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**The Core and the Hedonic Core:
Equivalence and Comparative Statics**

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The Core and the Hedonic Core: Equivalence and Comparative Statics

Abstract

For cooperative games in which players are identified with their attributes, we introduce the notion of the "hedonic core": there is a linear function on attributes that describes the payoff of each player or group of players. We show that for a class of large games with transferable utility, the hedonic core approximates the core. Equivalence of the core and the hedonic core has two implications: (i) Nontrivial groups of players whose attributes are close will have core payoffs that are close. (ii) The payoff received by a nontrivial group of players with given attributes must be similar in any two utility vectors in the core. Using the notion that a game "exhausts blocking opportunities", we show that if this condition is satisfied in each of two finite games drawn according to distributions of attributes that weight a particular attribute differently, the hedonic payoff to that attribute is larger (no smaller) in the game that gives it less weight.

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1. Introduction

When the productivity of a coalition depends only on the characteristics or attributes of its members, we might naturally conjecture that core payoffs must depend on the attributes in a systematic way. We prove this conjecture for large cooperative games with transferable utility. Our main assumptions are that players' attributes or endowments are drawn independently from a distribution with bounded support, and that the payoff of a coalition depends only on the total attributes of the coalition's members. We show that if the number of players is large, then players' payoffs in the core are closely approximated by a linear function of their attributes which we call a "hedonic payoff."

We represent feasible payoffs with a characteristic function that describes the total utility available to a coalition as a function of its total attributes. If this characteristic function is not homogeneous, the core might be empty. Our approximation result is therefore for the epsilon-core, and applies also to the core. We show that a linear function on attributes approximates payoffs in the epsilon core except possibly for coalitions that represent a small fraction of the player set. This fraction can be arbitrarily small for a sufficiently large game. For a survey of similar results for games derived from exchange economies, see Anderson (1991).

The equivalence of the core and the hedonic core has several consequences. First, while core payoffs are not unique, they are almost unique in the sense that with high probability any two core payoffs are close. Second, nontrivial groups of agents with similar attributes receive similar payoffs in the core. These conclusions are immediate in the continuum framework of Aumann and Shapley (1974), who assume that there are no scale effects in feasible payoffs. Our approximation theorem verifies that for purposes of characterizing the core, large finite games are similar to continuum games, and for this the homogeneity assumption is unnecessary.

Since the core and hedonic core are equivalent for large games, it is of interest to ask how the payoff to an attribute varies with how heavily it is represented in the player set. We show that scarcity leads to high payoffs: If an attribute is represented more heavily in one game than another, the hedonic payoff to that attribute is lower. This result requires that each game "exhausts blocking opportunities" in a sense we define below.[1]

In Section 2 we define the hedonic core and epsilon hedonic core, and describe an example showing that the core and hedonic core might not coincide for small player sets. In Section 3 we discuss the approximation of the core by the hedonic core. In Section 4 we discuss comparative statics of the hedonic core. Section 5 comments more generally on the idea of hedonic prices and discusses economic applications of this model.

2. Core Payoffs and Hedonic Core Payoffs

A player set will be represented by a set of indices, $N = \{1,...,n\}$. A coalition, say S, is a nonempty subset of N. A player i will be described his or her vector of attributes $A^i \in \mathbf{R}^T_+ \backslash \{0\}$. The attributes represent characteristics like work skills or resource endowments and A^i_t represents the amount of the t-th attribute that player i possesses. The attributes of a coalition S are simply the sum of members' attributes, $A^S = \Sigma_{i \in S} A^i$. It will be useful

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[1] Notions of exhaustion go back at least as far as Buchanan (1965), writing on club economies, and these continue in the subsequent literature. Most notions of exhaustion apply to the optimal sizes of groups: after a group reaches some minimal size, the average return increases very slowly or declines. Such definitions apply to the "technology" for producing utility or to the characteristic function. For Wooders' account of her own contributions on this subject, see Wooders (1992). For our equivalence proposition we use a uniform convergence assumption on the characteristic function. This condition is satisfied by many assumptions of previous authors, but is broader. For the comparative static result we require "exhaustion of blocking opportunities", which is a condition on the game as well as the technology. (Under our condition on the characteristic function, blocking opportunities can be exhausted in large finite games, as discussed below.) The definition follows Scotchmer (1992), but see also "exhaustion of gains to scale" defined in Scotchmer and Wooders (1988).

to refer to the composition of a player's or coalition's attributes, $a^S \equiv A^S/|A^S|$, where $|\bullet|$ is the norm defined by $|A^S| = \Sigma_t A_t^S$. A composition is in the simplex $\Delta = \{a \epsilon \mathbf{R}_+^T \mid \Sigma_t a_t = 1\}$.

Our premise is that the utilities achievable by a coalition depend only on the coalition's attributes, and not otherwise on the number or identities of its members. Thus, we assume that feasible utilities are described by a function $V: \mathbf{R}_+^T \backslash \{0\} \rightarrow \mathbf{R}$ where V(A) is the total consumption or profit available to a coalition or individual with attributes A. We assume that V is superadditive: $V(A^S + A^{S'}) \geq V(A^S) + V(A^{S'})$.

A game is an ordered pair, (N,V), where N is the player set and V is a function as above. We define the core in the usual way: A payoff is a vector $U = (U_1, ..., U_n)$ in $\mathbf{R}^n$ such that $\Sigma_{i \epsilon N} U^i \leq V(A^N)$. A payoff $U \epsilon \mathbf{R}^n$ is in the core of a game (N,V) if no coalition $S \subset N$ could block U. A coalition $S \subset N$ can block U if $\Sigma_{i \epsilon S} U^i < V(A^S)$. Since the core may be empty, we will also discuss ε-cores. Payoffs $U \epsilon R^n$ are in the ε-core if no coalition $S \subset N$ could ε-block. A coalition $S \subset N$ can ε-block if $\Sigma_{i \epsilon S} U^i < V(A^S) - \varepsilon |A^S|$.[2] We will let $C_\epsilon(N,V)$ denote the ε-core of a game (N,V), for $\varepsilon \geq 0$, so that $C_0(N,V)$ represents the core.

A hedonic payoff for the game (N,V) is a vector $w \epsilon \mathbf{R}^T$ such that $w \bullet A^N \leq V(A^N)$, where A^N represents the attributes of the coalition of the whole, N. The vector w is called a hedonic payoff because it represents payoffs to attributes rather than to players. The payoff to player i in the hedonic core is $w \bullet A^i$. A coalition $S \subset N$ can block a hedonic payoff w if $w \bullet A^S < V(A^S)$, and can ε-block if $w \bullet A^S < V(A^S) - \varepsilon |A^S|$. The ε-hedonic core of a game (N,V), for $\varepsilon \geq 0$, denoted by $C_\epsilon^H(N,V)$, is the set of all hedonic payoffs that cannot be

[2] The usual definition of the epsilon core is that no coalition can increase its utility by more than epsilon per capita. We have defined it differently for convenience, but under our assumptions the two definitions are equivalent.

ε-blocked by any coalition $S \subset N$. We refer to $C_0^H(N,V)$ as the hedonic core.

It is easy to verify that for $\varepsilon > 0$, and for a sufficiently large game that depends on ε, the ε-hedonic core is nonempty. (See the appendix, Proposition A.1.) It is also easy to verify that if w is in the ε-hedonic core, then the payoffs $(w \cdot A^1, \ldots, w \cdot A^n)$ are in the ε-core, which is consequently nonempty.[3] The proposition in Section 3 addresses the converse: whether every payoff in the ε-core corresponds to a payoff in the ε-hedonic core.

Figure 1 shows an example of a game with three players and two attributes. The simplex Δ is the line above which the concave function V is drawn. For purposes of representing our arguments in diagrams such as Figure 1, it is convenient to define, for fixed payoffs $(U_1, \ldots, U_n)$, the total payoff of a coalition S, $U^S \equiv \Sigma_{i \in S} U^i$, and its payoff normalized by its attributes, $u^S \equiv U^S/|A^S|$. Throughout the paper we use the notation $p^S = (a^S, u^S)$ and $p^i = (a^i, u^i)$, where u^S or u^i are derived from a payoff U in the core or ε-core.

In Figure 1 the compositions of the three players' attributes are labeled $a^i = A^i/|A^i|$, $i = 1,2,3$. The composition of the game is $a^N = (A^1 + A^2 + A^3)/|A^1 + A^2 + A^3|$, and it is a convex combination of a^1, a^2, and a^3. We assume for this example that V is homogeneous, and together with superadditivity, this implies that V is concave as drawn. On the simplex the function V represents feasible payoffs per unit attribute that a coalition possesses; e.g., singleton coalitions can achieve $V(A^i)/|A^i| = V(a^i)$, $i = 1,2,3$. Similarly a two-person coalition {2,3} can achieve a total of $V(a^{\{2,3\}})$ per unit attribute they possess. The linear function w represents a payoff in the hedonic core, and player i's payoff in the hedonic core, normalized by the size of his attributes, is the point on the line above his composition a^i, namely $w \bullet a^i$.

[3] In general it is harder to prove nonemptiness of the epsilon core than under our assumptions; see, e.g., Wooders and Zame (1984).

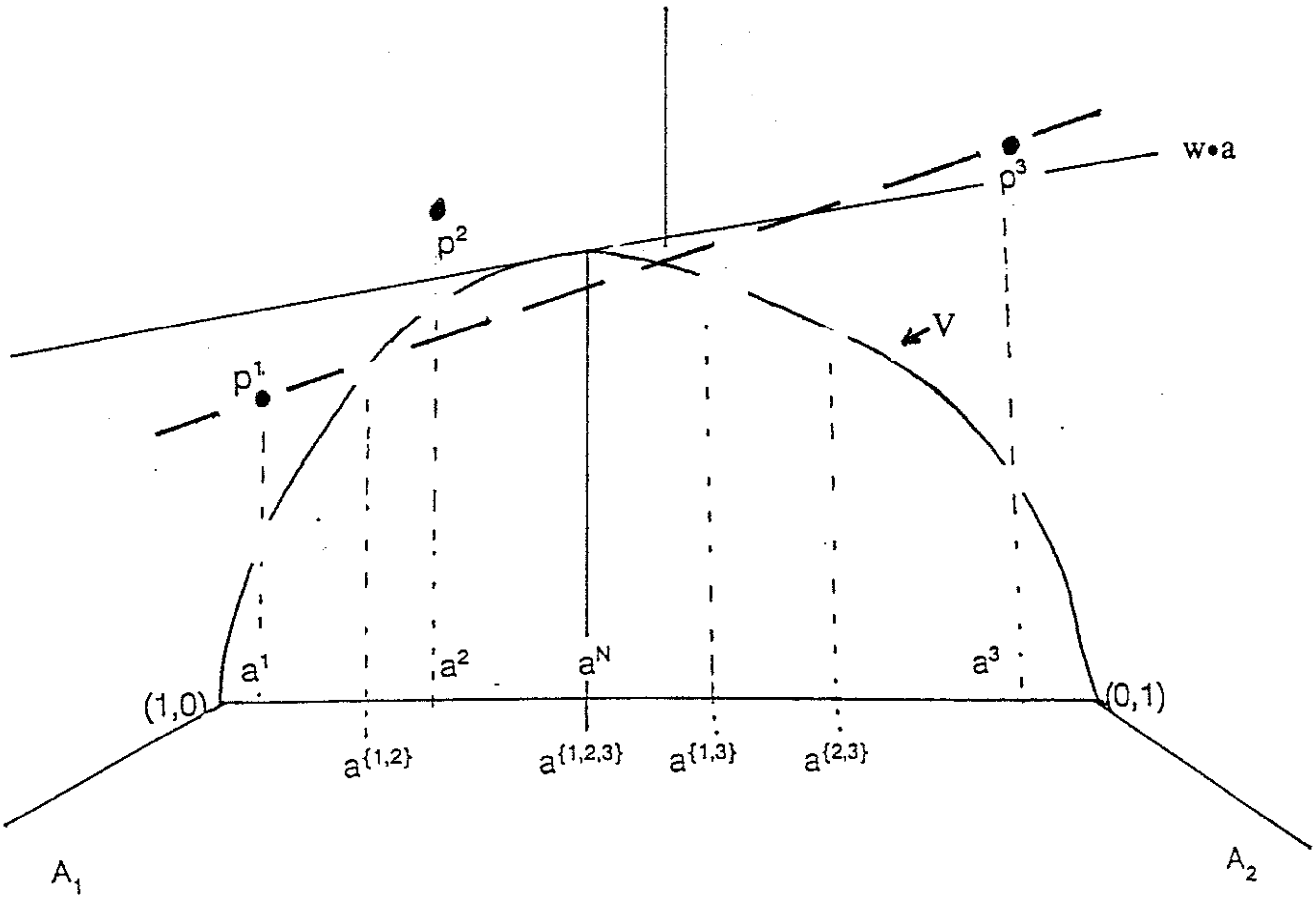

FIGURE 1

Figure 1 also shows payoffs in the core, normalized by attributes. The points p^i are $(a^i,u^i) \equiv (A^i/|A^i|\ ,U^i/|A^i|)$, i=1,2,3. The height of the point p^i above the simplex is u^i. These core payoffs are not hedonic payoffs, since they do not lie on a line. They are, however, in the core. No singleton coalition could block because each point p^i lies strictly above v. The coalition of the whole cannot block because their total payoff $u^{\{1,2,3\}}$ = $w \bullet a^{\{1,2,3\}}$ is no smaller than $V(a^{\{1,2,3\}})$. By Lemma 1 below, the coalition {1,3} could block only if the line that connects p^1 with p^3 is below $V(a^{\{1,3\}})$ at the composition $a^{\{1,3\}}$. As drawn, no combination of two players could block.

3. Equivalence of the Epsilon Core and Epsilon Hedonic Core

The following example shows that in general we cannot approximate the epsilon core uniformly (for all players) by a linear function. This example shows that the failure to converge uniformly is due to the weak blocking condition of the epsilon core, and not due to the fact that V may not be homogeneous.

<u>Example:</u> Suppose there are n (a positive integer) players. nd the aggregate payoff to a coalition depends only on the number of members. There is only one attribute, and every player possess it in the same amount, say $A^i = 1$. The attributes of a coalition S are the number of players in S, which we will denote by n^S. A payoff U is in the <u>ε-core</u> if for all $S \subset N$, $\Sigma_{i \in S} U^i \geq V(n^S) - n^S \varepsilon$. Assume that payoffs can be described by a function $V: \mathbf{R}_+ \backslash \{0\} \rightarrow \mathbf{R}_+$ that satisfies $\lim_{n \to \infty}[V(n)/n] = \sup_{n>0}[V(n)/n] = v < \infty$. Two such V functions are given in Figure 2, which depicts the standard example from "club theory", and Figure 3, where V is homogeneous. We will index a sequence of games by n, the number of players. The nth game is (N^n,V). Given $\varepsilon > 0$, for each n sufficiently large, the following payoffs are in the ε-core:

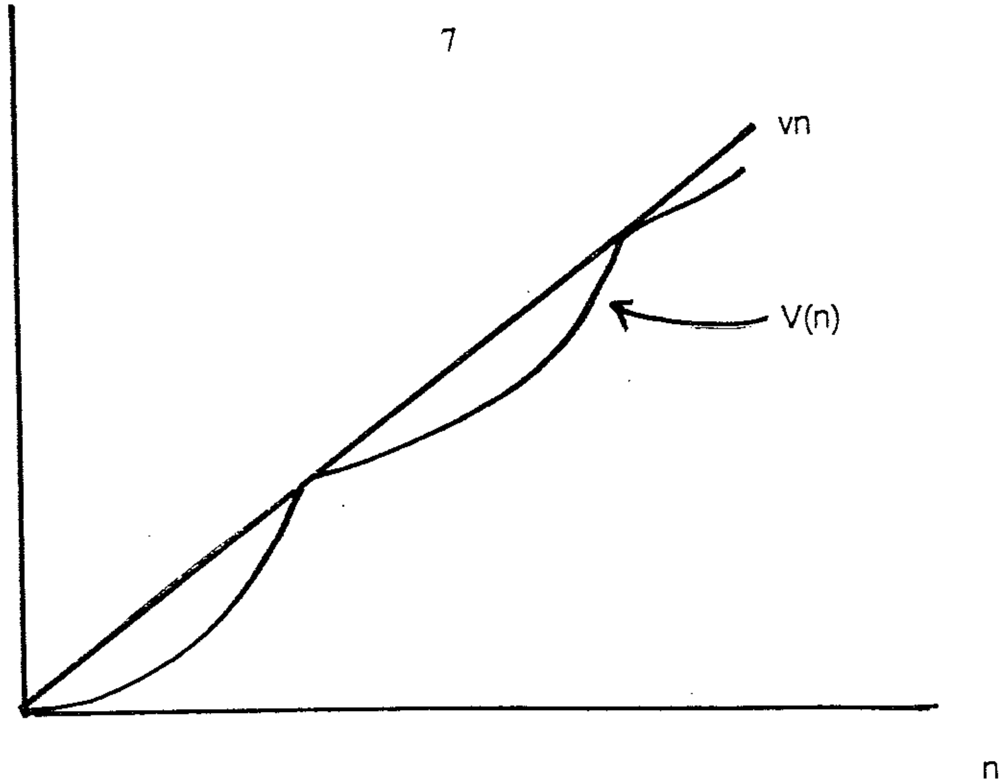

FIGURE 2

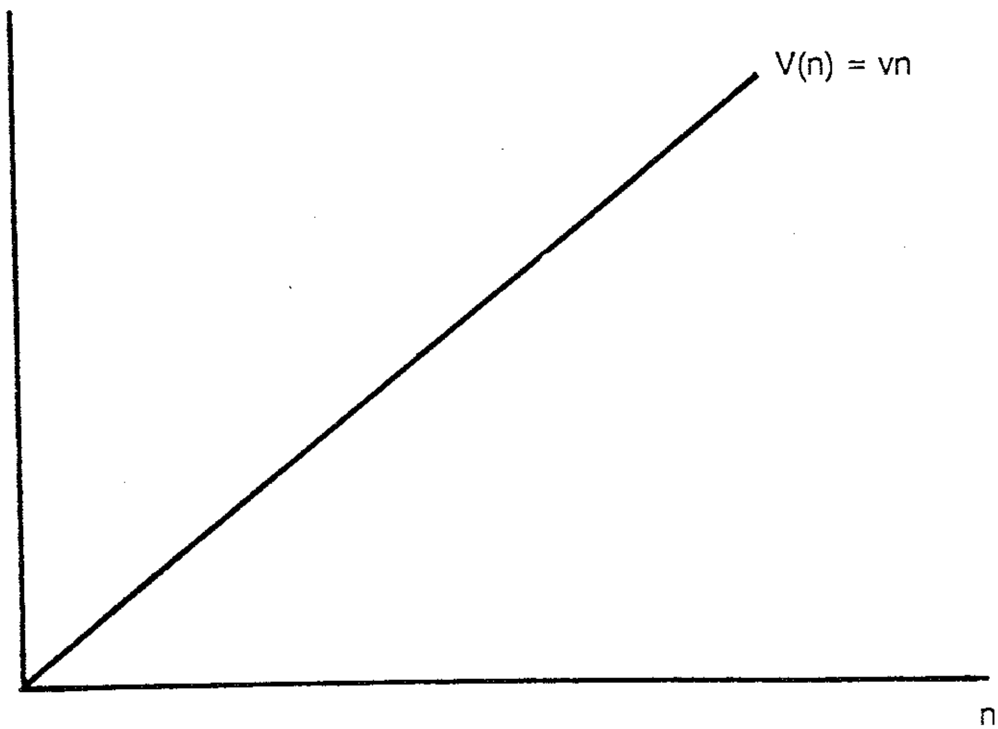

FIGURE 3

$$U^1 = (\varepsilon/2)(n-1), \quad U^i = [V(n) - (n-1)(\varepsilon/2)]/[n-1], \quad i=2,\ldots,n$$

The payoffs are feasible because $U^1 + \Sigma_{i\neq 1} U^i = V(n)$. To see that no coalition can ε-block, first consider coalitions not containing agent 1. Blocking by such a coalition would require $V(n^S) > \Sigma_{i\in S} U^i + n^S\varepsilon = (n^S/(n-1))\,[V(n) - ((n-1)\varepsilon/2)] + n^S\varepsilon$, which implies that $V(n^S)/n^S > V(n)/(n-1) + \varepsilon/2$, and therefore that $v \geq V(n^S)/n^S > [V(n)/n]\,[n/(n-1)] + \varepsilon/2$. But since $[V(n)/n]\,[n/(n-1)] \rightarrow v$ as n becomes large, S cannot block. A coalition of size n^S containing player 1 also cannot block (when n is large) because the total payoff of the coalition is even larger when S contains player 1, which makes it even harder to improve by more than ε.

