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Authors
Daganzo, Carlos F.
Laval, Jorge A.

Publication Date
2003-05-01

Institute of Transportation Studies
University of California at Berkeley

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RESEARCH REPORT
UCB-ITS-RR-2003-2

May 2003
ISSN 0192 4095

Moving Bottlenecks: A Numerical Method that Converges in Flows

Carlos F. Daganzo and Jorge A. Laval
Institute of Transportation Studies and Department of Civil and Environmental Engineering
University of California, Berkeley CA 94720
(May 6, 2003)

ABSTRACT

This paper presents a numerical method to model kinematic wave (KW) traffic streams containing slow vehicles. The slow vehicles are modeled discretely as moving boundaries that can affect the traffic stream. The proposed scheme converges in flows, densities and speeds without oscillations, and therefore can be readily used in situations where one wishes to model the effect of the traffic stream on the bottlenecks too. The approach is more accurate than Godunov's method in situations where the latter can be applied.

1. INTRODUCTION

This paper is a sequel to Daganzo and Laval (2003), which proposed a way to treat moving bottlenecks numerically within the context of kinematic wave (KW) theory. This reference showed that if the moving bottlenecks are replaced by sequences of fixed bottlenecks that are restricted to be on a space-time lattice, then the difference between the exact and approximate vehicle counts at any point in space-time are uniformly bounded by a quantity that tends to zero as the lattice spacing is reduced. Good features of this procedure are: (i) if exact solution methods are used to solve the approximate problem, the solution error in vehicle count is uniformly bounded; and (ii) "off-the-shelf" software can be used to solve the approximate problems. The main disadvantage is that the approximate flows, densities and speeds do not converge to the exact ones, even as vehicle counts do; thus, to provide estimates of these quantities, one needs to average their values over multiple cells. We present below a new numerical scheme that overcomes this disadvantage, albeit by doing away with the good points.

2. COMPOSITE-RIEMANN PROBLEMS

A building block of the proposed method is the solution of a class of initial value problems containing two moving bottlenecks, i and i+1, with linear trajectories. These problems will be called composite Riemann problems, or CRP's. The space-time geometry of a typical CRP and its input data are depicted in Fig. 1. Given are the following constants: (i) the locations of the bottlenecks at time zero (x_i and x_{i+1}); (ii) the initial vehicle numbers (N_i) at the location of the bottlenecks;[1] (iii) downstream and upstream densities (k_d and k_u); (iv) the speeds and maximum passing rates associated with each bottleneck (v_i, Q_i); and (v) a time of interest, Δt, such that bottleneck trajectories do not cross:

$$\Delta t \le (x_{i+1} - x_i)/(v_i - v_{i+1}); \quad \text{if} \quad v_i > v_{i+1} \qquad \text{(no crossing condition)}. \qquad (1)$$

It is assumed that the initial density is uniform in the three intervals demarcated by the bottleneck positions. The density in the middle interval is $(N_i - N_{i+1})/(x_{i+1} - x_i)$. For maximum generality, bottlenecks are allowed to travel faster than the traffic stream—they do not have to be embedded in it. We then look for the vehicle numbers, $N_i(t)$ and $N_{i+1}(t)$, on the trajectories of the bottlenecks at time $t = \Delta t$.

This is a well-posed problem in kinematic wave theory. Its solution can be obtained with standard recipes. For problems with piecewise linear data, as is the case with the CRP, the exact solution can

[1] As is conventional, vehicles are assumed to travel in the direction of increasing x and to be numbered in decreasing order with x. Thus, the Moskowitz function of vehicle number $N(t, x)$ is non-decreasing in t at every x.

always be obtained with a finite number of calculations if the fundamental diagram is piece-wise linear. Appendix A shows a simple procedure that can be used when the flow-density relation is triangular.

