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There is a wide choice of algebraic forms which can be used to represent and estimate the production function (1,2). Once a given algebraic form is chosen, certain key parameters are then estimated to determine the empirical functional relationship between the factor inputs and value-added. The algebraic forms used in current econometric studies usually imply highly restrictive economic assumptions. The Cobb-Douglas for instance estimates distribution parameters from which can be deduced a scale parameter (3), but it assumes unit elasticity of factor substitution. This later point was generalized by the constant elasticity of substitution production function (CES) (4). Hybrides have also been presented (5). The homothetic isoquant production function (HIPF) is less restrictive than the algebraic forms used hitherto.

The objects of the paper are to describe the advantages of the HIPF, to derive an explicit algebraic form, to estimate this form for the U.S. private non-farm domestic economy over the period 1929-53, and to interpret the results and compare them with some results already published in the field.

For the HIPF the elasticity of factor substitution is constant for all isoquants along a ray from the origin and is not necessarily constant along one isoquant. The only a priori assumption involved in

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the estimation is homotheticity. This concept implies that, in the two-factor plane, along a ray from the origin the slopes of the isoquants (pertaining to a given isoquant map) are unique and have identical numerical values.

The general form of the HIPF can be written as:

$V = F(f(K,L)) = F(z)$,

where F is monotonic in $f(K,L) = z$, and f is homogeneous of first degree in K and L. This functional relationship is devised to fulfill the following purposes:

1. Determine the type of profile exhibited by the production surface V. Choosing a form for $V = F(z)$ enables us to estimate a returns to scale parameter which may take different values at different output levels. The curvature of the production surface V indicates the type of returns to scale. Introducing time we can estimate the shift in the surface V or a technological change parameter. More on this later.

2. Determine the shape of the production surface contours by deriving the standard or cannical isoquant. More on this later.

The slope of the isoquant is:

(1) $-(dK/dL) = (w/r) = (wL/rK)(K/L) = MRS_{LK}$,

where the slope is the marginal rate of substitution of capital for labor (MRS_{LK}), wL and rK are the shares in the output produced attributed to labor and capital respectively, w is the wage rate and r the rate of return on capital. The slope of an isoquant along a given ray from the origin (MRS_{LK}) is a function of the slope of that ray, i.e. of the factor proportion (K/L), such that:

(1a) $-(dK/dL) = -\psi(K/L) = (\partial f/\partial L)/(\partial f/\partial K)$.

The ψ function gives full information about the shape of the isoquant. This will be clarified in the course of the paper. The assumption of homotheticity implies that any external economies or diseconomies that arise must be "neutral" in character. In other words, a proportional increase or decrease of all inputs should not affect the marginal rate of factor substitution along the isoquants.

For the sake of empirical estimation we shall transform the differential equation (1a) into a more convenient form. Since $z = f(K,L)$, is homogeneous of first degree in K and L, we have:

(2) $z = Lf(K/L, 1) = L\bar{f}(x)$,

where $x = K/L$. Now from (1a),

(3) $\psi(x) = dK/dL = -(\partial z/\partial L)/(\partial z/\partial K) = x - (\bar{f}(x)/\bar{f}'(x))$*.

Furthermore, let

(4) $\phi(x) = 1/(x-\psi(x)) = \bar{f}'(x)/\bar{f}(x) = \frac{d}{dx}\log\bar{f}(x) = \frac{d}{dx}\log(z/L)$,

whence

(5) $d\log(z/L) = \phi(x)dx$,

which has solution

(6) $z = L\,e^{\int\phi(x)dx}$.

This procedure is valuable for the empirical computation since:

(7) $\phi(x) = 1/(x-\psi(x)) = 1/((K/L) - (dK/dL))$,

$$= 1/((K/L) + (wL/rK)\ (K/L)) = 1/((1+(wL/rK))(K/L)$$

can be calculated for each sample point.

In equation (6) $\int\phi(x)dx$ can be solved by the trapezoidal rule or any numerical method of integration, but for practical reasons it is

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* Since from (2) $\partial z/\partial L = \bar{f}(x) + L\bar{f}'(x)(\partial x/\partial L) = \bar{f}(x) - (K/L)\bar{f}'(x)$, and $\partial z/\partial K = \bar{f}'(x)$..

preferable to approximate the function by polynomial curve fitting. So equation (6) becomes:[2]

(8) $z = L e^{\int \phi(x)dx} \approx L e^{\sum_{i=0}^{s} a_i (K/L)^i}$

Based on observed values the fitting of a polynomial curve

(8a) $\int \phi(x)dx \approx a_0 + a_1 x + a_2 x^2 + \dots + a_s x^s$

gives the best fit for a time series of $x = K/L$ and $\phi(x)$. Amongst the family of curves of various degrees, the one which maximizes correlation is chosen. On the basis of the set of the estimated a_i coefficients, the polynomial evaluation gives a series of estimated $\hat{\phi}(x)$ values. We can now proceed to calculate from (8) a series for z based on the $\hat{\phi}(x)$ series and the series for L. Having computed a series of z values we obtain the HIPF itself.

In this paper the following form of F(z) is used:

(9) $V = z^{\lambda}$,

where $\lambda = (dV/dz)(z/V)$ indicates the type of returns to scale. Since V can be observed and z calculated, (9) can be fitted and an estimate of λ obtained.

Let us review the determination of the various parameters involved in the estimation.

Technological Change and Returns to Scale.

If we postulate a technical progress index $C e^{\gamma t}$, then by regression of:

(10) $V = C e^{\gamma t} z^{\lambda}$,

in the form:

(11) $\log V = \log C + \gamma t + \lambda \log z$,

we obtain the technological change γ and returns to scale λ parameters respectively.

