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THE EFFICIENCY OF SEARCH STRATEGIES: A NUMERICAL AND ANALYTICAL COMPARISON OF STRATEGIES ASSUMING NO OR RESTRICTED KNOWLEDGE OF THE UNDERLYING DISTRIBUTIONS

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I. Introduction

Since Stigler's well known article[1] a variety of papers have been published on the economics of information. Important and well known applications of this theory are consumer search for information in price and quality of goods, search of unemployed workers for a job in the labor market[2] and the search of unwed individuals searching for a marriage partner.[3] In economics less well known is the application of search models in the design of randomized algorithms.[4] In this application an algorithm searches within a prespecified set of algorithms solving the same problem such that the total loss of the algorithm finally chosen to solve the problem plus the search cost is minimized.[5] Independent of the majority of the literature on optimal search several articles have been published analyzing the optimal policy of search if we assume that only ordinal utility can be assigned to the objects found.

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[1] G. Stigler, "The economics of information," Journal of Political Economy LXIX (June 1961), 213-25. Besides this the earlier paper of McQueen, J. and Miller, R.G., "Optimal Persistence Policies," Operations Research, Vol. 8, pp. 362-380 (1960) gave a more efficient search strategy in a different context.

[2] See for example S.A. Lippman, "Search Unemployment: Mismatches, layoffs, and unemployment insurance", Working paper No. 286, University of California, Los Angeles, Western Management Institute, 1979.

[3] See for example G.S. Becker and E.M. Landes, "An Economic Analysis of Marital Instability", Journal of Political Economy 8, no. 6 (December 1977), pp. 1141-88.

[4] W. Janko, "Stochastische Modelle in Such-und Sortierverfahren", Duncker & Humblot, Berlin, 1976.

[5] To the authors knowledge that is the only application of this type of search models which is in practical use.

This work usually does not consider search costs[6] and concentrates on asymptotic considerations. The results are thus hard to compare with the results in the economics of information literature.

In this paper we shall try to compare the efficiency of search strategies. To compare ordinal utility methods with cardinal utility methods we have to violate some of the basic assumptions of ordinal utility theory by introducing search costs which are deductible from the ordinal utility indices. Although this makes the interpretation of the results more difficult these results yield some insight into the efficiency of drawing with and without recall. Linear utility functions are assumed and a limited number of observations of this discrete and uniformly distributed utility (loss) function are presumed without being able to observe the actual value of an observation.

For exploratory purposes we shall describe the search model in terms of a consumer searching for the lowest cost of the alternative offered plus search costs.

II. Search Strategies

Using ordinal utility means that we assume the existence of a weak preference ordering on the set of consequences of the alternatives only. After putting all consequences to which we are indifferent in quotient classes we are able to introduce an ordinal utility function on the quotient set. We assume here that the quotient set consists of a finite number of elements only and the ordinal utility index is constructed by sorting the equivalence classes. (This is possible in $O(n \log n)$ time with n equivalence classes.) We shall furthermore for reasons of clarity and simplicity assume here that

[6] S. Chow, S. Moriguti, H. Robbins and M.S. Samuels, Optimal Selection Based on Relative Rank, Israel Journal of Mathematics, vol. 2, pp. 81-90 (1964); Y.S. Chow, H. Robbins and D. Siegmund, Great Expectations: The Theory of Optimal Stopping, Houghton Mifflin Company, Boston (1971).

every equivalence class contains only one element.[7] The resulting problem is then to draw with or without replacement from a finite set of the first n natural numbers, {1,2,...,n}, which represent the ordinal indices. Drawing an offer means that we are able to determine the rank of the utility of the offer within the offers we already got. We are not able to determine the ordinal utility index of the utility of the offer in the quotient set until we have drawn all offers. The efficiency of rank oriented search strategies can be compared with distribution oriented search strategies only if we assume search costs of zero or if we introduce search costs which are deductible from the ordinal utility index. We shall choose this latter more realistic possibility and assume fixed search costs c for every observation throughout. We shall compare the results we get investigating these rank-oriented strategies without recall with the results in a search with recall. For reasons of simplicity we shall assume a linear cardinal utility function of the observer. The results shall be compared with a strategy where the distribution of offers is unknown in the Bayesian sense. In this case we shall restrict ourselves on sequential strategies with and without recall. From the set of distributions involved we will primarily derive conclusions using the uniform distribution. For the consumer search example chosen we assume whenever it is plausible -- that the goods are described by the characteristic, 'price'. Sampling from this distribution is assumed to be costly. The cost of observation is constant and equal to c. Once an observation is drawn at cost c, the price can be observed without cost.

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[7] We such avoid the necessity to consider the problem of drawing from multisets of ordinal indexes (with and without replacement).

III. Rank Oriented Models of Search

1. The strategy with unlimited memory

Rank oriented stopping strategies have been studied extensively in literature.[8] None of the strategies tries to include search costs c into the considerations and are largely oriented to achieve asymptotic results. For various reasons and as it is economically hardly justfied to neglect the search costs this strategies did not gather much attention -- at least by economists. To clarify the efficiency of this type of strategies we shall investigate these strategies. The problem we consider first has been called the pigeon problem, the dowry problem, the beauty-contest problem and the secretary problem, among other names.

Suppose now that a consumer is offered n price quotations. The consumer can observe the offers only one at a time as he has no prior information on their true rank in sorting order (according to their negative utility $\ell(i)$). (With sorting order we mean that $\ell(1) \leq \ell(2) \leq \ldots \leq \ell\ (i) \leq \ell(i+1) \leq \ldots \leq \ell(n)$ is valid.) The only information the searcher can rely on, is the information about the relative rank of the latest offer observed by the searcher within the offers already observed before. We assume in the following considerations that the searcher will not be able to observe -- whether intentionally or unintentionally -- an offer already observed previously. We will furthermore in the derivation assume that the searcher, once he has decided not to accept

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[8] Besides the papers and the book mentioned in the introduction two references shall be mentioned without being exhaustive:
M. DeGroot, Optimal Statistical Decisions, McGraw-Hill Company, New York, 1970, and
J. Gilbert and F. Mosteller, Recognizing the maximum of a sequence, J. Am. Statistical Association, Vol. 61, pp. 35-73.

a particular object, can never go back and select it at a later stage.[9] Now let us assume that Y_m denotes the random variable of the relative rank of the m-th observed offer within the observations already drawn. The random variables

$$Y_1, Y_2, \ldots, Y_n$$

are independently distributed and we get for the probability

$$W(Y_m=j) = 1/m \quad \text{for } j = 1,2,\ldots,m$$

Now let R denote the true rank of the offer observed and let R_m be the random variable "true rank" when we assume that the m-th element drawn has the relative rank j within the offers observed so far. Let $E(R_m/Y_m=j)$ be the expected true rank assuming $Y_m=j$. Now obviously the following relation is valid:

$$E(R_m/Y_m = j) = \sum_{b=j}^{n-m+j} \ell(b)W(R_m=b/Y_m=j)$$

where $\ell(b)$ is the loss function.

