

# UC Berkeley

## Other Recent Work

### Title

Uniqueness, Stability, and Comparative Statistics in Rationalizable Walrasian Markets

### Permalink

<https://escholarship.org/uc/item/2pm4n029>

### Authors

Brown, Donald J.  
Shannon, Chris

### Publication Date

1997-07-01

UNIVERSITY OF CALIFORNIA AT BERKELEY

Department of Economics

Berkeley, California 94720-3880

Working Paper No. 97-256

Uniqueness, Stability, and Comparative Statics  
in Rationalizable Walrasian Markets

Donald J. Brown  
Yale University

Chris Shannon  
University of California, Berkeley

July 1997

Keywords: tâtonnement stability, comparative statics, monotonicity, monotone demand, refutability, exchange economies

JEL: C62, D51

**Abstract**

This paper studies the extent to which qualitative features of Walrasian equilibria are refutable given a finite data set. In particular, we consider the hypothesis that the observed data are Walrasian equilibria in which each price vector is locally stable under tâtonnement. Our main result shows that a finite set of observations of prices, individual incomes and aggregate consumption vectors is rationalizable in an economy with smooth characteristics if and only if it is rationalizable in an economy in which each observed price vector is locally unique and stable under tâtonnement. Moreover, the equilibrium correspondence is locally monotone in a neighborhood of each observed equilibrium in these economies. Thus the hypotheses that equilibria are locally stable under tâtonnement, equilibrium prices are locally unique and equilibrium comparative statics are locally monotone are not refutable with a finite data set.



# Uniqueness, Stability, and Comparative Statics in Rationalizable Walrasian Markets

Donald J. Brown

*Department of Economics, Yale University*

Chris Shannon

*Department of Economics, University of California, Berkeley*

July 1997

## Abstract

This paper studies the extent to which qualitative features of Walrasian equilibria are refutable given a finite data set. In particular, we consider the hypothesis that the observed data are Walrasian equilibria in which each price vector is locally stable under tâtonnement. Our main result shows that a finite set of observations of prices, individual incomes and aggregate consumption vectors is rationalizable in an economy with smooth characteristics if and only if it is rationalizable in an economy in which each observed price vector is locally unique and stable under tâtonnement. Moreover, the equilibrium correspondence is locally monotone in a neighborhood of each observed equilibrium in these economies. Thus the hypotheses that equilibria are locally stable under tâtonnement, equilibrium prices are locally unique and equilibrium comparative statics are locally monotone are not refutable with a finite data set.

## 1 Introduction

The major theoretical questions concerning competitive equilibria in the classical Arrow–Debreu model — existence, uniqueness, comparative statics, and stability of price adjustment processes — have been largely resolved over the last forty years. With the exception of existence, however, this resolution has been fundamentally negative. The conditions under which equilibria can be shown to be unique, comparative statics globally determinate or tâtonnement price adjustment globally stable are quite restrictive. Moreover, the Sonnenschein–Debreu–Mantel theorem shows in striking fashion that no behavior implied by individual utility maximization beyond homogeneity and Walras’ Law is necessarily preserved by aggregation in market excess demand. This arbitrariness of excess demand implies that monotone equilibrium comparative statics and global stability of equilibria under tâtonnement will only result from the imposition of a limited set of conditions on preferences and endowments. Based on these results, many economists conclude that the general equilibrium model has no refutable implications or empirical content.

Of course no statement concerning refutable implications is meaningful without first specifying what information is observable and what is unobservable. If only market prices are observable, and all other information about the economy such as individual incomes, individual demands, individual endowments, individual preferences and aggregate endowment or aggregate consumption is unobservable, then indeed the general equilibrium model has no testable restrictions. This is essentially the content of Mas-Colell's version of the Sonnenschein–Debreu–Mantel theorem. Mas-Colell [6] shows that given an arbitrary nonempty compact subset  $C$  of strictly positive prices in the simplex, there exists an economy  $\mathcal{E}$  composed of consumers with continuous, monotone, strictly convex preferences such that the equilibrium price vectors for the economy  $\mathcal{E}$  are given exactly by the set  $C$ .

In many instances, however, it is unreasonable to think that only market prices are observable; other information such as individual incomes and aggregate consumption may be observable in addition to market prices. Brown and Matzkin [2] show that if such additional information is available, then the Walrasian model does have refutable implications. They demonstrate by example that with a finite number of observations — in fact two — on market prices, individual incomes and aggregate consumptions, the hypothesis that these data correspond to competitive equilibrium observations can be rejected. They also give conditions under which this hypothesis is accepted and there exists an economy rationalizing the observed data.

This paper considers the extent to which qualitative features of Walrasian equilibria are refutable given a finite data set. In particular, we consider the hypothesis that the observed data are Walrasian equilibria in which each price vector is locally stable under tâtonnement. Based on the Sonnenschein–Debreu–Mantel results and the well-known examples of global instability of tâtonnement such as Scarf's [8], it may seem at first glance that this hypothesis will be easily refuted in a Walrasian setting. Surprisingly, however, we show that it is not. Our main result shows that a finite set of observations of prices, individual incomes and aggregate consumption vectors is rationalizable in an economy with smooth characteristics if and only if it is rationalizable in a distribution economy in which each observed price is locally stable under tâtonnement. Moreover, the equilibrium correspondence is locally monotone in a neighborhood of each observed equilibrium in these economies, and the equilibrium price vector is locally unique.

The conclusion that if the data is rationalizable then it is rationalizable in a distribution economy, i.e., an economy in which individual endowments are collinear, is not subtle. If we do not observe the individual endowments and only observe prices and income levels, then one set of individual endowments consistent with this data is collinear, with shares given by the observed income distribution. Since distribution economies by definition have a price-independent income distribution, this observation may suggest that our results about stability and comparative statics derive simply from this fact. Kirman and Koch [4] show that this intuition is false, however. They show that the additional assumption of a fixed income distribution places no restrictions on excess demand: given any compact set  $K \subset \mathbb{R}_{++}^{\ell}$  and any smooth

function  $g : \mathbb{R}_{++}^\ell \rightarrow \mathbb{R}$  which is homogeneous of degree 0 and satisfies Walras' Law, and given any fixed income distribution  $\alpha \in \mathbb{R}_{++}^n$ ,  $\sum_{i=1}^n \alpha_i = 1$ , there exists an economy  $\mathcal{E}$  with smooth, monotone, strictly convex preferences and initial endowments  $\omega_i = \alpha_i \omega$  such that excess demand for  $\mathcal{E}$  coincides with  $g$  on  $K$ . Hence any dynamic on the price simplex can be replicated by some distribution economy.

