Illiquid Assets and Optimal Portfolio Choice

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Abstract

The presence of illiquid assets, such as human wealth, housing and proprietorships substantially complicates the problem of portfolio choice. This paper is concerned with the problem of optimal asset allocation in a continuous time model when one asset cannot be traded. This illiquid asset, which depends on an uninsurable source of risk, provides a liquid dividend. In the case of human capital we can think about this dividend as labor income. The agent is endowed with a given amount of the illiquid asset and with some liquid wealth which can be allocated in a market where there is a risky and a riskless asset. The main point of the paper is that the optimal allocations to the two liquid assets and consumption will critically depend on the endowment and characteristics of the illiquid asset, in addition to the preferences and liquid wealth of the agent. We provide what we believe to be the first analytical solution to this problem when the agent has power utility of consumption and terminal wealth. We also derive the value that the agent assigns to the illiquid asset. The risk adjusted valuation procedure we develop can be used to value both liquid and illiquid assets, as well as contingent claims on those assets.

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1 Introduction

The problem of optimal asset allocation is of great importance both in the theory as well as in the practice of finance. Since the seminal work of Markowitz (1952) scholars and practitioners have looked at the issue of how much money should an investor optimally allocate to different assets or asset classes. The single period model of Markowitz (1952) was extended to a multiperiod setting by Samuelson (1969) and then to continuous time by Merton (1969, 1971). The traditional approach assumes that all assets can be traded at all times. This paper is concerned with optimal asset allocation in a continuous time model when one asset cannot be traded.

Typical examples of assets in which trading is problematic include human wealth, housing and proprietorships. When the asset allocation problem is solved without taking into account the existence of these “illiquid”assets the allocation is certainly suboptimal. Consider the following example [1]. Two individuals with the same wealth, the same preferences and the same horizon would invest in the same portfolio using the traditional asset allocation framework. However, if one of the individuals is a stock broker with his human wealth highly correlated with the stock market, and the other is a tenured university professor with his human wealth independent of the stock market, it would be reasonable to expect that they would have different allocations. This is the problem we address in this paper.

There are many definitions of illiquid assets. To make the problem tractable, in this paper we assume that illiquidity prevents the trading of the asset over the time horizon we consider (though this time horizon could become infinite). The illiquid asset, however, provides a liquid ”dividend” that is related to the level of an observable state variable associated with the illiquid asset. In the case of human wealth the dividend could be labor income, in the case of the housing dividend could be the housing services, and in the case of a proprietorship it could be distributed profits from the business. The uncertainty that drives the illiquid asset cannot be fully diversified in the market. Since the asset is not traded, the state variable associated with the illiquid asset can not be interpreted as a price. In the finite horizon case this state variable becomes the price of the asset only at the terminal date.

We assume that the agent in endowed with a given amount of the illiquid

[1]This example was presented by Robert Merton in a talk in Verona, Italy in June 2003.

asset and with some liquid wealth which can be allocated in a market where there are two liquid assets: a risky asset and a risk free asset. The main point of this paper is that the allocations to the two liquid assets and consumption will critically depend on the endowment and characteristics of the illiquid asset, in addition to the preferences and liquid wealth of the agent.

In the process of establishing the optimal allocation to the liquid assets we also derive the value that the agent assigns to the illiquid asset. As expected, the value that the agent assigns to the illiquid asset will always be lower than the value it would have if it were traded. Moreover, this value depends not only on the level of the state variable associated with the illiquid asset, but also on the preferences of the agent. Interestingly, under our assumptions about preferences and price dynamics this value does not depend on the endowment of the liquid risky asset.

The problem of asset allocation in the presence of illiquid assets has been the subject of intense research in the finance literature since the late 60's. Recognizing the complexity of the subject simplifying assumptions have been introduced to make the problem tractable.