Since a hedonic payoff would treat the players equally, and since $U^1 \rightarrow \infty$ as $n \rightarrow \infty$, while $U^i \rightarrow v - \varepsilon/2$ for all other players, the ε-core payoffs do not converge uniformly to hedonic core payoffs. □

Because this example shows that in general we cannot get uniform convergence for the epsilon core, the next proposition shows "almost" uniform convergence: a linear function of attributes closely approximates payoffs in the ε-core for all coalitions except ones that are a small fraction of the player set. The fraction can be arbitrarily small provided the player set is large enough. In the example, the fraction of the player set represented by player 1 is not fixed. It becomes small as the player set becomes large. For the case that V is homogeneous so that the core is nonempty, similar methods show that under an additional condition convergence of the core to the hedonic core is uniform.

We will form a large game by drawing players independently according to a distribution F on $\mathbf{R}^T_+$.[4] Let $a^F = E_F A^i/|E_F A^i|$, where $E_F A^i$ is the mean of a random draw from F. We assume that F is nondegenerate in the sense that a^F is in the interior of $\mathbf{R}^T_+$. We will assume that there are bounds $\underline{A} < \bar{A}$ such that $0 < \underline{A} \leq |A^i| \leq \bar{A}$ for all players i whose attributes are drawn from F, but we need no other restrictions on the sizes of these bounds or on F. For example, F could have full support on $\{A \in \mathbf{R}^T_+ \mid \underline{A} \leq |A| \leq \bar{A}\}$ or we could have a "types" game in which all the measure is on a finite set of points.

It will be convenient to define a function v: $\Delta \rightarrow \mathbf{R}$ that represents the supremum of average payoffs as the total amount of a coalition's attributes varies, holding the composition $a \in \Delta$ fixed:

$$v(a) \equiv \sup_{r>0} V(ra)/r, \quad a \in \Delta$$

We assume that v is finite. Since V is superadditive, v is concave (see the appendix). From v it is convenient to define the concave, homogeneous function $\hat{V}: \mathbf{R}^T_+ \backslash \{0\} \rightarrow \mathbf{R}$ as $\hat{V}(A) = |A|\, v(A/|A|)$. If V is homogeneous, then $\hat{V} = V$. Since $\hat{V}$ is differentiable except on a set of Lebesgue measure zero, it will have a unique supporting hyperplane at each $a \in \Delta$ except on a set of measure zero. A supporting hyperplane to $\hat{V}$ at a^N is also a supporting hyperplane at A^N, and the coefficients of the linear function that describe the hyperplane, say w^N, also describe hedonic core payoffs if $V(A^N) = \hat{V}(A^N)$. However, even if V is homogeneous of degree one, so that $V = \hat{V}$ and the hedonic core is nonempty, a payoff in the hedonic core need not correspond to a supporting hyperplane. The linear function described by the hedonic core payoffs can lie below $\hat{V}$ at $A \in \mathbf{R}^T_+$ or at $a \in \Delta$ provided the game does not have a coalition with that composition.

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[4] Hildenbrand introduced the idea of studying a sequence of economies in which the characteristics of players are drawn independently according to a distribution or a sequence of distributions, e.g., see his 1974 book. Anderson (1985) uses this notion to study core convergence. Wooders and Zame (1984) use it to study nonemptiness of approximate cores.

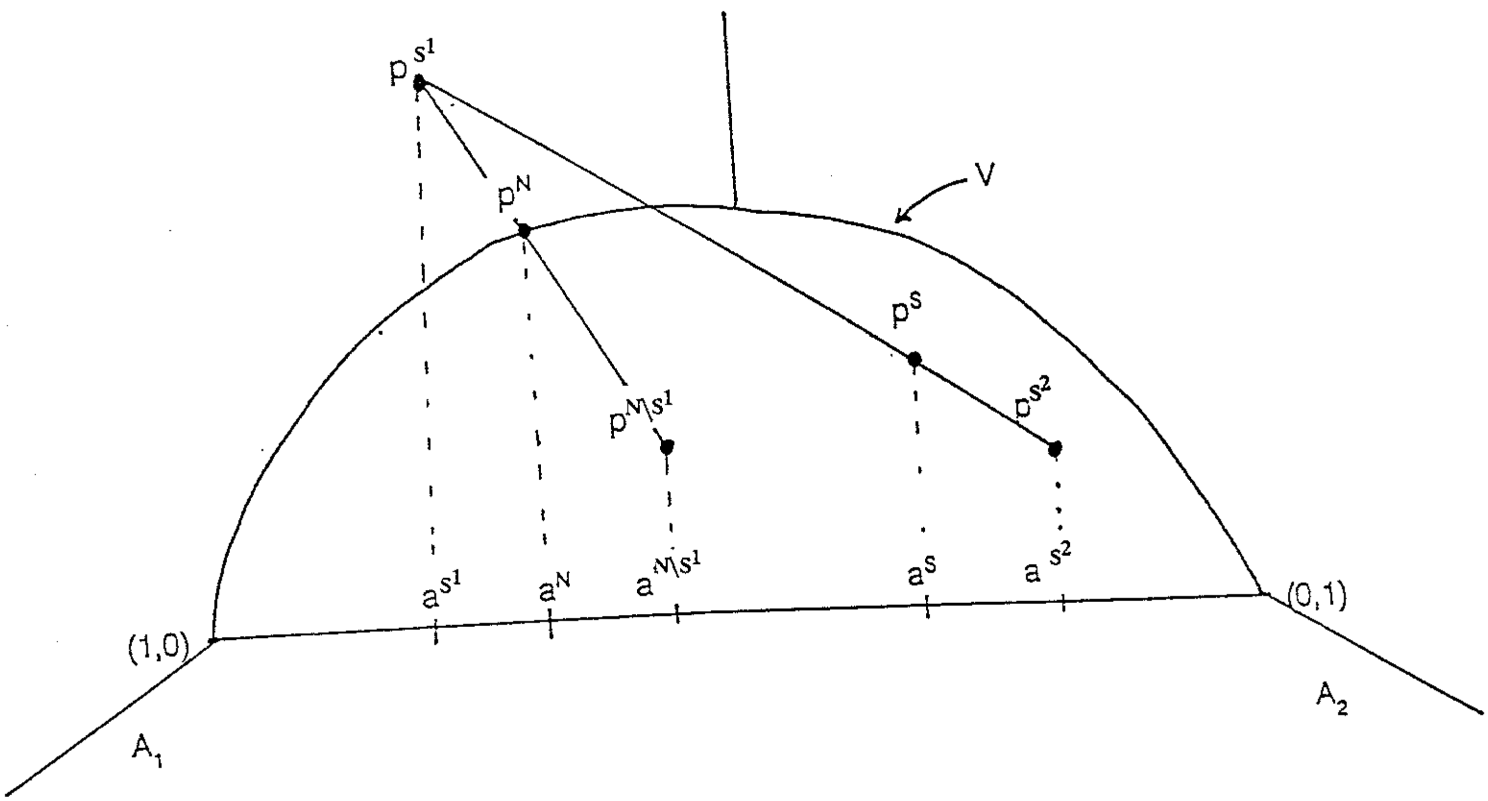
p^{s^1}
p^N
V
p^S
$p^{N\setminus s^1}$
p^{s^2}
a^{s^1}
a^N
$a^{N\setminus s^1}$
a^S
a^{s^2}
(1,0)
(0,1)
A_1
A_2

FIGURE 4

The following Lemma, illustrated by Figure 4, is useful in proving the propositions and in illustrating them. It can easily be verified.

Lemma 1: Let U be a payoff. (i) If $S^1 \cup S^2 = S$, the point $p^S \equiv (a^S,u^S)$ is a convex combination of the points $p^{S1} \equiv (a^{S1},u^{S1})$ and $p^{S2} \equiv (a^{S2},u^{S2})$. More precisely, $p^S = (|A^{S1}|/|A^S|)\, p^{S1} + (|A^{S2}|/|A^S|)\, p^{S2}$. (ii) In particular, if $\lambda = |A^{N\setminus S}|/|A^N|$, then $a^N = (1-\lambda)a^S + \lambda a^{N\setminus S}$, and $u^N = (1-\lambda)u^S + \lambda u^{N\setminus S}$.

Proposition 1: Let $\alpha \epsilon (0,1]$, $\delta > 0$ be given. Suppose

(1) $V \geq 0$,

(2) $V(ra)/r$ converges uniformly to $v(a)$ on Δ,[5] and

(3) $\hat{V}$ is differentiable at a^F.

Then there exist n, r, $\varepsilon_0 > 0$ such that if $|N| \geq n$, $\|a^N - a^F\| \leq r$, $\varepsilon \epsilon [0,\varepsilon_0]$ and $|S| \geq \alpha|N|$, then

$$U \epsilon C_\epsilon(N,V),\ w \epsilon C^H_\epsilon(N,V) \text{ implies } |w \bullet A^S - U^S| < \delta|A^S|.$$

The following corollary observes that the condition $\|a^N - a^F\| < r$ will hold with high probability for a sufficiently large game, and therefore we can avoid assuming it by stating the Proposition probabilistically.

Corollary 1: Let $\alpha \epsilon (0,1]$, $\delta,\theta > 0$. If (1)-(3) in Proposition 1 hold, then there exist n, $\varepsilon_0 > 0$ such that if $|N| \geq n$, $\varepsilon \epsilon [0,\varepsilon_0]$, and $|S| \geq \alpha|N|$, then with probability at least θ, $U \epsilon C_\epsilon(N,V)$, $w \epsilon C^H_\epsilon(N,V)$ implies $|w \bullet A^S - U^S| < \delta|A^S|$.

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[5] For all $\varphi > 0$, $\exists\ \underline{r}$ such that if $r > \underline{r}$, then for all a in the simplex, $v(a) - [V(ra)/r] < \varphi$.

Corollary 2: Under the hypotheses of Proposition 1, if $U, U' \in C_\epsilon(N,V)$, U is "close" to U' in the sense that if $|S| \geq \alpha|N|$, then $|U^S - U'^S| < 2\delta|A^S|$.

The appendix contains the proof. Here we give the idea of the proof.

The functions $\hat{V}$ and v are concave, hence differentiable almost everywhere on their domains, and Proposition 1 uses the assumption that $\hat{V}$ is differentiable at a^F with supporting hyperplane w^F. We show that every utility vector U in the ε-core is close to hedonic payoffs $(w^F \bullet A^1, \ldots, w^F \bullet A^n)$. We give the intuition for this result below. For payoffs w in the ε-hedonic core, $(w \bullet A^1, \ldots, w \bullet A^n)$ is in the ε-core and therefore close to w^F and to U. The proposition follows.

Part I: $w^F \bullet a^S - u^S \leq \delta$.

Suppose not. In Figure 5 we have drawn the case that $u^S \leq w^F \bullet a^S - \delta$. If a^S were close to a^F, we would have $u^S < v(a^S)$, and for a large enough player set, which implies that $V(A^S)/|A^S|$ is close to $v(a^S)$, coalition S could ε-block. Thus we only need to argue for the case that a^S is bounded away from a^F as in Figure 5. If a^N is close to a^F, a^S is also bounded away from a^N. Since $|A^S|$ is a nontrivial fraction of the player set $|A^N|$, it follows that $a^{N\backslash S}$ is also bounded away from a^F and a^N, as in Figure 5.

We will construct a blocking coalition, $N\backslash S_1$. From N we will remove a coalition of players, $S_1 \subset N\backslash S$, who are receiving high utility. We need the high-utility excluded coalition S_1 to have the properties that a^{S1} is close to $a^{N\backslash S}$, that $a^{N\backslash S1} \in D$, where D is shown in Figure 5, and that $u^{S1} \geq u^{N\backslash S}$. If these properties hold, then $u^{N\backslash S1}$ will lie below the dark line in Figure 5 in the domain D. (If $u^{S1} = u^{N\backslash S}$ and $a^{S1} = a^{N\backslash S}$, then $u^{N\backslash S1}$ will lie exactly on the dark line, according to Lemma 1.) If $u^{N\backslash S1}$ lies strictly below the dark line in the region D,

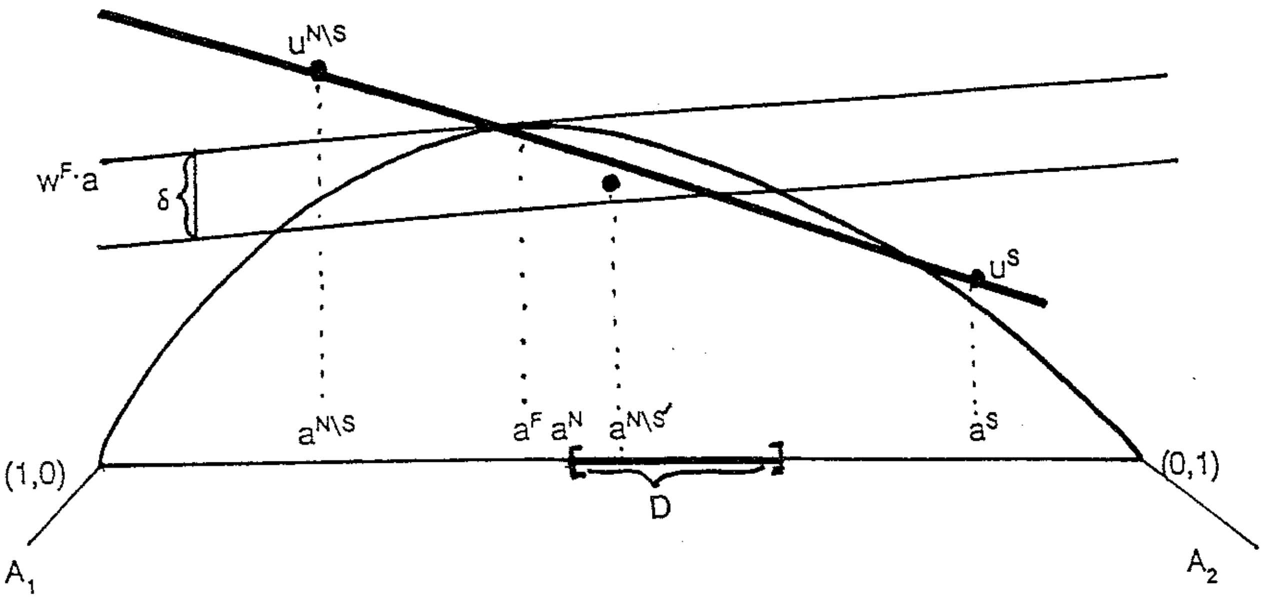

FIGURE 5

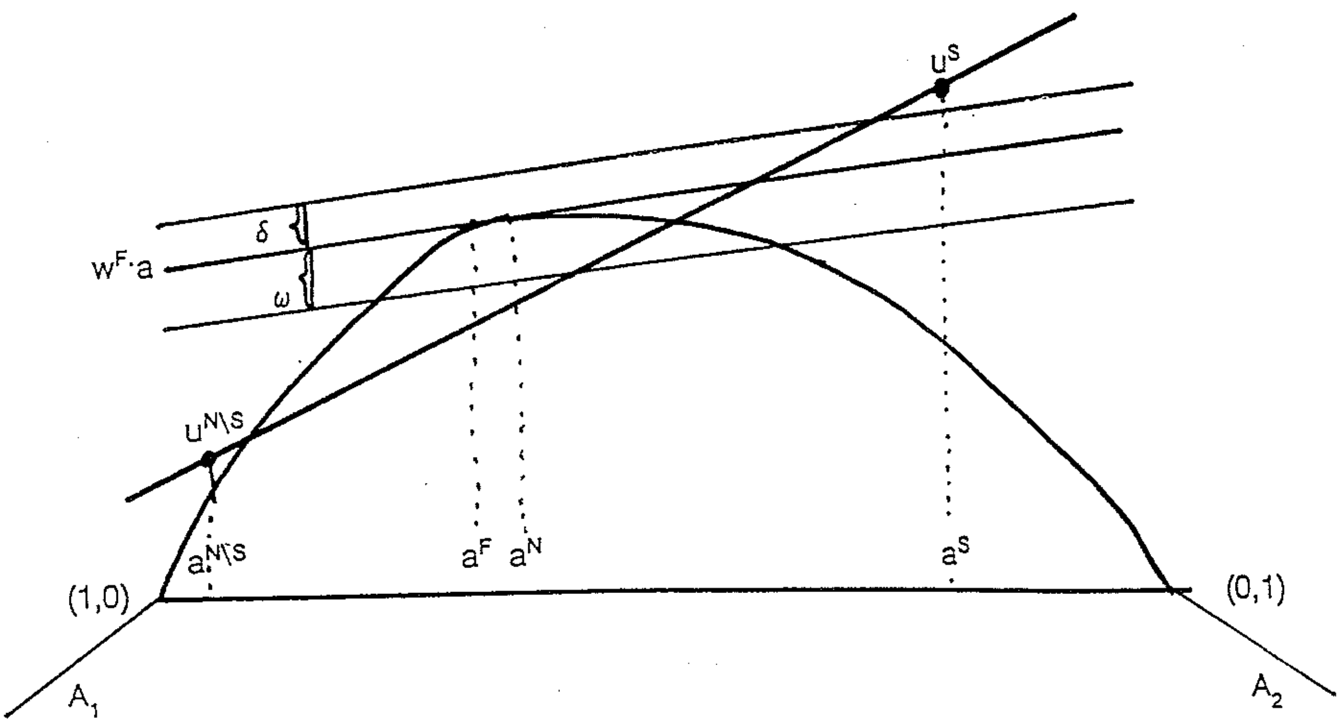

FIGURE 6

then it lies strictly below v. We can then choose ε and φ small enough so that $u^{N\backslash S1}$ lies strictly below $v(a^{N\backslash S1})-\varepsilon-\varphi$. Then for a large enough game, the coalition $N\backslash S_1$ can ε-block, as follows. By uniform convergence of V, a large enough coalition with composition $a^{N\backslash S1}$ can achieve utility greater than $v(a)-\varphi$ per unit attribute. Since the utility per unit attribute that $N\backslash S_1$ achieves, namely $u^{N\backslash S1}$, is less than $v(a^{N\backslash S1})-\varepsilon-\varphi$, $N\backslash S_1$ can improve its utility by more than $\varepsilon|A^{N\backslash S1}|$ and can therefore ε-block.

