A.3 Proof of Proposition 3

We show that $\sigma(r^\star - \rho^H) - \xi > 0$ is a necessary condition for aggregate assets $b^\star$ to be strictly positive in a dynamically efficient steady state, i.e., $(\sigma^H, \rho^H) \in \Gamma_{\Psi,T}$. As $b^\star = k^\star$ holds, this implies that for $k^\star > 0$ the steady state real interest rate must exceed $\rho^H + \frac{\xi}{\sigma}$.

The household's wealth, as given by equation (13b), can be re-written to yield

$$b^\star(a) = \frac{w^\star}{r^\star - \xi}\{\theta \exp\left[(\sigma(r^\star - \rho^H) - \xi)a\right] + (1-\theta)\exp[(r^\star - \xi)a] - 1\} \ , \tag{A.21}$$

with

$$\theta = \frac{1-\exp[-(r^\star - \xi)T]}{1-\exp[-(r^\star - \sigma^H(r^\star - \rho))T]} \ . \tag{A.22}$$

Assuming a dynamically efficient steady states implies that $r^\star - \xi > 0$ and we obtain from (A.22)

$$\theta \begin{cases} < 1, & \text{if } \sigma(r^\star - \rho^H) - \xi < 0 \\ = 1, & \text{if } \sigma(r^\star - \rho^H) - \xi = 0 \\ > 1, & \text{if } \sigma(r^\star - \rho^H) - \xi > 0 \end{cases} . \tag{A.23}$$

Thus, we can directly infer from (A.21) that $b^\star(a) = 0$ for all $a \in [0, T]$ for $\sigma(r^\star - \rho^H) - \xi = 0$. As all households hold no assets, the aggregate capital stock equals zero. To show that $\sigma(r^\star - \rho^H) - \xi < 0$ precludes strictly positive capital stocks, we analyze the second derivative of $b^\star(a)$

$$\begin{aligned} \frac{d^2\, b^\star(a)}{d\, a^2} = \frac{w^\star}{r^\star - \xi} \{ & \theta (\sigma(r^\star - \rho^H) - \xi)^2 \exp\left[ (\sigma(r^\star - \rho^H) - \xi) a \right] \\ & + (1 - \theta)(r^\star - \xi)^2 \exp[(r^\star - \xi) a] \} . \end{aligned} \tag{A.24}$$

For $\sigma(r^\star - \rho^H) - \xi < 0$, $\theta < 1$ holds, which implies that $\frac{d^2 b^\star(a)}{d\, a^2} > 0$. Hence, the household's wealth profile is strictly convex. Together with the boundary conditions $b^\star(0) = 0 = b^\star(T)$ this implies that all households possess non-positive wealth at all times. This, in turn, precludes $k^\star > 0$.

Further, it is obvious from (A.21) and (A.24) that $\sigma(r^\star - \rho^H) - \xi > 0$ does not contradict strictly positive wealth of the individual households and, therefore, is a necessary condition for $k^\star > 0$. □

### A.4 Proof of Proposition 4

(i) Both economies exhibit the same technology and rate of population growth by assumption and, thus, the market equilibria on the capital and the labor market imply that the equations of motion for the aggregate capital per effective labor (22c) and (10b) coincide. The remaining difference in the macroeconomic system dynamics is governed by the Euler equations (10a) and (22a) and by the transversality condition (21).

"⇒": Suppose the two economies are observationally equivalent, i.e., coincidence in the initial levels of consumption and capital imply coincidence at all future times. For this to hold the Euler equations (10a) and (22a) have to coincide giving rise to (23).

"⇐": If condition (23) holds, then also the Euler equations (10a) and (22a) coincide and the system dynamics of both economies is governed by the same system of two ordinary first order differential equations. The solution is uniquely determined by some initial conditions on $c$ and $k$. Thus, if the two economies coincide in the levels of consumption and capital at one point in time they also do so for all future times. In consequence, the two economies are observationally equivalent. Moreover, the capital stock is an equilibrium of $\Gamma^\star$ implying

$k^\star < k^{gr}$. As a consequence, the transversality condition for the ILA economy is satisfied and, thus, the described path is indeed an optimal solution.