This paper shows that rationalizable data is always rationalizable in an economy in which market excess demand has a very particular structure. Using recent results of Quah [7], we show that if the data is rationalizable then it is rationalizable in an economy in which each individual demand function is locally monotone at each observation. The strong properties of local monotonicity, in particular the fact that local monotonicity of individual demand is preserved by aggregation in market excess demand and the fact that local monotonicity implies local stability in distribution economies, allow us to conclude that if the data is rationalizable in a Walrasian setting, then it is rationalizable in an economy in which each observation is locally stable under tâtonnement. Thus global instability, while clearly a theoretical possibility in Walrasian markets, cannot be detected in a finite data set consisting of observations on prices, incomes, and aggregate consumption.

The paper proceeds as follows. In Section 2 we discuss conditions for rationalizing individual demand in economies with smooth characteristics, and show that if the observed data can be rationalized by an individual consumer with smooth characteristics then it can be rationalized by a smooth utility function which generates a locally monotone demand function. In Section 3 we discuss the implications of locally monotone demand and use these results together with the results from Section 2 to show that local uniqueness, local stability, and local monotone comparative statics are not refutable in Walrasian markets.

## 2 Rationalizing Individual Demand

Given a finite number of observations  $(p^r, x^r)$ ,  $r = 1, \dots, N$ , on prices and quantities, when is this data consistent with utility maximization by some consumer with a non-satiated utility function? We say that a utility function  $U : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$  rationalizes the data  $(p^r, x^r)$ ,  $r = 1, \dots, N$ , if  $\forall r$

$$p^r \cdot x^r \geq p^r \cdot x \Rightarrow U(x^r) \geq U(x), \forall x \in \mathbb{R}_+^\ell.$$

Using this terminology, we can restate the question above: given a finite data set, when does there exist a non-satiated utility function which rationalizes these observations? The classic answer to this question was given by Afriat [1].

**Theorem (Afriat).** *The following are equivalent:*

- (a) *there exists a non-satiated utility function which rationalizes the data*
- (b) *the data satisfies Cyclical Consistency*

(c) *there exist numbers  $U^i$ ,  $\lambda^i > 0$ ,  $i = 1, \dots, N$  which satisfy the “Afriat inequalities”:*

$$U^i - U^j \leq \lambda^j p^j \cdot (x^i - x^j), \quad i, j = 1, \dots, N$$

(d) *there exists a concave, monotone, continuous, non-satiated utility function which rationalizes the data.*

In particular, the equivalence of (a) and (d) shows that the hypothesis that preferences are represented by a concave utility function can never be refuted based on a finite data set, since if the data is rationalizable by any non-satiated utility function then it is rationalizable by a concave, monotone, and continuous one. Moreover, Afriat showed explicitly how to construct such a utility function which rationalizes a given data set. For each  $x \in \mathbb{R}_+^\ell$ , define

$$U(x) = \min_r \{U^r + \lambda^r p^r \cdot (x - x^r)\}.$$

This utility function is indeed continuous, monotone and concave, and rationalizes the data by construction.

As Chiappori and Rochet [3] note, however, this utility function is piecewise linear and thus neither differentiable nor strictly concave. Such a utility function does not generate a smooth demand function, and for a number of prices does not even generate single-valued demand. This utility function is thus incompatible with many standard demand-based approaches to the question of rationalizability or estimation, as well as with our questions about comparative statics and asymptotic stability.

Whether or not a given set of observations can be rationalized by a smooth utility function which generates a smooth demand function will obviously depend on the nature of the observed data. Two situations in which such a rationalization is impossible are depicted in Figures 1 and 2. The observations in Figure 1 cannot be rationalized by any smooth utility function, and those in Figure 2 cannot be rationalized by any demand function. If the data satisfies SARP, then situations like that in Figure 2 are eliminated; Chiappori and Rochet [3] show that if in addition situations like that in Figure 1 are ruled out then the data can be rationalized by a smooth, strongly concave utility function.

\*\*\*Insert Figures 1 and 2 here\*\*\*

More formally, the data satisfies the **strong strong axiom of revealed preference (SSARP)** if it satisfies SARP and if for all  $i, j = 1, \dots, N$ ,

$$p^i \neq p^j \Rightarrow x^i \neq x^j.$$

Chiappori and Rochet [3] show that given a finite set of data satisfying SSARP, there exists a strictly increasing,  $C^\infty$ , strongly concave utility function defined on a compact subset of  $\mathbb{R}_+^\ell$  which rationalizes this data. Although SSARP is a condition on the observable data alone, it can be equivalently characterized by the “strict Afriat

inequalities". That is, the data satisfy SSARP if and only if there exist numbers  $U^i, \lambda^i > 0, i = 1, \dots, N$  such that

$$U^i - U^j < \lambda^j p^j \cdot (x^i - x^j), i, j = 1, \dots, N, i \neq j.$$

Our work makes use of two modifications of the results of Chiappori and Rochet [3]. Since we assume the data is composed of observations on prices and incomes rather than prices and consumption bundles, our notion of rationalization must take that into account. Accordingly, given a finite number of observations  $(p^r, I^r), r = 1, \dots, N$ , we say that a utility function  $U : \mathbb{R}_+^\ell \rightarrow \mathbb{R}$  rationalizes the data if for each  $r$  there exists  $x^r \in \mathbb{R}_{++}^\ell$  such that  $p^r \cdot x^r = I^r$  and

$$p^r \cdot x^r \geq p^r \cdot x \Rightarrow U(x^r) \geq U(x), \forall x \in \mathbb{R}_+^\ell.$$

Since our data consists of prices and income levels, we also find it more natural to first recast the question of rationalizability in terms of indirect utility, and develop a set of dual Afriat inequalities characterizing data which can be rationalized by a consumer with smooth characteristics. An important benefit of this dual characterization is that it allows us to conclude not only that the data can be rationalized by a smooth demand function but by a demand function which is locally monotone in a neighborhood of each observation  $(p^r, I^r)$ .

