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Walrasian Comparative Studies

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Abstract

We present a finite system of polynomial inequalities in unobservable endogenous variables and market data that observations on market prices, individual incomes, and aggregate endowments must satisfy to be consistent with the equilibrium behavior of some pure exchange economy.

We also derive a corresponding family of polynomial inequalities for production economies. For these economies, we give necessary and sufficient conditions in terms of observations on market prices, aggregate endowments, individual profit shares, and individuals' income from their endowments for these observations to be consistent with the equilibrium behavior of some production economy.

We use quantifier elimination to derive for both the two-person model of pure exchange and the two-sector general equilibrium model comparative statics relationships between two observations on market data that are necessary and sufficient for the existence of utility functions and production functions which are consistent with equilibrium behavior in the pure exchange economy and the two-sector model.

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Walrasian Comparative Statics[1]

by Donald J. Brown & Rosa L. Matzkin

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1. Introduction

The core of the general equilibrium research agenda has centered around the questions of the existence and uniqueness of competitive equilibria and the stability of the price adjustment mechanism. Despite the resolution of these concerns, i.e. the existence theorem of Arrow and Debreu; Debreu's results on local uniqueness; Scarf's example of global instability of the tâtonnement price adjustment mechanism and the Sonnenschein-Debreu-Mantel theorem, general equilibrium theory continues to suffer the criticism that it lacks falsifiable implications or in Samuelson's terms: "meaningful theorems." The disappointing attempts of Walras, Hicks and Samuelson to derive comparative statics for the general equilibrium model are chronicled in Ingaro and Israel (9). Moreover, there has been no substantive progress in this field since Arrow and Hahn's discussion of monotone comparative statics for the Walrasian model (3).

In this paper we propose a methodology, i.e. quantifier elimination, for deriving refutable propositions within general equilibrium theory. In addition, we apply this methodology to derive comparative statics for two simple general equilibrium models: two-person pure exchange and the two-sector general equilibrium model.

Comparative statics is the primary source of refutable propositions in economic theory. This mode of analysis is most highly developed within the theory of the household and the theory of the firm, e.g. Slutsky's equation, Shepard's lemma, etc. As is well known from the Sonnenschein-Debreu-Mantel theorem, the Slutsky restrictions on

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individual excess demand functions, in general, do not extend to market excess demand functions--see Shafer and Sonnenschein (13) for a discussion of this important theorem. In particular, utility maximization subject to a budget constraint imposes no testable restrictions on the set of equilibrium prices, as shown by Mas-Colell (10).

In any comparative statics analysis, we must first identify the exogenous and endogenous variables. For exchange economies, where preferences or utility functions are stationary, the exogenous variables are the individual endowments and the endogenous variables are equilibrium prices. This suggests that the equilibrium manifold, i.e. the graph of the Walrasian correspondence--first introduced by Balasko (5), is the natural source of testable implications of general equilibrium theory. This differs from the traditional focus in equilibrium analysis where the primary construct for questions of existence, uniqueness and stability of price adjustment is the market excess demand function.

Our second innovation is methodological, where we introduce a new technique for deriving refutable comparative statics from the first order theory of competitive economies by eliminating unobservable exogenous or endogenous variables from equilibrium conditions expressed in terms of finite systems of polynomial inequalities. The coefficients or parameters in these families of inequalities are polynomials in the observable endogenous and exogenous variables of the model, and the unknowns are unobservables in the theory such as utility levels, marginal utilities of income or random shocks to tastes or technology.

The first implicit use of quantifier elimination (as this method is called in model theory--see Van Den Dries (15) for a discussion) to derive refutable propositions in economics occurs in the theory of demand, where Afriat (1) proved the equivalence of the Afriat inequalities (which contain unobservable utility levels and marginal utilities of income) and his axiom of revealed preference, "cyclical consistency," (which involves only observables, i.e. prices, consumption bundles and incomes). This equivalence and its extension to the Walrasian model are the fundamental elements of our analysis.

Quantifier elimination, when applicable, is a more powerful technique than the implicit function theorem, convex analysis, or lattice programming applied to the first order conditions of some optimization problem characterizing the equilibrium state of an economic agent or economic system. Typically these methods only give necessary conditions that the observed data must satisfy to be consistent with the given theory, i.e. sufficient conditions for refutability; but quantifier elimination gives necessary and sufficient conditions for refutability. An example from demand theory is the weak axiom of revealed preference which is implied by utility maximization subject to a budget constraint, but is not "equivalent" to it. That is, the data may satisfy the weak axiom but fail to satisfy the strong axiom of revealed preference, in which case there is no utility function that rationalizes the data. Moreover, if the data does satisfy the strong axiom then there must be some utility function which rationalizes the data. It is in this sense that the strong axiom of revealed preference or any of its equivalents, such as cyclical consistency, are equivalent to utility maximization subject to a budget constraint for any finite set of observations.

Quantifier elimination produces one of three mutually exclusive outcomes. First, the given equilibrium conditions may be inconsistent, i.e. quantifier elimination reduces the given system of polynomial inequalities to $1 \equiv 0$. Second, the given equilibrium conditions may have no testable implications, i.e. quantifier elimination reduces the given system of polynomial inequalities to $1 \equiv 1$. Finally, in the case of interest, the given equilibrium conditions have refutable implications, i.e. quantifier elimination reduces the given system to an equivalent family of finite sets of polynomial inequalities in the observables such that the equilibrium conditions have a solution iff the data satisfies at least one of the sets of polynomial inequalities in the observable endogenous and exogenous variables.

We present a finite system of polynomial inequalities that unobservable utility levels, marginal utilities of income and consumption bundles together with observations on prices, individual incomes, and aggregate endowments must satisfy to be consistent with the equilibrium behavior of some exchange economy. These conditions are both necessary and sufficient for the given observations to lie on the equilibrium manifold of that exchange economy. As such, they are an extension to an exchange economy of the necessary and sufficient conditions of Afriat (1) for market data on prices, incomes, and individual's demands to be consistent with utility maximization subject to a budget constraint. Our conditions are simply the Afriat inequalities; the budget restrictions; and the market clearing conditions. The nonlinearity of our system of polynomial inequalities derives from not observing individual demands, e.g. see Chiappori's paper on household labor supply (6).

Moreover, for the case of two agents and two observations, we use quantifier elimination to obtain from our conditions refutable comparative statics between observables that are necessary and sufficient for the data to be consistent with the general equilibrium model of pure exchange. This relationship we call the Axiom of Revealed Equilibrium, ARE.

For general production economies, our necessary and sufficient conditions for observations on prices, consumers' incomes, and total endowments to lie on the equilibrium manifold of some exchange economy are easily extended to production. This is done by augmenting our system of Afriat inequalities and market clearing conditions with the family of linear inequalities given by the Weak Axiom of Profit maximization (WAPM)--see Varian (17) for a discussion of WAPM.

Within the family of production economies, Robinson Crusoe economies deserve special attention since they embody many of the essential features of representative agent economies. We show that a necessary condition for the equilibrium manifold of a production economy to be consistent with the Robinson Crusoe (or representative agent) model is that the Walrasian correspondence be cyclically monotone. See Aubin and Frankowska (4) for a discussion of monotone and cyclically monotone correspondences.

In applied policy studies in fields such as public finance or international trade, the two-sector general equilibrium model of production and exchange is often used for comparative statics analysis--see Shoven and Whalley (14) for a discussion. The special structure of the two-sector model, where there are only two factors of production and firms have constant returns to scale production functions, can be used to convert the two-sector model into two models of exchange: a pure exchange model in factors space (for the production sector) and a pure exchange model in goods space (for the consumption sector)--this observation is due to Shoven and Whalley, see section 2.5 in (14). They exploit this structure to reduce the dimension in computing equilibria in an important class of applied general equilibrium models.

For us, their observation suggests that our necessary and sufficient conditions for the prices of goods, consumer's income, and total consumption of goods to lie on the equilibrium manifold of an exchange economy for some pair of continuous, concave and monotone utility functions on the goods space can be extended to the two-sector model.

To this end, we derive additional necessary and sufficient conditions on prices of factors, costs of producing goods and total factor endowments to lie on the equilibrium manifold of an "exchange" economy for some pair of homothetic, continuous, concave, and monotone production functions on the factors space. These conditions are derived in the context of pure exchange economies where we replace the Afriat inequalities for continuous, concave and monotone utility functions with his inequalities, for the subset of these functions which are also homothetic--see Varian (18) for a discussion.