Among non tradable assets, human wealth is certainly the most relevant source of risk in the individual allocation problem which is difficult to insure or diversify. Bodie, Merton and Samuleson (1992) consider a long horizon investor with a riskless stream of labor income and show that an investor with riskless non tradable human wealth should tilt his financial portfolio toward stocks relative to an investor who owns only tradable stock. Jagannathan and Kocherlakota (1996) show that this advice is economically sound as long as the human wealth is relatively uncorrelated with stock returns. Zeldes (1989) performs a numerical study of a discrete time model of optimal consumption in the presence of stochastic income. Koo (1995) and Heaton and Lucas (1997) introduce risky labor income and portfolio constraints in an infinite horizon portfolio choice problem and, using a numerical simulation, focus on how the presence of background risks from sources such as labor, influences consumption and portfolio choice. Both papers find that investors hold most of their financial wealth in stocks. Koo (1995) shows numerically that an increase in the variance of permanent income shocks decreases both the optimal portfolio allocation to stocks and the consumption labor income ratio of power utility investors. In a discrete time framework Viceira (2001) considers an approximate solution and finds that positive correlation between labor income innovations and unexpected financial returns reduces the investor's willingness to hold liquid risky asset because of its poor properties as an hedge against unexpected declines in labor income. Consistently, Heaton and Lucas (2000) find that entrepreneurs have significantly safer portfolios of financial assets than other investors with similar wage and wealth. Campbell and Viceira (2002) provide a comprehensive discussion of the empirical testing and of the economic implications of including human wealth in the household portfolio choice problem.

A related literature deals with portfolio choice in the presence of assets which cannot be traded. To our knowledge the first treatment is due to Myers (1972, 1973) which solves the static version of the problem. In a dynamic context, the problem we solve can be seen as the limit of large transaction costs of the Grossmann and Laroque (1990) model for illiquid durables. Svensson and Werner (1993) provide a treatment with exponential preferences. Longstaff, Liu and Kahl (2003) formulate and provide a numerical solution to the optimal dynamic allocation problem of an investor with power utility whose portfolio includes a stock which cannot be sold. Among the possible sources of background risk in household portfolios, housing is certainly one important asset class that is relatively illiquid and undiversified. Analyzing risk and return is however complicated because of the unobservable flow of consumption of housing services. Flavin and Yamashita (1998) consider housing both as an asset and as a source of consumption, and obtain the optimal portfolio allocations by simulation.

There is a large strand of the literature in stochastic optimization which addresses the continuous time portfolio allocation problem in incomplete markets both with through direct partial differential equation approach and with the martingale-measure duality approach. Duffie, Fleming, Soner and Zariphopolou (1997) study an asset allocation problem for an investor which maximizes HARA utility from consumption in a market composed by a risky and a riskless asset and receives an income which cannot be replicated by other securities. This study proves existence, uniqueness and regularity of the value function, while the optimal consumption path and the allocation strategy are implicitly specified throughout a feedback expression. Koo (1998) analyzes the same problem in the presence of constraints.

The stochastic optimization problem we discuss is strictly related to the utility based pricing of contingent claims whose underlying assets are non traded. Most of these references, (Davis (1999), Hobson and Henderson (2002), Henderson (2002) and Musiela and Zariphopolou (2004a)), assume that the agent has exponential preferences and no consumption and dividends. Musiela and Zariphopolou (2004b) and Elliott and Van der Hoek

(2004) discuss a discrete time general approach to the valuation of risky assets in an incomplete market similar to the one we discuss in Section 2.

Our results are based on the duality approach pioneered by Cox and Huang (1989), He and Person (1991), and Karatzas et al (1991). He and Pages (1993) and El Karoui and Jeanblanc (1998) deal with a constrained version of the problem when labor income risk can be diversified in the market.

When the agent receives an uninsurable random endowment the mathematical formulation of the stochastic control problem becomes difficult. Existence results under very general conditions on the price processes and on the utility function have been obtained by Cuoco (1997) attacking directly the primal problem, while exact results on the duality approach have been established by Cvitanic, Schachermayer and Wang (2001) in the case of maximization of utility from terminal wealth and extended by Karatzas and Zitkovic (2003) to the problem with intertemporal consumption and constraints.

As far as we are aware, our paper contains the first analytical solution to this problem when the agent has power utility of consumption and terminal wealth. The analytical solution obtained allows us to quantify the impact of the assets characteristics and the agent preferences on optimal asset allocation and consumption. In particular, we show that the higher is the correlation between the liquid and the illiquid asset, the lower will be the allocation to the risky liquid asset. So, in the example given above the professor would optimally invest a higher proportion of his liquid wealth in the risky liquid asset than the stock broker. Since the human wealth of the stock broker is highly correlated with the stock market, and his human wealth is non tradable, he will tend to invest a smaller fraction of his liquid wealth in the market portfolio. The value to the investor of the illiquid asset also depends on the correlation between the two assets. Interestingly, this value is higher for low and for high correlation and it reaches a minimum for an intermediate correlation. This is due to the fact that for low as well as for high correlations it is easier to hedge the endowment in the illiquid asset with positions in the liquid risky asset.

