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Some Properties of a Multi-Lane Extension of the Kinematic Wave Model

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Abstract

This paper extends an existing continuum multi-lane formulation for traffic flow, provides a discrete formulation for its numerical solution, and show initial results. The new formulation enables a natural treatment of boundary conditions such as merges, diverges, lane-drops and moving bottlenecks. The proposed model needs few extra parameters and is parsimonious. The look-ahead distance, for example, induces that non-local conditions affect the flow at any time-space point, causing smooth regime changes and fast waves. We find that as the look-ahead distance tends to zero, the solution tends to the KW one. The example of a lane-drop is analyzed.

1 Introduction

Among the numerical extensions of the kinematic wave (KW) model [1, 2] the multi-lane extension has received little attention in the literature. The first attempt [3] used a numerical scheme now known to be unable to properly treat boundary conditions in the KW model. Rather, existing multi-lane KW theories have been developed to solve specific problems [4–6] where traffic rules are simple enough so that graphical KW solutions are still tractable. Additionally, no more than 2×2 problems (two user types and two lane types) can be solved. In fact, in [4, 5] numerical methods are developed to solve the specific problems.

This paper proposes a framework to solve general $1 \times L$ problems using finite-difference schemes, easily expandable for $N \times L$ problems. During the last decade much has been learned for dealing with complex boundary conditions in the KW model. Particularly, [7] pointed out that the flow at the origin of the Riemann problem in the Godunov scheme [8], $g(k_u, k_d)$, is the cell transmission (CT) rule [9]:

$$g(k_u, k_d) = \min\{\lambda(k_u), \mu(k_d)\}, \quad (1)$$

where k_u and k_d are the densities upstream and downstream of the time-space point of interest. Two monotonic functions $\lambda(\cdot)$ and $\mu(\cdot)$ (see Fig. 1a,) define the fundamental diagram and represent the ability of a (1-pipe) road segment for sending resp. receiving flow. We shall call $\lambda(k)$ and $\mu(k)$ the *demand* and *supply* functions of a segment.

Through flows and lane changing flows are computed using discrete lane-choice models and a FIFO rule for resolving conflicts. The continuum formulation generalizes [3] where each lane is represented by an independent conservation equation and lane changing flows dictate common boundary conditions.

This paper is organized as follows: §§2 and 3 present the continuum resp. discrete formulation followed by an example of a lane-drop bottleneck in §4 and a discussion in §5.

2 The continuum formulation

Consider a freeway with L identical lanes where identical vehicles flow. Let $k \triangleq [k_1, \dots, k_L]$ and $q(k) \triangleq [q_1(k), \dots, q_L(k)]$ be vectors in \mathbb{R}^L ; $k_\ell(t, x)$ is the density in lane ℓ while $q_\ell(k)$ gives the *outflow* on $\ell = 1, \dots, L$. The outflow is the sum of all possible partial flows $q_{\ell\ell'}(k)$, $\ell' = 1, \dots, L$, emanating from ℓ , ie

$$q_\ell(k) = \sum_{\ell'} q_{\ell\ell'}(k). \quad (2)$$

We call $q_{\ell\ell}$ the through flows and $q_{\ell\ell'}, \ell' \neq \ell$, the lane changing flows, which satisfy $q_{\ell\ell'} = -q_{\ell'\ell}$. In a segment without entrances or exits, the conservation equation in terms of outflows becomes

$$\partial_t k + \partial_x q(k) = 0 \quad (3)$$

where the operators ∂_t and ∂_x give the vector of time resp. space partial derivatives of k . A general solution for (3) has not been identified yet [10]. However, if the problem is expressed in terms of the through flows the problem simplifies to a set of scalar PDE's with common boundary conditions. To see this, note that the ℓ th component of (3) is

$$\frac{\partial k_\ell}{\partial t} + \frac{\partial q_\ell(k)}{\partial x} = 0, \quad \forall \ell.$$

that combined with (2) gives

$$\frac{\partial k_\ell}{\partial t} + \frac{\partial q_{\ell\ell}(k_\ell)}{\partial x} = - \sum_{\ell' \neq \ell} \frac{\partial q_{\ell\ell'}(k)}{\partial x}, \quad \forall \ell. \quad (4)$$

The RHS of (4) is called here the lane changing flow “gradient” and denoted Φ_ℓ . When Φ_ℓ is known, (4) becomes a set of L independent scalar conservation laws, each one as in ordinary KW theory.