Elasticity of Factor Substitution.

The elasticity of factor substitution σ can be derived (9,p.341) as follows:

$$(12)\quad \sigma = \frac{d(K/L)/(K/L)}{d(dK/dL)/(dK/dL)} = \frac{d(\log K/L)}{d\log|dK/dL|} .$$

Hence, if log(K/L) is expressed as a function of $\log|dK/dL|$, σ is the rate of change of the former with respect to the latter. Usually, σ is treated as constant over the observations, which implies that σ is constrained to obey:

$$(12a)\quad \log(K/L) = a + \sigma \log|dK/dL| ,$$

where a is a constant. This is not so for the HIPF where σ can vary over the observations.

From (7) it follows that:

$$(12b)\quad \psi(x) = x - (1/\phi(x)).$$

Thus:

$$(12c)\quad \psi'(x) = 1 + (\phi'(x)/\phi^2(x)).$$

In terms of (1a), (12) becomes:

$$(12d)\quad \sigma = \psi/x\psi'.$$

Substituting (12c) in (12d) gives:

$$(12e)\quad \sigma = \frac{x\phi^2(x) - \phi(x)}{x\phi^2(x) + x\phi'(x)} .$$

In his review of Minhas' book, Leontief showed that it is difficult to identify which industry is capital intensive and which is labor-intensive if the CES production function is used. For the CES, the factor-intensity, (K/L) and the factor-price ratio, (w/r), maintain a log-linear relation for each industry. Since two straight lines, unless parallel to each other by coincidence, are bound to intersect, the CES estimation procedure loads the dice against unambiguous factor intensities. On the other hand, if one adopts the HIPF estimates, the σ coefficient is free to vary at different (K/L) values, this awkward necessity of crossing-over, need not occur, i.e. there is no inherent bias towards ambiguity. In view of the fact that the unambiguous factor intensity assumption plays an important role in many theorems of trade and development it is desirable to have an unbiased test for such a crucial assumption.

Distributive Shares.

The distributive shares or share in the output (V) accruing to the factors of production are derived theoretically as follows: $\partial F/\partial K = \partial F(f(K,L))/\partial K$ and $\partial F/\partial L = \partial F(f(K,L))/\partial L; (\partial F/\partial L)/(\partial F/\partial K) =$ w/r and $wL/(wL + rK) = \epsilon =$ labor's share, $1 - \epsilon =$ capital's share. ϵ can be calculated for each observation.

We can thus determine five parameters: technological change, returns to scale, elasticity of factor substitution, distributive shares, by estimation and theoretical derivation.

The Standard or Canonical Isoquant.

Given homothetic isoquants every isoquant can be derived from any other by appropriate scaling up or down, so that the whole map can be represented by a single isoquant. This definition follows from the following points:

1: In $V = F(f(K,L))$, $f(K,L)$ is a linear homogeneous production function (LHPF):

2. The HIPF shares with the LHPF the homotheticity of the isoquants.

3. Every HIPF is a monotonic transformation of some LHPF.

Since for a given isoquant map the isoquants are homothetic to each other, any isoquant pertaining to that map is a blow-up or scale-down of some other ones. The standard isoquant can be made to represent full efficiency. A comparison between any other isoquant and the standard isoquant for given output would indicate the departure from full efficiency.

Mr. M.J. Farrell (11) has proposed a method of measuring productive efficiency which uses an "efficient isoquant" estimated as part of the convex hull of the observed points; the same method could be applied using the HIPF standard isoquant to derive a measure of efficiency. The relation between Farrell's production function and the HIPF is, quite apart from the different estimation procedure, not simple. In the application in (11) to a cross-section of U.S. states' agricultural production, he assumes constant returns to scale, which makes his function a special case of the HIPF. On the other hand, Farrell and Fieldhouse (12) have applied the same method to agricultural units in England, without assuming constant returns; this function is in some ways more general than the HIPF, since it need not satisfy the homotheticity requirements, and in some ways less so since some complicated convexity assumptions are made. Certainly their work examplifies the range of uses of the HIPF.

Application of the HIPF to Market Conditions of Differentiated Competition.[3]

Traditionally, the estimation of production functions has been based on a framework of perfect competition. In view of the form of the functions used, these assumptions were necessary. The HIPF applies to market conditions of differentiated competition or monopolistic competition.

Figure 1 examplifies the firm and industry equilibrium,[4] dd is the demand for the firm, DD the industry demand. The industry comprises n identical firms. Since we deal with an industry under conditions of differentiated competition the value of output observable is the value of sales.

Value added = V = rK + wL = sK + wL + (S/E)

where sK is the "true rent" on capital and (S/E) the monopolistic margin is the ratio of sales to the elasticity of demand.

On Figure 1, OABC is the value of sales: S $\doteq$ pQ = V + cost of raw materials:

(13) S = (rK + wL) + (S-V) = wL + sK + (S/E) + (S-V)

where Q is industry output.

To derive the HIPF we need to solve (6); therefore we need calculated values for $\phi(x)$ from (7) where as with differentiated competition we have:

(14) $w/s = M_{LP}/M_{KP} = M_{LS}/M_{KS} = (wL/sK)(K/L)$

where for convenience we have assumed constant returns to scale so that the marginal private physical product (M_P) equals the marginal social physical product (M_S) (3).

sK the $M_{KP} \doteq M_{KS}$ or "true rent" on capital, can be estimated as follows:

(15) sK = rK + (S/E) = (V-wL) + (S/E).