**Definition.** Let  $\omega \in \mathbb{R}_{++}^\ell$  be given. An individual demand function  $f(p, I)$  is **locally monotone** at  $(\bar{p}, \bar{I})$  if there exists a neighborhood  $\mathcal{W}$  of  $(\bar{p}, \bar{I})$  such that

$$(p - q) \cdot (f(p, I) - f(q, I)) < 0$$

for all  $(p, I), (q, I) \in \mathcal{W}$  such that  $p \cdot \omega = q \cdot \omega$  and  $p \neq q$ .

Our first result can then be stated as follows.

**Theorem 1.** *Let  $(p^r, I^r), r = 1, \dots, N$  be given. There exists a smooth, strictly quasiconcave, monotone utility function rationalizing this data such that the implied demand function is locally monotone at  $(p^r, I^r)$  for each  $r = 1, \dots, N$  if and only if there exist numbers  $V^i, \lambda^i$  and vectors  $q^i \in \mathbb{R}^\ell$  such that the "dual strict Afriat inequalities" hold:*

(a) for  $i \neq j$ ,

$$V^i - V^j > q^j \cdot (p^i - p^j) + \lambda^j (I^i - I^j), i, j = 1, \dots, N$$

(b)  $\lambda^j > 0, q^j \ll 0, j = 1, \dots, N$

(c)  $p^j \cdot q^j = -\lambda^j I^j, j = 1, \dots, N.$



While condition (c) here may appear somewhat mysterious, it is really just an expression of Roy's identity in this context. To see this, note that the unobservable vector  $(q^j, \lambda^j)$  corresponds to the gradient of the rationalizing indirect utility function  $V$  evaluated at  $(p^j, I^j)$ ; this is essentially the content of (a). If we define  $x^j = -\frac{1}{\lambda^j} q^j$ , then  $x^j$  is indeed demand at the price-income pair  $(p^j, I^j)$  by Roy's identity, and (c) becomes the budget equation

$$(c') \quad p^j \cdot x^j = I^j, \quad j = 1, \dots, N.$$

The proof of Theorem 1 relies on two intermediate results. The first is a version of Lemma 2 in Chiappori and Rochet [3] modified to apply to our dual Afriat inequalities.

**Lemma 1.** *If there exist numbers  $V^i, \lambda^i$  and vectors  $q^i, i = 1, \dots, N$  satisfying the dual strict Afriat inequalities, then there exists a convex  $C^\infty$  function  $W : \mathbb{R}_{++}^{\ell+1} \rightarrow \mathbb{R}$  which is strictly increasing in  $I$  and strictly decreasing in  $p$  such that  $W(p^i, I^i) = V^i$ ,  $DW(p^i, I^i) = (q^i, \lambda^i)$ , and  $\frac{\partial^2 W}{\partial^2 I}(p^i, I^i) = 0$  for every  $i = 1, \dots, N$ .*

**Proof:** For each  $(p, I) \in \mathbb{R}_{++}^{\ell+1}$  define

$$Y(p, I) = \max_t \{V^t + (q^t, \lambda^t) \cdot [(p, I) - (p^t, I^t)]\}.$$

Then  $Y$  is convex, continuous, strictly increasing in  $I$  and strictly decreasing in  $p$ . Moreover,  $Y(p^r, I^r) = V^r$  for each  $r$ . To see this, note that by definition  $\forall r \exists s$  such that

$$Y(p^r, I^r) = V^s + (q^s, \lambda^s) \cdot [(p^r, I^r) - (p^s, I^s)].$$

By the dual strict Afriat inequalities, if  $r \neq s$

$$V^r - V^s > q^s \cdot (p^r - p^s) + \lambda^s (I^r - I^s) = (q^s, \lambda^s) \cdot [(p^r, I^r) - (p^s, I^s)].$$

So  $\forall s \neq r$

$$V^r > V^s + (q^s, \lambda^s) \cdot [(p^r, I^r) - (p^s, I^s)].$$

Thus  $Y(p^r, I^r) = V^r$  for each  $r = 1, \dots, N$ .

This argument shows that for every  $s \neq r$ ,

$$Y(p^r, I^r) > V^s + (q^s, \lambda^s) \cdot [(p^r, I^r) - (p^s, I^s)].$$

Since  $Y$  is continuous,  $\forall r$  there exists  $\eta_r > 0$  such that

$$Y(p, I) = V^r + (q^r, \lambda^r) \cdot [(p, I) - (p^r, I^r)]$$

for all  $(p, I) \in B((p^r, I^r), \eta_r)$ , i.e.,  $Y$  is piecewise linear.

Following Chiappori and Rochet [3], we will smooth  $Y$  by convolution. Let  $\eta = \min_r \eta_r$ , and  $y = (p, I)$ . Define

$$\rho(y) = \begin{cases} \exp\left[-\frac{1}{1-\|y\|^2}\right] \left[\int \exp\left[-\frac{1}{1-\|z\|^2}\right] dz\right]^{-1}, & \text{if } \|y\| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\rho_\eta(y) = \frac{1}{\eta} \rho\left(\frac{y}{\eta}\right).$$

Then  $\rho_\eta$  is nonnegative, symmetric, and  $C^\infty$ , and  $\rho_\eta(y) = 0$  if  $y \notin B(0, \eta)$ .

Now define

$$W(y) \equiv Y * \rho_\eta(y) = \int Y(y - \xi) \rho_\eta(\xi) d\xi.$$

Then  $W$  is  $C^\infty$  and convex, as

$$\begin{aligned} W(\gamma y + (1 - \gamma)y') &= \int Y(\gamma y + (1 - \gamma)y' - \xi) \rho_\eta(\xi) d\xi \\ &\leq \int [\gamma Y(y - \xi) + (1 - \gamma)Y(y' - \xi)] \rho_\eta(\xi) d\xi \\ &= \gamma \int Y(y - \xi) \rho_\eta(\xi) d\xi + (1 - \gamma) \int Y(y' - \xi) \rho_\eta(\xi) d\xi \\ &= \gamma W(y) + (1 - \gamma)W(y') \end{aligned}$$

for  $\gamma \in [0, 1]$ . Moreover,  $W$  is strictly decreasing in  $p$  and strictly increasing in  $I$ . To see this, suppose  $p < p'$ . Then

$$W(p, I) - W(p', I) = \int [Y((p, I) - \xi) - Y((p', I) - \xi)] \rho_\eta(\xi) d\xi > 0.$$

Similarly,  $W(p, I') - W(p, I) > 0$  for  $I' > I$ .