Notice the similarity of (4) with the model in [3] where Φ_ℓ is interpreted as a flow rather than a gradient, and did not consider partial flows explicitly. The following section is a proposition for their computation.

Partial flows

The basic assumption is that the demand for movements $\ell\ell'$ (called here partial demands) and the total supply in any lane can be obtained knowing the current state of the freeway, ie knowing k . This is reasonable since demands respond to the speed on different lanes and supply reflects the available space on the freeway. Both variables, speeds and available space, are derived quantities of k . We call partial demands $\lambda_{\ell\ell'}(k)$ and total supply $\mu_{\ell}(k)$.

The problem of allocating the available supply is well known in the literature [11], but only for single-pipe methods (eg, the merge model in [12] uses a fixed split independent of the demand.) Here, supply in lane ℓ' is split by analogy of a deterministic server of capacity $\mu_{\ell'}$ facing multiple customer types arriving at rates $\lambda_{\ell\ell'}$ with identical service times and priorities. The splitting coefficients are taken as the probability of a departure being type $\ell\ell'$, equal to $\frac{\lambda_{\ell\ell'}}{\sum_n \lambda_{n\ell'}}$ since we assumed FIFO. Hence, in direct analogy with (1) we postulate

$$q_{\ell\ell'}(k) = \min(\lambda_{\ell\ell'}(k), \frac{\lambda_{\ell\ell'}}{\sum_n \lambda_{n\ell'}} \mu_{\ell'}(k)). \quad (5)$$

We also impose an entropy-like condition stating that all the available supply should be used if possible. Next, we formalize the computation of partial demands.

Partial demands

Lane-changing decisions are assumed given by a discrete-choice process [13] that repeats continuously in time. The choice function in this case gives the proportion of users that *would like* to change lanes. As opposed to traditional problems treated with this technic, the choice for changing lanes does not guaranty that the lane changing will actually occur.

Let's focus first on a single decision (of a single driver) at some point (t, x, ℓ) and then derive its continuum formulation. The driver faces, at most, three alternatives: (i) change to the left lane $\ell' = \ell - 1$; (ii) stay in the same lane; and (iii) change to the right lane $\ell' = \ell + 1$. We assume that all the attributes necessary to make a

decision can be obtained from $k(x)$. Thus, the probability of choosing each alternative can be denoted $P_{\ell\ell'}(\theta; k(x), x)$ and needs to satisfy

$$\sum_{\ell'} P_{\ell\ell'}(\theta; k(x), x) = 1, \quad \forall \ell, x$$

where the location x is also included as an attribute and θ is the set of parameters of the discrete choice model, omitted hereafter for clarity.

To derive a formulation in continuous time, let $1/\tau$ be the frequency of lane-changing decisions (alternatively, τ can be interpreted as the time to complete a lane-changing maneuver given that there is enough supply.) We define the choice function in continuous time as $P_{\ell\ell'}(k(x), x)/\tau$. If we let $\lambda(k_{\ell})$ be the total demand at (t, x, ℓ) we express partial demands as

$$\lambda_{\ell\ell'}(k) = \lambda(k_{\ell}) P_{\ell\ell'}(k(x), x)/\tau. \quad (6)$$

Various specifications for $P(\cdot)$ could be explored. This paper proposes the use of a parsimonious model described in the example section.

3 The discrete formulation

Based on formulation (4)-(6) we now present the discrete model. We partition the freeway in cells (i, ℓ) as shown in Fig. 1b, where i is the *section* along the roadway and ℓ is the lane index. The numerical grid $(t_j \triangleq j\Delta t, x_i \triangleq i\Delta x)$ has spacial and temporal dimensions Δx and Δt related by

$$\Delta x = u\Delta t,$$

for stability, where u is the fastest characteristic in (3). Let $k_{i\ell}^j$ and $q_{i\ell\ell'}^j$ be the numerical approximation of $k_{\ell}(j\Delta t, i\Delta x)$ resp. $q_{\ell\ell'}(j\Delta t, i\Delta x)$. (The time index will be omitted as much as possible.) Let $\lambda_{i\ell\ell'}$ be the partial demand for changing from (i, ℓ) to $(i+1, \ell')$ during time step j and let $\mu_{i\ell}$ be the total supply in (i, ℓ') .

