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UNIVERSITY OF CALIFORNIA
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Generalized Quantum Theory and Mathematical Foundations
of Quantum Field Theory

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Physics

by

Michael Anthony Maroun

June 2013

Dissertation Committee:

Dr. Michel L. Lapidus, Chairperson

Dr. Ernest Ma

Dr. Shan-wen Tsai

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The Dissertation of Michael Anthony Maroun is approved:

Committee Chairperson

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right impetus I needed to reach my potential and complete this scientific work. I remember one of the more esoteric and philosophical conversations that we had on the way to lunch a few years ago. My memory is not precise enough to the word but it can be paraphrased as follows.

Michel: “What happened to you? I haven’t heard from you in a week.”

Me: “Oh well, I had an existential crisis!”

Michel: “Ah I see, but you haven’t answered your emails.”

Me: “Well, it was a crisis. Didn’t you ever have one of those types of crises when you were a student? You know the types where you wonder if any of this is even worth it.”

Michel: “Of course, I did.”

Me: “What did Choquet say then?”

Michel: “Well, I don’t know. I never bothered my adviser with such things.”

Me: (laughing aloud) “Oh I see!”

The best thing about the conversation was that any other person would have taken away the completely wrong message from the exchange— that their adviser did not care. But I took away the correct message, and that was something very simple; I should be polite and answer my emails. So above all, I thank you for caring. No acknowledgement would be complete without giving thanks to your family as well, Madame Odile Lapidus, Julie and Michaël, for their kind patience and in particular to Madame Odile Lapidus for saving this scientific work from being completely lost in Reggio di Calabria, Italy on November 12th, 2011.

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-Michael A. Maroun, University of California, Riverside (June 2013)

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ABSTRACT OF THE DISSERTATION

Generalized Quantum Theory and Mathematical Foundations
of Quantum Field Theory

by

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Doctor of Philosophy, Graduate Program in Physics

University of California, Riverside, June 2013

Dr. Michel L. Lapidus, Chairperson

This dissertation is divided into two main topics. The first is the generalization of quantum dynamics when the Schrödinger partial differential equation is not defined even in the weak mathematical sense because the potential function itself is a distribution in the spatial variable, the same variable that is used to define the kinetic energy operator, i.e. the Laplace operator. The procedure is an extension and broadening of the distributional calculus and offers spectral results as an alternative to the only other two known methods to date, namely a) the functional calculi; and b) non-standard analysis. Furthermore, the generalizations of quantum dynamics presented within give a resolution to the time asymmetry paradox created by multi-particle quantum mechanics due to the time evolution still being unitary. A consequence is the randomization of phases needed for the fundamental justification Pauli master equation. The second topic is foundations of the quantum theory of fields. The title is phrased as “foundations” to emphasize that there is no claim of uniqueness but rather a proposal is put forth, which is markedly different than that of constructive or axiomatic field theory. In particular, the space of fields is defined as a space of generalized functions with involutive symmetry maps (the CPT invariance) that affect the topology of the field space. The space of quantum fields is then endowed the Fréchet property and interactions change the topology in such a way as to cause some field spaces to be incompatible with others. This is seen in the consequences of the Haag theorem. Various examples and discussions are given that elucidate a new view of the quantum theory of fields and its (lack of) mathematical structure.

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Preface

The work herein will attempt to be as self-contained as possible given the dual target reader of both physicists and mathematicians alike. Nonetheless, certain assumptions must be made.

First a word about jargon, it is essentially impossible to avoid the use of jargon. A mathematician can easily be confused by a phrase like *spontaneous symmetry breaking* and a physicist can easily be equally confused by a phrase such as *restriction to compact subsets*. The only reasonable solution is to assume the reader has available to them either an electronic device or a physical text such as an encyclopaedia or dictionary of science and mathematics available to them as an aid. The reader should also be cautioned that this work, in being a novel scientific work, will on occasion use terminology or jargon in a deliberately non-standard fashion. Attention will be drawn to this fact whenever such an instance occurs. Whether such usage is justified or not, only time can tell.

Physics

For the physicists, it is assumed that they are familiar with basic mathematical logic notation, such as \forall , \cup , \in , \subset for which a quick reference page is included in the appendix. This immediately brings up simple delicate issues of logical significance, which are critical to the solid foundations of any physical model. This is because the logical formalism gives one knowledge in advance as to when one would expect a theory to break down. Nature does as it pleases and it is a marvel of modern humanity that using Newton's equations, one could place a person on the moon. Indeed, it is a marvel of science that any mathematical formalism has any correspondence to physical reality whatsoever. But it remains the fact that this is true, has been true, and in all

sensibility one should expect it to remain true, insofar as one is able to create novel enough models to encapsulate new natural phenomena or, and here is an important point, reconcile obvious inconsistencies.

Historical examples include, the state of physics whence Ernest Rutherford had realized that all the positive charge of the atom lies condensed together in a confined region aptly named the nucleus, an immediate paradox arises. Given the electrostatic repulsion of the positively charged protons to each other, how can the nucleus exist at all? The answer, of course, was the existence of an entirely new and unseen force of nature, the aptly-named strong nuclear force.

A more modern and equally obvious paradox is, for example, Albert Einstein's theory of relativity compels one to conclude that time is another coordinate axis. But one can move forward and backward on the spatial axes but not on the time axis. Typical current arguments to attempt to explain this, generally rely upon entropy. But this in itself brings to the table a new paradox. How can a collection of atoms (even ideal, and weakly or non-interacting), which all obey the Schrödinger equation, a time reversible equation, come together and undergo irreversible processes? A partial answer to this question will be proposed in this work. The problem also commonly goes by the name *time asymmetry* or the *arrow of time* problem. A related problem (that has been already partially addressed by E. Wigner, A. Bohm, et al.), is the fact that there are particles and/or atomic states which decay, but decaying processes are not (spontaneously) reversible¹.

Now return to the nuances of mathematical logic. As an example of simple but subtle logical delicacy, one says for example that kinetic energy is *non-negative* as opposed to saying that it is positive. This is because an object can be at rest, in which case the kinetic energy is zero. But in the strict logic of mathematics, zero is neither a positive number nor is it a negative number. It is at this level of subtlety that the physicist reader should be engaged, with the knowledge that the author intends to bring up such seemingly bureaucratic details only when absolutely necessary for understanding the avenue toward a novel physical model requiring a novel (but careful) approach to the mathematical modelling of reality. This is the challenge that is brought to the physicist reader. Succinctly, if K is the kinetic energy

¹One can often reverse a fundamental process by supplying external work, through careful 'preparation and registration of states'. See Arno Bohm et al. for the meaning of such a phrase.

then one writes $K \geq 0$, making K non-negative and if one wishes to exclude objects that are at rest, then one may say $K > 0$, making K (strictly) positive.

Mathematics

For the mathematician, it is assumed that they understand that the world around them exists and that it only exists because it can be interacted with and measured repeatedly to behave in the same manner, time after time. One cannot *prove* that the sun will rise tomorrow. Yet, I can assure the reader that tomorrow this will happen.

On the other hand, the life cycle of the sun is finite and there will be a day in the future where the sun will not rise, at least not in any manner as it is understood today. How does one know such a day will ever come? Again, this is only because astronomers and astrophysicists have tirelessly studied other examples of the sun's same *species*, called stars, and have carefully classified their different behaviors.

Further, it is important to note that if a statistically significant number of stars have been studied in comparison to the total number then the results of the observation can be relied upon. Is it certain that one has observed all possible behaviors? In no manner can one be completely certain of this.

However, one can be assured that the universe has provided a vast number of stars to observe, and the exceptions are known to be rare, though of great interest to fundamental theory. In any case, the likelihood of an event is a statement that does makes sense, and it is thus of great use and importance.

Even a fair die has a non-zero probability of landing on a corner or edge. But that probability is so low, and more importantly, the configuration is so unstable that no reasonable person would place a wager on it. It would be akin to hitting an integer whilst throwing darts at the real line or even any compact subset. It really makes no difference if that subset itself contained an integer or not.

Although such things *can* happen, it does not mean that it will. In fact, the physical definition of impossible is if the time expectation value for an event to happen at least once is of the order of, or many times greater than the age of the known universe, then one says that the event is effectively impossible.

A corollary, which has bearing upon the the day-to-day physical world, is that

every truly randomized deck of playing cards is utterly unique. A combinatorist can tell immediately that given a particular randomly ordered deck of cards as a *standard*, the number of combinations of other orderings is so high that even if every living person in current existence could randomize and check their deck of cards against the standard one in just a single second, and could do so every second since the beginning of the universe, then they would still find that the probability of the standard deck order recurring is so low that the mean time to recurrence is longer than the age of the current known universe.

There is one well-known exception and that is when humanity deliberately interferes, so as to cause the effectively impossible by improbability to become true with certainty. A typical example includes the mass production of unstable particles or the cooling of an atomic sample to within pico-Kelvin of absolute zero. The probability of this occurring on its own is next to zero but it is still possible. This would be no different than ordering by hand a new deck of playing cards to deliberately match the chosen *standard* in the example above.

Lastly, there is the selection and preference of one answer over another. For example, in elementary mathematics problems one may be asked to optimize some physical quantity given some physical constraints. The end result of the analysis may be a quadratic equation which gives two answers one positive and one negative. Now because it is known that the physical quantity sought, such as area or length, cannot be negative in an absolute sense, then one uses this physical criterion to select out a single unique answer when there would be no uniqueness otherwise. There is no hard and fast rule, which makes this always the case but it is frequently so and thus noteworthy, if not for its exceptions.

History and Philosophy

It is the author's strong belief that one cannot separate science from its own history or for that matter from the personal philosophy of the scientist conducting the science. This belief stems from the knowledge and experience in science that choices need to be made, choices such as what aspects to discard, what aspects to emphasize, and so forth. These very simple choices forever bias one's work and hence one's conclusions.

Simply put, one should be both very careful and precise in the presentation of scientific results, by being careful to present only what is necessary but at the same time being precise enough to state all assumptions made in arriving at a given conclusion. Every attempt in this work has been made to adhere to these two partially contradictory principles. Thus to uncover the ontological, one must expect to study the tenets and motivations.

In order to justify research into the foundations of a theory that is already in widespread use (thus implying a certain degree of supposed understanding), one must seriously consider the philosophical tenets of the theory that is in use, and find good reason to question the *presumed* foundations. That is to say, a scientist should point out inconsistencies (that ultimately arise through experimental data) in the assumptions behind any theory, which in physical dogma almost always result in new physical predictions. The very history of theoretical physics holds the answer to the question of whether or not one should or should not have good grounds for doubt in an already successful theory.

A quick historical example of this point is Maxwell's theory of Electricity and Magnetism. Though the theory was finally fully published in a single publication around 1872 and the physical phenomena of electricity and magnetism were unified and thus shown to be the result of a single abstract object whose name is commonly accepted today as the *electromagnetic field*, this theory could not explain the discrete nature of electric current. Maxwell was obviously aware of and understood Gauss's mathematical abstractions of vector fields but preferred himself the quaternions of the Irish mathematician William Rowan Hamilton.

It was Michael Faraday and Oliver Heaviside that focused on the notion of vector field and force field lines. At this point in history, mathematics and physics were closely intertwined at least in part due to the efforts of Faraday and Heaviside. But soon there after, J. J. Thomson in 1896 and then Ernest Rutherford in 1911 showed independently that negative charges— respectively positive charges, were quite discrete and corpuscular in nature.

But the success of Maxwell's theory was unabated by this experimental information that hinted at the incompleteness of Maxwell's theory, because the theory was so successful in applications to inventions such as the telegraph, the radio (the 'original'

wireless), and so forth. Maxwell's theory relies so heavily upon the continuum that its discretization, and quantization are either mathematically ill-defined or when they are defined mathematically rigorously, the mathematical prescription leads to contradictions when that prescription is applied *to any system mathematically inequivalent to the system for which the prescription was originally well defined*. It is this last statement that will be explored in this treatise.

Finally the two ultimate aspects of the philosophy of science and especially theoretical physics that must be mentioned as they are fundamental to the underlying motivation behind this scientific work, are as follows.

1. The Heisenberg Perspective: *Observation is king!* This summarizes the all important viewpoint that measurement and interaction with the physical world and one's own environment *is the definition of reality*, and thus truth. Truths hence are subject to revision, and are highly relative to the disposition of an observer. Indeed, Einstein's relativity teaches one that the lengths of rigid rods and the time elapsed on standard clocks made by different observers with different relative velocities to each other will measure different lengths and times. But the critical aspect here is that these differences can be reconciled within the framework of the theory of relativity, and so there is a transformation that links the two points of view that point to a common truth, namely the physical invariants. The only true notion of a global truth is the deep connection between physical invariants and their associated symmetries, commonly referred to in the current literature as Noether's theorem. However, one should bear in mind that these symmetries and their conserved quantities may not be as global as one might imagine. In particular, critical phenomena, and discontinuities in the environment (read boundary conditions) can result in remarkable behavior.
2. The Scientific Method: There really is no such notion of a single all encompassing scientific method. Yet here, the phrase will be taken to mean, *the formation of a question and the subsequent hypothesis under which a scientific investigation is taken*. For example, there is a collection of particles with new quantum numbers. One asks the question as to whether they arise from a single underlying phenomena. One then proposes the hypothesis that the quantum numbers can be used to classify the particles into a mathematical group structure. One

then undergoes the arduous but valorous task of building mathematical abstractions with which to attempt answers. However, it should be remarked that there *most definitely is such thing as a bad question*. This is why there is no single notion of "the scientific method", because there is no a priori method for distinguishing good questions from bad ones. The only way to make such a distinction is to undergo an investigation process, which still may not uncover the merit or lack of merit in a question but it may lead to virtuous modification of the initial hypothesis.

Thus in this scientific work, the question being posed is, "What is the mathematical nature of quantum fields?". The hypothesis is that the fields are a member of a vector space dual to a test function space, *which changes with each interaction and environment*. This hypothesis, which cannot be fully answered in this work at present, has the virtue of being able to account for the following two points.

- (a) all the current failures up to now in attempts at a single mathematical formalism
- (b) the validity of Haag's theorem while still believing that an answer is attainable, i.e. that there is in fact a mathematical formalism.

The very last philosophical perspective that is important to mention is what will be termed, *the Feynman* perspective. The idea here, as inspired by Richard Feynman and his writings on science, is that concepts such as energy conservation will probably always be true from all of its past success to nearly any imaginable future fate of the universe. If it is found that energy is not conserved then one concludes that there is energy present only it has taken a form or gone to a region which is not being measured or observed carefully. Sometimes, one must ask whether it is even practical to track all such forms of energy changes.

A perfect example is that of an inelastic collision where the kinetic energy is not conserved. The total energy if purely kinetic initially, must be conserved. But inelastic systems, like a car crash, have energy fluxes that cause the final kinetic energy to be less than the initial total due to shifts in the energy during the collision to other forms such as, internal thermodynamic energy, thermal heat energy, sound energy, energy of stress deformation etc. Thus the task of an experimental physicist is monumental. So, it is to no surprise that current good experiments take great time

and cost large sums of money for extensive need of staff and/or equipment. Imagine the task of trying to measure and account for all the various energy fluxes that take place in a single car crash.

“The earlier mathematics could always be made sound in that way, but in the renormalization theory we have a theory that has defied all the attempts of the mathematician to make it sound. I am inclined to suspect that the renormalization theory is something that will not survive in the future, and that the remarkable agreement between its results and experiment should be looked on as a fluke.”

– Paul Adrien Maurice Dirac

[“The Evolution of the Physicist’s Picture of Nature”, *Scientific American*, 1963.]

Chapter 1

Introduction

1.1 Motivation and Bias

In attempts to maintain the standards mentioned in the preface, one starts with the elucidation of biases and assumptions about today's physical theory by asking the following questions. What is the purpose of a mathematical model of a real life system? How accurate can one expect the model to be? It is taken for granted that the answers to these questions are truly understood by the physicist or applied mathematician. Indeed, many researchers simply build on what is already accepted with no regard as to why such things are accepted, or better yet whether such things *should be* accepted despite obvious contradictions elsewhere. The answer to the question regarding the purpose of modelling real life systems is to furnish humanity with predictive power and thus to be able to control one's destiny more than zero control at all. That is to say, to make predictions of physical processes with known success (rates— at the least).

To address the question of how accurate is accurate enough, the answer is that a good mathematical model should be adjustable in order to produce more and more agreement with the various types and level of measurements of physical processes, i.e. the output of experiments. It is then immediate that one can safely conclude that whenever a previously successful model fails, that it is due in part to a failure of one or more of the basic assumptions regarding the original construction of the model. This conclusion comes after one has determined that no amount of fine adjustment to the

theory or experiments can account for disagreement and¹ when said data disagreement is accompanied by traces of contradiction, either in the disagreeing data set itself or elsewhere.

The key here is that the model is a mathematical one and that it has a logical structure so that failures can be directly connected to its ability to model. For example in the case of the quantum mechanics of a single particle, its failure when applied to many body systems is that the Schrödinger evolution assumes that there is a system with describable states, with a single (reduced) mass whose internal and external dynamic degrees of freedom are encapsulated by the potential and that the states are reversible under an unitary time evolution. If any one of these three simultaneously needed conditions fails to be true then so does the entire application of the Schrödinger theory. In both the Newton theory and the Schrödinger theory there is no prescription for the potential. This is not a shortcoming, rather a strength because it falls under the description of a tunable theory. If there is data disagreement then perhaps the potential is wrong. The Newton theory does offer one advantage and that is that it gives a rule for multi-particle interaction, i.e. Newton's third law, which states,

$$\vec{F}_{12} = -\vec{F}_{21}.$$

This law is equivalent to momentum conservation and therefore it can be ported directly to the Schrödinger theory. This leads to a condition on both Newton and Schrödinger multi-particle potentials that takes the form of symmetry. However, it should be noted that in the classical theory the Legendre transformation on the classical phase space does not always exist. Consequently, there are Hamiltonian systems with no associated Lagrangian, and perhaps vice versa. This manifests itself in an ambiguity of the definition of the canonical momentum. In this work, the Hamiltonian point of view is taken because the (simultaneous) existence of a Lagrangian

¹It is worth noting that in this work the word "and", when used as a conjunction in the common English language such as in a list, e.g. 'for reasons 1, and 2', will be preceded by the Oxford comma. But when the word "and" is used to carry meaning in a logical fashion, e.g. 'both condition A and condition B must be met for something to be true', then there will be no Oxford comma. There is also the possibility that the word "and" is used to make something more specific when logic is in play but the conjunction is not playing a technical role, e.g. 'is a horse, and thus a mammal'. In these cases, an Oxford comma will be used because the word "and" itself is playing no logical role, rather a linguistic role of helping to elucidate a thing.

demands rather stronger geometric conditions upon the classical phase space, but energy conservation will be demanded and is most readily seen in the Hamiltonian.

1.1.1 A Concrete Physical Example: Renormalization

Thus in order to see clearly the failure of a theory, it will be demanded that all such physical theories be mathematically consistent. Inconsistencies of a theory with a proper logical foundation either indicate a dire need for an increase in mathematical understanding to arrive at the conclusion that an inconsistency does not exist, or they indicate a paradox in the fundamental philosophy behind a physical theory. It is generally regarded among the founders of quantum field theory that the presence and need for renormalization is an indication of the ignorance of the founders, and indeed all of humanity in understanding some basic attribute of nature. In modern times, this philosophical assumption regarding the incomplete understanding of nature and thus incompleteness of quantum field theory itself takes the form of what is commonly referred to as *effective field theories*.

But renormalization is far worse than this. In fact, it is the pinnacle of absurd. It essentially is equivalent to asking the question, “What would the charge of an electron be if one could observe the charge of the electron unabated by the influences of its physical environment?”. No such circumstances can ever arise, and thus one must discard this perspective. A more suitable question and most likely the question that is commonly understood to be the underlying issue with renormalization is to ask just how the electron charge changes as a function of its environment. But the quantum field theory of fundamental particles does not answer this question. Typically, the best one does is to connect the effective charge to the strength of the coupling constant as a function of energy. Instead one ought to ask, what interactions must be present in order that the coupling *does not* depend upon energy, and furthermore can this be done for all values of the energy, or are there critical phenomena? This question will be partially answered in this work indirectly by demanding that there be no renormalization at all in the resulting theory.

1.1.2 The Unique Status of Condensed Matter Theory

The closest circumstances, in which one can study a quantum field's behavior with respect to its environment and the changes accompanied therein, is the quantum field theory of condensed matter systems, ironically sometimes incorrectly regarded as effective field theories. Here the composition of the environment is explicitly known, and more so it is alterable in experiment. Thus, condensed matter phenomena is testable, and these tests have successfully lead to a certain degree of self-consistency. This self-consistency is the essence of the highly successful theory of universality. Universality asserts, in the broadest sense, that all critical attributes remaining the same, the scale of a system does not change the basic physics. The failure of universality is due to critical phenomena, far from equilibrium behavior, spontaneous symmetry breaking, and boundary effects that do not fit into the classification of the current criteria for universality because for example, it is tacitly assumed that in a condensed matter setting, one is necessarily far from any boundaries, i.e. material edges. So both the philosophy of condensed matter theory and the failures of predictability of condensed matter theory are clearly delineated. The difficulty is in discovering each situation and its nuances in order to then construct a suitably robust model. In any case, the successes of condensed matter theory implies that effective or not, i.e. fundamental or not, quantum field theory has physical meaning in its own right and a natural regime² of both precision and accuracy. This alone is sufficient reason to demand a mathematically rigorous definition of quantum field.