For two observations, using quantifier elimination, we derive refutable comparative statics between the observables for the homothetic case of pure exchange. This relationship we call the Homothetic Axiom of Revealed Equilibrium, or HARE. The conjunction of ARE and HARE give a complete characterization of the testable implications of the two-sector model of production and exchange for two observations.

This paper is organized as follows. Section II presents necessary and sufficient conditions for data on market prices, individual incomes and total endowments to be consistent with the general equilibrium model of pure exchange. Section III applies these conditions to capital markets, assets markets, and labor markets. In section IV we introduce the axioms for the consistency of observed data with the GE model for the 2 x 2 case. Finally, in section V, we extend the results of sections II and IV to productions economies. All proofs are given in the Appendix.

II. Competitive restrictions in the model of pure exchange

We consider a world with K commodities and T traders, where the intended interpretation is a pure trade model. The commodity space is R^K and each agent has R^K_+ as her consumption set. Each trader is characterized by an endowment vector $\omega_t \in R^K_{++}$ and a utility function $V_t:R^K_+ \to R$. Utility functions are assumed to be continuous, monotone, and concave.

An allocation is a consumption vector, x_t, for each trader such that $x_t \epsilon R^K_+$ and $\Sigma^T_{t=1} x_t = \Sigma^T_{t=1}\omega_t$.

The price simplex, $\Delta = \{p \epsilon R^K_+ \mid \Sigma^K_{i=1}p_i = 1\}$. We shall restrict attention to strictly positive prices, $S, = \{p \epsilon \Delta \mid p_i > 0$ for all $i\}$.

A competitive equilibrium consists of an allocation $\{x_t\}^T_{t=1}$ and prices p such that each x_t is utility maximizing for agent t subject to her budget constraint. The prices p are called equilibrium prices.

Suppose that we observe a finite number N of individual endowment vectors $\{\omega^r_t\}^T_{t=1}$ and market prices p^r, where $r = 1,...,N$. Does there exist a set of utility functions $\{V_t\}^T_{t=1}$ such that for each r, p^r are the equilibrium prices for the exchange economy defined by $<\{V_t\}^T_{t=1},\{\omega^r_t\}^T_{t=1}>$? Notice that the $\{V_t\}^T_{t=1}$ are independent of r. That is, do the ordered pairs $<\{\omega^r_t\}^T_{t=1},p^r>$ lie on the equilibrium manifold defined by the $\{V_t\}^T_{t=1}$? Here the equilibrium manifold is simply the graph of the Walras correspondence. The next example shows that there may be no such exchange economy, where traders have monotone utility functions.

Example 1: In this example there are two goods and two consumers. In figure I, we superimpose the two Edgeworth boxes, which are defined by the endowment vectors ω^1 and ω^2. The first box, (I), is ABCD and the second box, (II), is AEFG. The first agent lives at the A vertex in both boxes and the second agent lives at vertex C in box (I) and at vertex F in box (II). The individual endowments $\omega^1_1, \omega^1_2; \omega^2_1, \omega^2_2$ and the two price vectors p^1 and p^2 define the budget sets of each consumer. The sections of the budget hyperplanes that intersect with each Edgeworth box is the set of potential equilibrium allocations. From the figure it is clear that no monotone utility function can generate p^1 and p^2 as equilibrium prices. All pairs of allocations in box (I) and (II) that lie on the given budget lines violate the weak axiom of revealed preference (WARP) for the first agent (the agent living at vertex A). Hence there are no utility functions in the specified class such that the observed data lie on an equilibrium manifold.

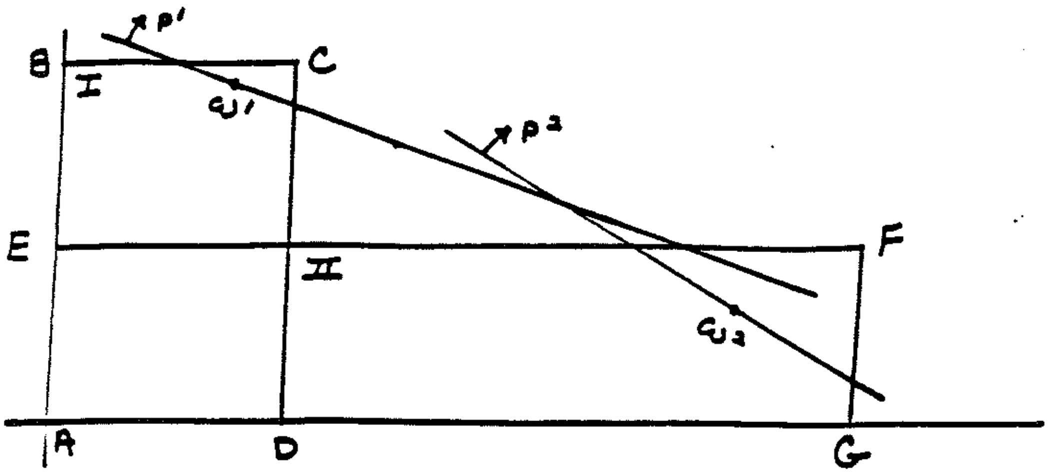

Figure I

In theorem 1, we give a characterization of the set of market prices and individual endowments for which there exists a set of continuous, concave, and monotone utility functions; such that each price vector is an equilibrium price vector of the exchange economy defined by the corresponding family of individual endowments and utility functions.

Theorem 1: Let $<p^r, \{\omega_t^r\}_{t=1}^T>$ for $r = 1,...,N$ be given. Then there exists a set of continuous, concave and monotone utility functions $\{V_t\}_{t=1}^T$ such that for each $r = 1,...,N$, p^r is an equilibrium price vector for the exchange economy $<\{V_t\}_{t=1}^T, \{\omega_t^r\}_{t=1}^T>$ iff there exists numbers $\{\bar{V}_t^r\}_{t=1,...,T; r=1,...,N}$ and $\{\lambda_t^r\}_{t=1,...,T; r=1,...,N}$ and vectors $\{x_t^r\}_{t=1,...,T; r=1,...,N}$ satisfying:

(1.1) $\bar{V}_t^r - \bar{V}_t^s - \lambda_t^s p^s \cdot (x_t^r - x_t^s) \leq 0$ $\quad r,s = 1,...,N;\ t = 1,...,T$

(1.2) $\lambda_t^r > 0,\ x_t^r \geq 0$ $\quad r = 1,...,N;\ t = 1,...,T$

(1.3) $p^r \cdot x_t^r = p^r \cdot \omega_t^r$ $\quad r = 1,...,N;\ t = 1,...,T$

(1.4) $\Sigma_{t=1}^T x_t^r = \Sigma_{t=1}^T \omega_t^r$ $\quad r = 1,...,N$

Theorem 1 can be used to test the consistency of data on market prices and consumers' endowments with the general equilibrium model, of consumers maximizing utility and markets clearing. In fact, a weaker set of data suffices to test this hypothesis. By inspecting Theorem 1, it is clear that the only necessary data are the market prices, the incomes of the consumers, and the aggregate demand. We make this result explicit in the following theorem. In the statement of this theorem I_t^r denotes the income of consumer t in observation r and ω^r denotes the aggregate endowment in observation r.

Theorem 2: Let $<p^r,\{I_t^r\}_{t=1}^T,\omega^r>$ for $r = 1,...,N$ be given. Then there exists a set of continuous, concave and monotone utility functions $\{V_t\}_{t=1}^T$ such that for each $r = 1,...,N$; p^r is an equilibrium price vector for the exchange economy $<\{V_t\}_{t=1}^T,\{I_t^r\}_{t=1}^T, \omega^r\}>$ iff there exists numbers $\{\bar{V}_t^r\}_{t=1,...,T;r=1,...,N}$ and $\{\lambda_t^r\}_{t=1,...,T;r=1,...,N}$ and vectors $\{x_t^r\}_{t=1,...,T;r=1,...,N}$ satisfying:

(2.1) $\bar{V}_t^r - \bar{V}_t^s - \lambda_t^s p^s \cdot (x_t^r - x_t^s) \leq 0$ $\quad r,s = 1,...,N;\ t = 1,...,T$

(2.2) $\lambda_t^r > 0,\ x_t^r \geq 0$ $\quad r = 1,...,N;\ t = 1,...,T$

(2.3) $p^r \cdot x_t^r = I_t^r$ $\quad r = 1,...,N;\ t = 1,...,T$

(2.4) $\Sigma_{t=1}^T x_t^r = \omega^r$ $\quad r = 1,...,N$

In practice, one might use cross-sectional data to obtain the necessary variation in market prices and individual endowments. If you imagine cities or states having the same distribution of tastes but having different income distributions and consequently different market prices, then these observations can serve as market data for our model. Also, in the stylized economies in our examples, you could think of each "trader" as an agent type, consisting of numerous small consumers each having the same tastes and endowments.