The proposed numerical scheme consists of the following independent loops indexed by section and lane, where we use the symbol “:=” to eliminate the time index:

$$q_{i\ell\ell'} := \min\{\lambda_{i\ell\ell'}, \frac{\lambda_{i\ell\ell'}}{\sum_n \lambda_{in\ell'}} \mu_{i+1, \ell'}\}, \quad (7a)$$

$$k_{i\ell} := k_{i\ell} + \frac{\Delta t}{\Delta x} (E_{i\ell} - S_{i\ell}), \quad (7b)$$

where $S_{i\ell} \triangleq \sum_{\ell'} q_{i\ell\ell'}$ and $E_{i\ell} \triangleq \sum_{\ell'} q_{i-1,\ell'\ell}$ are the flows that exit and enter (i, ℓ) in time-step j , respectively. Eqn. (7a) represents the discrete analog of (5) (evaluated $\forall \ell'$), while (7b) ensures the conservation of vehicles. Total supplies are readily obtained from the flow-density relation. Partial demands are computed using

$$\lambda_{i\ell\ell'}(k) \triangleq \lambda(k_{i\ell}) P_{\ell\ell'}(k, x_i) \Delta t / \tau, \quad (8)$$

as one needs to account for the demand in the entire time interval, ie, an approximation for the following time integral of (6):

$$\frac{1}{\tau} \int_{t_j}^{t_j + \Delta t} \lambda(k_{\ell}(s, x_i)) P_{\ell\ell'}(k(s, x_i), x_i) ds.$$

Since Δt is small one can safely assume that $\lambda(\cdot)$ and $P_{\ell\ell'}(\cdot)$ remain constant during the integration interval, resulting in (8) as claimed.

To see the correspondence between (7a) and the continuum model (4) we rewrite (7b) as

$$\frac{k_{i\ell}^{j+1} - k_{i\ell}^j}{\Delta t} + \frac{q_{i\ell} - q_{i-1,\ell\ell}}{\Delta x} = - \frac{\sum_{\ell' \neq \ell} q_{i\ell\ell'} - q_{i-1,\ell'\ell}}{\Delta x}.$$

4 Example: Lane-drops

Consider a 1.2-mile, L -lane freeway segment that has a lane-drop at $D = 1.2$ mi (see Fig. 2a.) At $t = 0$ the freeway starts flowing at capacity. We are interested in comparing the propagation of the resulting wave with the KW solution.

We assume that each lane obeys a triangular fundamental diagram with free-flow speed $u = 60$ mph, jam density for one lane $\kappa = 150$ vpmpl and wave speed $w = 15$ mph. It follows that

$$\begin{aligned} \lambda_{i\ell\ell'} &= uk_{i\ell} P_{\ell\ell'}(k, x_i) \Delta t / \tau, \\ \mu_{i\ell'} &= w(\kappa - k_{i\ell'}). \end{aligned}$$

In this example we use $\tau = 12$ sec and the simplest lane-choice model where

$$\frac{P_{\ell\ell'}(k, x)}{\tau} = \begin{cases} \pi & \text{when } \Delta v_{\ell\ell'}(t, x) > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

The parameter π is a (fixed) lane changing proportion (in % per time unit) and $\Delta v_{\ell\ell'}(t, x)$ is

the average speed difference w.r.t the target lane, measured across some look-ahead distance, L_a , ie

$$\Delta v_{\ell\ell'}(t, x) \triangleq \frac{1}{L_a} \int_x^{x+L_a} (v_{\ell'}(t, s) - v_{\ell}(t, s)) ds$$

in the continuum, where $v_{\ell}(t, x)$ is the speed on lane ℓ at (t, x) .

With these specifications, procedure (7) was applied for $\Delta t = 1$ sec and all combinations of $L_a = \{.2, .4, .6, .8\}$ and $\pi = \{10, 20, \dots, 70\}$ % per min. The data set analyzed herein consists of six cumulative count curves (N -curves) measured at the evenly-spaced locations shown in Fig. 2a. Next we show some properties of the parameters of the model.

Effects of π and L_a : smoothing

For ease of exposition, the following illustration uses the schematic of Fig. 2b-c based on typical N -curves from our sample, and by the end of the section maps of the numerical solution are shown.

We are interested in the difference in the numerical solution compared to the exact KW solution, defined as $\delta(t)$ in part (b) of the figure, and in the duration of the discrepancy, σ in part (c). Recall that the total flow in section i ,

$$q_i \triangleq \sum_{\ell, \ell'} q_{i\ell\ell'}, \quad (11)$$

is the only comparable to the 1-pipe KW flow. Only data where $\delta > 0$ is considered, ie, around the time that the shock passes.