The monopolistic margin (S/E) is derived as follows (see Figure 1):

(16) $Q(p-MC) = Qp - Q(d(pQ)/dQ)$

(17) $\qquad = Qp - (Qp + Q^2(dp/dQ)) \doteq -(Qp/E) \doteq -(S/E)$

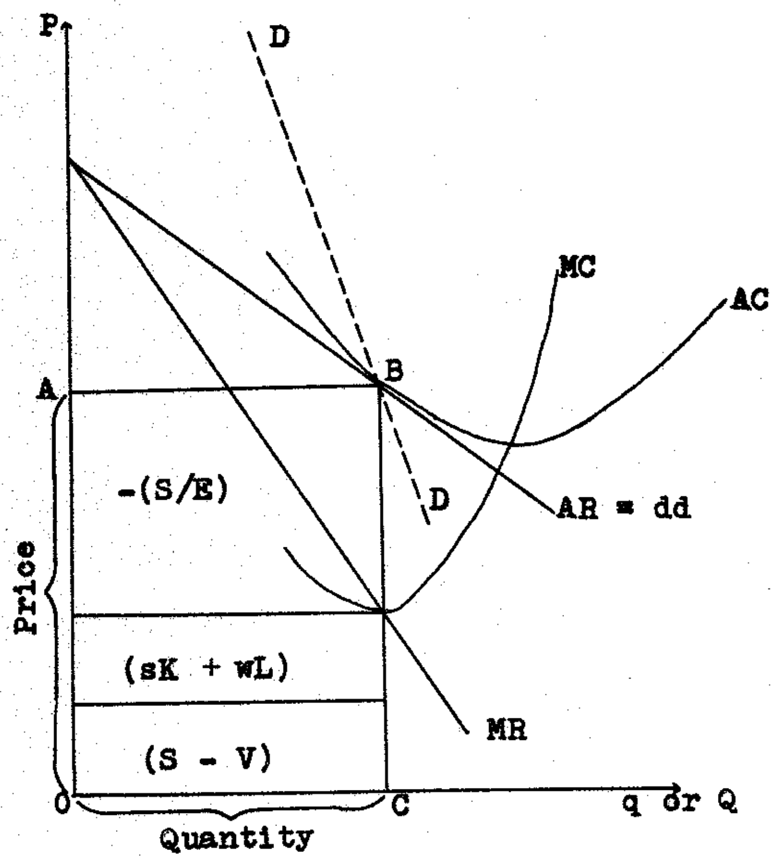

Figure 1

where MC is marginal cost, p is the price, and E can be approximated on the basis of sample surveys.

So far we have assumed that the entire monopolistic margin was absorbed by the entrepreneurs. If due to trade union pressures, for instance, part of this margin is absorbed by labor then our analysis requires an explicit knowledge of the size of labor's share in the margin. Theoetically, the case of administered pricing can also be solved knowing the markup ratio: $(1-(Mc/p))$. Under conditions of differentiated competition as well as perfect competition the HIPF can be solved simply on the basis of (7). This application of the HIPF is advantageous particularly for a simple industry which cannot be rid of the materials component. This approach would refute the objection that: "An aggregate production function is never related to a material component"(13).

The HIPF and Other Production Functions.

The HIPF is a fairly general formulation of the input-output relationships relevant to the production process. It covers as special cases the straight-line, Cobb-Douglas and generally the CES production functions. If in (6) we replace $\phi(x)$ by its value in equation (7) we obtain:

$$(18)\quad z = L\, e^{\int d(K/L)/[(K/L) - \psi(K/L)]}$$

Since the forms taken by the above mentioned functions are well-known we can easily derive mathematically the value of the MRS_{LK} and of $\psi(K/L)$ as in (1) and (1a) respectively. Substituting in (18) we can see that if the data so indicates an estimation based on the HIPF would be adequate.

Since the straight-line and Cobb-Douglas production functions are special cases[5] of the CES, it suffices to show that the results hold for the CES.

<u>Constant Elasticity of Substitution Production Functions</u>.

The general class of CES production functions takes the form (4):

$$V = (\delta K^{-\rho} + (1-\delta)L^{-\rho})^{-1/\rho}$$

where the elasticity of substitution $\sigma = 1/(1+\rho)$. The MRS_{LK} is:

(19) $dK/dL = -((1-\delta)/\delta)(K/L)^{1+\rho} = \psi(K/L)$

Then:

(20) $$\int_{z=Le} \frac{d(K/L)}{\left(\left(\frac{1-\delta}{\delta}\right)\left(\frac{K}{L}\right)^{1+\rho} + \left(\frac{K}{L}\right)\right)} \quad \frac{1}{\rho}\log\frac{(K/L)^{\rho}}{\left(\left(\frac{1-\delta}{\delta}\right)\left(\frac{K}{L}\right)^{\rho} + 1\right)} + k$$

$$= L\,e$$

or:

(21) $$z = L\left\{\frac{(K/L)^{\rho}}{\left(\frac{1-\delta}{\delta}\right)\left(\frac{K}{L}\right)^{\rho} + 1}\right\}^{1/\rho} e^{k}.$$

Equation (21) reduces to:

(22) $z = (((1-\delta)/\delta)L^{-\rho} + K^{-\rho})^{-1/\rho} e^{k}$.

Setting $e^{k} = \delta^{-1/\rho}$ equation (21) reduces to:

$$z = (\delta K^{-\rho} + (1-\delta)L^{-\rho})^{-1/\rho}.$$

We see that if the function to be estimated is of the CES type, the HIPF estimates it correctly up to a constant.