Given N demand observations, $<p^r,x_t^r,\omega_t^r>$ for $r = 1,...,N$ on prices, consumption bundles, and endowment vectors such that for all r, $p^r \cdot x_t^r = p^r \cdot \omega_t^r$, one may want to determine whether the observations could have been generated by the maximization of a common concave and monotone utility function. Afriat (1) showed that there exists a concave and monotone utility function generating the observations if and only if there exists numbers $\{\bar{V}_t^r\}_{r=1}^N$ satisfying (1.1). Matzkin and Richter (11) provided a different set of inequalities that tests whether the observations are consistent with the existence of a strictly concave and strictly monotone utility function. Existence of a solution to the Matzkin and Richter inequalities is equivalent to the requirement that the demand observations satisfy the Strong Axiom of Revealed Preference (SARP). Matzkin and Richter inequalities can be used to derive a variation of Theorem 1 in which the utility functions of the consumers are required to be strictly increasing and strictly monotone. To obtain the corresponding inequalities, just replace condition (1.1) in Theorem 1 by

(1.1′) $\bar{V}_1^r - \bar{V}_t^s - \lambda_t^s p^s \cdot (x_t^r - x_t^s) < 0$ if $x_t^r \neq x_t^s, \ldots, N; t=1,\ldots,T$

(1.2′) $\bar{V}_t^r = \bar{V}_t^s$ if $x_t^r = x_t^s$ $r,s=1,\ldots,N; t=1,\ldots,T$

It is also possible to require that the traders possess strictly concave, strictly monotone, and C^∞ utility functions; this would use the strong version of SARP (denoted SSARP), i.e. SARP and the condition that $p^s \neq \alpha p^r$ implies $x^s \neq x^r$ for $\alpha > 0$, developed by Chiappori and Rochet (7). In this latter case, the inequalities characterize points on the smooth equilibrium manifold first defined by Balasko. Additive separability of utility functions (a common assumption in the intertemporal model of exchange) can be tested with conditions derived by Varian in (18), using Afriat's method--see Theorem 6 in Varian's paper.

Homotheticity of concave and monotone utility functions can also be tested, by using the conditions developed by Diewert (8). In this case, condition (1.1) is replaced by

(1.1″) $\bar{V}_t^r - \lambda_t^s p^s \cdot x_t^r \leq 0$ and

(1.2″) $\bar{V}_t^r - \lambda_t^r p^r \cdot x_t^r = 0.$

III. General equilibrium restrictions on capital markets, assets markets, and labor markets

In this section we consider general equilibrium restrictions for markets other than the standard case of pure exchange. In particular, we demonstrate such restrictions in capital markets, assets markets, and labor markets.

Capital markets

Suppose we have a two period model where in each period there are two consumption goods available. Also assume there are only two agents, each of whom has an additively separable (time invariant) utility function. If we observe both the individual endowments in each period and each period's spot prices, then we can test the hypothesis that the spot prices are equilibrium prices at which the two agents borrow and lend to maximize their discounted sum of utilities subject to their intertemporal budget constraint, and that these prices clear the markets for consumption goods each period.

To demonstrate the refutability of the given hypotheses, we construct an example in figure II which is a slight variant of the example illustrated in figure I. Again we draw two Edgeworth Boxes and superimpose one on the other. As before box I is ABCD; box II is AEFG; the first agent lives at vertex A in both boxes and the second agent lives at vertex C in box I and at vertex F in box II. The principal difference between figure II and figure I is that now we have a band of possible budget lines for the first agent in both box I and box II. These bands correspond to the borrowing and lending opportunities of the first agent, where the two solid price lines correspond to the case of no borrowing or lending. We see from figure II that the model is refuted, since it is as if the first agent is maximizing the same preferences in each period over the intersection of a budget line (in one of the bands) with the relevant Edgeworth box. But in every case this behavior violates WARP.

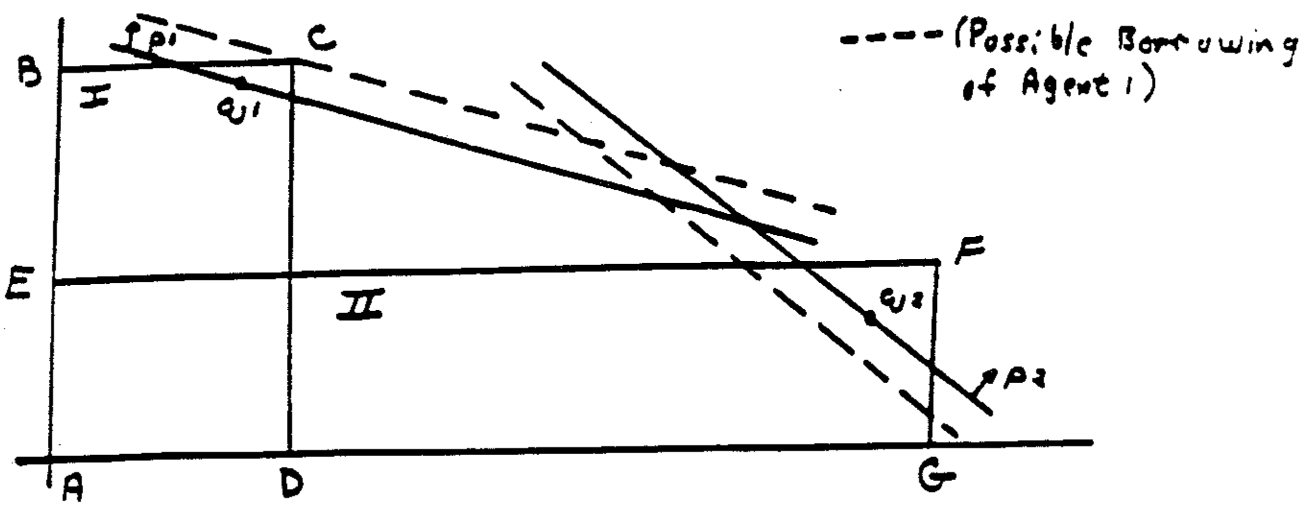

Figure II

Assets markets

We next turn to asset markets and suppose there are two marketed assets. Since states of the world are unobservable, we cannot simply replicate the geometry of the previous examples where the two consumption goods in each Edgeworth box are reinterpreted as consumption in state one and consumption in state two. Instead we assume that the marketed assets have limited liability and we know their distribution of returns (possibly from historical data).

Our hypothesis is that agents maximize concave monotone indirect utility functions over feasible portfolios of assets, where a portfolio is feasible iff it has a nonnegative distribution of returns. Suppose we observe the number of outstanding shares of the two assets, the income distribution, and the asset prices. Then we may ask if the general equilibrium model for these asset markets is refutable. Again, the answer is yes, but notice there is no assumption of complete markets. In fact, since we haven't specified a state space, the notions of complete or incomplete markets are not well defined in this model.

In figure III we see the "Edgeworth Boxes" for the securities markets in two different periods. These boxes are in portfolio space and their peculiar shape derives from allowing unlimited short sales of the marketed assets, subject to producing feasible portfolios. Of course, we now have the familiar picture with respect to asset prices q^1 and q^2 and initial portfolios θ^1 and θ^2 which refutes the competitive hypothesis, i.e. the weak axiom fails for the agent living at vertex A in the two boxes for all possible equilibrium allocations. Notice that despite these prices being arbitrage free, they do not support competitive allocations of assets.