1.1.3 Some Attributes of a Satisfactory Theory

More and more theoretical physicists are needing to draw on properties of mathematics that are outside the standard scope of field theory³. This indicates the general lack of robustness that quantum field theory suffers from. Returning to the issue of renormalization, it is now the case that one wants a straight forward theory like one more akin to the Schrödinger theory or the Newton theory. A small number of basic parameters are entered into the theory and the system is evolved. The result is a

²The word regime will be used throughout the entire work to mean a physical domain of applicability, like that of an energy scale, or type of material etc. This is to avoid use of the word *domain*, which will carry the usual mathematically technical meaning

³From henceforth, 'field theory' will uniquely mean the theory of quantum fields.

prediction that is already at the desired level of accuracy, without the necessity for iterative calculations, i.e. perturbative corrections—some terms that improve accuracy and some terms that diverge from the experimental value, as is the case in field theory. One is reminded of the historical situation of the theory of equants, deferents and epicycles that were used in modified Ptolemaic theories to attempt to describe the retrograde motion of Mars. The theory tried to answer the absurd question, “What would the orbit of Mars be if all the planets rotated in perfect circles with earth nearest the center?”. This question is a perfect example of one that is seemingly answerable and gives rise to a model that appears to account for the observed oddities of nature but whose underlying assumptions are false or by virtue of presumption of a false scenario not truly answerable at all despite the closeness to which one can get to some of the observed phenomena. The tell tale trace is of course that one can only account for *some* and not all of the observed phenomena.

1.2 Basic Assumptions

Having touched lightly on the underlying issues that are present in the modern physical theory, it is with this bias that this work assumes that there is in fact only one electron mass and only one electron charge, namely the very measured values of these quantities in low energy table top laboratory experiments. By the quantum nature of particles it is not unreasonable to expect that this value is the average value of some common state, a state that is in principal highly influenced by its environment, and thus possibly energy scale dependent. But in any case, any changes in these values must come from direct physical interactions, and as such ought to be directly calculable with a proper model of interaction—not a set of running physical ‘constants’. From this perspective, field theory is grossly insufficient even in the case of condensed matter systems, as well as the theory of large numbers of atoms and molecules, i.e. quantum statistical mechanics. More to the point, there is no justification for renormalization of electron charge in a condensed matter system because the energy scale of interaction is not changing. The notable exception is when such systems are investigated over a large range of temperatures, then it is through the thermal energy that the scale has changed drastically. But still, no interacting theory, accounting

for temperature as an interaction i.e. photons in the environment, takes the place of renormalization in condensed matter theory just as one would expect.

1.2.1 The Shadow of Dimensionality

One more shadow of the deficiency of current quantum field theoretic formalism is the catastrophic effects brought on by changes to dimensionality, which all of the current mathematical attempts to describe the quantum theory of fields exhibit. Though this statement is made in gross generality here it will be proved on a case by case basis in the proceeding chapters. For now, let one accept it as true, in order to focus here and now on the point of what attributes a satisfactory theory should carry.

A good physical model should be expressible in general terms that leave the dimensions of the physical space unspecified. This is needed because even in the simplest systems of 3 dimensions, one can easily construct systems that are constrained so as to give rise to a 2 dimensional or 1 dimensional subsystem. Working backward, one should expect to remove such constraints and return to higher dimensions. However, this is not the case in quantum field theory to date. A serious setback of current mathematical models of quantum field theory are that they exhibit new divergences whenever the dimensions of the physical space are altered. A good mathematical foundation would undoubtedly cure this, as easily as the derivative is generalized to more than just one dimension. The logical structure of mathematics guarantees that one can easily pass to generalizations, some easier than others. Nonetheless, it is made possible by the logical structure. The lack of a rigorous mathematical foundation, or the incorrect choice of one, is the major obstruction to generalizations of quantum statistical mechanics and hence its presumed descendent, quantum field theory.

There is another issue and that is that dimensional independence is actually required for dynamical purposes, and here is the key point, whenever the number of degrees of freedom cannot influence the dynamics. It is pointed out elsewhere that large dimensions often lead to triviality in various branches of mathematics because of the large number of degrees of freedom available to the system. On the other hand, the difference between the usual topology of \mathbb{R}^d versus \mathbb{R}^{d-1} is mundane. Other than the change in the number of degrees of freedom, which may for example affect the spectral signature, there should not be critical phenomena, except when the change in

dimension is accompanied with or replaced by a non-trivial change in topology. This is akin to changing the boundary conditions of a system. In particular, one notices that every infinite dimensional Hilbert space is isomorphic to every other infinite dimensional Hilbert space. But on the other hand, there are vital difference between physical dynamics on $L^2(\mathbb{R})$ and $L^2([0, 1])$, owing to their important mathematical differences.

Lastly, the existence of anomalous dimensions, or fractal dimensions, i.e. non-integer dimensions as measured empirically in laboratories, implies that dimensionality is a dynamic entity. Thus it should be expected that a useful mathematical formalism should be able to keep track of how changes in dimension affect the resulting physical system's behavior and vice versus. As of recent in the high energy theoretical physics literature, there has been a multitude of suggestions that there are extra integer numbers of dimensions. However, there is no experimental evidence for this conclusion as of yet or perhaps ever. On the other hand, fractal dimensions of non-integer value have been measured and experimentally verified for porous and aggregated media. This fact furthers the view that quantum reality necessitates a dynamic view of the dimension of space. In this work, the view point that dimensions are dynamic and possibly non-integer in nature is taken. But there is general disbelief in higher integer dimensions, i.e. thus far there seems to be an upper bound of 4 on the dimensions of space-time.

1.2.2 Problems With Current Attempts

Lastly, one should consider why a theory like string theory, or M-theory is almost equally unhelpful. The reason is that the ansatz that physical particles have extended geometric shapes, such as a string, is rather arbitrary. If particles are not 0 dimensional points then why should they be 1 dimensional lines, 2 dimensional surfaces, etc.? The nice attribute of a point is that it automatically comes with rotational symmetry. But extended geometric objects embedded in higher dimensions may or may not have this symmetry. In fact they never have this symmetry unless highly specific conditions are met. One obviously sufficient condition for extended objects is that they are surfaces of the type S^{n-1} , with $n \in \mathbb{N}$, if one agrees that S^0 is a point.

In the absence of physical evidence for the need for such geometry, the presence of geometry in theories that attempt to go beyond quantum field theory and the standard model, are in actuality at least partial by-products of the historical-social fact that the only other major paradigm that physicists are extremely familiar with, aside from the quantum theory, is the theory of general relativity that incorporates geometry in a fundamental way. Thus, it is natural for a lost physicist to rely on geometry as the only other familiar grounds under which nature is partially understood. But this is a deficiency in the variety of knowledge of physical models, not a fundamental attribute of nature. On inspection of the foundational works of string theory, one finds that the founders had no real reason or evidence for making particles extended geometric objects other than curiosity, and some coincidental mathematical formulae. The error in assuming that coincidences are not coincidental, namely that there is a pattern behind the coincidence that makes it inevitable, is that in the absence of experiment there is no guiding criterion under which to cast assumptions that are grounded in reality. In other words, it is quite easy to be right for all the wrong reasons. If one then assumes that the reasons are grounded in reality because one was accidentally right, then almost inevitably this assumption leads to great folly.

1.3 Mathematical Considerations

1.3.1 Example of a Newtonian Particle

Every measurement is a rational number. It has been tacitly presumed since antiquity that measurements of physical phenomena are approximations or truncated results of the consequence of measuring the more sophisticated real physical world, which is assumed to be continuous. A typical argument for the continuity of the motion (velocity as a function of time) of a macroscopic object goes as follows. The object began at rest, from zero (the origin), moved forward in the positive direction, *and traversed every point in between along a route identical to an interval of the real line, as well as in a time increasing along a real line.* Therefore, by an epsilon-delta argument and the intermediate value theorem, the object moved continuously through the interval of choice.

One of many possible problems here start with the fact that one cannot prove

the object actually traversed every point along an interval on the real line nor each point in time for any finite time interval. That is to say in experimental practice, one cannot even be certain that the position of the object as a function of time is continuous, and so no proof of the continuity of the velocity is possible.

1.3.2 The Implicit Continuum Assumption

Instead, one can only show, to within experimental error of length measurements and time measurements, that the object moved, to a good approximation, continuously in the sense that the size, s , of the object and the error in measurement of length, Δx , have a relation akin to: $s \gg_x \Delta x$. While the time for the event to occur (and all experimentally measurable subevents), t , and the error in time measurement, Δt , must also have the relationship: $t \gg_t \Delta t$. Thus an epsilon-delta argument can only apply insofar as one is willing to admit that the implication, $|t - t_i| < \delta \Rightarrow |x(t) - x_i| < \varepsilon$, where t_i is any time in between zero and the arrival time of the object and $x_i := x(t_i)$ is any point between zero and the arrival point of the object, is dependent upon the validity of the binary relations, \gg_t and \gg_x .

Arguments of this type (as well as more compelling ones) are used in justifications for the acceptance of quantum theory. However quantum theory makes the assumption that, *if an infinite number of experiments were conducted, then the resulting continuous probability distribution, $|\psi|^2$, would match that of the prepared state of the system, ψ* . Thus the presumption since antiquity that experiments are truncated approximations of continuous reality is even still assumed in the current canonical quantum theory.

1.3.3 The Absence of the Continuum

But let it stand here as a reminder that the opposite may in fact be true. It may be the case that the rational number measurements represent real physical phenomena exactly (within the measurements' errors), and that the real numbers are an idealized machination for the purpose of, above all, simplifying one's problem. Now at this point, one may worry that there could be an infinity of complexity that opens up causing a discomfort due to the loss of the comfort provided by the simplifying as-

sumption that the real (complex in the case of quantum theory where one deals with $\psi : \mathbb{C} \rightarrow \mathbb{C}$), numbers are the exact mathematical construct and not a simplifying assumption. Fortunately, the situation is in far better shape.

As pointed out by Cassels on page 33, in [Cassels, J.W.S. (1986). *Local Fields*⁴. London Mathematical Society Student Texts. 3. Cambridge University Press.], there is a theorem by Alexander Ostrowski that asserts that any field (of numbers, i.e. in the mathematical sense) that is complete with respect to an Archimedean absolute value is *both algebraically and topologically* isomorphic to the real or complex numbers. What is more is that there is yet another, even more remarkable theorem also attributed to Ostrowski, that asserts that the only completions with respect to a bona fide valuation of the rational numbers, *which exhibit the canonical algebraic and topological properties*, are (up to isomorphisms⁵) the fields of the real/complex numbers or the p-adically completed rational numbers.

This is truly remarkable, and it is a vital fact for the mathematical modelling of reality because it asserts that the countability of events, macroscopic objects, or quantum particles (fields) is both the beginning and the end of fundamental mathematical concerns of such modelling. For if one can count then one has the set of natural numbers. If one can put together two (or more) of such objects, one has addition and thus the integers. If one can count many sub-collections of different types then one has combinations, thus multiplication and consequently the rational numbers. Lastly, the study of the changes that take place in reality require the notion of dynamics, which at its heart is simply the comparing of the rate of change of one object with respect to another, i.e. a notion of derivative and hence implicitly limits and continuity—ergo the need for completing sequences (when and if they exist).

One should note that the need for inverses was the driving fact behind the leap to larger sets. This is noteworthy as there are the rare physical phenomena that are truly reversible and do in fact concern inverses, though most of physical reality owing to causality is indeed irreversible and one should not presume the existence of inverses. It is this last fact that can explain why truncations, discretizations, and

⁴Here the word “fields” is in the mathematical sense, and whenever this occurs, specification will be made.

⁵Since there is an entire class of such fields (mathematical) they have been collectively referred to as a place.

finite difference equations often describe some physical systems better than other methods.

Lastly, it is important to mention that objects like the Cantor set have remarkable mathematical properties that may model very well the idea and paradox of the micro to macro properties of matter, as well as, the wave-particle duality. It is well-known that a solid sheet of metal is mostly empty space microscopically, while macroscopically it gives the sense of the continuum. The Cantor set has the cardinality of the continuum but the sum of the length of its elements is zero. This is just the type of paradoxical attribute that may be needed in quantum field theory. In fact, it is a conjecture that measures supported on sets with Cantor-like properties may very well be the measures nature intended for quantum field theory (at least of materials) and thus could be the tool that renders all of condensed matter quantum field theory, finite and mathematically rigorous. Needless to say, this idea is nothing more than a fleeting conjecture and is in no manner proved in this work.

1.4 The Axioms

The basic axioms that are assumed and enforced upon this investigation are enumerated below. The Wightman axioms are flatly rejected on grounds of a major inconsistency that has arisen in the literature. In 1972, Konrad Osterwalder and Robert Schrader gave axioms for Euclidean Green's functions and proved correspondence properties with the Wightman axioms.

However in 1988, Carl M. Bender and Hugh F. Jones[BenJon] demonstrated the triviality of an Euclidean four dimensional, ϕ^4 theory through, at the time, a novel then-dubbed 'non-perturbative' calculation. This implies an inconsistency since the 3 Euclidean space dimensions with 1 time dimension (heretofore 3+1 dimensional Minkowski space-time) ϕ^4 theory is experimentally known to be non-trivial.

In the same token, the Bender-Jones results imply the Euclidean 4 space dimensional ϕ^4 theory is trivial. One is then faced with the following paradox. One must either conclude that the Osterwalder-Schrader correspondence is incorrect at least when going between 3+1 dimensional Minkowski space-time and 4 dimensional Euclidean space, or the Bender-Jones result of triviality is somehow in error.

It is the opinion here that the Bender-Jones results are true on account of the recurring phenomena of the triviality of the spectrum of the Laplace operator in (Euclidean) dimensions equal to or greater than 4 for many Schrödinger potentials possessing physical relevance. For example the Dirac delta potential explored throughout this work shows marked dimension sensitivity.

Furthermore, there are many recurring mathematical facts that result as a consequence of dimensionality being large, i.e. a system having (too) many degrees of freedom available to it. Famous examples include the triviality of knot theory in dimensions 4 or greater, the return to the origin with certainty in 1 and 2 dimensional random walks but a drastic decrease in the probability to approximately only 19% in 4 dimensions— with the probability approaching zero as the number of dimensions approaches infinity.

The Fundamental Axioms

- (0) *The physical phenomena are represented by mathematical idealizations that are members of a bona fide mathematical set, class, or category. This is listed in increasing abstraction, noting that one has the strict logical containments: set \subset class \subset category.*
- (1) *The existence of an approximate or weak binary relation that is a type of quasi-addition, for the purpose that a notion of physical superposition exists. For these purposes, take **notion** to mean closure, such that for example, if the mathematical idealization is a member of some set then there exists a binary operation such that any two elements of the set is again an element of the set (read: class, category etc. as appropriate).*

*By **weak**, one means something of the sort, in the example that follows. Make the simple observation that the irrational numbers do not exhibit set closure under the usual notion of addition of real numbers. This is because for example the irrational number $\pi - 1$ and the irrational number $-\pi$ have a sum which is rational, i.e. $(\pi - 1) + (-\pi) = -1 \in \mathbb{Q}$, and is thus not irrational.*

Nonetheless, this lack of closure can be trivially avoided by disallowing irrational numbers and truncating every Cauchy sequence, thus restricting oneself to the rational numbers. So the approximation or weak binary relation is the usual

addition of real numbers as applied to irrational numbers but the equivalence is not strong equality but replaced with, for example, Landau 'big O' and/or 'little O' notation. This axiom assumes that the analogue of this example can always be performed with suitable choices of 'weak' and notion of collection.

- (2) There exists a space of test functions, whose topological dual are a generalized function space to be regarded as the space of quantum fields. It is convenient to regard the generalized functions as the **naked** or **unsmearred** fields and the particles as the smeared fields, i.e. the generalized functions paired with an appropriate test function.
- (3) Physical symmetries do not manifest as constraints on the fields no more than the solutions of a differential equation respect the symmetries of the differential equation of motion that gave rise to them. That is to say as an example, Newton's equation of motion is time reversal invariant but there are many physically relevant solutions of the equation of motion that are not time reversal invariant. Consequently, one cannot assume that the fields, which are solutions of the equations of motion, should carry the invariances manifestly themselves. Instead, one assumes here and throughout that the symmetries manifest themselves as involutive maps on the test function space and it is further assumed that the symmetries lift (for example by the Hahn-Banach theorem) to the fields through a canonical prescription such as passing the action of the symmetry on the field, onto the test function.

Chapter 2

The Generalized Quantum Theory

In this chapter, several examples of simple but mathematically subtle quantum physical systems are given. The application of the methods of the distributional calculus becomes mandatory in analysis of quantum evolutions where the classical Schrödinger differential equation is undefined. In particular, one will see that contrary to the usual claims in the literature in physics, there are solutions to so-called *singular* potentials that can be obtained without recourse to renormalization. This makes renormalization *sufficient but not necessary* in order to obtain solutions.

Indeed, it is quite common for physicists to invoke renormalization in singular quantum systems to argue that renormalization is fundamental and unavoidable, since quantum theory is well accepted as being fundamental. Though there is no reason to reject the canonical quantum theory largely due in part to its great successes, this line of reasoning is extremely misleading because it fails to separate a fundamental theory from its mathematically idealized model. In particular, there are certain missteps in mathematical reasoning that one can inadvertently take, which lead to false conclusions. There is the ever famous example from grade school shown below.

$$-1 = i^2 = (\sqrt{-1})^2 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

From which, one can falsely conclude that $1 = -1$. The misstep occurs in the incorrect application of the square root operation $\sqrt{\cdot}$ to negative real numbers. The operation is only defined for use in the following correct way.

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}, \quad \forall a, b \geq 0$$

In a similar fashion, one may conclude that renormalization may be sufficient to realize physically interesting solutions, but it is by no means necessary. In this context, the word *singular* is used to mean a Hamiltonian operator with an effective potential that is not bounded from below (or from above), i.e. not even semi-bounded. Recall from the section entitled "Physics" in the Preface, that the kinetic energy is not bounded but it is semi-bounded from below, i.e. $K \geq 0$. It is then immediate that ignoring this subtle detail of the inequality would be as erroneous as the grade school example above. It is also worth noting that potentials (like the simple harmonic oscillator), which are also semi-bounded from below, are in a certain sense 'compatible' with the kinetic energy operator.

There is a natural operator that manifests itself in quantum mechanics that encapsulates the properties of kinetic energy. To see this, one notes the following. Obviously, one cannot have less motion than no motion at all and indeed the Laplace operator, $-\nabla^2$, has exactly the same property for its spectrum, i.e. $\sigma(-\nabla^2) \geq 0$, when viewed on all of \mathbb{R}^d . Throughout, the notation, $\sigma(H)$, will be used to mean the spectrum of the operator, H . Quick analysis of the Helmholtz equation shows:

$$(2.0.0.1) \quad (-\nabla^2 + \lambda) \psi = 0,$$

and hence

$$\lambda \in [0, \infty) \quad \text{whenever; } \psi \in L^2(\mathbb{R}^d).$$

Also, note that to elucidate a spectrum one must specify the vector space over which the linear operator acts on. Implicit in the above conditions are the boundary conditions, as well as the domain and codomain, i.e. $\psi \xrightarrow{\|x\| \rightarrow \pm\infty} 0$. Consequently, (minus) the Laplace operator is bounded from below¹ (but not from above and hence not a bounded operator) and $\lambda = 0, \iff \psi = 0$, (when ψ is required to be a function) which simply asserts the nice obvious fact that static systems are not dynamically interesting. One then writes, $\sigma(-\nabla^2) = [0, \infty)$. Incidentally, one should expect in general, by definition of the spectrum of any operator, that the spectrum is a closed subset of the complex plane, i.e. $\sigma(H) \subset \mathbb{C}$. Two more comments are in order here that come as results from the mathematical field of functional analysis. The first is that there are operators, A , for which no such λ -eigenvalue exists but that have

¹Operators with this property will also be referred to as *semibounded* operators.

non-empty spectra, i.e. $\sigma(A) \neq \emptyset$. Here the notation “=” for sets means: for any two sets, \mathcal{A} and \mathcal{B} , $\mathcal{A} = \mathcal{B}$, if and only if (abbreviated iff) both $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \supseteq \mathcal{B}$ are true. In contrast, if the set \mathcal{A} is completely contained in the set \mathcal{B} , one simply writes $\mathcal{A} \subset \mathcal{B}$.

2.1 The Postulates and Philosophy of Quantum Theory

Here it is desirable to set the record straight regarding just exactly what a quantum mechanical theory is. It is assumed that there is a state, Ψ , which strictly speaking represents an entire ray on a Hilbert space that encapsulates the physical attributes of the system under quantum investigation. That is to say, the ray carries with it a set of quantum numbers that track each and every individual degree of freedom of the quantum system. The state, Ψ , does not represent a particle in any way except for the following special circumstances. In general, Ψ , simply describes the states of a quantum system.

A quantum system could be a molecule, an atom, a protein, the intrinsic angular momentum (spin) states of any one or more of the previous examples, and so forth. It is a special attribute of the hydrogen atom that it is the stable composite structure of only one proton and one electron. The proton has a mass of almost 2000 times that of the electron and so it can be chosen to be at rest and all the relative configurations of the hydrogen system can be regarded as internal configurations of the electron. It is this elementary example that often gives the false impression that, Ψ , describes the state of electrons. In general, it does not. It merely describes the state of the quantum system, whichever such system it may be.

Likewise as is the case with hydrogen, atoms like sodium can be approximated as being stationary with the exception of its one valence electron, which is again regarded as being responsible for all the changes in the internal quantum configurations of the sodium atom. In the regime in which this approximation is true, then the hydrogen-like Ψ states will be applicable to sodium as well, modified by the appropriate change in the atomic number, Z .

It is important to point out that there are very simple physical systems, i.e. one dimensional quantum mechanics, for which there are no unique answers. As a very simple example, consider the equation $x^2 + 1 = 0$. If one seeks solutions for x over the set of real numbers then there are no solutions- no such real-valued x , will make the statement true.

On the other hand, if one seeks values of x that are complex, i.e. find x such that $x \in \mathbb{C}$, then there are in fact two solutions. An example of this was already seen in the Helmholtz equation 2.0.0.1, where ψ was required to be a *function (on all of \mathbb{R}^d)*. In particular, the quartic potential, even in only one dimension with lower degree polynomial terms, has no global solutions on the real line. It is this surprising fact that is the source of a great deal of technical difficulties in what physicists call the “ ϕ^4 theory”.

In the same spirit, there are solutions to differential equations, which are not solutions in the classical sense- meaning the solutions are not smooth functions of a real variable. There are the well-known weak solutions as generalizations, and yet there are even more general solutions, the generalized function solutions. These seemingly obvious mathematical facts are brought up because theoretical physicists often speak of “solving an equation”, such as the Lippmann-Schwinger equation but the vector space over which solutions are to be considered critically affect one’s very notion of the existence of the various types of objects to be regarded as solution.