To find a suitable coalition S_1, we prove Lemma A.2 in the appendix. Assuming that $|A^{N\backslash S}|$ (or, equivalently, $|N\backslash S|$) is large, as when $|A^N|$ is large and $|A^{N\backslash S}|/|A^N|$ is bounded below (as it must be if a^S is bounded away from $a^N \sim a^F$), Lemma A.2 implies that, given an integer k and $r>0$, then $N\backslash S$ can be partitioned into k subcoalitions such that (i) all subcoalitions have ℓ or $\ell+1$ players, where $k\ell+q$ is the number of players in $N\backslash S$, and $0\leq q<k$, and (ii) the composition of each subcoalition is close to $a^{N\backslash S}$, in particular, if S_1 is one of the subcoalitions, $a^{S1} \in B(a^{N\backslash S},r)$, where $B(a^{N\backslash S},r)$ is the ball around $a^{N\backslash S}$ in the simplex defined by $\{a'\in\Delta \mid \|a^{N\backslash S}-a'\| < r\}$.

By Lemma A.2, a^{S1} can be assumed close to $a^{N\backslash S}$. Removing the coalition S_1 from N will move the composition $a^{N\backslash S1}$ away from a^N toward a^S as in Figure 5. But if $|A^{S1}|/|A^{N\backslash S}|$ were large - e.g., if S_1 comprised the entire coalition $N\backslash S$ - then the composition $a^{N\backslash S1}$ would be too close to a^S and outside the region D. To ensure that $|A^{S1}|/|A^{N\backslash S}|$ is not too large, the integer k cannot be too small. To find a sufficiently large k, in the proof we consider the "worst case scenario", that $a^{N\backslash S}$ (hence a^{S1}) is as close as possible to a^F, but still bounded away as explained above (due to the fact that $|A^S| \geq \alpha|A^N|$).

We do not need to worry that k might be too large. For any fixed k, $|A^{S1}|/|A^N|$ will be bounded below, and so $|a^{N\backslash S1}-a^S| < |a^N-a^S|$; in fact $|a^{N\backslash S1}-a^S| \leq \beta|a^N-a^S|$ for some $\beta<1$.

If $|N|$ is large, $a^F \sim a^N$. Thus by choosing $|N|$ large we can assure that $a^{N\setminus S1}$ is bounded away from a^F. Then choosing the left boundary of D close to a^F will ensure $a^{N\setminus S1} \in D$. As explained above, the set D must satisfy the property that for $a \in D$, $v(a)-\varepsilon-\varphi$ lies above the dark line, but this can be ensured because we are free to chose φ and ε.

To ensure that $u^{S1} \geq u^{N\setminus S}$, we only need to observe that not all subcoalitions chosen by Lemma A.2 could have utility smaller than average. We can choose for S_1 the subcoalition with the highest average utility.

Part II: $u^S - w^F \bullet a^S < \delta$.

Suppose not. Then u^S is as shown in Figure 6. Since $u^N \leq v(a^N) \leq w^F \bullet a^N$, hedonic payoffs strictly larger than w^F are infeasible for the players on average. Members of S in Figure 6 are receiving significantly more payoff to their attributes than w^F on average, and therefore members of N\S are receiving less than w^F on average. Since coalition S is a significant fraction of the player set, each member of N\S is providing a significant "subsidy" to S; for some $\omega > 0$, $w^F \bullet a^{N\setminus S} - u^{N\setminus S} > \omega$. But then we can apply Part I, using N\S in place of S, to argue that such utilities cannot be in $C_\epsilon(N,V)$.

4. Comparative Statics

The most intuitive notion in economics is that scarcity leads to high rents. Scarcity is usually interpreted to mean scarcity of a commodity; e.g., economists often use partial equilibrium models in which a reduction in the supply of a commodity increases its own price. This result is harder to obtain in general equilibrium where all prices change in response to the supply reduction, but there are known conditions under which it is true (Arrow and Hahn (1971)). Scarcity can also mean scarcity of agents. In the context of one-to-one or many-to-many matching models, Crawford (1991) shows that a larger number of players on one side of the match will reduce their equilibrium payoffs.[6] Scotchmer and Wooders (1988) gave conditions under which an increase in the number of agents of one type will lead to a decrease in the utility received by each player of that type in the core.[7] The comparative statics below are on attributes: If we draw our player sets from two distributions that weight a particular attribute differently, the hedonic core will reward that attribute more in the game where it is scarce.

The comparative static result follows if the games we compare are large enough to exhaust blocking opportunities in a sense we now define (but still finite). We first notice that w in the hedonic core or ε-hedonic core need not correspond to a supporting hyperplane: there might be points in $\mathbf{R}_+^T$ where $\hat{V}$ lies above the linear function w. But our

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[6] Crawford also reviews the literature. In matching models the set of equilibria or stable payoffs can be large, and therefore the comparison is on "extreme" payoffs: those that favor one side of the market rather than the other. Proposition 1, our approximation result, is proved for the case that $\hat{V}$ is differentiable at a^F, which forces uniqueness of payoffs in the hedonic core in the limit game as the size of the game becomes large. However, the hypotheses in our comparative statics theorems may be satisfied even when the hedonic core is not unique, and therefore the comparative statics do not require uniqueness.

[7] Scotchmer and Wooders (SW, 1988) discuss games with "types" of players, and show in their Proposition 2 that if w and w' are in the equal-treatment cores of (N,V) and (N',V) respectively, then, under their conditions, $(w-w')\cdot(A^N-A^{N'}) \leq 0$, where $A^N=N$ and $A^{N'}=N'$, which implies the monotonicity result we have just mentioned. In Proposition 7, SW (1989) draw attention to games with concave, homogeneous V, such as those derived from exchange economies, and show monotonicity for large, finite games. Proposition 2 below generalizes that result to differences in attributes. In Propositions 4.1 and 4.2, Wooders (1992) restates the comparative static result for games with concave homogeneous V, but gives a weaker version of it. Instead of comparing all equal treatment payoffs in the cores of large finite games as in SW Proposition 7 and in Proposition 2 below, Wooders restricts attention to payoffs in C(f) which is the set of bounding hyperplanes to (in our notation) V at f. For finite games, a core payoff need not be a bounding hyperplane (see our Figure 7), and therefore C(f) is only a selection from the set of equal-treatment core payoffs for a finite game with players $f \in Z^T_+$. SW (1988) described comparative statics of the limit game in their Figure 5.

Without homogeneity of V, the core may be empty. Our main contribution in Proposition 3 is to extend the monotonicity result to approximate cores, showing that the "fudge factor" in the monotonicity result can be arbitrarily small. Contemporaneously with this paper, Wooders (1992, Proposition 4.3) gives a monotonicity result for the weak epsilon core that is much weaker than ours in that it does not imply that the "fudge factor" can be small. In Proposition 4.4 Wooders restates Proposition 5 of SW, which gave a monotonicity result for the strong epsilon core. However we caution the reader that the original argument of SW contained an error which has not been corrected, and Wooders does not give a proof.

concept of exhaustion ensures that, if w is in the hedonic core or ε-hedonic core, the set of compositions $a\epsilon\Delta$ where $\hat{V}$ lies above w is small.

The comparative statics are particularly simple when V is homogeneous, so that the core and hedonic core are nonempty. For simplicity, we discuss that case first, and then modify the arguments for the approximate core, which is relevant when V may not be homogeneous.

For each $w\epsilon\mathbf{R}^T$ we define $\Delta_w = \{a\epsilon\Delta \mid v(a) > w\cdot a\}$, and say that the game (N,V) <u>δ-exhausts blocking opportunities</u> if $\mathrm{diam}(\Delta_w) > \delta$ implies that there exists a coalition $S\subset N$ with $a^S\epsilon\Delta_w$. Then, assuming that V is homogeneous, S can block because $v(a^S) = V(A^S)/|A^S| > w\bullet a^S$. Intuitively exhaustion means that the compositions of feasible coalitions fill in the simplex. In Figure 7, w could not represent payoffs in the hedonic core if the game δ-exhausts blocking opportunities, since there would then be a coalition such as S that could block.

The following proposition compares hedonic core payoffs in two games that exhaust blocking opportunities. If one attribute is more heavily represented in one player set than in the other, and if the representation of all the other attributes is proportionately smaller, then the payoff to the more prevalent attribute in the hedonic core must be smaller. This can be seen in Figure 8, in which we assume that V is homogeneous so that any coalition with composition a^S can achieve payoff $v(a^S)$. The directional derivative of the linear function $A\rightarrow w\bullet A$ in the direction from (0,1) to (1,0) is $(w_1-w_2)/\sqrt{2}$, and similarly for the linear function $A\rightarrow w'\bullet A$. The game (N',V) has a higher fraction of attribute 1 than (N,V) and a smaller fraction of attribute 2. (With T attributes, there must be proportionately less of each other attribute.) In Figure 8, it could not be the case that one linear function lies entirely below the other between a^N and $a^{N'}$, because then, since the games δ-exhaust

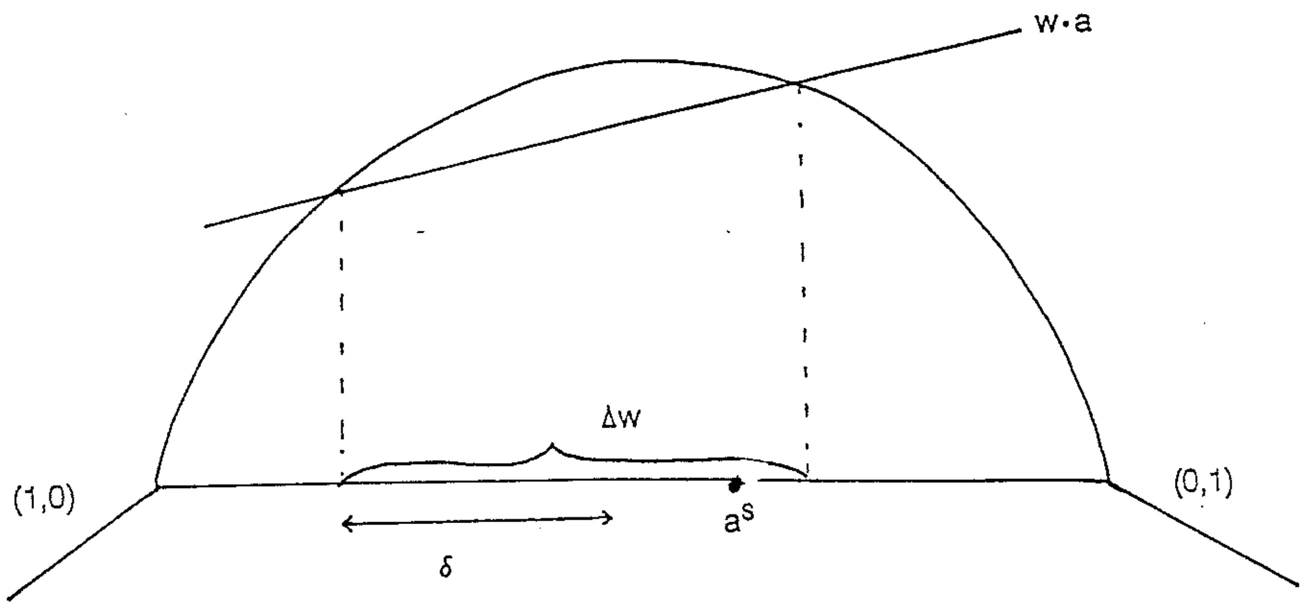

FIGURE 7

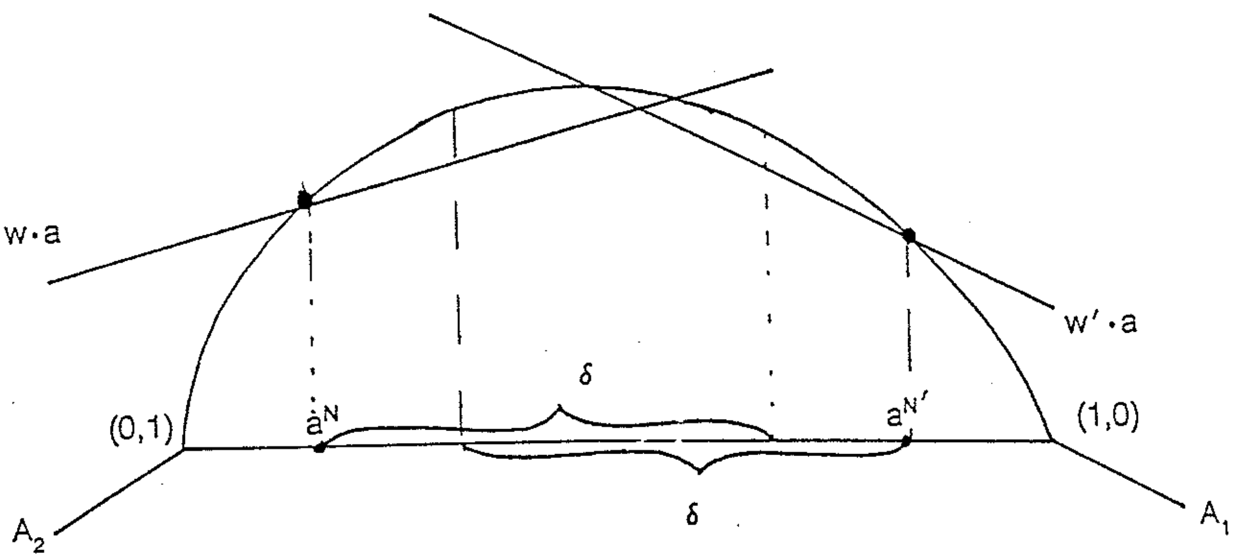

FIGURE 8

blocking opportunities, some coalition could block w or w'. The slopes of the linear functions in Figure 8 imply that $w_1-w_2 > w_1'-w_2'$. Since one cannot lie entirely below the other, we cannot have $w << w'$ (or $w' << w$); hence $w_1 \geq w_1'$ and $w_2 \leq w_2'$. The generalization of this diagram, stated in Proposition 2, is that $w_1 \geq w_1'$, but with more than two attributes we cannot say in general whether w_t is greater or smaller than w_t' for $t \neq 1$.

Proposition 2: Suppose (i) that V is homogeneous, (ii) that $w \epsilon C_0^H(N,V)$ and $w' \epsilon C_0^H(N',V)$, (iii) that (N,V) and (N',V) δ-exhaust blocking opportunities, (iv) that $\|a^N - a^{N'}\| > \delta$, and (v) for $\beta > 0$, that $a_t^{N'} = k\, a_t^N$, $t \neq j$, and $a_j^{N'} = a_j^N + \beta$. Then $w_j \geq w_j'$.

Proof: First, from δ-exhaustion of blocking opportunities, we show that (i) $w' \bullet a^{N'} \leq w \bullet a^{N'}$, and (ii) $w \bullet a^N \leq w' \bullet a^N$. Suppose (i) did not hold. Then $w \bullet a^{N'} < w' \bullet a^{N'} = V(a^{N'})$, so $a^{N'} \epsilon \Delta_w$. Since $w \bullet a^N = V(a^N)$, $(1-\lambda)a^N + \lambda a^{N'} \epsilon \Delta_w$ for $\lambda \epsilon (0,1]$, as V is concave. Therefore $\text{diam}(\Delta_w) > \delta$, which implies there exists $S \subset N$ which can block w, and this is a contradiction. The same argument applies for (ii). Multiply the second inequality by k and subtract from the first inequality to get $w_j' ((1-k)a_j^N + \beta) \leq w_j ((1-k)a_j^N + \beta)$. Since $\beta > 0$, and $a^{N'} \epsilon \Delta$, $k \epsilon [0,1)$, and the result follows. □

This simple argument needs to be developed in three directions. First, if we choose the players in the games (N,V) and (N',V) independently according to distributions F and F', hypothesis (v) of Proposition 2 is unlikely to be satisfied. We therefore need to show a similar comparative static result when the players are chosen independently from distributions F and F', and when we apply the hypotheses directly to a^F and $a^{F'}$ rather than to a^N and $a^{N'}$. Second, the core will typically be empty when V is not homogeneous.

Therefore we need to show that the comparative static result holds for the epsilon hedonic core developed above. Third, we need to define our exhaustion hypothesis in a way that guarantees blocking opportunities are exhausted for finite games. V might have the property that there is a sequence of sets Δ_w^n with diameter at least δ for which the sequence of their measures converges to zero. In that case we cannot guarantee that a finite game will satisfy the definition with high probability. To avoid this problem, we redefine exhaustion in a way that makes the proof of the comparative static result less elegant, but can be satisfied with high probability in a large enough (finite) game.

We accomplish these tasks in the following order. First we modify the definition of exhaustion such that the definition is satisfied for large finite games and so as to accommodate the case that V may not be homogeneous. Second, we present Proposition 3, which reports the comparative static result for the approximate core with hypotheses directly on a^F and $a^{F'}$.

The idea behind exhaustion is that enlarging the game cannot introduce a coalition S for which hedonic payments w are feasible, in the sense that $w \bullet A^S \leq V(A^S)$, if the hedonic payments w were not feasible for some coalition in the game before it was enlarged. This would be guaranteed if for every value v(a), $a \in \Delta$, there were a coalition S for which $A^S/|A^S| = a$ and $V(A^S)/|A^S| = v(a)$. Then the total payoff v(a) and the hedonic payments represented by the supporting hyperplane at v(a) would already be feasible for some coalition in the game. This condition is, of course, too strong, since it cannot be satisfied by a finite game. Therefore in our notion of exhaustion we only require that each v(a) or each supporting hyperplane is "almost" achievable by some coalition in the game. "Almost" has two aspects. The first aspect, which is captured in the above definition of exhaustion, says that the compositions of feasible coalitions are distributed throughout the simplex. Thus for every $a \epsilon \Delta$, there is a coalition with composition a^S close to the composition a. Therefore, when V is homogeneous, the feasible payoff $V(A^S)/|A^S| = v(a^S)$ is close to v(a) and therefore close to $w^a \bullet a = v(a)$. When V is not homogeneous, "almost" has a second aspect. Not only must the compositions of feasible coalitions be distributed throughout the simplex, but in addition such coalitions must be large enough so that their feasible payoffs $V(A^S)/|A^S|$ are "close" to $v(a^S)$. The modification that follows incorporates the latter aspect.

We will say that the game (N,V) δ-exhausts ε-blocking opportunities if for every $a \epsilon \Delta$, there exists a coalition S for which (i) $a^S \epsilon B(a,\delta/2)$, where $B(a,\delta/2)$ is a ball in the simplex (as defined above) of diameter δ around a, and (ii) $V(A^S)/|A^S| > v(a^S)-\varepsilon$.

Remarks on the definition of exhaustion: (1) Given a probability $\pi > 0$, a large enough finite game will satisfy our notion of exhaustion with at least probability π provided the vertices of Δ are in the supports of F and F′ and V(ra)/r converges uniformly to v(a). (2) Inclusion of the vertices is a stronger condition than required for Proposition 3, which only requires that conditions (i) and (ii) of the definition hold for some subset of Δ including a^F and $a^{F'}$. (3) If $w \epsilon C^H_\varepsilon(N,V)$ then the set $\Delta(w,\varepsilon) \equiv \{a \epsilon \Delta \mid v(a) - w \bullet a > 2\varepsilon\}$ does not contain a ball of diameter δ. (If it did, there would exist a coalition S that could ε-block the hedonic payoffs w, since $V(A^S)/|A^S| - \varepsilon > v(a^S) - 2\varepsilon > w \bullet a^S$.) This weaker condition is all that is needed for Lemma 2 (boundedness).

Proposition 3 requires that payoffs in the approximate hedonic core are bounded, and we therefore precede Proposition 3 with Lemma 2. Lemma 2 uses the notion of distance from a point in Δ to the boundary of Δ, defined as $\partial\Delta = \{a \epsilon \Delta | a_t = 0, \text{ some } t\}$. Using $\|\bullet\|$ to denote Euclidean distance, distance to the boundary is defined as $d(a,\partial\Delta) =$

$\min_{a' \in \partial\Delta} \|a-a'\|$, and in fact $d(a,\partial\Delta) = \min_t a_t (T/(T-1))^{.5}$.

