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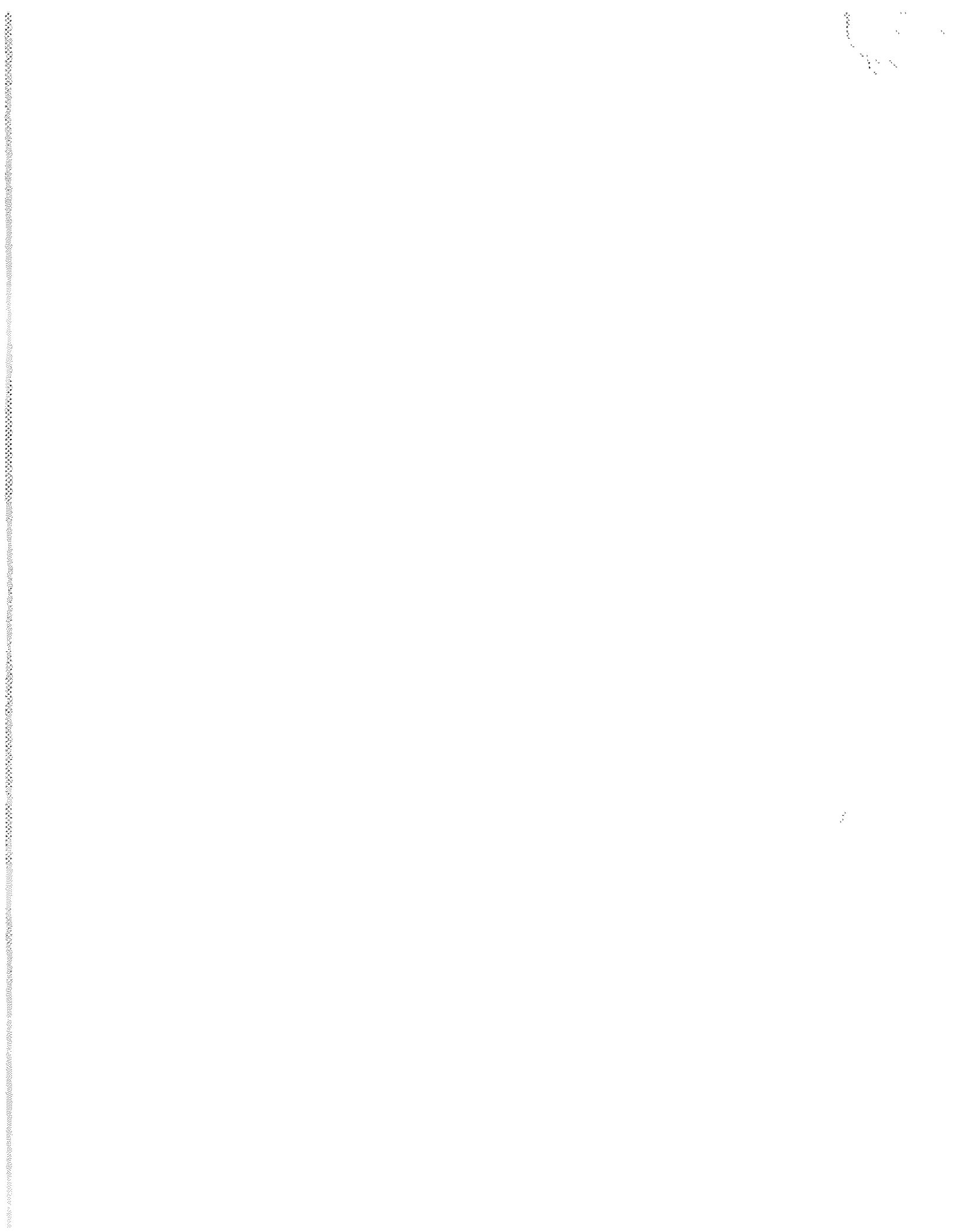


AN EQUILIBRIUM APPROACH TO THE THEORY OF FUTURES MARKETS

PETER BERCK and STEPHEN G. CECCHETTI

In the Keynes-Hicks (K-H) view of futures markets, hedgers pay speculators a risk premium for the insurance services the speculators offer. As a theoretical proposition, the K-H view therefore depends on the premise that speculators increase their risk when they buy futures. But empirical evidence, such as that presented in the Cootner [3, 4]-Telser [10] debate, lends only weak support to the K-H proposition or its premise. This paper examines the robustness of the K-H proposition in an equilibrium model of cash and futures markets that admits diversification and inflation protection as speculative motives and, therefore, need not yield the K-H proposition. In this general model, we present a simple criterion--in terms of the correlation of futures price with anticipated consumption net of other asset holdings--for the validity of the K-H proposition and its premise. The comparative statics of changes in supply and demand in the assets and cash markets on open interest, storage, risk premium, and the spread are also considered.