There is one important feature of the CES production function which is brought into light by considering equation (22). The so-called distribution parameter δ must work jointly with the elasticity of factor substitution parameter ρ and the factor intensity ratio (K/L) to decide the factor shares. A CES production function cannot be completely specified by knowing ρ and δ alone. The units to be taken for V, K and L must also be specified. In other words δ values of the CES are not input unit free. This can be seen as follows:

$$(23)\quad K(\partial V/\partial K)/V = (\partial V/\partial K)/(V/K) = \delta/(\delta+(1-\delta)(K/L)^{\rho})$$

which gives the share of capital under marginal productivity factor pricing. When $\rho = 0$, the Cobb-Douglas case, the denominator becomes independent of (K/L) and goes to one. Only in the Cobb-Douglas case do the relative shares equal the elasticity of output vis-à-vis input. In the general CES case, the ratio $(K/L)^{\rho}$ is not independent of the units of measurement, and therefore δ is not. This shows that too much emphasis should not be put on δ as a structural parameter. This feature becomes particularly relevant when international comparisons are made, since then the problem of units arises specially.

As far as the forms of the functions mentioned above are concerned a priori we can plot $|\psi(K/L)|$ against (K/L) to get an idea of the type of function involved as shown in Figure 2a. Figure 2b traces the relationship between these functions and their corresponding unit-output isoquants as set out in Table 1.

Production Functions Which are HIPF but not CES.

The following production functions are LHPF but not CES (14):

$$(24)\quad z = L\log((K/L)+1).$$

The MRS_{LK} is:

$$(25)\quad \frac{dK}{dL} = \frac{-f_L}{f_K} = \frac{K}{L} - \left(\frac{K}{L}+1\right)\log\left(\frac{K}{L}+1\right)$$

$$(26)\quad z = .05K + L\log((K/L)+1).$$

The MRS_{LK} is:

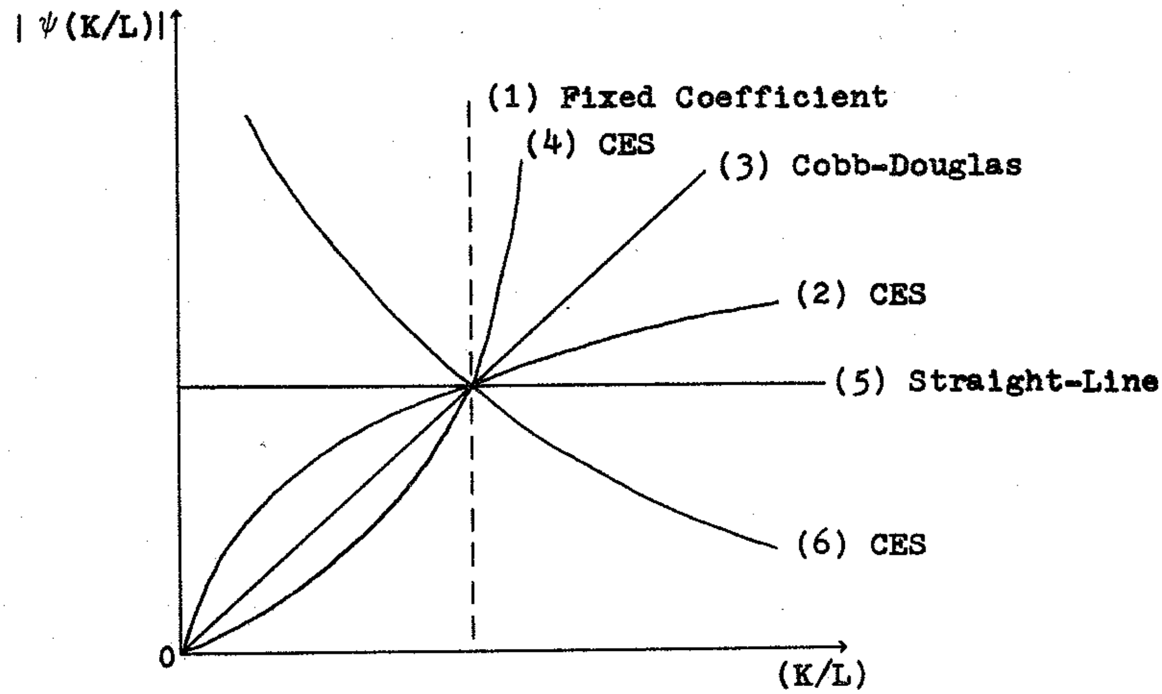

Figure 2a

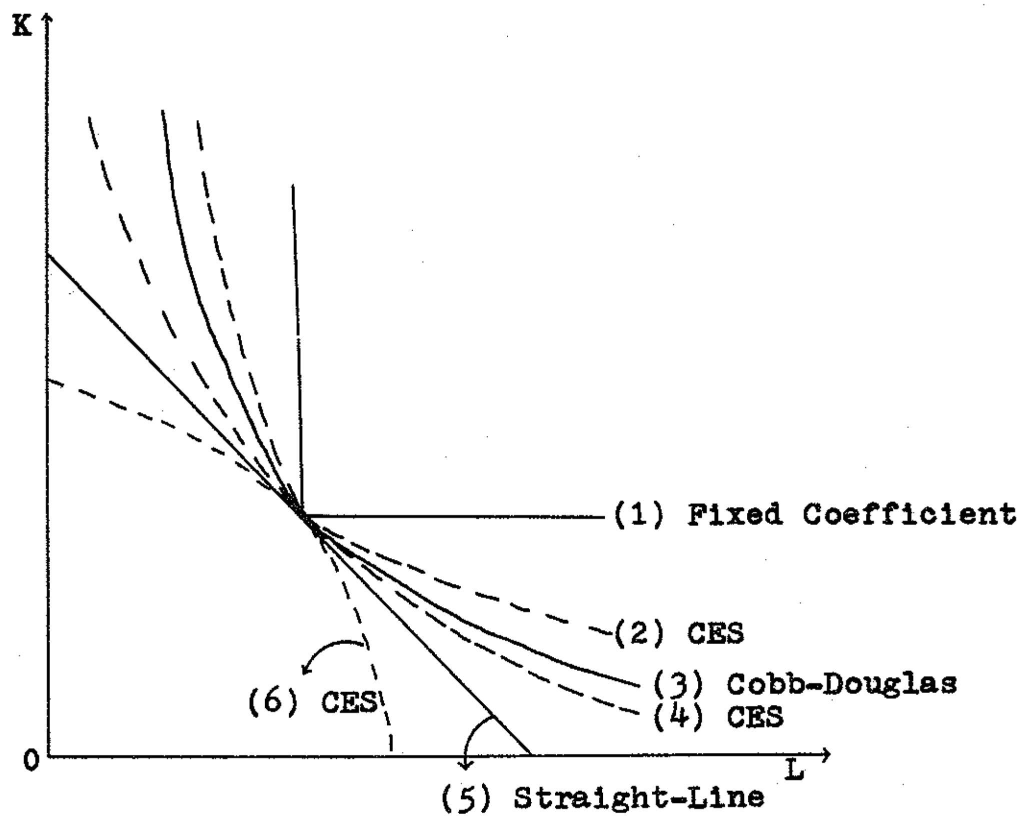

Figure 2b

Table 1

Values of ρ	Values of $\sigma = 1/1+\rho$	Types of Function	Loci
$\rho = \infty$	$\sigma = 0$	Fixed Coefficient	(1)
$\infty > \rho > 0$	$1 > \sigma > 0$	CES	(2)
$\rho = 0$	$\sigma = 1$	Cobb-Douglas	(3)
$-1 < \rho < 0$	$\infty > \sigma > 1$	CES	(4)
$\rho = -1$	$\sigma = \infty$	Straight-Line	(5)
$\rho < -1$	$-\infty < \sigma < 0$	CES	(6)

$$(27)\ \frac{dK}{dL} = \frac{-f_L}{f_K} = \frac{\frac{K}{L} - (\frac{K}{L}+1)\ \log(\frac{K}{L}+1)}{.05\ \frac{K}{L}+1.05}\ .$$

These functions are LHPF since in each case dK/dL is zero homogeneous in K, L. These functions are not CES since (z/L) and ($\partial z/\partial L$) do not exhibit any log-linear relationship.

Test of the HIPF.

1. The production surface profile.

Whatever production function is used to approximate the input-output relationship the problem of reliability of the statistical data remains. This issue is further complicated by the conceptual aspects of the type of data desirable. Much has already been written on the subject and space does not allow us to elaborate here. So we shall proceed with the empirical estimation of the HIPF for the United States private non-farm domestic economy over the period 1929-1953. The data used for this purpose are given and briefly described in Table 3. The results of testing the HIPF are given in Table 2. The polynomial curve fitting is done following an IBM program using an orthogonal polynomial (15, Section 8,4). For one or two estimations we have cross-checked this program by calculating the powers of x in (8a) and finding the best linear regression. The results by both methods were identical for all practical purposes. Since we shall see later that the estimate to be preferred is the third one, Figure 3 gives a graph[6] of the polynomial for this estimation.[7]

We have tried three different estimates of capital stock since the problem of adjusting capital stock for capacity has been a pernicious one. Several major studies deal with the adjustment of

Table 2

Test of the HIPF

Polynomial Curve Fitting*			Type of K Used	Regression of $V = Ce^{\gamma t}z^{\lambda}$, in the form $\log_e V = \log_e C + \gamma t + \lambda \log_e z$			
Degree	Residual Sum of Squares	SE		$\log_e C$	γ	λ	R^2
6	.0018	.0101	K_1	-1.0486 (.0320)	.0244 (.0014)	1.1332 (.0543)	.9926
5	.0036	.0139	K_2	-1.1094 (.0099)	.0244 (.0022)	1.1511 (.0885)	.9686
3	.0005	.0048	K_3	- .8569 (.0068)	.0231 (.0015)	1.1211 (.0573)	.9849

*The maximum degree fitted was 11.

In order to check whether it is worthwhile to increase the degree of the polynomial, for the range of degrees considered, a t test on each coefficient establishes whether the coefficient is significantly different from zero. The significance level is one percent (or better).

The unbiased standard error (SE) is the square root of the ratio of the residual sum of squares over the number of degrees of freedom (i.e. the number of observations less the number of unknown coefficients in the polynomial).

The distributive shares are in all cases: average ϵ = .5811 and average $(1-\epsilon)$ = .4189.

The estimations of the distributive shares are obtained as follows: for each observation, a value for ϵ is calculated and the mean of all calculated ϵ's is taken as average factor share.

Values for σ are as follows: for the first, second and third estimations σ is .5801, 1.2841 and .8494 respectively. Estimations for σ were obtained on a yearly basis and then averaged over the whole period. These estimations are obviously sensitive to the type of capital data used.