Lemma 2: (Appendix) Let ε, δ, $r \geq 0$ and suppose $d(a^N,\partial\Delta) \geq r > \delta$. Let Ψ be the set of feasible hedonic payoffs w such that $\Delta(w,\varepsilon)$ contains no open ball in Δ of diameter δ. Then Ψ is bounded and the bound is independent of a^N.

Proposition 3: Suppose $a^F, a^{F'} \in \text{int}(\Delta)$ and satisfy, for $\beta>0$, $a_t^{F'} = k\, a_t^F$, $t \neq j$, and $a_j^{F'} = a_j^F + \beta$. Given $\gamma>0$, there exist $r>0$, $\delta>0$, $\varepsilon_o>0$ such that if (i) $\|a^F-a^N\|$, $\|a^{F'}-a^{N'}\| < r$, (ii) $0 \leq \varepsilon < \varepsilon_o$, (iii) $\exists$ $M>0$ such that $\|w\|$, $\|w'\| \leq M$ for $w \in C_\varepsilon^H(N,V)$ and $w' \in C_\varepsilon^H(N',V)$, and (iv) (N,V), (N',V) δ-exhaust ε-blocking opportunities, then $w \in C_\varepsilon^H(N,V)$, $w' \in C_\varepsilon^H(N',V)$ imply that $w_j' < w_j + \gamma$.

Remarks on Proposition 3: (1) It is easy to see that $k \in (0,1)$. (2) Condition (i) will hold with high probability if $|N|$ and $|N'|$ are large. (3) Lemma 2 gives a condition under which (iii) holds.

Proof: Let $\theta>0$ satisfy $d(a^F,\partial\Delta)$, $d(a^{F'},\partial\Delta) > \theta$, $\|a^F-a^{F'}\| > \theta$.

Let $r = \min \{ (\gamma/8)\, [\theta((T-1)/T)^{.5}(1-k)+\beta] / 2M,\ \theta/3 \}$

$$\varepsilon_o = [\gamma/4(1+k)]\ [\theta((T-1)/T)^{.5}(1-k)+\beta]\ (\|a^F-a^{F'}\|-\theta) \ / \ (\|a^F-a^{F'}\|+2\theta).$$

$$\psi = [\gamma/2(1+k)]\ [\theta\, ((T-1)/T)^{.5}(1-k) + \beta].$$

Then if $\varepsilon < \varepsilon_o$,

$$(1) \qquad \frac{2\varepsilon}{\psi} \leq \frac{\|a^F-a^{F'}\|-\theta}{\|a^F-a^{F'}\|+2\theta} < 1$$

It follows that ψ-$2\varepsilon_o>0$. Let m be a Lipschitz constant for v on $K=\{a\epsilon\Delta \mid d(a,\partial\Delta)\geq(\theta/3)\}$. Let $\delta>0$ satisfy $\delta < \min\{\ 2(\psi-2\varepsilon_o)/(M+m),\ 2\theta/3\ \}$. We have chosen δ and K so as to ensure that if $\|a^{N'}-a^{F'}\| < r$, $B(a^{N'},\delta/2)\subset K$. This condition is used in the following lemma.

Lemma 3: If (N,V) and (N',V) δ-exhaust ε-blocking opportunities, and if $w\epsilon C^H_\varepsilon(N,V)$, $w'\epsilon C^H_\varepsilon(N',V)$, then $w'\bullet a^{N'} \leq w\bullet a^{N'} + \psi$ and $w\bullet a^N \leq w'\bullet a^N + \psi$.

Proof: We will prove only the first of these inequalities. Suppose, on the contrary, that $w'\bullet a^{N'} - \psi > w\bullet a^{N'}$. We show that the set $\Delta(w,\varepsilon)$ then contains a δ-ball, namely $B(a^{N'},\delta/2)$, and thus, by exhaustion, there exists a coalition that can ε-block. We have $v(a^{N'}) - \psi \geq w'\bullet a^{N'} - \psi > w\bullet a^{N'}$ or $v(a^{N'}) - w\bullet a^{N'} > \psi$. Let ν: $K\rightarrow\mathbf{R}$ be defined by $\nu(a) = v(a) - w\bullet a$. Then (M+m) is a Lipschitz constant for ν. As noted above, if $a\epsilon B(a^{N'},\delta/2)$, $a\epsilon K$. Hence, $|\nu(a) - \nu(a^{N'})| \leq (M+m)\ \|a-a^{N'}\| < (M+m)\ \delta/2 < \psi-2\varepsilon_o$. Since $\nu(a^{N'}) > \psi$, $\nu(a) > 2\varepsilon_o > 2\varepsilon$ for $a\epsilon B(a^{N'},\delta/2)$. Thus, $B(a^{N'},\delta/2) \subset \Delta(w,\varepsilon)$, and by exhaustion, $B(a^{N'},\delta/2)$ contains the composition of a coalition that could ε-block w. □

We can now complete the proof that $w_j' < w_j+\gamma$. The completion follows the technique in Proposition 2; that is, we multiply the second inequality of Lemma 3 by k, and then subtract it from the first inequality. But since our hypotheses are on a^F and $a^{F'}$ rather than on a^N and $a^{N'}$, the resulting inequality will involve "nuisance terms", and we must show that these "nuisance terms" are small.

Let $e_j\epsilon\mathbf{R}^T_+$ be the jth unit vector (all zeros except for a one in the jth place). We can write $a^{N'} = ka^N + (a^N_j + \beta - ka^N_j)e_j + d$ for an appropriate vector d. The first inequality in Lemma 3 can therefore be written

$$w' \bullet [\, ka^N + (a_j^N + \beta - ka_j^N)\, e_j + d \,] \le w \bullet [\, ka^N + (a_j^N + \beta - ka_j^N)\, e_j + d \,] + \psi$$

Multiplying the second inequality by k and subtracting, we get

$$w' \bullet [(a_j^N + \beta - ka_j^N)\, e_j + d] - k\psi \le w \bullet [(a_j^N + \beta - ka_j^N)\, e_j + d] + \psi$$

$$\Rightarrow \quad (w'-w) \bullet (a_j^N + \beta - ka_j^N)\, e_j \le (w-w')\bullet d + k\psi + \psi \quad \Rightarrow$$

$$(2) \quad (w'-w) \bullet e_j \le (w-w')\bullet d \,/\, (a_j^N + \beta - ka_j^N) + (k\psi+\psi) \,/\, (a_j^N + \beta - ka_j^N)$$

If the righthand side of (2) were equal to zero, we would have the result, as in Proposition 2, that $w_j' \le w_j$. The righthand side is not zero, but it is smaller than γ. To show this, we show that each of the two terms in the righthand side of (2) is smaller than $\gamma/2$.

Our hypotheses imply that $a^{F'} = ka^F + (a_j^F + \beta - ka_j^F)e_j$. Since we have assumed that a^N is close to a^F and $a^{N'}$ is close to $a^{F'}$, it follows that the vector d must be small. In fact,

<u>Claim:</u> $\|d\| < 4r$.

<u>Proof of Claim:</u> Let $b^N = ka^N + (a_j^N + \beta - ka_j^N)e_j$.

$$\|a^{F'}-b^N\| = \|k(a^F-a^N) + [(a_j^F-a_j^N)-k(a_j^F-a_j^N)]\, e_j\,\| \le k\|a^F-a^N\| + |(a_j^F-a_j^N)-k(a_j^F-a_j^N)|\,\|e_j\|$$

$$\le \|a^F-a^N\| + \|a_j^F-a_j^N\| + k\|a_j^F-a_j^N\| \le 3\,\|a^F-a^N\| < 3r$$

$$\|d\| = \|a^{N'}-b^N\| \le \|a^{N'}-a^{F'}\| + \|a^{F'}-b^N\| < 4r \qquad \square$$

<u>Claim:</u> $(w-w')\bullet d \,/\, (a_j^N + \beta - ka_j^N) < \gamma/2$

<u>Proof:</u> First, $d(a^N,\partial\Delta) \ge \theta \Rightarrow a_j^N \ge \theta\,((T-1)/T)^5$.

$$\|d\| < 4r \le (\gamma/2)\,[\,\theta((T-1)/T)^5(1-k) + \beta\,] \,/\, 2M \le (\gamma/2)\,[a_j^N(1-k)+\beta]/\,2M.$$

$$(w-w')\bullet d \,/\, (a_j^N + \beta - ka_j^N) \le \|w-w'\|\bullet\|d\| \,/\, [a_j^N(1-k)+\beta]$$

$$\le 2M\,\|d\| \,/\, [a_j^N(1-k)+\beta] < \gamma/2 \quad \square.$$

<u>Claim:</u> $[k\psi+\psi] \,/\, (a_j^N + \beta - ka_j^N) \le \gamma/2$.

Proof: $\psi = [\gamma/(2(1+k))]\ [\theta((T-1)/T)^{.5}(1-k) + \beta] \leq [\gamma/(2(1+k))]\ [a_j^N(1-k)+\beta]$

$\Rightarrow (k\psi+\psi)/(a_j^N + \beta - ka_j^N) = (1+k)\psi/[a_j^N(1-k)+\beta] \leq \gamma/2$ □

This completes the proof of Proposition 3.

5. Concluding Remarks

There is a large literature on hedonic pricing of land and of jobs. The literature on land begins with the observation that land prices capitalize the amenities available by living on the land; e.g., police protection, good schools or environmental quality. Although empirical estimates of the hedonic price relationship often assume linearity, this assumption has no theoretical foundation. Similarly, there is large empirical literature that seeks to explain equilibrium wages as a function of the attributes of jobs. For example, jobs that expose workers to environmental risks should pay higher wages in equilibrium in order to compensate. As in the literature on land prices, there is no theoretical reason to believe this relationship should be linear. In contrast, we have shown that when the hedonic prices are on the attributes of workers (in contrast to the attributes of jobs), there is a *linear* hedonic price function in equilibrium, provided the economy is large.

Our approximation theorem can be interpreted as a "core-convergence" theorem in the sense that core payoffs converge to competitive payoffs, since we can interpret hedonic core payoffs as market prices.[8] If $p \in R^T$ are the market prices of attributes, the condition that no firm could make positive profit is $V(A)-p \cdot A \leq 0$ for all A. All firms in equilibrium

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[8] Subsequent to circulation of this paper, Wooders has also circulated papers discussing core convergence and hedonic pricing; see her 1991 working paper, and her seminar paper "Equivalence of Effective Small Groups and Competition" circulated at Berkeley in April 1992. Wooders conjectures that by approaching core convergence through an equivalence between games and markets, one can inherit for games the core convergence results known for markets. Perhaps our convergence result can be simplified through this approach, but as yet this has not been done. (Wooders discusses the case of homogeneous V.)

make zero profit only if $V(A^N)-p\cdot A^N=0$. But these two conditions define a hedonic core payoff $w=p$. A zero-profit price-taking equilibrium may not exist when V is not homogeneous. Payoffs in the ε-hedonic core would then be equilibrium prices if we defined equilibrium such that profit could not be increased by more than $\varepsilon|A^S|$ for any set of workers S.

Three natural applications of our model are as follows:

Coalition Production: A coalition of players with attributes A^S produces profit in amount $V(A^S)$ and sells a vector of outputs at fixed prices. The average payoff to coalitions might vary with size, e.g., as described by Figure 2 where V is not homogeneous. The comparative static results in Section 4 can be interpreted to mean that a worker with good management skills will be rewarded less in equilibrium if the proportional representation of management skills in the population grows. The hedonic price for management skills will fall. For a broader discussion of coalition production in which players cooperate in forming firms and then sell their output at fixed prices established in a market, see Ichiishi (1991).

Exchange Economies with Transferable Utility: Player i's utility can be represented as $f^i(y) = y_1 + g^i(y_2,...,y_k)$, where $y\in\mathbf{R}^k_+$ is a vector of private goods. Suppose the support of F is on T points, each on a coordinate axis, so that each point represents a "type" of player. The vector A^S represents the numbers of players of each type, and $V(A^S)$ is the sum of their utilities, where the private goods have been distributed so as to maximize the sum of utilities. In this example, V is homogeneous of degree one and the core is nonempty. The comparative static result implies that if the proportion of one type of player increases, their payoff in the core will decrease. To link this result with comparative statics on commodities, we might want to think of a "type" as the group of people endowed with a particular commodity. If that type of player becomes more abundant, the aggregate supply of the commodity increases. However aggregate demand also changes since players demand commodities as well as supply them.

Club Economies: The basic point of the club model is that groups of players confer externalities on each other, and that "optimal" groups of each composition are multiples of a particular optimal finite size, as depicted, for example, in Figure 2. The earliest type of externality considered was that agents could pool private resources to produce public goods, and noncontributors could be excluded from using the public goods. A more general type of externality is simply that agents want to be in groups with other agents who possess different skills or talents. (For an early treatment of such games see Drèze and Greenberg (1980), and for a summary see Scotchmer (1992).) Such a model is formally identical to the coalition production model, where agents pool their skills to produce output efficiently, and the efficient pooling of skills may require complementarities rather than homogeneity in abilities. In the coalition production example, the output is real-valued profit, whereas in the club example, output is utility, which is real-valued if there is a quasilinear good.

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APPENDIX

A The ε-Hedonic Core

See Corollary →

Proposition A.1 *If $A^N \in int\boldsymbol{R}^T_+$ and $\varepsilon \geq \overline{A}[v(a^N) - V(A^N)/|A^N|]$ then $C^H_\varepsilon(N,V) \neq \emptyset$.*

Proof: Let w_0 be a supporting hyperplane to $\hat{V}$ at A^N i.e. $w_0 \in \mathbf{R}^T$ and

1. $w_0 \cdot A^N = \hat{V}(A^N)$
2. $w_0 \cdot A \geq \hat{V}(A)$ for every $A \in \mathbf{R}^T_+$

By 2,

$$w_0 \cdot a = w_0 \cdot \frac{A}{|A|} \geq \frac{\hat{V}(A)}{|A|} = v(a) \quad \text{for every } a \in \Delta.$$

Let

$$w = w_0 - \frac{\hat{V}(A^N) - V(A^N)}{|A^N|}\mathbf{1}$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{R}^T$.

Claim: $w \in C^H_\varepsilon(N,V)$

1. (feasibility)
$$w \cdot A^N = w_0 \cdot A^N - (\hat{V}(A^N) - V(A^N)) = V(A^N)$$

2. (no coalition can improve by more than ε per 'capita.')
Suppose there exists a coalition S such that

$$V(A^S) > w \cdot A^S + \varepsilon|A^S| = w_0 \cdot A^S - \frac{|A^S|}{|A^N|}\left[\hat{V}(A^N) - V(A^N)\right] + \varepsilon|A^S|.$$

This implies

$$\begin{aligned} v(a^S) &\geq \frac{V(A^S)}{|A^S|} > w_0 \cdot a^S - \frac{1}{|A^N|}\left[\hat{V}(A^N) - V(A^N)\right] + \varepsilon \\ &\geq v(a^S) - \frac{1}{|A^N|}\left[\hat{V}(A^N) - V(A^N)\right] + \varepsilon \\ &= v(a^S) - \left[v(a^N) - \frac{V(A^N)}{|A^N|}\right] + \varepsilon. \end{aligned}$$

But this implies that

$$v(a^N) - \frac{V(A^N)}{|A^N|} > \varepsilon,$$

which contradicts the choice of ε. ■

Corollary *Given $\varepsilon > 0$, if $V(ra)/r$ converges uniformly to $v(a)$ on the simplex, then there exists n such that if $|N| \geq n$ and $a^N \in int\mathbf{R}^T_+$ then $C^H_\varepsilon(N, V) \neq \emptyset$.*

Lemma A.1 *v is concave.*[1]

Proof: Suppose $a = \lambda_1 a^1 + \lambda_2 a^2$, $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$. $v(a) = \sup_{r>0} V(ra)/r$, so choose $n_1, n_2 \in \mathbf{N}$ such that

$$\frac{V((n_i\lambda_i)a^i)}{n_i\lambda_i} > v(a^i) - \varepsilon/2\lambda_i, \quad i = 1, 2$$

Let $n = n_1 n_2$, then

$$na = n(\lambda_1 a^1 + \lambda_2 a^2) = n_2(n_1\lambda_1)a^1 + n_1(n_2\lambda_2)a^2$$

so by superadditivity,

$$V(na) \geq n_2 V((n_1\lambda_1)a^1) + n_1 V((n_2\lambda_2)a^2)$$

$$\begin{aligned} v(a) &\geq \frac{V(na)}{n} \geq \frac{1}{n_1}V((n_1\lambda_1)a^1) + \frac{1}{n_2}V((n_2\lambda_2)a^2) \\ &= \lambda_1 \frac{V((n_1\lambda_1)a^1)}{n_1\lambda_1} + \lambda_2 \frac{V((n_2\lambda_2)a^2)}{n_2\lambda_2} \\ &> \lambda_1(v(a^1) - \varepsilon/2\lambda_1) + \lambda_2(v(a^2) - \varepsilon/2\lambda_2) \\ &= \lambda_1 v(a^1) + \lambda_2 v(a^2) - \varepsilon \end{aligned}$$

but ε was arbitrary, so $v(a) \geq \lambda_1 v(a^1) + \lambda_2 v(a^2)$. ■

Corollary *$\hat{V}$ is concave.*

Proof: Let $A_1, A_2 \in \mathbf{R}^T_+ \setminus \{0\}$ and $\lambda \in [0, 1]$ be given. Let $a_i = A_i/|A_i|$ and let $A = (1-\lambda)A_1 + \lambda A_2$ then $|A| = (1-\lambda)|A_1| + \lambda|A_2|$.

$$\begin{aligned} \hat{V}((1-\lambda)A_1 + \lambda A_2) &= \hat{V}(A) = |A|v(A/|A|) \\ &= |A|v\left(\frac{(1-\lambda)A_1}{|A|} + \frac{\lambda A_2}{|A|}\right) \\ &= |A|v\left(\frac{(1-\lambda)|A_1|}{|A|}a_1 + \frac{\lambda|A_2|}{|A|}a_2\right) \\ &\geq |A|\frac{(1-\lambda)|A_1|}{|A|}v(a_1) + |A|\frac{\lambda|A_2|}{|A|}v(a_2) \\ &= (1-\lambda)|A_1|v(a_1) + \lambda|A_2|v(a_2) = (1-\lambda)\hat{V}(A_1) + \lambda\hat{V}(A_2) \end{aligned}$$

■

[1] We include this lemma for completeness. Aumann has previously remarked on the concavity of v— along with everything else true and interesting in game theory. The proof given here follows the methods in Scotchmer and Wooders (1988).