The paper is organized in five parts, the first of which is this introduction. The second part presents the pricing of assets in a two-period world when both asset payoffs and consumption prices are uncertain. The equilibrium model is constructed in the third part. It is an extension of Stein's [9] model: the second period cash market is made explicit and the asset-pricing equations of part 2 replace Stein's ad hoc supply of speculation equation. The model is then solved for the equilibrium values of storage and other variables. Section 4 presents a series of lemmas and two theorems that are



the comparative statics results and show when the K-H proposition is indeed true. Section 5 presents a discussion and some further conclusions.

2. ASSET-PRICING EQUATIONS

We consider the portfolio problem of dividing wealth, W , among assets, z , at purchase prices, s , to maximize expected utility when both second-period asset prices and consumption prices are uncertain. Expanding the first-order conditions for this problem in a Taylor series (as Arrow [1] and Pratt [8] did for the direct utility function) gives both a risk premium and an asset-pricing equation in terms of expected real return. The expected real return is approximately the expected nominal return less the covariance of the asset's return with the cost of a unit step along the income-expansion path, which is the inflation premium. The risk premium can be characterized by the covariance of the asset's return with the entire asset portfolio net of anticipated consumption. When applied to pricing futures, these equations are found to differ from constant consumption price asset-pricing equations in their inclusion of terms related to second-period consumption. They differ from the equations of Grauer and Litzenberger [5] in their use of a Taylor expansion and their admission of nonhomothetic utility functions as well as in their emphasis on second-period consumption rather than on covariance of marginal utility and real price.

The wealth holder faces uncertainty in the form of unknown prices for the assets he holds and the goods he will consume. By convention, these prices are a stochastic N -vector, p , whose first M elements price assets and whose

remaining elements price consumption goods. Similarly, z and s are N -vectors with their last $N-M$ elements zero. The utility indicator is

$$u(z) = E [v(p'z, p)] \quad (2.1)$$

where v is the indirect utility function (of stochastic income, $y = p'z$, and price, p) and E is the expectation operator. The problem is to maximize $u(z)$ subject to $s'z = W$.

Assuming that the first and i^{th} assets are both included in an optimal portfolio and that $s_1 \neq 0$, the first-order condition for this problem is: the ratio of asset prices equals the rate of asset substitution, or,

$$\frac{E v_y p_i}{E v_y p_1} = \frac{s_i}{s_1} \quad (2.2)$$

A futures contract is an asset that (in its perfect form) costs nothing ($s = 0$) and pays off its closing or settlement price, p_F , less its opening price, p_F^0 . Applying this definition to equation (2.2) gives a more general form (not restricted by homotheticity) of the Grauer-Litzenberger [5] asset-pricing equation for futures.

$$p_F^0 = \frac{E p_F v_y}{E v_y} \quad (2.3)$$

This first-order condition can be extended to the case of an imperfect future (one for which $s \neq 0$) by pricing with respect to a nominal bond. A nominal bond is an asset that costs a dollar and pays off n dollars in all states of nature. Carrying out the algebra, one obtains

$$p_F^0 = \frac{E p_F v_y}{E v_y} - n s_F. \quad (2.4)$$

Equation (2.4) is a general pricing equation that allows for both costs that occur in the first period (brokerage fee for futures and purchase price for stocks) and a return based on an opening (or striking) price.

These pricing equations can be interpreted if v_y and p_F are expanded in a Taylor expansion about their means. The interpretation will be in terms of the bundle of goods purchased at expected prices which we will call the anticipated bundle, $x(Ey, Ep)$. It is this anticipated bundle that makes the choice of assets dependent on "real" magnitudes. The algebra proceeds by expanding v_y as follows:

$$v_y(y, p) \doteq v_y + (v_{yy}z' + v'_{yp}) \tilde{p}, \quad (2.5)$$

where $\tilde{p} = p - \bar{p}$ and the right-hand side is evaluated at the expected prices, $\bar{p} \equiv Ep$. Differentiating Roy's identity ($-x \equiv v_p/v_y$) with respect to y and substituting in equation (2.5),

$$v_y(y, p) \doteq v_y + [v_{yy}(z - x) - v_y x_y]' \tilde{p}, \quad (2.6)$$

where the right-hand side is again evaluated at \bar{p} . Rearranging equation (2.4),

$$E [v_y(p_F - p_F^0 - n s_F)] = 0, \quad (2.7)$$

expanding p_F about its mean, defining absolute risk aversion ($r = -v_{yy}/v_y$), and using equation (2.5) gives the pricing equation in terms of the first and second moments of p :

$$p_F^0 = \bar{p}_F - x_y' E \tilde{p} \tilde{p}_F - rz' E \tilde{p} \tilde{p}_F + rx' E \tilde{p} \tilde{p}_F - ns_F. \quad (2.8)$$

The price of a future is its expected (nominal) price less the covariance of the cost of income expansion with the futures price, less the coefficient of risk aversion times the covariance of the portfolio and the futures price, plus the coefficient of risk aversion times the covariance of the cost of anticipated consumption and the futures price, less the nominal rate of interest times the "up front" costs.