For the unconditional expectation $E(R_m)$ we get therefore:

$$E(R_m) = \sum_{i=1}^{m} E(R_m/Y_m=i)W(Y_m=i)$$

Assuming that the true rank is equal to the loss index, that means $l(i)=i$, we get

$$E(R_m/Y_m=j) = \sum_{r=j}^{n-m+j} W(R_m=r/Y_m=j)r \quad .$$

[9]This is usually called "searching without recall."

An offer with true rank r has the relative rank j iff the j-1 offers are drawn out of r-1 offers with the true ranks 1,2,...,r-1 and m-j offers were drawn out of the n-r offers with the true rank r+1,r+2,...,n.

Therefore we get

$$W(R_m=r/Y_m=j) = \frac{\binom{r-1}{j-1}\binom{n-r}{m-j}}{\binom{n}{m}}$$

and from the definition above

$$E(R_m/Y_m=j) = \sum_{r=j}^{n-m+j} \frac{r\binom{r-1}{j-1}\binom{n-r}{m-j}}{\binom{n}{m}} = \frac{(n+1)}{m+1}\, j$$

Using this result we get for the expected true rank of the m-th offer observed

$$E(R_m) = E(\frac{n+1}{m+1}\, Y_m) = \frac{n+1}{m+1}\, E(Y_m)$$

Using the principle of backward induction[10] we get for the expected rank when only one offer is left:

$$E(R_n) = E(\frac{n+1}{n+1}\, j) = \frac{1}{n}\sum_{i=1}^{n} j = \frac{n+1}{2}$$

Assuming now that we have drawn n-1 observations we should only observe the n-th observations if the expected rank plus the search cost, which must in our derivation be expressable in units of ranks is lower than the expected rank at the (n-1)-th step; we get therefore

[10]See for example, Y.S. Chow, H. Robbins and D.Siegmund, Great Expectations:...., 1971.

$$v_{n-1} = E(\min(\frac{n+1}{n} Y_{n-1} , v_n+c)) = \frac{1}{n-1} \sum_{j=1}^{n-1} \min(\frac{n+1}{n} j, v_n+c)$$

where v_k denotes the expected loss (expressed in rank units) at the k-th observation and $v_n = E(R_n) = \frac{n+1}{2}$.

Similarly we get for the expected loss at the i-th observation:

$$v_i = E(\min(\frac{n+1}{i+1} Y_i, v_{i+1} + c)) = \frac{1}{i} \sum_{j=1}^{i} \min(\frac{n+1}{i+1} j, v_{i+1} + c)$$

Computing successively the values of v_{n-1}, $v_{n-2},\ldots,v_1$, we get the expected value of the strategy v_1.

This can be simplified if we use the for practical purposes indeed necessary reservation index s_i for stopping with a relative rank $\leq s_i$ at the i-th observation.

We get the reservation index vector, which is an integer valued function of the number of observations drawn by the following considerations: Stopping with the i-th observation implies

$$\frac{n+1}{i+1} Y_i \leq v_{i+1} + c$$

or in terms of the implicit relative rank which implies stopping at the ith observation

$$s_i = [\frac{i+1}{n+1} (v_{i+1} + c)] \quad \text{for } i=n-1,\ldots,2,1.$$

Using s_i and the fact that the relation

$$v_i = \frac{n+1}{i+1} E(Y_i/Y_i \leq s_i) + W(Y_i > s_i) (v_{i+1} + c)$$

is valid we get:

$$v_i = \frac{1}{i} \left(\frac{n+1}{i+1} (1+2+\ldots+s_i) + (i-s_i)(v_{i+1} + c)\right) =$$

$$= \frac{1}{i} \left(\frac{n+1}{i+1} \frac{s_i(s_i+1)}{2} + (i-s_i) (v_{i+1} + c)\right)$$

Using $v_n = \frac{n+1}{2}$ we can easily calculate the reservation rank vector and the expected value of this strategy v_1.

For the following calculation we used search costs of c=1 rank units and price offers for goods of identical utility assuming that the price represents the loss.

n	Search Cost Plus Price	Relative Rank to Stop (Sample size until which relative rank above has to be applied)																	
3	2.0	(1	2	3															
		1	2	3)															
4	2.5	(1	2	4															
		1	3	4)															
5	3	(1	2	3	5														
		2	3	4	5)														
6	3.5	(1	2	3	6														
		2	4	5	6)														
7	4	(1	2	4	7														
		2	4	6	7)														
8	4.5	(1	2	3	4	8													
		2	4	6	7	8)													
9	5	(1	2	3	5	9													
		2	4	7	8	9)													
10	5.5	(1	2	3	4	5	10												
		2	5	7	8	9	10)												
15	7.03	(1	2	3	4	5	6	8	15										
		2	5	9	11	12	13	14	15)										
20	8.44	(1	2	3	4	5	6	8	10	20									
		3	6	10	13	15	17	18	19	20)									
25	9.81	(1	2	3	4	5	6	7	8	9	13	25							
		3	7	11	15	18	20	21	22	23	24	25)							
30	10.99	(1	2	3	4	5	6	7	8	9	11	15	30						
		3	7	12	16	20	23	25	26	27	28	29	30)						
35	12	(1	2	3	4	5	6	7	8	9	11	13	18	35					
		3	8	12	17	21	25	28	30	31	32	33	34	35)					
40	12.98	(1	2	3	4	5	6	7	8	9	11	12	15	20	40				
		3	8	13	18	23	27	31	33	34	36	37	38	39	40)				
45	13.95	(1	2	3	4	5	6	7	8	9	10	11	12	14	17	23	45		
		4	9	14	19	24	29	33	36	38	39	40	41	42	43	44	45)		
50	14.89	(1	2	3	4	5	6	7	8	9	10	11	12	13	15	19	25	50	
		4	9	15	20	25	31	35	39	41	43	44	45	46	47	48	49	50)	
55	15.75	(1	2	3	4	5	6	7	8	9	10	11	12	13	15	17	21	28	55
		4	9	15	21	27	32	37	42	45	47	48	49	50	51	52	53	54	55)