First, note that when $\{\pi = 0, L_a \geq 0\}$ or $L_a = 0$ the model gives the exact KW solution, reassuring that the discrete model is properly formulated. This was expected for $\pi = 0$ because in the absence of lane changing (11) is the KW solution for we are using triangular equilibrium relations. It follows that L_a is responsible for the smoothing of the back-of-the-queue (BOQ.)

When L_a remains fixed (Fig. 2b) points a and d remain fixed and δ increases with π , but σ remains unchanged. When π is held fixed (part (c) of the figure,) the initial separation point a goes to the right to point b in the figure, and $d \rightarrow f$ as $L_a \rightarrow 0$. This implies a reduction in δ and σ as $L_a \rightarrow 0$.

To see why smoothing occurs, take $L = 2$, assume that lane 2 terminates and that $\Delta t = \tau$ for clarity in notation. The first lane changing takes place from lane 2 to lane 1 at $(t = (D - L_a)/u, x = D - L_a)$, which is the point where the leader first sees the lane drop, call it section i . It can be verified that the lane changing flow $q_{i21}^j = \frac{Q\pi}{1+\pi}$ causes a queue in cell $(i, 1)$ carrying a flow $q_{i21}^j = Q - q_{i21}^j = \frac{Q}{1+\pi}$ at a speed $v = uw/[u\pi + w(1 + \pi)]$. Notice that q_i^{j+1} is still $2Q$ but $q_{i-1}^{j+1} = \frac{Q}{1+\pi} + Q = Q\frac{2+\pi}{1+\pi} \leq 2Q$. Thus, “small” bottlenecks caused by lane changing induce local flow reductions *before* the actual wave passes, smoothing-out the BOQ.

Several δ -maps obtained with the numerical method introduced here are shown in Figs. 3 and 4, for $L = 2$ and 3 lanes, resp. There is one (t, x) plane for every combination run, shown as rectangles in the figure. In both figures it is apparent how L_a dictates σ (as shown in the top left (t, x) plane of Fig. 3,) and how δ stabilizes at a location that decreases with L_a . The magnitude of the discrepancy is also similar, as seen in the scale box of the figures. The main difference is the duration of the smoothing σ , much larger for $L = 3$. This is explainable since anticipation creates “fast waves” (see Fig. 4) that are magnified with the number of lanes. For example, if lane 3 terminates, the first lane changing will occur on lane 3 (to 2) at a distance L_a from the lane-drop, and this induces in turn lane changing from lane 2 to 1 up to $2L_a$ mi upstream of the first lane changing.

5 Discussion

The evidence in the above paragraph confirms the intuition that the inclusion of L_a induces the propagation of fast waves caused by anticipation. As opposed to conventional single-pipe waves, fast waves propagate across adjacent lanes in the form of “small” bottlenecks induced by lane-changers.

Further research is need to investigate other functional forms for the choice function $P(\cdot)$, but not deviate too much from the simple specification (10) used in this paper. These functions

should be easy to calibrate for the BOQ is readily observable in the field and we have identified how the parameters affect the BOQ.

The analogy with a queueing system for splitting supplies could be used in more complicated situations, such as special lanes, different vehicle length and speeds. It may be advantageous to generalize the FIFO rule by extending the incremental transfer principle introduced in [5].

The particular traffic rules (ie, the heuristic to compute the Φ_ℓ 's) are part of boundary conditions of the problem and cannot alter the stability of (4) as long as rules are stable.

We showed the case of a lane-drop, but the model has been successfully tested with moving bottlenecks by including the method in [14] for single-pipe models. In fact, the modelling of moving bottlenecks is simpler in the multi-lane framework since (i) their capacity is automatically defined as the capacity of the remaining lanes; and (ii) passing is handled naturally, with no need for modifying the method. Current research shows, encouragingly, that losses in capacity comparable to the elusive capacity drop are observed when cars changing lanes are modelled as moving bottlenecks. When this concept is applied to vehicles passing a moving bottleneck, we observe capacities that depend on the speed of the original moving bottleneck, and the relations match the empirical observations in [15].

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References

- [1] M J Lighthill and G B Whitham. On kinematic waves. i flow movement in long rivers. ii a theory of traffic flow on long crowded roads. *Proc. Roy. Soc.*, 229(A):281–345, 1955.
- [2] P I Richards. Shockwaves on the highway. *Opns. Res.*, (4):42–51, 1956.