In particular, the scattering solutions to the Schrödinger equation *are not elements of a square integrable Hilbert space*. The so-called ‘delta normalization’ is not a satisfactory replacement either. Rather, the theory of Hilbert spaces is so rich that there in fact exist a great many variety of Hilbert spaces of entire functions for example, which encapsulate the free scattering states. One must bear all of this in mind as the quantum theory is generalized beyond the confines of the classical theory of partial differential equations and the confines of L^2 spaces, as one has in mind the notion of rigged Hilbert space as well.

Indeed, this was the goal of Feynman with his functional integral approach. There being many more integrable functions than differentiable ones, this would at first seem quite wise. However, it is wrought with great technical difficulty because there is no integration measure with which to perform the functional integration. This is in sharp

contrast to the diffusion equation, where Wiener measure exists and plays a critical role in the functional integral solutions to the diffusion equation, best known as the Feynman-Kac formula.

For example, Edward Nelson's approach to making the Feynman (functional) integral mathematically sound relied upon using the beautiful art of analytic continuation into the complex plane to side step the issue of the non-existence of the functional measure. For the state-of-the-art mathematical rigor and most general results on the Feynman integral the interested reader is directed to the monograph by G. W. Johnson and M. L. Lapidus, *The Feynman Integral and Feynman's Operational Calculus*, Oxford University Press, 2000.

2.2 One Dimensional Quantum Mechanics

Consider the following one dimensional quantum systems, that is the Schrödinger equation with various potentials of interest. There are 3 in total that are of particular interest along with their higher dimensional analogues. But for now, one dimension will suffice.

2.2.1: $V(x) = -\alpha \delta(x) \quad x \in \mathbb{R}, \quad \alpha > 0$

2.2.2: $V(x) = -\alpha \delta'(x) \quad x \in \mathbb{R}, \quad \alpha > 0$

2.2.3: $V(x) = -\frac{\alpha}{|x|} \quad x \in \mathbb{R}, \quad \alpha > 0$

The general trend to look for is to inspect the time independent Green function solutions of such systems carefully or to identify conditions, which must be met in order that such a solution exists. Bound state eigenvalues will have the property that they will cause the Green function to cease to exist, for example a divergence resulting from evaluating a function at its simple pole. This is a consequence of the fact that the Schrödinger operator is not invertible at the values of the energy eigenvalues, when they exist and are thus members of the spectral set of solutions. Recall, for the theory of finite dimensional linear equations, this condition is equivalent to the vanishing of the determinant of the eigenvalue matrix, i.e. eigenvalues that prevent one from inverting the original (bounded) linear operator (finite rank square matrix).

In the classical analysis of the Schrödinger equation, being a linear second order differential equation in the simplest case, singular points of the Green function be them, poles, discontinuities of any of its first or second derivatives, or poles of those derivatives, will have significant physical meaning in that they will correspond to energy eigenvalues. These will typically occur at single points in energy space and so any bound state spectrum that is *discrete* (i.e. as in every member of the discrete spectral set, can be enclosed by a ball of radius, $r = \varepsilon > 0$, such that there exists an ε small enough so that no other energy eigenvalue can be found within this ball.) cannot be discovered by the Lebesgue integral, except possibly integrating around epsilon balls enclosing a discontinuity.

2.2.1 Dirac Delta Function (Measure)

The first system is somewhat trivial. It has been treated in many texts most notably, [Gri]. There are 3 main methods that will be used on this well-known system that will allow one to view the system in a manner that elucidates the more general view of solving such Schrödinger systems. The methods in order of their presentation are:

- 1: The Fourier transform
- 2: The canonical distributional calculus
- 3: The convolution method

It is the last method, 3, that will allow one to extend the theory of quantum mechanics in a mathematically rigorous means, while simultaneously extending solution methods in the mathematical theory of partial differential equations.

2.2.1.1 Method 1: The Fourier Transform

Here, the solution will be obtained via the Fourier transform. Indeed consider the following Schrödinger operator,

$$H_{1\delta} \psi = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \alpha \delta(x) \right] \psi = -|E| \psi, \quad \text{such that, } E < 0.$$

The idea is to seek the inverse of the operator, $H_{1\delta}$ above. So one writes,

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \alpha \delta(x) + |E| \right] G(x) = \delta(x).$$

Then after an application of the Fourier transform one arrives at,

$$\frac{2\pi^2\hbar^2}{m} k^2 \widehat{G}(k) - \alpha G(0) + |E| \widehat{G}(k) = 1.$$

At this point since one has already assumed the existence of $G(x)$, it makes sense to then assume $G(0)$ exists and is a finite complex number. One then must check for self-consistency at the end to verify that the assumptions were correct. In any case, there is then a nice simple algebraic equation for $\widehat{G}(k)$ given by,

$$\widehat{G}(k) = \frac{1 + \alpha G(0)}{|E| + \frac{2\pi^2\hbar^2}{m} k^2}$$

Applying the inverse Fourier transform one arrives at,

$$G(x) = [1 + \alpha G(0)] \frac{e^{-\sqrt{\frac{2m|E|}{\hbar^2}} |x|}}{\sqrt{\frac{2\hbar^2|E|}{m}}}$$

At this point one notices that $G(x)$ is an $L^2(\mathbb{R})$ function and as well its Fourier transform $\widehat{G}(k) \in L^2(\mathbb{R})$ too. So, it is immediate that the use of the Fourier transform was justified.

The self-consistency condition is then

$$G(0) = [1 + \alpha G(0)] \frac{1}{\sqrt{\frac{2\hbar^2|E|}{m}}}.$$

One then finds,

$$G(0) = \frac{1}{\sqrt{\frac{2\hbar^2|E|}{m}} - \alpha}.$$

Inserting this expression into the original expression for $G(x)$, one arrives at,

$$G(x) = \left[\frac{1}{\sqrt{\frac{2\hbar^2|E|}{m}} - \alpha} \right] e^{-\sqrt{\frac{2m|E|}{\hbar^2}} |x|}$$

Now there is a slight issue and that is $G(x)$ will remain $L^2(\mathbb{R}) \forall x$ iff $\alpha \neq \sqrt{\frac{2\hbar^2|E|}{m}}$. Otherwise, $G(x) \rightarrow \infty \forall x$. The expression for the function $G(x)$ contains the characteristic equation for the operator and hence the only element of the spectrum with $E < 0$ is indeed the pole,

$$|E| = \frac{m\alpha^2}{2\hbar^2},$$

as it is clearly not an element of the resolvent set, $\varrho(H_{1\delta}) = \overline{\sigma(H_{1\delta})}$. Finally, one concludes that there is a single unique bound state with energy given by,

$$E = -\frac{m\alpha^2}{2\hbar^2},$$

with its single unique normalized bound state,

$$\psi(x) = \sqrt{\frac{m\alpha}{\hbar^2}} e^{-\frac{m\alpha}{\hbar^2}|x|}.$$

2.2.1.2 Method 2: The Canonical Distributional Calculus

To be completely mathematically thorough, one should check that the Schrödinger equation in this context makes sense as a generalized equation, namely as an identity for generalized functions. One needs only the classic theory of distributions as outlined by Laurent Schwartz². The pairing $\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ is given by the duality bracket,

$$\langle T, \phi \rangle := \int T \phi \in \mathbb{C},$$

where $T \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$.

The Schrödinger equation now reads,

(2.2.1.1)

$$\begin{aligned} -\frac{\hbar^2}{2m} \left[\frac{m\alpha}{\hbar^2} \right]^{\frac{5}{2}} e^{-\frac{m\alpha}{\hbar^2}|x|} [2h(x) - 1]^2 + \frac{\hbar^2}{2m} \left[\frac{m\alpha}{\hbar^2} \right]^{\frac{3}{2}} e^{-\frac{m\alpha}{\hbar^2}|x|} \frac{d}{dx} [2h(x) - 1] \\ - \alpha \sqrt{\frac{m\alpha}{\hbar^2}} e^{-\frac{m\alpha}{\hbar^2}|x|} \delta(x) + \frac{m\alpha^2}{2\hbar^2} \sqrt{\frac{m\alpha}{\hbar^2}} e^{-\frac{m\alpha}{\hbar^2}|x|} = 0, \end{aligned}$$

where $h(x)$ is the Heaviside function. It is defined as a piecewise function on \mathbb{R} as,

$$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

If a value at $x = 0$ is assigned to be $h(0) := \frac{1}{2}$ then one has the equality, $2h(x) - 1 = \text{sgn}(x)$ point-wise on \mathbb{R} , where $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(x) = 0$ if $x = 0$. The function $\text{sgn}(x)$ is sometimes called the “sign” function or the “signum” function to avoid verbal confusion with the “sine” function.

²See both the Chapter on Distribution Theory and Fundamental Physics as well as the appendix Mathematical Definitions’ section on the Schwartz space

Now there are a couple of technical issues that need to be addressed here. The first is the meaning of the product $T \cdot \psi$. This question has long been answered in the theory of distributions since $T \in \mathcal{S}'(\mathbb{R})$ and the function ψ is not only $L^2(\mathbb{R})$ but it is in fact $L^p(\mathbb{R})$ and everywhere continuous on \mathbb{R} and in particular the value of $\psi(0) \in \mathbb{R}$ exists.

So by the duality bracket above, one concludes that $T := \delta(x) \cdot \psi(x) := \psi(0) \delta(x) \in \mathcal{S}'(\mathbb{R})$ since $\delta(x) \in \mathcal{S}'(\mathbb{R})$. The second and last technical issue is the meaning of the expression $[2h(x) - 1]^2$. Though this function is not continuous, one can give meaning to the expression at each point on \mathbb{R} by usual point multiplication on \mathbb{R} for the product of two functions $f(x) \cdot g(x)$ at each $x \in \mathbb{R}$, with the exception of the points of discontinuity in this case only one value, the value at $x = 0$. Hence the product of the piecewise defined function $2h(x) - 1$ with itself is the piecewise function, $\lambda(x)$,

$$[2h(x) - 1]^2 := \lambda(x) := \begin{cases} \text{undefined} & x = 0 \\ 1 & \text{otherwise} \end{cases}$$

The thing to notice is that $\lambda(x)$ is equivalent to the unit constant function Lebesgue almost everywhere. That is, except on the set of (Lebesgue) measure zero, the point $x = 0$, and thus it is identical to the constant unit function integrated against the test function itself, i.e. it is simply Lebesgue measure in the sense of distributions.

This is true in the sense of measures as well, where one utilises the fact that every measure is necessarily a distribution, as in an element of the topological dual of the vector space of infinitely differentiable functions on \mathbb{R} , $C_c^\infty(\mathbb{R})$, with compact support. Lastly, in the sense of distributions $\frac{d}{dx} [2h(x) - 1] = 2\delta(x)$. Using these facts, one can now write for equation 2.2.1.1,

$$(2.2.1.2) \quad \begin{aligned} & -\frac{\hbar^2}{2m} \left[\frac{m\alpha}{\hbar^2} \right]^{\frac{5}{2}} e^{-\frac{m\alpha}{\hbar^2}|x|} \lambda(x) + \frac{\hbar^2}{2m} \left[\frac{m\alpha}{\hbar^2} \right]^{\frac{3}{2}} e^{-\frac{m\alpha}{\hbar^2}|x|} 2\delta(x) \\ & -\alpha \sqrt{\frac{m\alpha}{\hbar^2}} e^{-\frac{m\alpha}{\hbar^2}|x|} \delta(x) + \frac{m\alpha^2}{2\hbar^2} \sqrt{\frac{m\alpha}{\hbar^2}} e^{-\frac{m\alpha}{\hbar^2}|x|} = 0. \end{aligned}$$

The last remaining issue is to then define the meaning of $\psi(x) \cdot \lambda(x)$, which is accomplished in exactly the same manner as $\psi(x)\delta(x)$ since $\psi(x)$ is everywhere continuous and integrable on all of \mathbb{R} , while $\delta(x)$ and $\lambda(x)$ are both measures on \mathbb{R} . This makes $\psi(x)$ a measurable function with respect to both measures (even though the two measures are not absolutely continuous with respect to each other). Hence

$\psi(x)\lambda(x) = \psi(x)$, in the sense of measures, i.e. the equality is true Lebesgue almost everywhere.

Therefore the equation 2.2.1.2 can now be written,

$$(2.2.1.3) \quad -\frac{\hbar^2}{2m} \left[\frac{m\alpha}{\hbar^2} \right]^{\frac{5}{2}} e^{-\frac{m\alpha}{\hbar^2}|x|} + \frac{\hbar^2}{m} \left[\frac{m\alpha}{\hbar^2} \right]^{\frac{3}{2}} \delta(x) - \alpha \sqrt{\frac{m\alpha}{\hbar^2}} \delta(x) + \frac{m\alpha^2}{2\hbar^2} \sqrt{\frac{m\alpha}{\hbar^2}} e^{-\frac{m\alpha}{\hbar^2}|x|} = 0.$$

The above equality is justified only in the sense of distributions. This means to verify the identity one must multiply both sides by a test function and integrate both sides. But it is also true for measures and so this results in the fact that one need only integrate both sides of 2.2.1.3, the test function simply being unity. It is then immediate that,

$$-\frac{\hbar^2}{m} \left[\frac{m\alpha}{\hbar^2} \right]^{\frac{3}{2}} + \frac{\hbar^2}{m} \left[\frac{m\alpha}{\hbar^2} \right]^{\frac{3}{2}} - \alpha \sqrt{\frac{m\alpha}{\hbar^2}} + \alpha \sqrt{\frac{m\alpha}{\hbar^2}} = 0.$$

So the two methods are self-consistent and lead to the same set of conclusions.

2.2.1.3 Method 3: The Convolution Method

This method can also be reasonably called the smearing method and it makes use of convolution by delta sequences of test-function-type to smooth out undesirable behaviors in, for example, both the potential functions and the solution functions when they are of $L^2(\mathbb{R})$ type or perhaps worse, of a generalized function type. The trick relies on the theorems that ensure that (a) the convolution of a distribution function with a test function is a smooth function, known as the *Regularization Theorem* 3.1.2.1 and (b) the *Approximation Theorem* 3.1.2.2, which ensures that any distribution can be approximated by a sequence of test functions. These theorems are stated formally in the section on the distributional calculus as they are, by now, classical results in the calculus of distributions.

The basic idea is to convolve the solution function with a delta sequence of test functions. This creates a smoothed out version of the solution but in the limit, the sequence will tend to the solution function. The method is quite robust because of its solid foundation on rigorous mathematical theorems. It also gives rise to a new and profound change in one's view of physical systems. The shift in view is to regard

generalized function solutions of equations as pure, naked, or unsmeared (quantum) fields while the smeared versions should be regarded as the localizations of the fields and hence are aptly called the particles. This view point will be revisited in chapter 4, *The Quantum Theory of Fields*.

First, convolve the free solution function of the equation, which is of $L^2(\mathbb{R})$ type when the potential function, here the Dirac delta function (measure), is treated as zero consistent with the Lebesgue integral measure on the space $L^2(\mathbb{R})$. Hence the notion of free solution. The convolution is with respect to a test function that is a delta sequence. Here the test function space is simply the space of Schwartz functions, $\mathcal{S}(\mathbb{R})$. The point is that in the limit one obtains again the free solution function. At this point, it is assumed that the energy is unknown. One defines the following convenient constant.

$$(2.2.1.4) \quad a^2 := \frac{2m|E|}{\hbar^2} > 0.$$

The convolution calculation yields,

$$\begin{aligned} (\psi * \phi_n)(y) &= \left(\sqrt{a} e^{-a|x|} * n a e^{-\pi n^2 a^2 x^2} \right)(y) \\ &= \frac{\sqrt{a}}{2} e^{\frac{1}{4\pi n^2}} \left[e^{ay} \operatorname{erfc} \left(\frac{1}{2\sqrt{\pi n}} + \sqrt{\pi n} a y \right) \right. \\ &\quad \left. + e^{-ay} \operatorname{erfc} \left(\frac{1}{2\sqrt{\pi n}} - \sqrt{\pi n} a y \right) \right]. \end{aligned}$$

From the property of differentiation of the convolution, one has

$$\frac{\partial^2}{\partial y^2} (\psi * \phi_n)(y) = (\psi * \phi_n)''(y) = (\psi'' * \phi_n)(y) = (\psi * \phi_n'')(y)$$

This operation yields,

$$(\psi * \phi_n)''(y) = -2a^2 n \sqrt{a} e^{-\pi n^2 a^2 y^2} + a^2 (\psi * \phi_n)(y).$$

The crux is to now apply the smeared or smoothed function, $(\psi * \phi_n)(y)$ to the Schrödinger operator. Inserting this expression into the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} - \alpha \delta(y) + |E| \right) (\psi * \phi_n)(y) = 0,$$

yields,

$$\left(\frac{\hbar^2 a^2}{m} n \sqrt{a} e^{-\pi n^2 a^2 y^2} - \frac{\hbar^2 a^2}{2m} (\psi * \phi_n)(y) \right) - \alpha \delta(y) (\psi * \phi_n)(y) + |E| (\psi * \phi_n)(y) = 0$$

From equation 2.2.1.4, the identity $-\frac{\hbar^2}{2m}a^2 = -|E|$ can be used to simplify the above giving,

$$2|E|n\sqrt{a}e^{-\pi n^2 a^2 y^2} - |E|(\psi * \phi_n)(y) - \alpha\delta(y)(\psi * \phi_n)(y) + |E|(\psi * \phi_n)(y) = 0,$$

which simplifies to,

$$(2.2.1.5) \quad 2|E|n\sqrt{\frac{a}{2}}e^{-\pi n^2 a^2 y^2} - \alpha\delta(y)(\psi * \phi_n)(y) = 0$$

Now mathematically, above is the sum of two measures each singular with respect to one another. The Dirac measure is symmetric, while Lebesgue measure is not. To reconcile the two mathematically, it makes sense to (a) integrate the measures and (b) integrate them over a set which, (i) contains the support of δ , (ii) is not of measure zero with respect to Lebesgue measure and (iii) is symmetric about the support of δ causing the Lebesgue measure to become symmetric and share symmetry with δ .

Since the support of the Dirac potential measure is at the origin, the integration is done over the compact subset $\mathbb{R} \supset [-r, r]$, with $r > 0$ some length scale. This is as much as can be determined mathematically. Now comes the physics. Physically, it is known that the only length scale for this system is that given by the parameter a^{-1} , which is induced by the energy that presumably depends upon the coupling α . Since it is the dependence of E on α that is to be determined, only a is available. Thus, the integration is done over the compact subset $\mathbb{R} \supset [-\frac{s}{a}, \frac{s}{a}]$, with $s > 0$ and a^{-1} the physical length scale.

Integrating equation 2.2.1.5 gives,

$$\begin{aligned} 2|E|n\sqrt{a} \int_{-\frac{s}{a}}^{\frac{s}{a}} e^{-\pi n^2 a^2 y^2} dy - \alpha \int_{-\frac{s}{a}}^{\frac{s}{a}} \delta(y)(\psi * \phi_n)(y) dy &= 0 \\ 2\sqrt{\frac{1}{a}}|E| \operatorname{erf}(ns\sqrt{\pi}) - \alpha(\psi * \phi_n)(0) &= 0 \\ 2\sqrt{\frac{1}{a}}|E| \operatorname{erf}(ns\sqrt{\pi}) - \alpha\sqrt{a}e^{\frac{1}{4\pi n^2}} \operatorname{erfc}\left(\frac{1}{2\sqrt{\pi}n}\right) &= 0 \end{aligned}$$

Now in the limit that n tends to infinity, the smeared system above becomes exactly the bound state system under investigation. Taking the limit of the last equality

above and noting that,

$$\begin{aligned}\lim_{n \rightarrow \infty} \operatorname{erf}(x n) &= 1 && \forall x > 0 \\ \lim_{n \rightarrow \infty} e^{\frac{1}{x^2 n^2}} \operatorname{erfc}\left(\frac{1}{x n}\right) &= 1 && \forall x > 0,\end{aligned}$$

gives the equation,

$$\frac{2}{\sqrt{a}} |E| - \alpha \sqrt{a} = 0.$$

Rearranging terms this shows that,

$$|E|^2 - \frac{m \alpha^2 |E|}{2 \hbar^2} = |E| \left(|E| - \frac{m \alpha^2}{2 \hbar^2} \right) = 0.$$

Now whether or not $E = 0$ is in the spectrum of the bound state has to be handled carefully. The compactness of the resolvent indicates that zero certainly belongs in the spectrum of this Schrödinger operator. However, for this particular system, zero is a bound state eigenvalue iff its corresponding eigenvector is zero so that there is a one-to-one correspondence between the eigenvalues and the eigenvectors. Therefore, one can finally conclude that the only eigenvalue belonging to the only non-zero eigenvector is,

$$E = -\frac{m \alpha^2}{2 \hbar^2}.$$

Nota bene, the arbitrary scale, s did not enter the calculation anywhere! Evidently, either the system itself or some aspect of it was scale invariant. This is not at all apparent from the operator, in fact

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \alpha \delta(x),$$

scales as (taking $x = s y$),

$$H_s = -\frac{\hbar^2}{2m s^2} \frac{\partial^2}{\partial y^2} - \alpha \frac{1}{s} \delta(y).$$

But the resolvent,

$$R(E) = H - E,$$

scales as,

$$R_s(E) = H_s - E.$$

So if $\alpha = \frac{\alpha'}{s}$ then one would have $s^2 R_s(E) = R(E)$ for bound states. This, out of no coincidence, bears vital resemblance to renormalization theory and the beta function³ of renormalization group analysis. The connections will be slowly elucidated culminating in a conclusion for the self-energy of a free charged particle. Also, notice that the resolvent operator as a function of energy, scales in the same manner that the energy does as a function of the coupling constant. Incidentally, the aspect of the system that is scale invariant and that the entire system must be based upon is the Lebesgue measure (over all of \mathbb{R}) with which the Hilbert space, $L^2(\mathbb{R})$, is defined with.

