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Abstract

Engineering models generally find that most consumers are unwilling to adopt energy efficient appliances, even though the financial returns are positive. It is commonly thought that this is either due to market imperfections such as an incomplete credit market, very high intertemporal consumer discount rates, or irrational behavior. This paper presents a more sanguine explanation based on a model of rational dynamic choice in an uncertain environment. A random utility model (RUM) with consumer preferences that depend on the quality mix of energy–using appliances predicts that under plausible conditions – including the consumer's intertemporal discount rate equal to the real market rate of return on risk free investments – it may well be optimal for consumers ***never*** to adopt an energy efficient appliance. Essential model parameters include purchase prices of new appliances, periodic costs of use, including energy, failure rates for appliances per period, quality mixes of the service flows generated by different appliance types, consumers' permanent incomes and other demographic variables, and the random components of the RUM preferences. Empirical implementation of this model is straightforward with a McFadden andTrain mixed multinomial logit econometric model using grouped time series, cross-sectional, or panel data.

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Energy Efficiency and Appliance Replacement

1. Introduction

State and federal policies and programs to encourage the adoption of energy saving appliances and other technologies in American homes is one important means of mitigating total energy consumption and greenhouse gas emissions in the United States. Engineering models of energy use and energy cost savings over time generally have found that most consumers are unwilling to adopt more energy efficient appliances even though the financial returns are positive. It is commonly argued that this must be due to market imperfections, high consumer discount rates, or irrational behavior.

This paper presents a more sanguine explanation for these observations based on rational dynamic consumption choice in an uncertain economic environment. A dynamic extension of the random utility model where consumer preferences depend on the quality mix of energy–using appliances predicts that under plausible conditions – including consumer intertemporal discount rates equal to the real market rate of return on a risk free investment – it may well be optimal for consumers never to adopt an energy efficient appliance.

The essential model parameters include the purchase prices of new appliances, the periodic costs of appliance use, the failure rates for appliances per period leading to a new replacement decision, the quality mixes of the service flows generated by different appliance types, consumers' permanent incomes and other demographic characteristics, and the random components of RUM preferences. The model is simple enough to be empirically implemented within any econometric model that has the same flavor as the McFadden and Train (2000) multinomial logit model using grouped time series, cross-section, or a mixture of cross-section/time-series panel data, given a reasonable choice for the periodic preferences of consumers and other essential model variables and parameters.

Section two develops the model and derives the optimal decision rule for rational dynamic expected utility maximizing households. The third section then briefly discusses choice of functional form and issues related to the practical application of this model.

2. Dynamic Appliance Choice

The purpose of this section is to develop a very simple model of purchase and replacement decisions for energy-using appliances. The discussion focuses on generic incandescent versus florescent electric light bulbs to focus ideas. However, the basic model can be generalized to any number of appliances that can provide substitute services – e.g., varieties of refrigerators, freezers, central heating and/or air conditioning units, and so forth.

There are two (or more) types of appliances available on the market. We denote the traditional appliance type as x^0, a generic incandescent bulb, and the alternative appliance type as x^1, a generic florescent bulb. The feasible choice set for appliance ownership, x, by the household in each time period are $x_t \in \{x^0, x^1, 0\}$, where $x^i = 1$ indicates ownership of a working light bulb of type $i = 0{,}1$, at the beginning of the t^{th} time period, $t = 1, 2, \ldots$ with $x_t^0 \cdot x_t^1 = 0 \ \forall \ t$, In other words, in each period the household can own and use an incandescent light bulb, a florescent light bulb, or neither, but not both.

A scalar index, q_0, measures the quality of the periodic service flow from an incandescent bulb, and this is higher than the quality of the flow of electric light services produced by a florescent bulb, q_1, $q_0 > q_1$. This can be generalized to any number of quality measures. For example, florescent bulbs cause headaches for some individuals when they are used for reading, some types of florescent bulbs make noise when they are on, and so forth. Alternatively, one could also think of the mix of characteristics of refrigerator-freezers – the freezer is on the top, bottom, or a side-by-side, whether there is an ice maker, the height, width, depth of the unit, the number of cubic feet of storage space in the refrigerator compartment and/or in the freezer compartment, the number and location of shelves, the noise of the compressor during operation, and so forth – as the vector of quality attributes that are part of the selection criteria incorporated by households in their purchase decisions. A scalar quality index captures the essential flavor of the economic issues involved while simplifying the analysis. The perceived quality of an appliance also may well be subjective and random across households, or even a source of uncertainty for the household. However, we abstract away from these considerations to simplify the basic model setup. We denote the quality of the existing stock of energy-using appliances by $q \in \{q_0, q_1, 0\}$, with $q = q_0$ if and only if $x = x^0$, and so forth.