(ii) Let $r^\star$ be the steady state interest rate of $\Gamma^\star$. Thus, all combinations of $(\rho^R, \sigma^R)$ which satisfy

$$r^\star = \rho^R + \frac{\xi}{\sigma^R} \ , \tag{A.25}$$

yield ILA economies which are observationally equivalent in the steady state. As for all $\Gamma^\star$, $r^\star < r^{gr}$ holds, also the transversality condition (21) is satisfied. □

### A.5 Proof of Corollary 2

(i) For the steady state, equation (10a) returns $\frac{1}{\sigma^H}\left[\frac{\Delta c(t)}{c(t)} + \nu\right] = r(t) - \rho^H - \frac{\xi}{\sigma^H}$ which, by Proposition 3, is strictly positive. Thus, by equation (24) $\rho^R - \rho^H > 0$.

(ii) From the respective Euler equations (10a) and (22a) we obtain the condition that

$$r - \frac{\xi}{\sigma^R} = \rho^R > \rho^H = r - \frac{1}{\sigma^H}\left[\frac{\Delta c(t)}{c(t)} + \nu + \xi\right] \tag{A.26}$$

$$\Leftrightarrow \frac{\sigma^H}{\sigma^R} < \frac{1}{\xi}\left[\frac{\Delta c(t)}{c(t)} + \nu + \xi\right] \tag{A.27}$$

which is equivalent to equation (26). □

### A.6 Proof of Proposition 6

The optimization problem (29) subject to condition (30) is equivalent to a resource extraction model (or an isoperimetrical control problem). We denote consumption at time $t$ of an individual of age $a$ by $\mathcal{c}(a) \equiv c(t, t-a)$ and define the stock of consumption left to distribute among those older than age $a$ by

$$y(a) = \bar{c}(t) - \int_0^a \mathcal{c}(a')\gamma \exp[-\nu a']\, da' \ . \tag{A.28}$$

Then, the problem of optimally distributing between the age groups is equivalent to optimally 'extracting' the consumption stock over age (instead of time). The equation of motion of the stock is $\frac{dy}{da} = -\mathcal{c}(a)\gamma \exp[-\nu a]$, the terminal condition is $y(T) \geq 0$, and the present value Hamiltonian reads

$$\mathcal{H} = \frac{\mathcal{c}(a)^{1-\frac{1}{\sigma^H}}}{1-\frac{1}{\sigma^H}}\gamma \exp\left[(\rho^S - \rho^H - \nu)a\right] - \lambda(a)\mathcal{c}(a)\gamma \exp[-\nu a] \ , \tag{A.29}$$

where $\lambda(a)$ denotes the co-state variable of the stock $y$. The first order conditions yield

$$\lambda(a) = \mathcal{c}(a)^{-\frac{1}{\sigma^H}} \exp\left[\left(\rho^S - \rho^H\right) a\right] \ , \tag{A.30a}$$

$$\dot{\lambda}(a) = 0 \ , \tag{A.30b}$$

which imply that

$$\mathcal{c}(a) = \mathcal{c}(0) \exp\left[\sigma^H(\rho^S - \rho^H)a\right] \ . \tag{A.31}$$

As $\lambda(T)$ is obviously not zero, transversality implies that $y(T) = 0$. Therefore, we obtain from equation (A.28), acknowledging $Q_T(\nu) = 1/\gamma$,

$$\mathcal{c}(0) = \bar{c}(t) \ \frac{Q_T(\nu)}{Q_T(\nu + \sigma^H(\rho^H - \rho^S))} \ , \tag{A.32}$$

which, together with equation (A.31), returns equation (31). □

### A.7 Proof of Proposition 7

(i) The equivalence of the unconstrained social planner problem and of the optimization problem in the ILA economy pointed out in relation to equations (32) and (33) implies the Euler equation of the unconstrained social planner economy

$$\frac{\dot{c}(t)}{c(t)} = \sigma^H \left[r(t) - \rho^S\right] - \xi \ . \tag{A.33}$$

For both economies the Euler equation implies that a time varying consumption rate also implies a time varying interest rate (and obviously so does a time varying capital stock).