16.51	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	19	23	30	60)			
	4	10	16	22	28	34	39	44	48	50	52	53	54	55	56	57	58	59	60)			
17.27	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	17	20	25	33	65)		
	4	10	16	22	29	35	41	46	50	54	56	57	58	59	60	61	62	63	64	65)		
18.02	(1	2	3	4	5	6	7	8	9	10	11	12	13	15	16	18	22	26	35	70)		
	4	11	17	23	30	36	42	48	53	57	59	61	62	64	65	66	67	68	69	70)		
18.75	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	16	17	20	23	28	38	75)	
	5	11	17	24	31	37	44	50	55	60	63	65	66	68	69	70	71	72	73	74	75)	
19.48	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	17	19	21	25	30	40	80)
	5	11	18	25	32	38	45	52	58	62	66	68	70	72	73	74	75	76	77	78	79	80)
20.20	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	18	20	22	26	32	43
	5	12	18	25	32	39	46	53	60	65	69	72	74	76	77	78	79	80	81	82	83	84
	85)																					
	85)																					
20.83	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	19	21	24		
	5	12	19	26	33	40	48	55	62	67	72	75	78	80	81	82	83	84	85	86		
	28	34	45	90)																		
	87	88	89	90)																		
21.44	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	20	22		
	5	12	19	27	34	41	49	56	63	70	75	79	82	84	85	86	87	88	89	90		
	25	29	36	48	95)																	
	91	92	93	94	95)																	
22.06	(1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	18	19	21	23		
	5	12	20	27	35	42	50	58	65	72	78	82	85	87	89	90	92	93	94	95		
	26	31	38	50	100)																	
	96	97	98	99	100)																	

The matrix $\binom{1\ 2\ 3\ 4\ 5\ 10}{2\ 5\ 7\ 8\ 9\ 10}$ would say that until a sample size of 2 you should stop with relative rank 1, with the sample sizes 3, 4, 5 you should stop with a relative rank equal to or better than 2, and so on. If you do this your expected cost for searching and purchasing would be 5.5 rank units assuming search costs of c=1 rank units.

2. The strategy with extremely limited memory

As we can see by the numerical examples given the unlimited acceptance of offers of any relative rank using their true rank as a representation of their (cardinal) utility involves for a large number of offers some calculation and permanent sorting and memorizing of offers already observed, although these offers are not valid anymore. The other extreme would be to accept only offers with the possible true rank 1. These offers must naturally also have the relative rank of 1. As DeGroot[11] shows, this is the other extreme to the strategy considered above. The strategy considered below is equivalent to a strategy which maximizes the probability to find the offer with true rank 1. If search costs are 0 it is well known that asymptotically the observer should initially observe n/e offers and then stop with an offer which is relatively at least as good as the best of these n/e offers already observed in a learning phase. If we introduce search costs of c the behavior of this strategy has to be reconsidered. The problem is formulated as a rank maximizing problem for the decreasing sorting order $\ell(n)$, $\ell(n-1),\ldots,\ell(1)$.

Lemma: The probability, that the true rank of the offer chosen is v and the number of the initially drawn subset of the total sample (learning set) is k, is equal to

$$W(N=v,\ L=\ell) = \frac{k}{\ell-1}\ \frac{(v-1)!\,(n-\ell)!}{(v-\ell)!\,n!}$$

[11] M. DeGroot, Optimal Statistical Decisions,...loc.cit., p. 337.

There is a probability that the offer with true rank one was already observed within the first k offers.

The sum $$\sum_{\ell=k+1}^{n} \sum_{v=\ell}^{n} \frac{(v-1)!\,k(n-\ell)!}{(v-\ell)!\,(\ell-1)n!}$$

therefore is equal to $1 - \frac{k}{n}$. It is important to recognize that this strategy will be formulated as a rank maximizing strategy. Using a different sorting order this does not cause a problem. If we denote the event, that the best offer is not observed within the sample of the first k offers with T, we get the following theorem.

Theorem:[12] The conditional expectation E(R/T) of the true rank of the offer accepted by the above policy assuming that the best offer is not within the first k offers observed is given by:

$$E(R/T) = \frac{1}{2}\left(\frac{(2k+1)(n+1)}{(k+1)} - \frac{(2n+1)\,k}{n}\right)\frac{n}{n-k}$$

The variance of this mean value $\sigma^2_{(R/T)}$ is equal to

$$\sigma^2_{(R/T)} = \left(\frac{2n+1}{3}\right) E(R/T) - E^2(R/T) + \frac{n(n+1)k}{3(k+2)}$$

The expected number of offers observed such is

$$E(L/T) = \frac{n\ k}{(n-k)} \sum_{\ell=k}^{n-1} \frac{1}{\ell}$$

and the variance of this mean value is

$$\sigma^2_{(L/T)} = nk + E(L/T)(1-E(L/T))$$

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[12]The proof of this theorem is lengthy and can be found in Appendix A.

As we usually can assume that the relation $k \ll n$ is valid we can use in many cases the approximation

$$E(L/T) = \frac{nk}{n-k} \sum_{\ell=k}^{n-1} \frac{1}{\ell} \simeq \frac{nk}{(n-k)} \log \frac{n}{k} + 0.5$$

As the reader can easily verify the unconditional expectation of $E_k(R)$ can be interpreted as a continuous and differentiable and concave function of k and we get for the maximum

$$k_{max} \simeq [(\frac{n^2+n}{2n+1})^{0.5}] \text{ for } k \ll n \text{ and } 0 \leq k \leq n \text{ using the equation } \frac{dE_k(R)}{dk} = 0.$$

The search process considered has two phases:

- the learning phase, which consists of drawing k observations initially and memorizing the best observation o*;
- the actual search phase, which leads to stopping as soon as an observation is made which is better than o*.