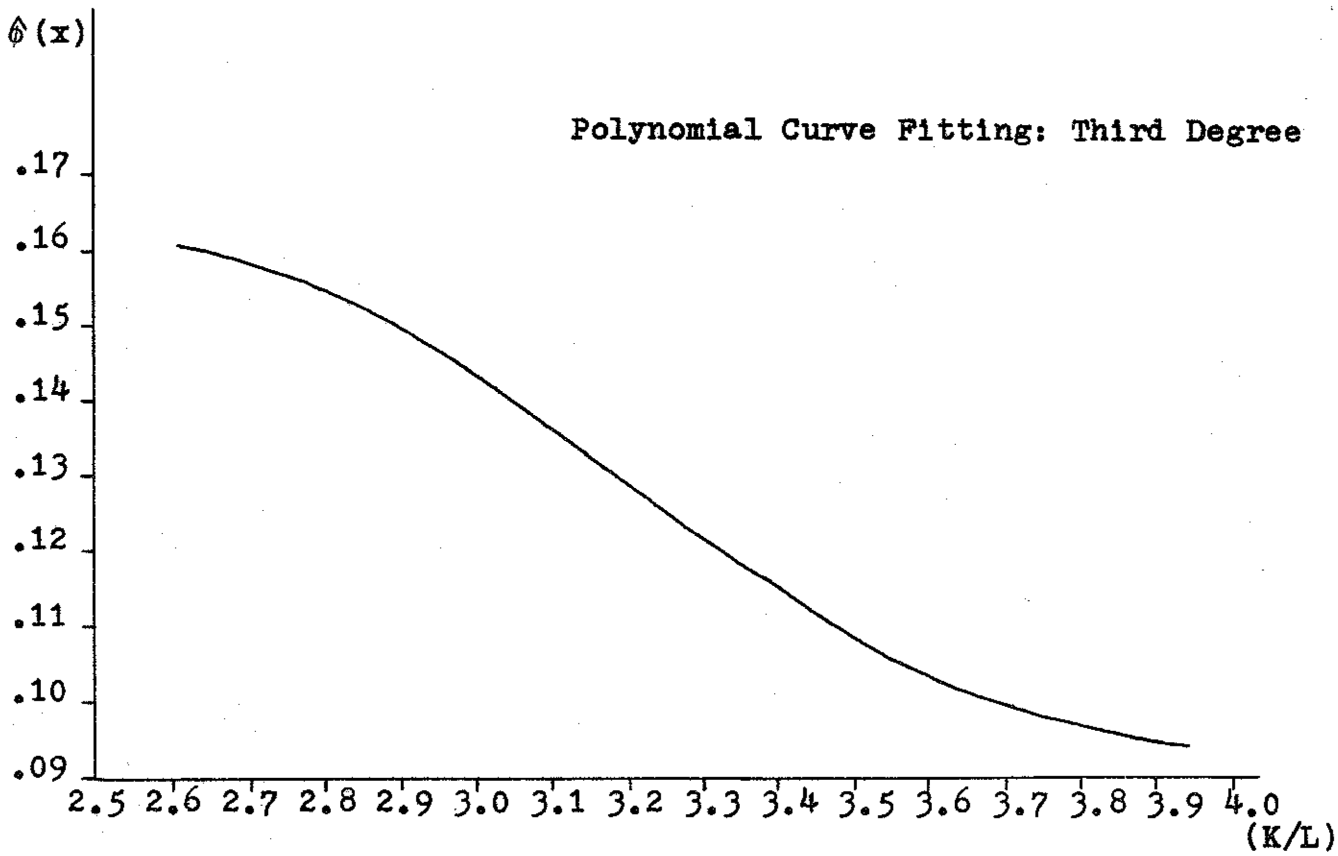
ϕ(x)
Polynomial Curve Fitting: Third Degree
.17
.16
.15
.14
.13
.12
.11
.10
.09
2.5 2.6 2.7 2.8 2.9 3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9 4.0
(K/L)

Figure 3

Table 3

Test of the HIPF Production Function
(Data for the U.S. Private Non-Farm
Economy, 1929-1953)

Year	(1) V	(2) K_1	(3) K_2	(4) K_3	(5) L	(6) wL
1929	79.330	150.096	162.266	257.432	89.467	45.991
1930	70.757	147.075	148.895	255.647	81.854	43.033
1931	64.279	130.628	135.355	239.308	72.386	39.485
1932	52.384	110.103	107.070	218.125	62.069	33.124
1933	49.816	133.549	97.633	240.198	61.248	31.444
1934	57.090	106.985	108.600	212.239	62.366	34.299
1935	66.137	108.571	123.994	212.875	66.023	37.109
1936	74.901	115.300	140.836	219.400	73.426	41.878
1937	82.261	123.230	152.137	227.612	77.568	46.590
1938	75.105	114.429	139.857	219.220	70.460	43.005
1939	82.756	117.651	152.398	223.326	75.131	47.291
1940	92.303	124.233	154.136	231.355	79.694	50.956
1941	106.532	137.770	158.722	246.574	89.276	58.939
1942	116.396	152.012	162.407	261.544	87.056	66.405
1943	124.646	158.143	162.365	266.789	101.633	73.040
1944	133.887	158.235	160.808	265.455	100.124	76.137
1945	132.223	153.833	158.217	259.680	94.920	72.978
1946	128.915	154.941	157.376	260.695	96.671	73.200
1947	128.908	164.524	166.042	271.879	100.072	76.208
1948	135.459	173.851	181.663	283.774	101.304	78.696
1949	131.806	176.454	184.613	289.334	96.784	77.408
1950	147.207	188.044	197.733	305.006	100.352	83.824
1951	156.145	202.702	208.756	324.122	104.801	89.603
1952	162.155	213.138	218.828	338.136	106.168	93.926
1953	168.629	222.095	227.790	350.699	109.195	100.647

SOURCES: (1) J.W.Kendrick: Productivity Trends in the U.S., Princeton, Princeton University Press, 1961; Table A-III, p.298, 1929 prices.

(2) Idem, Table A-XV, pp.320-22. Capital stock (K) (private non-farm non-residential) is adjusted for capacity by the percentage of the labor force employed (from U.S. Economic Almanac, 1953-54 and 1956). 1929 prices.

(3) Idem, capital stock is adjusted for capacity as described in the core of this paper.

(4) Idem, Table A-XV, pp.320-22. Capital stock private non-farm economy. Private non-residential capital is adjusted for capacity by the percentage of the labor force employed as in (2). Residential capital is not adjusted for capacity.