3. THE SOLUTION METHOD

The general procedure works on a rectangular space-time lattice with a fixed spatial spacing, Δx, and variable time spacing Δt. We assume that at the current discrete time point we know the vehicle numbers at all the lattice points and bottleneck locations, that the density is constant between points and that the bottleneck speeds are also constant during the ensuing time step. The objective is predicting the vehicle numbers at the new bottleneck locations and all lattice points. The recipes of Sec. 2 do the job if we treat mesh lines as moving bottlenecks with zero speed and maximum passing rate equal to the road capacity, Q_{max}. We simply need to ensure that the time step is short enough to ensure that (1) applies (bottlenecks do not cross) and also that the domain of dependence for the "end-point" of bottleneck i includes at most one bottleneck root other than its own; i.e., that the new counts are CRP counts. If we let the range of feasible wave velocities be denoted $[-w, v_f]$, and also assume that bottleneck speeds are in the range, the CRP condition is:

$$\Delta t \le (x_{i+2} - x_i)/(w + v_i)\ ,\ \Delta t \le (x_i - x_{i-2})/(v_f - v_i)\ , \quad \text{and}$$
$$\Delta t \le max\{(x_{i+1} - x_i)/(w + v_i)\ ;\ (x_i - x_{i-1})/(v_f - v_i)\} \quad \text{for all } i. \qquad (2)$$

This conditions ensures that bottleneck i interacts at most with one other companion bottleneck; either i or i+1. Thus, we propose choosing the largest Δt consistent with (1), (2) and with the assumption of constant bottleneck speed. An iteration of the procedure is as follows:

Step 1. Calculate the (uniform) densities between points i and i+1 with, $k_{i,i+1} = (N_i - N_{i+1})/(x_{i+1} - x_i)$; if $x_{i+1} = x_i$ put $k_{i,i+1} = 0$. Do for all i.

Step 2. Determine the new bottleneck speeds, the new step size with (1) and (2), and the final bottleneck positions.

Step 3. Predict the new counts by solving a CRP for each i (with its appropriate companion, if any.)

Step 4. Update the bottleneck positions and predict the new maximum passing rates. If two bottlenecks coincide and will pass, reverse their indices. Update time. Stop or go to step 1.

4. DISCUSSION

We note that step 1 involves an averaging approximation. This introduces some error into the result. This error is smaller than in the best first order methods. For example the technique compares favorably with Godunov's method when there are no moving bottlenecks and $v_f > w$. This happens

because the time steps with the proposed method can be longer than those of conventional methods, which have to satisfy Courant's condition. In particular, the proposed procedure is exact if the flow-density relation is triangular with $v_f = 2w$, whereas Godunov's method yields a significant truncation error. Note too that the recipe is numerically stable because it is the composition of two contraction mappings in the space of N-curves: an averaging operation (step 1) and the exact solution of a KW problem (steps 2-4); for more information on these mappings, see Daganzo (2001).

This procedure avoids the stalling problems that arise when bottleneck interactions are not allowed and bottlenecks pass each other. Figure 2a shows how the series of time steps $\{t_i - t_{i-1}\}$ obtained with a no-interaction procedure stalls near the crossing point of two bottlenecks. Figure 2b shows how the proposed procedure performs in the same situation. The new procedure can only stall if three or more bottlenecks coincide at a point in space-time, or two bottlenecks coincide with a lattice line. This, however, can be easily avoided by locally perturbing the trajectory of the bottlenecks.

Figures 3 and 4 show the result of the method when applied to examples 1 and 2 of Daganzo and Laval (2003). Note the smooth flows and the fine match even though the average time step is considerably longer now (3 secs in example 1 and 1.5 secs in example 2.) Close examination of Fig. 4, however, confirms that the proposed method introduces some numerical error in the N-values. This is due to the small time steps that need to be introduced when bottlenecks cross, which magnify the averaging errors of step 1. These errors can be reduced if one uses longer steps and CRP's with three-bottleneck interactions.