Lemma A.2 *Let $\{A^i\}$ be a sequence in $\boldsymbol{R}_+^T$ with $0 < \underline{A} \leq A^i \leq \overline{A}$ for all i. Let $k \in \boldsymbol{N}$ and $r > 0$ be given, then there exists an n such that if $n_1, \ldots, n_k \in \boldsymbol{N}$ with $n_1, \ldots, n_k \geq n$ and $S \subset \boldsymbol{N}$ with $|S| = \sum_1^k n_i$ then there exist $S_1, \ldots, S_k \subset S$ such that $\bigcup_1^k S_i = S$, $|S_i| = n_i$ and $a^{S_i} \in B(a^S, r)$.*

Proof: The proof is by induction on k.
$\boldsymbol{k = 2}$ Suppose not, then there exists $r > 0$ such that for every n there exist $n_1, n_2 \geq n$ and $S \subset \mathrm{N}$ with $|S| = n_1 + n_2$ such that there do not exist $S_1, S_2 \subset S$ with $S_1 \bigcup S_2 = S$, $|S_i| = n_i$ and $a^{S_i} \in B(s^S, r)$.
Idea: Choose $S_1, S_2 \subset S$ with $S_1 \bigcup S_2 = S$, $|S_i| = n_i$ and with $||a^{S_1} - a^{S_2}||$ minimal. Consider the hyperplane through a^{S_i} perpendicular to $p = a^{S_1} - a^{S_2}$. a^{S_1} is a convex combination of

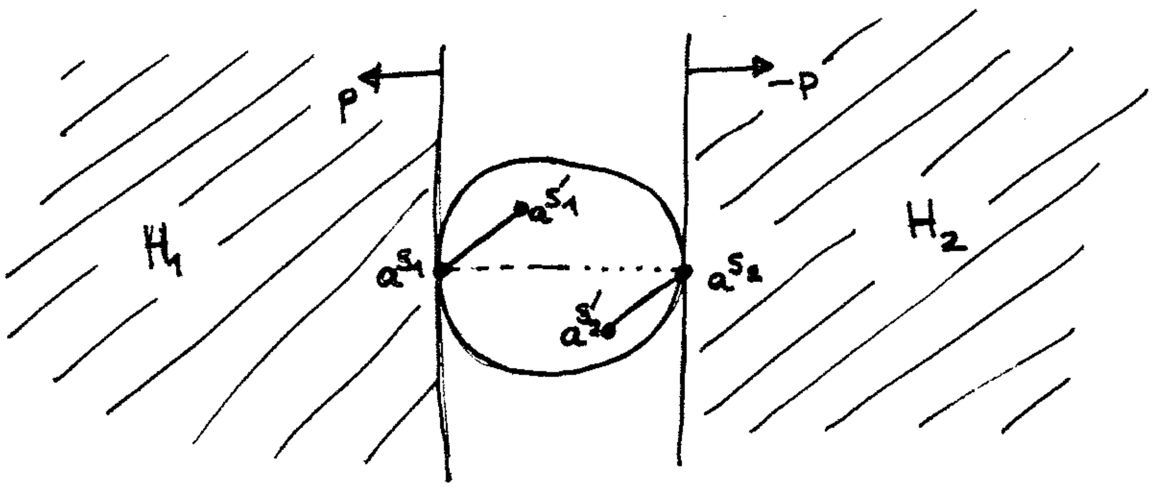


Figure 1: Trading Places

the points a^i, $i \in S_1$ so there must be an $i_1 \in S_1$ such that $a^{i_1} \in H_1$, the closed half space to the left of a^{S_1}. Similarly, there is an $i_2 \in S_2$ such that $a^{i_2} \in H_2$, the closed half space to the right of a^{S_2}. We trade i_1 for i_2 to get coalitions S_1', S_2' with compositions $a^{S_1'}, a^{S_2'}$. In going from a^{S_1} to $a^{S_1'}$ we move away from the hyperplane through a^{S_1} toward a^{S_2}. *If* n is large, $||a^{S_i} - a^{S_i'}||$ is small and so $||a^{S_1'} - a^{S_2'}|| < ||a^{S_1} - a^{S_2}||$ which is a contradiction.

Choose n such that $\frac{2\overline{A}\sqrt{2}}{n\underline{A}} - \frac{(\underline{A})^2 r^2}{2\sqrt{2}(\overline{A})^2} < 0$ and let $n_1, n_2 \geq n$ and S be as above, then given $S_1, S_2 \subset S$ with $S_1 \bigcup S_2 = S$, and $|S_i| = n_i$ we have $||a^{S_1} - a^{S_2}|| \geq r$ otherwise $a^{S_1}, a^{S_2} \in B(a^S, r)$. Choose $S_1, S_2 \subset S$ with $S_1 \bigcup S_2 = S$, $|S_i| = n_i$ and with $||a^{S_1} - a^{S_2}||$ minimal. Let $p = a^{S_1} - a^{S_2}$ and let

$$H_1 = \{x \mid p \cdot x \geq p \cdot a^{S_1}\}$$
$$H_2 = \{x \mid (-p) \cdot x \geq (-p) \cdot a^{S_2}\}$$

Choose $i_2 \in S_1$ such that $a^{i_1} \in H_1$ and choose $i_2 \in S_2$ such that $a^{i_2} \in H_2$. Let $S_1' = (S_1 \setminus \{i_1\}) \bigcup \{i_2\}$, $S_2' = (S_2 \setminus \{i_2\}) \bigcup \{i_1\}$.

$$|A^{S_1'}| = |A^{S_1 \setminus \{i_1\}}| + |A^{i_2}| = |A^{S_1}| - |A^{i_1}| + |A^{i_2}|$$

$$a^{S_1'} = \frac{A^{S_1'}}{|A^{S_1'}|} = \frac{A^{S_1} - A^{i_1} + A^{i_2}}{|A^{S_1'}|} = \frac{1}{|A^{S_1'}|}\left[|A^{S_1}|a^{S_1} - |A^{i_1}|a^{i_1} + |A^{i_2}|a^{i_2}\right]$$

Claim: $||a^{S_i} - a^{S_i'}|| \leq \frac{2\overline{A}\sqrt{2}}{n_i\underline{A}}$ *where* $n_i = |S_i| = |S_i'|$.

$$a^{S_1} = \frac{1}{|A^{S_1'}|}\left[|A^{S_1}|a^{S_1} - |A^{i_1}|a^{S_1} + |A^{i_2}|a^{S_1}\right]$$

$$\begin{aligned}
||a^{S_1} - a^{S_1'}|| &= \frac{1}{|A^{S_1'}|}\left\||A^{i_1}|(a^{i_1} - a^{S_1}) + |A^{i_2}|(a^{S_1} - a^{i_2})\right\| \\
&\leq \frac{1}{|A^{S_1'}|}\left[|A^{i_1}|||a^{i_1} - a^{S_1}|| + |A^{i_2}|||a^{S_1} - a^{i_2}||\right] \\
&\leq \frac{1}{|A^{S_1'}|}\left[|A^{i_1}|\sqrt{2} + |A^{i_2}|\sqrt{2}\right] \\
&= \frac{\sqrt{2}(|A^{i_1}| + |A^{i_2}|)}{|A^{S_1'}|} \leq \frac{2\overline{A}\sqrt{2}}{n_1\underline{A}}
\end{aligned}$$

Similarly, $||a^{S_2} - a^{S_2'}|| \leq \frac{2\overline{A}\sqrt{2}}{n_2\underline{A}}$. □

Claim: $p \cdot (a^{S_1'} - a^{S_1}) \leq -\underline{A}r^2/n_1\overline{A}$

$$\begin{aligned}
p \cdot (a^{S_1'} - a^{S_1}) &= p \cdot \left[\frac{1}{|A^{S_1'}|}\left(|A^{i_1}|(a^{S_1} - a^{i_1}) + |A^{i_2}|(a^{i_2} - a^{S_1})\right)\right] \\
&= \frac{1}{|A^{S_1'}|}\left[|A^{i_1}|(p \cdot a^{S_1} - p \cdot a^{i_1}) + |A^{i_2}|(p \cdot a^{i_2} - p \cdot a^{S_1})\right] \\
&\leq \frac{1}{|A^{S_1'}|}\left[0 + |A^{i_2}|(p \cdot a^{S_2} - p \cdot a^{S_1})\right] \\
&= -\frac{|A^{i_2}|}{|A^{S_1'}|}||p||^2 \leq -\frac{\underline{A}r^2}{n_1\overline{A}}
\end{aligned}$$

□

By symmetry, $(-p) \cdot (a^{S_2'} - a^{S_2}) \leq -\underline{A}r^2/n_2\overline{A}$.

Claim: $||a^{S_1'} - (a^{S_1} + a^{S_2})/2|| < ||p/2||$

$$\begin{aligned}
||a^{S_1'} - (a^{S_1} + a^{S_2})/2||^2 &= ||a^{S_1'} - a^{S_1} + (a^{S_1} - a^{S_2})/2||^2 = ||a^{S_1'} - a^{S_1} + p/2||^2 \\
&= \left[a^{S_1'} - a^{S_1} + p/2\right] \cdot \left[a^{S_1'} - a^{S_1} + p/2\right] \\
&= ||a^{S_1'} - a^{S_1}||^2 + p \cdot (a^{S_1'} - a^{S_1}) + ||p/2||^2 \\
&= ||a^{S_1'} - a^{S_1}||\left[||a^{S_1'} - a^{S_1}|| + \frac{p \cdot (a^{S_1'} - a^{S_1})}{||a^{S_1'} - a^{S_1}||}\right] + ||p/2||^2 \\
&\leq ||a^{S_1'} - a^{S_1}||\left[||a^{S_1'} - a^{S_1}|| - \frac{\underline{A}r^2}{n_1\overline{A}||a^{S_1'} - a^{S_1}||}\right] + ||p/2||^2 \\
&\leq ||a^{S_1'} - a^{S_1}||\left[||a^{S_1'} - a^{S_1}|| - \frac{\underline{A}r^2}{n_1\overline{A}\frac{2\overline{A}\sqrt{2}}{n_1\underline{A}}}\right] + ||p/2||^2 \\
&\leq ||a^{S_1'} - a^{S_1}||\left[\frac{2\overline{A}\sqrt{2}}{n_1\underline{A}} - \frac{(\underline{A})^2r^2}{2\sqrt{2}(\overline{A})^2}\right] + ||p/2||^2 < ||p/2||^2
\end{aligned}$$

as $n_1 \geq n$ and $a^{S_1'} \neq a^{S_1}$ by the previous Claim. □

Similarly, $\|a^{S_2'} - (a^{S_1} + a^{S_2})/2\| < \|p/2\|$. Therefore,

$$\begin{aligned}\|a^{S_1'} - a^{S_2'}\| &\leq \|a^{S_1'} - (a^{S_1} + a^{S_2})/2\| + \|(a^{S_1} + a^{S_2})/2 - a^{S_2'}\| \\ &< \|p/2\| + \|p/2\| = \|p\| = \|a^{S_1} - a^{S_2}\|\end{aligned}$$

and this is a contradiction.

Induction Hypothesis *Now suppose for $k = 1, 2, \ldots, m$ that for every $r > 0$ there exists $n(k, r)$ such that if $n_1, n_2, \ldots, n_k \geq n(k, r)$ and $S \subset \boldsymbol{N}$ with $|S| = \sum_1^k n_i$ then there exist $S_1, \ldots, S_k \subset S$ such that $\bigcup_1^k S_i = S$, $|S_i| = n_i$ and $a^{S_i} \in B(a^S, r)$.*

Let $n(m+1, r) = \max\{n(m, r/2), n(2, r/2)\}$ and suppose $n_1, n_2, \ldots, n_{m+1} \geq n(m+1, r)$ and $S \subset \mathbf{N}$ with $|S| = \sum_1^{m+1} n_i$, then since the result is true for $k = 2$, there exist $S', S_{m+1} \subset S$ with $S' \bigcup S_{m+1} = S$, $|S'| = \sum_1^m n_i$, $|S_{m+1}| = n_{m+1}$ and $a^{S'}, a^{S_{m+1}} \in B(a^S, r/2)$. By the induction hypothesis there exist $S_1, S_2, \ldots, S_m \subset S'$ with $\bigcup_1^m S_i = S'$, $|S_i| = n_i$ and $a^{S_i} \in B(a^{S'}, r/2)$ for $i = 1, 2, \ldots, m$. For $i \leq m$,

$$\|a^{S_i} - a^S\| \leq \|a^{S_i} - a^{S'}\| + \|a^{S'} - a^S\| < r/2 + r/2 = r$$

therefore $a^{S_i} \in B(a^S, r)$ for $i = 1, 2, \ldots, m+1$. ■

Lemma A.3 *Suppose $V(ra)/r$ converges uniformly to $v(a)$. Given $\phi > 0$ there exists r_0 such that for any $\varepsilon \geq 0$ if $U \in \boldsymbol{R}^{|N|}$ and S is a coalition with*

1. $|A^S| \geq r_0$
2. $U^S \leq (v(a^S) - \phi - \varepsilon)|A^S|$

then $V(A^S) > U^S + \varepsilon|A^S|$ so that $U \notin C_\varepsilon(N, V)$.

Proof: $V(ra)/r$ converges uniformly to $v(a)$ so there exists r_0 such that $r \geq r_0 \Longrightarrow V(ra) > (v(a) - \phi)r$. Thus conditions 1 and 2 give

$$V(|A^S|a^S) - U^S > (v(a^S) - \phi)|A^S| - (v(a^S) - \phi - \varepsilon)|A^S| = \varepsilon|A^S|.$$ ■

Corollary *Suppose $V(ra)/r$ converges uniformly to $v(a)$. Given $\phi, \alpha > 0$ there exists n_0 such that for any $\varepsilon \geq 0$ if*

1. $|N| \geq n_0$
2. $|S| \geq \alpha|N|$
3. $U \in C_\varepsilon(N, V)$

then $u^S > v(a^S) - \phi - \varepsilon$.

Proof: Let r_0 be as in the Lemma. Take $n_0 = r_0/\alpha\underline{A}$, then

$$|A^S| \geq |S|\underline{A} \geq \alpha|N|\underline{A} \geq \alpha n_0 \underline{A} = r_0$$

but $U \in C_\varepsilon(N, V)$ so by the Lemma,

$$u^S = U^S/|A^S| > v(a^S) - \phi - \varepsilon.$$

■

Lemma A.4

1. $|A^S| \geq \alpha|A^N| \Longrightarrow |S| \geq \alpha|N|\underline{A}/\overline{A}$

2. $|S| \geq \alpha|N| \Longrightarrow |A^S| \geq \alpha|A^N|\underline{A}/\overline{A}$

Proof:

1. $\alpha \leq \frac{|A^S|}{|A^N|} \leq \frac{|S|\overline{A}}{|N|\underline{A}} \Longrightarrow |S| \geq \alpha|N|\underline{A}/\overline{A}$

2. $\frac{|A^S|}{|A^N|} \geq \frac{|S|\underline{A}}{|N|\overline{A}} \geq \frac{\alpha|N|\underline{A}}{|N|\overline{A}} = \alpha\underline{A}/\overline{A}$ ■

Proposition A.2 *Let $\alpha \in (0, 1], \delta > 0$ be given. Suppose*

1. $V \geq 0$

2. $V(ra)/r$ *converges uniformly to* $v(a)$

3. $\hat{V}$ *is differentiable at* a^F

Then there exist $n, r, \varepsilon_0 > 0$ such that if $|N| \geq n, \|a^N - a^F\| \leq r$, $\varepsilon \in [0, \varepsilon_0]$ and $|S| \leq \alpha|N|$ then

$$U \in C_\varepsilon, w \in C_\varepsilon^H \Longrightarrow |w \cdot A^S - U^S| < \delta|A^S|.$$

Corollary *Let $\alpha \in (0, 1], \delta > 0, \theta \in (0, 1)$ be given. Under the conditions in the Proposition there exist $n, \varepsilon_0 > 0$ such that if $|N| \geq n$ and $\varepsilon \in [0, \varepsilon_0]$ then with probability at least $1 - \theta$,*

$$U \in C_\varepsilon, w \in C_\varepsilon^H \Longrightarrow |w \cdot A^S - U^S| < \delta|A^S|.$$

Proof: Let $D\hat{V}(a^F) = w^F$. We first show that there exist $n, r, \varepsilon_0 > 0$ such that if $|N| \geq n$, $\|a^N - a^F\| \leq r$, $\varepsilon \in [0, \varepsilon_0]$ and $U \in C_\varepsilon(N, V)$ then $|w^F \cdot A^S - U^S| < \delta|A^S|$. For this it suffices to show $|w^F \cdot a^S - u^S| < \delta$. If $|S| \geq \alpha|N|$ then by Lemma A.4,

$$|A^S| \geq \gamma|A^N| \tag{1}$$

where $\gamma = \alpha\underline{A}/\overline{A}$. Note that

$$\gamma \in (0, 1) \tag{2}$$

as $\underline{A} < \overline{A}$ and $\alpha \in (0, 1]$.

Part I: We show $u^S = U^S/|A^S| > w^F \cdot a^S - \delta$
Suppose to the contrary that $u^S \leq w^F \cdot a^S - \delta$.

We first show that it suffices to consider a^S outside a small neighborhood of a^F.

Idea: $\hat{V}$ is homogeneous of degree 1 and $D\hat{V}(a^F) = w^F$ so $v(A^F) = \hat{V}(a^F) = w^F \cdot a^F$ by Euler's theorem for homogeneous functions. v is continuous at a^F so if a^S is close to a^F (which we denote by $a^S \sim a^F$) then $v(a^S) \sim v(a^F) = w^F \cdot a^F \sim w^F \cdot a^S$. If $u^S \leq w^F \cdot a^S - \delta$ and ϕ, ε are small, then

$$u^S \leq w^F \cdot a^S - \delta \sim v(a^S) - \delta < v(a^S) - \phi - \varepsilon.$$

But then $U \notin C_\varepsilon$ by the Corollary to Lemma A.3 (assuming $|N|$ is large and $|S| \geq \alpha|N|$).

Claim 1 *It suffices to consider a^S such that $\|a^S - a^F\| \geq r_0$ for some $r_0 > 0$.*

By the Corollary to Lemma A.3, there exists n_0 such that if $|N| \geq n_0$ then $u^S > v(a^S) - \delta/4 - \varepsilon$. v is concave and therefore continuous on the interior of its domain (Roberts and Varberg page 93) so

$$a^S \sim a^F \Longrightarrow v(a^S) \sim v(a^F) = w^F \cdot a^F \sim w^F \cdot a^S.$$

If $\varepsilon \in [0, \delta/4]$ then $\delta/4 + \varepsilon \leq \delta/2 < \delta$ and

$$u^S > v(a^S) - \delta/4 - \varepsilon > v(a^S) - \delta \sim w^F \cdot a^S - \delta$$

Therefore there exists $r_0 > 0$ such that if $|N| \geq n_0$ and $\|a^S - a^F\| < r_0$ then $u^S > w^F \cdot a^S - \delta$. □

We can assume r_0 is so small that

$$\frac{1}{1-\gamma} < \frac{2\sqrt{2}}{r_0} \tag{3}$$

Overview Comment We will construct a region D, in the simplex, which will contain the composition of a blocking coalition. Let $\rho^F = (a^F, v(a^F))$. Let $a \in \partial\Delta = \{a \in \Delta \mid a_t = 0 \text{ for some } t\}$ be given and let $\rho^a = (a, w^F \cdot a - \delta)$. Let $L(\rho^F, \rho^a, \lambda)$ be the height of the line through ρ^F and ρ^a at the point $a^F + \lambda(a - a^F)$. Further, let

$$f(\rho^F, \rho^a, \lambda) = v(a^F + \lambda(a - a^F)) - L(\rho^F, \rho^a, \lambda).$$

Because $\hat{V}$ is differentiable at a^F, w^F is a supporting hyperplane for $\hat{V}$ at a^F (Roberts and Varberg page 115), i.e. $w^F \cdot a^F = \hat{V}(a^F)$ and $w^F \cdot A \geq \hat{V}(A)$ for every $A \in \mathbf{R}^T_+ \setminus \{0\}$. On the simplex this gives $w^F \cdot a^F = v(a^F)$ and $w^F \cdot a \geq v(a)$ for every $a \in \Delta$. Therefore for every $a \in \partial\Delta$ there exists $\lambda_a \in (0,1)$ such that $f(\rho^F, \rho^a, \lambda_a) > 0$. Since v is concave, f is concave in λ and continuous for $\lambda \in [0,1)$. Therefore for every $a \in \partial\Delta$ there is a neighborhood of λ_a on which $f(\rho^F, \rho^a, \cdot)$ is positive. We show that there is such a neighborhood, $[\lambda_0, \lambda_1]$ say, that is independent of a. We will use the fact that f positive on this interval to show the existence of a coalition which can ε-block. From this interval we also construct the region $D = \{a^F + \lambda(a - a^F) \mid \lambda \in [\lambda_0, \lambda_1], a \in \Delta\}$.