One can say a little more about the first two terms of (2.8) for the homothetic utility function, $v = h[f(p)y]$. By the usual Taylor series expansion, expected real return (Efp_F) is approximately $f(\bar{p}) [p_F - x_y' E \tilde{p} \tilde{p}_F]$. Thus, the futures price is approximately the expected closing price divided by the deflator less the risk term.

Expression (2.8) differs in two ways from the result without uncertain consumption prices. First, one subtracts $x_y' E \tilde{p} \tilde{p}_F$ from the expected return. This term adjusts the mean return downward so that high nominal payoffs when prices are high are accurately reflected as low real payoffs. Second, from the covariance of the market portfolio with expected return, one subtracts the covariance of consumption, a term which adjusts the covariance of return so that it becomes the covariance with the portfolio net of anticipated consumption.

3. FUTURES MARKET EQUILIBRIUM

A reasonable representation of a futures market requires equilibrium in the futures (or asset) market itself, in the first-period cash market, and in the second-period cash market. This section makes use of the asset-pricing equation (2.8) to give a symmetrical view of hedgers and speculators. What distinguishes these agents is that the speculators ("wealth holders") own the wealth of the economy but do not own physical stocks, while the hedgers ("storers") own the stocks but do not own other of the economy's wealth.¹ We shall discuss below the institutional factors that keep these agents from diversifying their portfolios through the obvious means of selling shares in stocks.

The remaining two markets are quite simple. There is a linear demand for the stored good in both the first and second periods. In the first period, the good is split between storage and satisfying the demand; in the second period, this storage is added to whatever may be stochastically produced and is then consumed. The solution of the model proceeds by substituting the other equations into the equation for the futures market until it becomes an equation in storage alone. Given equilibrium storage, it is easy to find the open interest, spread, and risk premium.

The storers of commodities--especially agricultural commodities--are (or, traditionally, were) closely held firms dependent on loans for operating capital and on futures markets for insurance against price changes. They hold title to storage facilities and they pay off mortgages. In addition to these fixed investments, they choose three other assets for their portfolios: (1) a commodity, S , purchased at price p_c , stored at (possibly uncertain) cost $c(s, t)$, where t is factor prices, and sold for price p_s ; (2) loans at the

gross nominal rate of interest, n ; and (3) futures, z_F , without margin or brokerage, at opening price, p_F^0 , and closing price, p_F , which (assuming no basis risk) is the same as the second-period cash price, p_S .² Since the purchase of commodities is financed with a loan, the choice of storage is also the choice of loan size; and there are no financial constraints on the storer's portfolio choice problem:

$$\max_{z_F, S} \text{Ev}[S(p_F - np_c) - c(S, t) + z_F(p_F - p_F^0), p] \quad (3.1)$$

The storer's choice problem leads to the economy's demand for storage, to its supply of futures, and to a relationship between the cash and futures prices. To find the relationship between prices, take the derivative of Ev with respect to S and z_F and set them equal to zero and to each other:

$$\text{Ev}_y \cdot (p_F - np_c - c') = \text{Ev}_y \cdot (p_F - p_F^0) = 0, \quad (3.2)$$

where c' is marginal costs.

On rearranging, noting that np_c is nonstochastic, using the definition of covariance, and dividing by Ev_y ,

$$p_F^0 = np_c + Ec' + \frac{\text{cov}(c', v_y)}{\text{Ev}_y}. \quad (3.3)$$

This relationship holds with strict equality whenever even one agent holds any hedged stock.³