The main contribution of the paper is to provide for a risk adjusted valuation procedure that can be used to value both liquid and illiquid assets. The procedure reduces to risk neutral valuation for the liquid assets, while the risk adjustment for valuing illiquid assets depends on the preferences of the investor and on the volatility of the uninsurable risk.

The paper is organized as following. In Section 2 we develop a single period binomial example which contains the main ingredients of the approach we pursue in the rest of the paper. Section 3 develops the continuous time optimal allocation problem. Section 4 provides for an illustrative example to show the main characteristics of the continuous time solution. Section 5 concludes and the Appendix contains some of the more technical results.

2 A binomial example

Consider a single period, finite state model for the optimal asset allocation problem in the presence of an illiquid asset. Trading can occur at time $t = 0$ and $t = T$. The market is composed of three assets:

- A risk free bond b_t, which pays an equal amount in any state of the world at time T. R_f is the riskless interest rate over the period $[0, T]$. Hence its final cash flow is given in terms of its initial price by:

  $$b_T = b_0 (1 + R_f)$$

  Without loss of generality we can assume $b_0 = 1$

- A liquid risky asset with price s_t, which evolves along a binomial tree, such that at time T:

  $$\begin{aligned} s_T (up) &= s_0 u_s \\ s_T (down) &= s_0 d_s \quad with \quad u_s > 1 > d_s \end{aligned}$$

- An illiquid risky asset with a level of h_t which also evolves along a binomial tree, such that at time T:

  $$\begin{aligned} h_T (up) &= h_0 u_h \\ h_T (down) &= h_0 d_h \quad with \quad u_h > 1 > d_h \end{aligned}$$

Note that the illiquid asset can only be traded at time T, so its level at time 0, h_0, represents an observable state variable associated with the illiquid asset and not the price of the asset.

The state space Ω of the market at time T is completely specified by the knowledge of the states reached by the illiquid asset and by the liquid asset. Thus there are 4 possible states:

$$\omega_1 = (u_s, u_h) \quad \omega_2 = (u_s, d_h) \quad \omega_3 = (d_s, u_h) \quad \omega_4 = (d_s, d_h)$$

each of them occurs with probability $\mathbb{P}(\omega = \omega_i) = p_i$, $i = 1, ..., 4$.

The single period problem in the absence of consumption can be stated as follows: find the allocation at time 0 which maximizes the expected utility of terminal wealth (at time T) . We denote by π_0^s and π_0^b the initial dollar amounts invested in each of the two liquid securities. Since the illiquid wealth cannot be traded we assume that the investor has a claim of h_T at time T. Given an initial liquid endowment $l_0 > 0$ and an initial level of the state variable associated with the illiquid asset of $h_0 \geq 0$, the optimal allocations to the liquid asset π_0^s and cash π_0^b have to be selected in order to maximize the final expected utility of wealth. For tractability we assume a power utility function with a coefficient of relative risk aversion of γ.

$$\sup_{\left(\pi_0^s,\pi_0^b\right)\in\mathcal{A}(l_0,h_0)} E^{\mathbb{P}}\left[U_\gamma\left(w_T\right)\right] = \sup_{\left(\pi_0^s,\pi_0^b\right)\in\mathcal{A}(l_0,h_0)} E^{\mathbb{P}}\left[\frac{w_T^{1-\gamma}}{1-\gamma}\right]$$

where U_γ is the CRRA utility function with risk aversion parameter $\gamma \in (1, +\infty)$ and $\mathcal{A}(l_0, h_0)$ is the set of admissible plans $\left(\pi_0^s, \pi_0^b\right)$ such that $w_T = l_T + h_T$ can be reached given an initial liquid wealth $l_0 > 0$ and the level of the initial illiquid asset (state variable) h_0. Any admissible strategy $\left(\pi_0^s, \pi_0^b\right)$ must obey the financing condition:

$$\pi_0^s + \pi_0^b = l_0$$

and the proportion invested in each asset cannot be changed in the interval $[0, T]$.