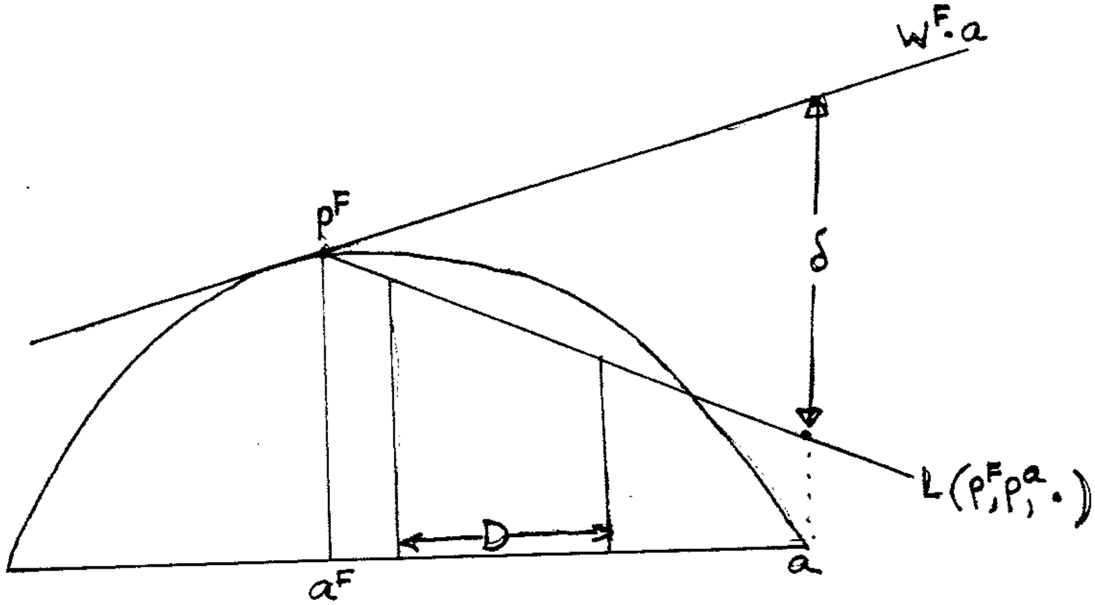


Figure 2: Clearly $f(\rho^F, \rho^a, 0) = 0$.

Construction of the Function f

Let $a \in \Delta$, $\tilde{a} \in \Delta \setminus \partial\Delta$ be distinct. Consider all points of the form $\tilde{a} + \lambda(a - \tilde{a})$ for $\lambda \in \mathbf{R}$.

$$[\tilde{a} + \lambda(a - \tilde{a})]_t = \tilde{a}_t + \lambda(a_t - \tilde{a}_t)$$

so the t^{th} coordinate of $\tilde{a} + \lambda(a - \tilde{a})$ is a linear function of λ which is positive for $\lambda = 0$. Therefore there exists a unique $\lambda_b \geq 1$ such that $\tilde{a} + \lambda_b(a - \tilde{a}) \in \partial\Delta$. Note that $\lambda_b = 1$ if $a \in \partial\Delta$. (Further,

$$0 \leq \lambda < \lambda_b \Longrightarrow \tilde{a} + \lambda(a - \tilde{a}) \in \Delta \setminus \partial\Delta$$

$$\lambda > \lambda_b \Longrightarrow \tilde{a} + \lambda(a - \tilde{a}) \notin \Delta.)$$

Let $a^b = \tilde{a} + \lambda_b(a - \tilde{a})$ and let

$$I(\tilde{a}, a) = \{\lambda \in \mathbf{R} \mid \tilde{a} + \lambda(a^b - \tilde{a}) \in \Delta\}.$$

$I(\tilde{a}, a)$ is a closed interval with right endpoint 1 and $0 \in \text{int} I(\tilde{a}, a)$ as $\tilde{a} \in \Delta \setminus \partial\Delta$. Let $\rho = (a, h)$, $\tilde{\rho} = (\tilde{a}, \tilde{h})$ where $h, \tilde{h} \in \mathbf{R}$. Let $L(\tilde{\rho}, \rho, \cdot) : I(\tilde{a}, a) \longrightarrow \mathbf{R}$ be the height of the line through $\tilde{\rho}$ and ρ above the simplex at the point $\tilde{a} + \lambda(a^b - \tilde{a})$ i.e.[1]

$$L(\tilde{\rho}, \rho, \lambda) = \tilde{h} + \lambda\lambda_b(h - \tilde{h}) = (1 - \lambda\lambda_b)\tilde{h} + \lambda\lambda_b h.$$

Let $f(\tilde{\rho}, \rho, \cdot) : I(\tilde{a}, a) \longrightarrow \mathbf{R}$ by

$$f(\tilde{\rho}, \rho, \lambda) = v(\tilde{a} + \lambda(a^b - \tilde{a})) - L(\tilde{\rho}, \rho, \lambda).$$

---

[1]

$$a^b = \tilde{a} + \lambda_b(a - \tilde{a}) \Longrightarrow a = \tilde{a} + (1/\lambda_b)(a^b - \tilde{a})$$

we want: $L(\tilde{\rho}, \rho, 0) = \tilde{h}$, $L(\tilde{\rho}, \rho, 1/\lambda_b) = h$ so $m(\text{slope}) = \lambda_b(h - \tilde{h})$

Since v is concave, f is concave in λ and continuous for $\lambda \in [0,1)$ (A concave function is continuous on the interior of its domain, Roberts and Varberg, page 93.) Let $\rho^F = (a^F, v(a^F))$ (Note that $a^F \in \Delta \setminus \partial\Delta$.) and given $a \in \partial\Delta$, let $\rho^a = (a, w^F \cdot a - \delta)$ then

$$\begin{aligned} f(\rho^F, \rho^a, \lambda) &= v(a^F + \lambda(a - a^F)) - L(\rho^F, \rho^a, \lambda) \\ &= v(a^F + \lambda(a - a^F)) - [v(a^F) + \lambda(w^F \cdot a - \delta - v(a^F))] \\ &= v((1-\lambda)a^F + \lambda a) - [(1-\lambda)v(a^F) + \lambda(w^F \cdot a - \delta)]. \end{aligned}$$

Choice of the Interval $[\lambda_0, \lambda_1]$ and Construction of the Region D

Claim: *For every $a \in \partial\Delta$ there exists $\lambda_a \in (0,1)$ such that $f(\rho^F, \rho^a, \lambda_a) > 0$.*

Fix $a \in \partial\Delta$ and let $V_a : I(a^F, a) \longrightarrow \mathbf{R}$ by

$$V_a(\lambda) = \hat{V}(a^F + \lambda(a - a^F)) = v(a^F + \lambda(a - a^F)).$$

V_a is concave and

$$DV_a(0) = D\hat{V}(a^F) \cdot (a - a^F) = w^F \cdot (a - a^F).$$

Let $L : I(a^F, a) \longrightarrow \mathbf{R}$ by

$$L(\lambda) = V_a(0) + \lambda DV_a(0) = v(a^F) + \lambda w^F \cdot (a - a^F).$$

L is the tangent line to V_a at $\lambda = 0$. Let m_a be the slope of the line $L(\rho^F, \rho^a, \cdot)$.

$$L(\rho^F, \rho^a, 0) = v(a^F) = L(0)$$

and

$$L(\rho^F, \rho^a, 1) = w^F \cdot a - \delta < w^F \cdot a = v(a^F) + w^F \cdot (a - a^F) = L(1)$$

as $w^F \cdot a^F = v(a^F)$. Therefore $DV_a(0) > m_a$. Let $\lambda' \in (0,1)$, let $a' = a^F + \lambda'(a - a^F)$ and let $\rho' = (a', v(a'))$. $L(\rho^F, \rho', \cdot)$ is a secant line for V_a so if λ' is close to 0 then a' is close to a^F and m', the slope of $L(\rho^F, \rho', \cdot)$, is close to $DV_a(0)$. Therefore there exists a' such that $m' > m_a$. Suppose $a' = a^F + \lambda_a(a - a^F)$ then $L(\rho^F, \rho', \lambda) > L(\rho^F, \rho^a, \lambda)$ for $\lambda > 0$. In particular, for $\lambda = \lambda_a$ we get $v(a^F + \lambda_a(a - a^F)) = v(a') > L(\rho^F, \rho^a, \lambda_a)$ so $f(\rho^F, \rho^a, \lambda_a) > 0$. □

Given $a \in \partial\Delta$, let

$$g(a) = \sup\{\lambda \in [0,1] \mid f(\rho^F, \rho^a, \lambda) \geq 0\}$$

Claim 2 *$f(\rho^F, \rho^a, \lambda) > 0$ for $\lambda \in (0, g(a))$.*

Since $f(\rho^F, \rho^a, \lambda_a) > 0$, $\lambda_a \leq g(a)$ and by convexity, $f(\rho^F, \rho^a, \lambda) > 0$ for $\lambda \in (0, \lambda_a]$. Let $\lambda_n \uparrow g(a)$ with $\lambda_n \geq \lambda_a$ and $f(\rho^F, \rho^a, \lambda_n) \geq 0$. Then by convexity, $f(\rho^F, \rho^a, \lambda) > 0$ for $\lambda \in [\lambda_a, \lambda_n)$ and therefore $f(\rho^F, \rho^a, \lambda) > 0$ for $\lambda \in [\lambda_a, g(a))$. □

Claim 3 $\lambda^* \equiv \inf\{g(a) \mid a \in \partial\Delta\} > 0$

$f(\rho^F, \rho^a, \lambda_a) > 0$ and $\lambda_a \in (0,1)$ so $g(a) > 0$. Suppose there exists a sequence, a_n, such that $g(a_n) \longrightarrow 0$. Assume without loss of generality that $a_n \longrightarrow a \in \partial\Delta$. For n large, $g(a_n) < \lambda_a \Longrightarrow f(\rho^F, \rho^{a_n}, \lambda_a) < 0$ which implies that $f(\rho^F, \rho^{a_n}, \lambda_a) \not\longrightarrow f(\rho^F, \rho^a, \lambda_a)$. But this is a contradiction as f is continuous in a (as $\lambda_a < 1$ and v is continuous on $\Delta \setminus \partial\Delta$). □

We now choose $[\lambda_0, \lambda_1]$ and the region $D = \{a^F + \lambda(a - a^F) \mid \lambda \in [\lambda_0, \lambda_1], a \in \Delta\}$. We also choose γ_0, γ_1 such that if $\gamma_0 \leq \|a^D - a^F\| \leq \gamma_1$ then $a^D \in D$.

Choose $r_I > 0$ such that $r_I < \inf\{\|a - a^F\| : a \in \partial\Delta\}$ and let

$$\alpha_0 = \sup\{\|a - a^I\| : a \in \partial\Delta, \|a^I - a^F\| \leq r_I\}$$

$$\alpha_1 = \inf\{\|a - a^I\| : a \in \partial\Delta, \|a^I - a^F\| \leq r_I\}.$$

Note that

$$\|a^I - a^F\| \leq r_I \Longrightarrow a^I \notin \partial\Delta \tag{4}$$

so that $\alpha_1 > 0$. Choose $\lambda_1 \in (0, \lambda^*)$ and let $\gamma_1 = \lambda_1\alpha_1$. Let $f : [0,1] \times [1/(1-\gamma), 2\sqrt{2}/r_0] \longrightarrow \mathbf{R}$ by $f(x,y) = (1-x)/(y-x)$. (Recall $\gamma \in (0,1)$ by (2) and $1/(1-\gamma) < 2\sqrt{2}/r_0$ by (3).) f is continuous and therefore uniformly continuous so there exists $\psi > 0$ such that

$$x \leq \psi \Longrightarrow |f(x,y) - f(0,y)| \leq \gamma_1/2\sqrt{2}. \tag{5}$$

$$\frac{\partial f}{\partial x} = \frac{(y-x)(-1) - (1-x)(-1)}{(y-x)^2} = \frac{x - y + 1 - x}{(y-x)^2} = \frac{1-y}{(y-x)^2} < 0$$

therefore f is decreasing in x. Choose $k \in \mathbf{N}$ such that

$$2\overline{A}/k\underline{A} \leq \min\{\psi, 1\} \tag{6}$$

and let

$$L = \min_y \left[f(0,y) - f\left(\frac{\underline{A}}{2k\overline{A}}, y\right)\right] > 0.$$

Choose $\lambda_0 \in (0, \lambda_1)$ and let $\gamma_0 = \lambda_0\alpha_0$. Assume λ_0 is so small that

$$\gamma_0 \leq Lr_0/8. \tag{7}$$

Claim 4 *There exists $M > 0$ such that $f(\rho^F, \rho^a, \lambda) \geq M$ for $\lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta$.*

As a function of a and λ, f is continuous on $\partial\Delta \times [\lambda_0, \lambda_1]$ as v is continuous on $\Delta \setminus \partial\Delta$ and $\lambda_1 < \lambda^* \leq 1$. Thus it suffices to show $f(\rho^F, \rho^a, \lambda) > 0$ for $a \in \partial\Delta$, $\lambda \in [\lambda_0, \lambda_1]$. And since f is concave in λ, it suffices to show $f(\rho^F, \rho^a, \lambda_i) > 0$ for $i = 0, 1$ and $a \in \partial\Delta$. Suppose there exist $a_n \in \partial\Delta$ such that $f(\rho^F, \rho^{a_n}, \lambda_i) \longrightarrow 0$. Assume without loss of generality that $a_n \longrightarrow a$, then $f(\rho^F, \rho^a, \lambda_i) = 0$. But this contradicts Claims 2 and 3. □

Next we partition $N \setminus S$ by Lemma A.2 into subcoalitions $S_1, \ldots, S_k$ such that $\gamma_0 \leq \|a^{N\setminus S_i} - a^F\| \leq \gamma_1$. The second inequality holds because we chose k in Lemma A.2 sufficiently large. Since k is large $a^{N\setminus S_i} \sim a^N$ and since $|N|$ is large $a^N \sim a^F$ so $a^{N\setminus S_1} \sim a^F$. But no matter how large k is, removing S_1 from N moves the mean, a^N, at least some distance away from $a^N \sim a^F$ to $a^{N\setminus S_1}$. This is because $a^{S_1} \sim a^{N\setminus S}$ is bounded away from $a^N \sim a^F$ as a^S is bounded away from a^F and S is a significant portion of the population ($|S| \geq \alpha|N|$). Therefore $a^{N\setminus S_1} \neq a^F$ and the first inequality holds because we chose γ_0 sufficiently small.

Claim: *$D = \{a^F + \lambda(a - a^F) \mid \lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta\}$ is compact.*

Let $f : [\lambda_0, \lambda_1] \times \partial\Delta \longrightarrow \Delta$ by $f(\lambda, a) = a^F + \lambda(a - a^F)$ then f is continuous and $D = f([\lambda_0, \lambda_1] \times \partial\Delta)$. □

Since $\hat{V}$ is concave, it is Lipschitz on the compact set $D \subset \Delta \setminus \partial\Delta$, which is in the interior of $\mathbf{R}^T_+ \setminus \{0\}$ (Roberts and Varberg page 93). Let $C > 0$ be a Lipschitz constant. Choose $r > 0$ such that

$$r \leq \min\left\{\frac{\gamma_1}{8}, \frac{Lr_0}{16}, \frac{r_0}{2}, \frac{M}{8C}, \frac{r_I}{2}\right\}. \tag{8}$$

Assume further that r is so small that

$$2r\|w^F\| \leq \delta/2 \tag{9}$$

and, since v is continuous at a^F, that

$$\|a^N - a^F\| \leq r \Longrightarrow |v(a^N) - v(a^F)| \leq \min\left\{\frac{\delta}{4}, \frac{M}{8}\right\}. \tag{10}$$

Assume $\|a^N - a^F\| \leq r$. To complete the proof we derive a contradiction.

Claim 5 *a^S is bounded away from a^N: $\|a^S - a^N\| \geq r_0 - r$*

$$\begin{aligned}\|a^S - a^N\| &= \|(a^S - a^F) + (a^F - a^N)\| \\ &\geq \|a^S - a^F\| - \|a^F - a^N\| \geq r_0 - r\end{aligned}$$

By Claim 1. □

by (8) $r_0 - r \geq r_0/2 > 0$ so $a^S \neq a^N$, which implies $a^{N\setminus S} \neq a^S$.

Claim: *$|A^{N\setminus S}|$ is neither trivially small nor very large: $\frac{1}{\sqrt{2}}(r_0 - r) \leq \frac{|A^{N\setminus S}|}{|A^N|} \leq 1 - \gamma$.*

By (1) $|A^S|/|A^N| \geq \gamma$, which implies $|A^{N\setminus S}|/|A^N| \leq 1 - \gamma$.

$$a^N - a^S = \frac{|A^S|}{|A^N|}a^S + \frac{|A^{N\setminus S}|}{|A^N|}a^{N\setminus S} - a^S = \frac{|A^{N\setminus S}|}{|A^N|}(a^{N\setminus S} - a^S).$$

Therefore

$$\frac{|A^{N\setminus S}|}{|A^N|} = \frac{\|a^N - a^S\|}{\|a^{N\setminus S} - a^S\|} \geq \frac{\|a^N - a^S\|}{\sqrt{2}} \geq \frac{r_0 - r}{\sqrt{2}}.$$ □

Let $|N\setminus S| = kq+l$ where $q, l \in \mathbf{Z}$ and $0 \le l < k$. By Lemma A.4, $|N\setminus S| \ge \frac{r_0 - r}{\sqrt{2}}|N|\underline{A}/\overline{A}$ so that $|N|$ large implies that q is large. Choose n_1 such that if $|N| \ge n_1$ then Lemma A.2 applies with k and r as above and $n_i = q$ or $q+1$. Let $S_1, \ldots, S_k$ be the partition of $N \setminus S$ from Lemma A.2 and assume $u^{S_1} \ge u^{S_i}$ for $i = 1, \ldots, k$.