Insofar as risk in marginal cost c' cannot be insured, high marginal costs are expected to accompany low income and high v_y . So, when c' is

unexpectedly high, v_y is also high. Thus, the covariance term should be positive; and it is most likely that p_F^0 exceeds $np_c + Ec'$. When there is no risk of fire or shortage of transportation (or whatever would make c' stochastic) and when, as assumed previously, there is no basis risk, then,

$$p_F^0 = np_c + c'(S,t), \quad (3.4)$$

which can be inverted to give the demand for storage,

$$S = c'^{-1}(p_F^0 - np_c). \quad (3.5)$$

Finding the supply of futures requires the use of the approximate pricing equation (2.8). The storers, whom we denote A, have asset bundle, z^A , which is the vector with the F^{th} element

$$c'^{-1}(p_F^0 - np_c) + z_F^A$$

and all other elements zero.⁴ Solve for z_F^A to yield the supply of futures:

$$z_F^A = \frac{E p_F - p_F^0 + r^A x^A E p p_F - x_y^A E p p_F}{r^A \sigma^2} - c'^{-1}(p_F^0 - np_c), \quad (3.6)$$

where $\sigma^2 = E \tilde{p}_F \tilde{p}_F$.

The demand side of the futures market consists of a financier (wealth holder), whom we denote as B. He is not in the physical storage business;

therefore, S is zero in his portfolio. He may choose any futures holding, z_F^B . The remainder of his portfolio is the economy's endowment of claims on real capital, z^* . The wealth holder does not hold shares in the physical storage firms because syndicating so many firms would involve a very large transactions cost, much larger than the transactions costs of creating a single futures market. Although futures are imperfect claims on the firm (they fail to diversify the risks of physical storage), they are good enough claims to dominate the use of the more costly shares. The argument for the financier to hold the economy's real capital is the familiar portfolio argument of the capital asset-pricing model: the supply of real capital is completely inelastic within a given period, and someone must own it.⁵ The wealth holder's demand for futures can be found by letting z in the pricing equation (2.8) be $z_F^B + z^*$ and solving for z_F^B :

$$z_F^B = \frac{E p_F - p_F^0 + r^B (x^B - z^*)' \tilde{E} p p_F - x_y' \tilde{E} p p_F}{\sigma^2 r^B}. \quad (3.7)$$

The risk premium, open interest, and quantity stored are equilibrium notions. They are determined by finding the prices and quantities that simultaneously clear the asset market, the first-period cash market, and the second-period cash market. To find these quantities and examine how they change with changes in exogenous parameters, we will use the Taylor series approximations and impose homotheticity on the assumed identical consumption preferences of the agents. The asset market clears when the net futures position of the economy is zero:

$$z_F^A + z_F^B = 0.$$

Adding equation (3.6) to equation (3.7) and solving for $p_F^0 - E p_F$ expresses equilibrium in the futures market:

$$p_F^0 - E p_F = \left[\left(\frac{r^A r^B}{r^A + r^B} \right) (x^A - S + x^B - z^*) - x_y \right] E \tilde{p}_F. \quad (3.8)$$

The first-period cash market clears when the storers' bid price and the consumers' demand price are the same. Let

$$p_C = a_1 - b_1(S_0 - S) \quad (3.9)$$

be the consumers' bid price in the first period. By equation (3.9), the asset-pricing equation for hedgers [equation (2.8)] and equation (3.4), the first-period cash market clears when

$$n[a_1 - b_1(S_0 - S)] + c' = E p_F - x_y' E \tilde{p}_F + r^A (x^A - S - z_F)' E \tilde{p}_F. \quad (3.10)$$

The second-period cash market is cleared when storage equals demand,

$$p_S = a_2 - b_2 S + \epsilon, \quad (3.11)$$

where ϵ is a random variable with expectation zero and variance σ^2 .

The equilibrium system [equations (3.8) through (3.11)] can be solved by substituting the other equations into equation (3.8). Combining the cash futures spread, equation (3.4), and the first-period demand equation (3.9) yields

$$p_F^0 = n[a_1 - b_1(S_0 - S)] + c'. \quad (3.12)$$

Noting that $E p_F = E p_S$ and making use of the second-period demand equation gives

$$E p_F = a_2 - b_2 S. \quad (3.13)$$

Combining the last two equations gives an expression in S that is the negative of the risk premium and is the expression on the right-hand side of the futures market-clearing equation (3.8); making the substitution into equation (3.8) gives the futures market-clearing equation:

$$c' + n a_1 - n b_1 S_0 - a_2 + (n b_1 + b_2) S^* = \left[\left(\frac{r^A r^B}{r^A + r^B} \right) (x^A - S^* + x^B - z^*) - x_y \right] E \tilde{p} \tilde{p}_F. \quad (3.14)$$

Note that x^A , x^B , and x_y are functions of $E p_F$ and by equation (3.13) of S ; so that, with these substitutions, equation (3.14) contains only the single endogenous variable, S , with S^* as its solution. Open interest,

$$z_F = z_F^B,$$

is computed⁶ by solving equations (3.6) and (3.7) for

$$(E p_F - p_F^0 - x_y' E \tilde{p} \tilde{p}_F),$$

eliminating the common term, and solving for z_F^* as follows:

$$z_F^* = \frac{1}{\sigma^2 (r^A + r^B)} [r^B (x^B - z^*) - r^A (x^A - S^*)]' E \tilde{p} \tilde{p}_F. \quad (3.15)$$

It is these future market-clearing equations, not the first-order conditions (2.2) and (2.8), that determine the quantities and prices in the futures market.