Then, at time T the liquid wealth in units of bond will be:

$$\frac{l_T(\omega)}{(1+R_f)} = \frac{w_T(\omega)}{(1+R_f)} - \frac{h_T(\omega)}{(1+R_f)} = \pi_0^b + \pi_0^s \frac{s_T(\omega)}{(1+R_f)\, s_0} \tag{1}$$

The only decision variables in this simple setup are π_0^b and π_0^s. In order to determine the optimal value of these variables we use the duality approach: we first determine the optimally allocated final wealth w_T and then we determine the optimal strategy π_0^b, π_0^s.

2.1 Determination of the optimally allocated wealth using the duality approach

Consider a market composed only by the liquid asset and the bond. In this market no arbitrage requires that exists a probability measure $\mathbb{Q}$, such that the current price of the risky liquid asset is given by:

$$s_0 = E^{\mathbb{Q}}\left[\frac{s_T}{1+R_f}\right] = E^{\mathbb{P}}\left[\xi^{\mathbb{Q}} s_T\right] = \sum_{i=1}^{4} q_i \frac{s_T(\omega_i)}{1+R_f} \tag{2}$$

where $\xi^{\mathbb{Q}}$ is the stochastic discount factor:

$$\xi_i^{\mathbb{Q}} = \frac{q_i}{p_i(1+R_f)} \quad i = 1, .., 4$$

Since the illiquid risky asset can not be traded at time 0, the absence of arbitrage does not imply a relation similar to (2) for h_t. Then, $\mathbb{Q}$ is not uniquely defined since we have four states of the world and only three equations (the normalization, one for the stock and one for the bond). Thus, illiquidity generates market incompleteness and there exists a non trivial set $\mathcal{M}$ of measures such that (2) is verified. These set of measures are usually called equivalent martingale measures [2]. The no arbitrage condition alone will not provide an unique $\mathbb{Q}$. However, as we discuss below, the utility maximizer agent selects a unique pricing measure $\mathbb{Q}^*$ that applies both to the valuation of liquid and illiquid assets.

Our first objective is to determine the optimally allocated wealth[3], w_i^T, in each of the possible states $1, .., 4$ at time T. The maximization problem can be written as:

$$\sup_{\left(\pi_0^s, \pi_0^b\right) \in \mathcal{A}(l_0, h_0)} \sum_{i=1}^{4} p_i U_\gamma\left(w_i^T\right) \tag{3}$$

The constraints on w_T^i implied by (1) in each state ω_i depend on the allocations π_0^s and π_0^b, thus it is not possible to fix the final wealth without knowing the initial allocation. Cox and Huang (1989) show how to obtain the optimal terminal wealth without knowing the optimal allocations by solving a dual optimization problem. The basic idea is that the constraint on any admissible vector of final wealths $\left\{w_i^T\right\}_{i=1,..,4}$, which can be reached without violating (1), can be characterized independently of the allocation strategy π_0^s, π_0^b. The

[2] Strictly speaking this name is appropriate only in the multiperiod setting, where eq.(2) becomes a martingale condition for discounted prices under $\mathbb{Q}$: $E_t^{\mathbb{Q}}\left[e^{-r(t+k)} S_{t+k}\right] = e^{-rt} S_t$ where r is the continuously compounded risk free rate of interest.

[3] The wealth that the investor has at time T, conditional on his having followed the optimal allocation strategy (see Merton (1992)).

set of feasible final wealths $\left\{w_i^T\right\}_{i=1,..,4}$ is the set of possible claims attainable trading only in the liquid assets with an initial liquid endowment l_0:

$$\frac{w_i^T - h_i^T}{1+R_f} = l_0 + \pi_0^S \left(\frac{s_i^T}{(1+R_f)\, s_0} - 1 \right) \quad i = 1,..,4$$

therefore by definition of martingale measure, equation (2), any attainable $\left\{w_i^T\right\}_{i=1,..,4}$ has to satisfy:

$$\sum_{i=1}^{4} q_i \left[\frac{w_i^T - h_i^T}{1+R_f} \right] = l_0 \quad \forall \mathbb{Q} \in \mathcal{M}$$