For observational equivalence to hold, consumption and interest rate of the unconstrained utilitarian OLG economy have to coincide with that of the ILA economy, implying the following equality of the Euler equations

$$\begin{aligned} &\sigma^H \left[r(t) - \rho^S\right] - \xi = \sigma^R \left[r(t) - \rho^R\right] - \xi \\ &\quad \Leftrightarrow \sigma^R\rho^R - \sigma^H\rho^S = (\sigma^R - \sigma^H)r(t) \ . \end{aligned} \tag{A.34}$$

For a time varying interest rate this equation can only be satisfied if $\sigma^R = \sigma^H$ and $\rho^H = \rho^S$.

If $\sigma^R = \sigma^H$ and $\rho^H = \rho^S$ hold, the equivalence of the two problems was explained in relation to equations (32) and (33).

(ii) Existence of an observationally equivalent ILA economy implies that, first, the ILA economy has to be in a steady state as well and, second, that the steady state Euler equations have to

coincide implying

$$r = \rho^R - \frac{\xi}{\sigma^R} = \rho^S - \frac{\xi}{\sigma^H}$$
$$\Rightarrow \rho^R - \rho^S = \xi \, \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H} \; .$$

The same reasoning applies when starting from the ILA economy steady state and assuming an observationally equivalent unconstrained utilitarian OLG economy.

If equation (34) is satisfied and the unconstrained utilitarian OLG economy is in a steady state, equation (A.33) implies

$$r^S = \rho^S + \frac{\xi}{\sigma^H} \; . \tag{A.35}$$

Using equation (34) to substitute $\rho^S$ on the right hand side yields

$$r^S = \rho^R - \xi \, \frac{\sigma^R - \sigma^H}{\sigma^R \sigma^H} + \frac{\xi}{\sigma^H} = \rho^R + \frac{\xi}{\sigma^R} = r^R \; . \tag{A.36}$$

Thus, also the ILA economy is in a steady state (see Section 4) with coinciding interest rate. As the interest rates coincide, so does the capital stock and so do the consumption paths. Starting with the ILA steady state with interest rate $r^R$ yields a coinciding unconstrained utilitarian OLG steady state by the same procedure. □

### A.8 Proof of Proposition 8

(i) According to the proof of Proposition 7, the Euler equation of the unconstrained social planner solution is (A.33). In a steady state with interest rate $r^\star$ it is satisfied for any (obviously non-empty) set of preference parameters $\sigma^H$ and $\rho^S$ satisfying

$$\rho^S + \frac{\xi}{\sigma^H} = r^\star \; . \tag{A.37}$$

Moreover, by virtue of Proposition 3, $\rho^S = r^\star - \frac{\xi}{\sigma^H} > \rho^H$ holds. Note that for all decentralized economies $\Gamma^\star$ $r^\star < r^{gr}$. Hence, the same reasoning as in the proof of Proposition 4 can be applied to make sure that the budget constraints of the decentralized OLG and the unconstrained utilitarian social planner OLG coincide. The condition $r^\star < r^{gr}$ also implies that the social planner's transversality condition is satisfied.

(ii) Using (31), we can write the intratemporal allocation of consumption across the generations alive in steady state in the unconstrained utilitarian OLG as

$$c_S^\star(a) = \frac{c(t, t-a)}{\exp[\xi t]} = c^\star \frac{Q_{_T}(\nu)}{Q_{_T}(\nu + \sigma^H(\rho^H - \rho^S))} \exp[-\sigma^H(\rho^H - \rho^S)a] \; . \tag{A.38}$$

The intratemporal allocation of consumption in the decentralized OLG economy is given by (13a) and can be written as

$$c_d^\star(a) = c^\star \frac{Q_T(\nu)}{Q_T(\nu + \xi - \sigma^H(r_d^\star - \rho^H))} \exp[(\sigma^H(r_d^\star - \rho^H) - \xi)a] \ , \tag{A.39}$$

where $r_d^\star$ is the steady state interest rate of the decentralized OLG in which the households exhibit the same preference parameters as in the unconstrained utilitarian OLG economy.