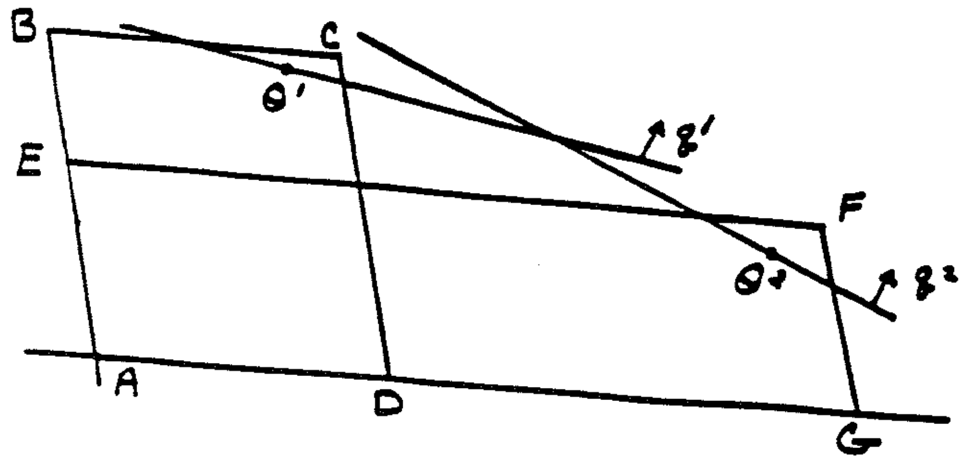

Figure III

To derive the shape of the "Edgeworth Boxes" in figure III, we consider the special case of a finite state space with k states of nature. Suppose there are two marketed assets, a_1 and a_2, which define the assets' returns matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{k1} & a_{k2} \end{pmatrix}$$

and a_{ij} is the payoff of asset j in state i. A portfolio of asset holdings $\theta = (\theta_1, \theta_2)$ define a state contingent claim $A\theta = \theta_1 a_2 + \theta_2 a_2$.

We allow short sales but impose the no bankruptcy condition that $A\theta \geq 0$ as the definition of a feasible portfolio θ. Assuming limited liability of both marketed assets, i.e. both a_1 and a_2 have nonnegative payoffs in every state of nature, then the set of feasible portfolios $X = \{\theta \in R^2 \mid A\theta \geq 0\}$ contains the positive quadrant of R^2. X is a nonempty convex cone and therefore defines a partial order on R^2, which we denote as $\cdot\geq$, where $x \cdot\geq y$ iff $x - y \in X$. Given the partial order $\cdot\geq$ on R^2, the order intervals are the subsets of R^2 denoted $[x,y]$ where $y \cdot\geq x$ and $[x,y] = \{z \in R^2 \mid z \cdot\geq x\} \cap \{z \in R^2 \mid y \cdot\geq z\}$. Edgeworth Boxes are simply order intervals of the form $[0,x]$, where x is the "total endowment." In the case at hand we assume that each agent's initial asset holdings are marketed, hence the sum of their initial holdings correspond to some portfolio $\bar{\theta}$, and the "Edgeworth Box" in portfolio space, R^2, is $[0,\bar{\theta}]$ with respect to the partial order $\cdot\geq$.

Finally, given a continuous, monotone concave utility function U over assets, the indirect utility function V over portfolios is defined by $V(\theta) = U(A\theta)$. If asset prices are given by the price vector q, then the value of a portfolio θ is simply $q \cdot \theta$. Let $\bar{\theta}$ be the portfolio of initial asset holdings. Then consumers solve the optimizing problem:

$$\begin{aligned} &\max V(\theta) \\ &s.t.\ q \cdot \theta \leq q \cdot \bar{\theta}. \\ &\quad \theta \in X \end{aligned}$$

This maximum is achieved if the prices q are arbitrage free, i.e. they lie in the interior of the (negative) polar cone of X. An equilibrium in this two person exchange economy consists of asset prices q and portfolios $\hat{\theta}^1, \hat{\theta}^2$ such that $\hat{\theta}^1 + \hat{\theta}^2 = \bar{\theta}^1 + \bar{\theta}^2$ and they solve the agents' portfolio optimization problems.

Labor markets

For our final example, i.e. labor markets, we turn to a recent paper of Chiappori (6) on household labor supply. He considers a household with, say, two consumers. There are three goods in his model: a consumption good and labor/leisure for each of the two agents. Agents receive income from selling their labor and we observe both their labor/leisure choices and their wage rates--the consumption good is chosen as numeraire. In addition, the household receives nonlabor income which we observe and we observe the total household consumption. Chiappori asks, is there a division of total consumption and nonlabor income which is Pareto optimal for some pair of concave monotone utility functions for the two consumers?

In fact, it follows from the Second Welfare Theorem that an equivalent formulation of Chiappori's question is, does there exist a division of nonlabor income and total consumption such that the individual consumptions and labor/leisure choices comprise a competitive allocation at the observed prices for some pair of concave monotone utility functions? We could, of course, write out the system of equations which are necessary and sufficient to refute this model. Again they would be the Afriat conditions, the market clearing conditions, and the equations characterizing the division of the nonlabor income. In fact, this is equivalent to what Chiappori does--see Proposition (2) in (6).

What is important to notice is that our model admits the possibility of nonobservables, such as individual consumption in the case of Chiappori's model, or asset holdings or the borrowing and lending decisions in our previous two examples.

IV. Axiom of Revealed Equilibrium (ARE) and Homothetic Axiom of Revealed Equilibrium (HARE)

In section II of this paper we proposed a finite system of polynomial inequalities characterizing finite sets of observations that are consistent with the general equilibrium model of pure exchange. The unknowns are the unobservable consumption vectors, utility levels of these consumptions and marginal utilities of income. The coefficients (or parameters) in these inequalities are polynomials in the observable data, i.e. prices, incomes, and the aggregate endowments.

We proved that the data is consistent with the general equilibrium model iff this system of polynomial inequalities has a solution. The Tarski-Seidenberg Theorem gives an algorithm which, in a finite number of steps, eliminates the unknowns from the given system and produces an equivalent family of polynomial inequalities in the parameters--equivalent in the sense that the original family of polynomial inequalities has a solution iff the observed data satisfies at least one of the derived family of polynomial inequalities in the parameters.

As an example, recall that the quadratic equation $ax^2 + bx + c = 0$ has a real solution iff $b^2 - 4ac \geq 0$. In general, it is the existence of an equivalent system of polynomial inequalities in the data that is important and not the application of the algorithm in any particular instance. (For a discussion of the Tarski-Seidenberg Theorem see Van Den Dries (15) and for a discussion of the complexity of the Tarski-Seidenberg algorithm, see the volume edited by Arnon and Buchberger (2).) But, in our case, we actually want the derived family of polynomial inequalities in the data since they constitute an axiom -- a set of algebraic conditions on the data -- for refuting the general equilibrium model. To this end, we recall Afriat's Theorem as presented by Varian in (16).

<u>Afriat's Theorem</u>: The following conditions are equivalent:

(1) There exists a non-satiated utility function that rationalizes the data;

(2) the data satisfies "cyclical consistency;" that is, $p^r \cdot x^r \geq p^r \cdot x^s, p^s \cdot x^s \geq p^s \cdot x^t, \ldots, p^\ell \cdot x^\ell \geq p^\ell \cdot x^r$ implies $p^r \cdot x^r = p^r \cdot x^s, \ldots, p^\ell \cdot x^\ell = p^\ell \cdot x^r$.

(3) there exists numbers $U^i, \lambda^i > 0$, $i=1,\ldots,n$ that satisfy the Afriat inequalities:
$U^i \leq U^j + \lambda^j p^j \cdot (x^i - x^j)$ for $i,j=1,\ldots,n$

(4) there exists a concave, monotonic, continuous, non-satiated utility function that rationalizes the data.

The equivalence of conditions (2) and (3) can be shown to be a consequence of the Tarski-Seidenberg Theorem, where the unknowns U^i, λ^i have been eliminated from (3) to obtain (2), an equivalent family of polynomial inequalities in the data: p^j, x^j.

Varian shows that "cyclical consistency" is equivalent to GARP, the Generalized Axiom of Revealed Preference, an axiom due to Varian. We use the Chiappori-Rochet inequalities, hence the appropriate axiom is the Strong Axiom of Revealed Preference

(SSARP).