- [3] PG Michalopoulos, DE Beskos, and Y Yamauchi. Multilane traffic dynamics: Some macroscopic considerations. *Trans. Res. B*, (18):377–395, 1984.
- [4] C F Daganzo. A continuum theory of traffic dynamics for freeways with special lanes. *Trans. Res. B*, 2(31):83–102, 1997.
- [5] C F Daganzo, W Lin, and J Del Castillo. A simple physical principle for the simulation of freeways with special lanes and priority vehicles. *Trans. Res. B*, 2(31):105–125, 1997.
- [6] C F Daganzo. A behavioral theory of multi-lane traffic flow part i: Long homogeneous freeway sections. *Trans. Res. B*, 2(36):131–158, 2002.
- [7] J P Lebacque. Les modeles macroscopic du trafic. *Annales des Ponts.*, (67):24–45, 1993.
- [8] S Godunov. A difference scheme for numerical computation of discontinuous solutions of equations of fluid dynamics. *Mat. Sb.*, 89 (47):271–306, 1959.
- [9] C F Daganzo. The cell-transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. Technical Report UCB-ITS-RR-93-7, Inst. Trans. Studies, Univ. of California, Berkeley, CA, 1993.
- [10] RL LeVeque. Numerical methods for conservation laws. *Birkhauser Verlag*, 1993.
- [11] J P Lebacque and JB Lesort. Macroscopic traffic flow models: a question of order. In A. Ceder, editor, *14th Int. Symp. on Transportation and Traffic Theory*, pages 3–25, Pergamon, New York, N.Y., 1999.
- [12] J P Lebacque. The godunov scheme and what it means for first order traffic flow models. In J. B. Lesort, editor, *13th Int. Symp. on Transportation and Traffic Theory*, pages 647–678, Elsevier, New York, 1996.
- [13] C F Daganzo. *Multinomial Probit*. Academic Press, 1979.
- [14] J A Laval and C F Daganzo. A hybrid model of traffic flow: Impacts of roadway geometry on capacity. submitted for publication to: *Trans. Res. Rec., TRB*, 2003.
- [15] JC Muñoz and C F Daganzo. Moving bottlenecks: a theory grounded on experimental observation. In M.A.P. Taylor, editor, *15th Int. Symp. on Transportation and Traffic Theory*, pages 441–462, Pergamon-Elsevier, Oxford,U.K., 2002.

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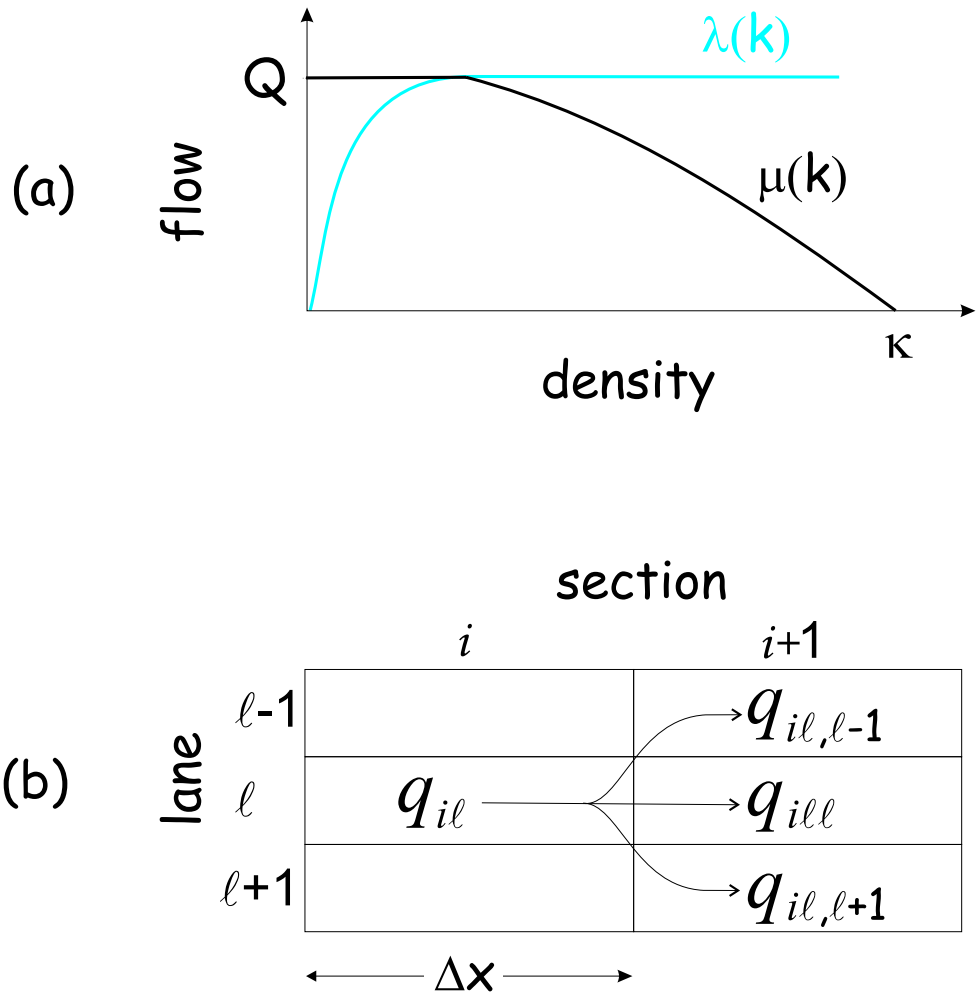