The decisive choice the searcher has to make is the choice of the number of observations k he wants to draw in the learning phase. To choose an optimal value k^+ of k with respect to fixed search costs c he has to act such that an increase of k^+ by one observation increases the expected rank of the observation stopped with by less than c per unit of search cost. We get the increased search costs implied by an increase of the learning phase (with k observations) by one observation (to a total of k+1 observations) by considering the difference $E_{k+1}(L/T)-E_k(L/T)$ $= \Delta E_k(L/T)$. The gain in ranks is equal to

$$E_{k+1}(R/T) - E_k(R/T) = \Delta_k E(R/T)$$

conditional on the event T that the best offer is not observed within the learning phase. So we have to consider the case of the complimentary event $\bar{T}$. $\bar{T}$ denotes the event that the best offer was already observed in the learning phase. We assume that we continue in this case until we have observed all offers realizing then that the best offer in the learning phase must have been the best offer of all offers. We get therefore the expected rank of our rank maximizing strategy using a learning phase of k observations by

$$E_k(R/Pol) = E_k(R/T) + n\,\frac{k}{n}$$

Similar considerations hold for the value of the total expected search costs of our policy:

$$E_k(L/Pol) = cE_k(L/T) + cn\,\frac{k}{n}$$

We denote the differences of expected search cost

$$E_{k+1}(L/Pol) - E_k(L/Pol)$$

by $\Delta E_{k+1}(L/Pol)$ and the difference of expected rank

$$E_{k+1}(R/Pol) - E_k(R/Pol)$$

by $\Delta E_{k+1}(R/Pol)$.

Therefore we should choose an optimal value k^+ such that

$$\frac{\Delta E_{k^+}(R/Pol)}{\Delta E_{k^+}(L/Pol)} \leq c \leq \frac{\Delta E_{k^+ +1}(R/Pol)}{\Delta E_{k^+ +1}(L/Pol)}$$

is valid.

(It should be remembered that for $k=1,2,\ldots,[(\frac{n^2+n}{2n+1})^{0.5}]$ the value of $E_k(R)$ is increasing monotonically; for larger k these values decrease. Furthermore $E_k(L/T)$ and $E_k(R/T)$ are increasing monotonically for all values of k (k=1,2,...,n-1)). Our strategy was formulated as a rank maximizing strategy; it suffices now to think of a decreasing sorting order

$$\ell(n),\ \ell(n-1),\ \ldots,\ \ell(1)$$

of the prices for the different offers. We thus get the ranks for ascending sorting order E(PR/Pol) from E(R/Pol) simply through

$$E(PR/Pol) = n+1 - E(R/Pol).$$

Obviously it suffices to use $\Delta E(R/Pol)$ as the relation

$$/\Delta E(R/Pol)/ = /\Delta E(PR/Pol)/$$

is valid.

Now let us assume c=1. Using this value and using above considerations to get optimal values of k for various values of n we get:

k^+	for n between 3 and 1000		
1	3	-	25
2	25	-	56
3	57	-	103
4	104	-	167
5	168	-	248
6	249	-	348
7	349	-	468
8	469	-	607
9	608	-	767
10	768	-	948
11	948	-	

Using these values of k^+ and calculating the expected search costs and the expected rank of the price we get for selected values of n and c=1:

n	E(PR/Pol)+k	cE(L/Pol)+ck
20	6.5	4.734
30	7.5	8.346
40	9.167	8.85
50	10.83	9.248
60	11	12.989
70	12.25	13.401
80	13.5	13.763
90	14.75	14.084
100	16	14.373
200	22.167	24.454
500	36.278	42.13
1000	53.167	61.668

These results are obviously valid for the case that we assume that we are searching from the finite set {0,1,2,...,n-1} of price offers without visiting one offer twice[14] and without recall if we reduce the value of E(PR/Pol) + k by one. The effect of not allowing recall in this case is not the only difference. In the strategies described here we made the following assumptions:

1) The number of offers which will ultimately become available is known precisely.

2) There is no recall of offers once they have been passed over.

3) There is no possibility to evaluate any offers on a priori grounds apart from ranking it relative to other offers already observed.

The difference between the first strategy given and the latter one is that in the latter one we simply maximize the probability of selecting the best alternative or, to put it into other words, the second best is not more acceptable than the worst. The difference between these strategies and the strategies so far described primarily in the economics of search literature lies in the abilities we require

[14]Sampling without replacement.

from our searcher. In our case we only require that he is able to sort the alternatives observed. Besides we require however for the calculation of the optimal policy that search costs c can be expressed in true rank units or in other words and that the difference in utility between two alternatives is proportional (at least in the average) to the difference in true rank. But we are not restricted in the application of these models to those situations where we observe a random variables value with a distribution with known or unknown parameters. Even the true rank is in this policies not supposed to be a directly observable quantity. We only assume sortability of the observed alternatives. But we indeed switch with the way we introduce search costs to cardinal utility functions, which are not any more invariant to monotone transformations.

3. Possible Extensions

The two models presented only constitute two extreme examples of a pattern of strategies with variable memory limits. Besides it was felt that the assumption that the number of alternatives, which will ultimately become available is known precisely is a rather strong restriction to the applicability of these models. There are at least two ways to avoid this assumption. One way is to assume the existence of a known distribution on the number N of offers or at least to assume a Bayesian case with a proper or improper prior. Another version is to assume a constant flow of incoming and outgoing offers according to stochastic laws like for example the convolution of poisson arrival times with normally distributed values of the offers. We will discuss possible extensions of these models to handle the case of unknown value of N in another paper.