The periodic use cost of an incandescent bulb, e_0, is higher than that of a florescent bulb, e_1, $e_0 > e_1$. On the other hand, the market price of a new incandescent bulb, p_0, is lower than that of a new florescent bulb, p_1, $p_0 < p_1$. Denote the real market rate of return on risk free investments by r. We define the *financial payoff period*, τ, for a florescent bulb relative to an incandescent bulb by the minimal number of time periods that a florescent light must be used in place of an incandescent one in order that the consumer at least breaks even:

$$\tau \equiv \min\left\{t \in \{0,1,2,\cdots\} \equiv \mathbb{N} : p_1 \le p_0 + \left(\frac{(1+r)^{t+1}-1}{r(1+r)^t}\right)(e_0 - e_1)\right\}, \tag{1}$$

where $\sum_{s=1}^{t}(1+r)^{-s} = [(1+r)^{t+1}-1]/r(1+r)^t$. To fix ideas, we make the following assumption on the relative magnitudes of the market prices and energy costs.

A1. $$p_0 + e_0 - e_1 < p_1 < p_0 + (1+r)(e_0 - e_1)/r\,.$$

In other words, the payoff period for a florescent bulb is more than one period but finite, $0 < \tau < \infty$. The household therefore must have a planning horizon that is at least two periods long before a florescent bulb would be financially viable. On the other hand, if both types of bulbs lasted forever and consumers had infinite planning horizons, and if there are no quality differences between appliance types, market barriers or inefficiencies, or some form of irrational or quasi-rational decision process on the part of households, then the florescent bulb would be the financially optimal choice in every period. A1 therefore captures the financial tension in any model that focuses on purchase prices and periodic use costs of energy-efficient appliances versus traditional appliance types.

Appliances do not last forever. This is particularly true for electric light bulbs. To capture this aspect of the decision problem within a simple framework, let the time-invariant probability that a working incandescent bulb fails by the end of period t be denoted by $1\text{-}\pi_0$, so that the probability that it is still in working condition at the beginning of the next period is π_0. Similarly, let the probability that a working florescent bulb fails by the end of period t be denoted by $1\text{-}\pi_1$, so that the probability that it is in working condition at the beginning of the next period is π_1. We assume that florescent bulbs fail more often than incandescent bulbs, $\pi_0 \geq \pi_1$, although this assumption is not essential.

We assume a random utility model (RUM) with preferences that depend on the type of energy using appliance in use by the household, x, a Hicks composite commodity representing the periodic consumption of all other goods, y, the quality of the service flows generated by the energy using appliance, q, and a vector of random variables, ε, known to the household but unknown to and unobservable by the analyst. Let the support for ε be denoted by $\boldsymbol{\mathcal{E}} \subseteq \mathbb{R}^n$. The periodic utility function is denoted by $u(x,y,q,\varepsilon)$. We require the following property.

A2. $u(\cdot,\varepsilon)$ is strictly increasing and jointly concave in (x,y,q) $\forall$ $\varepsilon \in \boldsymbol{\mathcal{E}}$.

To simplify the derivations below, we also assume that the appliance choice in each period comprises a sufficiently small part of household wealth that consumption smoothing arising from the permanent income hypothesis applies (see LaFrance 2002). We denote total periodic personal consumption expenditures by m and use the sobriquet *income* throughout to denote this value. The periodic budget constraint has the following form. If the household has a working bulb of type i at the end of period t–1 and chooses not to replace it with a new bulb of either type at the beginning of period t, then income equals the sum of the cost of using the appliance, e_i, and the expenditure on other goods, y, i.e., $m = e_i + y$. If for any reason, the household replaces an existing bulb with a new one of type j at the beginning of the t^{th} period, then income is the sum of the cost of that type of new bulb, the cost during the period of using that type of bulb, and total expenditure on all other goods, so that $m = p_j + e_j + y$. In particular, we assume that the household can freely dispose of an existing appliance whether or not it is in good working order at the beginning of each decision period. We also abstract away from the maintenance and repair decisions that are associated with durable appliance ownership and use. The model can be extended readily in these directions if this were considered to be important in a given application.

Preferences also clearly depend on household attributes in addition to income, such as ethnicity, education, family size, age and gender of family members, the occupation and employment status of adult members of the household, and so forth. However, these arguments of the utility function are suppressed to simplify the notational burden. Nevertheless, these are important indicators for this type of choice problem and it should be kept in mind that they are essential explanatory variables in any empirical implementation of the model developed here.