The procedure can be used with inhomogeneous lattices and can be applied to inhomogeneous highways. The procedure is also well suited to deal with endogenous bottlenecks, since any reasonable rule can be used in step 4 to modify the maximum passing rates and in step 2 to modify the bottleneck trajectories; including rules that account for road geometry and the presence of other nearby bottlenecks.

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APPENDIX

The results in this appendix, are based on the following two facts, specific to CRP's:

Fact a. Consideration of all possible wave-maps reveals that in a CRP, there can be at most one flow change on each bottleneck. Further consideration reveals that if there is a singularity where the bottleneck changes status (from active to inactive or vice versa) the singularity must be at one of the four locations marked by dots in Figure A1. (The slopes of the slanted lines in this figure that are not bottleneck trajectories are either $-w$ or v_f ; they determine the location of the dots uniquely.)

Fact b. Consider now an arbitrary point "P" on one of the bottleneck trajectories and let k_j be the "jam density". The vehicle number at this point can be obtained from the vehicle numbers at three anchor points {PU, PM, PD} (see figure) by:

$$N_P = min\{N_{PU}\ ;\ N_{PD} + t_{PD}\, wk_j\ ;\ N_{PM} + t_{PM}\, Q_i\ \} \qquad \text{(A1)}$$

if (as occurs in Fig. A1) the CRP anchors satisfy the following conditions: (1) PU is on the other bottleneck trajectory (or on the boundary) and a fast forward wave goes from PU to P without crossing any bottleneck trajectories; (2) PD is on the boundary (or on the other bottleneck trajectory) and a fast backward wave goes from PD to P in time t_{PD} without crossing any bottleneck trajectories; (3) the middle anchor is on the same bottleneck trajectory as P and the time separation between PM and P is t_{PM}; and (4) all singularities in the domain of dependence of P are in the union of the domains of dependence of PU, PM and PD.

Fact (b) can be proven using Newell's minimum principle for triangular flow-density relations (Newell, 1993), modified to allow for homogeneous (constant-speed-constant-maximum-passing-rate) bottlenecks. The proof is beyond the scope of this note. The two facts can be used to obtain the solution in three simple steps that avoid the singularities. The procedure is described with reference to the point labels of Fig. A2:

Step1. Use (A1) to determine N_B, using {D,A,G} as anchors. The passing rate along AB is $(N_B - N_A)/t_B$ at all points. Repeat for segment DE, using as anchors {H,D,A}. The passing rate is also constant along DE. This determines $N_i(t)$ and $N_{i+1}(t)$ up to points E and B.

Step 2. Again, use (A1) to determine N_C and N_F, with{E, B, I} and {J, E, B} as anchors. The passing rates along segments BC and EF are again constant and equal to $(N_C - N_B)/(t_C - t_B)$ and $(N_F - N_E)/(t_F - t_E)$, respectively. Thus, $N_i(t)$ and $N_{i+1}(t)$ are now determined up to F and C.

Step 3. Using {F,C,K} and {L,F,C} as anchors, determine N_M and N_N, and the ensuing (constant) passing rates. This yields $N_i(t)$ and $N_{i+1}(t)$ from points F and C onward.