Thus, it is possible to compute the optimal attainable wealth through the following constrained optimization problem where the allocation strategy does not appear:

$$\begin{aligned} & \sup_{\left\{w_i^T\right\}_{i=1,..4}} \sum_{i=1}^{4} p_i U_\gamma \left(w_i^T\right) \\ s.t. \quad : \quad & \sum_{i=1}^{4} q_i \left[\frac{w_i^T - h_i^T}{1+R_f} \right] = l_0 \quad \forall \mathbb{Q} \in \mathcal{M} \end{aligned} \tag{4}$$

Following the **dual** approach we introduce the Lagrangian:

$$\begin{aligned} & \mathcal{L}\left(\left\{w_i^T\right\}_{i=1,..4}, \left\{q_i\right\}_{i=1,..4}, \lambda \right) = \\ = \quad & \sum_{i=1}^{4} p_i U_\gamma \left(w_i^T\right) - \lambda \left[\sum_{i=1}^{4} q_i \left(\frac{w_i^T - h_i^T}{1+R_f} \right) - l_0 \right] \quad \lambda > 0,\ \mathbb{Q} \in \mathcal{M} \end{aligned}$$

and applying the first order conditions we find the optimal wealths $\left\{w_i^T\right\}_{i=1,..4}$ for a fixed $\mathbb{Q}$ and λ:

$$\begin{aligned} p_i U_\gamma' \left(w_i^T\right) &= \frac{\lambda}{(1+R_f)} q_i \\ w_i^T (\lambda, \mathbb{Q}) &= \left(\frac{\lambda}{(1+R_f)} \frac{q_i}{p_i} \right)^{-\frac{1}{\gamma}} = \left(\lambda \xi_i^{\mathbb{Q}}\right)^{-\frac{1}{\gamma}} \end{aligned} \tag{5}$$

Then we express the maximized Lagrangian as a function of λ and $\mathbb{Q}$ and minimize over the possible $\mathbb{Q} \in \mathcal{M}$:

$$\begin{aligned} & \inf_{\mathbb{Q}\in\mathcal{M}} \mathcal{L}\left(\left\{w_i^T(\lambda,\mathbb{Q})\right\}_{i=1,..,4}, \{q_i\}_{i=1,..,4}, \lambda\right) \qquad (6) \\ = & \inf_{\mathbb{Q}\in\mathcal{M}} E^{\mathbb{P}}\left[\frac{\gamma}{1-\gamma}\left(\frac{\lambda}{(1+R_f)}\frac{q_i}{p_i}\right)^{\frac{\gamma-1}{\gamma}}\right] + \lambda\left(l_0 + E^{\mathbb{P}}\left[\frac{q_i}{p_i}\frac{h_T}{1+R_f}\right]\right) \end{aligned}$$

Any martingale measures $\mathbb{Q}$ that belongs to $\mathcal{M}$ is constrained by (2) and by the normalization, hence:

$$\begin{aligned} 1 &= \mathbb{Q}\left(s^T = s_0 u_s\right) u_s + \mathbb{Q}\left(s^T = s_0 d_s\right) d_s = (q_1+q_2)\, u_s + (q_3+q_4)\, d_s \\ 1 &= q_1 + q_2 + q_3 + q_4 \end{aligned}$$

thus:

$$\begin{aligned} q_1 + q_2 &= \frac{1-d_s}{u_s - d_s} \\ q_3 + q_4 &= \frac{u_s - 1}{u_s - d_s} \end{aligned}$$

Note that no arbitrage provides a constraint on the marginal probabilities, but these restrictions do not uniquely determine the optimal measure $\mathbb{Q}^*$. To simplify the analysis we define the probabilities of moving up or down for the illiquid asset conditional on the state of the liquid risky asset:

$$\begin{aligned} f_1 &\triangleq \mathbb{Q}\left(h^T = h_0 u_h \mid s^T = s_0 u_s\right) = \frac{q_1}{q_1+q_2} \\ f_2 &\triangleq \mathbb{Q}\left(h^T = h_0 d_h \mid s^T = s_0 u_s\right) = \frac{q_2}{q_1+q_2} \\ f_3 &\triangleq \mathbb{Q}\left(h^T = h_0 u_h \mid s^T = s_0 d_s\right) = \frac{q_3}{q_3+q_4} \\ f_4 &\triangleq \mathbb{Q}\left(h^T = h_0 d_h \mid s^T = s_0 d_s\right) = \frac{q_4}{q_3+q_4} \\ f_1 + f_2 &= 1, \qquad f_3 + f_4 = 1 \end{aligned}$$