Furthermore, for each  $r = 1, \dots, N$ ,

$$\begin{aligned} W(p^r, I^r) &= \int Y((p^r, I^r) - \xi) \rho_\eta(\xi) d\xi \\ &= \int_{B(0, \eta)} (V^r - (q^r, \lambda^r) \cdot \xi) \rho_\eta(\xi) d\xi \\ &= V^r - (q^r, \lambda^r) \cdot \int_{B(0, \eta)} \xi \rho_\eta(\xi) d\xi \\ &= V^r \end{aligned}$$

since  $\rho_\eta(\xi) = \rho_\eta(-\xi)$  for all  $\xi$ .

Similarly,

$$\begin{aligned} DW(p^r, I^r) &= \int_{B(0, \eta)} DY((p^r, I^r) - \xi) \rho_\eta(\xi) d\xi \\ &= \int_{B(0, \eta)} (q^r, \lambda^r) \rho_\eta(\xi) d\xi \\ &= (q^r, \lambda^r) \end{aligned}$$

and

$$\begin{aligned} D^2W(p^r, I^r) &= \int_{B(0,\eta)} D^2Y((p^r, I^r) - \xi) \rho_\eta(\xi) d\xi \\ &= \int_{B(0,\eta)} 0 \rho_\eta(\xi) d\xi = 0. \end{aligned}$$

In particular,  $\frac{\partial^2 W}{\partial I^2}(p^r, I^r) = 0$  for each  $r$ .  $\square$

The intuition behind this result is straightforward. The dual Afriat function  $Y(p, I)$  gives a convex indirect utility function rationalizing the data which has kinks whenever two or more of the dual Afriat equations are equal. Since the data satisfy the strict dual Afriat inequalities, none of these kinks occurs at an observation, so smoothing the function  $Y$  in a sufficiently small neighborhood of each kink gives a smooth function which is equal to the original dual Afriat function in a neighborhood of each observation, and thus in particular is locally linear in each such neighborhood.

The second important result we use, due to Quah [7], gives conditions on indirect utility analogous to the Mijutschin-Polterovich conditions on direct utility under which individual demand is locally monotone.

**Theorem (Quah).** *Let  $h : \mathbb{R}_+^{\ell+1} \rightarrow \mathbb{R}$  be a smooth, convex, strictly quasiconvex indirect utility function satisfying the property<sup>1</sup>*

$$\varepsilon(\bar{p}, \bar{I}) \equiv \frac{\bar{I} h_{II}(\bar{p}, \bar{I})}{h_I(\bar{p}, \bar{I})} < 2.$$

*Then  $h$  generates a demand function which is locally monotone in a neighborhood of  $(\bar{p}, \bar{I})$ .*

In particular, if the indirect utility function  $h$  is linear in income in a neighborhood of  $(\bar{p}, \bar{I})$ , then  $\varepsilon(\bar{p}, \bar{I}) = 0$  and Quah's theorem shows that  $h$  generates a demand function which is locally monotone in a neighborhood of  $(\bar{p}, \bar{I})$ .

The proof of Theorem 1 then combines these two ideas and exploits the duality between direct and indirect utility.

**Proof of Theorem 1:** Let  $M = \max_{s,t} \|p^s - p^t\|^2$ . By assumption there exist two distinct observations, so  $M > 0$ . Since there exists a solution to the strict dual Afriat inequalities, there exists  $\varepsilon > 0$  sufficiently small so that  $\forall i \neq j$ ,

$$V^i - V^j > q^j \cdot (p^i - p^j) + \lambda^j (I^i - I^j) + \varepsilon M. \quad (*)$$

Define

$$\tilde{q}^j = q^j - \varepsilon p^j, \quad j = 1, \dots, N$$

---

<sup>1</sup>Here subscripts denote partial derivatives.

and

$$\tilde{V}^j = V^j - \frac{1}{2}\varepsilon\|p^j\|^2, j = 1, \dots, N.$$

Then  $\forall i \neq j$ , we claim that

$$\tilde{V}^i - \tilde{V}^j > \tilde{q}^j \cdot (p^i - p^j) + \lambda^j(I^i - I^j). \quad (**)$$

To verify (\*\*), let  $i \neq j$ . Then

$$\begin{aligned} & \tilde{V}^i - \tilde{V}^j - \tilde{q}^j \cdot (p^i - p^j) - \lambda^j(I^i - I^j) \\ &= \left( V^i - \frac{1}{2}\varepsilon\|p^i\|^2 \right) - \left( V^j - \frac{1}{2}\varepsilon\|p^j\|^2 \right) - (q^j - \varepsilon p^j) \cdot (p^i - p^j) - \lambda^j(I^i - I^j) \\ &= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j(I^i - I^j) - \frac{1}{2}\varepsilon\|p^i\|^2 + \frac{1}{2}\varepsilon\|p^j\|^2 + \varepsilon p^j \cdot (p^i - p^j) \\ &= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j(I^i - I^j) + \frac{1}{2}\varepsilon(\|p^j\|^2 - \|p^i\|^2) - \frac{1}{2}\varepsilon 2p^j \cdot (p^j - p^i) \\ &= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j(I^i - I^j) + \frac{1}{2}\varepsilon(p^j - p^i) \cdot (p^j + p^i - 2p^j) \\ &= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j(I^i - I^j) - \frac{1}{2}\varepsilon(p^j - p^i) \cdot (p^j - p^i) \\ &= (V^i - V^j) - q^j \cdot (p^i - p^j) - \lambda^j(I^i - I^j) - \frac{1}{2}\varepsilon\|p^j - p^i\|^2 \\ &> 0 \end{aligned}$$

by (\*).