List of Figures

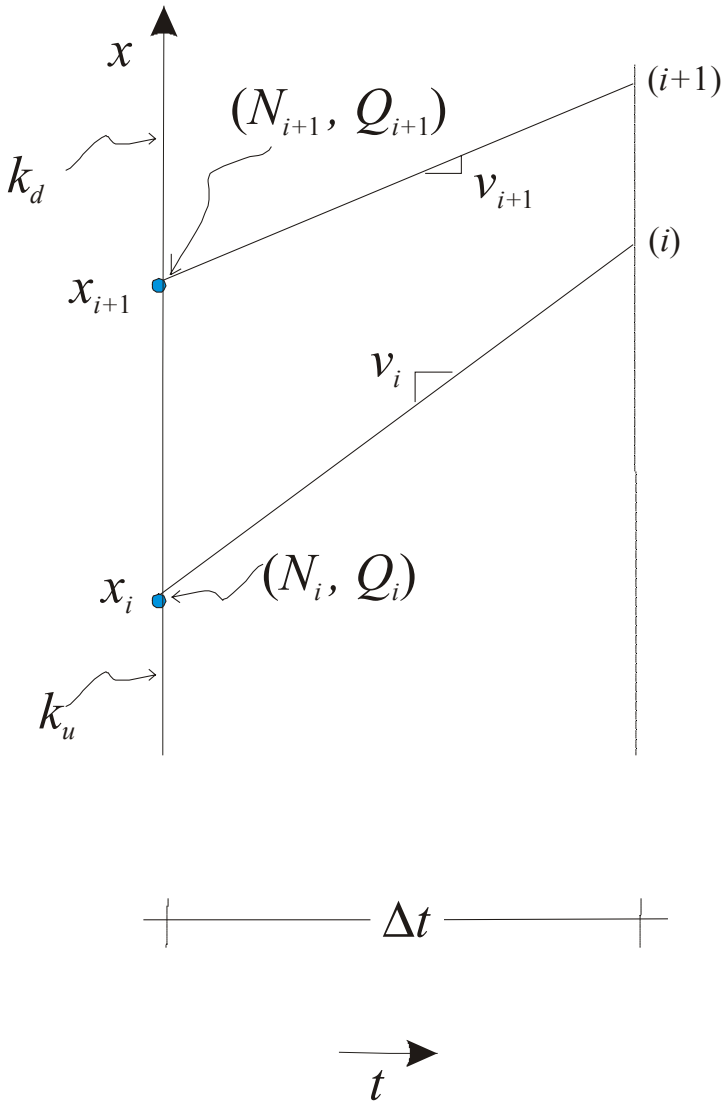
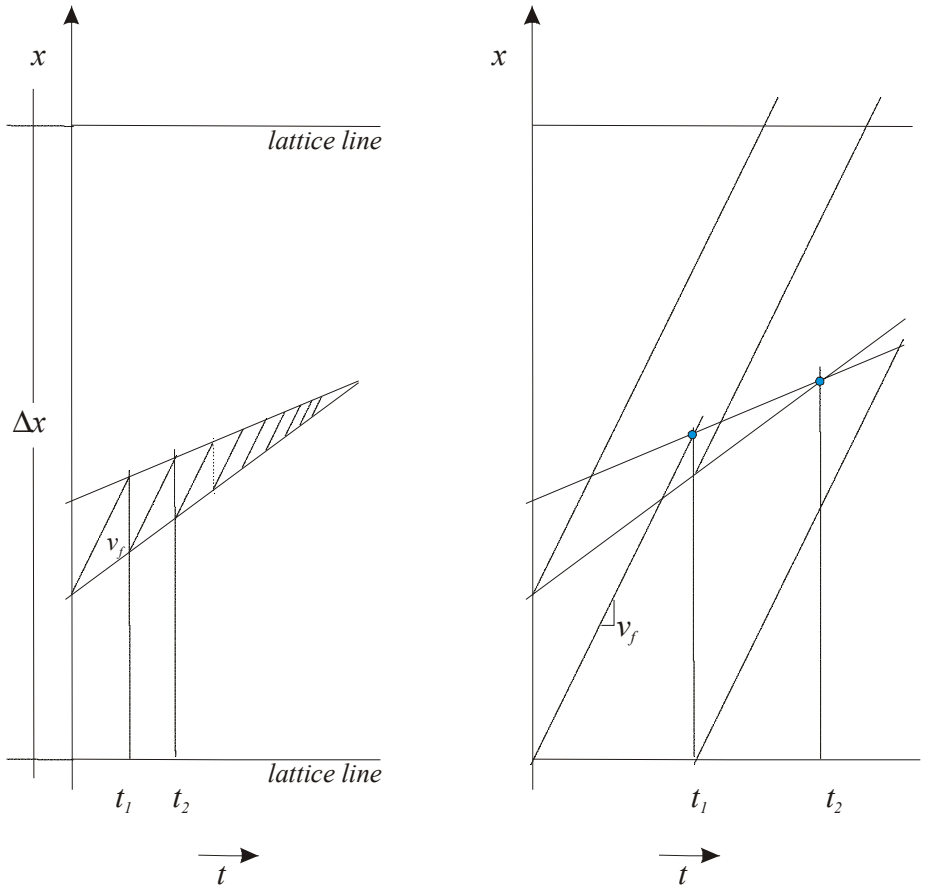