$\Rightarrow$: Suppose that the allocation of consumption across all generations alive at each point is identical. For this to be the case, the following two equations have to hold simultaneously for all $a \in [0, T]$

$$\exp[-\sigma^H(\rho^H - \rho^S)a] = \exp[(\sigma^H(r_d^\star - \rho^H) - \xi)a] \ , \tag{A.40a}$$

$$\sigma^H(\rho^H - \rho^S) = \xi - \sigma^H(r_d^\star - \rho^H) \ . \tag{A.40b}$$

Minor mathematical transformations show that this only holds for

$$\rho^s = r_d^\star - \frac{\xi}{\sigma_H} \ . \tag{A.41}$$

This is the condition for the unconstrained utilitarian OLG and the decentralized OLG to be observationally equivalent in steady state.

$\Leftarrow$: Now suppose that the unconstrained utilitarian OLG and the decentralized OLG are observationally equivalent in steady state, i.e., equation (A.41) is satisfied.

Inserting $\rho^S$ as given by (A.41) into (A.38) yields

$$c_S^\star(a) = c^\star \frac{Q_T(\nu)}{Q_T(\nu + \xi - \sigma^H(r_d^\star - \rho^H))} \exp[(\sigma^H(r_d^\star - \rho^H) - \xi)a] \ , \tag{A.42}$$

which is identical to (A.39). Hence, observational equivalence in steady state is also sufficient for identical allocations across the generations alive in both economies. □

### A.9 Proof of Proposition 9

We show that the constrained social planner can implement the steady state social optimum with a tax/subsidy regime on interest and wages only if the steady states of the first-best optimum and the decentralized OLG economy coincide. This implies that the first-best solution is, in general, not implementable, as every first-best solution converges to a non-implementable steady state.

We show that for a given steady state, the intratemporal distribution of consumption coincides in the constrained and the unconstrained utilitarian OLG economy if and only if $\tau_r^\star = 0$. To see this consider an unconstrained utilitarian OLG economy in steady state. The household

problem in the constrained utilitarian OLG economy is identical to the household problem in the decentralized economy if we substitute $r(t)$ by $r^{\mathrm{e}}(t)$ and $w(t)$ by $w^{\mathrm{e}}(t)$. Solving for individual consumption and wealth in the steady states yields analogously to equations (13a) and (13b):

$$c^{\mathrm{e}\star}(a) \equiv \left.\frac{c^{\mathrm{e}}(t,s)}{\exp[\xi t]}\right|_{k=k^\star} = w^{\mathrm{e}\star}\frac{Q_T(r^{\mathrm{e}\star}-\xi)}{Q_T\big(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H)\big)}\exp\left[\big(\sigma^H(r^{\mathrm{e}\star}-\rho^H)-\xi\big)a\right] , \tag{A.43a}$$

$$\begin{aligned} b^{\mathrm{e}\star}(a) \equiv \left.\frac{b^{\mathrm{e}}(t,s)}{\exp[\xi t]}\right|_{k=k^\star} &= w^{\mathrm{e}\star}Q_a\big(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H)\big)\ \exp[(r^{\mathrm{e}\star}-\xi)a] \\ &\times\left[\frac{Q_a(r^{\mathrm{e}\star}-\xi)}{Q_a\big(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H)\big)}-\frac{Q_T(r^{\mathrm{e}\star}-\xi)}{Q_T\big(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H)\big)}\right] , \end{aligned} \tag{A.43b}$$

where $r^{\mathrm{e}\star} = r^{\mathrm{e}}(t)$ and $w^{\mathrm{e}\star} = w^{\mathrm{e}}(t)/\exp[\xi t]$, both evaluated at the steady state. Following the aggregation rule (9), we derive for aggregate steady state consumption and wealth:

$$c^{\mathrm{e}\star} = w^{\mathrm{e}\star}\frac{Q_T(r^{\mathrm{e}\star}-\xi)}{Q_T(\nu)}\frac{Q_T(\nu+\xi-\sigma^H(r^{\mathrm{e}\star}-\rho^H))}{Q_T\big(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H)\big)} , \tag{A.44a}$$

$$\begin{aligned} b^{\mathrm{e}\star} &= \frac{w^{\mathrm{e}\star}}{r^{\mathrm{e}\star}-\xi}\left[\frac{Q_T(\xi+\nu-r^{\mathrm{e}\star})}{Q_T(\nu)}-1\right]-\frac{w^{\mathrm{e}\star}}{r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H)} \\ &\times\frac{Q_T(r^{\mathrm{e}\star}-\xi)}{Q_T(\nu)}\frac{Q_T(\xi+\nu-r^{\mathrm{e}\star})-Q_T(\xi+\nu-\sigma^H(r^{\mathrm{e}\star}-\rho^H))}{Q_T\big(r^{\mathrm{e}\star}-\sigma^H(r^{\mathrm{e}\star}-\rho^H)\big)} . \end{aligned} \tag{A.44b}$$