Claim 6 *$a^{N\setminus S_1}$ is not too far away from a^F: $\|a^{N\setminus S_1} - a^F\| < \gamma_1 - 2r$*

By (8)

$$\frac{1}{1-\gamma} \le \frac{|A^N|}{|A^{N\setminus S}|} \le \frac{\sqrt{2}}{r_0 - r} \le \frac{\sqrt{2}}{r_0 - r_0/2} = \frac{2\sqrt{2}}{r_0}.$$

And

$$\frac{\underline{A}}{2k\overline{A}} \le \frac{q\underline{A}}{k(q+1)\overline{A}} \le \frac{q\underline{A}}{|N\setminus S|\overline{A}} \le \frac{|A^{S_1}|}{|A^{N\setminus S}|} \le \frac{(q+1)\overline{A}}{|N\setminus S|\underline{A}} \le \frac{(q+1)\overline{A}}{kq\underline{A}} \le \frac{2\overline{A}}{k\underline{A}} \le \min\{\psi, 1\}$$

by (6). Let $\beta = |A^{N\setminus S}|/|A^N|$, $\beta' = |A^{N\setminus(S\cup S_1)}|/|A^{N\setminus S_1}|$ then

$$\beta = \frac{|A^{N\setminus S}|}{|A^N|} = f\left(0, \frac{|A^N|}{|A^{N\setminus S}|}\right)$$

and

$$\beta' = \frac{|A^{N\setminus(S\cup S_1)}|}{|A^{N\setminus S_1}|} = \frac{|A^{N\setminus S}| - |A^{S_1}|}{|A^N| - |A^{S_1}|} = \frac{1 - \frac{|A^{S_1}|}{|A^{N\setminus S}|}}{\frac{|A^N|}{|A^{N\setminus S}|} - \frac{|A^{S_1}|}{|A^{N\setminus S}|}} = f\left(\frac{|A^{S_1}|}{|A^{N\setminus S}|}, \frac{|A^N|}{|A^{N\setminus S}|}\right).$$

So by (5), $|\beta' - \beta| \le \gamma_1/2\sqrt{2}$. Further, since f is decreasing in x,

$$|\beta' - \beta| = \beta - \beta' \ge \min_y \left[f(0,y) - f\left(\frac{\underline{A}}{2k\overline{A}}, y\right)\right] = L. \tag{11}$$

Note that

$$a^N = (1-\beta)a^S + \beta a^{N\setminus S}, \qquad a^{N\setminus S_1} = (1-\beta')a^S + \beta' a^{N\setminus(S\cup S_1)}.$$

Let $b^N = (1-\beta')a^S + \beta' a^{N\setminus S}$ (new weights, old points) then

$$\begin{aligned}
\|a^{N\setminus S_1} - a^F\| &\le \|a^{N\setminus S_1} - b^N\| + \|b^N - a^N\| + \|a^N - a^F\| \\
&= \beta'\|a^{N\setminus(S\cup S_1)} - a^{N\setminus S}\| + |\beta' - \beta|\|a^{N\setminus S} - a^S\| + \|a^N - a^F\| \\
&\le \|a^{N\setminus(S\cup S_1)} - a^{N\setminus S}\| + |\beta' - \beta|\sqrt{2} + r \\
&< r + \frac{\gamma_1}{2\sqrt{2}}\sqrt{2} + r = 2r + \frac{\gamma_1}{2} \le \frac{\gamma_1}{4} + \frac{\gamma_1}{2} \quad \text{by (8)} \\
&= \gamma_1 - \frac{\gamma_1}{4} \le \gamma_1 - 2r
\end{aligned}$$

□

Claim 7 *$a^{N\setminus S_1}$ is not too close to a^F: $\|a^{N\setminus S_1} - a^F\| > Lr_0/4$*

$$\begin{aligned}
\|a^{N\setminus S_1} - a^F\| &\ge \|a^{N\setminus S_1} - a^N\| - \|a^N - a^F\| \\
&\ge \|a^N - b^N\| - \|b^N - a^{N\setminus S_1}\| - \|a^N - a^F\| \\
&= |\beta' - \beta|\|a^{N\setminus S} - a^S\| - \beta'\|a^{N\setminus S} - a^{N\setminus(S\cup S_1)}\| - \|a^N - a^F\| \\
&> |\beta' - \beta|\|a^S - a^N\| - \|a^{N\setminus S} - a^{N\setminus(S\cup S_1)}\| - \|a^N - a^F\| \\
&> L(r_0 - r) - r - r \quad \text{by (11) and Claim 5} \\
&> \frac{Lr_0}{2} - \frac{Lr_0}{8} - \frac{Lr_0}{8} = \frac{Lr_0}{4} \quad \text{by (8)}
\end{aligned}$$

□

To complete the proof we show that $N \setminus S_1$ can ε-block. By Claim 4 we know that $f(\rho^F, \rho^a, \lambda) \geq M$ for $\lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta$. If $a^{N\setminus S_1} = a^F + \lambda(a - a^F)$ for some $\lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta$ and if ϕ, ε are small then $f(\rho^F, \rho^a, \lambda) - \phi - \varepsilon > 0$, i.e. $v(a^F + \lambda(a - a^F)) - \phi - \varepsilon > L(\rho^F, \rho^a, \lambda)$. To show that $N \setminus S_1$ can block, by the Corollary to Lemma A.3, it suffices to show $u^{N\setminus S_1} \leq L(\rho^F, \rho^a, \lambda)$. We do this only approximately: in the previous discussion we replace ρ^F by a point ρ^I, close to ρ^F. This will suffice.

$p^S = (a^S, u^S)$, $p^{N\setminus S} = (a^{N\setminus S}, u^{N\setminus S})$, $p^N = (a^N, u^N)$ and by Lemma 1, $p^N = (1-\beta)p^S + \beta p^{N\setminus S}$ where $\beta = |A^{N\setminus S}|/|A^N|$. Let $\tilde{\rho} = (\tilde{a}, \tilde{u}) = (a^{N\setminus(S\cup S_1)}, u^{N\setminus S})$ and let

$$\rho^I = (a^I, u^I) = (1-\beta)p^S + \beta\tilde{\rho}. \tag{12}$$

$\tilde{\rho} \sim p^{N\setminus S}$ by Lemma A.2 and so $\rho^I \sim p^N$ as both are the same convex combination of points that are close. We show that $p^N \sim \rho^F$ and therefore $\rho^I \sim \rho^F$. Then by continuity $f(\rho^I, \rho^a, \lambda) \geq M/2$ for $\lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta$. Hence if ϕ and ε are small then $f(\rho^I, \rho^a, \lambda) - \phi - \varepsilon > 0$ for $\lambda \in [\lambda_0, \lambda_1], a \in \Delta$. This will yield a contradiction to the Corollary to Lemma A.3.

Claim 8 $\|a^I - a^F\| < 2r$

$a^I = (1-\beta)a^S + \beta a^{N\setminus(S\cup S_1)}$, $a^N = (1-\beta)a^S + \beta a^{N\setminus S}$.

$$\begin{aligned} \|a^I - a^F\| &\leq \|a^I - a^N\| + \|a^N - a^F\| = \beta\|a^{N\setminus(S\cup S_1)} - a^{N\setminus S}\| + \|a^N - a^F\| \\ &\leq \|a^{N\setminus(S\cup S_1)} - a^{N\setminus S}\| + \|a^N - a^F\| < r + r = 2r \end{aligned}$$ □

Let $\phi = \varepsilon_0 = \min\{\delta/8, M/16\}$ and assume $\varepsilon \in [0, \varepsilon_0]$, then

$$\phi + \varepsilon \leq \min\left\{\frac{\delta}{4}, \frac{M}{8}\right\}. \tag{13}$$

By the Corollary to Lemma A.3, there exists n_2 such that

$$|N| \geq n_2, |S| \geq \alpha|N|, U \in C_\varepsilon \Longrightarrow u^S > v(a^S) - \phi - \varepsilon. \tag{14}$$

Assume $|N| \geq \max\{n_0, n_1, n_2\}$.

Claim 9 $|v(a^F) - u^N| < \min\{\delta/2, M/4\}$

$U \in C_\varepsilon(N, V)$ so U is feasible i.e. $U^N = \sum_{i\in N} U^i \leq V(A^N)$ which implies that $u^N \leq V(A^N)/|A^N| \leq v(a^N)$. By (14), $u^N > v(a^N) - \phi - \varepsilon$. Therefore $v(a^N) - \phi - \varepsilon < u^N \leq v(a^N)$. Further, $\|a^N - a^F\| \leq r$ so by (10), $|v(a^N) - v(a^F)| \leq \min\{\delta/4, M/8\}$. Thus

$$\begin{aligned} |v(a^F) - u^N| &\leq |v(a^F) - v(a^N)| + |v(a^N) - u^N| \\ &< \min\left\{\frac{\delta}{4}, \frac{M}{8}\right\} + \phi + \varepsilon \\ &\leq \min\left\{\frac{\delta}{4}, \frac{M}{8}\right\} + \min\left\{\frac{\delta}{4}, \frac{M}{8}\right\} \quad \text{by (13)} \\ &= \min\left\{\frac{\delta}{2}, \frac{M}{4}\right\} \end{aligned}$$ □

Claim 10 *$f(\rho^I, \rho^a, \lambda) \geq M/2$ for $\lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta$.*

$$\begin{aligned}
|f(\rho^I, \rho^a, \lambda) - f(\rho^F, \rho^a, \lambda)| &= |v(a^I + \lambda(a - a^I)) - L(\rho^I, \rho^a, \lambda) - v(a^F + \lambda(a - a^F)) \\
&\quad + L(\rho^F, \rho^a, \lambda)| \\
&\leq |v((1-\lambda)a^I + \lambda a) - v((1-\lambda)a^F + \lambda a)| \\
&\quad + |L(\rho^F, \rho^a, \lambda) - L(\rho^I, \rho^a, \lambda)| \\
&\leq C(1-\lambda)\|a^I - a^F\| + |(1-\lambda)v(a^F) + \lambda(w^F \cdot a - \delta) \\
&\quad - [(1-\lambda)u^I + \lambda(w^F \cdot a - \delta)]\,| \\
&\leq C\|a^I - a^F\| + (1-\lambda)|v(a^F) - u^I| \\
&= C\|a^I - a^F\| + (1-\lambda)|v(a^F) - u^N| \quad \text{by (12)} \\
&< C2r + \frac{M}{4} \leq \frac{M}{4} + \frac{M}{4} = \frac{M}{2} \quad \text{by Claim 9 and (8)}
\end{aligned}$$

So by Claim 4, $f(\rho^I, \rho^a, \lambda) \geq M/2$ for $\lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta$. □

Therefore $f(\rho^I, \rho^a, \lambda) - \phi - \varepsilon \geq M/2 - M/8 > 0$ for $\lambda \in [\lambda_0, \lambda_1], a \in \partial\Delta$ by (13).

We now show that $a^{N\setminus S_1}$ can ε-block.
$p^{N\setminus(S\cup S_1)} = (a^{N\setminus(S\cup S_1)}, u^{N\setminus(S\cup S_1)})$, $p^{N\setminus S_1} = (a^{N\setminus S_1}, u^{N\setminus S_1})$. Recall $\beta' = |A^{N\setminus(S\cup S_1)}|/|A^{N\setminus S_1}|$. We have

$$p^{N\setminus S_1} = (1-\beta')p^S + \beta' p^{N\setminus(S\cup S_1)}. \tag{15}$$

Let

$$\rho^D = (a^D, u^D) = (1-\beta')p^S + \beta'\tilde{\rho}. \tag{16}$$

Note that $a^D = a^{N\setminus S_1}$, and $\beta > \beta'$ by (11). Extend the line segment from $\tilde{a}$ to a^S until it

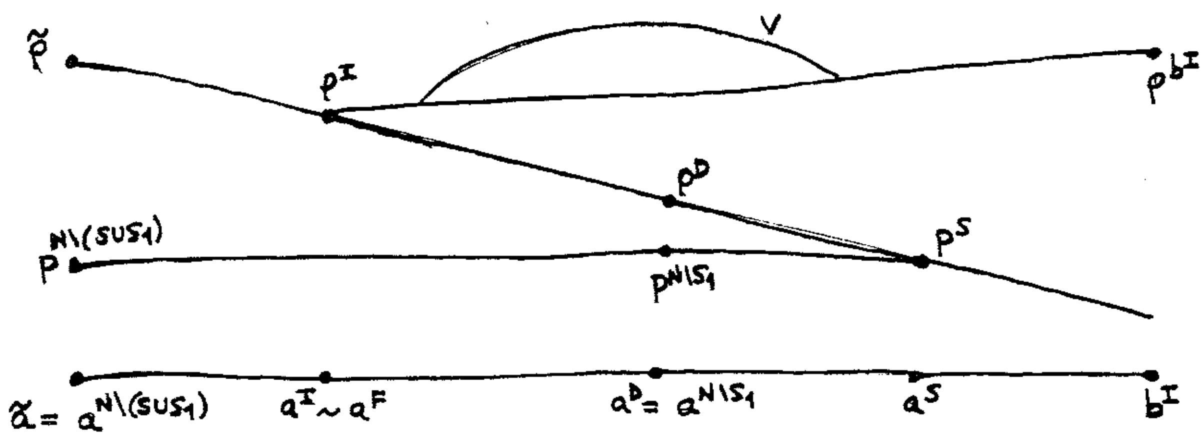


Figure 3: Who lives where.

meets $\partial\Delta$ in a point b^I and choose λ_D such that $a^{N\setminus S_1} = a^D = (1-\lambda_D)a^I + \lambda_D b^I$. We show $\lambda_D \in [\lambda_0, \lambda_1]$ and so, by the remark after (12), $f(\rho^I, \rho^{b^I}, \lambda_D) - \phi - \varepsilon > 0$. By the application of Lemma A.2, $N \setminus S = \cup_1^k S_i$. We assume that $u^{S_1} \geq u^{S_i}$ for $i = 1, 2, \ldots, k$. Therefore

$u^{N\setminus(S\cup S_1)} \leq u^{N\setminus S}$ and so $p^{N\setminus(S\cup S_1)}$ lies below $\tilde{\rho}$ and hence $p^{N\setminus S_1}$ lies below ρ^D. We show $L(\rho^I, \rho^{b^I}, \cdot)$ lies above p^S and hence above ρ^D, i.e. $L(\rho^I, \rho^{b^I}, \lambda_D) \geq u^D \geq u^{N\setminus S_1}$. But then

$$\begin{aligned} 0 &< f(\rho^I, \rho^{b^I}, \lambda_D) - \phi - \varepsilon \\ &= v(a^D) - L(\rho^I, \rho^{b^I}, \lambda_D) - \phi - \varepsilon \\ &\leq v(a^{N\setminus S_1}) - u^{N\setminus S_1} - \phi - \varepsilon. \end{aligned}$$

This contradicts the Corollary to Lemma A.3 and thus completes the proof.

Claim: $\gamma_0 \leq \|a^D - a^I\| \leq \gamma_1$

$$\begin{aligned} \|a^D - a^I\| &\leq \|a^D - a^F\| + \|a^F - a^N\| + \|a^N - a^I\| \\ &< \gamma_1 - 2r + r + \beta\|a^{N\setminus S} - a^{N\setminus(S\cup S_1)}\| \quad \text{by Claim 6} \\ &< \gamma_1 - r + r = \gamma_1 \end{aligned}$$

$$\begin{aligned} \|a^D - a^I\| &\geq \|a^D - a^F\| - \|a^F - a^I\| > \|a^{N\setminus S_1} - a^F\| - 2r \quad \text{by Claim 8} \\ &> \frac{Lr_0}{4} - 2r \geq \frac{Lr_0}{4} - \frac{Lr_0}{8} = \frac{Lr_0}{8} \geq \gamma_0 \quad \text{by Claim 7, (8) and (7)} \quad \square \end{aligned}$$

Claim 11 *There exists $b^I \in \partial\Delta$, $\lambda_D \in [\lambda_0, \lambda_1]$ such that $a^D = (1-\lambda_D)a^I + \lambda_D b^I$ and $f(\rho^I, \rho^{b^I}, \lambda_D) - \phi - \varepsilon > 0$.*

By Claim 8, $\|a^I - a^F\| < 2r$ and by (8), $2r \leq r_I$. Therefore $\|a^I - a^F\| \leq r_I$ and $a^I \notin \partial\Delta$ by (4). Choose $t > 0$ such that $b^I = a^I + t(a^D - a^I) \in \partial\Delta$ (n.b. $a^D \neq a^I$ as $\|a^D - a^I\| \geq \gamma_0 > 0$) then $a^D = (1 - 1/t)a^I + (1/t)b^I$ so let $\lambda_D = 1/t$.

$$\gamma_0 = \lambda_0\alpha_0 \leq \|a^D - a^I\| = \lambda_D\|b^I - a^I\| \leq \gamma_1 = \lambda_1\alpha_1$$

This implies $\lambda_D \in [\lambda_0, \lambda_1]$ by definition of α_0, α_1. By the remark after Claim 10, $f(\rho^I, \rho^{b^I}, \lambda_D) - \phi - \varepsilon > 0$. $\square$

By (12) we know that

$$a^I = (1-\beta)a^S + \beta\tilde{a}. \tag{17}$$

And by (16) we know that

$$a^D = (1-\beta')a^S + \beta'\tilde{a}. \tag{18}$$

Subtracting (17) from (18) we get, $a^D - a^I = (\beta' - \beta)(\tilde{a} - a^S)$ and substituting into (17) we get

$$a^S = a^I + \beta(a^S - \tilde{a}) = a^I + \frac{\beta}{\beta - \beta'}(a^D - a^I). \tag{19}$$

By the definition of λ_D (see the proof of Claim 11), $\beta/(\beta - \beta') \leq 1/\lambda_D$ which implies that $\lambda_D\beta/(\beta-\beta') \leq 1$. Let $\lambda_S = \lambda_D\beta/(\beta - \beta')$ ($\lambda_S \geq 0$ as $\beta > \beta'$).

Claim 12 $a^S = (1-\lambda_S)a^I + \lambda_S b^I$

$$\begin{aligned} a^S &= a^I + \frac{\beta}{\beta - \beta'}(a^D - a^I) \quad \text{by (19)} \\ &= a^I + \frac{\beta}{\beta - \beta'}\left[(1-\lambda_D)a^I + \lambda_D b^I - a^I\right] \\ &= a^I + \frac{\beta}{\beta - \beta'}\left[\lambda_D(b^I - a^I)\right] = a^I + \lambda_S(b^I - a^I) \\ &= (1-\lambda_S)a^I + \lambda_S b^I \end{aligned}$$

□

Claim: $u^N = u^I > w^F \cdot a^I - \delta$

$$\begin{aligned} |u^I - w^F \cdot a^I| &\leq |u^N - v(a^F)| + |w^F \cdot a^F - w^F \cdot a^I| \quad \text{as } v(a^F) = w^F \cdot a^F \\ &\leq |u^N - v(a^F)| + \|w^F\|\|a^F - a^I\| \\ &< \frac{\delta}{2} + \|w^F\| 2r \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned}$$

by Claims 9 and 8, and (9). □

Claim 13 $L(\rho^I, \rho^{b^I}, \lambda_S) > u^S$

$$\begin{aligned} L(\rho^I, \rho^{b^I}, \lambda_S) &= (1-\lambda_S)u^I + \lambda_S(w^F \cdot b^I - \delta) \\ &> (1-\lambda_S)(w^F \cdot a^I - \delta) + \lambda_S(w^F \cdot b^I - \delta) \\ &= w^F \cdot [(1-\lambda_S)a^I + \lambda_S b^I] - \delta \\ &= w^F \cdot a^S - \delta \geq u^S \quad \text{by Claim 12 and by assumption} \end{aligned}$$

□

$\beta > \beta'$ so take $\lambda'_D = 1 - \beta'/\beta$ then

Claim: $\rho^D = (1-\lambda'_D)\rho^I + \lambda'_D p^S$

$$\begin{aligned} (1-\lambda'_D)\rho^I + \lambda'_D p^S &= \frac{\beta'}{\beta}\rho^I + \left(1 - \frac{\beta'}{\beta}\right)p^S \\ &= \frac{\beta'}{\beta}\left[(1-\beta)p^S + \beta\tilde{\rho}\right] + \left(1 - \frac{\beta'}{\beta}\right)p^S \quad \text{by (12)} \\ &= (1-\beta')p^S + \beta'\tilde{\rho} = \rho^D \quad \text{by (16)} \end{aligned}$$

□

Therefore

$$u^D = (1-\lambda'_D)u^I + \lambda'_D u^S = (1-\lambda'_D)u^N + \lambda'_D u^S$$

as $u^I = u^N$. Further

$$\begin{aligned} a^D &= (1-\lambda'_D)a^I + \lambda'_D a^S \\ &= (1-\lambda'_D)a^I + \lambda'_D[(1-\lambda_S)a^I + \lambda_S b^I] \quad \text{by Claim 12} \\ &= (1-\lambda'_D\lambda_S)a^I + \lambda'_D\lambda_S b^I. \end{aligned}$$

Therefore $\lambda'_D \lambda_S = \lambda_D$ by Claim 11 (as $a^I \neq b^I$ since $a^I \notin \partial\Delta$ by the proof of Claim 11).

$$\begin{aligned} u^D &= (1-\lambda'_D)u^N + \lambda'_D u^S < (1-\lambda'_D)u^N + \lambda'_D L(\rho^I, \rho^{b^I}, \lambda_S) \quad \text{by Claim 13} \\ &= (1-\lambda'_D)u^N + \lambda'_D \left[(1-\lambda_S)u^N + \lambda_S(w^F \cdot b^I - \delta)\right] \\ &= (1-\lambda'_D\lambda_S)u^N + \lambda'_D\lambda_S(w^F \cdot b^I - \delta) \\ &= L(\rho^I, \rho^{b^I}, \lambda'_D\lambda_S) = L(\rho^I, \rho^{b^I}, \lambda_D) \end{aligned}$$

i.e. $L(\rho^I, \rho^{b^I}, \cdot)$ is above u^D at a^D. We have chosen S_1 so that $u^{S_1} \geq u^{S_i}$ for $i = 1, \ldots, k$ so,

$$\begin{aligned} u^{N\setminus S_1} &= (1-\beta')u^S + \beta' u^{N\setminus(S\cup S_1)} \quad \text{by (15)} \\ &\leq (1-\beta')u^S + \beta' u^{N\setminus S} = u^D \quad \text{by (16)} \\ &< L(\rho^I, \rho^{b^I}, \lambda_D). \end{aligned}$$

Further, by Claim 11,

$$\begin{aligned} 0 &< f(\rho^I, \rho^{b^I}, \lambda_D) - \phi - \varepsilon \\ &= v((1-\lambda_D)a^I + \lambda_D b^I) - L(\rho^I, \rho^{b^I}, \lambda_D) - \phi - \varepsilon \\ &< v(a^D = a^{N\setminus S_1}) - u^{N\setminus S_1} - \phi - \epsilon. \end{aligned}$$

Thus $u^{N\setminus S_1} < v(a^{N\setminus S_1}) - \phi - \varepsilon$. This contradicts (14), which applies as $S_1 \subset N \setminus S \Longrightarrow N \setminus S_1 \supset S \Longrightarrow |N \setminus S_1| \geq |S| \geq \alpha|N|$.