The approach that we take to analyze the dynamic appliance choice problem is stochastic dynamic programming. As is standard practice for this technique, we will begin with a finite planning horizon for the household, perform a backwards recursion from the last period in the planning horizon to the first by applying the Bellman equation (see Dreyfus and Law 1977), and then let the length of the planning horizon increase without bound to derive a stationary closed loop decision rule (Stokey and Lucas 1989). The final result is a criterion for choosing an incandescent or florescent bulb at the first replacement time as a function of the cost of each type of new bulb, the periodic cost of using each type of bulb, the quality of services provided by each type of bulb, the failure and survival rates of each type of bulb, household income (and other demographic factors), the intertemporal discount rate of the household, and the random component(s) of the RUM preference model.

Before we begin formally developing the complete solution to this model, we will establish some of the preliminary properties of the decision problem. First, consider the decision faced at the beginning of an arbitrary period, t, in which an existing bulb of type $i = 0{,}1$ is in working condition at the end of the previous period, $t-1$. The household has no need to replace the appliance at the beginning of the t^{th} period since the household can choose to receive the services of the existing bulb. For both types of bulb, it seems reasonable to assume that in a well-organized market economy, using one type of light bulb or the other is always preferred to no light source at all,

$$u(x^i, m - e_i, q_i, \varepsilon) > u(x^i, m - p_i - e_i, q_i, \varepsilon) > u(0, m, 0, \varepsilon),\ i = 0,1\,. \tag{2}$$

In other words, even for a single period, the household prefers to continue to use an existing bulb of type i to purchasing a new one of the same type (this is purely due to monotonicity of preferences in y) and also prefers to purchase a new appliance of either type to foregoing the convenience of having at least some source of electric lighting. Moreover, given A1 and $q_0 > q_1$, if there is only one period remaining in the planning horizon, the household will always opt for the relatively high energy–using incandescent type of bulb, unless perhaps it already owns an existing florescent bulb that is in good working order. We formalize this with the pair of conditions:

$$u(x^1, m - e_1, q_1, \varepsilon) > u(x^0, m - p_0 - e_0, q_0, \varepsilon)\,; \tag{3}$$

and $$u(x^0, m - e_0, q_0, \varepsilon) > u(x^1, m - e_1, q_1, \varepsilon) > u(x^1, m - p_1 - e_1, q_1, \varepsilon)\,. \tag{4}$$

In other words, it is not optimal for the household to replace a working bulb of either type with a new one of either type since the cost of the new bulb can be delayed at least one more period. The energy savings generated by a florescent bulb also must not be sufficient in a single period to overcome the perceived lower quality of the energy services provided, or the financial tension exhibited by A1 would not be observed in any real-world data.

We are now ready to derive the optimal decision rule for light bulb purchases, use, and replacement by an individual household. Let the length of the planning horizon be T periods, and consider the decision problem in the last period. It the end of the second to last period, we observe whether the existing bulb used in that period is still in working condition for the final period. If it is, then by A1, A2, and inequalities (2)–(4), the optimal decision is to use the existing bulb in the last period, generating a final period utility flow of $u(x^i, m - e_i, q_i, \varepsilon)$, $i = 0,1$. On the other hand, if the bulb has gone bad during the second to last period, then the optimal decision is to replace it with an incandescent bulb, generating a final period utility flow of $u(x^0, m - p_0 - e_0, q_0, \varepsilon)$, regardless of the type of bulb that was used in period T–1. Denote the state of nature at the end of the previous period by $x_{t-1}^{(+)} \in \{x^0, x^1, 0\}$. Also denote the optimal value function for the final period utility flow by $v_{T|T-1}(x_{T-1}^{(+)}, \boldsymbol{p}, \boldsymbol{e}, \boldsymbol{q}, \boldsymbol{\pi}, m, \varepsilon)$. Then we have

$$v_{T|T-1}(x_{T-1}^{(+)}, \boldsymbol{p}, \boldsymbol{e}, \boldsymbol{q}, \boldsymbol{\pi}, m, \varepsilon) = \begin{cases} u(x^0, m - e_0, q_0, \varepsilon), & x_{T-1}^{(+)} = x^0 \\ u(x^1, m - e_1, q_1, \varepsilon), & x_{T-1}^{(+)} = x^1 \\ u(x^0, m - p_0 - e_0, q_0, \varepsilon), & x_{T-1}^{(+)} = 0 \end{cases}. \tag{5}$$

Next, we step back to the second to the last period, and derive the optimal decision in that period, conditional on the state of nature for the previous period's light bulb, and incorporating the impact that this period's choice has on the optimal decision in the final period. We again have three possible states of nature which is revealed at the end of the second-to-last period, $x_{T-2}^{(+)} \in \{x^0, x^1, 0\}$. If the existing bulb is still good, then the household can keep it or replace it with a new one. However, the latter decision will not be optimal in light of inequalities (2)–(4), since the added cost of buying a good bulb can be deferred at least until the last period.