Lastly, it is worth noting that the scale invariance comes from the fact that the scale of the system is rigidly set. That is to say, no unitless quantity can be formed out of the fundamental constants of the system, the coupling α , the mass m , and the Planck constant \hbar . This is easily checked by forming a product of the three quantities' units raised to three unknown powers and finding that the system of 3 equations (stemming from the 3 fundamental units kg, m, s or alternately N, m, s) with 3 unknown powers has no consistent solutions. As a consequence, the energy is quite fixed. This was actually tacitly utilized in the calculation when the only available length scale a^{-1} was inserted in the convolving delta sequence $\phi_n(x)$.

2.2.2 Derivative of the Dirac Delta Function

Immediately, one should notice that there are serious issues with this system. First, one can in fact form a unitless quantity out of the 3 fundamental constants of the system (the coupling α , the mass m , and the Planck constant \hbar), since the coupling now carries units of Newtons times meters cubed, (Nm^3). Worse yet, no quantity with the units of energy can be formed out of the 3 fundamental constants of the system! This has a deep consequence. This means that there must be a fourth quantity with which to form a function of energy with. In light of the fundamental units of kg, m, and s, owing to the time independent nature of the system and the uniqueness of the single particle mass, the only available quantity is some unspecified length scale. This will turn out to be connected to, depending upon one's point of view, either

³This function is not to be confused with the mathematical special function called the beta function (as well) defined one way as, $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

the length scale that makes the Schwartz test function arguments unitless or the size of the set over which the system's measures are integrated. Lastly mathematically speaking, the equation is not a well-formed differential equation (even in the weak sense), since $\delta'(x)$ is not even a measure.

A deeper look into the system shows that method 2 from the previous section will be met with a serious mathematical obstruction, namely, the arising of products of distributions. This is because the Fourier transform indicates that the function $(A + B \cdot \text{sgn}(x)) e^{-a|x|}$, is the form of the solution, where A and B are real constants and the function $\text{sgn}(x)$ ⁴ is defined as,

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

This functional form of the solution implies that the quantity, $\text{sgn}(x) \delta'(x)$, will arise and from the above definition it is clear that the sign function is not a smooth function on \mathbb{R} and thus will lead to products of delta functions.

As a brief mathematical aside, it is interesting to consider the connections between the Colombeau algebraic method for multiplying distributions and the elements of nonstandard analysis that arise in the usual literature's treatment of this system. In particular, one wonders whether there are transfer principles between the algebraic methods of Colombeau and the nonstandard analysis. Finally, if so then what are the mathematical consequences of their existence?

Returning to the topic at hand, one proceeds as usual with the techniques of the third method from the previous subsection on the Dirac delta function. The Hamiltonian is,

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \alpha \delta'(x),$$

which leads to the ill-defined equation,

$$(2.2.2.1) \quad -\frac{\hbar^2}{2m} \psi''(x) - \alpha \delta'(x) \psi(x) + |E| \psi(x) = 0.$$

Again, one is searching for bound states so the energy is presumed negative (certainly non-positive), and where ostensibly $\psi \in L^2(\mathbb{R})$.

⁴It is sometimes called the signum function or sign function but sign can be verbally confused with the sine function.

Using the Fourier transform to obtain the solution, one arrives at the wave function,

$$\psi(x) = \text{sgn}(x) e^{-a|x|},$$

where again $a^2 := \frac{2m|E|}{\hbar^2}$. Convoluting this with a Gaussian delta sequence as before with the length parameter a^{-1} , i.e. $\phi_n(x) := a n e^{-\pi n^2 a^2 x^2}$, returns the expression,

$$(\psi * \phi_n)(y) = \frac{\sqrt{a}}{2} e^{\frac{1}{4\pi n^2}} \left[e^{a y} \text{erfc} \left(\frac{1}{2\sqrt{\pi} n} + \sqrt{\pi} n a y \right) - e^{-a y} \text{erfc} \left(\frac{1}{2\sqrt{\pi} n} - \sqrt{\pi} n a y \right) \right].$$

Applying this to the equation 2.2.2.1 results in,

$$\frac{\hbar^2}{m} a \sqrt{a} n (-2\pi a^2 n^2 y) e^{-\pi n^2 a^2 y^2} - \alpha \delta'(y) (\psi * \phi_n)(y) = 0.$$

Next, notice that taking the distributional limit produces,

$$\frac{\hbar^2}{m} \sqrt{a} \delta'(y) - \alpha \sqrt{a} e^{-a y} \delta'(y) = 0.$$

This distributional equation must be bracketed with a Schwartz test function for the equality to be in the usual sense rather than true distributionally. The (odd) anti-symmetric nature of the wave function requires an odd test function. After pairing, one obtains the constraint,

$$\frac{m \alpha}{\hbar^2} = \pm 1.$$

This fact can also be seen from the distributional equality as well by working separately with $y > 0$ and $y < 0$. This is exactly what is needed because although it was known that the fundamental parameters of the system formed an unitless quantity, the magnitude of the scale was not known. Evidently, it is simply unity. Calling the unknown scale b and in light of the new information, it is necessary from a physical viewpoint that the inverse length scale of the system a and the unspecified length scale multiply to the magnitude of this unitless scale, i.e.

$$(2.2.2.2) \quad a b = \pm \left[\frac{m \alpha}{\hbar^2} \right]^{\pm 1}.$$

However because the quantity is unitless, it is not know whether it is the given product or its reciprocal, but it must be one of the two and not both. The only sign ambiguity comes from the ambiguity in b because the system is defined over all of \mathbb{R} . Given these

conditions and the fact that $E < 0$, one obtains for the possible energy eigenvalues of the bound states either,

$$E_{-1} = -\frac{m \alpha^2}{2 \hbar^2 b^2} \quad \text{or} \quad E_1 = -\frac{\hbar^6}{2 m^3 \alpha^2 b^2}$$

Several comments are in order. Since E_{-1} is the energy eigenvalue for the primitive (distributional anti-derivative), then it is not an eigenvalue of this operator. Consequently, one is led to the following conclusion, there is a single bound state eigenfunction of the system and its corresponding bound state eigenvalue is,

$$E = -\frac{\hbar^6}{2 m^3 \alpha^2 b^2}.$$

Furthermore, the sign of b is irrelevant because only the square of it appears in the energy. Without the appearance of the unspecified length scale b no combination of products and powers of the fundamental parameters of the system could form an energy. This was known from the onset. For this system, b plays the role of the unitless scale, s , in the previous system with Dirac measure as the potential.

2.2.3 The Singular One Dimensional Coulomb Potential

It is interesting to note that the Dirac Delta measure as a potential is highly singular in 3 dimensions, while the Coulomb potential is extremely well-behaved (owing to the presence of the centripetal effective potential, which causes the potential to be bounded from below). On the other hand as was noted above, the Dirac delta measure, as a potential in one dimension, behaved quite nicely. But as will be seen here, the Coulomb potential in one dimension is highly singular, as it is not bounded from below. Informally speaking, this creates a sort of tension between the kinetic energy operator, which is bounded from below, with that of the 1 dimensional Coulomb potential, which is not bounded from below. The Hamiltonian is,

$$(2.2.3.1) \quad H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{\alpha}{|x|},$$

with $\alpha > 0$. The stationary Schrödinger differential equation in 1 dimension is then,

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{\alpha}{|x|} \right] \psi(x) = E\psi(x).$$

Again, one searches for bound states of the form $E < 0$ since again $V(x) < 0, \forall x \in \mathbb{R}$. The equation of interest for the bound states is then:

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) - \frac{\alpha}{|x|} \psi(x) + |E| \psi(x) = 0.$$

One would like to use the Fourier transform once again. To accomplish this multiply the above equation by $|x|$.

A comment is in order first. One should ask whether any particular procedure will cause a loss of solutions or the artificial generation of solutions, which would not exist otherwise. In general, because one can never know a priori whether a given system will yield any solutions at all, it is reasonable to proceed with assumptions and then to check for consistency as was done in the first example above.

In any case, the resulting equation owing to the suggested manipulation gives:

$$|x| \psi''(x) + \beta \psi(x) - a^2 |x| \psi(x) = 0.$$

with as usual

$$(2.2.3.2) \quad a^2 := \frac{2m|E|}{\hbar^2} > 0 \quad \text{and} \quad \beta := \frac{2m\alpha}{\hbar^2} > 0.$$

Using the Fourier transform here would be very effective because it would substitute a first order linear differential equation in place of the above second order linear differential equation. This in and of itself is useful. However, there is a sort of conservation of trouble. The first order equation is wrought with choices of branch cuts in the complex plane, since a real function raised to an arbitrary real power will not necessarily be real or even unique.

After applying the Fourier transform and re-arranging terms, the following first order equation presents itself,

$$\widehat{\psi}'(k) = 2\pi i \frac{(4\pi i k - \beta)}{4\pi^2 k^2 + a^2} \widehat{\psi}(k).$$

The solutions is obviously,

$$(2.2.3.3) \quad \widehat{\psi}(k) = C \frac{(2\pi k + i a)^{\frac{\beta}{a}}}{(4\pi^2 k^2 + a^2)^{1 + \frac{\beta}{2a}}},$$

with C an arbitrary constant, which depends on initial conditions. Hence the inverse Fourier transform, which is valid for all $L^2(\mathbb{R})$ functions by its extension gives for the solution the integral equation below.

$$\psi(x) = \int_{\mathbb{R}} C \frac{(2\pi k + i a)^{\frac{\beta}{a}}}{(4\pi^2 k^2 + a^2)^{1+\frac{\beta}{2a}}} e^{2\pi i x k} dk$$

And so it is apparent that the inversion of this function must be handled with care. Things are not quite so bad, as the range of possibilities are reduced by the inequalities 2.2.3.2. In particular, there is the special case whenever the expression,

$$(2.2.3.4) \quad \frac{\beta}{2a} \in \mathbb{N}$$

then the equation 2.2.3.3 gives a guaranteed rational function of k . The denominator in 2.2.3.3 then guarantees that the degree of the polynomial of that denominator will always be greater than the degree of the numerator's polynomial, then the function $\hat{\psi}(k)$ is necessarily, $L^2(\mathbb{R})$. This implies that the constraint or assumption on the energy 2.2.3.4 actually determines, in this special case, the energy eigenvalues. One simply has that $\frac{\beta}{2a} = n \in \mathbb{N}$, and therefore,

$$(2.2.3.5) \quad E = -\frac{m\alpha^2}{2\hbar^2 n^2} \quad n \in \mathbb{N}.$$

But these are not the complete list of energy eigenvalues because the inversion of 2.2.3.3 as an integral equation is more robust when defined through the Cauchy integral formula. The reason this works out is because the expression inside of the parenthesis of the numerator of 2.2.3.3 is a polynomial root of the quadratic polynomial inside of the parenthesis of the denominator. Therefore, rational powers can be made sense of in terms of the Cauchy integral formula with appropriate choices of branch cuts. In these instances, the numerator can be forced to be analytic and the principal value integral will give results that are also in $L^2(\mathbb{R})$, provided the rational number is such that the degree of the denominator remains larger than that of the numerator.

From this perspective, one can obtain any energy that is a rational multiple of $-\frac{2m\alpha^2}{\hbar^2}$, including $E \rightarrow -\infty$, i.e. $\frac{\beta}{2a} = 0$, which corresponds to $n = 0$. In this last extreme case, it is clear that there exists an associated $L^2(\mathbb{R})$ eigenfunction, owing

to the well-defined inversion of the solutions 2.2.3.3. This is a consequence of the operator not being bounded from below. It is interesting to note that the associated quadratic form to the operator 2.2.3.1, is in fact bounded from below. This shows that the quadratic form is a generalization that can extract physically reasonable attributes from a system that would otherwise not possess them. The lowest lying energy state for the quadratic form is the same as that of the delta function, which is the same as the ground state for a finite energy constraint namely, $E = -\frac{m\alpha^2}{2\hbar^2}$. To see why this is, notice the following differential equation below for the unknown function, $f(x)$,

$$(2.2.3.6) \quad x f'(x) + f(x) = 0,$$

is solved by *both* the function $|x|^{-1}$ and the distribution $\delta(x)$, owing to the distributional identity below for $g(x) \in C^1(\mathbb{R})$.

$$g(x) \delta'(x) = g(0) \delta'(x) - g'(0) \delta(x),$$

in the sense of distributions. Thus, there is no coincidence here. All the energy eigenvalues are accounted for and have corresponding eigenfunctions. This completes the analysis of the system, as well as the analysis of one dimensional quantum mechanics.

2.3 Time Asymmetry and Causality from Non-Unitarity

One, as of yet, unanswered question in theoretical physics is, “How does irreversible phenomena come about?”. In the theory of special relativity, causality is completely preserved. If an event A precedes an event B that is time-like separated then event A precedes event B in all reference frames. However, when the events are space-like separated there is no notion of simultaneity. That is to say if there is a frame where events A and B that are space-like separated occur simultaneously then there exist frames for which event A precedes event B and yet another frame for which event B precedes event A. Now it is important to note the following fact in quantum theory, the unitary evolution of an unbounded quantum system’s localized state requires spatial evolution. This is rather surprising in comparison to the classical theory.

But it occurs pervasively. The only exceptions being highly prepared states that are continuously confined by non-zero work for the environment designed specifically to counteract this natural trend. This process is often referred to as *spreading of the quantum wave packet*.

One may wonder how special relativity and quantum theory can coexist but there is no paradox. This is because the Schrödinger evolution, though not manifestly relativistically invariant, must hold true in each reference frame. Therefore, there exists a stationary frame for which the Schrödinger evolution is exactly true and unmodified by relative motion. The difficulty is in just how to exactly connect the two points of view, i.e. how to connect the stationary frame to the frame moving with relative motion. In particular, there is still open debate on this topic under the context of the *Unruh effect*. The issue is over just how the relative (accelerating) motion of the observer affects the measured observables and in particular just how those observables manifest themselves, i.e. as differences in temperature or as radiation or perhaps both. It shall be taken as a matter of convenience that the Schrödinger evolution is the only relevant dynamics since it will be assumed that the reference frame is a stationary one, and can always be chosen to be as such.

The system to be analyzed is that of a Hamiltonian for a highly singular but rotationally invariant potential. This system models well the notion of screened Coulomb interactions for atomic-molecular systems. It will be shown in the argument that such a Hamiltonian gives rise to a non-unitary evolution because the Hamiltonian fails to be even essentially self-adjoint. More so even though it is known that non-symmetric operators can have real spectra, the meaning of an imposed unitary time evolution is neither precisely determined nor defined. In particular, the lack of an unique self-adjoint operator manifests itself in this context as the fact that the Hamiltonian is dissipative and thus the time evolution is not invertible. Recall, that an invertible time evolution is always given by an unitary operator and conversely any unitary operator is necessarily invertible. This phenomena and its ensuing argument is not unique to the screened Coulomb interaction.

The argument is given for the (screened) electromagnetic interaction via its multipole expansion, but is essentially unchanged for other potentials with polynomial interactions higher than degree 2, since both of these fundamental interactions have

the property of non-convexity of the potential. This includes but is not limited to the anti-screened effective potential of the strong nuclear force. One first needs the following three simplifying observations about a real physical system.

- 1 *The multi-pole numerators are bounded.* This follows from the fact that the stable bound system (an atom or molecule) is localizable, and thus has a finite average size with a finite non-zero variance.
- 2 *The spin interactions can be omitted.* The spin operator is a bounded linear operator and this splits the energy levels only in a bounded finite manner. Thus without loss of generality, it may be neglected.
- 3 *The orbital interaction can also be omitted.* Though in general, the orbital linear operator is not bounded, it is in fact bounded by the depth of the stable basic bound states' effective potentials, the finite number of constituents (for an atom, its number of electrons and nucleons) and by the total energy of the incoming scatterer.

In what follows, it is also assumed that there is a term in the multipole expansion after which all higher order terms can be neglected. That is, each sum stops at some $m := \alpha_{\max} \geq 3$. Hence the energy operator has the form below,

$$H = -\frac{1}{2}\nabla^2 + \sum_{\alpha=1}^m \frac{\beta_{\alpha}(x)}{r^{\alpha}},$$

where the functions $\beta_{\alpha}(x)$ are the α -number of contractions of the multipole moments with the coordinate tensors in the numerator of the multipole expansion, $x \in \mathbb{R}^3$, $r > |x_{\max}| \in \mathbb{R}^+$, and the unbounded rotationally invariant operator H acts on the Sobolev-Hilbert space $H^{2,2}(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$. Now because the $\beta_{\alpha}(x)$ are bounded (both in domain and range) due to conditions, (**1**), (**2**), and (**3**), which is emphasized by the existence of $r > |x_{\max}|$, it suffices to only consider the,

$$\text{sgn}(\beta_{\alpha}(x)) \cdot \sup_{x \in \mathbb{R}^3} \{|\beta_{\alpha}(x)|\} =: \beta_{\alpha}$$

where the constants β_{α} are negative for at least one α . Thus, there is at least one local minimum and all such local minima correspond to resonances. Then the operator to

analyze is

$$(2.3.0.7) \quad H_{\text{sup}} = -\frac{1}{2}\nabla^2 + \sum_{\alpha=1}^m \frac{\beta_\alpha}{r^\alpha}.$$

The constants, β_α , are not all positive, i.e. the potential is not convex, which causes ‘wiggles’ in the graph of the potential that manifest as resonances. Therefore the operator H_{sup} is not even essentially self-adjoint on $H^{2,2}(\mathbb{R}^+)$. That is to say, the Hamiltonian operator is not self-adjoint nor does it have an unique self-adjoint extension. This conclusion follows from arguments in, for example [JohLap], or [ReeSim1]. Though the analysis does not prohibit the existence of some arbitrary self-adjoint extension, there is no single unique way of producing an unique self-adjoint extension. This is because either no such extension exists or because extensions exist but there is an arbitrary possibly infinite number of them. But what can be assured is that given $m := \alpha_{\text{max}} \geq 3$, uniqueness of the time evolution operator ensures non-unitarity due to the dissipative nature of the unbounded linear operator, H_{sup} . This follows from the fact that the Hilbert space is always reflexive, and the unbounded operator H_{sup} is both symmetric and dissipative on $H^{2,2}(\mathbb{R}^+)$. The consequence is that global (on all of \mathbb{R}^+) unitarity is broken and the resulting time evolution is a time irreversible semigroup.

2.3.1 Mathematical Justifications

One first begins by showing that the operator H_{sup} is dissipative. To this end, one should first know the meaning of such a term. There are two main useful and equivalent definitions.

Definition 2.3.1.1 (Dissipative Operator). *A linear operator A on a Hilbert space⁵ is called dissipative if,*

$$1 \quad \forall \lambda > 0 \text{ and } \forall \psi \in D(A), \text{ one has that the inverse of the resolvent obeys, } \|(\lambda\mathbb{1} - A)\psi\| \geq \lambda\|\psi\|.$$

⁵The definitions can be extended to Banach spaces but it suffices for the current purposes to consider Hilbert spaces. Additionally, it is convenient that every Hilbert space is not only reflexive but the dual of any Hilbert space is again a Hilbert space with the Reisz canonical isomorphism.

2 $Re \langle A\psi, \psi \rangle \leq 0, \quad \forall \psi \in D(A) \subset \mathcal{H}$ where \mathcal{H} is a Hilbert space and the canonical identification with the Riesz dual has been assumed.

It is also worth noting that A is *maximally dissipative* if it is dissipative and if the inverse resolvent $\lambda \mathbb{1} - A$ is surjective $\forall \lambda > 0$. To this end, one should also beware of the following.

Definition 2.3.1.2 (Symmetric Operator). *An operator A is symmetric if,*

$$(A\psi, \phi) = (\psi, A\phi) \quad \forall \psi, \phi \in D(A)$$

Notice that symmetric operators need not be self-adjoint because the domain of the adjoint operator $D(A^*)$ need not coincide with the domain, $D(A)$, itself. However, every symmetric operator is closable.

The following theorem for the operator of interest H_{sup} gives the essential mathematical properties, which represent the physical consequences in the above section.

Theorem 2.3.1.3 (Maximal Dissipation of H_{sup} and Non-Unitary Time Evolution). *The operator, H_{sup} , acting on the densely defined subspace, $D(H_{\text{sup}})$ of the Hilbert space $H^{2,2}(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$ is maximally dissipative, where $r, u \in \mathbb{R}^+, m := \alpha_{\text{max}} \geq 3$, and thus gives rise to a time evolution that is a contraction semigroup.*

Proof. The representation of the operator in position space is given below and is equivalent to the operator in the representation given by the isomorphic transformation $r = u^{-1}$ on \mathbb{R}^+ .

$$\begin{aligned} H_{\text{sup}} &= -\frac{1}{2}\nabla^2 + \sum_{\alpha=1}^m \frac{\beta_{\alpha}}{r^{\alpha}} \\ &= -\frac{1}{2}r^2 \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\beta_1}{r} + \frac{\beta_2}{r^2} + \dots + \frac{\beta_m}{r^m} \\ (2.3.1.1) \quad &= -\frac{1}{2} \frac{\partial^2}{\partial u^2} + \beta_1 u + \beta_2' u^2 + \dots + \beta_m u^m, \end{aligned}$$

where $\beta_2' = \beta_2 + l(l+1)$ such that $l \geq 0$. Incidentally, equation 2.3.1.1 makes clear the fact that the study of the cut-off multipole potential with a cut-off greater than or equal to m is equivalent to the study of a rotationally symmetric polynomial potential

with degree greater than 2. The associated quadratic form is then,

$$\begin{aligned}
Q &= \int_{\mathbb{R}^+} [\phi^*(r) H \phi(r)] r^2 dr \\
&= \int_{\mathbb{R}^+} [\varphi^*(u) H \varphi(u)] du \\
&= \int_{\mathbb{R}^+} \left[\frac{1}{2} \varphi^{*'}(u) \varphi'(u) + \beta_1 u \varphi^*(u) \varphi(u) + \beta_2 u^2 \varphi^*(u) \varphi(u) \right. \\
&\quad \left. + \dots + \beta_m u^m \varphi^*(u) \varphi(u) \right] du
\end{aligned}$$

Without loss of generality, one can choose ϕ such that

$$\int_{\mathbb{R}^+} \phi^*(r) \phi(r) r^2 dr = \int_{\mathbb{R}^+} \varphi^*(u) \varphi(u) du = 1,$$

with $\varphi(u) := u^{-2} \phi(u^{-1})$. The integral over \mathbb{R}^+ can be broken into two integrals over $(0, 1)$ by,

$$\int_{\mathbb{R}^+} du = \int_0^1 du + \int_1^\infty du = \int_0^1 du + \int_0^1 \frac{du}{u^2}$$

Now from the inequalities with $p \geq 1$ and $p = 0$,

$$\begin{aligned}
\int_0^1 u^p |\varphi(u)|^2 du &\leq 1 && \text{and} \\
\int_0^1 r^{p+2} |\phi(r)|^2 dr &\leq 1
\end{aligned}$$

one has that $Q \leq 0$. Subject to the constraint that,

$$\int_0^1 |\varphi(u)|^2 du + \int_0^1 r^2 |\phi(r)|^2 dr = 1$$

Further, note that if the graph, $\Gamma(H_{\text{sup}}) = \{(\phi, H_{\text{sup}}\phi) \mid \phi \in D(H_{\text{sup}}) \wedge H_{\text{sup}} : D(H_{\text{sup}}) \rightarrow H^{2,2}(\mathbb{R}^+)\}$ is not a closed set then the operator would not be closed (or even closable). But by definition of the Sobolev-Hilbert space here, one means the vector space defined as the closure of the set of all $L^2(\mathbb{R}^+)$ functions whose weak

second derivatives are also in the space $L^2(\mathbb{R}^+)$. Thus the graph is closed and one can take the operator H_{sup} as the closed operator associated to the closed graph.