4. Rank oriented search with recall

The models given above concentrate on observations without recall. At least limited recall is in many practical cases available. So we should compare the results derived in a search without recall with the strategies selecting alternatives according to their rank with unlimited[15] recall. A nonsequential rule is investigated. In a sequential policy we would indeed be able to determine a reservation rank but we would not be able to determine in general whether or whether not an offer observed has a true rank lower or equal to this reservation rank. Therefore we shall consider here only a nonsequential strategy. This strategy shall simply consist in drawing a sample of a priori fixed size and determining the best alternative drawn. This is the alternative to be chosen. To reduce the side effects we have to determine the optimal number of observations in observations without replacement. We assume again that the n alternatives will be sorted in decreasing order of the price of the offers:

$$l(n) \geq l(n-1) \geq \ldots \geq l(1)$$

Our aim is to maximize the rank in this sorting order; that means $\ell(n)$ has rank 1, $\ell(n-1)$ rank 2 and so on.[16]

Lemma: $X_k = \max\{y_1, y_2, \ldots, y_k\}$ and the set $\{y_1, y_2, \ldots, y_k\}$ is a random sample of offers drawn without replacement from $\{1, 2, \ldots, n\}$. The probability for drawing an offer with rank x is

$$P(X_k = x) = \frac{\binom{x-1}{k-1}}{\binom{n}{k}} \quad , \; k \leq x \leq n$$

[15] Limited recall is to the author's knowledge rarely considered in the literature so far.

[16] Sampling with replacement was already considered by G. Stigler, loc. cit.

The expected rank observed in a sample of k alternatives is

$$E(X_k) = \frac{(n+1)k}{k+1}$$

and its variance $\sigma^2_{X_k}$ is equal to

$$\sigma^2_{X_k} = E(X_k)\ (u-E(X_k))$$

with $u = \frac{(n+2)\ (k+1)}{k+2} -1$

Using now the difference $E(X_{k+1})-E(X_k)$ we get for the optimal value of k the equation:

$$0 = c(k^2+3k) - (n-1)$$

or

$$k_{1,2} = \frac{-3c \mp \sqrt{c^2 - 4c(n-1)}}{2c}$$

For the special case c=1 we get for the optimal value k^+

$$k^+ = [-1.5 + \sqrt{5/4 + n}]$$

which for large n is approximately equal to $\sqrt{n}$.

The expected true rank will then be

$$E(n+1 - X_k) = \frac{n+1}{k^+ + 1}$$

For larger values of n this will not deviate much from $\sqrt{n}$ giving a total cost of approximately $2\sqrt{n}$ for larger n and c=1. Using our numerical calculations it can be seen that this is less than the policies assuming no recall. This is not only true for the expected cost of the offer plus the search cost but also for the variance of the distribution of cost of the alternative stopped with.

IV. Models of search with some knowledge of the underlying distribution function

Rank oriented models do not assume that we can exactly quantify the value of an alternative. They do not even assume that we know anything about the distribution of the utility, loss or other observable quantities characterizing an alternative's utility. We only assume that we are able to rank the alternatives drawn. However the necessity to introduce search cost in our considerations seem to diminuish the practical value of these strategies. It seems not to be easily possible anymore to confine ourself to ordinal utility functions. In the models to be considered now we assume that it is possible to observe a value - which we call price - characterizing the alternative. Also we assume that the price distribution is known. Using the Bayesian approach we only assume knowledge of the distribution and not of the parameters of the price distribution. We shall consider here the multinomial distribution only. We shall give analytical results for the special case of the discrete uniform distribution.

M. Rothschild has treated the problem of searching for the lowest price when the distribution of prices is multinomial.[17] It is well known that the family of Dirichlet distributions is a conjugate family for observations which

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[17] M. Rothschild, Searching for the Lowest Price When the Distribution of Prices is Unknown, J.P.E., Vol. 82 (1976), pp. 689-711; despite its title this paper assumes knowledge of the type of distribution function as it treats the multinomial case only. The technical results of this paper can in part be found in the excellent paper of Randolph.[18]

have a multinomial distribution.[18] It must be emphasized that using the multinomial distribution we assume prior knowledge of the maximal value of observable offers. Let the prior density function of the multinomial probabilities $x = (x_0, x_1, \ldots, x_{k-1})$ be equal to

$$f(x/\alpha) = \frac{\Gamma(\alpha_0 + \ldots + \alpha_{k-1})}{\Gamma(\alpha_0)\ldots\Gamma(\alpha_{k-1})} \; x_0^{\alpha_0 - 1} \cdots x_{k-1}^{\alpha_{k-1} - 1}$$

for $x_i > 0$ and $\sum_{i=1}^{k} x_i = 1$.

To simplify notation we assume without loss of generality that $\alpha_i > 0$ for $i = 0,1,\ldots,k-1$. The posterior distribution function after making the n observations and letting n_j be the number of observations of j and $n = \sum_{j=0}^{k-1} n_j$ is then

$$f(x/\alpha+n) = \frac{\Gamma(\alpha_0 + \ldots + \alpha_{k-1} + n)}{\Gamma(\alpha_0 + n_0)\ldots\Gamma(\alpha_{k-1} + n_{k-1})} \quad x_1^{\alpha_0 + n_0 - 1} \cdots\cdots\cdots\cdots x_k^{\alpha_{k-1} + n_{k-1} - 1}$$

The multinomial probability $p(X_{n+1}=j)$ is than for $j \varepsilon \{0,1,2,\ldots,k-1\}$ equal to

$$p^{n+1}(j) = \frac{\alpha_j + n_j}{\alpha + n}$$

Using the tuple $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-1})$ we can determine the probability vector $p_0, p_1, \ldots, p_{k-1}$ but not vice versa. The value of $\alpha = \sum_{i=0}^{k-1} \alpha_i$ plays such an important role in the use of our prior distribution; it can be interpreted as the "degree of belief" we are having in our prior distribution. Assuming α to be very large

[18] See for example M. DeGroot, Optimal Statistical Decisions, McGraw Hill Company, N.Y., 1970 and P.H. Randolph, Optimal Stopping Rules for Multinomial Observations, Metrika, vol. 14, pp. 48-61 (1968).