4. COMPARATIVE STATICS

Under reasonable assumptions, the equilibrium futures model of Section 3 yields the intuitive results usually attributed to Keynes [7] and Hicks [6] and further explored by Stein [9]. This section sets out sufficient conditions for the risk premium ($E p_F - p_F^0$) to be positive, for an increase in the cost of storage to decrease storage, and for increases in speculation to decrease the risk premium. It also examines the spread ($p_F^0 - p_C$) and the effects on other variables of changes in either storage costs or demand. The general finding, embodied in a string of lemmas and two propositions is that the K-H intuition leads (with additional assumptions) to the same results as does our equilibrium model.

The K-H hypothesis that risk premium is positive holds [by equation (3.8)] whenever

$$RS^* \sigma^2 > [R(x^A + x^B) - x_y] E \tilde{p} \tilde{p}_F - Rz^* E \tilde{p} \tilde{p}_F, \quad (4.1)$$

where

$$R = \frac{r^A r^B}{r^A + r^B}.$$

The most obvious case is that in which the cost to wealth holders and storers of anticipated consumption is not correlated with futures prices (by homotheticity, x_y is also not correlated) and the portfolio, $z^* p$, is



nonnegatively correlated with the futures price. A well-diversified economy with a relatively small food sector (the United States, perhaps) might well meet these conditions. When a food sector is small, it is unlikely that shocks in it will be transmitted to the industrial sector; therefore, the prices of food and manufacturing should not move together. The small proportion of national income claimed by wealth holders (about one-third) also helps to ensure that consumption effects do not dominate the economy. In more agrarian economies, the covariance between agricultural prices and manufacturing prices would be higher but could be offset by the miniscule share of capital in gross national product. The usual case for agricultural commodities is certainly that of Keynes and Hicks.

Finding the effects of three actions: (1) increasing first-period demand, modeled by increasing a_1 , (2) increasing second-period demand, modeled by increasing a_2 , and (3) increasing costs, modeled by increasing t requires three additional assumptions.

Assumption 1:

$$E \left(\frac{dx}{dp_F} \tilde{pp}_F \right) = \sigma^2 \frac{d(x)_F}{dp_F},$$

where $(x)_F$ is the consumption of the stored good. The assumption is negligible covariance of the futures prices and change in the cost of the anticipated bundle exclusive of the stored good incident on a change in p_F . In essence, the assumption is that change in the price of the consumption good does not lead to the anticipated consumption of "riskier" goods.

Assumption 2:

$$\frac{1}{b_2} = y^T \frac{d(f_{p_F}/f)}{dp_F}$$

where y^T is national income and f is the deflator as defined in Section 2.

The second-period demand curve has a slope proportional to the demand curve of any agent evaluated at an expected price. This is an extension of the "identical homothetical" assumption to all consumers.

Assumption 3: r^A and r^B are constant.

With these three assumptions, it is possible to place bounds on the derivative of open interest with respect to optimal storage.

Lemma 1: If Assumptions 1, 2, and 3, then any action that increases equilibrium storage increases open interest but on a less than one-for-one basis.

Proof: Equation (3.15) gives open interest as a function of the actions (da_1, da_2, dt) through $x(E_{p_F})$ which is a function of S by equation (3.13) and through S directly. Taking the total derivative of equation (3.15), gives

$$\frac{dz_F}{dS^*} = \frac{1}{\sigma^2(r^A + r^B)} \left[r^B \frac{dx^B}{dp_F} \frac{dp_F}{dS^*} - r^A \left(\frac{dx^A}{dp_F} \frac{dp_F}{dS^*} - e_F \right) \right]' \tilde{E}_{pp_F} \quad (4.2)$$

where e_F is the unit vector in the F^{th} direction.