Next, one can start by taking the dense domain, $D(H_{sup})$, to first be only $C_c^\infty(\mathbb{R}^+)$. The operator is densely defined due to density of $C_c^\infty(\mathbb{R}^+)$ in $H^{2,2}(\mathbb{R}^+) \subset L^2(\mathbb{R}^+)$. Now consider, $\psi_k \in D(H_{sup})$, such that,

$$\psi_k = \begin{cases} e^{\frac{-k}{u(1-u)}} & 0 < u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then one can obtain,

$$\begin{aligned} \left(\frac{\partial^2}{\partial u^2} \psi_k, \psi_k \right) &= \int_{\mathbb{R}^+} \psi_k''(u) \psi_k(u) \\ &= \int_0^1 \frac{k e^{\frac{-2k}{u(1-u)}}}{(1-u)^4 u^4} (6u^4 - 12u^3 + (8+4k)u^2 - (2+4k)u + k) \\ &< 0 \end{aligned}$$

This gives the desired dissipative nature of the operator since for any ψ_k of such a nature, there exists a constant, k , large enough such that although $\psi_k > 0$, there is still however $(\psi_k'', \psi_k) < 0$, where ψ_k is given by,

$$\psi_k'' = \begin{cases} \frac{k e^{\frac{-2k}{u(1-u)}}}{(1-u)^4 u^4} (6u^4 - 12u^3 + (8+4k)u^2 - (2+4k)u + k) & 0 < u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The zeros, the extrema and the convexity of this function show that the result is independent of the constants β_j , since k can always be chosen large enough even if the positive values of the β_j are arbitrarily large but necessarily finite. All higher powers of polynomials are even smaller owing to the restriction to the interval $[0, 1]$. This establishes the existence of a non-empty set of vectors that comprise the domain such that the operator is dissipative.

Lastly, the operator is surjective since every element of $D(H_{sup})$ can be combined in such a way to produce one of the Laguerre polynomials by sequences of elements of $C_c^\infty(\mathbb{R}^+)$ and thus in the limit every element of $H^{2,2}(\mathbb{R}^+)$ is reached as the Laguerre polynomials are a basis for this Hilbert space. Since H_{sup} is closed, the operator

$\lambda \mathbb{1} - H_{\text{sup}}$, with $\lambda > 0$ is also closed. The same operator will also be surjective if there is a polynomial to multiply and an anti-derivative of the example ψ_k such that,

$$\|\lambda \mathbb{1} - H_{\text{sup}}\| > 0 \quad \forall \lambda > 0.$$

Again, because there exists a k large enough, regardless of λ , as long as it is positive, the strict inequality will be satisfied. This thus gives maximality of the dissipative operator and hence by the Lumer-Phillips theorem the time evolution is necessarily given by a contraction semigroup and not an unitary operator. ■

A consequence of the above theorem is that ideal gases comprised of weakly interacting diatomic molecules, for example, undergo time irreversible processes. It is in this sense that the macroscopic phenomena of entropy is born. There is a bit of technical subtlety and that is about what constitutes macroscopic. Here it is simply meant that there is Avogadro's number of interacting systems, which require a statistical ensemble perspective. The two viewpoints of a collection of pure quantum systems and the aggregate of an ensemble of such systems is reconciled by the understanding that the operator of interaction, H_{sup} does not accommodate the time reversible unitary evolution.

This has a deep implication for the early universe. It implies that screened (or anti-screened) effective potentials give rise to causality. Moreover, in the early universe, prior to any effective potentials and in the presence of only pure unitary evolutions, there is no notion of causality. Time simply did not exist during this period of pure interactions. Said another way, the 'wiggles' in the graph of such effective potentials allow for localizations to occur, which are unitarily incompatible with interactions of other particles in different spatial regions. That is to say, there may be local convexity of an effective potential that has its own unitary evolution that is not unitarily related to some other local convex region of the potential. Consequently, in the absence of global convexity of the potential the system undergoes independent local evolutions that allow for inhomogeneity to arise. Obviously, this phenomena is vital to the understanding of the evolution of the early universe, where the formation of an arrow of time from first principle is clearly needed and no appeal to entropy *a priori* is justified. The mathematics gives a simple and elegant model that furnishes a self-consistent solution to the fundamental problem.

Lastly, the existence of non-unique self-adjoint operators implies that there is no way to select out such a distinguished self-adjoint operator. The only exceptions being systems with precisely identified boundary conditions. But in the absence of boundary conditions, this implies that a priori every operator is equally likely. A consequence that one can imply from this is that the quantum multi-particle states, which are all elements of the same Hilbert space (the tensor product of the single particle state Hilbert space) have their phases randomly distributed. This viewpoint provides a natural justification for the Pauli master equation. Thus these screened Hamiltonians, which are not self-adjoint give rise to time irreversible dynamics, which in turn justify the quantum statistical mechanics, for which time irreversible systems must exist in. So it is not unreasonable to conclude that all of these advantages of non-(not even essentially)-self-adjoint operators unify the origins of time irreversibility.

Chapter 3

Distribution Theory and Modern Fundamental Physics

In this chapter, it is shown how the abstraction of the mathematical sub-discipline of the calculus of distributions¹ can give extremely relevant physical information. Furthermore, novel techniques and modifications of the standard theory of distributions is made and it is shown how these calculations and new definitions can be immediately applied to theoretical physics.

First, the reader is introduced to the very basics of the distributional calculus and from the immediate onset, certain strong conclusions about the form of physically relevant potentials, which possess the attribute of being dubbed *singular potentials*, can then be made. The guiding concept will be the all important delta (measure) function potential in the Schrödinger equation. With this in mind, it is appropriate to begin with an elucidation of the calculus of distributions and the role the delta measure plays in the calculus of distributions sans the backdrop of physics.

3.1 The Basic Calculus of Distributions

For the reader unfamiliar with the theory of distributions references [AlG] and [FriJos] are excellent sources. Nonetheless, here an attempt will be made to quickly summarize the main aspects. From the outset, one constructs a topological vector space of

¹A distribution is also called a generalized function, though generalized function should be (and will be in this work) reserved specifically to objects that are even more general than distributions

functions that generally possess some sort of key property that is desirable. It is simplest to begin with the vector space of Schwartz² functions, $\mathcal{S}(\mathbb{R}^d)$, also called the space of smooth functions of rapid decay, or rather ambiguously simply the space of test function. The Schwartz space is the vector space of functions, which decay at infinity faster than the power of any polynomial and all of the derivatives of the reciprocal of the polynomial. The precise definition is given in the appendix.

Next, one considers the topological dual to the the vector space of Schwartz functions, known as the tempered distributions, $\mathcal{S}'(\mathbb{R}^d)$. That is, first one considers the algebraic dual analogous to the transpose dual vector space from the mathematics of vector spaces of finite dimension that one encounters in basic linear algebra. The topological requirement is that the algebraically canonical injection of the set of Schwartz functions into its dual space must be a continuous map. By canonical injection one means a precise identification with elements of a set to those of any superset containing the same elements. As an example, consider the canonical injection of the integers into the rational numbers. Because the rational numbers contain all of the integers the map is an obvious one, shown below. With this in mind, one can see that

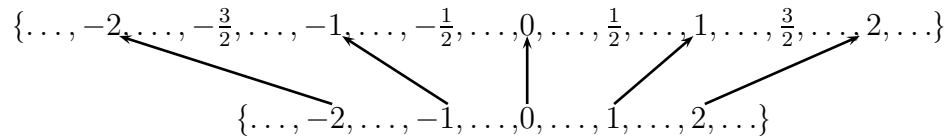


Figure 3.1: The canonical injection of the integers, \mathbb{Z} , into the rational numbers, \mathbb{Q}

the dual space, $\mathcal{S}'(\mathbb{R}^d)$, is necessarily larger than the space of Schwartz functions, i.e. one has the complete containment, $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$. This follows from the fact that $\mathcal{S}'(\mathbb{R}^d)$ contains all of the elements of $\mathcal{S}(\mathbb{R}^d)$ but it also contains limiting sequences of

²The vector space is named after French mathematician Laurent-Moïse Schwartz (1915-2002), recipient of the 1950 *International Medal for Outstanding Discoveries in Mathematics* for his work leading to the theory of distributions. The prize is otherwise known as the *Fields Medal*, so-named in honor of the Canadian mathematician John Charles Fields. Laurent Schwartz should not be confused with German mathematician, Karl Hermann Amandus Schwarz (1843-1921) who shares his name with Augustin-Louis Cauchy (1789-1857), and Viktor Yakovlevich Bunyakovsky (1804-1889) for their celebrated inequalities for sums and integrals respectively with Hermann Schwarz independently rediscovering the inequality for integrals first discovered by Viktor Bunyakovsky.

elements of $\mathcal{S}(\mathbb{R}^d)$. For example, one can define the following sequence.

$$\delta_n(x) := \sqrt{\frac{n}{\pi}} e^{-n\|x\|_e^2}, \quad x \in \mathbb{R}^d,$$

where $\|\cdot\|_e$ is the Euclidean norm of an element in \mathbb{R}^d . For each fixed value of n , one has: $\delta_n(x) \in \mathcal{S}(\mathbb{R}^d)$. But in the limit, this sequence is no element of $\mathcal{S}(\mathbb{R}^d)$. One may be tempted to say that it is the (additive) identity vector zero on the vector space. But this is not so. Since for $x = 0$, one finds that the sequence diverges. So there cannot be a classical derivative at the origin. Furthermore, it is clear that this sequence does not converge to any type of classical function, its closest analogue being the zero function but unlike zero it is not smooth on all of \mathbb{R}^d .

Now on the other hand, the Lebesgue integral of each will be zero. So to strike a balance, the usual Schwartz³ bracket is introduced. The Schwartz bracket is the map of the following type.

Definition 3.1.0.4 (Schwartz Bracket). *The following bilinear map is called the Schwartz bracket and is standard in the theory of distributions.*

$$\langle \cdot, \cdot \rangle : \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C} \supset \mathbb{R}$$

One can also note that the dual space $\mathcal{S}'(\mathbb{R}^d)$ constitutes linear functionals acting on the space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. The two notions of Lebesgue integration and action of a linear functional are connected as follows. Since by design every element of $\mathcal{S}'(\mathbb{R}^d)$ is the limit of a sequence of elements of $\mathcal{S}(\mathbb{R}^d)$, one can make the following assignment, for $T \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$.

$$T[\phi] := \langle T, \phi \rangle := \lim_{n \rightarrow \infty} \langle T_n, \phi \rangle := \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} T_n(x) \phi(x)$$

³Again, here the bracket is named after Laurent-Moïse Schwartz. Though due to the historical order of mathematical publications, one is also made aware of the fact that the Russian mathematician Sergei Lvovich Sobolev (1908-1989) had developed a theory of distributions himself. The bracket ought aptly be named the Schwartz-Sobolev bracket. In any case, due to the cold war an award for Sobolev in 1950 would not have been possible, especially with the award proceedings being held in the United States for the International Congress of Mathematics (ICM).

Now it is clear that since the action of the Lebesgue integral over \mathbb{R}^d is the action of a linear functional then so is the action of $T[\phi]$. Notice as well that the action of the linear functional is the same as the action of the Lebesgue integral *in the limit*. It is a simple exercise to check that one *cannot interchange the limit with the integral in general*.

Lastly, the important issue of convergence of the sequence T_n to the element $T \in \mathcal{S}'(\mathbb{R}^d)$ is elucidated by the noticing that the topology of $\mathcal{S}'(\mathbb{R}^d)$ is induced by that of $\mathcal{S}(\mathbb{R}^d)$. Furthermore, every such issue can be answered by, in an informal sense, *pushing the question on to the test functions*. So one has that $T_n \rightarrow T$ in $\mathcal{S}'(\mathbb{R}^d)$ if $T_n[\phi] \rightarrow T[\phi]$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

This is known as the pointwise convergence and the convergence is in the weak topology of $\mathcal{S}'(\mathbb{R}^d)$. So, one commonly speaks of this as *convergence in the weak sense* or similarly *convergence in distribution*. Likewise, the second name motivates an even greater abuse but convenient nomenclature for notation.

Namely, the expression $T = \lim_{n \rightarrow \infty} T_n$ is often referred to as T is equal to $\lim_{n \rightarrow \infty} T_n$ *in the sense of distributions*. For if one were to apply the usual Schwartz bracket to both sides of the equality the resulting equality would be precise. Below is a collection of results about the actions of common operations on distributions.

The Derivative $\left\langle \frac{\partial^{|k|}}{\partial x^{k_1} \partial x^{k_2} \dots \partial x^{k_i}} T, \phi \right\rangle = (-1)^{|k|} \left\langle T, \frac{\partial^{|k|}}{\partial x^{k_1} \partial x^{k_2} \dots \partial x^{k_i}} \phi \right\rangle$

The Fourier transform $\langle \widehat{T}, \phi \rangle := \langle T, \widehat{\phi} \rangle$

Translations $\langle T(x - a), \phi \rangle := \langle T, \phi(x + a) \rangle, \quad x, a \in \mathbb{R}^d$

Scaling $\langle T(ax), \phi \rangle := \frac{1}{a} \langle T, \phi(\frac{x}{a}) \rangle, \quad x \in \mathbb{R}^d, \quad a > 0$

where $|k| := \sum_{l=1}^i k_l$ with $k_l \in \mathbb{N}_0 \quad \forall \quad i \geq l \in \mathbb{N}$.

So far up to here, everything in this section is completely standard in the mathematical literature and the reader is referred to the two references pointed out in the beginning of the section for more detail. However, in what follows, the material is new and it is convenient to take an aside on some consequences as they apply to physical systems and the calculations used to try to elucidate them.

3.1.1 A Formula for Nelson's Approach to the Feynman Integral

The idea of equality in the sense of distributions seems rather mundane but to see just how dramatic of a difference it can make, it is wise to consider the Feynman integral as elucidated by Edward Nelson in [Nel1]. In this scheme, the Feynman integral is defined through analytic continuation and none of the integrals converge, rather they are oscillatory integrals and they attain their unique values through analytic continuation in the complex plane. Below is a formula that is very useful for evaluating the Feynman integral of this type.

$$(3.1.1.1) \quad \int_0^\infty e^{i a x^s + i b x^p} = \sum_{n=0}^\infty \frac{(i b a^{s-1})^n e^{i \frac{\pi}{2} \left(\frac{pn+1}{s} \right)} \Gamma \left(\frac{pn+1}{s} \right)}{|s| n! a^{s-1}}$$

At first glance, this formula should be extremely troubling. The reason is that the left hand side is symmetric on double simultaneous interchange of a with b and of s with p . However looking at the right hand side, one notices that this symmetry is not present! This appears as if formula 3.1.1.1 cannot possibly be correct. The answer is surprisingly that it is quite correct. The reason is that the equality is not a strong equality but rather only equal in the sense of distributions. This means that one must multiply both sides of the equality 3.1.1.1 by a test function and integrate. Upon doing this, it is immediately obvious that the symmetry of the formula is restored. In this circumstance, Schwartz functions suffice because the convergence is rapid enough for the Borel bounded function in the integrand, i.e. the integrand's modulus is simply 1.

3.1.2 Classical Theorems

The classical theorems here are quoted from the two sources [FriJos] and [Rud]. The two classical theorems to be quoted the regularization and approximation theorems played a critical role in establishing an alternative method to determine bound state spectra of various distributional potentials. This in essence extends the usual theory of partial differential equations and builds a firm connection between the canonical functional calculus and the theory of the distributional calculus. There are quite

more general theories for accomplishing this and they go by the name of the theory of pseudo-differential operators. Starting with Lars Hörmander and continuing with others most notably, William Donoghue, Gerd Grubb, and Francois Trèves, there has been an increasing amount of literature that devotes connections between functional calculi and distributional calculi or generalized PDE theory. In any case, below are the relevant theorems for the content treated in this work.

Theorem 3.1.2.1 (Regularization Theorem). *If $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$ then the convolution,*

$$(\phi * T) = \int_{\mathbb{R}^d} T(x-y)\phi(x) = \langle T(x), \phi(x+y) \rangle = T(\phi(x+y)) \in C^\infty(\mathbb{R}^d)$$

Proof. Let $\varrho \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\text{supp}\{\psi\}} \varrho = 1$, with ψ an arbitrary member of $C_c^\infty(\mathbb{R}^d)$, then the function defined as,

$$\varrho \otimes \phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} : (x, y) \mapsto \varrho(x)\phi(y-x),$$

has compact support. The test function ψ can also be regarded as an element of the space of bounded distributions. Hence, one has that,

$$\langle \psi \otimes T, \varrho \phi \rangle = \langle T, \langle \psi, \phi \rangle \rangle = \langle \phi \otimes T, \psi \rangle = \langle \phi * T, \psi \rangle = \langle \psi, (\phi * T) \rangle$$

Now since, $\psi(x+y) \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, while simultaneously being regarded as a bounded distribution, this implies that $(\phi * T)$ must be its test function and thus can be no worse than an element of $C^\infty(\mathbb{R}^d)$. ■

Likewise, the following theorem guarantees that despite how pathological a distribution or generalized function may be, there is always a sequence of functions from any of its dense subspaces that approximate any element of the space of generalized functions. In particular, the theorem below focuses on the density of the smooth functions.

Theorem 3.1.2.2 (Approximation Theorem). *Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ and define $\phi_n(x) := n^d \phi(nx)$, $n \in \mathbb{N}$ such that $\phi \geq 0$ and*

$$\int_{\mathbb{R}^d} \phi = 1$$

then for $T \in \mathcal{D}'(\mathbb{R}^d)$ one has that the sequence of smooth functions defined as $f_n := T * \phi_n$,

$$\lim_{n \rightarrow \infty} f_n = T$$

Proof. From the above regularization theorem it is clear that the sequence $f_n \in C^\infty(\mathbb{R}^d) \forall n \in \mathbb{N}$. But the ϕ_n are compactly supported, all on the same compact subset and thus denote a *partition of unity*. A direct calculation with some $\psi \in C_c^\infty(\mathbb{R}^d)$ gives,

$$\begin{aligned} T(\psi(-x)) &= (T * \psi)(0) = \lim_{n \rightarrow \infty} (T * (\phi_n * \psi))(0) = \lim_{n \rightarrow \infty} (T * \phi_n) * \psi(0) \\ &= \lim_{n \rightarrow \infty} (f_n * \psi)(0) = \lim_{n \rightarrow \infty} f_n(\psi(-x)), \end{aligned}$$

from which one clearly concludes,

$$T(\psi(-x)) = \lim_{n \rightarrow \infty} f_n(\psi(-x)),$$

and hence $T = \lim f_n$ in the sense of distributions. ■

3.2 Technical Difficulties with Distributions in Functional Integrals

It is common in the physics literature to claim that no new physics can be garnered from the path taken in this work because the spectra for Dirac distributions and its derivatives have already been calculated via the Feynman integral. The mathematical truth of the matter is that this is utterly false. This is because there can to date be no sensible mathematical meaning attached to the exponential of a distribution in simple terms that lend to the typical calculations used throughout the theoretical physics literature.

There are notable exceptions. The exceptions are Hida distributions, named in honor of one of the primary founders of this stochastic theory approach⁴ to the Feynman integral, Takeyuki Hida.

⁴There are many such stochastic calculus approaches to the Feynman integral generally revolving around Wiener measure and its generalizations, named in honor of Norbert Wiener.

Another notable exception is the formal product integral where exponentiation of operators of multiplication are also elucidated under suitable conditions. But in either case, they are not the physicist's Feynman integral. The main mathematical difference is that in the stochastic and product integral theories, there exists bona fide pushforward measures, which make the ensuing formalism mathematically well-defined as an integral operator on function spaces.

However, for the usual notion of Feynman integral, there is the exact opposite result. There is a famous theorem due to Robert Horton Cameron that asserts that there is in fact no *countably additive \mathbb{C} -valued measure*, which accomplishes the task of being a measure for the Feynman integral as is defined with Dirac's ansatz of complex exponentiating the classical action. The key mathematical fact is that bona fide \mathbb{C} -valued measures over Borel sigma algebras that are countably additive must have finite total variation.

3.2.1 Elucidation of the Mathematical Obstacles

Here in this subsection, it is shown just how the mathematical obstructions to the idea of the Feynman integral involving the exponential of the Dirac measure or the exponential of tempered distributions arise. The trouble begins immediately by trying to give meaning to the very expression,

$$e^{\delta(x)}.$$

One must consider all the issues. As has been emphasized throughout, it is desirable to have a rigorous mathematical procedure for a Feynman functional integral, which gives the correct Schrödinger evolution and spectral properties for potentials, $V(x)$, that are distributions in the space variable. But a correct answer is not enough to conclude that a given procedure is mathematically rigorous. Furthermore, the procedure should be independent of dimension and agree with the physically known answers as outlined in the Preface and Introduction.