Hence, if the bulb is good, then the household continues to use it this period.[1] The two-period discounted value of total expected utility flows therefore satisfies

$$v_{T-1|T-2}(x^i, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) = u(x^i, m - e_i, q_i, \varepsilon)$$

$$+(1+\rho)^{-1}\left[\pi_i u(x^i, m - e_i, q_i, \varepsilon) + (1-\pi_i) u(x^0, m - p_0 - e_0, q_0, \varepsilon)\right],\ i = 0,1, \tag{6}$$

where $\rho > 0$ is the household's intertemporal discount rate.

On the other hand, if the bulb has gone bad in the second-to-last period, then the household compares the two-period expected utility flow

$$u(x^0, m - p_0 - e_0, q_0, \varepsilon)$$

$$+(1+\rho)^{-1}\left[\pi_0 u(x^0, m - e_0, q_0, \varepsilon) + (1-\pi_0) u(x^0, m - p_0 - e_0, q_0, \varepsilon)\right],$$

to the two-period expected utility flow

$$u(x^1, m - p_1 - e_1, q_1, \varepsilon)$$

$$+(1+\rho)^{-1}\left[\pi_1 u(x^1, m - e_1, q_1, \varepsilon) + (1-\pi_1) u(x^0, m - p_0 - e_0, q_0, \varepsilon)\right].$$

Grouping and rearranging terms, the optimal decision is to choose x^0 or x^1 as

$$\left(\frac{1+\rho+\pi_1-\pi_0}{1+\rho}\right) u(x^0, m - p_0 - e_0, q_0, \varepsilon) + \left(\frac{\pi_0}{1+\rho}\right) u(x^0, m - e_0, q_0, \varepsilon)$$

$$\gtrless\ u(x^1, m - p_1 - e_1, q_1, \varepsilon) + \left(\frac{\pi_1}{1+\rho}\right) u(x^1, m - e_1, q_1, \varepsilon)\,. \tag{7}$$

At this point, it is worth noting that $\pi_1 = \pi_0 \equiv \pi$ reduces the comparison to

[1] It is easy to show formally that replacing a good bulb in any period leads to a contradiction of the monotonicity of preferences in (x,y,q).

$$u(x^0, m - p_0 - e_0, q_0, \varepsilon) + \left(\frac{\pi}{1+\rho}\right) u(x^0, m - e_0, q_0, \varepsilon)$$

$$\gtrless u(x^1, m - p_1 - e_1, q_1, \varepsilon) + \left(\frac{\pi}{1+\rho}\right) u(x^1, m - e_1, q_1, \varepsilon)\,. \tag{8}$$

Hence, the probability of appliance failure in each period has exactly the same effect as an increase in the intertemporal discount rate. Both terms on the left-hand-side of (8) are greater than the corresponding terms on the right-hand-side by hypothesis, the household will purchase an incandescent bulb in period T–1 if replacement is required. Conversely, if $\pi_1 \neq \pi_0$, then either choice can be optimal in the two-period case. As long as $\tau > 1$, $q_0 > q_1$, and $\rho \geq r$, it is easy to see that $\overset{*}{x}_{T-1} \mid_{x_{T-2}^{(+)}=0} = x^0$. In other words, replacing with a florescent bulb can only be optimal if there is at least τ+1 periods remaining in the planning horizon, We denote the state-dependent two-period discounted expected total utility flow by $v_{T-1|T-2}(x_{T-2}^{(+)}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon)$.

Next, we continue in the same manner following the backward recursion to the initial period in the planning horizon, and then take the limit of this problem as $T \to \infty$. This is a stationary decision problem with Markov transition probabilities from one state to the next. This implies that the state-dependent infinite horizon discounted present value of expected utility flows converges to an optimal value function that does not depend explicitly on t (Dreyfus and Law 1977; Stokey and Lucas 1989). In each period t, if the appliance has failed at the end of the previous period, then the appliance choice decision problem is

$$v(0, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) = \max_{i=0,1} \Big\{ u(x^i, m - p_i - e_i, q_i, \varepsilon)$$

$$+(1+\rho)^{-1} \big[\pi_i v(x^i, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) + (1-\pi_i) v(0, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) \big] \Big\}, \tag{9}$$

subject to

$$v(x^0, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) = u(x^0, m - e_0, q_0, \varepsilon)$$