Figure 1: (a) Demand and supply functions; (b) discretized freeway representation.

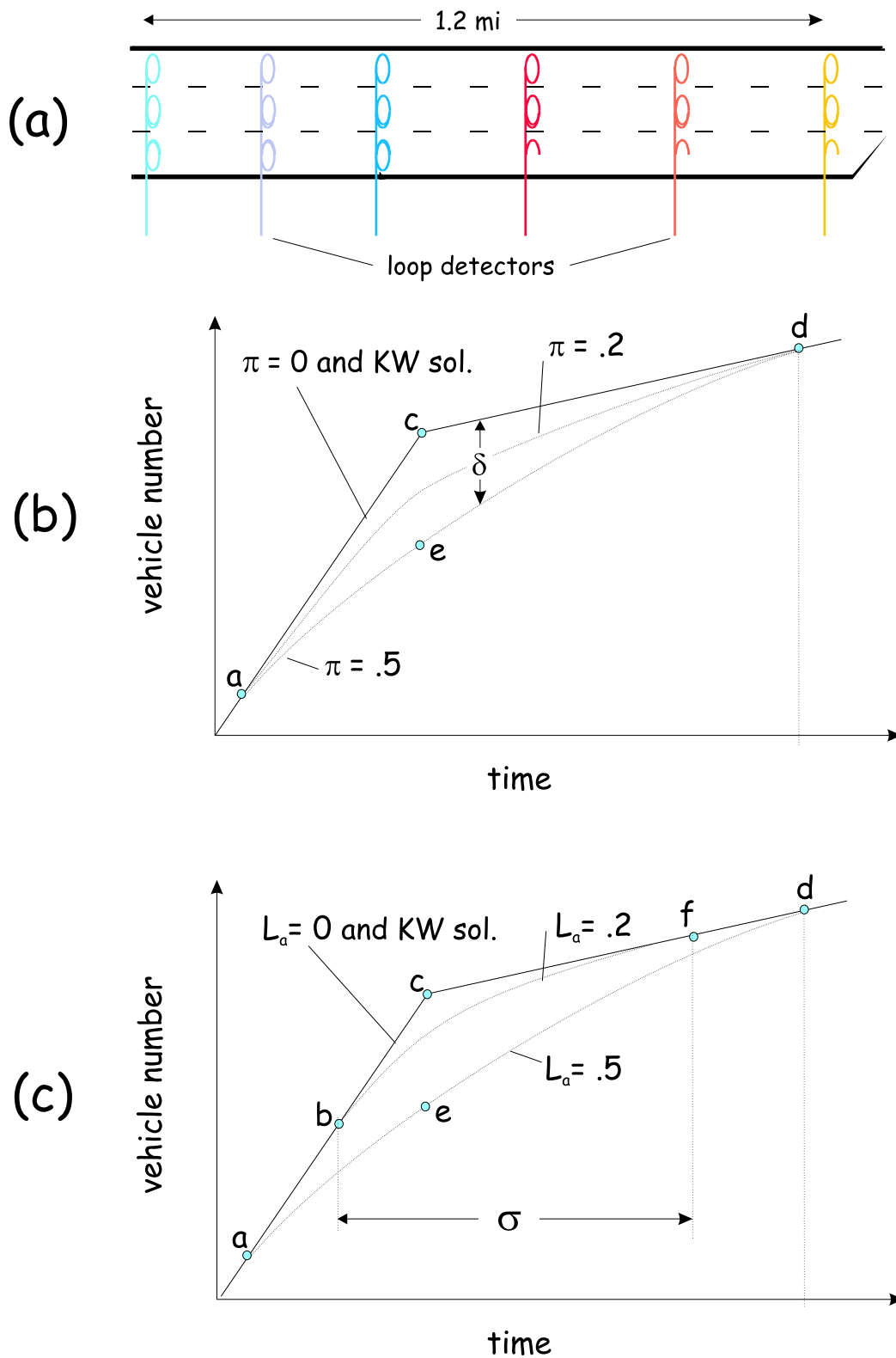


Figure 2: (a) Lane-drop; effects of the parameters (b) L_a constant; (c) π constant.