In this work, there is no claim to a complete resolution to this problem and it is currently not known to have a robust answer, rather here is presented a procedure for the potential function as a singular measure, the Dirac delta function only. It has the following two important properties that make it interesting yet tractable.

1. δ is a tempered distribution, i.e. $\delta \in \mathcal{S}'(\mathbb{R}^d)$.
2. δ is not a continuous probability measure since it has (singular) support at a single point with a mass measuring unity, but nonetheless it is a bona fide measure.

This second point is important because although Lebesgue measure is singular with respect to the Dirac measure, the set of measurable functions coincide whenever the value of the Lebesgue measurable functions at the singular support of the Dirac measure (usually the origin) exists and is finite.

Consider now the general Trotter-Lie product formula for unitary groups represented through the product of exponentials,

$$s - \lim_{n \rightarrow \infty} \left(e^{-\frac{it}{n}A} e^{-\frac{it}{n}B} \right)^n = e^{-it(A+B)},$$

where A and B are arbitrary self-adjoint operators and $A+B$ is the form sum of the two operators with $t \in [0, T]$, $T > 0$. It is not known whether the above equation holds for arbitrary non-negative self-adjoint operators. Note that the slightly different problem of a non-negative self-adjoint operator A and a self-adjoint operator B with small negative part, i.e. such that $B = B_+ - B_-$ with B_- being *A-form bounded*, is believed to still be open as well. For a more in depth summary, see P. Exner et al. in [ExnNeiZag].

In general, the exponential of a distribution is not defined. Attempts have been made to give it meaning by the assignment,

$$\langle \exp \circ \delta, \varphi \rangle := \langle e^\delta, \varphi \rangle := e^{\langle \delta, \varphi \rangle} = e^{\varphi(0)},$$

where $\langle \cdot, \cdot \rangle$ is the usual Schwartz bracket with $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\delta \in \mathcal{S}'(\mathbb{R}^d)$. But this definition leads to inconsistencies. If the Schrödinger potential, $V(x) = \delta(x)$, then the full Green function for the solution of the PDE should be given by $G = G_0 + G_\delta$ where G_0 is the solution to the equation for $V = 0$ and G_δ is the solution to the potential problem. However instead, the above definition gives $G = G_0 \cdot G_\delta$, which is not physical and incompatible with linearity.

One can instead seek an alternative definition for the exponential of δ . Consider the notion of a *log-Vector Space*. The idea is expounded in what follows. By considering the domains and codomains (images) of the two maps, $\exp : \mathbb{R} \rightarrow \mathbb{R}_0^+$, and

$\delta : \mathbb{V} \rightarrow \mathbb{R}$, one can start to make sense of the meaning of the composition, $\exp \circ \delta$. The modification must come in the meaning of the map, \exp . One must have the map, $\exp : \mathbb{W} \rightarrow \mathbb{V}$. Then one makes the identification of the vector space \mathbb{V} with that of the compactly supported smooth functions, i.e., $\mathbb{V} = C_c^\infty(\mathbb{R}^d)$. It is then immediate that one can identify the set $\mathbb{W} := \log\text{-}\mathbb{V} \subset \mathcal{M}$, where \mathcal{M} is the space of meromorphic functions intersecting \mathbb{R} and vanishing outside compact subsets of \mathbb{C} . The subset, $\log\text{-}\mathbb{V}$, is selected based on the degree of the polynomial in the denominator and its multiplicity. The main draw back is that this idea was constructed upon and thus is heavily dependent upon one dimension. A consequence is that in more than one dimension this procedure requires rotational symmetry to make sense.

Alternatively, one may try to construct a resolvent-esque function of δ in order to sidestep the difficulty with the exponential map altogether. While the previous $\log\text{-}\mathbb{V}$ formalism resolves the incompatibility of the domain and codomain, it does not resolve the larger issue of the meaning of a function of the measure. One finds inspiration from the the theorem on imaginary resolvents [JohLap], and one may wisely choose to attempt to avoid the exponential entirely. Consider the formal expression as a distribution,

$$\frac{1}{1 + \delta(x)}.$$

Proposition 3.2.1.1. *One then has in the sense of distributions,*

$$\frac{1}{1 + \delta(x)} = \mathbb{1}, \quad x \in \mathbb{R}$$

where $\mathbb{1}$ is Lebesgue measure. That is for each $\varphi \in C_c^\infty(\mathbb{R})$, one has that,

$$\left\langle \frac{1}{1 + \delta(x)}, \varphi(x) \right\rangle = \int_{\mathbb{R}} \varphi(x).$$

Proof. The proof of this proposition is straightforward in the usual distributional

calculus. Let $\varphi'(x) \in C_c^\infty(\mathbb{R})$ compactly supported on $[a, b]$ with $b > a \in \mathbb{R}$, then:

$$(3.2.1.1) \quad \left\langle \frac{1}{1 + \delta(x)}, \varphi'(x) \right\rangle := \lim_{n \rightarrow \infty} \left\langle \frac{1}{1 + \delta_n(x)}, \varphi'(x) \right\rangle$$

$$(3.2.1.2) \quad = \lim_{n \rightarrow \infty} \left\{ \int_{-\infty}^{-\frac{1}{n}} \varphi'(x) dx + \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{\varphi'(x)}{1 + \frac{1}{2}n} dx + \int_{\frac{1}{n}}^{\infty} \varphi'(x) dx \right\}$$

$$(3.2.1.3) \quad = \int_{\mathbb{R}} \varphi'(x) dx + \lim_{n \rightarrow \infty} \left\{ \frac{\varphi(\frac{1}{n}) - \varphi(\frac{-1}{n})}{1 + \frac{1}{2}n} \right\}$$

$$(3.2.1.4) \quad = \varphi(b) - \varphi(a)$$

The proof is easily generalized to \mathbb{R}^d . ■

This proposition is enlightening but not fruitful. The reason is clear. Evidently, the resolvent-esque function cannot be made of use as it is no different from the tools already available, i.e. Lebesgue measure. This is not surprising, as the function formed is not a proper resolvent. Therefore, more drastic means must be taken and one must still face the issue of a function of δ .

As is the case with operators, one can expect quite a bit of ambiguity. One resolution is as follows. First, start with the assumption of a spectral theorem (called the *spectral assumption*) and proceed as if it exists. Second, make use of the coincidental special case that the action of δ as a characteristic set function is idempotent since both $0^2 = 0$ and $1^2 = 1$. This is also the case with δ -normalized⁵ test functions with compact support, since they too take values of either 0 or 1. This motivates the definition of the spectrum of a distribution as it appears in W. Donoghue.

Definition 3.2.1.2. *The closed set, $\sigma(T) := \text{supp}(\widehat{T}) \subseteq \mathbb{C}$ is the spectrum of the distribution, $T \in \mathcal{D}'$, where \widehat{T} is its distributional Fourier transform given by $\langle \widehat{T}, \varphi \rangle := \langle T, \widehat{\varphi} \rangle$, $\forall \varphi \in \mathcal{D}$.*

⁵Here, delta normalized means dividing the test function by its value at the support of the singular measure and nothing more. Most other notions of delta function normalization are either not mathematically rigorous or if they are they are akin to normalizing Lebesgue measure of the real line to unity. This in and of itself is fine. However, it does not resolve the troubles in physics and other applications because as soon as one begins to compute moments and fluctuations the divergences then re-appear.

Notice that the Fourier transform for distributions is *not* given by an integral. This being the case, one can now use the preceding definition to investigate spectra of polynomials whose complex coefficients have modulus one.

Example 3.2.1.3 (Spectrum of the Unit Function as a Distribution). *From the previous definition, one has,*

$$\sigma(1) := \text{supp}(\widehat{1}) = \text{supp}(\delta_0) = \{0\}.$$

So the spectrum of the unit constant function is the single point, $x = 0$, the support of δ_0 . This makes sense in view of the fact that the constant function is smooth and bounded.

It is well known that tempered distributions are given by the Fourier transform of polynomials. In light of this and the Stone-Weierstrass theorem, one can build up sequences limiting to any continuous function that automatically comes along with an *associated spectral sequence*.

Using the idempotence of δ as mentioned previously, one can give meaning to functions of δ by the following definition, which contains the so-called *spectral assumption*.

Definition 3.2.1.4 (Analytic Linearization). *Let f be an analytic function about $x = a$, with power series given by*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n,$$

then the expression $f(\delta_a)$, i.e. Dirac measure with singular support at the point $x = a$, is defined as,

$$f(\delta_a) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} \delta_a$$

where $\mathbb{1}$ is again Lebesgue measure. This is the analytic linearization of the distribution $f(\delta_a)$.

To see just what this assumption implies one gives the following example.

Example 3.2.1.5 (Analytic Linearization: The Exponential). *From the previous definition, one has,*

$$e^{\delta_0} := \mathbb{1} + (e - 1) \delta_0.$$

since

$$\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$$

Through the usual methods of analytic continuation, one can push this notion of linearization further. Assign the expression below, the following meaning.

$$\frac{1}{1 + \delta_0} = \mathbb{1} - \delta_0 (1 - 1 + 1 - 1 + \dots) = \frac{1}{2} \{ \mathbb{1} + (\mathbb{1} - \delta_0) \} = \mathbb{1} - \frac{1}{2} \delta_0$$

For the special case of the measure- δ regarded as an idempotent set function, the spectral assumption as an ansatz is sensible from the viewpoint that δ plays the role of the identity in the convolution algebra. So the spectral assumption is simply re-interpreting the operation in the Taylor expansion as convolution instead of point-wise multiplication and the two viewpoints are compatible in this light.

There is one last observation worth making and that is that the idea of *analytic linearization* is not in contradiction to Proposition 3.2.1.1. In fact, just as the divergent series can be made to conditionally converge to the average value⁶ of $\frac{1}{2}$, the results of *analytic linearization* give that it is an average between the Lebesgue measure and Lebesgue measure minus a Dirac delta measure, as would be expected from the naive Taylor series.

3.3 Extensions: The Dirac-Schwartz Bracket

There are several physical and mathematical problems to address regarding powers of the inverse of the Euclidean norm of a vector in \mathbb{R}^d , if it is to be regarded as a generalized function for the purposes of making use of it in mathematically rigorous calculations, due to its prominence in physical phenomena. To begin with, regard the following proposition.

⁶The value can actually be made to converge to any real number by Riemann's reordering theorem but the arithmetic average value is a distinguished value that one can agree upon its uniqueness by definition of the arithmetic average value.

Proposition 3.3.0.6. *Given powers of the inverse Euclidean norm, $|x|^{-\alpha} > 0$, with $x \in \mathbb{R}^d$, $\alpha > 0$, and any normalized test function (i.e. $\int \phi = 1$), $\phi \in \mathcal{D}(\mathbb{R}^d)$, whose support lies in the unit ball centered at the origin of \mathbb{R}^d . One has,*

$$\left\langle \frac{1}{|x|^\alpha}, \phi \right\rangle = \int_{\mathbb{R}^d} \frac{1}{|x|^\alpha} \phi(x) = \begin{cases} \infty & \text{(i) } \alpha = d \\ \infty > b > 0 & \text{(ii) } \alpha < d \\ c \in \mathbb{R} & \text{(iii) } \alpha > d \wedge \alpha \neq d + 2pn, n \in \mathbb{N} \\ \infty & \text{(iv) } \alpha = d + 2pn, n \in \mathbb{N} \end{cases}$$

The proof can be accomplished by observing the results of a direct calculation, where $p > 0$.

Proof.

$$(3.3.0.5) \quad \int_{\mathbb{R}^d} \frac{1}{|x|^\alpha} \phi(x) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^1 \frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}} \left(\frac{e^{-1}}{2p}\right) \Gamma(\frac{d}{2p}) U(\frac{d}{2p}; 0; 1)} r^{d-\alpha-1} e^{\frac{-1}{1-r^{2p}}}$$

$$(3.3.0.6) \quad = \frac{\Gamma(\frac{d-\alpha}{2p}) U(\frac{d-\alpha}{2p}; 0; 1)}{\Gamma(\frac{d}{2p}) U(\frac{d}{2p}; 0; 1)}$$

Here, one notes the fact that the pole of the gamma function, $\Gamma(x)$, at $x = 0$ turns up each time $\alpha = d$, i.e. whenever the power of the inverse Euclidean norm is equal to the number of dimensions of the Euclidean space. This explains the first case, (i), being divergent. For $d > \alpha$, the function is completely well behaved. This justifies the second case (ii). Next, it can be pointed out that if $\alpha > d$ while $\alpha \neq d + 2pn$, $n \in \mathbb{N}$, then the scalar can well be negative. This is because both; the gamma function of negative (non-integer) arguments oscillate around its poles which occur at the non-positive integers; and the Tricomi function $U(s, tx)$ for fixed t and x has its zeros along the negative real axis. Consequently, the sign can only be determined when all the parameters, p , α and d are simultaneously specified. Hence the result for case (iii). Finally, if $\alpha = d + 2pn$, $n \in \mathbb{N}$, then the poles of the gamma function are reached and the expression diverges. This gives the last case (iv), since in this case all of α , d and p are greater than zero so this is a case of concern whenever $\alpha > d$. ■

In reflecting upon Proposition 3.3.0.6, there is cause for both serious mathematical and physical concern. Mathematically, one notes that although $|x|^{-\alpha} > 0$ and $\phi(x) >$

0 for all $x \in \mathbb{R}^d$, case (iii) allows for negative values of the Schwartz bracket. Thus evidently, $|x|^{-\alpha}$ is not a positive generalized function, with the exception of when the conditions for case (ii) are met. Recall a generalized function, T , is positive if $T(\phi) > 0, \forall \phi \in \mathcal{D}(\mathbb{R}^d)$. Though this would not be remotely the first counter intuitive result in analysis, it would be nice if the equalities obeyed by the functions themselves were at all related to their behavior in the distributional calculus.

Moreover, one would like to treat more singular objects that are not distributions in the canonical theory as it exists beyond the single example in Proposition 3.3.0.6. Some desirable extensions to more generalized functions are listed below,

$$(3.3.0.7) \quad \delta(x^2)$$

$$(3.3.0.8) \quad e^{\frac{1}{|x|}}$$

$$(3.3.0.9) \quad \delta(x) \log |x| \dots$$

and so forth. There are both physical and mathematical reasons for wanting to treat such singular objects in a single consistent fashion. Before a resolution is proposed, the physical concerns are also addressed in the next paragraph's discussion.

Physically, there is cause for great concern, as Proposition 3.3.0.6 implies that the energy density of a free static Coulombic system, which is physically constrained to be strictly positive, can be negative. Apparently, the value being divergent has been something physicists have lived with for almost a century but there is no reason to accept the divergence either. Finally, these combined troubles motivate the following novel definition for the pairing of test functions with (more) generalized functions.

Definition 3.3.0.7 (The Dirac-Schwartz Bracket).

Case 1: If the generalized function T is a standard distribution then,

$$\langle T, \phi \rangle_{\mathcal{D}, \mathcal{S}} := \langle T, \phi \rangle,$$

where the right hand side is the usual Schwartz bracket and is defined $\forall \phi \in C_c^\infty(\mathbb{R}^d)$.

Case 2: However, if T has a finite number, N , of singularities that are simple poles at the points $x_j \in \mathbb{R}^d$ with $j \in \{1, 2, \dots, N\}$ then,

$$(3.3.0.10) \quad \langle T, \phi \rangle_{\mathcal{D}, \mathcal{S}} := \sum_{j=1}^N \int_{\text{supp}(\phi_j)} T \phi_j,$$

where the ϕ_j are defined as all $\phi \in C_c^\infty(\mathbb{R}^d)$ such that the $x_j \in \partial\text{supp}(\phi) \forall j$. Here, the set $\partial\text{supp}(\phi)$ is the boundary of the support of the function ϕ . Thus in this case the bracket is just the finite sum of the Lebesgue integrals over the compact subsets of \mathbb{R}^d over which the ϕ are supported and the singularities are on the boundaries.

Case 3: Finally, there is the case where T has a finite number of essential singularities in which case the Dirac-Schwartz bracket is defined much the same as above except that it is restricted to only all $\phi \in C_c^\infty(\mathbb{R}^d)$ for which each individual integral in equation 3.3.0.10 converges.

A few comments regarding the definition are in order. The first condition is vital because among other things it ensures that (i) the Dirac-Schwartz bracket reduces to the usual Schwartz bracket; and (ii) it prevents the Dirac delta (evaluation functional) from being always zero. Thus, the term, “ T is a generalized function with singularities”, is rather subtle. It implicitly means that one can identify what a singularity is. This is somewhat open and ambiguous, say for example in the case $\delta(x^2)$, but has certain obvious resolutions. The two foremost are (a) if T itself is a well-defined function, and (b) if $T = fS$ where S is a known distribution and f is a well-defined function.

But on the other hand, there is the more subtle issue of $\delta(x^2)$. Here, the problem is that the multiplicity of the zeros of the function x^2 , in a certain sense, overemphasize the natural singular support. It would probably more appropriate if this expression were assigned to the evaluation functional by ignoring the multiplicity of the zeros of the argument function. This being not the case, among the two choices of 0 or ∞ , evidently 0 was chosen as a side-effect of obtaining better results for other functional forms like $|x|^{-\alpha}$, $\alpha > 0$.

The finiteness of the sets can be generalized to infinite sets and sequences ϕ_j . However in said cases, the problem of convergence of the sum then becomes an issue and that would need to be addressed. This is brought to light because of special interest is the application of the sequence ϕ_j to a set of singularities that have properties like that of the Cantor set. Then, each ϕ_j is supported on the deleted segment and obviously the sum is unaffected by the disjointness of the supports. The main issue being that of convergence.

While on the topic of convergence, it is also worth noting that the added condition of convergence is just the type of condition to restrict the test function space even more and give rise to a larger set of generalized functions, when and if needed. There are several benefits. The first is that the handling of the singularities is relegated to the action of the bracket given attributes of T , as opposed to being forced to exclude by hand elements of $C_c^\infty(\mathbb{R}^d)$ in an ad hoc fashion. Evidently, the Dirac-Schwartz bracket provides a natural mechanism under which to decide just how the usual test function space $C_c^\infty(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$ do not quite coincide. This plays directly into the intuition that the attributes of the interaction T due to its singular structure will affect and hence partially determine the topology of the test function space Dd , so that both of the following possibilities arise, (a) $\mathcal{D}(\mathbb{R}^d) \not\cong C_c^\infty(\mathbb{R}^d)$ and (b) if $T_1 \neq T_2$, then it may happen that $\mathcal{D}(\mathbb{R}^d)_{T_1} \not\cong \mathcal{D}(\mathbb{R}^d)_{T_2}$, as well. This will be discussed further in chapter 4.

The main benefit, to be checked below, is that it now allows one to attribute finite values to a larger set of generalized functions, which reduces the size of the space of test functions and thus increases the allowed set of generalized function. Recall that from set theoretic considerations, a larger class of distributions necessitates a smaller test function space. Ergo, one has the following examples.

Example 3.3.0.8. *The reader can verify that $\forall \phi \in C_c^\infty(\mathbb{R}^d)$. One has,*

$$\langle \delta(x^2), \phi \rangle = \begin{cases} 0 & 0 \notin \text{supp}(\phi) \\ \infty & 0 \in \text{supp}(\phi) \setminus \partial \text{supp}(\phi) \\ 0 & \text{the conditions of } \langle \cdot, \cdot \rangle_{\mathcal{D}, \mathcal{S}} \text{ are met} \end{cases}$$

Whereas in contrast, one has always,

$$\langle \delta(x^2), \phi \rangle_{\mathcal{D}, \mathcal{S}} = 0.$$

The same is true of $\delta(x) \log|x|$.

Example 3.3.0.9. *Again, it is the case that,*

$$\langle e^{\frac{1}{|x|}}, \phi \rangle = \begin{cases} \infty > c_1 > 0 & 0 \notin \text{supp}(\phi) \\ \infty & 0 \in \text{supp}(\phi) \setminus \partial \text{supp}(\phi) \\ \infty > c_2 > 0 & \text{the conditions of } \langle \cdot, \cdot \rangle_{\mathcal{D}, \mathcal{S}} \text{ are met} \end{cases}$$

In contrast, the Dirac-Schwartz bracket gives,

$$\left\langle e^{\frac{1}{|x|}}, \phi \right\rangle_{\mathcal{D}'^s} = c_2 \quad \text{where, } \infty > c_2 > 0.$$

In particular, the minimal example of $x \in \mathbb{R}$, $\phi_j \in C_c^\infty(\mathbb{R})$ and defined by,

$$\phi_j(x) = \begin{cases} 0 & -\infty < x \leq -1 \\ \phi_1(x) = e^{\frac{1}{x(1+x)}} & -1 \leq x \leq 0 \\ \phi_2(x) = e^{\frac{-1}{x(1-x)}} & 0 \leq x \leq 1 \\ 0 & 1 \leq x < \infty \end{cases}$$

results in $c_2 = 2\Gamma(-1, 1) > 0$ where $\Gamma(s, x)$ is the upper incomplete gamma function.

Finally, one has that for $\alpha > 0$, $\langle |x|^{-\alpha}, \phi \rangle_{\mathcal{D}'^s} > 0$ for all $d > 0$ and suitable $\mathcal{D}(\mathbb{R}^d) \ni \phi > 0$. Further examples would bring the content into the realm of application and so the remainder is reserved for that purpose in chapter 4.

Chapter 4

The Quantum Theory of Fields

What is a quantum field? The fact of the matter is that to date, there is no single satisfactory definition for a quantum field. Herein, it is presented as a matter of evidence to the reader to decide whether such a question is justified and if it is at all partially answered in this work. In all the literature on the topic there is no definition of quantum field that encompasses a range from one to four dimensions, nor for all known physically relevant interaction types. Notice that one is content to give a definition for simply 4 dimensions, let alone 5, 10, 11, or 26 dimensions for, if nothing else, purposes of comparison. Every attempt has been made to introduce formulas, theorems, definitions and equations, which depend on dimension as little as possible.

There are generally two phenomena, which in all likelihood cannot be independent of dimension. The first is that of the inherent large number of degrees of freedom that one incurs in many extra dimensions, as noted earlier in the Introduction, chapter 1. This often leads to triviality for the spectral properties of a given interaction. By spectral properties, one means not only the energy dispersion but also the range of allowed relative probabilities, in the physics literature it is often referred to as the selection rules. But one means more than just selection rules since this term generally implies transitions of countably many states. One should also bear in mind the idea of continuous and/or asymptotic states.