Use Assumption 1 to reduce

$$\left(\frac{dx^B}{dp_F} \tilde{E}_{pp_F} \right) \frac{dp_F}{dS^*} \quad \text{to} \quad \sigma^2 \frac{d(x^B)_F}{dp_F} \frac{dp_F}{dS^*} \quad (4.3)$$

and Assumption 2 to reduce

$$\sigma^2 \frac{d(x^B)_F}{dp_F} \frac{dp_F}{dS^*} \quad \text{to} \quad \sigma^2 \frac{y^B}{y^T} \quad (4.4)$$

Similar use of Assumptions 1 and 2 reduces the term for the storer, A, and yields

$$\frac{dz_F}{dS^*} = \frac{r^B y^B}{(r^A + r^B) y^T} + \frac{r^A (y^T - y^A)}{(r^A + r^B) y^T} \quad (4.5)$$

Both terms are positive so dz_F/dS^* is greater than zero. The derivative is less than one because it is a weighted average of terms less than one with weights $r^B/(r^A + r^B)$ and $r^A/(r^A + r^B)$.

Remark: $a^2_c/aS^2 \geq 0$ is a second-order condition for the hedger's maximization problem, and $a^2_c/aSat > 0$ is a property of cost functions.

Lemma 2: The spread changes with a_1 , a_2 , and t as follows:

$$(a) \quad \frac{d(p_F^0 - p_c)}{da_1} = \frac{a(p_F^0 - p_c)}{aS} \frac{dS}{da_1} + n - 1$$

$$(b) \quad \frac{d(p_F^0 - p_c)}{da_2} = \frac{a(p_F^0 - p_c)}{aS} \frac{dS}{da_2}$$

$$(c) \quad \frac{d(p_F^0 - p_c)}{dt} = \frac{a(p_F^0 - p_c)}{aS} \frac{dS}{dt} + \frac{a^2_c}{aSat}$$

where

$$\partial(p_F^0 - p_c)/\partial S = (n - 1) b_1 + \frac{\partial^2 c}{\partial S^2} > 0.$$

Proof: From equations (3.4) and (3.9),

$$p_F^0 - p_c = (n - 1) (a_1 - b_1 S_0 + b_1 S) + c'. \quad (4.6)$$

The lemma follows by taking the derivative with respect to a_1 , a_2 , and t .

Lemma 3: Any action that increases equilibrium storage also increases the gain to the long position, which is the risk premium.

Proof: Equation (3.8) gives the risk premium as a function of S and x ; as argued earlier, x is a function of p_F which, by equation (3.13), is a function of S so the right-hand side of equation (3.8) can be viewed as a function of the single endogenous variable, S . Since a_1 , a_2 , and c do not appear in equation (3.8), the effect of changing these variables on the risk premium comes solely through S . Carrying out the algebra,

$$\frac{d(Ep_F - p_F^0)}{dS} = \left[R \left(e_F - \frac{d(x^A + x^B)}{dp_F} \frac{dp_F}{dS} \right) + \frac{dx_y}{dp_F} \frac{dp_F}{dS} \right] \cdot \tilde{E}pp_F \quad (4.7)$$

and by Assumptions 1 and 2

$$\frac{d(Ep_F - p_F^0)}{dS} = R\sigma^2 \left(\frac{y^T - y^A - y^B}{y^T} \right) + \frac{\sigma^2}{y^2} > 0. \quad (4.8)$$

Lemma 4: An (a) increase in costs, (b) increase in first-period demand, or (c) decrease in second-period demand decreases optimal storage.

Proof: Recall the cost function $c(S,t)$ where t represents cost increasing factors such as factor prices. Taking the total derivative of equation (3.14) with respect to t and solving yields

$$\frac{dS^*}{dt} = \frac{\partial^2 c}{\partial S \partial t} \left(\frac{dp_F}{dS^*} \left[R \frac{d(x^A + x^B)}{dp_F} - \frac{dx_y}{dp_F} \right] - Re_F \right)' \left(\tilde{E}pp_F - b_1 n - b_2 - \frac{\partial^2 c}{\partial S^2} \right)^{-1} \quad (4.9)$$

Using the assumptions and results derived above for dz_F/dS^* , this expression simplifies to:

$$\frac{dS^*}{dt} = \frac{\partial^2 c}{\partial S \partial t} \left[- (nb_1 + b_2 + \partial^2 c / \partial S^2) - \left(\frac{y^T - y^A - y^B}{y^T} \right) \sigma^2 R - \frac{\sigma^2}{y^T} \right]^{-1} \quad (4.10)$$

which is negative because $y^T > y^A + y^B$, marginal cost increases with increasing factor costs, and all other symbols are positive.⁷

Again, using Assumptions 1 and 2, one can use the same method to show that $dS^*/da_2 > 0$, i.e., increasing second-period demand increases storage. Similarly, $dS^*/da_1 < 0$, i.e., increasing first-period demand decreases storage. Putting these four lemmas together, gives

Proposition 1: Under Assumptions 1, 2, and 3, if (a) first-period demand increases, (b) second-period demand decreases, or (c) storage costs increase, then, (a) S^* falls, (b) the risk premium decreases, and (c) open interest decreases—but by less than the decrease in S^* . Under Assumptions 1, 2,

and 3, a decrease in second-period demand decreases the spread. The effects of changes in first-period demand and costs on the spread are ambiguous.