means that observations have little impact on the value of our multinomial probabilities. (With $\alpha \to \infty$ new observations would not change our prior probabilities at all). Low values of α to the contrary make our prior multinomial probabilities very "vulnerable" to experience. Now let v_n be equal to $x_n + cn$ in the case of no recall and equal to $\min (x_1, x_2, \ldots, x_n) + cn$ in the case of recall. Let the prior density before we observe the (n+1)-th alternative be

$$f_n(x) = \Gamma(\alpha+n) \prod_{j=0}^{k-1} (x_j^{\alpha_j+n_j-1} / \Gamma(\alpha_j+n_j))$$

Than the multinomial prior probabilities are given by

$$p_n(j) = (\alpha_j+n_j)/(\alpha+n) \text{ for } j = 0,1,2,\ldots,k-1$$

The expected value from an additional observation is then equal to

$$T_{n+1} (v_n) = \sum_{j=0}^{[v_n]} (v_n - j)\, p_n(j)$$

We would stop therefore when

$$T_{n+1} (v_n) \leq c$$

and continue search when

$$T_{n+1} (v_n) > c$$

Using $T_{n+1}(v_n) = \sum_{j=0}^{[v_n]} (v_n-j)p_n(j) = (\alpha+n)^{-1} \sum_{j=0}^{[v_n]} (v_n-j)(\alpha_j+n_j)$

and bearing in mind that

$\alpha_j = \alpha\, p_0(j)$ ($p_0(j)$ are the prior probabilities of our search) we get

$$T_{n+1}(v_n) = (\alpha+n)^{-1}\alpha T(v_n) + (\alpha+n)^{-1} \sum_{j=0}^{[v_n]} (v_n-j)n_j$$

$T(v_n)$ is equal to the expression

$$\sum_{j=0}^{[v_n]} (v_n-j)\, p_0(j)$$

which is the expected value of an additional observation assumig a known multinomial distribution function with the probabilities $p_0(j)$. In sampling with recall v_n is the lowest offer observed in all observations, the values of n_j are therefore zero for $j=0,1,\ldots,[v_n]$. That means we get for the case of recall

$$T_{n+1}(v_n) = (\alpha+n)^{-1}\alpha\ T(v_n)$$

and for the case of no recall

$$T'_{n+1}(v_n) = (\alpha+n)^{-1}\alpha\ T(v_n) + (\alpha+n)^{-1} \sum_{j=0}^{[v_n]} (v_n-j)n_j$$

As $\lim_{\alpha\to\infty} T_{n+1}(v_n) = T(v_n)$ and $\lim_{\alpha\to\infty} T'_{n+1}(v_n) = T(v_n)$ we can see that the case of fixed prior probabilities based on an infinite coefficient of belief α is a very special case of the adaptive formulation.

We shall also consider the problem of the influence of α. With increasing α and observations with recall the value of $T_{n+1}(v_n)$ increases with α. Or to put it into other words: if two searchers have the same prior distribution and have observed the same alternatives the one with a higher degree of belief α in his prior probabilities continues searching when the one with the lower degree of (subjective) belief in his prior probabilities has already stopped the search. This statement can be more precisely expressed if we consider the discrete uniform distribution only: the maximum number of observations is in this case exactly equal to the solution of

$$k-1 = -0.5 + \sqrt{0.25 + 2c + \frac{2nkc}{\alpha}}$$

or $$k^2 - k\left(\frac{\alpha+2nc}{\alpha}\right) - 2c = 0$$

Solving this equation for n we get for the maximal value n^+ of n, which can be obtained:

$$n^+ = \frac{\alpha(k-1)}{2c} - \frac{\alpha}{k}$$

The dominating term in this expression is $\alpha(k-1)/2c$. Such the number of alternatives searched for will be primarily dependent on the degree of belief in our prior distribution (expressed through α), the value of k and the search costs. The higher the search costs are, the lower the maximal number of observations. The larger k, the larger is the maximal number of observations and the larger is the variance. It is seldom mentioned that with this search strategy k has to be fixed in advance. As k has to be fixed in advance and is <u>not</u> revised by the adaptive strategy this model of search will rather show a tendency to make the maximal number of observations larger than necessary. This because k has to be chosen such that it covers also the most unfortunate cases which eventually do not even occur.

The basic assumption in this adaptive strategy is that we want to converge to one "true" distribution[19] which is unknown. But this is not the case in many applications. In fact wherever the "true" unknown underlying distribution changes over time the strategy described leads to wrong results. It can be shown that the reservation price strategy is rather stable against moderate shifts of the reservation price if the entropy is relatively high within the scope of the (elementary) events $j(j=0,1,...,[v_n])$ concerned.[20] Simulation experiments have shown that the prior distribution converges relatively fast to the "true" distribution. However as mentioned earlier as the true distribution would itself be rather a function of the underlying economic environment, the consumers attitude and other factors, the "true" distribution itself will vary over time. But the larger the coefficient of confidence α, the slower the distribution will converge to the "true" distribution. With $\alpha+n\to\infty$ the "prior" will obviously not react any more to a change of the true distribution. What therefore should be done in this case is to start to reduce $\alpha+n$ by the initial observations if $\alpha+n \geq g(\alpha)$ for some $g(\alpha)$. $g(\alpha)$ has to be fixed individually according to the amount and the speed of the change of the underlying "true" distribution with respect to the speed the observations done in the same time interval. g should also be a function of the confidence coefficient α, (as this coefficient determines the speed of adaptation.)

V. Conclusions

Two classes of search strategies were discussed, which basically do not assume knowledge of the distribution of the observations value or utility. Rather we assumed in this first class of strategies that we are able to sort the offers observed. We saw that the amount of memory we admit allows to distinguish a pattern of different search strategies. From these possible strategies we considered only two extreme strategies, one with a very limited memory

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[19] "True" is within quotes as the author believes that there is nothing like one "true" distribution in the economic applications considered!

[20] If the prior distribution is uniform the expected number of observations is equal to the expected waiting time to stop $\frac{k}{[v_n+1]}$ plus the expected value of the alternative, which is equal to $[v_n]/2$.

and the other one with a "complete" amount of memory. All strategies introduced in the first part do not allow recall. One observation was allowed per offer and the number n of offers available was assumed to be known in advance. This last assumption at least could be dropped if we follow an alternate approach. It was shown that if we drop the assumption of no recall using the simple fixed sample size strategy we get a strategy which seems to have less costs. This proposition is however based only on numerical calculations.