The second phenomena depending on dimension are those of a mathematical nature whose very intrinsic definitions are sensitive to changes in dimension. An immediate and indeed prototypical example is that of the mathematical notion of the

capacity of a set. This was an idea put forth by Gustav Choquet in [Cho] and studied further by others such as [ArmGar], in the context of mathematical potential theory. In any case, it follows from the logical definitions that the only set of capacity zero in one dimension is the empty set. But this does not remain true in dimensions greater than one.

So to answer the question posed, one might begin by realizing what a quantum field is not. Starting with the Coulomb potential as an object of central interest due entirely to its successful application in the current canonical theory of quantum electrodynamics, one notes that the function $|x|^{-4}$ for $x \in \mathbb{R}^3$ cannot be a standard distribution. This is because this function is simply not smooth. Moreover, it is not smooth even in the sense of distributions and not even in a neighborhood of the origin. This would suggest that a quantum field is something more general than what is typically assumed, namely, an operator-valued distribution. Evidently, it must at least be an operator-valued (quite) generalized function.

As was seen in the section entitled Extensions: The Dirac-Schwartz Bracket in the chapter on Distribution Theory and Fundamental Physics, the self-energy of a free static charged particle should be finite and positive. Incidentally, its self-energy difference with a bound state should be negative and for the hydrogen atom this is known as the Lamb shift. In this chapter, the consequences of the mathematical extensions of the previous chapters for quantum electrodynamics will be investigated insofar as the machinery developed allows for mathematical rigor to be maintained.

4.1 Free Static Self-Energy of a Charged Particle

One should begin by computing the classical self-energy of a free static charged particle. Static suffices since there is always a frame in which a massive particle is at rest. The electric field generated by a particle of mass, m , and charge e is,

$$\mathbf{E} = \frac{e}{4\pi\epsilon_0|x|^2} \hat{x}, \quad x \in \mathbb{R}^3,$$

where \hat{x} is the radially outward unit vector, $|x|$ is the Euclidean norm, and ϵ_0 the permittivity of the vacuum. The classical static energy density is then given by,

$$\mathcal{H}(x) := \frac{1}{2}\epsilon_0|\mathbf{E}|^2 = \frac{e^2}{32\pi^2\epsilon_0|x|^4}$$

This object is highly singular but under the definition 3.3.0.7 of the Dirac-Schwartz bracket, it is finite and positive. Choosing a suitable test function of $x \in \mathbb{R}^3$ with compact support in the ball of radius, $b > 0$ (length cut-off of b),

$$(4.1.0.1) \quad \phi(x) = \begin{cases} e^{\frac{-b^2}{|x|(b-|x|)}} & |x| \leq b \\ 0 & \text{otherwise,} \end{cases}$$

one obtains for the self-energy (with respect to this ϕ):

$$(4.1.0.2) \quad \langle \mathcal{H}(x), \phi(x) \rangle_{\mathcal{D}'\mathcal{D}} = \frac{e^2}{32\pi^2 \varepsilon_0} \frac{8\pi}{b} e^{-2} K_1(2) = \frac{e^2}{4\pi\varepsilon_0 b} e^{-2} K_1(2),$$

where e is the charge of the particle, e is Euler's number, and $K_1(x)$ is the MacDonal function of order 1. Notice that this equation 4.1.0.2 has units of energy as it should since the bracket automatically integrates over the volume of space. Furthermore, this implies that the energy density Hamiltonian is itself, in fact, a generalized function.

Now, the size of the support, i.e. the cut-off length scale must be determined by the physics. The only available quantities other than those used already are the mass m of the particle (stated earlier but not used), the Planck constant \hbar , and the propagation speed of the Maxwell radiation– the speed of light, c . Hence, one arrives at the conclusion that $b \sim \frac{\hbar}{mc}$ and therefore equation 4.1.0.2 now reads,

$$(4.1.0.3) \quad \begin{aligned} \langle \mathcal{H}(x), \phi(x) \rangle_{\mathcal{D}'\mathcal{D}} &= \frac{e^2 m c}{4\pi\varepsilon_0 \hbar} e^{-2} K_1(2) = \frac{e^2 m c^2}{4\pi\varepsilon_0 \hbar c} e^{-2} K_1(2) \\ &= \alpha e^{-2} K_1(2) m c^2 \approx .018928788 \dots (\alpha m c^2), \end{aligned}$$

where here and throughout this chapter α is the fine structure constant, $\alpha := \frac{e^2}{4\pi\varepsilon_0 \hbar c}$.

The thing to notice is that the scale is not fixed. The fine structure constant is unitless and the proportionality with the cut-off can be scaled by any positive real number. Moreover, the function ϕ a particular one so the number, $.01892 \dots$, has no meaning. If one were to normalize ϕ then the result would be an energy density and not an energy. Furthermore, even the ionization energy of hydrogen is of order α^2 , which implies that the free self-energy is huge since $\alpha \sim \frac{1}{137} < 1$. The conclusion here from just a static naive classical perspective is that *the self-energy is not infinite so much as it is indeterminate*. This is analogous to the expression $\lim_{x \rightarrow \infty} \sin(x)$. Since the sine function is bounded the limit is not of arbitrarily large magnitude but rather it is indeterminate.

So under what conditions does one expect to determine the self-energy? The self-energy will be made ‘rigid’, metaphorically speaking, when the charged particle has the opportunity to interact with its environment. In fact, it is the very act of interacting that sets the scale of the energy. What is more is that the scale will vary for each interaction type.

4.2 Modification of the Coleman and Mermin-Wagner Theorems for $d \leq 2$

An extremely important consequence of the Dirac-Schwartz bracket is that one can successfully smooth out propagators in \mathbb{R}^d with $d \leq 2$ in such a manner that it is consistent with the smoothing of the propagators in other dimensions. This implies that one cannot argue that the presence of superficial divergences have physical implications. More to the point, the Dirac-Schwartz bracket gives a well-defined linear functional *that is inequivalent to the linear functionals defined through the standard distributional calculus*. Ergo, the theorem in [Col] by S. Coleman is modified accordingly because the expression $\delta(k^2)$ in any dimension Euclidean space is a distribution under the Dirac Schwartz bracket. The real argument then distils to whether or not the Dirac measure is present in the two-point correlation function.

This question is particularly subtle since recall that the Dirac-Schwartz bracket does not pinch values of the propagator viewed as a generalized function that arise from the Dirac measure but rather only pinches those quantities that are divergent in the standard distributional calculus. In the two-point correlation function, the Dirac measure is in principle always formally present. The key attributes are whether the field theory is defined for 1+1 Minkowski space-time or for 2 dimensional Euclidean space. This is another basis for which to reject the Osterwalder-Schrader correspondence theorems since it is clear the two cannot be remotely related because one will contain the Dirac measure as a bona fide element of the dual space while the other does not.

Likewise the Dirac Schwartz bracket (much like the quadratic form in quantum theory removing the negative infinite energy solutions), removes the infrared divergence from the correlation function. Thus at finite temperature, the Heisenberg model

does not preclude the possibility of spontaneous continuous symmetry breaking for local interactions. Clearly, long range interactions are favored by entropy arguments, there is no issue there. The issue is over whether or not such conditions are possible, not whether they are likely— as of course locally, entropy can surely decrease.

A simple physical example where continuous symmetries can be broken is the case where continuous translational invariance is spontaneously broken by boundary effects. Of course, it is a matter of philosophical debate as to whether there ever truly is such a thing as continuous translational symmetry. In condensed matter systems, the issue is over whether the medium can be regarded as having continuous symmetry, whereas in fundamental physics the issue is over whether or not space-time itself is in fact the same as the continuum. That is to say, all of these issues degenerate to questions of just how continuous is continuous in the sense of the physical realm. Of course mathematically, the definition is precise and something either does or does not meet the criteria (or it is undecidable).

Proposition 4.2.0.10. *For the rotationally invariant normalized version of the test function 4.1.0.1, φ in \mathbb{R}^2 , one has,*

$$0 < |\langle \log(r), \varphi \rangle_{\mathcal{D}, \mathcal{D}}| < |-\log(b) + 2K_2(2)|,$$

where $K_\nu(x)$ is again the MacDonald function.

Proof. The following inequalities hold on the interval $[0, \infty)$

$$1 > \frac{1}{(t+1)^3} > 0 \quad \text{and} \quad t > \log(t+1) > 0.$$

Consequently, one obtains after the change of variables $r = \frac{b}{t+1}$ with $b > 0$,

$$\begin{aligned} 0 < |\langle \log(r), \varphi \rangle_{\mathcal{D}, \mathcal{D}}| &= \left| -\log(b) + \int_0^\infty \frac{\log(t+1)}{(t+1)^3} e^{-t-\frac{1}{t}} \right| \\ &< \left| -\log(b) + \int_0^\infty t e^{-t-\frac{1}{t}} \right| \\ &= |-\log(b) + 2K_2(2)| \end{aligned}$$

where the last inequality follows from the above inequalities and the Lebesgue dominated convergence theorem. ■

The above proposition 4.2.0.10 is extremely important because not only does it give a finite value but it indicates that the sign of the two-point correlation function or G_0 propagator in two dimensions, depends upon the state φ and the fundamental length scale b . This is in sharp contrast to the summary of findings in [MerWag].

Notice that if $b < 1$ then the first term is positive. It may also happen that there are more than one fundamental length scales available, ideally 2. When there are exactly 2 scales available, the physical parameters are set invariant of scale because there is a second scale with which to compare. This was seen in one dimensional quantum mechanics where the δ potential in subsection 2.2.1 and the singular Coulomb potential in subsection 2.2.3 had two length scales, $a^{-1} = \sqrt{\frac{\hbar^2}{2m|E|}}$ and $\beta^{-1} = \frac{\hbar^2}{2m\alpha}$ (though β was not expressly shown for the Dirac measure but it was present).

In contrast, the derivative of the Dirac delta potential had only one length scale. This made the energy difficult to derive and its value is somewhat arbitrary. The same is the case in field theories. The pairing of the fields with their smearing states φ or test functions in mathematical terms, gives an inherent scale. This scale can be normalized out to some degree but not entirely. In proposition 4.2.0.10 the first missing scale is the physical constant in front of the Laplace operator. The second is determined by the physical constants of the theory. However, again as in subsection 2.2.2, there may be no scale that is set by the physical parameters. Such is the case pure quantum electrodynamics (QED). The fine structure constant is unitless this makes the theory renormalizable, not the fact that $\alpha < 1$ as is commonly misstated since the Dyson series is known not to converge. See for example Gill and Zachary [GilZac]. This leads nicely into the next section.

4.3 Spectrum of the Free Unitary Operator

The spectrum of the free unitary operator, $\sigma(U(t))$ with $U(t) = e^{-\frac{i}{\hbar}tH_0}$, has a form, which exactly explains the types of special functions and number theoretic quantities that appear in the Dyson series for QED. Here the term *free* means a Hamiltonian system with no potential or boundary conditions other than the natural ones for the usual topology on \mathbb{R}^d . Recall that the infinite well has no potential but does have boundary conditions that force the quantum dynamics onto a compact subset

of \mathbb{R}^d and induce quantized energy levels, whereas the free particle has a continuous spectrum. The trick is to determine the resolvent for the unitary operator itself. Starting from,

$$(U(x) - \lambda) G(x) = \left(e^{\frac{i\hbar}{2m} t \nabla^2} - \lambda \right) G(x) = \delta(x), \quad x \in \mathbb{R}^3$$

Regard the following calculation.

$$\begin{aligned} \mathcal{F} \left[e^{\frac{i\hbar}{2m} t \nabla^2} G(x) - \lambda G(x) \right] &= 1 \\ &= e^{-i\hbar t \frac{4\pi^2 k^2}{2m}} \widehat{G}(k) - \lambda \widehat{G}(k), \quad k \in \mathbb{R}^3 \end{aligned}$$

where $\mathcal{F}[\cdot]$ is the usual Fourier transform and the spectral mapping theorem has been used on the unitary operator. This implies,

$$\begin{aligned} (4.3.0.4) \quad G(x) &= \int_{\mathbb{R}^3} \frac{e^{2\pi i x k}}{e^{-i\hbar t \frac{4\pi^2 k^2}{2m}} - \lambda} \\ &= \frac{2}{|x|} \int_0^\infty \frac{\sin(2\pi k |x|) k}{e^{-i\hbar t \frac{4\pi^2 k^2}{2m}} - \lambda} dk \end{aligned}$$

The above integral does not converge as the integrand is highly oscillatory but the expression does have an analytically continued value in terms of an infinite series of polylogarithms. This series converges under the conditions that $|\lambda| < 1$, which is not the case but for $|\lambda|$ the series conditionally converges to the Lerch transcendental function, defined as

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}.$$

Its connection to the polylogarithm is given by,

$$Li_s(z) = z \Phi(z, s, 1)$$

Now because the operator is unitary, it is known that the eigenvalues λ necessarily satisfy, $|\lambda| = 1$. This is the case for example with the Fourier transform, as well. In any case, the integral in 4.3.0.4 can be expressed as,

$$\mathcal{L}^{-1} \left[\lambda \Phi \left(\lambda, 1, 1 - \frac{2m|x|^2}{\hbar t} \right) \right] \left(\frac{\hbar t}{2m r^2} \right),$$

where $\mathcal{L}^{-1}[\cdot]$ is the inverse Laplace transform. The point is that the expression is given by a conditionally convergent series of polylogarithms, which if one recalls, are generalizations of the Riemann zeta function. There is a corresponding formula for relativistic free unitary operators as well also involving the Lerch transcendental function. Consequently, it is of no surprise that the Dyson series contains terms, which have Riemann zeta functions and polylogarithm functions in its perturbative corrections.

4.4 Induced Test Function Spaces

Before addressing the issue of the CPT invariance and the space of fields, one must first understand how test function spaces are produced in this view of quantum field theory. The idea is as follows. The CPT operators generate involution maps on the test function space, which in turn select out specific types of test functions with specific properties. As an extremely simple trivial example but one that makes the point quite clear, consider the parity operator, \mathcal{P} and its action on one dimensional test functions. Suppose one wanted only test functions which were invariant under the action of \mathcal{P} . Then the resolution is obvious. One only allows those test functions that are even functions, i.e. $\mathcal{P}\phi(x) = \phi(-x) = \phi(x)$ for all $\phi \in \mathcal{D}(\mathbb{R}^d)$ that are even. This example in its simplicity relays all the basic ideas. The remaining issues will be over consequences and mathematical technicalities of the details.

A more sophisticated example is below and follows closely that of Gel'fand and Shilov, [GelShi1].

Example 4.4.0.11. *Consider the Fourier transform acting on $\mathcal{D}(\mathbb{R}^d)$. The transform of such functions is not guaranteed to be another function in $\mathcal{D}(\mathbb{R}^d)$. For one to be certain of this fact, one must require that only the resulting transformed functions with the proper growth rate can be allowed, in order to be certain that an inverse exists.*

Recall that an involution is its own inverse in the sense that if \mathfrak{J} is an involution then $\mathfrak{J}[\mathfrak{J}[\phi]] = \phi$. Incidentally, the Fourier transform itself is not an involution because $\mathcal{F}[\mathcal{F}[\phi(x)]] = \phi(-x)$ for $\phi \in \mathcal{S}(\mathbb{R}^d)$. However, this does imply that the square of the Fourier transform is in fact an involution provided the domain is well-defined, for example is known to be in $\mathcal{S}(\mathbb{R}^d)$.

Because of the dense continuous embedding of $C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, it is tempting to think that the Fourier transform will also have the right decay properties. In general it does not, but by the Paley-Wiener theorem it is known that, for example in one dimension, that the growth rate is satisfied on the left or on the right but not necessarily both. It is however guaranteed that such a function will be a smooth function as in an element of $C^\infty(\mathbb{R}^d)$.

All of the following spaces will be dense, continuously embedded subspaces unless otherwise noted. There is a dense subset of $\mathcal{D}(\mathbb{R}^d)$ called say, \mathcal{E} and another space, which the Fourier transform will map to say, \mathcal{Z} , i.e. $\mathcal{F}[\cdot] : \mathcal{E} \rightarrow \mathcal{Z}$. Then one has the following topology. For a sequence $\psi_n \in \mathcal{E}$ and some $\phi \in \mathcal{D}(\mathbb{R}^d)$ and such that $\mathcal{F}[\psi_n] \in \mathcal{Z} \forall n$, one has $\|\psi_n - \phi\|_{\mathcal{E}} \xrightarrow{n \rightarrow \infty} 0$, where $\|\cdot\|_{\mathcal{E}}$ is the norm on the space \mathcal{E} , implies that $\|\mathcal{F}[\psi_n] - \phi\|_{\mathcal{Z}} \xrightarrow{n \rightarrow \infty} 0$, where $\|\cdot\|_{\mathcal{Z}}$ is the norm on the space \mathcal{Z} . In particular, due to Paley-Wiener this means that whenever $\|\mathcal{F}[\psi_n]\| \xrightarrow{n \rightarrow \infty} 0$, one has a sequence of positive numbers, $C_m > 0 \forall m$ and $a > 0$ such that $e^{a|x|} C_m \geq ||k|^m \mathcal{F}[\psi_n(k)]||$, where n does not depend upon a or m .

The above example shows how the test space $\mathcal{D}(\mathbb{R}^d)$ can be reduced to a space \mathcal{E} by action of a map. Furthermore, the topology on \mathcal{E} gives that of the space \mathcal{Z} through the, in the case of the quantum theory of fields, involution maps but in the above example the Fourier transform. It is then the case necessarily that the topological dual of the induced test space \mathcal{Z} denoted as \mathcal{Z}^\times is such that $\mathcal{D}'(\mathbb{R}^d) \subset \mathcal{Z}^\times$. In turn generally speaking, the topology on the dual spaces is induced by the test function spaces.

4.5 The Space of Quantum Fields, Its Metric, and CPT Invariance

Finally, the space of quantum fields can be defined. A quantum field is an operator-valued quite generalized function. Its definition changes with the topology of its test function space. Each interaction comes with a natural length scale and set of invariances that form involution maps on the space of test functions. These involution maps in turn dictate the topology on a reduced test function space, which in turn induces the topology on the field space. Furthermore, though it was not generally

discussed previously, the test function space and the space of fields are defined in general over the complex numbers so that the Hilbert spaces involved are complex. The reason being for the express application to quantum mechanics in rigged Hilbert spaces as it pertains to scattering theory and holomorphic functions. In these cases, the generalized functions are known to be hyper-functions.

With all of this in mind, one constructs a metric on the space of fields as follows. The construction defines a dynamic metric and bears strong commonalities with the theory of \mathcal{PT} symmetric Hamiltonians and their associated \mathcal{PT} inner products, as put forth by C. Bender et al. [BenBroJon]. As well, there are also commonalities with the theory of partial inner product spaces as put forth by J. P. Antoine and C. Trapani [AntTra].

Let the test function space be \mathcal{Z} and let its continuous dual space be \mathcal{Z}^\times . One calls the elements of \mathcal{Z}^\times representations of the quantum fields in the sense that each element of \mathcal{Z}^\times ostensibly is the generalized function that results as a type of operator of multiplication, when such a term can be made precise. The metric on \mathcal{Z}^\times is constructed from a sequence of seminorms.

Definition 4.5.0.12. *A seminorm is a map $\|\cdot\| : \mathcal{Z}^\times \rightarrow [0, \infty)$ such that for Φ and $\Psi \in \mathcal{Z}^\times$ with $a \in \mathbb{C}$,*

$$(i) \quad \|\Phi + \Psi\| \leq \|\Phi\| + \|\Psi\|$$

$$(ii) \quad \|a\Phi\| = |a| \|\Phi\|$$

Additionally, the seminorm is called a norm if also **(iii)** $\|\Phi\| = 0 \Leftrightarrow \Phi = 0$.

Now, choose a family of operators parametrized by the parameter, t (the physical time), \mathcal{O}_t acting on a fixed vector, ψ in the Hilbert space \mathcal{H} , such that the Gel'fand triple or rigged Hilbert space $\mathcal{Z} \subset \mathcal{H} \subset \mathcal{Z}^\times$ holds. The seminorm on \mathcal{Z}^\times is then given as,

$$|\langle \cdot, \mathcal{O}_t \psi \rangle_{\mathcal{D}, \mathcal{H}}| =: \|\cdot\|_{\mathcal{Z}^\times}(t).$$

Notice that the above satisfies the conditions of definition 4.5.0.12. This family parametrized by, t , gives rise to a translational invariant metric on the space of fields as follows. Define a metric between two fields Φ and $\Psi \in \mathcal{Z}^\times$ as,

$$g(\Phi, \Psi)(t) := \|\Phi - \Psi\|_{\mathcal{Z}^\times}(t).$$

This is fully dynamic in the sense that as the interaction of the system changes then so does the operator \mathcal{O} and thus so does the metric $g(\cdot, \cdot)$. In the most general terms this is all that needs to be said without detailed reference to any specific interaction type.

As a final comment it needs to be pointed out that the \mathcal{CPT} involutions are defined as acting on the test functions. Their action onto the fields are defined by the usual rules of the distributional calculus by passing actions on the fields, onto the test functions. That is to say, just as one has for the Fourier transform of a distribution that,

$$\langle \mathcal{F}[T], \phi \rangle = \langle T, \mathcal{F}[\phi] \rangle,$$

the same goes for the involutions \mathcal{C} , \mathcal{P} , and \mathcal{T} , i.e.

$$\langle \mathcal{C}\Psi, \phi \rangle_{\mathcal{D}, \mathcal{S}} := \langle \Psi, \mathcal{C}\phi \rangle_{\mathcal{D}, \mathcal{S}}.$$

This explains why a quantum field as a generalized function need not be manifestly Lorentz invariant itself. Its symmetries are manifest only (a) in its equation of motion and/or Hamiltonian/Lagrangian and (b) through the action on its test functions. This is because one needs the test function space topology to be defined first, that way the space of fields has its topology induced by it.