Now by Lemma 1, there exists a convex,  $C^\infty$  function  $W : \mathbb{R}_+^{\ell+1} \rightarrow \mathbb{R}$  which is strictly increasing in  $I$ , strictly decreasing in  $p$  and satisfies:

$$\begin{aligned} W(p^i, I^i) &= \tilde{V}^i, i = 1, \dots, N \\ DW(p^i, I^i) &= (\tilde{q}^i, \lambda^i), i = 1, \dots, N \\ \frac{\partial^2 W}{\partial I^2}(p^i, I^i) &= 0, i = 1, \dots, N. \end{aligned}$$

Define  $V(p, I) = W(p, I) + \frac{1}{2}\varepsilon\|p\|^2$ . Then  $V$  is  $C^\infty$ , convex in  $(p, I)$ , strictly convex in  $p$ , strictly increasing in  $I$  and strictly decreasing in  $p$  for  $\varepsilon$  sufficiently small. Moreover,

$$\begin{aligned} V(p^i, I^i) &= V^i, i = 1, \dots, N \\ DV(p^i, I^i) &= (q^i, \lambda^i), i = 1, \dots, N \\ \frac{\partial^2 V}{\partial I^2}(p^i, I^i) &= 0, i = 1, \dots, N. \end{aligned}$$

By Quah's theorem, the demand function

$$x(p, I) \equiv -\frac{1}{D_I V(p, I)} D_p V(p, I)$$

generated by this indirect utility function is locally monotone at  $(p^i, I^i)$  for each  $i = 1, \dots, N$ .

To convert this indirect utility function into a direct utility function, for each  $x \in \mathbb{R}_+^\ell$  define

$$U(x) \equiv \min_{(p, I) \in \Delta \times \mathbb{R}_+} V(p, I) \quad \text{s.t.} \quad p \cdot x \leq I.$$

Then  $U$  is smooth, strictly quasiconcave, monotone, and rationalizes the data. Moreover,  $x(p, I)$  is the demand function generated by  $U$ . To establish this claim, we must show that for each  $(p, I) \in \Delta \times \mathbb{R}_+$ ,

$$V(p, I) = \max_{x \geq 0} U(x) \quad (\dagger)$$

$$\text{s.t. } p \cdot x \leq I.$$

Let  $\bar{x}$  solve  $(\dagger)$ . Then  $\bar{x}$  solves the first order conditions

$$DU(\bar{x}) = \lambda p$$

$$p \cdot \bar{x} = I,$$

where  $\lambda > 0$ . By definition,

$$U(\bar{x}) \equiv \min_{(p', I') \in \Delta \times \mathbb{R}_+} V(p', I') \quad (\ddagger)$$

$$\text{s.t. } p' \cdot \bar{x} \leq I'.$$

so by the envelope theorem,  $DU(\bar{x}) = \gamma p(\bar{x})$ , where  $\gamma > 0$  is the Lagrange multiplier for  $(\ddagger)$  and  $(p(\bar{x}), I(\bar{x}))$  is the solution to  $(\ddagger)$ . Thus  $\gamma p(\bar{x}) = \lambda p$ , which implies that  $p(\bar{x}) = p$  since  $p, p(\bar{x}) \in \Delta$ . Then  $I(\bar{x}) = p(\bar{x}) \cdot \bar{x} = p \cdot \bar{x} = I$ , so  $U(\bar{x}) \equiv V(p(\bar{x}), I(\bar{x})) = V(p, I)$ . Finally, by Roy's identity  $x(p, I)$  is the demand function generated by  $U$ , which also implies that  $U$  rationalizes the data.  $\square$

The conditions which are necessary and sufficient for the existence of a smooth, monotone utility function rationalizing a given finite data set – the dual strict Afriat inequalities – are exactly the same conditions which are necessary and sufficient for the existence of such smooth preferences rationalizing the data for which in addition demand is locally monotone at each observation. Thus it follows immediately from Theorem 1 that any finite data set which can be rationalized by smooth preferences can be rationalized by smooth preferences giving rise to locally monotone demand.

*Corollary.* Let  $(p^r, I^r), r = 1, \dots, N$  be given. There exists a smooth, strictly quasiconcave, monotone utility function rationalizing this data if and only if there exists a smooth, strictly quasiconcave, monotone utility function rationalizing this data such that the implied demand function is locally monotone at  $(p^r, I^r)$  for each  $r = 1, \dots, N$ .

This result will provide the foundation for the study of rationalizing equilibria which is contained in the next section.

### 3 Rationalizing Walrasian Equilibria

In this section, we turn to the question of rationalizing observations as equilibria. Here we consider a finite number of observations  $(p^r, \omega^r, \{I_t^r\}_{t=1}^T), r = 1, \dots, N$  of

prices, income levels for each of  $T$  consumers, and aggregate consumption. Our main result shows that such a finite data set can never be used to refute the hypotheses that equilibria are locally unique or locally stable under tâtonnement, or that equilibrium comparative statics are locally monotone.

**Theorem 2.** *Let  $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$ ,  $r = 1, \dots, N$ , be given. This data can be rationalized by an economy in which each consumer has a smooth, strictly quasiconcave, monotone utility function if and only if it can be rationalized by an economy in which each consumer has a strictly quasiconcave, monotone utility function and in which each observed equilibrium  $p^r$  is locally unique and locally stable under tâtonnement and in which the equilibrium correspondence is locally monotone at  $(p^r, \omega^r)$  for each  $r$ .*

To establish this result, consider a given finite data set  $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$ ,  $r = 1, \dots, N$ . When can these observations be rationalized as Walrasian equilibria? Clearly a necessary condition for this rationalization is that the individual observations  $(p^r, I_t^r)$ ,  $r = 1, \dots, N$ , be rationalizable for each consumer  $t = 1, \dots, T$ . The data is then rationalizable as Walrasian equilibria if markets clear in each observation.

**Lemma 2.** *Let  $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$ ,  $r = 1, \dots, N$  be a finite set of observations. There exist smooth, strictly quasiconcave, monotone utility functions rationalizing this data and initial endowments  $\{\omega_t^r\}_{t=1}^T$  such that  $p^r$  is an equilibrium price vector for the economy  $\mathcal{E}^r$  if and only if:*

- (a) *the dual strict Afriat inequalities hold for each consumer  $t = 1, \dots, T$*
- (b) *"markets clear":*

$$\sum_{t=1}^T x_t^r = \omega^r, \forall r$$

where  $x_t^r = -\frac{1}{\lambda_t^r} q_t^r$  for each  $r$  and  $t$ .