If we assume that the true value (or rank) of the offers can be recognized at the time of the observation we should rather choose a strategy which revises its prior distributional assumption if necessary. Maintaining a high degree of flexibility, this is possible if we assume that the offers are multinomially distributed. This distribution leaves a great amount of freedom in the learning process and does not fix the form of a distribution function (as it is the case with the normal distribution function). We also saw that in this adaptive case we have to distingush policies with and without recall. This is not the case with strategies which do assume a precise prior knowledge of the offers utility (loss) distribution. In these cases there is no difference between strategies with and without recall, if we do not assume a prior limitation on the number of offers to be observed. In this case however the strategies have to be constructed using the principle of backward induction[21] to determine the value of the strategies and the stopping vector. The considerations made in establishing these strategies are in fact similar to the considerations we had to make in our rank oriented strategy (without memory limitations).

[21] It consists basically of dynamic programming arguments. The policy construction principle is described in Y.S. Chow, H. Robbins and D. Siegmund, Great Expectations:..., loc. cit., pp. 50-51 and shall therefore not be described here.

Basically the rank oriented strategies without recall, seem to be less efficient than strategies where we assume that the distribution is known and the offers utility (or loss) can be recognized. We hardly can compare Bayesian strategies with respect to their efficiency to non-Bayesian or rank oriented strategies. However, we saw that in the Bayesian strategies we have not to fear the risk of possibly infinitely many observations if we start with the uniform distribution. The need for the development of a strategy where we limit the number of observations prior to the search is such not felt in this case. In addition to this we have to distinguish between the policy with and without recall as these give different strategies. A very special property of the Bayesian multinomial policy is the considerable importance of the choice of the coefficient of confidence (α). As increasing values of α diminish the adaptability of the policy to a change in the "true" distribution, limited memory should be introduced. We would such be able to forget "elder" experiences and adopt ourself to the underlying distribution continuously. In the case of a discrete uniform prior distribution the policy with recall and the policy without recall converges with $\alpha\to\infty$ to the well known optimal policy with an underlying discrete uniform distribution.

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Appendix A

First we show that the probability that the true rank N finally chosen is v when the size of the learning set is k and the number L of observed alternatives is ℓ is equal to:

$$W(N=v,\ L=\ell) = \frac{k}{\ell-1}\ \frac{(v-1)!}{(v-\ell)!}\ \frac{(n-\ell)!}{n!}$$

If the ℓ-th alternative observed has a rank of v and we stop with this alternative implies that $\ell-1$ alternatives observed before were of a rank lower than v. As there are v-1 such alternatives we have $(v-1)!/(v-\ell)!$ ordered subsets of $(\ell-1)$ alternatives drawn from (v-1) alternatives. Now the second best alternative was within the first k alternatives observed. As this alternative is observed with equal probability as first, second, ..., k-th alternative, the probability for the second best to be within the first k observations is such $k/(\ell-1)$. Recognizing that there are exactly $n!/(n-\ell)!$ possibilities of drawing ℓ alternatives (sorted) out of n alternatives we get the lemma. We show now that the relation

$$\sum_{\ell=k+1}^{n} \sum_{v=\ell}^{n} \frac{(v-1)!\,k(n-\ell)!}{(v-\ell)!\,(\ell-1)n!} = 1 - \frac{k}{n}$$

is correct.

We show the correctness by calculation:

$$\sum_{\ell=k+1}^{n} \sum_{v=\ell}^{n} \frac{(v-1)!\,k(n-\ell)!}{v(v-\ell)!\,(\ell-1)} = \sum_{\ell=k+1}^{n} \frac{k(n-\ell)!\,(\ell-1)!}{(\ell-1)n!} \sum_{v=\ell}^{n} \binom{v-1}{\ell-1} =$$

$$= \sum_{\ell=k+1}^{n} \frac{k}{(\ell-1)n} \binom{n-1}{\ell-1}^{-1} \binom{n}{\ell} = \sum_{\ell=k+1}^{n} \frac{k}{(\ell-1)\ell} =$$

$$= k \sum_{\ell=k+1}^{n} \left(\frac{1}{\ell-1} - \frac{1}{\ell} \right) = k\left(\frac{1}{k} - \frac{1}{n} \right) = 1 - \frac{k}{n}$$

We now show the correctness of the formulas in our theorem 1.

$$E(R/T) = b \sum_{\ell=k+1}^{n} \sum_{v=\ell}^{n} v \frac{(v-1)!\,k(n-\ell)!}{(v-\ell)!\,(\ell-1)n!} = \frac{bk}{n!} \sum_{\ell=k+1}^{n} \frac{(n-\ell)!}{(\ell-1)} \sum_{v=\ell}^{n} \frac{v!}{(v-\ell)!} =$$

$$= \frac{bk}{n!} \sum_{\ell=k+1}^{n} \frac{(n-\ell)!\,\ell!}{(\ell-1)} \sum_{v=\ell}^{n} \binom{v}{\ell} = b\,\frac{k}{n!} \sum_{\ell=k+1}^{n} \frac{(n-\ell)!\,\ell!\,(n+1)!}{(\ell-1)(\ell+1)!\,(n-\ell)!} =$$

$$= bk(n+1) \sum_{\ell=k+1}^{n} \frac{1}{(\ell-1)(\ell+1)} = \frac{k(n+1)}{2}\, b \sum_{\ell=k+1}^{n} \left(\frac{1}{(\ell-1)} - \frac{1}{(\ell+1)}\right) =$$

$$= b\,\frac{k(n+1)}{2} \left(\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{n-1} - \frac{1}{(k+2)} - \frac{1}{(k+3)} - \ldots - \frac{1}{(n+1)} \right)$$

$$= b\,\frac{1}{2} \left((n+1) + \frac{k(n+1)}{k-1} - \frac{k(n-1)}{n} - k\right) =$$

$$= \left(\frac{(2k+1)(n+1)}{k+1} - \frac{(2n+1)k}{n} \right) \left(\frac{n}{2(n-k)} \right)$$