We next describe ARE (Axiom of Revealed Equilibrium). Let $c = a,b$ denote a typical consumer in the economy. Let $r = 1,2$ denote a typical observation. Let ω^r and p^r denote respectively, the vector of aggregate endowment and market prices in observation r. Let I_c^r denote the income of consumer c in observation r. For each $c = a,b$ and each $r = 1,2$, let $\bar{z}_c^r$ denote the solution to

$$\begin{aligned} &\text{Maximize} && p^s \cdot x \\ &\text{subject to} && p^r \cdot x = I_c^r \\ & && 0 \le x \le \omega^r, \end{aligned}$$

where $s \neq r$. And let $\underline{z}_c^r$ denote the solution to

$$\begin{aligned} &\text{Minimize} && p^s \cdot x \\ &\text{subject to} && p^r \cdot x = I_c^r \\ & && 0 \le x \le \omega^r, \end{aligned}$$

where $s \neq r$.

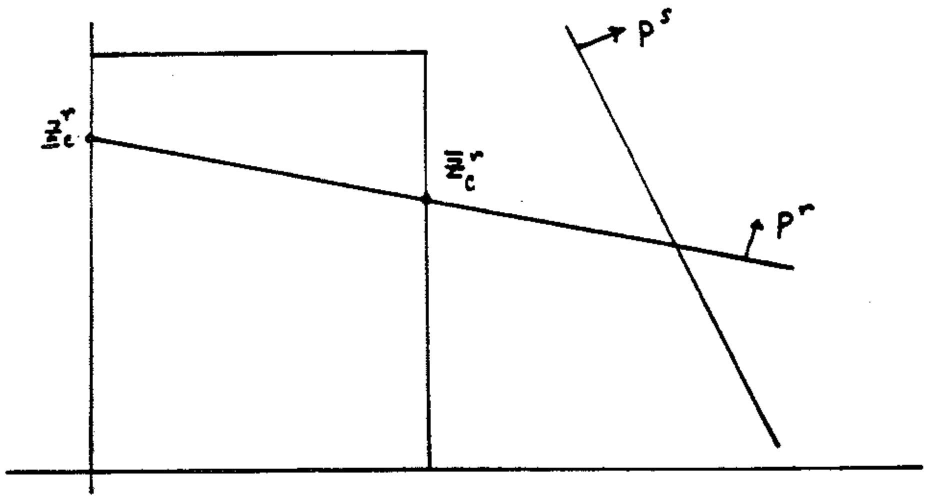

Figure IV

The observations $\{p^r\}_{r=1,2}$, $\{I_c^r\}_{r=1,2;c=a,b}$, $\{\omega^r\}_{r=1,2}$ satisfy the Axiom Revealed Equilibrium (ARE) if

(0) $\forall\, r = 1,2$

$$I_a^r + I_b^r = p^r \cdot \omega^r$$

(i) $\forall\, c = a,b\ \forall\, r,s = 1,2\ (r \neq s)$

$$[(p^s \cdot \bar{z}^r_c \leq I^s_c) \rightarrow (p^r \cdot \bar{z}^s_c > I^r_c)]$$

and

(ii) $\forall$ r,s = 1,2 $(r \neq s)$

$$[(p^s \cdot \bar{z}^r_a \leq I^s_a) \; \& \; (p^s \cdot \bar{z}^r_b \leq I^s_b)] \rightarrow (p^r \cdot \omega^s > p^r \cdot \omega^r)$$

Theorem 3: Let $\{p^r\}_{r=1,2}$, $\{I^r_t\}_{r=1,2;\, c=a,b}$, $\{\omega^r\}_{r=1,2}$ be given such that p^1 is not a scalar multiple of p^2. Then the data is consistent with the general equilibrium model of pure exchange, where the utility functions of the consumers are strictly increasing, strictly concave and C^∞, if and only if the data satisfy ARE.

We now turn to the case in which the utility functions are homothetic. Afriat's conditions for the existence of homothetic, concave, continuous and monotone utility functions; the budget constraints; and the market clearing conditions define a nonlinear family of polynomial inequalities. Observations on total endowments, income distributions, market prices and unobserved utility levels, marginal utilities of income and consumption vectors must satisfy these inequalities for the data to be consistent with the pure exchange model for some pair of homothetic, concave, continuous and monotone utility functions. Varian in (18) shows that the Afriat inequalities for homothetic utility functions are equivalent to the Homothetic Axiom of Revealed Preference (HARP). For two observations $\{p^r,x^r\}_1^2$, HARP reduces to:

(*) $\quad (p^r \cdot x^s)(p^s \cdot x^r) \geq (p^r \cdot x^r)(p^s \cdot x^s)$ for $r,s=1,2$ $(r \neq s)$.

If we substitute (*) for the Afriat inequalities defined earlier in (1.1") and (1.2"), we obtain a nonlinear family of polynomial inequalities where the unknowns are the consumption vectors x^r_t for $r=1,2$ and $t=a,b$. Applying Tarski-Seidenberg elimination, we derive an axiom that is equivalent to the satisfaction of those inequalities. We call this new axiom the Homothetic Axiom of Revealed Equilibrium (HARE).

Given observations $\{p^r\}_{r=1,2}$, $\{I^r_c\}_{r=1,2;c=a,b}$, $\{\omega^r\}_{r=1,2}$, we define the following terms:

$\gamma_a = I^1_a I^2_a$, $\quad \gamma_b = I^1_b I^2_b$, $\quad \gamma_\omega = (p^1 \cdot \omega^2)(p^2 \cdot \omega^1)$,

$\psi_1 = \gamma_b - \gamma_a - \gamma_\omega$,

$\psi_2 = (\gamma_b - \gamma_a - \gamma_\omega)^2 - 4\,\gamma_a\,\gamma_\omega$

$$r_1 = \frac{\gamma_a}{p^1 \cdot \bar{z}_a^2} \qquad r_2 = p^2 \cdot \omega^1 - \frac{\gamma_b}{p^1 \cdot \bar{z}_b^2}$$

$$t_1 = \frac{-\psi_1 - (\psi_2)^{\frac{1}{2}}}{2p^1 \cdot \omega^2} \qquad t_2 = \frac{-\psi_1 + (\psi_2)^{\frac{1}{2}}}{2p^1 \cdot \omega^2}$$

$$s_1 = \max\{ r_1, t_1\} \qquad s_2 = \min\{ r_2, t_2\} \ .$$

We say that the data $\{p^r\}_{r=1,2}$, $\{I_c^r\}_{r=1,2;c=a,b}$, $\{\omega^r\}_{r=1,2}$ satisfy the <u>Homothetic Axiom of Revealed Equilibrium</u> (HARE) if

(1) $\psi_1 < 0$,

(2) $\psi_2 > 0$,

(3) $r_1 \leq r_2$

(4) $s_1 \leq s_2$

(5) $p^2 \cdot \underline{z}_a^1 \leq s_2$

(6) $p^2 \cdot \bar{z}_a^1 \geq s_1$

(7) $I_a^1 + I_b^1 = p^1 \cdot \omega^1$ and $I_a^2 + I_b^2 = p^2 \cdot \omega^2$.

<u>Theorem 4</u>: Let $\{p^r\}_{r=1,2}$, $\{I_t^r\}_{r=1,2;\, c=a,b}$, $\{\omega^r\}_{r=1,2}$ be given such that p^1 is not a scalar multiple of p^2. Then the data is consistent with the general equilibrium model of pure exchange, where the utility functions of the consumers are concave, monotone increasing, and homothetic if and only if the data satisfy HARE.

<u>V. Production economies</u>

We now proceed to characterize observations that lie on the equilibrium manifold of production economies. Consumers now, in addition to utility functions and endowments, have shareholdings in the F firms, where θ_{tj} is the profit share of consumer t in firm j. The $\theta_{tj} \geq 0$ and $\Sigma_{j=1}^F \theta_{tj} = 1$. θ_t is the vector of shareholdings of consumer t. The technology of firm j is given by a closed, convex subset of R^K, Y_j. In addition we assume free disposal, i.e. $y' \leq y$ and $y \in Y_j \rightarrow y' \subseteq Y_j$. Each firm is assumed to be a price-taking, profit maximizer.