Figure 1: Data for a CRP



(a) (b)

Figure 2: Time steps of two numerical procedures: (a) conventional; (b) proposed (assume $w << v_f$).

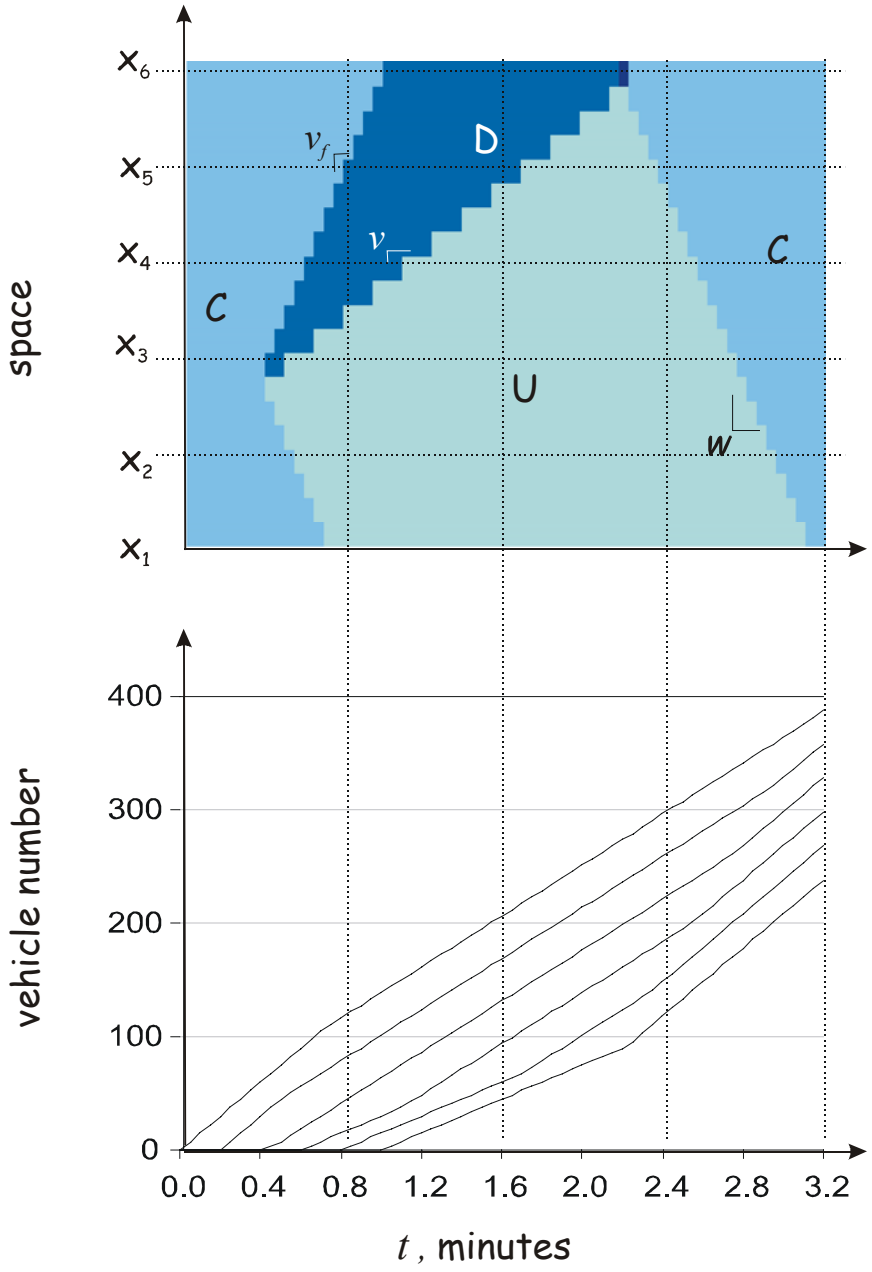


Figure 3: Numerical solution of example 1 (Δt = 3 sec).