**Proof:** Follows immediately from Theorem 1.  $\square$

Moreover, note that given a finite set of observations  $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$ ,  $r = 1, \dots, N$ , without observations of initial endowments we can without loss of generality assume that each observation of the income distribution  $\{I_t^r\}_{t=1}^T$  is derived from collinear individual endowments. More precisely, for each  $r$  and  $t$  define

$$\alpha_t^r = \frac{I_t^r}{p^r \cdot \omega^r}$$

and  $\alpha^r = (\alpha_1^r, \dots, \alpha_T^r)$ . Given utility functions  $\{U_t\}_{t=1}^T$ , the distribution economy  $\mathcal{E}_{\alpha^r}$  is the economy in which consumer  $t$  has preferences represented by the utility

function  $U_t$  and endowment  $\alpha_t^r \omega^r$ . Using this observation we can now restate Lemma 2 as follows.

**Lemma 2'.** *Let  $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$ ,  $r = 1, \dots, N$ , be a finite set of observations. There exist smooth, strictly quasiconcave, monotone utility functions rationalizing this data such that  $p^r$  is an equilibrium price vector for the distribution economy  $\mathcal{E}_{\alpha^r}$  if and only if:*

- (a) *the dual strict Afriat inequalities hold for each consumer  $t = 1, \dots, T$*
- (b) *"markets clear":*

$$\sum_{t=1}^T x_t^r = \omega^r, \forall r$$

where  $x_t^r = -\frac{1}{\lambda_t^r} q_t^r$  for each  $r$  and  $t$ .

The dual strict Afriat inequalities in (a) are exactly the conditions characterizing observations which can be rationalized by consumers with smooth characteristics, and we showed in Theorem 1 that such observations can always be rationalized by utility functions generating locally monotone demand. The results we derive regarding the refutability of local stability and local comparative statics then follow from the striking properties of locally monotone individual demand functions.

First, unlike almost every other property of individual demand such as the weak axiom or the Slutsky equation, local monotonicity aggregates. If  $f_t$  is an individual demand function which is locally monotone at  $(\bar{p}, \bar{I}_t)$  for each  $t = 1, \dots, T$ , then market excess demand

$$F(p) \equiv \sum_{t=1}^T f_t(p, \bar{I}_t) - \omega$$

is locally monotone at  $\bar{p}$ .

Furthermore, local monotonicity at equilibrium implies local stability of tâtonnement, at least in distribution economies, as the following result shows.

**Theorem (Malinvaud).** *Let  $\bar{p}$  be an equilibrium price vector for a distribution economy  $\mathcal{E}_{\alpha}$  with income distribution  $\{\alpha_t\}_{t=1}^T$ . If each consumer's demand function is locally monotone at  $(\bar{p}, \bar{I}_t)$ , where  $\bar{I}_t = \alpha_t(\bar{p} \cdot \omega)$ , then the tâtonnement price adjustment process is locally stable at  $\bar{p}$ .*

In addition, local monotonicity at equilibrium implies both local uniqueness<sup>2</sup> and monotone local comparative statics in distribution economies.

**Theorem (Malinvaud).** *Let  $\bar{p}$  be an equilibrium price vector for a distribution economy  $\mathcal{E}_{\alpha}$  with income distribution  $\{\alpha_t\}_{t=1}^T$ . If each consumer's demand function is*

<sup>2</sup>To see this, note that if  $F(\bar{p}) = 0$  and  $F$  is locally monotone at  $\bar{p}$ , then  $(p - \bar{p}) \cdot F(p) < 0$  for  $p$  sufficiently close to  $\bar{p}$ . In particular,  $F(p) \neq 0$  for such  $p$ .

locally monotone at  $(\bar{p}, \bar{I}_t)$ , where  $\bar{I}_t = \alpha_t(\bar{p} \cdot \omega)$ , then the equilibrium correspondence  $\mathcal{C}$  for the distribution economy  $\mathcal{E}_\alpha$ <sup>3</sup> is locally monotone in a neighborhood  $\mathcal{P} \times \mathcal{U}$  of  $(\bar{p}, \omega)$ , i.e., if  $(p', \omega') \in (\mathcal{P} \times \mathcal{U}) \cap \mathcal{C}$  then

$$(p' - p) \cdot (\omega' - \omega) < 0.$$

This conclusion of locally monotone comparative statics implies in particular that if the aggregate supply of a good increases, all else held constant, then its equilibrium price will fall, at least locally.

The main conclusion of the paper is now an immediate consequence of the results of section 2 and the properties of locally monotone demand functions in distribution economies: local uniqueness, local stability, and local monotone comparative statics are not refutable given a finite set of observations of prices, income levels, and aggregate consumption.<sup>4</sup>

**Proof of Theorem 2:** Let  $\langle p^r, \omega^r, \{I_t^r\}_{t=1}^T \rangle$ ,  $r = 1, \dots, N$ , be a finite set of observations which can be rationalized in an economy in which each consumer has a smooth, strictly quasiconcave, monotone utility function. Then conditions (a) and (b) of Lemma 2 must hold. By Theorem 1, there exist smooth, strictly quasiconcave, monotone utility functions rationalizing the data such that  $p^r$  is an equilibrium price vector for the distribution economy  $\mathcal{E}_{\alpha^r}$  and such that the market excess demand function for  $\mathcal{E}_{\alpha^r}$  is locally monotone at  $p^r$  for each  $r = 1, \dots, N$ . Thus  $p^r$  is a locally unique equilibrium in the economy  $\mathcal{E}_{\alpha^r}$ . By Malinvaud's results,  $p^r$  is locally stable under tâtonnement, and the equilibrium correspondence in the distribution economy  $\mathcal{E}_{\alpha^r}$  is locally monotone at  $(p^r, \omega^r)$  for each  $r = 1, \dots, N$ .  $\square$

## References

- [1] Afriat, S. (1967), "The Construction of a Utility Function from Demand Data," *International Economic Review*, 8, 67–77.
- [2] Brown, D. and R. Matzkin (1996), "Testable Restrictions on the Equilibrium Manifold," *Econometrica*, 64, 1249–1262.
- [3] Chiappori, P.-A. and J.-C. Rochet (1987), "Revealed Preferences and Differentiable Demand," *Econometrica*, 55, 687–691.
- [4] Kirman, A. and K.-J. Koch (1986), "Market Excess Demand in Exchange Economies with Collinear Endowments," *Review of Economic Studies*, 174, 457–463.

<sup>3</sup>Here the income distribution is assumed to be constant as aggregate endowment changes, so the equilibrium correspondence  $\mathcal{C}$  for a distribution economy  $\mathcal{E}_\alpha$  is the set of pairs  $(p, \omega)$  such that  $p$  is an equilibrium price for the economy in which consumer  $t$  has utility  $U_t$  and endowment  $\alpha_t \omega$ .