Part II: $u^S = U^S/|A^S| < w^F \cdot a^S + \delta$

Suppose $u^S \geq w^F \cdot a^S + \delta$. As in Part I, let $\beta = |A^{N\setminus S}|/|A^N|$ then by (1) we get $1 - \beta \geq \gamma$, which implies that $\beta \leq 1 - \gamma$.

Let $f(x) = (1-x)/x$ then $f'(x) = [-x - (1-x)]/x^2 = -1/x^2 < 0$. Therefore f is decreasing and so

$$\frac{1-\beta}{\beta} \geq \frac{1-(1-\gamma)}{1-\gamma} = \frac{\gamma}{1-\gamma} \Longrightarrow -\frac{1-\beta}{\beta} \leq -\frac{\gamma}{1-\gamma}.$$

w^F is a supporting hyperplane for $\hat{V}$ so $w^F \cdot a \geq v(a)$ for every $a \in \Delta \setminus \partial\Delta$. If $\|a^N - a^F\| \leq r$ as in Part I, then $a^N \notin \partial\Delta$ by (4) and so

$$w^F \cdot a^N \geq v(a^N) \geq u^N = (1-\beta)u^S + \beta u^{N\setminus S} \geq (1-\beta)(w^F \cdot a^S + \delta) + \beta u^{N\setminus S}.$$

This implies

$$\begin{aligned} \beta u^{N\setminus S} &\leq w^F \cdot a^N - (1-\beta)w^F \cdot a^S - (1-\beta)\delta \\ &= w^F \cdot [(1-\beta)a^S + \beta a^{N\setminus S}] - (1-\beta)w^F \cdot a^S - (1-\beta)\delta \\ &= \beta w^F \cdot a^{N\setminus S} - (1-\beta)\delta. \end{aligned}$$

Hence

$$u^{N\setminus S} \leq w^F \cdot a^{N\setminus S} - \frac{1-\beta}{\beta}\delta \leq w^F \cdot a^{N\setminus S} - \frac{\gamma}{1-\gamma}\delta.$$

(n.b. $\beta \neq 0$ otherwise $S = N \Longrightarrow u^S = u^N \leq w^F \cdot a^N = w^F \cdot a^S$ which would contradict our hypothesis.) Now if there is a lower bound for β, then the argument in Part I, with $\gamma\delta/(1-\gamma)$ in place of δ, will apply to $N \setminus S$ to produce a contradiction.

Claim: $\|p^N - p^{N\setminus S}\| \leq \sqrt{2 + (\|w^F\| + \varepsilon)^2}$

$U \in C_\varepsilon(N, V) \implies U^S \geq V(A^S) - \varepsilon|A^S|$ for *any* coalition $S \subset N$. Therefore $u^S \geq V(A^S)/|A^S| - \varepsilon \geq -\varepsilon$. This is where $V \geq 0$ is used (any lower bound would do).

$$\begin{aligned} u^N &\leq v(a^N) \leq w^F \cdot a^N \leq \|w^F\|\|a^N\| \leq \|w^F\| \\ u^{N\setminus S} &\leq w^F \cdot a^{N\setminus S} - \frac{\gamma}{1-\gamma}\delta \leq w^F \cdot a^{N\setminus S} \leq \|w^F\| \end{aligned}$$

This implies $-\varepsilon \leq u^N, u^{N\setminus S} \leq \|w^F\|$ and therefore $|u^N - u^{N\setminus S}| \leq \|w^F| + \varepsilon$. Thus

$$\|p^N - p^{N\setminus S}\|^2 = \|a^N - a^{N\setminus S}\|^2 + |u^N - u^{N\setminus S}|^2 \leq 2 + (\|w^F\| + \varepsilon)^2. \qquad \square$$

Claim: $\|p^S - p^N\| \geq \delta/\|(w^F, -1)\|$

$u^S \geq w^F \cdot a^S + \delta$ by hypothesis and $p^S = (a^S, u^S)$ so,

$$(w^F, -1) \cdot p^S = w^F \cdot a^S - u^S \leq -\delta.$$

$w^F \cdot a^N \geq v(a^N) \geq u^N$ and $p^N = (a^N, u^N)$ so,

$$(w^F, -1) \cdot p^N = w^F \cdot a^N - u^N \geq 0.$$

Therefore,

$$\delta \leq |(w^F, -1) \cdot p^S - (w^F, -1) \cdot p^N| = |(w^F, -1) \cdot (p^S - p^N)| \leq \|(w^F, -1)\|\|p^S - p^N\|. \qquad \square$$

Claim: $\beta = \|p^N - p^S\|/\|p^{N\setminus S} - p^S\|$

By Lemma 1, $p^N = (1-\beta)p^S + \beta p^{N\setminus S}$. Thus $p^N - p^S = \beta(p^{N\setminus S} - p^S)$ and so $\beta = \|p^N - p^S\|/\|p^{N\setminus S} - p^S\|$ ($p^S \neq p^{N\setminus S}$ as $u^S \geq w^F \cdot a^S + \delta$ and $u^{N\setminus S} \leq w^F \cdot a^{N\setminus S} - \gamma\delta/(1-\gamma)$). $\square$

$$\beta = \frac{\|p^N - p^S\|}{\|p^{N\setminus S} - p^S\|} \geq \frac{\delta/\|(w^F, -1)\|}{\sqrt{2 + (\|w^F\| + \varepsilon)^2}}$$

Therefore β is bounded below.

This shows that there exist n, r, ε_0 such that if $|N| \geq n$, $\|a^N - a^F\| \leq r$ and $\varepsilon \in [0, \varepsilon_0]$ then

$$|w^F \cdot A^S - U^S| < \frac{\delta}{2}|A^S|.$$

If $w \in C^H_\varepsilon$ then $(w \cdot A^1, \ldots, w \cdot A^{|N|}) \in C_\varepsilon$ so that

$$|w \cdot A^S - U^S| \leq |w \cdot A^S - w^F \cdot A^S| + |w^F \cdot A^S - U^S| < \frac{\delta}{2}|A^S| + \frac{\delta}{2}|A^S| = \delta|A^S|. \qquad \blacksquare$$

B Boundedness of Hedonic Payoffs

Proof of Lemma 2 (in text): If $T=1$, then $\Delta = \{1\}$, so $a^N=1$. Further, a feasible hedonic payoff is just a number $w\in\mathbf{R}$ such that $w\bullet A^N \leq V(A^N)$, which implies that $w = w\bullet a^N \leq V(A^N)/|A^N| \leq v(a^N) = v(1)$. This shows that if $T=1$ the set of feasible hedonic payoffs is bounded, so assume $T\geq 2$. Given $w\in\Psi$, let

$$w_\perp = (1/T)\ (\Sigma w_t)\ \mathbf{1} \quad \text{(where } \mathbf{1} \text{ is a vector of ones of length T)}$$

$$w_p = w - w_\perp$$

Claim: $w_p\bullet w_\perp = 0$

$$w_p\bullet w_\perp = (w-w_\perp)\bullet w_\perp = (1/T)\ (\Sigma w_t)^2 - T\ ((1/T)(\Sigma w_t))^2 = 0 \quad \square$$

Let $K = \{a\in\Delta \mid d(a,\partial\Delta) \geq (1/2)(r-\delta)\}$. K is a compact set in $\text{int}(\Delta)$ containing a^N. v is Lipschitz on K. (Although v isn't defined on all of R^T_+, $\hat{V}$ is, and one can extend K to a compact set in the interior of $\mathbf{R}^T_+$ to get the Lipschitz constant.) So let C be a Lipschitz constant. Assume $\|w_p\| > 0$ and let

$$a^\alpha = a^N - \alpha\ w_p\ /\ \|w_p\|\ .$$

Claim: $a^\alpha \in K$ for $|\alpha| \leq d(a^N,\partial K)$.

$$K = \{a\in\Delta \mid d(a,\partial\Delta) \geq (1/2)\ (r-\delta)\} = \{a\in\Delta \mid \min_t a_t\ (T/(T-1))^{.5} \geq (1/2)(r-\delta)\}$$

$$= \{a\in\Delta \mid a_t \geq (1/2)\ (r-\delta)\ ((T-1)/T)^{.5}\ \forall t\}.$$

$$\min_t\ a^N_t\ (T/(T-1))^{.5} = d(a^N,\partial\Delta) \geq r$$

$$\Rightarrow \quad a^N_t \geq r\ ((T-1)/T)^{.5} > (1/2)\ (r-\delta)\ ((T-1)/T)^{.5}\ \ \forall t\ .$$

Therefore a^N is in the interior of K.

$$\Sigma_t\ a^\alpha_t = \Sigma\ a^N_t - (\alpha/\|w_p\|)\ \Sigma\ [w_t - (w_\perp)_t] = 1 - (\alpha/\|w_p\|)\ [\Sigma w_t - \Sigma w_t] = 1\ .$$

Thus a^α starts off in K and stays in K as $|\alpha|$ increases until it hits the boundary ∂K, when $|\alpha| = \|a^\alpha - a^N\| \geq d(a^N,\partial K)$. $\square$

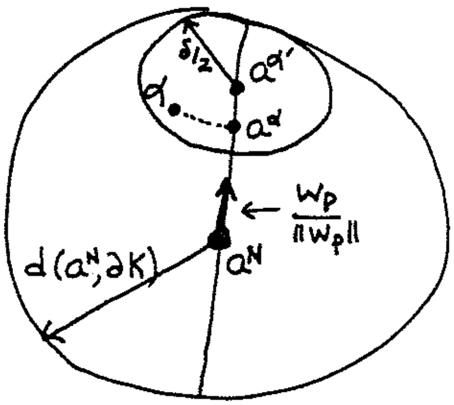
Claim: $d(a^N,\partial K) > \delta$.

$$d(a^N,\partial K) = d(a^N,\partial\Delta) - (1/2)\ (r-\delta) \geq r - (1/2)\ (r-\delta) = (r+\delta)/2 > \delta\ . \quad \square$$

Let $\alpha' = d(a^N,\partial K) - \delta/2$. Then $a^{\alpha'} \epsilon K$.

Given $a \epsilon B(a^{\alpha'},\delta/2)$, let $\alpha = (a^N\text{-}a)\bullet w_p/\|w_p\|$.

Claim: $w\bullet a = w\bullet a^\alpha$.



$$w\bullet a^\alpha = w \bullet [a^N - ((a^N\text{-}a)\bullet(w_p/\|w_p\|))\, w_p/\|w_p\|\,]$$
$$= w\bullet a^N - (a^N\text{-}a) \bullet w_p$$
$$= w\bullet a^N - (a^N\text{-}a) \bullet w = w\bullet a$$

as $w_\perp\bullet a$ is constant on Δ. □

Claim: $|\alpha\text{-}\alpha'| < \delta/2$, thus $\alpha > d(a^N,\partial K)\text{-}\delta$.

$(a\text{-}a^\alpha) \bullet(a^{\alpha'}\text{-}a^\alpha) = (a\text{-}a^\alpha) \bullet (\alpha\text{-}\alpha')\, w_p/\|w_p\| = [(\alpha\text{-}\alpha')/\|w_p\|]\, [(a\text{-}a^\alpha)\bullet w] = 0.$

Thus $\|a\text{-}a^{\alpha'}\|^2 = \|a\text{-}a^\alpha\|^2 + \|a^\alpha\text{-}a^{\alpha'}\|^2$.

$|\alpha\text{-}\alpha'|^2 = \|a^\alpha\text{-}a^{\alpha'}\|^2 = \|a\text{-}a^{\alpha'}\|^2 - \|a\text{-}a^\alpha\|^2 < (\delta/2)^2 - \|a\text{-}a^\alpha\|^2 \leq (\delta/2)^2.$

Thus $|\alpha\text{-}\alpha'| < (\delta/2) \Rightarrow \alpha > \alpha'\text{-}(\delta/2) = d(a^N,\partial K) - \delta$. □

Claim: $B(a^{\alpha'},\delta/2) \subset K$.

Suppose $a \epsilon B(a^{\alpha'},\delta/2)$.

$\|a^N\text{-}a\| \leq \|a^N\text{-}a^{\alpha'}\| + \|a^{\alpha'}\text{-}a\| < \alpha' + (\delta/2) = d(a^N,\partial K)$. □

Claim: $\|w_p\| < [C\, d(a^N,\partial K)+2\varepsilon] / [d(a^N,\partial K)\text{-}\delta] \leq [C\sqrt{2} + 2\varepsilon]/[(1/2)(r\text{-}\delta)]$.

Let $a\epsilon B(a^{\alpha'},\delta/2)$. Then $a\epsilon K$ so $|v(a^N)\text{-}v(a)| \leq C\|a^N\text{-}a\| \leq C\, d(a^N,\partial K)$,

which implies $v(a) \geq v(a^N) - C\, d(a^N,\partial K)$.

$w\bullet a = w\bullet a^\alpha = w\bullet a^N - \alpha\, w\bullet w_p / \|w_p\| \leq (V(A^N)/|A^N|) - \alpha\, [(w_p+w_\perp)\bullet w_p]/\|w_p\|$

$\leq v(a^N) - \alpha\|w_p\| < v(a^N) - [d(a^N,\partial K)\text{-}\delta]\, \|w_p\|$.

Suppose $\|w_p\| \geq [C\, d(a^N,\partial K)+2\varepsilon] / [d(a^N,\partial K)\text{-}\delta]$.

Then $[d(a^N,\partial K)\text{-}\delta]\|w_p\| \geq C\, d(a^N,\partial K)+2\varepsilon$ and $-C\, d(a^N,\partial K)\text{-}2\varepsilon \geq -[d(a^N,\partial K)\text{-}\delta]\, \|w_p\|$.

$w\bullet a < v(a^N) - [d(a^N,\partial K)\text{-}\delta]\|w_p\| \leq v(a^N) - C\, d(a^N,\partial K) - 2\varepsilon \leq v(a)\text{-}2\varepsilon$.

Therefore $B(a^{\alpha'},\delta/2) \subset \{a\epsilon\Delta \mid w\bullet a < v(a)\text{-}2\varepsilon \}$ (Contradiction) □

This shows that $\|w_p\|$ is bounded. To complete the proof it suffices to show that $\|w_\perp\|$ is bounded, since $\|w\|^2 = \|w_p+w_\perp\|^2 = \|w_p\|^2 + \|w_\perp\|^2$. $w_\perp \bullet a$ is constant on Δ and $(1/T)1\epsilon\Delta$, so $w_\perp \bullet a = w_\perp \bullet(1/T)1 = ((1/T)\Sigma w_t)1 \bullet (1/T)1 = (1/T)\Sigma w_t$. $\|w_\perp\| = (w_\perp \bullet w_\perp)^{.5} = [\,((1/T)\Sigma w_t)\,1 \bullet ((1/T)\Sigma w_t)\,1\,]^{.5} = (1/T)\,|\Sigma w_t|\,\sqrt{T} = (1/\sqrt{T})\,|\Sigma w_t| = \sqrt{T}\,|w_\perp \bullet a|$. Thus it suffices to show that $w_\perp \bullet a$ is bounded.

Claim: $w_\perp \bullet a$ is bounded above.

$w_\perp \bullet a = w_\perp \bullet a^N = w \bullet a^N - w_p \bullet a^N \le v(a^N) + |w_p \bullet a^N| \le \max_{a\epsilon K} v(a) + \|w_p\|$.

But $\|w_p\|$ is bounded above. □

Claim: $w_\perp \bullet a$ is bounded below.

Let $m = \min\{v(a)-2\varepsilon \,|\, a\epsilon K\}$, and $\|w_p\| \le M\ \forall w\epsilon\Psi$. Suppose $w_\perp \bullet a < m-M \le m-\|w_p\|$. $|w\bullet a - w_\perp \bullet a| = |w_p \bullet a| \le \|w_p\| \le M$. Thus, $w\bullet a \le w_\perp \bullet a + M < m \le v(a)-2\varepsilon\ \ \forall a\epsilon K$. Thus $K \subset \{a\epsilon\Delta \,|\, w\bullet a < v(a)-2\varepsilon\}$, but $a^N \epsilon K$, so $B(a^N, d(a^N,\partial K)) \subset \{a\epsilon\Delta \,|\, w\bullet a < v(a)-2\varepsilon\}$. This implies that $\delta > 2d(a^N,\partial K) \ge 2[r-(r-\delta)/2] = r+\delta > \delta$. (Contradiction) □

Corollary: Let $\varepsilon,\delta,r \ge 0$. Suppose $d(a^N,\partial\Delta) \ge r > \delta$ and (N,V) δ-exhausts ε-blocking opportunities. Then $C^H_\epsilon(N,V)$ is bounded.

Corollary: Let $\varepsilon_o,\delta,r,r' \ge 0$. Suppose (N,V), (N′,V) δ-exhaust ε_o-blocking opportunities and that $d(a^N,\partial\Delta) \ge r > \delta$, $d(a^{N'},\partial\Delta) \ge r' > \delta$. Then $\exists$ M>0 such that if $\varepsilon \le \varepsilon_o$, $\|w-w'\| \le M$ for $w\epsilon C^H_\epsilon(N,V)$, $w'\epsilon C^H_\epsilon(N',V)$.

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