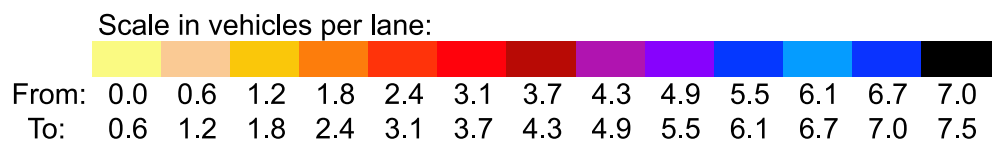
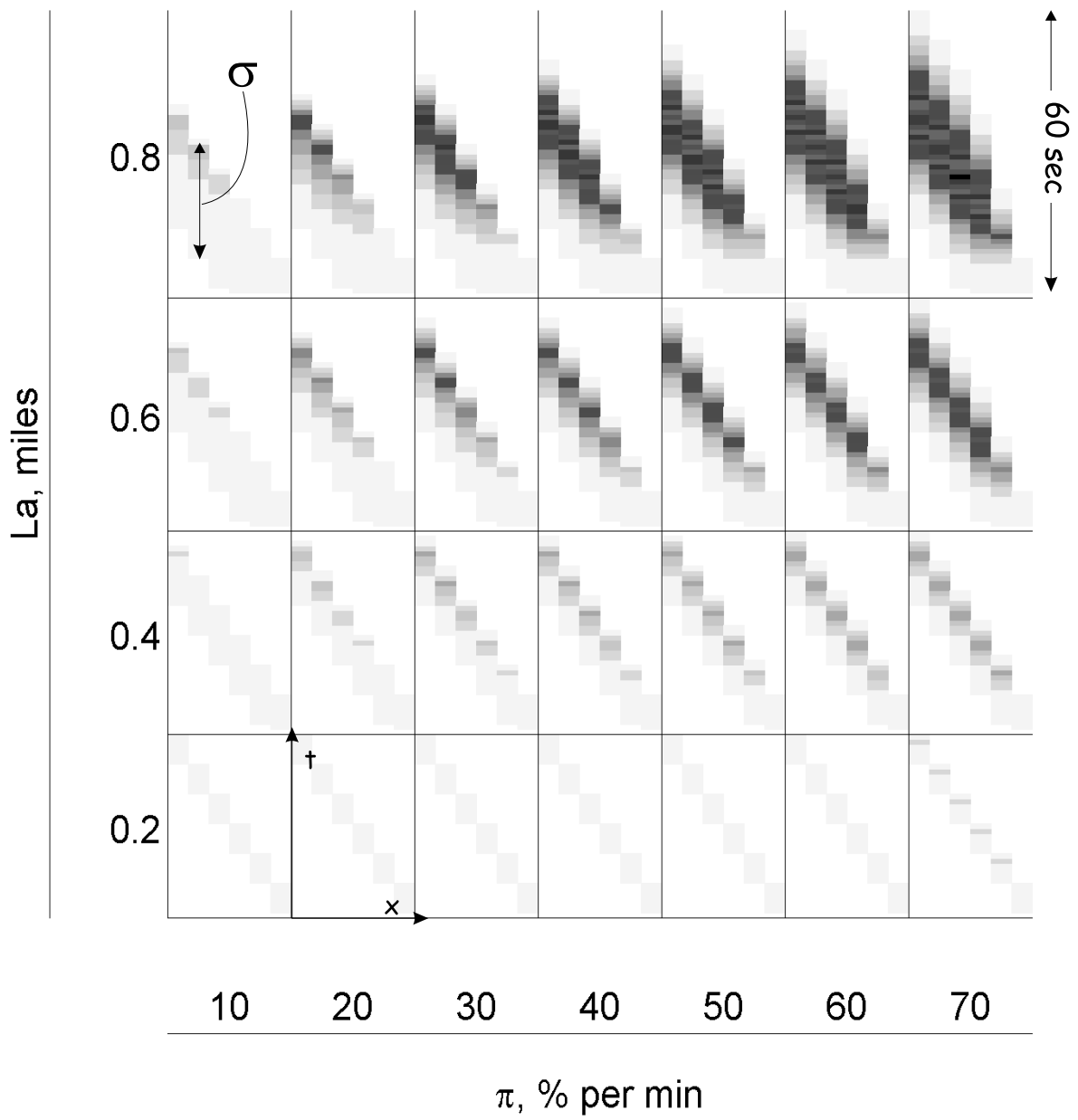


Figure 3: Numerical δ -maps, $L = 2$ lanes.