Given this description of our economy, we fix the utility functions and production sets and ask what refutable comparative statics are imposed on the Walras correspondence, if we have a finite set of observations on market prices, individual

Theorem 5: Let $<p^r, \{\omega_t^r\}_{t=1}^T, \{\theta_t^r\}_{t=1}^T>$ for $r = 1,...,N$ be given. Then there exists a set of continuous, concave and monotone utility functions $\{V_t\}_{t=1}^T$ and firms $\{Y_j\}_{j=1}^J$ such that for all $r = 1,...,N$; p^r is the equilibrium price vector for the production economy $<\{V_t\}_{t=1}^T, \{\omega_t^r\}_{t=1}^T, \{\theta_t^r\}_{t=1}^T, \{Y_j\}_{j=1}^J>$ iff there exists numbers $\{\bar{V}_t^r\}_{t=1,...,T; r=1,...,N}$ and $\{\lambda_t^r\}_{t=1,...,T; r=1,...,N}$, and vectors $\{x_t^r\}_{t=1,...,T; r=1,...,N}$ and $\{y_j^r\}_{t=1,...,T; r=1,...,N}$ satisfying:

(3.1) $\bar{V}_t^r - \bar{V}_t^s - \lambda_t^s p^s \cdot (x_t^r - x_t^s) \leq 0$ $\quad$ $r,s = 1,...,N;\ t = 1,...,T$

(3.2) $\lambda_t^r > 0$ $\quad$ $r = 1,...,N;\ t = 1,...,T$

(3.3) $p^r \cdot x_t^r - p^r \cdot \omega_t^r - \Sigma_{j=1}^F \theta_{t,j}^r\, p^r \cdot y_j^r \leq 0$ $\quad$ $r = 1,...,N;\ t = 1,...,T$

(3.4) $p^r \cdot y_j^s \leq p^r \cdot y_j^r$ $\quad$ $r, s = 1,...,N;\ j = 1,...,J$

(3.5) $\Sigma_{t=1}^T x_t^r + \Sigma_{j=1}^F y_j = \Sigma_{t=1}^T \omega_t^r$ $\quad$ $r = 1,...,N$

Proof: The argument is a slight modification of the proof of Theorem 1, where we invoke Varian (17) for that part of the proof pertaining to production.

As in Theorem 2, a smaller set of data suffices to derive the general equilibrium comparative statics. The necessary data in this case is $<p^r, \{I_t^r\}_{t=1}^T, \{p^r \cdot \omega_t^r\}_{t=1}^T, \omega^r, \{\theta_t^r\}_{t=1}^T>$ for each r.

In the case of Robinson Crusoe economies, we can say much more. Consider an economy with one consumer, a single firm and ℓ commodities. The intended interpretation is a representative agent economy. Suppose the consumer has a smooth concave utility function $u(x)$, then the gradient map $\nabla u(x)$ is cyclically monotone as shown by Rockafellar in Theorem 24.8 in (12). That is, for any set of pairs $(x^i, \nabla u(x^i))$, $i=0,1,...,n$ we have $(x^1-x^0)\cdot\nabla u(x^0)+(x^2-x^1)\cdot\nabla u(x^1)+...+(x^0-x^n)\cdot\nabla u(x^n) \geq 0$.

Necessary and sufficient conditions for a finite set of observations on market prices $\{p^i\}_1^n$, and aggregate endowments $\{\omega^i\}_1^n$ to lie on the equilibrium manifold of some Robinson Crusoe economy is given by the following family of linear inequalities. There exists $x^1,...,x^n$ in R_+^ℓ and $y^1,...,y^\ell$ in R^ℓ such that:

(1) Every finite family of $(x^i,p^i)_1^n$ is cyclically monotone

(2) $p^i \cdot y^j \leq p^i \cdot y^i$ for all i,j

(3) $x^i = y^i + \omega^i$ for $i=1,...,n$

Therefore, a necessary condition for the observations $(\omega^i,p^i)_1^n$ to be consistent with the Robinson Crusoe model is that every subset of the data is cyclically monotone.

Hence in Figure V, the data refutes the Robinson Crusoe model.

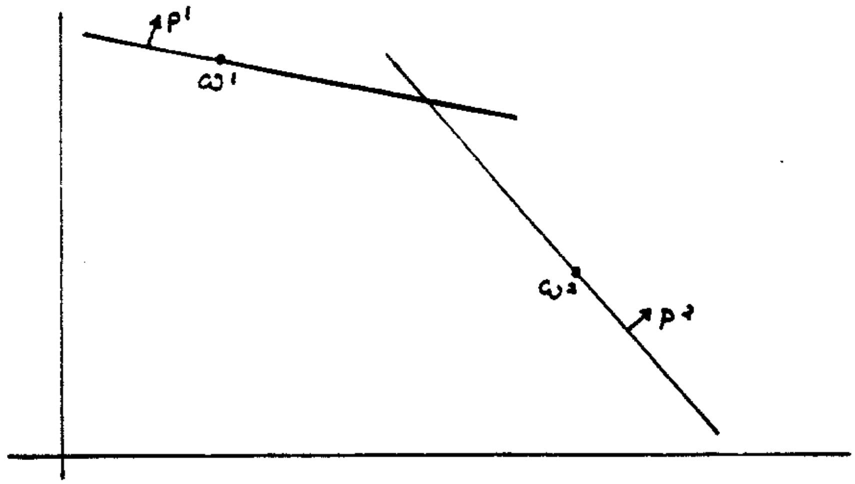

Figure V

The final model that we consider is the two-sector general equilibrium model of production and exchange where there are two factors, two goods, two households and two firms. The basic assumptions are that households are only endowed with factors, which they inelastically supply, and that the production functions of the firms have constant returns to scale. We assume that we have two observations on market prices of factors and goods; costs of production of goods; total factor endowments of the economy; and the income distribution of households.

Given the constant returns to scale assumption we can deduce the amounts produced of each good at the market prices, as a consequence of the zero profits condition. These quantities define the Edgeworth boxes for consumption. Given the market prices of goods and the income distributions of households, the consumption sector of the economy is in equilibrium iff the observations satisfy ARE.

We now turn to equilibrium in the production sector which can be construed as a pure exchange equilibrium in the space of factors where the total factor endowments determine the dimensions of the Edgeworth box. In this setting profit maximization is equivalent to firms maximizing output subject to a cost constraint at the given factor prices. Hence equilibrium in the production sector is formally equivalent to competitive equilibrium in a two person exchange economy with homothetic utility functions.

Our last theorem gives necessary and sufficient conditions for observations on market data to be consistent with the two-sector general equilibrium model. First, a bit of notation. Goods will be denoted e and f; $p^r = (p^r_e, p^r_f)$ are the prices of goods in period r. Factors are denoted ℓ and k; $q^r = (q^r_\ell, q^r_k)$ are the prices of factors in period r. Total factor endowments in period r are denoted (L^r, K^r). C^r_e and C^r_f will denote the costs of goods in period r at factor prices q^r. $(C^r_e/p^r_e, C^r_f/p^r_f) = \omega^r$ are the amounts of goods produced (hence available for exchange) in period r at market prices p^r and q^r. As before, consumers are denoted a and b, and their income in period r is I^r_a and I^r_b.

Theorem 6: If $D_1 = \langle \{p^r\}_{r=1,2}, \{I^r_c\}_{r=1,2;c=a,b}, \{\omega^r\}_{r=1,2} \rangle$, $D_2 = \langle \{q^r\}_{r=1,2}, \{C^r_t\}_{r=1,2;t=e,f}, \{(L^r,K^r)\}_{r=1,2} \rangle$ and p^1 is not a scalar multiple of p^2. Then the data, D_1 and D_2, are consistent with the two-sector general equilibrium model of production and exchange, where the utility functions of consumers are strictly increasing, strictly concave and C^∞ and the production functions of firms are homogenous of degree one, concave, continuous and monotone iff D_1 satisfies ARE and D_2 satisfies HARE.

Proof: Theorems 3 and 4.

APPENDIX

Proof of Theorem 1 : Suppose that there exists $\{\bar{V}_t^r\}, \{\lambda_t^r\}$ and $\{x_t^r\}$ s satisfying (1.1)-(1.4). Then, (1.1)-(1.3) imply, by Afriat's Theorem--see Varian (16)--that for each t, there exists a continuous, concave, and monotone utility function $V_t: R_+^K \to R$ such that for each r, x_t^r is one of the maximizers of V_t subject to the budget constraint: $p^r \cdot y \leq p^r \cdot \omega_t^r$. Hence, since $\{x_t^r\}_{t=1}^T$ define an allocation, i.e., satisfy (1.4), p^r is an equilibrium price vector for the exchange economy $<\{V_t\}_{t=1}^T, \{\omega_t^r\}_{t=1}^T>$, for each r = 1,...,N.