(5) Idem, Table A-X, pp.312-13, "...all classes of workers are included in the estimates of manhours: proprietors and self-employed, unpaid family workers, and employees of all categories including non-production as well as production workers."

(6) U.S. Income and Output, 1958, Table I-10, pp.134-35. rK = V - wL.

capital stock data for capacity.[8] The trouble is that they disagree on the basic concept of capacity. Furthermore the indexes of capacity proposed do not cover extended periods of time. Our first estimate of the HIPF is based on data for capital stock adjusted for capacity by the percentage of the labor force employed. This method has been criticized on various grounds, the major argument being against its implication of fixed factor proportion to some extent.

Our second estimation of the HIPF tried to adjust capital stock for capacity as follows:

(28) $K = \bar{K}(V/\bar{V})$,

where $\bar{K}$ is capital stock observed, K is capital stock adjusted for capacity, V is observed output, $\bar{V}$ is output at full capacity or peak output. The method we apply here is a modified version of the Wharton School measurement of capacity (16). We take $\bar{V}$ peak output as a 100. The time period 1929-1953 comprises as peaks the years 1929, 1937, 1944 and 1948. All expansion years (above the preceding peak) are taken as 100. The reason for this is that the behavior of capacity during expansion years cannot be specified a priori. The downturn and trough years are adjusted on the preceding peak where $\bar{V} = 100$. Of course, to take peak output as a 100 implies full use of capacity at the peak. This in itself is questionable since even at the peak there can be excess capacity. The concept used here therefore is really maximum attained output. It would imply fixed capital output ratio over the downswing. Neither of these two adjustments for capacity is satisfactory and are only used here <u>faute-de-mieux</u>. We observe in Table 2 that regardless of the adjustment used for the capital data the results are fairly good and do not vary significantly from one estimation to another.

The test of these production functions estimates against actual output is fairly good. For the first estimation out of 25 years the prediction error is less than 4 percent of the predicted value in 22 years (much less in many years) and not more than 6 percent in the remaining three years. Testing for the presence of autocorrelated disturbances the Durbin-Watson statistic is 1.1296, which is smaller than the lower bound as tabulated by Durbin and Watson. For the second estimation, out of 25 years the predicted error is not more than 5 percent of the predicted value in 22 years, the error for the other years is 6, 7, and 13 percent. The Durbin-Watson statistic is 1.0815. Finally, for the third estimate, out of 25 years the predicted error is less than 5 percent of the predicted value in all years but one where it is 5 percent. The Durbin-Watson statistic is 1.1528.

The Durbin-Watson statistics suggest that autocorrelated disturbances are present. From this point of view the best estimate is our third approximation. We shall therefore apply the generalized least squares to this estimate. For this purpose we proceed with a two-stage estimation procedure with autoregressive transformation coefficient of 1.4235 (17, 18, 19: Chapter 7). The regression of $V = C\ e^{\gamma t} z^{\lambda}$ with transformed variables in logarithmic form gives:

$$\log_e V = \log_e \underset{(.0030)}{-\ .7437} + \underset{(.0032)}{.0234t} + \underset{(.0990)}{1.1124} \log_e z$$

with $R^2 = .9831$. The Durbin-Watson statistic is now 1.4943 which is a marked improvement on the first estimate without adjustments. The adjusted data now show that there is no significant serial correlation at the 5 percent level on a one tail test or at the 10 percent level on the two tail test with h = 2. The variable h is equal to the number of independent variables. To read its table, n the number of observations must also be taken into account.

Other Functions' Tests.

Data for the U.S. non-farm output have been tested for the Cobb-Douglas and CES production functions. Other interesting results can be found in the literature (20, 21). Testing the Cobb-Douglas production function, $V = A K^a L^b e^{\gamma t}$ for the U.S. private non-farm economy (1909-1949) A.A. Walters (22) obtains the following results:

$\hat{a}$	$\hat{b}$	$\hat{a}+\hat{b}$	R^2	von Neumann's ratio
.227 (.123)	.993 (.125)	1.220 (.095)	.963	2.120

The CES production function: $V = \gamma(\delta K^{-\rho} + (1-\delta)L^{-\rho})^{-1/\rho}$ (with $\gamma = \gamma_0(10)^{\lambda t}$ the technological change parameter) for the U.S. non-farm output for the period 1929-49, is given in (4, p. 245, eq(37)) as:

$$V = .584(1.0183)^t (.519K^{-.756} + .481L^{-.756})^{-1.322}.$$

2. The Production Surface Contour.

A simple way to derive the standard or canonical isoquant is by the Cauchy-Lipschitz approximation (23). For our third estimate such an isoquant is shown on Figure 4.

Conclusions.

The class of homothetic isoquant production function (HIPF) is fairly general; the only assumption is homotheticity. This concept implies that along a ray from the origin crossing the isoquant map all the isoquants' slopes would be equal. Any homogeneous production function is homothetic but the reverse does not necessarily hold true.