Inserting equation (A.44a) into equation (A.43a), we obtain the following intratemporal distribution of consumption

$$c^{\mathrm{e}\star}(a) = c^{\mathrm{e}\star}\frac{Q_T(\nu)}{Q_T(\nu+\xi-\sigma^H(r^{\mathrm{e}\star}-\rho^H))}\exp\Big[\big(\sigma^H(r^{\mathrm{e}\star}-\rho^H)-\xi\big)a\Big] . \tag{A.45}$$

By virtue of equation (31), however, the steady state intertemporal distribution of consumption in the social optimum yields:

$$c^\star(a) = c^\star\frac{Q_T(\nu)}{Q_T(\nu-\sigma^H(\rho^S-\rho^H))}\exp\Big[\big(\sigma^H(\rho^S-\rho^H)\big)a\Big] . \tag{A.46}$$

Aggregate equivalence requires that $c^{\mathrm{e}\star} = c^\star$. Distributional equivalence at a point in time requires moreover that equation (A.45) and equation (A.46) coincide. Together these conditions imply that $\sigma^H(r^{\mathrm{e}\star}-\rho^H)-\xi = \sigma^H(\rho^S-\rho^H) \Leftrightarrow r^{\mathrm{e}\star} = \rho^S + \frac{\xi}{\sigma^H}$. Thus, by equation (A.37), it must be $r^{\mathrm{e}\star} = r^\star$ and therefore $\tau_r^\star = 0$. □

### A.10 Characteristics of the functions characterizing the steady state capital stock

**Lemma 1**

*The function $Q_T(r)$ defined in (12) satisfies:*

*(i) $Q_T(r) > 0$ for all $r \in \mathbb{R}$,*

*(ii)* $Q'_T(r) < 0$ *for all* $r \in \mathbb{R}$.

*The function*

$$q(r) \equiv \frac{Q'_T(r)}{Q_T(r)} = \frac{T}{\exp(rT) - 1} - \frac{1}{r} \ , \tag{A.47}$$

*satisfies*

*(iii)* $q(r) < 0$ *for all* $r \in \mathbb{R}$,

*(iv)* $\lim_{r\to\infty} q(r) = 0$ *and* $\lim_{r\to-\infty} q(r) = -T$ ,

*(v)* $q'(r) = q'(-r) > 0$ *for all* $r \in \mathbb{R}$,

*(vi)* $q'(r) > z^2 q'(zr)$ *for all* $r \in \mathbb{R}$, $z \in (0, 1)$,

*(vii)* $y^2 q'(yr) > z^2 q'(zr)$ *for all* $r \in \mathbb{R}$, $y > z \geq 1$,

*(viii)* $q''(r) < 0$ *for all* $r \in \mathbb{R}_{++}$.

**Proof:** (i) Obviously, $Q_T(r) > 0$ for all $r \neq 0$. In addition, $\lim_{r\to 0} Q_T(r) = T > 0$.
(ii) We obtain

$$Q'_T(r) = -\frac{1 - \exp[-rT](1 + rT)}{r^2} \ .$$

For all $r \neq 0$:

$$Q'_T(r) < 0 \quad \Leftrightarrow \quad \exp[-rT](1 + rT) < 1 \quad \Leftrightarrow \quad 1 + rT < \exp[rT] \ .$$

The last inequality holds as $x + 1 < \exp[x]$ for all $x \in \mathbb{R}$. In addition, $\lim_{r\to 0} Q'_T(r) = -\frac{T^2}{2} < 0$.
(iii) Follows directly from items (i) and (ii).
(iv) Follows directly from the definition (A.47).
(v) We obtain:

$$q'(r) = -\frac{1}{r^2} - \frac{T^2 \exp[-rT]}{(1 - \exp[-rT])^2} = \frac{1}{r^2} - \frac{T^2}{2(\cosh[rT] - 1)} \ .$$