In the K-H view of futures markets, an increase in the number (or wealth) of speculators should decrease the risk premium, $(E p_F - p_F^0)$. There are several ways in which this increase in speculation could occur. Agents in group B might become richer in some exogenous fashion such as a spurt in GNP growth, a change in trade policy, or a shift in labor laws that is unfavorable to workers. Another possibility is that agents formerly in the excluded group would be able to exercise their demands in the futures market. This could occur in a less-developed economy with the development of other organized capital markets and, particularly, with the syndicating of firms that were formerly privately held. For a country such as the United States, this would require that claims be created against the wealth of the excluded agents--presumably claims against their human capital--so that those agents could participate in the capital markets. The last possibility is, of course, farfetched. Any of these cases could correspond to an increase in speculation, and we model any of these changes by considering the case in which the portfolio owned by the wealth holder expands proportionately so that z^* becomes kz^* .

Lemma 5: Under Assumptions 1, 2, and 3, an increase in k (wealth) increases storage if, and only if, $E[(x^B - z^*)' \tilde{p} p_F] > 0$.

Proof: Note that, by the budget constraint, a proportional increase in z^* implies a proportionate increase in x^B . Solve equation (3.14) for S^* :

$$S^* = \frac{\left\{ R[x^A + k(x^B - z^*)] - x_y \right\}' E \tilde{p} p_F - c' - na_1 + nb_1 S_0 + a_2}{R\sigma^2 + nb_1 + b_2}. \quad (4.11)$$

Take the derivative with respect to k , evaluated at $k = 1$,

$$\frac{dS^*}{dk} = \frac{\left(\left\{ R \left[\left(\frac{dx^A}{dp_F} + \frac{dx^B}{dp_F} \right) \frac{dp_F}{dS^*} \frac{dS^*}{dk} + x^B - z^* \right] - \frac{dx_y}{dp_F} \frac{dp_F}{dS^*} \frac{dS^*}{dk} \right\}' E \tilde{p} p_F - \frac{dc'}{dS^*} \frac{dS^*}{dk} \right)}{R\sigma^2 + nb_1 + b_2}, \quad (4.12)$$

and use Assumptions 1 and 2 to simplify dS^*/dk :

$$\frac{dS^*}{dk} = \frac{(x^B - z^*)' E \tilde{p} p_F}{R \left(1 - \frac{y^A}{y_I} - \frac{y^B}{y_I} \right) + \frac{\sigma^2}{y} + nb_1 + b_2 + \frac{dc'}{dS^*}}. \quad (4.13)$$

Note that each term of the denominator is positive so that

$$\text{sign} \left(\frac{dS^*}{dk} \right) = \text{sign} [(x^B - z^*)' E \tilde{p} p_F]. \quad (4.14)$$

Lemma 6: Under Assumptions 1, 2, and 3, an increase in k decreases the risk premium if and only if

$$E[(x^B - z^*)' \tilde{p} p_F] > 0.$$

Proof: The risk premium is:

$$E p_F - p_F^0 = (-1) [c'(S, t) + na_1 - nb_1 S_0 - a_2 + (nb_1 + b_2) S]$$

$$\frac{d(Ep_F - p_F^0)}{dk} = - \left(\frac{\partial^2 c}{\partial S^2} + nb_1 + b_2 \right) \frac{dS}{dk}.$$

Since $\partial^2 c / \partial S^2$, n , b_1 , and b_2 are positive, the proposition follows.

Lemma 7: Under Assumptions 1, 2, and 3, an increase in k increases open interest if and only if $E[(x^B - z^*)' \tilde{p}\tilde{p}_F] > 0$.

Proof: From equation (3.15),

$$\frac{dz_F}{dk} = \frac{\partial z_F}{\partial S} \frac{dS}{dk} + \frac{r^B}{(r^A + r^B)\sigma^2} E(x^B - z^*) \tilde{p}\tilde{p}_F.$$

By Lemma 1, $\partial z_F / \partial S > 0$ and, by Lemma 5, dS/dk has the same sign as $E(x^B - z^*) \tilde{p}\tilde{p}_F$; the lemma follows.

Proposition 2: Under Assumptions 1, 2, and 3, if $(x^B - z^*)' E\tilde{p}\tilde{p}_F > 0$, then, with an increase in wealth, (a) S^* rises, (b) the risk premium decreases, and (c) open interest rises.