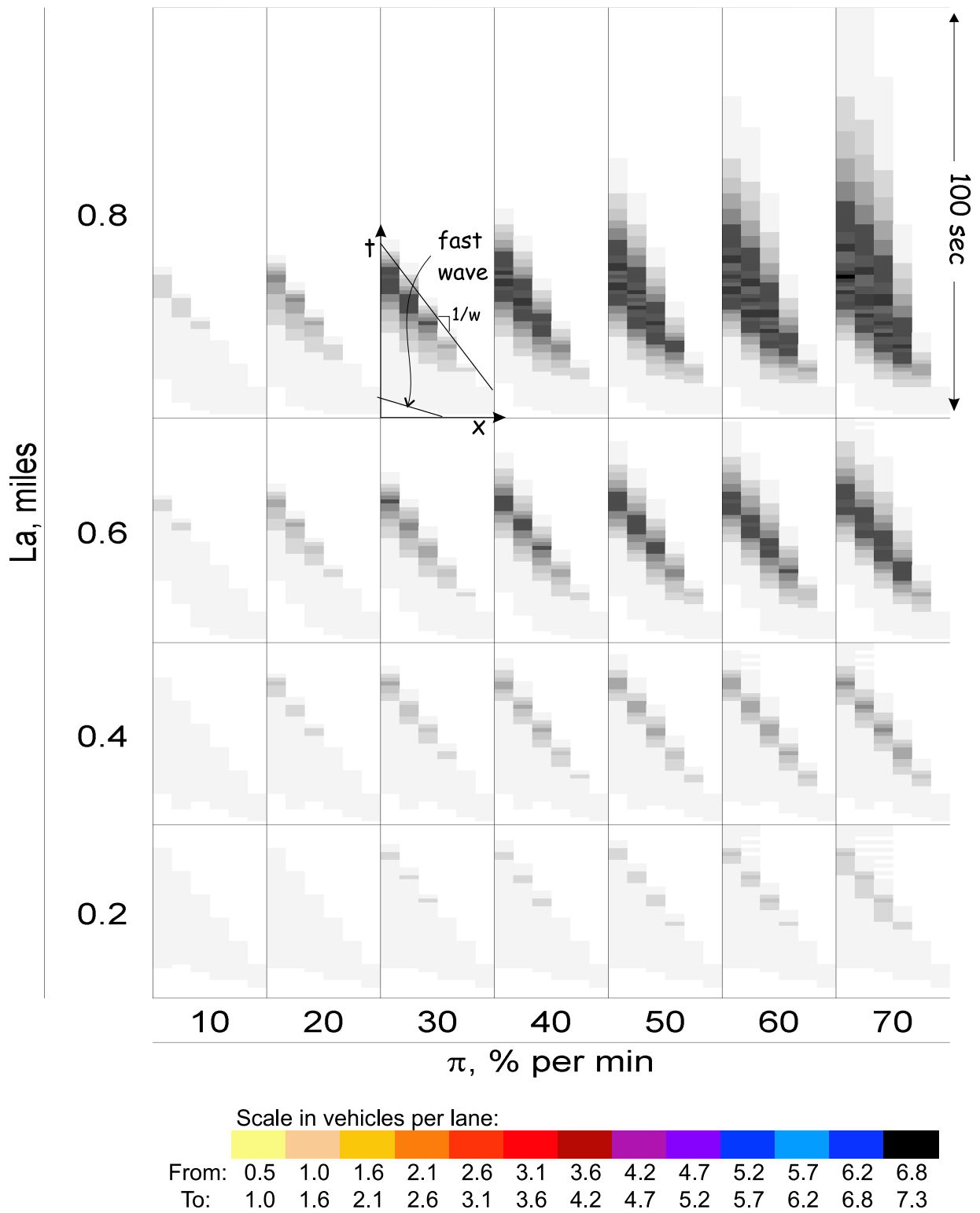


Figure 4: Numerical δ -maps, $L = 3$ lanes.