For all $r \neq 0$:

$$q'(r) > 0 \quad \Leftrightarrow \quad 2(\cosh[rT] - 1) > r^2 T^2 \quad \Leftrightarrow \quad \cosh[rT] > 1 + \frac{r^2 T^2}{2} \ .$$

The last inequality holds as $\cosh[x] > 1 + \frac{x^2}{2}$ for all $x \in R$. In addition, $\lim_{r\to 0} q'(r) = \frac{T^2}{12} > 0$.
(vi) The statement holds if and only if:

$$\begin{aligned} q'(r) - z^2 q'(zr) = \frac{z^2 T^2}{2(\cosh[zrT] - 1)} - \frac{T^2}{2(\cosh[rT] - 1)} > 0 \\ \Leftrightarrow z^2(\cosh[rT] - 1) > \cosh[zrT] - 1 \ . \end{aligned}$$

To see that the last inequality holds, we employ the infinite series expansion of $\cosh[x]$:

$$z^2(\cosh[x]-1)-(\cosh[zx]-1)=z^2\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-1\right)-\left(\sum_{n=0}^{\infty}\frac{(zx)^{2n}}{(2n)!}-1\right)$$
$$=z^2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}-\sum_{n=1}^{\infty}\frac{(zx)^{2n}}{(2n)!}=\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\left(z^2-z^{2n}\right)>0\ .$$

The inequality holds, as the first summand is zero and all other terms are strictly positive for all $z\in(0,1)$. (vii) The statement holds if and only if:

$$y^2q'(yr)-z^2q'(zr)=\frac{z^2T^2}{2(\cosh[zrT]-1)}-\frac{y^2T^2}{2(\cosh[yrT]-1)}>0$$
$$\Leftrightarrow z^2(\cosh[yrT]-1)>y^2\cosh[zrT]-1\ .$$

Employing the infinite series expansion of $\cosh[x]$, we obtain

$$z^2(\cosh[yx]-1)-y^2(\cosh[zx]-1)=z^2\left(\sum_{n=0}^{\infty}\frac{(yx)^{2n}}{(2n)!}-1\right)-y^2\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-1\right)$$
$$=z^2\sum_{n=1}^{\infty}\frac{(yx)^{2n}}{(2n)!}-y^2\sum_{n=1}^{\infty}\frac{(yx)^{2n}}{(2n)!}=\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}z^2y^2\left(y^{2(n-1)}-z^{2(n-1)}\right)>0\ .$$

The inequality holds, as the first summand is zero and all other terms are strictly positive for all $y>z\geq 1$.
(viii) We obtain:

$$q''(r)=-\frac{2}{r^3}+\frac{2T^3\sinh[rT]}{(2\cosh[rT]-2)^2}=-2T^3\left(\frac{1}{(rT)^3}+\frac{\sinh[rT]}{(2\cosh[rT]-2)^2}\right)$$

Then, the statement holds if and only if $(\cosh[x]-2)^2>x^3\sinh[x]$. To see this, we employ the infinite series expansion of $\cosh[x]$ and $\sinh[x]$

$$\left(2\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}-2\right)^2-x^3\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=\left(2\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\right)^2-\sum_{n=0}^{\infty}\frac{x^{2n+4}}{(2n+1)!}$$
$$=4\left(\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\right)^2-\sum_{n=0}^{\infty}\frac{x^{2n+4}}{(2n+1)!}$$

Both series exhibit all even powers of $x$ starting with $x^4$:

$$x^4\left(\frac{4}{2!2!}-1\right)+x^6\left(\frac{2\cdot 4}{2!4!}-\frac{1}{3!}\right)+x^8\left(\frac{2\cdot 4}{2!6!}+\frac{4}{4!4!}-\frac{1}{5!}\right)+\cdots\geq 0\ .$$

The inequality holds as the first term is zero and all other terms are strictly positive for all $x\in\mathbb{R}_{++}$. □

**Lemma 2**

*For all $\xi, \nu \in \mathbb{R}_{++}$ the function $J$ defined in* (A.2) *satisfies*

*(i)* $J(r) > 0$.