The converse is immediate, since, given continuous, concave and monotone utility functions, V_t; the equilibrium price vectors p^r and allocations $\{x_t^r\}_{t=1}^T$ satisfy (1.3) and (1.4) by definition. The existence of $\{\lambda_t^r\}_{t=1}^T$ such that (1.1) and (1.2) hold follows from the Kuhn-Tucker Theorem, where $\bar{V}_t^r = V_t(x_t^r)$.

Proof of Theorem 3:[2] Suppose that there exist strictly monotone, strictly concave C^∞ utility functions V_a and V_b such that p^r is an equilibrium price vector for (I_a^r, I_b^r, ω^r) for $r=1,2$. Let x_c^r be the unique maximizer of V_c subject to the budget constraint determined by (p^r, I_c^r) for $r=1,2$ and $c=a,b$. Then

4.1) $\quad 0 \leq x_c^r \leq \omega^r \qquad r=1,2;\ c=a,b$

4.2) $\quad x_a^r + x_b^r = \omega^r \qquad r=1,2$

Let us show that ARE is satisfied. First note that

$$I_a^r + I_b^r = p^r \cdot x_a^r + p^r \cdot x_b^r = p^r \cdot \omega^r \text{ for } r=1,2.$$

Next suppose that $p^r \cdot \bar{z}_c^s \leq I_c^r$. Then for all x such that $0 \leq x \leq \omega^s$ and $p^s \cdot x = I_c^s$, $p^r \cdot x \leq I_c^r$. In particular, $p^r \cdot x_c^s \leq I_c^r$. Since x_c^s and x_c^r satisfy SSARP, $p^s \cdot x_c^r > I_c^s$. Hence

[2] Although we have developed ARE using the Tarski-Seidenberg algorithm to eliminate quantifiers, we present here a more direct proof of the equivalence between the equilibrium inequalities and ARE.

$p^s \cdot \bar{z}_c^r > I_c^s$ and we have shown that conditions (0) and (i) of ARE are satisfied.

Next suppose that when $s \neq r$, $p^r \cdot \bar{z}_a^s \leq I_a^r$ and $p^r \cdot \bar{z}_b^s \leq I_b^r$. Then $p^r \cdot x_a^s \leq I_a^r$ and $p^r \cdot x_b^s \leq I_b^r$. By SSARP $p^s \cdot x_a^r > I_a^s$ and $p^s \cdot x_b^r > I_b^s$. Since $x_a^r + x_b^r = \omega^r$, it follows that $p^s \cdot \omega^r = p^s \cdot (x_a^r + x_b^r) > I_a^s + I_b^s = p^s \cdot \omega^s$. So $p^s \cdot \omega^r > p^s \cdot \omega^s$ and we have shown that condition (ii) of ARE is satisfied. This completes the proof of necessity.

To prove sufficiency, we show that ARE implies that there exists $\{x_c^r\}_{r=1,2;c=a,b}$ satisfying SSARP, the budget equations, and the market clearing equations.

Consider the following cases:

Case 1: $p^r \cdot \bar{z}_a^s > I_a^r$ and $p^s \cdot \bar{z}_b^r > I_b^s$ for $s \neq r$

Case 2: $p^r \cdot \bar{z}_a^s > I_a^r$, $p^r \cdot \bar{z}_b^s > I_b^r$, and $p^r \cdot \omega^s > p^r \cdot \omega^r$ for $s \neq r$

Note that ARE implies that either Case 1 is true, or Case 1 is not true and Case 2 is true.

When Case 1 is true, it follows from the definitions of $\bar{z}_a^s$ and $\bar{z}_b^r$ that there exists x_a^s and x_b^r such that

4.3) $0 \leq x_a^s \leq \omega^s$ and $0 \leq x_b^r \leq \omega^r$,

4.4) $p^s \cdot x_a^s = I_a^s$ and $p^r \cdot x_b^r = I_b^r$, and

4.5) $p^r \cdot x_a^s > I_a^r$ and $p^s \cdot x_b^r > I_b^s$.

Let $x_b^s = \omega^s - x_a^s$ and $x_a^r = \omega^r - x_b^r$. Then, $\{x_c^r\}_{r=1,2;c=a,b}$ satisfy the equilibrium equalities by definition and SSARP by (4.5). By (4.4) and (0) in ARE, x_b^s and x_a^r satisfy the budget equations.

When Case 1 is not true and Case 2 is true[3], it follows from (0) in ARE that

4.6) $I_a^r = p^r \cdot \omega^r - I_b^r < p^r \cdot \omega^s - I_b^r$.

Since $p^r \cdot \bar{z}_b^s > I_b^r$ and $\bar{z}_b^s = \omega^s - \underline{z}_a^s$, it follows that

$$p^r \cdot \underline{z}_a^s < p^r \cdot \omega^s - I_b^r.$$

So, since $p^r \cdot \bar{z}_a^s > I_a^r$ and $I_a^r < p^r \cdot \omega^s - I_b^r$, there must exist x_a^s satisfying

4.7) $0 \leq x_a^s \leq \omega^s$,

[3] The original proof of this case contained an error. We thank Susan Snyder for pointing it out to us.

4.8) $p^s \cdot x_a^s = I_a^s$, and

4.9) $I_a^r < p^r \cdot x_a^s < p^r \cdot \omega^s - I_b^r$.

Let $x_b^s = \omega^s - x_a^s$. Then x_b^s and x_a^s satisfy the equilibrium equality. By (0) in ARE and (4.8) $p^s \cdot x_b^s = I_b^s$. And by (4.9)

$$p^r \cdot x_b^s = p^r \cdot (\omega^s - x_a^s) = p^r \cdot \omega^s - p^r \cdot x_a^s > p^r \cdot \omega^s - p^r \cdot \omega^s + I_b^r = I_b^r.$$

So,

4.10) $p^r \cdot x_b^s > I_b^r$.

Let x_a^r be any vector such that

$$0 \le x_a^r \le \omega^r \qquad \text{and} \qquad p^r \cdot x_a^r = I_a^r.$$

Let $x_b^r = \omega^r - x_a^r$. It then follows that

$$p^r \cdot x_b^r = p^r \cdot \omega^r - p^r \cdot x_a^r = I_b^r \quad \text{and} \quad 0 \le x_b^r \le \omega^r.$$

Finally, by (4.9) and (4.10), $\{x_c^r\}_{r=1,2;c=a,b}$ satisfy SSARP.

Proof of Theorem 4:[4] We first prove the necessity of HARE. Suppose that there exist monotone, concave, and homothetic utility functions rationalizing the data. Let $\{x_c^r\}_{r=1,2;c=a,b}$ be obtained by maximizing such utility functions over the budget constraints determined by $\{p^r\}_{r=1,2}$ and $\{I_t^r\}_{r=1,2}$. Then,

5.1) $x_c^r \ge 0 \qquad r=1,2;\ c=a,b$

5.2) $p^r \cdot x_c^r = I_c^r \qquad r=1,2;\ c=a,b$

5.3) $x_a^r + x_b^r = \omega^r \qquad r=1,2$

5.4) $(p^1 \cdot x_a^2)(p^2 \cdot x_a^1) \ge \gamma_a$ and

5.5) $(p^1 \cdot x_b^2)(p^2 \cdot x_b^1) \ge \gamma_b$.

Expression (5.4) and (5.5) characterize demand data that is consistent with the maximization of concave, homothetic, and monotone utility functions (See Varian (18).)

By (5.3) and (5.5)

$$\gamma_b \le p^1 \cdot (\omega^2 - x_a^2)\, p^2 \cdot (\omega^1 - x_a^1).$$

So that

[4] As in the proof of Theorem 3, we provide a direct proof instead of deriving the axiom using the Tarski-Seidenberg algorithm.

$$\psi_1 = \gamma_b - \gamma_a - \gamma_\omega$$
$$\leq (p^1 \cdot \omega^2)(p^2 \cdot \omega^1) - (p^1 \cdot \omega^2)(p^2 \cdot x_a^1) - (p^1 \cdot x_a^2)(p^2 \cdot \omega^1) + (p^1 \cdot x_a^2)(p^2 \cdot x_a^1) - \gamma_a - \gamma_\omega$$
$$= -(p^1 \cdot x_a^2)\, p^2 \cdot (\omega^1 - x_a^1) - (p^1 \cdot \omega^2)(p^2 \cdot x_a^1) - \gamma_a$$
< 0, which proves (1).