$$+(1+\rho)^{-1} \big[\pi_0 v(x^0, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) + (1-\pi_0) v(0, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) \big], \tag{10}$$

and

$$v(x^1, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) = u(x^1, m - e_1, q_1, \varepsilon)$$

$$+(1+\rho)^{-1} \big[\pi_1 v(x^1, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) + (1-\pi_1) v(0, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e}, \boldsymbol{\pi}, m, \varepsilon) \big], \tag{11}$$

where the identities (10) and (11) follow from the fact that it always will be optimal to delay the cost of replacing a good bulb at least one more period (see Dixit and Pindyck 1994). Substituting the right-hand-sides of these two equations into the corresponding places inside the $\{\cdot\}$ in (9) and rearranging terms, we find that the optimal stationary decision rule is to immediately replace a failed light bulb with x^0 if and only if

$$u(x^0, m - p_0 - e_0, q_0, \varepsilon) + \left(\frac{\pi_0}{1+\rho-\pi_0} \right) u(x^0, m - e_0, q_0, \varepsilon) \geq$$

$$\left(\frac{1+\rho-\pi_1}{1+\rho-\pi_0} \right) u(x^1, m - p_1 - e_1, q_1, \varepsilon) + \left(\frac{\pi_1}{1+\rho-\pi_0} \right) u(x^1, m - e_1, q_1, \varepsilon), \tag{12}$$

where we break a tie by assuming the traditional technology choice. Conversely, a failed appliance is immediately replaced with x^1 if (and only if)

$$u(x^1, m - p_1 - e_1, q_1, \varepsilon) + \left(\frac{\pi_1}{1+\rho-\pi_1}\right) u(x^1, m - e_1, q_1, \varepsilon) >$$

$$\left(\frac{1+\rho-\pi_0}{1+\rho-\pi_1}\right) u(x^0, m - p_0 - e_0, q_0, \varepsilon) + \left(\frac{\pi_0}{1+\rho-\pi_1}\right) u(x^0, m - e_0, q_0, \varepsilon)\,. \tag{13}$$

Suppose $\pi_1 = \pi_0 \equiv \pi$. Recall that $u(x^0, m - p_0 - e_0, q_0, \varepsilon) > u(x^1, m - p_1 - e_1, q_1, \varepsilon)$, since otherwise all households would choose the energy efficient appliance even in a one–period framework, which we do not observe in reality. If we also have the inequality $u(x^0, m - e_0, q_0, \varepsilon) > u(x^1, m - e_1, q_1, \varepsilon)$, so that the pure energy savings are insufficient to overcome a perceived lower quality of the energy efficient type, then consumer's would always choose the more energy intensive appliance type regardless of intertemporal discount rates, realizations for random preference variables, or probabilities of appliance failures per period. All of these are hypotheses that can be tested or used to modify the underlying appliance choice predictions.

3. Discussion

Inequalities (12) and (13) define the essential empirical framework that would be used to implement this model in a practical setting. The choice of the functional form for preferences should include considerations such as the potential for interactions between the RUM variables ε, the type indicator, x, and the quality index, q. Moreover, appliance purchases are generally not independent of income, so a quasi-linear utility function is not likely to be an appropriate choice. If one is using aggregate data, then a member of Gorman's extended class of functional forms in income is warranted (LaFrance, Beatty, and Pope 2004, 2005). In addition, as discussed above, preferences also depend on household attributes such as ethnicity, education, family size, age and gender of family members, the occupation and employment status of adult members of the household, and so forth. These arguments of the utility function are important indicators for this type of choice problem and essential explanatory variables in an empirical implementation of the model. The quality of energy-using appliances is almost certainly multi-dimensional, with elements that may be uncertain to the household at the date of purchase. This adds an additional level of economic uncertainty to the model structure. The ideal data set for econometric estimation is a panel of cross-section/time-series observations on individual households' appliance choices. An aggregate time series data set with summary information on the income distribution, demographic variables, and other key model variables and parameters also should be adequate to estimate this model. Grouped cross-section/time-series data sets, at say, the county level also could be employed, as well as a cross-sectional snapshot of the population's appliance choices.

It would not be difficult to add additional attributes such as credit constraints, limited liability (bankruptcy laws), or other market imperfections, household uncertainty about the qualities of various appliance types and future prices, energy costs, and incomes, as well as multiple varieties of appliances, changing technologies over time, or even quasi-rational expectations and behavior to this simple model. However, it is very likely that the decision rule identified in (12) and (13) above is sufficiently rich to capture the crude economic forces at work that affect household appliance choices over time.

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