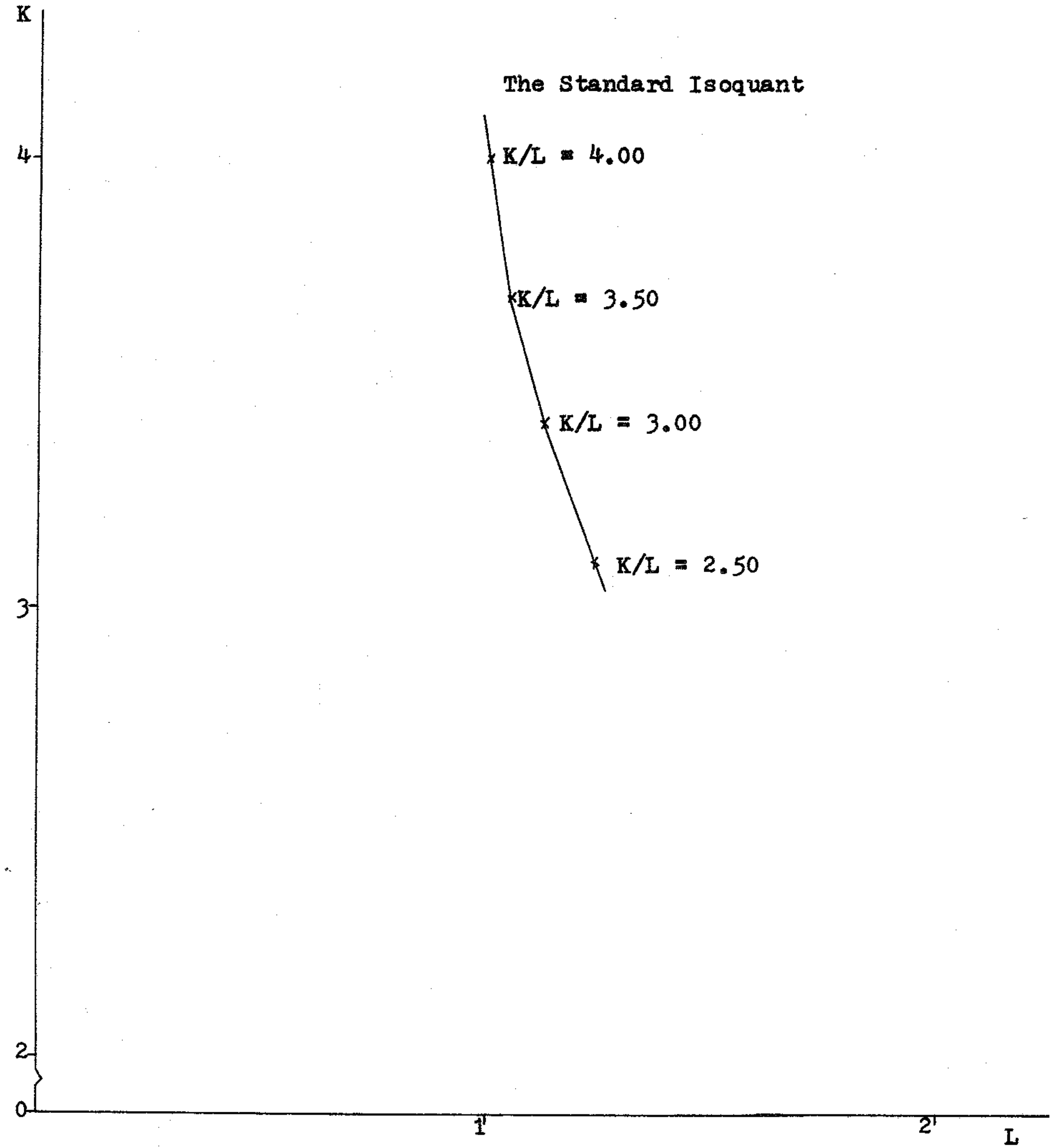

Figure 4

The mathematical formulation of this function is simple and the function can be estimated empirically on the basis of either polynomial curve fitting or numerical integration, a speedy task thanks to electronic computers. The parameters which can be estimated are technological change, returns to scale, the elasticity of factor substitution and the factors' shares in the product distribution.

If the data should so indicate the empirical fit of the HIPF would estimate adequately a straight line, Cobb-Douglas, constant elasticity of substitution production function or whatever form is relevant.

The HIPF is applicable under market conditions of differentiated competition as well as perfect competition. Tests of the HIPF for the U.S. private non-farm economy over the period 1929-1953 give interesting results within the framework of data limitations.

FOOTNOTES

[1]Polynomial curve fitting estimation of trends is particularly interesting for extension of prediction theory to non-stationary time-series (6,7). Another advantage of polynomials is that to establish changing relationships between variables these curves can advantageously be used for cross-section (country) analysis. This method is specially important for the analysis of growth trends and patterns (8).

[2]Eq. (8) is symmetrical with respect to either K or L. The reader can verify by substituting K for L in (8) in which case $\phi(x)$ is replaced by: $\Phi(K/L) = \phi(K/L) - (L/K)$.

[3]It seems that the HIPF would also apply to condition of monopoly (2).

[4]In order to reach equilibrium in the industry the following assumptions must hold for each firm:

(a) Zero profit for each firm.

(b) Equal elasticity of demand for each firm, although this elasticity may vary over the sample points through time.

(c) The ratio of raw materials costs to price is equal for each firm.

(d) Wage earners receive their marginal physical product.

With some amendments these conditions can be relaxed.

[5]The reader can verify the results for these special cases.

[6]The polynomials of degree 6 and 5 for the first and second estimates respectively also give a smooth curve.

[7]One of the crucial questions concerning the polynomial estimation procedure is whether the resultant isoquant will have the correct convexity. Our empirical calculations show that:

$$\psi' = \frac{\phi'}{\phi^2} + 1 < 0$$

for all the values of x in the observable range. Since:

$$\psi' = \frac{d}{dL}\left(\frac{dK}{dL}\right)\frac{dL}{dx} = \left(\frac{L^2}{L\psi - K}\right)\frac{d^2K}{dL^2} \quad ,$$

$\psi' < 0$ implies $\frac{d^2K}{dL^2} > 0$ which means that the convexity condition is fulfilled.

[8]National Industrial Conference Board, Wharton School of Finance and Commerce, McGraw-Hill, etc.

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