Finally, the fact that the interaction type itself modifies the test function space was seen in chapter 3 where the definition of the Dirac-Schwartz bracket gave a natural mechanism for reducing the size of the test function space $C_c^\infty(\mathbb{R}^d)$. The canonical example is the photon propagator, which appears as the Coulomb potential in position space or as a generalized Fourier transform in momentum space where the singularity is worsened from order 1 to order 2, i.e. $\frac{1}{k^2}$. Nonetheless, the Dirac-Schwartz bracket gives a well-defined action on all such generalized functions.

Thus one can see the consequences of Haag's theorem [Haa] at work. If one were to define a notion of a free field theory, then any ensuing interactions would be topologically inequivalent making the interaction picture invalid. This is not surprising considering the mathematical requirements for a perturbation theory to be well-defined. When more than one interaction is in play for a given physical system, then it is a natural assumption that the full test function space is given by the tensor product of each of the well-defined test function spaces. This is in direct analogy to the prescription in multi-particle quantum theory.

Chapter 5

Quantum Space-Time

Though a full theory of dynamic space-time cannot be made precise with only the tools developed here, it is nonetheless useful to set some groundwork on the expected attributes of a fully quantum theory of gravitation. Here and throughout, it will be assumed that the strong principle of equivalence is valid to all scales and energies. This of course need not be true but as of current there is insufficient evidence to conclude the contrary. In any case, one is certainly free to suppose it were true. This supposition leads directly to fundamental principle that can be stated in words in such a manner so as to bring together both quantum and classical notions of gravitation, which under typical descriptions appear to be in competition with each other in a sort of pseudo-paradox.

The pseudo-paradox that arises in trying to reconcile the theory of gravitation, i.e. Einstein's fully geometric theory, with that of quantum theory is the apparent contradiction between Einstein's field equations requiring an energy density (energy per unit volume) and the idea that quantum particles are not necessarily local phenomena in the sense that waves are non-local.

The difficulties can be resolved as follows. Introduce the concept of *the strong quantum equivalence principle* as follows.

- The gravitational mass and the inertial mass are one in the same to all energy scales.
- The curvature of space-time and the distribution of energy are directly proportional at all scales.

- Curvature is a physical observable and thus has an associated operator.
- Finally, the distribution of quantum energy is dynamic and the smearing of the fields determines a notion of density. Moreover, interactions produce localizations and thus determine the smeared density.

These four criteria are sufficient to produce a theory that can give rise to predictions that can presumably be tested and either verified or else falsified indicating a modification is needed.

5.1 Dimensionality

In the subsection entitled “The Shadow of Dimensionality” from section 2 “Basic Assumptions” of chapter 1 the Introduction, it was pointed out that there exists real physically observable dimensions that are not integers. This phenomena sometimes also goes by the name of *anomalous dimensions* or *fractal dimensions*. The existence of such things necessitates a notion of *dynamic dimensionality*. Therefore, one may expect that the dimensions are a physical observable. It is then natural that one constructs an associated quantum evolution. Throughout this chapter the following definition is made,

$$\ell := \sqrt{\frac{G\hbar^2}{c^3}},$$

where G is Newton’s gravitational constant, \hbar is Planck’s constant divided by 2π , and c is the speed of light. This quantity shall be referred to as *the Planck length*.

5.1.1 Counting the Dimensions of Space

In the current theory of quantum fields, the gamma function keeps track of the dimensional regularization procedure. This is no coincidence. In fact, the gamma function plays a very special role in the counting of dimensions. To see this, it is first necessary to define a few mathematically technical quantities.

Definition 5.1.1.1 (d-Dimensional Hausdorff Measure). *Given a metric space, i.e. the pair $(\mathcal{M}, g(\cdot, \cdot))$, with \mathcal{A} any subset of \mathcal{M} , and $\mathcal{B}(x_j, r_j)$ a sequence of balls*

centered (respectively) at the points $x_j \in \mathcal{A}$ with radii $r_j > 0$, then the quantity,

$$H^d(\mathcal{A}) = \lim_{\varepsilon \rightarrow 0} \left(\inf \left\{ \sum_{j=1}^{\infty} (r_j)^d \mid \bigcup_{j=1}^{\infty} \mathcal{B}(x_j, r_j) \supseteq \mathcal{A}, r_j < \varepsilon \right\} \right),$$

is called the d -dimensional Hausdorff measure of the set \mathcal{A} , where $\inf \emptyset := \infty$.

This now puts one in a position to understand the notion of Hausdorff dimension.

Definition 5.1.1.2 (Hausdorff Dimension). *Given the d -dimensional Hausdorff measure above, one now defines the Hausdorff dimension as,*

$$\dim_{\text{H}}(\mathcal{M}) := \inf \{d \geq 0 \mid H^d(\mathcal{M}) = 0\}.$$

Notice that these definitions lead to the following consequences.

$$(5.1.1.1) \quad H^d(\mathbb{R}^n) = \begin{cases} 0, & \text{if } n < d \\ 1, & \text{if } n = d \\ \infty, & \text{if } n > d. \end{cases}$$

Likewise, one has the intuitive result,

$$\dim_{\text{H}}(\mathbb{R}^d) = d.$$

One can define the following function that has just the right behavior as indicated by the (outer) measure H^d . Define,

$$H^d(\mathbb{R}^n) =: 0^{d-n}$$

Now an inspection of several definitions of the gamma function is in order. Consider the various definitions for the gamma function given as follows.

$$(5.1.1.2) \quad \Gamma(s) := \int_{\mathbb{R}^+} x^{s-1} e^{-x} dx \quad s > 0$$

$$(5.1.1.3) \quad := \frac{\Gamma(s+1)}{s} \quad \forall s \in \mathbb{C} \setminus \{-\mathbb{N}_0\}$$

$$(5.1.1.4) \quad := \frac{e^{-\gamma s}}{s} \prod_{k=1}^{\infty} \frac{k}{k+s} e^{\frac{s}{k}}$$

where γ is the Euler-Mascheroni constant. The last equation 5.1.1.4 can be rearranged as,

$$e^{\gamma s} \Gamma(s + 1) = \prod_{k=1}^{\infty} \frac{k}{k + s} e^{\frac{s}{k}}.$$

For large values of k in the product the exponential factor is ≈ 1 and so asymptotically this product is very close to,

$$\prod_{k=1}^{\infty} \frac{k}{k + s}.$$

But this product is trivial and hopefully familiar, it is simply,

$$H^{s+n}(\mathbb{R}^n) = 0^s = \prod_{k=1}^{\infty} \frac{k}{k + s},$$

since if s is greater than 0 the denominator will be large and the product will converge to 0. But if s is less than 0 then each term will be greater than 1 and the product will diverge. It is no wonder the gamma function appears in the calculation of d -dimensional volumes and surface areas. It is essentially always present in the measuring of size and dimension.

5.2 Dimension Operators

As was previously mentioned in the introduction, the view of dimensions in this work is a dynamic one in the sense that various physical processes are known to strongly affect the measurement of apparent dimensions. In particular, one might wonder how dimensions can be dynamic. Here it is proposed that dimension is an observable and consequently has an associated quantum operator. If the eigenvalue is d then the operator is named D . If the dimensions are fixed and constant then every measurement measures the number of dimensions being the same with negligible or zero fluctuation. A quantum operator that accomplishes this is given below.

$$D = -\frac{\ell^2}{4} \frac{\partial^2}{\partial \nu^2} + \ell \sqrt{\#} \delta(\nu),$$

where $\ell \sqrt{\#}$ is the coupling constant in front of the Dirac delta measure, and $\nu \in \mathbb{R}$. Then the equation

$$(5.2.0.5) \quad D \vartheta(\nu) = d \vartheta(\nu),$$

will have a single eigenvalue $d = \#$ with a single eigenvector $\vartheta(\nu) \in L^2(\mathbb{R})$. The full quantum theory carries over and many quantities of interest can be calculated. If for some reason in the universe there is an arbitrary number of dimensions of space that are unbounded but all integer then the operator,

$$D = -\frac{\ell^2}{2} \frac{\partial^2}{\partial \nu^2} + \frac{1}{2} \left(\frac{\nu^2}{\ell^2} + 1 \right),$$

returns eigenvalues of any natural number (in its pure states the actual sequential set of natural numbers \mathbb{N}), i.e. $d \in \mathbb{N}$ with pure states $\vartheta(\nu)_d$ given by the Hermite polynomials. Finally, it is worth mentioning that there is a natural way to generate non-integer dimensions as those measured in experiments that observe fractal dimensions. It involves the operator,

$$D = -\frac{\ell^2}{2} \frac{\partial^2}{\partial \nu^2} + V(\nu),$$

where,

$$V(\nu) = \begin{cases} C & -\nu_0 < \nu < \nu_0 \\ 0 & \text{other,} \end{cases}$$

with $C > 0$. This summarizes a basic method for treating dimensional operators within the constructs of conventional quantum theory. There are many more possibilities obviously.

5.3 Gravitation

Lastly, a few remarks are made about just how this view of quantum field theory can and will fit into Einstein's canonical theory of gravity. The first thing to note is that the smearing of the fields against test functions automatically comes with a non-zero volume of space, namely, the set of compact support¹. So particles are no longer just points.

The process is an iterative one and it is envisioned as follows. The smearing of the fields in 3d space gives smooth compact manifolds. The sum of these smeared manifolds is the total stress-energy momentum tensor. The smeared regions change

¹Since the support is a set, it is unclear that this rearrangement of the phrase is any less meaningful or correct.

with the interactions and singular or critical phenomena change the topology of the smeared surfaces. This in turn changes the stress-energy momentum tensor. The procedure is iterative in the following way.

- 1) Compute the smeared regions for a given fixed interacting system.
- 2) Compute the stress-energy tensor as a result of (1).
- 3) Solve the Einstein equations to determine the Einstein metric.
- 4) Use the Einstein metric's 3-space metric to re-compute the smeared fields.
- 5) The above step (4) defines the gravitational correction to the stress-energy tensor.

This iterative procedure gives in principle the full gravitational affects on the fields. The only remaining issue is to consider just how this system gives rise to quantized gravity fields. The idea is rather subtle. The gravitational quanta are the discrete corrections to the stress-energy tensor due to the discrete nature of the non-gravitational fields. Ultimately, the form of all non-gravitational fields must give rise to energy density differences. These differences are due to the non-gravitational interactions and alter both the density and energy content of the stress-energy tensor.

These energy differences in turn come in discrete amounts because the fields are themselves countable, which in turn must alter the space-time curvature tensor accordingly. This is why and how gravitation does not require the quantization of space-time itself but rather is automatically quantized in the form of discrete corrections to the curvature.

Ergo, it is curvature that is quantized and not space-time itself. This is the essence of the strong quantum principle of equivalence, with the added condition that the mass of all (free, possibly asymptotically free fundamental) fields is the same as the gravitational mass that appears in the Newtonian macroscopic equation,

$$\vec{F} = m \vec{a}.$$

In other words, the relation between energy density and space-time curvature is the physical principle that holds all the way down to the smallest length scales. Said another way, it is the underlying physical principle that survives (gravitational) quantization.

So summarizing the view of fields here, the Einstein field equations for gravity below,

$$G^{\mu\nu} = 8\pi G T^{\mu\nu},$$

where G is Newton's constant of gravitation, and where the curvature is built from the torsionless Levi-Civita connection,

$$\Gamma_{\nu\sigma}^{\mu} = \frac{1}{2}g^{\mu\rho} (\partial_{\sigma}g_{\rho\nu} - \partial_{\rho}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\rho}),$$

has its curvature tensor $G^{\mu\nu}$ determined by the stress-energy tensor $T^{\mu\nu}$, which in turn is determined by the particles (smeared fields). The particles are then acted on by the gravitational interaction, in at least so far as the Jacobian (given by minus the determinant of the metric tensor) influences the pairing integrals of the fields with their test functions. This process then closes full cycle. See reference [MisThoWhe] for more details on canonical Einstein gravitation theory.

5.3.1 Energy of Space-Time

It is increasingly possible that due in large part to the observed accelerating expansion of the universe, it may be the case that space-time itself carries energy and that this energy is repulsive with respect to itself. This sometimes goes by the name of dark energy. However, this term need not be invoked for what follows.

The idea is very simple and is built off of those ideas summarized in Misner, Thorne, and Wheeler [MisThoWhe]. Think of space-time as a set of intricate springs that start compressed and are expanding with an accelerating expansion rate. The idea is a metaphor but the conclusion is that space itself contains its own fundamental energy just as Einstein viewed mass as bearing its own fundamental energy in the form of $E = mc^2$.

In what follows, an attempt is made to construct a very simple and succinct way of viewing space-time itself as having energy. There are two types of energy that can arise. The energy that arises from the measure of volumes, or the energy that arises from the separation of two points in space. The formulas given start in 1 dimension and go up to 4 but the general pattern is mostly obvious.

In one dimension, the difference between the separation energy and the volume energy cannot be distinguished obviously because the metric in 1 dimension is the

same as the distance formula. In any case, here are the formulas, where G is again Newton's constant of gravitation, \hbar Planck's constant, and c the speed of light.

$$\begin{aligned} & \underline{1D} \\ E_{\text{sep}} &= \frac{c^4}{G} |x_2 - x_1| \\ E_{\text{vol}} &= E_{\text{sep}} \end{aligned}$$

$$\begin{aligned} & \underline{2D} \\ E_{\text{sep}} &= \frac{c^4}{G} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ E_{\text{vol}} &= \sqrt{\frac{c^{11}}{G^3 \hbar}} \left| \int_{y_1}^{y_2} \int_{x_1}^{x_2} dx dy \right| \end{aligned}$$

$$\begin{aligned} & \underline{3D} \\ E_{\text{sep}} &= \frac{c^4}{G} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ E_{\text{vol}} &= \frac{c^7}{G^2 \hbar} \left| \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} dx dy dz \right| \end{aligned}$$

$$\begin{aligned} & \underline{4D} \\ E_{\text{sep}}^2 &= \sqrt{\frac{c^{15}}{G^3 \hbar}} (t_2 - t_1)^2 - \sqrt{\frac{c^{11}}{G^3 \hbar}} [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \\ E_{\text{vol}} &= \sqrt{\frac{c^{17}}{G^5 \hbar^3}} \left| \int_{t_1}^{t_2} \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \sqrt{-g} dx dy dz dt \right| \end{aligned}$$

where $g < 0$ is the determinant of the metric tensor in its matrix representation.

It seems rather appropriate to end with a possibly interesting connection between the distributional calculus and the energy density.

Definition 5.3.1.1 (Spectrum of a Quantum Field). *The closed set, $\sigma(\Psi) := \text{supp}(\hat{\Psi}) \subseteq \mathbb{C}$ is the spectrum of the field, $\Psi \in \mathcal{Z}^\times$, where $\hat{\Psi}$ is its Fourier transform.*

Let the Einstein energy density be $T^{\mu\nu}$, with $\mu, \nu \in \{0, 1, 2, 3\}$ and $i, j \in \{1, 2, 3\}$, then $T^{ij} = \sum_p \text{supp}(\hat{\phi}_p)$, where $\hat{\phi}_p \in \mathcal{Z}(\mathbb{R}^3)$ and the supports of the $\hat{\phi}_p$ are smooth and compact.

Appendix A

Table of Notation and Symbols

\subset	proper subset	\supset	proper superset
\subseteq	subset or equal	\supseteq	superset or equal
\otimes	the tensor product	\oplus	the direct sum
$B(\cdot, \cdot)$	Beta function	b	anomalous length scale
$C^\infty(\mathbb{R}^d)$	Space of smooth (infinitely differentiable) functions	$C_c^\infty(\mathbb{R}^d)$	space of smooth compactly supported functions
$\mathcal{D}(\mathbb{R}^d)$	space of test functions	$\mathcal{D}'(\mathbb{R}^d)$	topological dual of $\mathcal{D}(\mathbb{R}^d)$
e	charge of the electron	e	Euler's number
γ	Euler-Mascheroni constant	γ^μ	Dirac Matrices
$\Gamma(s)$	Gamma function	$\Gamma(s, x)$	Incomplete Gamma function
G	Newton's gravitational constant	g	anomalous magnetic moment of the electron
$g(\cdot, \cdot)$	metric or distance between 2 points	$g^{\mu\nu}$	Pseudo-metric, Einstein metric, or quadratic form
$J_\nu(x)$	Bessel function	\mathbf{J}	total angular momentum
$K_\nu(x)$	MacDonald function	ℓ	Planck length
$\text{Li}_s(x)$	polylogarithm	$L^p(\mathbb{R}^d)$	p^{th} -integrable Banach space
μ_L	Lebesgue measure	μ_H	Haar measure
\mathbb{N}	set of natural numbers	\mathbb{N}_0	set of natural numbers +zero
$\mathcal{S}(\mathbb{R}^d)$	space of Schwartz functions	$\mathcal{S}'(\mathbb{R}^d)$	the topological dual of $\mathcal{S}(\mathbb{R}^d)$
$u(\cdot), \bar{u}(\cdot)$	fermion/anti-fermion fields	U	an arbitrary open set
$U(a, b, x)$	Tricomi function	$U(t)$	Schrödinger unitary operator

Appendix B

The Mathematical Definitions

In this appendix is a collection of terms that have precise mathematical definitions and meanings. The definitions are collected here to ease the reading of this work for three reasons. The first reason is to let the non-mathematical reader be unobstructed by mathematical technicalities. The second reason is to remove unnecessary content from the body for those already familiar with the terminology. The third and final reason is to have a common reference for those familiar with slightly different versions of precise mathematical definitions. Indeed, it is unfortunate that there is no single mathematical standard for many of the terms used throughout this work. The terms are arranged in alphabetical order. As a consequence, there is the unfortunate fact that certain theorems and/or definitions use terms that are defined after they first appear in the appendix rather than before.

B.1 Schwartz Space

The space of rapidly decreasing integrable infinitely differentiable functions on \mathbb{R}^d .

Definition B.1.0.2 (Schwartz Space). *The Schwartz space or space of rapidly decreasing functions on \mathbb{R}^d is the function space*

$$\mathcal{S}(\mathbb{R}^d) = \{\phi \in C^\infty(\mathbb{R}^d) \mid \|\phi\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta\},$$

where α, β are multi-indices (i.e. $\alpha, \beta \in \mathbb{N}_0^d$), $C^\infty(\mathbb{R}^d)$ is the set of smooth functions

from \mathbb{R}^d to \mathbb{C} , and

$$\|\phi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta \phi(x)|.$$

Here, *sup* denotes the supremum, and again one uses the multi-index notation.

B.2 The Fourier Transform

By the term Fourier transform in this work, there are several meanings that are all interrelated. The most basic version of the Fourier transform is an isometric automorphism (an isomorphism of a space into itself, that is a one-to-one, onto map, i.e. a map whose graph passes both the vertical and horizontal line tests, that maps from a set into itself) on the Schwartz vector space of functions $\mathcal{S}(\mathbb{R}^d)$. One chooses a particular version of the transform so that it is deliberately isometric (preserves the length of vectors). With this choice, the Fourier transform is a unitary operator on $\mathcal{S}(\mathbb{R}^d)$.

Definition B.2.0.3 (The Basic Fourier Transform). *The linear unitary operator denoted, $\mathcal{F}[\cdot]$ on $\mathcal{S}(\mathbb{R}^d)$, such that $\mathcal{F}[\cdot] : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ and its action on an element $\phi \in \mathcal{S}(\mathbb{R}^d)$ is given by:*

$$\mathcal{F}[\phi(x)](k) := \int_{\mathbb{R}^d} \phi(x) e^{-2\pi i k x} =: \hat{\phi}(k),$$

with $x, k \in \mathbb{R}^d$ and $d \in \mathbb{N}$.

Now one can extend the above notion of Fourier transform in a unique way through a version of a theorem given by Hahn and Banach described below.

Definition B.2.0.4 (The Extended Fourier Transform). *By a version of the Hahn-Banach theorem, one extends uniquely the above basic Fourier transform to the vector space of functions $L^2(\mathbb{R}^d)$, so that,*

$$\mathcal{F}[\cdot]_e := \mathcal{F}[\cdot] : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

is an isometric automorphism on the Hilbert space of functions $L^2(\mathbb{R}^d)$. The extension is accomplished by completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, which is not a Hilbert

space, under the unique Hilbert norm defined through the unique Hilbert inner product on $L^2(\mathbb{R}^d)$, given by

$$\left\{ \int_{\mathbb{R}^d} \psi(x) \phi(x) \right\}^{\frac{1}{2}} < \infty, \quad \forall \phi, \psi \in L^2(\mathbb{R}^d).$$

This then gives the precise topology under which one regards the automorphism, $\mathcal{F}[\cdot]_e$, as continuous.

Appendix C

List of Some Delta Sequences

Below is a list of some standard delta sequences, as well as some that are more obscure.

Recall that a sequence $\delta_n(x)$ is a delta sequence if,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \delta_n(x) f(x) = f(0).$$

All of the following below are delta sequences, $\delta_n(x)$, for $x \in \mathbb{R}$. With suitable changes in the normalization, the identities can be extended to $x \in \mathbb{R}^d$ rather trivially.

$$\begin{aligned} \text{(C.0.0.1)} \quad \delta_n(x) &= \frac{1}{2} n e^{-n|x|} \\ \text{(C.0.0.2)} \quad &= n e^{-\pi n^2 x^2} \\ \text{(C.0.0.3)} \quad &= \frac{-n}{\text{Li}_2(-n)} \log(1 + n e^{-2n|x|}) \\ \text{(C.0.0.4)} \quad &= \frac{\sin(2\pi n x)}{\pi x} \\ \text{(C.0.0.5)} \quad &= \begin{cases} n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & \text{otherwise} \end{cases} \\ \text{(C.0.0.6)} \quad &= \begin{cases} n - |x|, & -\frac{1}{n} < x < \frac{1}{n} \\ 0, & \text{otherwise} \end{cases} \\ \text{(C.0.0.7)} \quad &= \frac{\sin((2n+1)\pi x)}{\sin(\pi x)} \\ \text{(C.0.0.8)} \quad &= \frac{n J_2(e^{-n|x|})}{1 - 2 J_1(1)} \\ \text{(C.0.0.9)} \quad &= \begin{cases} \frac{e n}{\sqrt{\pi} U(\frac{1}{2}, 0, 1)} e^{\frac{-1}{1-n^2 x^2}}, & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

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