Using (5.4), (5.5), the definitions of $\bar{z}_a^2$ and $\bar{z}_b^2$, and (5.3), it follows that

$$(5.6) \qquad p^2 \cdot x_a^1 = (p^2 \cdot x_a^1)\frac{(p^1 \cdot x_a^2)}{(p^1 \cdot x_a^2)} \geq \frac{\gamma_a}{(p^1 \cdot x_a^2)} \geq \frac{\gamma_a}{(p^1 \cdot \bar{z}_a^2)} = r_1$$

and

$$p^2 \cdot (\omega^1 - x_a^1) = p^2 \cdot x_b^1 = \frac{(p^2 \cdot x_b^1)(p^1 \cdot x_b^2)}{p^1 \cdot x_b^2} \geq \frac{\gamma_b}{p^1 \cdot x_b^2} \geq \frac{\gamma_b}{p^1 \cdot \bar{z}_b^2}$$

so that

$$(5.7) \qquad p^2 \cdot x_a^1 \leq p^2 \cdot \omega^1 - \frac{\gamma_b}{p^1 \cdot \bar{z}_b^2} = r_2.$$

Using again (5.4) and (5.5) we get that

$$p^1 \cdot x_a^2 \geq \frac{\gamma_a}{p^2 \cdot x_a^1}$$

and

$$p^1 \cdot x_b^2 \geq \frac{\gamma_b}{p^2 \cdot (\omega^1 - x_a^1)}$$

so that adding up and using (5.3) we get

$$p^1 \cdot \omega^2 \geq \frac{\gamma_a}{p^2 \cdot x_a^1} + \frac{\gamma_b}{p^2 \cdot (\omega^1 - x_a^1)}$$

or

$$(p^1 \cdot \omega^2)\,(p^2 \cdot x_a^1)^2 + (\gamma_b - \gamma_a - \gamma_\omega)\,(p^2 \cdot x_a^1) + \gamma_a\, p^2 \cdot \omega^1 \leq 0.$$

Consider the quadratic function $f(t) = (p^1 \cdot \omega^2)t^2 + (\gamma_b - \gamma_a - \gamma_\omega)t + \gamma_a \gamma_\omega$. Notice that $f(0) = \gamma_a\, p^2 \cdot \omega^1 > 0$, $f'(0) = (\gamma_a - \gamma_b - \gamma_\omega) = \psi_1 < 0$ and $f(p^2 \cdot x_a^1) \leq 0$.

Hence the quadratic equation $f(t) = 0$ must have real roots t_1 and t_2. Therefore, $\psi_2 = (\gamma_b - \gamma_a - \gamma_\omega)^2 - 4\gamma_a(p^1 \cdot \omega^2) \geq 0$, which proves (2).

Moreover, since the roots of the quadratic equation $f(t) = 0$ are real numbers t_1 and t_2, it follows that $f(0) > 0$, $f'(0) < 0$, $f''(t) = 2p^1 \cdot \omega^2 > 0$ for all t, and $f(p^2 \cdot x_a^1) \leq 0$, it follows that

(5.8) $t_1 \leq p^2 \cdot x_a^1 \leq t_2$.

Since by (5.6) and (5.7)

$$r_1 \leq p^2 \cdot x_a^1 \leq r_2,$$

it follows that $r_1 \leq r_2$, which proves (3) and

$$s_1 = \max\{ r_1, t_1\} \leq p^2 \cdot x_a^1 \leq \min\{ r_2, t_2\} = s_2,$$

which proves (4). Moreover, since $s_1 \leq p^2 \cdot x_a^1 \leq s_2$, it follows by the definitions of $\overline{z}_a^1$ and $\underline{z}_a^1$ that

$$p^2 \cdot \underline{z}_a^1 \leq s_2 \quad \text{and} \quad p^2 \cdot \overline{z}_a^1 \geq s_1,$$

which prove (5) and (6).

Finally, (7) is satisfied by (5.2) and (5.3). This completes the proof of necessity.

To show the sufficiency of HARE, we first note that by (2) t_1 and t_2 are real numbers and by (1) and inspection, $0 < t_1 < t_2$. By (4), (5), and (6), there exists x_a^1 such that

$$p^1 \cdot x_a^1 = I_a^1,$$
$$0 \leq x_a^1 \leq \omega^1, \text{ and}$$
$$s_1 \leq p^2 \cdot x_a^1 \leq s_2.$$

Take such x_a^1. It follows that

(5.9) $t_1 \leq p^2 \cdot x_a^1 \leq t_2$ and

(5.10) $r_1 \leq p^2 \cdot x_a^1 \leq r_2$.

The latter expression implies that

$$p^2 \cdot x_a^1 \leq p^2 \cdot \omega^1 - \frac{\gamma_b}{p^1 \cdot \overline{z}_b^2} = p^2 \cdot \omega^1 - \frac{\gamma_b}{p^1 \cdot (\omega^2 - z_a^2)} \quad \text{and}$$

$$p^2 \cdot x_a^1 \geq \frac{\gamma_a}{p^1 \cdot \bar{z}_a^2}$$

so that

(5.11) $p^1 \cdot z_a^2 \leq p^1 \cdot \omega^2 - \dfrac{\gamma_b}{p^1 \cdot (\omega^1 - x_a^1)}$

(5.12) $p^1 \cdot \bar{z}_a^2 \geq \dfrac{\gamma_a}{p^2 \cdot x_a^1}$

Let $f(t) = (t - t_1)(t - t_2)$. For all $t \in [t_1, t_2]$ $f(t) \leq 0$. Hence, (5.9) implies that

$$(p^2 \cdot x_a^1 - t_1)(p^2 \cdot x_a^1 - t_2) \leq 0$$

or for $t_1 = \dfrac{-\psi_1 - (\psi_2)^{1/2}}{2p^1 \cdot \omega^2}$ and $t_2 = \dfrac{-\psi_1 + (\psi_2)^{1/2}}{2p^1 \cdot \omega^2}$ we have

$$(p^2 \cdot x_a^1)^2 + (p^2 \cdot x_a^1)\frac{\psi_1}{p^1 \cdot \omega^2} + \gamma_a \frac{p^2 \cdot \omega^1}{p^1 \cdot \omega^2} \leq 0.$$

Rearranging terms, we derive that

$$\frac{\gamma_a}{p^2 \cdot x_a^1} \leq p^1 \cdot \omega^2 - \frac{\gamma_b}{p^2 \cdot (\omega^1 - x_a^1)}.$$

Hence, by (5.11) and (5.12) it follows that there exists x_a^2 such that

$$0 \leq x_a^2 \leq \omega^2,$$

$p^2 \cdot x_a^2 = I_a^2$, and

$$\frac{\gamma_a}{p^2 \cdot x_a^1} \leq p^1 \cdot x_a^2 \leq p^1 \cdot \omega^2 - \frac{\gamma_b}{p^2 \cdot (\omega^1 - x_a^1)}.$$

Hence,

5.13) $\gamma_a \leq p^1 \cdot x_a^2 \, p^2 \cdot x_a^1$

and

$$\gamma_b \leq p^1 \cdot (\omega^2 - x_a^2) \; p^2 \cdot (\omega^1 - x_a^1).$$

With (5.13) we have completed the proof that $\{p^r\}_{r=1,2}$, $\{I_a^r\}_{r=1,2}$, and $\{x_a^r\}_{r=1,2}$ satisfy HARP.

Let $x_b^1 = \omega^1 - x_a^1$ and $x_b^2 = \omega^2 - x_a^2$. Since $0 \leq x_a^1 \leq \omega^1$, $0 \leq x_a^2 \leq \omega^2$, $p^1 \cdot x_a^1 = I_a^1$, and $p^2 \cdot x_a^2 = I_a^2$, it follows, using (7), that

$$0 \leq x_b^1 \leq \omega^1, \quad 0 \leq x_b^2 \leq \omega^2, \quad p^1 \cdot x_b^1 = I_b^1, \quad p^2 \cdot x_b^2 = I_b^2,$$

and by above

$$\gamma_b \leq p^1 \cdot x_b^2 \;\; p^2 \cdot x_b^1.$$

Hence, also $\{p^r\}_{r=1,2}$, $\{I_a^r\}_{r=1,2}$, and $\{x_b^r\}_{r=1,2}$ satisfy HARP.

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