$$E(L/T) = b \sum_{\ell=k+1}^{n} \sum_{v=\ell}^{n} \ell \frac{(v-1)!\,k(n-\ell)!}{(v-\ell)!\,(\ell-1)n!} =$$

$$= \frac{k}{n!}\, b \sum_{\ell=k+1}^{n} \frac{\ell(n-\ell)!}{(\ell-1)} \sum_{v=\ell}^{n} \frac{(v-1)!}{(v-\ell)!} =$$

$$= \frac{k}{n!}\, b \sum_{\ell=k+1}^{n} \frac{\ell(n-\ell)!\,(\ell-1)!}{(\ell-1)} \sum_{v=\ell}^{n} \frac{(v-1)!}{(v-\ell)!\,(\ell-1)!} =$$

$$= \frac{k}{n!}\, b \sum_{\ell=k+1}^{n} \frac{\ell(n-\ell)!\,(\ell-1)!}{(\ell-1)} \sum_{v=\ell}^{n} \binom{v-1}{\ell-1} = \frac{k}{n!}\, b \sum_{\ell=k+1}^{n} \frac{\ell(n-\ell)!\,(\ell-1)!}{(\ell-1)} \binom{n}{\ell} =$$

$$= kb \sum_{\ell=k+1}^{n} \frac{1}{\ell-1} = \frac{nk}{(n-k)} \sum_{\ell=k}^{n-1} \ell^{-1} .$$

We have to show

$$E(R^2/T) = \frac{k}{n!} b \sum_{\ell=k+1}^{n} \frac{(n-\ell)!\,\ell!}{(\ell-1)} \sum_{v=\ell}^{n} \binom{v}{\ell} v \quad .$$

The following relation is valid:

$$\sum_{v=\ell}^{n} v\binom{v}{\ell} = \sum_{v=k}^{n} ((v+1)\binom{v}{\ell} - \binom{v}{\ell}) = \sum_{v=k}^{n} ((\ell+1)\binom{v+1}{\ell+1} - \binom{v}{\ell}) =$$

$$= (\ell+1) \sum_{v=\ell}^{n} \binom{v+1}{\ell+1} - \sum_{v=\ell}^{n} \binom{v}{\ell} = (\ell+1) \binom{n+2}{\ell+2} - \binom{n+1}{\ell+1} =$$

$$= (\frac{\ell+(\ell+1)n}{(\ell+2)} \binom{n+1}{\ell+1} \quad .$$

Using this relation we get:

$$E(R^2/T) = \frac{k}{n!} b \sum_{\ell=k+1}^{n} \frac{(n-\ell!\,\ell!\,(n+1)!}{(\ell-1)(\ell+1)!\,(n-\ell)!} \quad - \quad \frac{(\ell+1)n+\ell}{(\ell+2)}$$

or

$$E(R^2/T) = \frac{k}{n!} b \sum_{\ell=k+1}^{n} \frac{(n-\ell)!\,\ell!\,(n+1)!}{(\ell-1)(\ell+1)!\,(n-\ell)!} \quad (\frac{(n+2)(\ell+1)}{(\ell+2)} - 1) =$$

$$= (n+1)(n+2)kb \sum_{\ell=k+1}^{n} \frac{1}{(\ell-1)(\ell+2)} - bk(n+1) \sum_{\ell=k+1}^{n} \frac{1}{(\ell-1)(\ell+1)} =$$

$$= \frac{k(n+1)(n+2)}{3} b \sum_{\ell=k+1}^{n} (\frac{1}{(\ell-1)} - \frac{1}{(\ell+2)}) - \frac{k(n+1)b}{2} \sum_{\ell=k+1}^{n} (\frac{1}{(\ell-1)} - \frac{1}{(\ell+1)}) =$$

$$= \frac{2(n+2)}{3} b (\frac{(n+1)k}{2} ((\frac{1}{k} + \frac{1}{(k+1)} - \frac{1}{(n+1)} - \frac{1}{n}) + (\frac{1}{(k+2)} - \frac{1}{(n+2)}))) -$$

$$- b \frac{1}{2} (\frac{(2k+1)(n+1)}{(k+1)} - \frac{(2n+1)k}{n}) \quad .$$

From this we get

$$E(R/T) = b\,\frac{(n+1)k}{2}\left(\frac{1}{k} + \frac{1}{(k+1)} - \frac{1}{(n+1)} - \frac{1}{n}\right) =$$

$$= \frac{b}{2}\left(\frac{(2k+1)(n+1)}{k+1} - \frac{(2n+1)k}{n}\right) .$$

We get

$$E(R^2/T) = \frac{2(n+2)}{3}\left(E(R/T) + \frac{b(n+1)k}{2(k+2)} - \frac{b(n-1)k}{2(n+2)}\right) - E(R/T) =$$

$$= \frac{2(n+2)}{3}\left(E(R/T) + \frac{b(n+1)k(n-k)}{2(n+2)(k+2)}\right) - E(R/T)$$

and using the lemma

$$E((R - E(R/T))^2/T) = E(R^2/T) - E^2(R/T)$$

we get finally:

$$\sigma^2(R/T) = \left(\frac{2n+1}{3}\right) E(R/T) - E^2(R/T) + \frac{n(n+1k}{3(k+2)} .$$

Similarly we get the variance of E(L/T).

$$E(L^2/T) = \sum_{\ell=k+1}^{n} \sum_{v=\ell}^{n} \ell^2 \frac{(v-1)!\,k(n-\ell)!}{(v-\ell)!\,(\ell-1)n!}\, b =$$

$$= \frac{k}{n!}\, b \sum_{\ell=k+1}^{n} \frac{\ell\cdot\ell(n-\ell)!\,(\ell-1)!}{(\ell-1)} \sum_{v=\ell}^{n} \binom{v-1}{\ell-1} = kb \sum_{\ell=k+1}^{n} \frac{\ell}{(\ell-1)} =$$

$$= bk \sum_{\ell=k}^{n-1} \left(1 + \frac{1}{\ell}\right) = nk + E(L/T).$$

Using the lemma above we get

$$\sigma_L^2 = nk + E(L/T) - E^2(L/T) .$$