*For all $\xi, \nu \in \mathbb{R}_{++}$ and $\sigma^H \in (0,1]$ the function $J$ satisfies*

*(ii)* $\frac{d}{dr}\left(\frac{J'(r)}{J(r)}\right) > 0$ *for all* $r \geq \xi$,

*(iii)* $\lim_{r\to\infty} \frac{J'(r)}{J(r)} = \sigma^H T$.

*For all $\xi, \nu \in \mathbb{R}_{++}$ and $\sigma^H > 1$ the function $J$ satisfies*

*(iv)* $\frac{d}{dr}\left(\frac{J'(r)}{J(r)}\right) > 0$ *for all* $r \geq \nu + \xi$ *and* $\rho^H < \frac{\sigma^H - 1}{\sigma^H}(\nu + \xi)$,

*(v)* $\lim_{r\to\infty} \frac{J'(r)}{J(r)} = T$.

**Proof:** (i) Follows immediately from $Q_T(r) > 0$ for all $r \in \mathbb{R}$ as shown in Lemma 1.
(ii) Using the definition (A.47), we obtain

$$\frac{J'(r)}{J(r)} = q(r-\xi) - \sigma^H q\left(\nu + \xi - \sigma^H\left(r - \rho^H\right)\right) - \left(1-\sigma^H\right) q\left(r - \sigma^H\left(r - \rho^H\right)\right) \ , \quad \text{(A.48a)}$$

and

$$\begin{aligned} M(r) \equiv \frac{d}{dr}\left(\frac{J'(r)}{J(r)}\right) &= \frac{J''(r)}{J(r)} - \left(\frac{J'(r)}{J(r)}\right)^2 \qquad \text{(A.48b)} \\ &= q'(r-\xi) + \left(\sigma^H\right)^2 q'\left(\nu + \xi - \sigma^H\left(r - \rho^H\right)\right) - \left(1-\sigma^H\right)^2 q'\left(r - \sigma^H\left(r - \rho^H\right)\right) \ . \end{aligned}$$

For $\sigma^H \in (0,1]$ set $x = r - \xi$ and restrict attention to all $x \geq 0$

$$\begin{aligned} M(x) &= q'(x) + \left(\sigma^H\right)^2 q'\left(\nu + \left(1-\sigma^H\right)\xi - \sigma^H\left(x - \rho^H\right)\right) \\ &\quad - \left(1-\sigma^H\right)^2 q'\left(\left(1-\sigma^H\right)x + \left(1-\sigma^H\right)\xi + \sigma^H\rho^H\right) \\ &> q'(x) - \left(1-\sigma^H\right)^2 q'\left(\left(1-\sigma^H\right)x + \left(1-\sigma^H\right)\xi + \sigma^H\rho^H\right) \\ &\geq q'(x) - \left(1-\sigma^H\right)^2 q'\left(\left(1-\sigma^H\right)x\right) \geq 0 \ . \end{aligned}$$

The first inequality holds due to part (v), the second inequality due to part (viii) and the last inequality due to part (vi) of Lemma 1.
(iii) Follows directly from equation (A.48a) and part (iv) of Lemma 1.
(iv) For $\sigma^H > 1$ and $\rho^H < \frac{\sigma^H - 1}{\sigma^H}(\nu + \xi)$ consider only $r \geq \nu + \xi$

$$\begin{aligned} M(r) &= q'(r-\xi) + \left(\sigma^H\right)^2 q'\left(\sigma^H r - \sigma^H\rho^H - (\nu+\xi)\right) - \left(\sigma^H - 1\right)^2 q'\left(\left(\sigma^H - 1\right)r + \sigma^H r\right) \\ &> \left(\sigma^H\right)^2 q'\left(\sigma^H r - \sigma^H\rho^H - (\nu+\xi)\right) - \left(\sigma^H - 1\right)^2 q'\left(\left(\sigma^H - 1\right)r + \sigma^H r\right) \\ &> \left(\sigma^H\right)^2 q'\left(\sigma^H r\right) - \left(\sigma^H - 1\right)^2 q'\left(\left(\sigma^H - 1\right)r\right) \geq 0 \end{aligned}$$

The first inequality holds due to part (v), the second inequality due to part (viii) and the last inequality due to part (vii) of Lemma 1.

(v) Follows directly from equation (A.48a) and part (iv) of Lemma 1. □

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