<sup>4</sup>Note that this result is still valid if in addition individual demands or net trades are observed.



- [5] Malinvaud, E. (1972), *Lectures on Microeconomic Theory*. London: North-Holland.
- [6] Mas-Colell, A. (1977), "On the Equilibrium Price Set of an Exchange Economy," *Journal of Mathematical Economics*, 4, 117-126.
- [7] Quah, J. (1995), "The Monotonicity of Individual and Market Demand," working paper, October 1995.
- [8] Scarf, H. (1960), "Some Examples of Global Instability of the Competitive Equilibrium," *International Economic Review*, 1, 157-172.

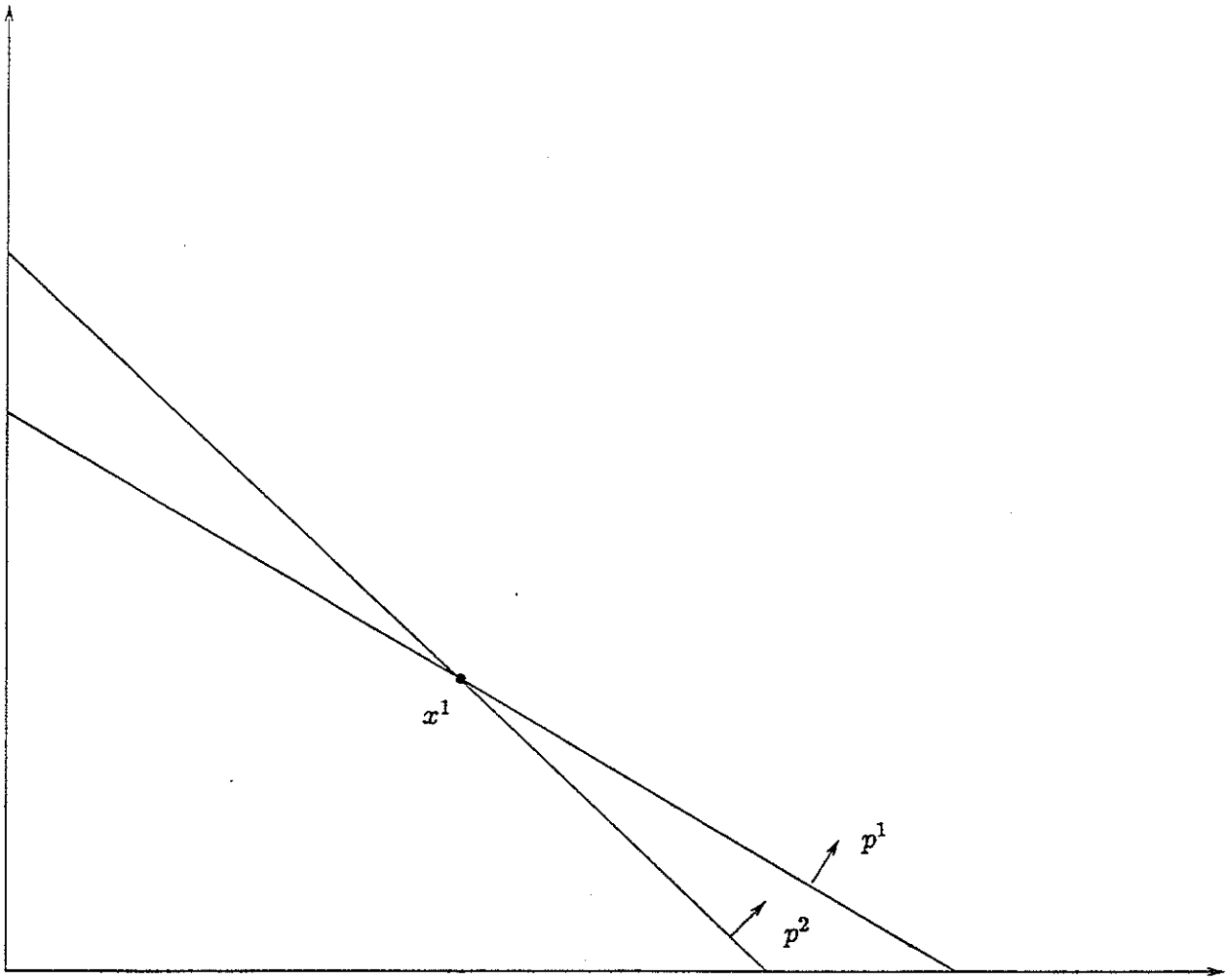


Figure 1

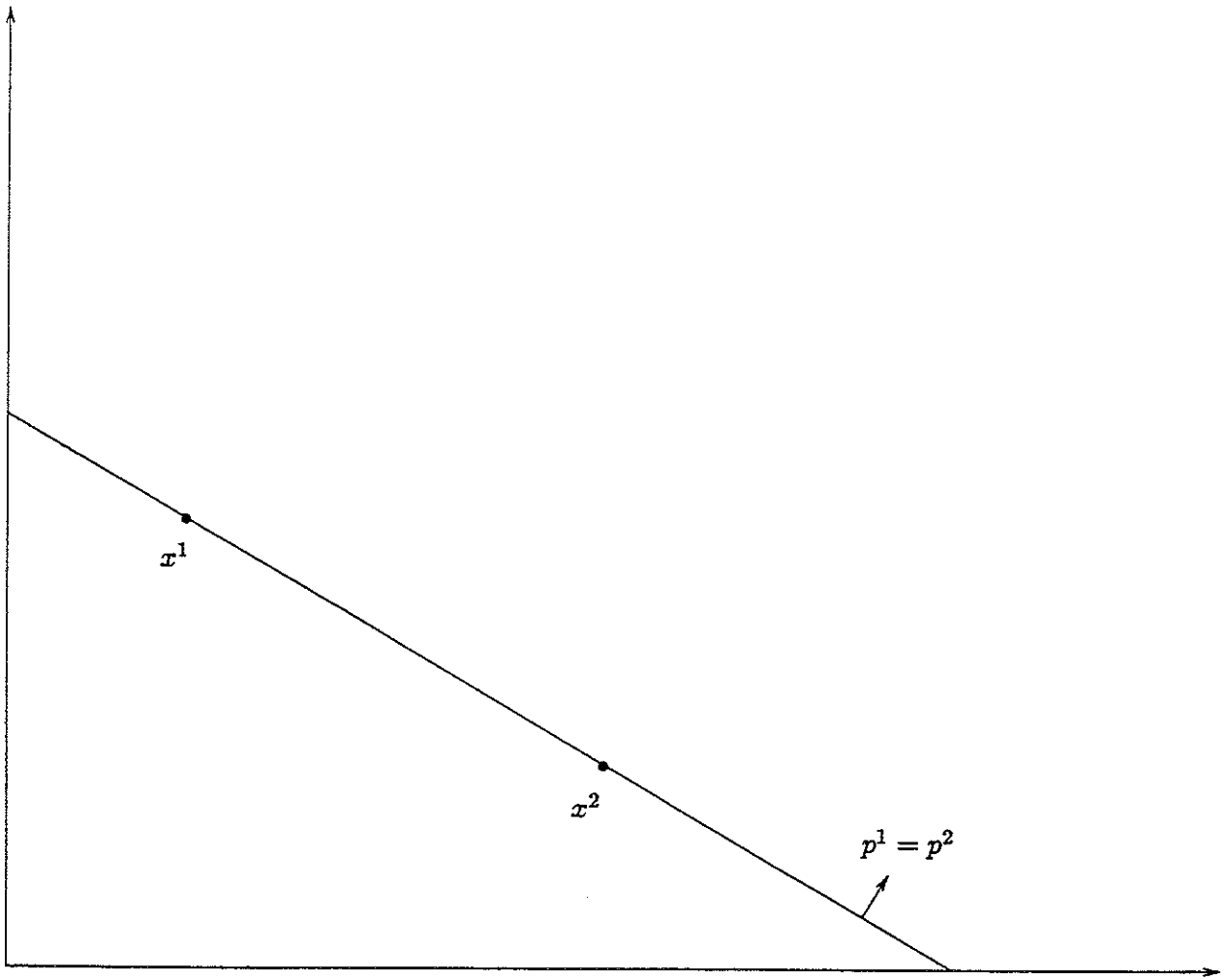


Figure 2

**Working Paper Series**  
**Department of Economics**  
**University of California, Berkeley**

*Individual copies are available for \$3.50 within the USA and Canada; \$6.00 for Europe and South America; and \$7.50 for all other areas. Papers may be obtained from the Institute of Business and Economic Research: send requests to IBER, F502 Haas Building, University of California, Berkeley CA 94720-1922. Prepayment is required. Make checks or money orders payable to "The Regents of the University of California." \* indicates paper available on-line at <http://haas.berkeley.edu/~iber/>*

- 97-255 "Clubs and the Market: Large Finite Economies." Bryan Ellickson, Birgit Grodal, Suzanne Scotchmer, and William R. Zame. April 1997.
- 97-254 "Clubs and the Market: Continuum Economies." Bryan Ellickson, Birgit Grodal, Suzanne Scotchmer, and William R. Zame. April 1997
- 97-253 "Doing It Now or Later." Ted O'Donoghue and Matthew Rabin. January 1997.
- 97-252 "Fairness in Repeated Games." Matthew Rabin. January 1997.
- 97-251 "Psychology and Economics." Matthew Rabin. January 1997.
- 97-250 "First Impressions Matter: A Model of Confirmatory Bias." Matthew Rabin and Joel Schrag. January 1997.
- 96-249\* "Determinacy of Competitive Equilibria in Economies with Many Commodities." Chris Shannon. July 1996.
- 96-248 "Learn, Earn, and Serve." (Commencement Address). Michael J. Boskin. May 1996.
- 96-247 "Spatial Effects upon Employment Outcomes: The Case of New Jersey Teenagers." Katherine M. O'Regan and John M. Quigley. March 1996.