Proof: The result follows from Lemmas 5-7.

Combining the results of Proposition 2 and the condition for a positive risk premium, one concludes that the K-H intuition holds only when $E(x^B - z^*) \tilde{p}\tilde{p}_F$ is small and positive.

5. CONCLUSIONS

A model of futures markets in a two-period world requires equality of supply and demand in three markets: the first period cash market, the second period cash market, and the market for futures contracts. This formulation considers a futures contract as a standard financial asset and it naturally gives rise to a supply of speculation. Speculation depends on expected return and covariance of the future with the market portfolio net of anticipated consumption. The size of the risk premium embodied in these contracts depends inter alia on how large the anticipated consumption covariance or consumption hedging effect is. Since the correlation of commodity prices and the market portfolio can be as large as the correlation of any asset price and the market portfolio and since consumption has the same value as income, there is every reason to believe that the anticipated consumption effects will be large, at least in a two-period model. In a multiperiod model, one might expect these effects to be attenuated by the small percentage of wealth devoted to consumption in each period; but, when prices in period two convey information about prices in future periods, the effects would again be large.

The size and direction of these consumption effects partially determines whether or not the K-H proposition holds. So long as the futures price correlates nonpositively with the value of anticipated consumption and correlates nonnegatively with the market portfolio, the risk premium will be positive.

The last subject examined by this paper is the effect of increasing speculation. Increasing speculation can occur either through some exogenous force making speculators richer or through the repair of a market failure. The market failure is the exclusion of some agents, in the U. S. laborers and

in developing countries the poor of all types, from the organized capital markets. In this model (which has only a single market failure), Proposition 3 shows that the capital market failure leads to too little storage in equilibrium.

Other uses of parts of this model include the home purchase decision and the "food security" of the developing nations. The home purchase decision is fundamentally a question of hedging the future consumption of housing. Since a home purchase is risk reducing in the sense of correlation with the portfolio net of anticipated consumption, the prospective homeowner would be willing to pay a "risk premium" over the expected cost of housing services for its purchase. The developing nations' problem of food security is whether to invest in agriculture to grow food or whether to invest in other sectors and trade for food. The advantage of growing food is that it is risk reducing when anticipated consumption it is included in the notion of risk; the disadvantage is that it is usually not the investment with the highest expected value. The problem is formally analogous to the decision problem at the beginning of section 2 and illustrates that the portfolio problem, with uncertain consumption, has applications to many fields of economics besides finance.

FOOTNOTES

¹Technically, this model has only a single agent in each group; the generalization to n identical agents per group is tedious and trivial.

²Brokerage and margin costs are small and they are neglected in the analysis for the algebraic simplicity. Basis risk is the risk that $p_F \neq p_S$, usually because of delivery point or grade. There is no convenience yield--gain to having inventory to sell from--in a two-period model.

³Blau [2] and, later, Grauer and Litzenberger established a similar expression from an arbitrage argument. The argument gives the expression with inequality. To obtain equality, note that z_F is unbounded and S is bounded from below by zero. Thus, by the Kuhn-Tucker theorem, if $S > 0$, then both first-order conditions must hold with equality as must equations (3.3) through (3.5).

⁴ z_F^A and z_F^B are used both as the vector quantity with a non-zero F^{th} place and as a scalar quantity, the F^{th} element.

⁵It is assumed that the agents in this model take the return of the nominal bond as given. In essence, the market for these bonds is cleared by some force (the monetary authority) external to the system. Because the covariance of a nominal bond and any other nominal return--most particularly, $\text{cov}(n, p_F)$ --is zero, the quantity of the bonds held affects neither the pricing equation nor the futures market equilibrium. Therefore, we ignore the market for, and quantity of, nominal bonds.

⁶By homotheticity and identical preference the term in parentheses is the same for all agents.

7

In general,

$$\frac{dS^*}{dc'} = \left\{ \begin{aligned} & -(nb_1 + b_2) - \sigma^2 R \left(\frac{y^T - y^A - y^B}{y^T} \right) - \frac{\sigma^2}{y^T} \\ & + R \frac{dp_F}{dS^*} \sum_{i \neq F} \frac{d[(x^A)_i + (x^B)_i]}{dp_F} \text{cov}(p_i, p_F) \end{aligned} \right\}^{-1} .$$

Because the $\text{cov}(p_i, p_F)$ is of unknown sign, the expression cannot be signed without further assumptions. A weaker form of Assumption 1 that leaves dS^*/dc' unambiguously negative is:

$$\sum_{i \neq F} \frac{d[(x^A)_i + (x^B)_i]}{dp_F} \text{cov}(p_i, p_F) \geq 0.$$

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