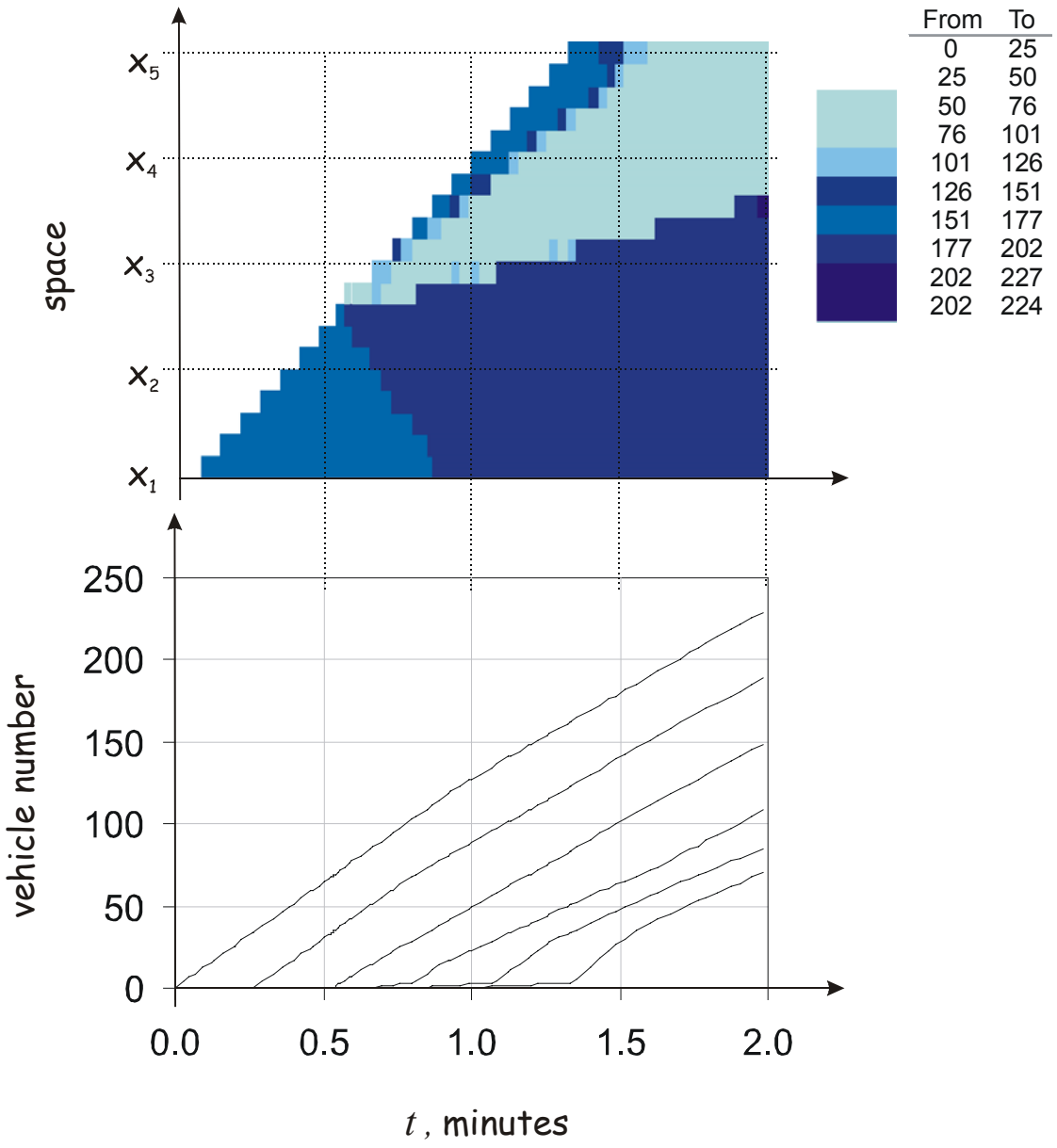


Figure 4: Density map (top) and cumulative counts (bottom) for example 2.