- 96-246 "Externality Pricing in Club Economies." Suzanne Scotchmer. March 1996.
- 96-245 "Cost Optimization in the SIS Model of Infectious Disease with Treatment." Steven M. Goldman and James Lightwood. January 1996.
- 95-244 "The Northridge Earthquake: A Natural Experiment in Market Structure." Sean Ennis. November 1995.
- 95-243 "Decentralization in Replicated Club Economies with Multiple Private Goods." Robert P. Gilles and Suzanne Scotchmer. October 1995.
- 95-242 "Patent Breadth, Patent Life and the Pace of Technological Improvement." Ted O'Donoghue, Suzanne Scotchmer, and Jacques-François Thisse. October 1995.
- 95-241 "Moral Preferences, Moral Constraints, and Self-Serving Biases." Matthew Rabin. August 1995.
- 95-240 "The Time Series Relationship between Investment and Cash Flow in the Scientific Sector: A Panel Data Study Comparing French, Japanese, and the United States Firms." Lee Branstetter, Bruno Crepon, Bronwyn H. Hall, and Jacques Mairesse. July 1995.
- 95-239 "Teenage Employment and the Spatial Isolation of Minority and Poverty Households." Katherine M. O'Regan and John M. Quigley. July 1995.
- 95-238 "Strict Monotonicity in Comparative Statics." Aaron S. Edlin and Chris Shannon. July 1995.
- 95-237\* "Simulation Methods for Probit and Related Models Based on Convenient Error Partitioning." Kenneth E. Train. June 1995.
- 95-236 "Regulation, Competition and Cross-subsidization in Hospital Care: Lessons from the Economics of Regulation." Stephen Earl Foreman and Theodore E. Keeler. May 1995.
- 95-235 "Edgeworth's Conjecture with Infinitely Many Commodities." Robert M. Anderson and William R. Zame. May 1995.
- 95-234\* "Fairly Priced Deposit Insurance and Bank Charter Policy." Roger Craine. May 1995.
- 94-233 "Determinacy in Infinite Horizon Exchange Economies." Chris Shannon. December 1994.
- 94-232 "Increasing Returns in Infinite Horizon Economies." Chris Shannon. October 1994.
- 94-231 "Debt Deflation and Financial Instability: Two Historical Explorations." Barry Eichengreen and Richard S. Grossman. October 1994.
- 94-230 "Financing Infrastructure in Developing Countries: Lessons from the Railway Age." Barry Eichengreen. October 1994.
- 94-229 "Unemployment and the Structure of Labor Markets: The Long View." Barry Eichengreen. October 1994.