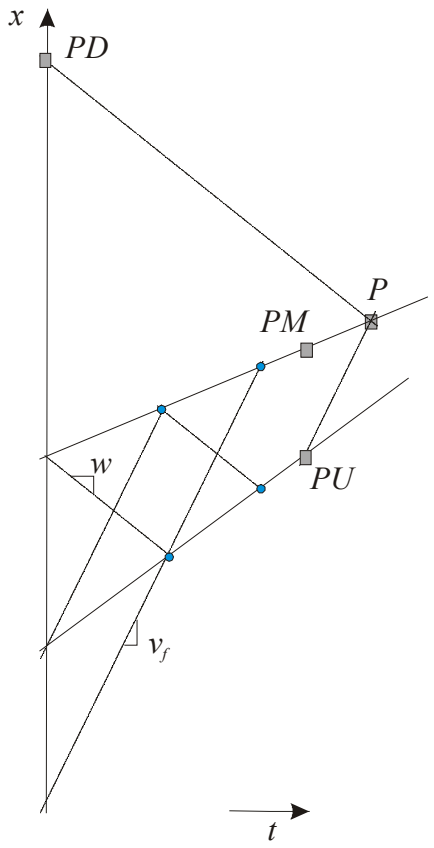


Figure A1: Basic properties of CRP's: dots denote possible singularities; squares denote possible anchors for point P.

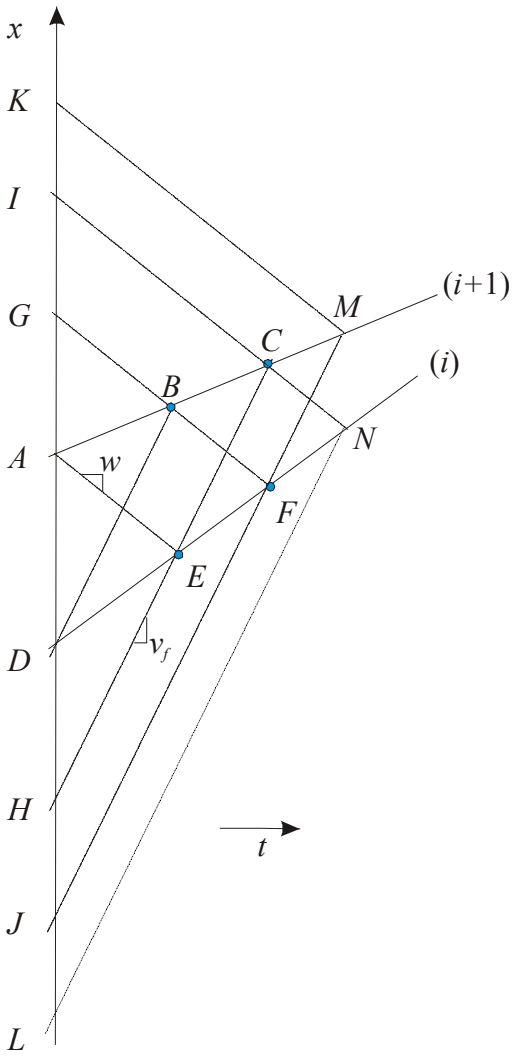


Figure A2: Points used in the solution of a CRP. (All the slanted lines that are not bottleneck trajectories have slopes - w or v_f.) Points B, E, C and F are potentially singular.