Using the above no arbitrage conditions and the definitions of conditional probabilities, we can substitute the q_i's in (6) in terms of the f_i's. Then we can optimize over the set of martingale measures by simply imposing the first order conditions with respect to f_i, which imply:

$$\frac{f_i}{p_i} \propto \left(h_i^T\right)^{-\gamma}$$

Then normalization of conditional probabilities implies

$$\begin{aligned} f_i^* &= \frac{p_i h_i^{-\gamma}}{\sum_{i=1}^{2} p_i h_i^{-\gamma}} \qquad i = 1,2 \\ f_i^* &= \frac{p_i h_i^{-\gamma}}{\sum_{3=1}^{4} p_i h_i^{-\gamma}} \qquad i = 3,4 \end{aligned} \tag{7}$$

Hence the optimal measure $\mathbb{Q}^*$ is given by:

$$q_1^* = \frac{1-d_s}{u_s-d_s} f_1^*, \quad q_2^* = \frac{1-d_s}{u_s-d_s} f_2^*, \quad q_3^* = \frac{u_s-1}{u_s-d_s} f_3^* \quad q_4^* = \frac{u_s-1}{u_s-d_s} f_4^*$$

Finally λ^* is easily determined by the condition that at optimality the strict equality in the budget constraint holds:

$$E^{\mathbb{Q}^*}\left[\frac{w^T(\lambda^*, \mathbb{Q}^*) - h^T}{1+R_f}\right] = l_0$$

Substituting in this equation the expression of the optimal wealth (5) and the optimal measure $\mathbb{Q}^*$ from (7), and simplifying we obtain the expression for λ^*

$$\lambda^* = \left\{ E^{\mathbb{Q}^*}\left[\left(\xi^{\mathbb{Q}^*}\right)^{-1/\gamma}\right]^{-1} \left(l_0 + E^{\mathbb{Q}^*}\left[\frac{h_T}{1+R_f}\right]\right)\right\}^{-\gamma}$$

The saddle point theorem (see e.g. Duffie (1996)) proves that the optimal solutions $(w_T^*, \mathbb{Q}^*, \lambda^*)$ found in the **dual** problem are optimal also for the **direct** problem because at optimality the following min-max equality holds:

$$\begin{aligned} E^{\mathbb{P}}\left[U_\gamma\left(w_T^*\right)\right] &= \sup_{\left\{w_i^T\right\}_{i=1,..4}} \inf_{\lambda>0,\ \mathbb{Q}\in\mathcal{M}} \mathcal{L}\left(\left\{w_i^T\right\}_{i=1,..,4}, \left\{q_i\right\}_{i=1,..,4}, \lambda\right) \\ &= \inf_{\lambda>0,\ \mathbb{Q}\in\mathcal{M}} \sup_{\left\{w_i^T\right\}_{i=1,..4}} \mathcal{L}\left(\left\{w_i^T\right\}_{i=1,..,4}, \left\{q_i\right\}_{i=1,..,4}, \lambda\right) \\ &= E^{\mathbb{P}}\left[\frac{\gamma}{1-\gamma}\left(w_T^*\right)^{(\gamma-1)/\gamma}\right] + \lambda^*\left(l_0 + E^{\mathbb{Q}^*}\left[\frac{h_T^*}{1+R_f}\right]\right) \end{aligned} \tag{8}$$

The key results of this section are:

1. The optimal $\left(w_i^T\right)^*$ is given as a function of $(\lambda^*, \mathbb{Q}^*)$ by:

$$\left(w_i^T\right)^* = \left(\frac{\lambda^*}{(1+R_f)} \frac{q_i^*}{p_i}\right)^{-\frac{1}{\gamma}}, \tag{9}$$

2. $\{(q_i^*/p_i)(1+R_f)\}_{i=1,..,4}$ is the optimal stochastic discount factor in the market, such that discounted expected prices of liquid assets equal current price under $\mathbb{Q}^*$ ($\mathbb{Q}^*$ is the optimal martingale measure);

3. There exists a value of the Lagrangian parameter $\lambda^* > 0$, such that discounted liquid optimal wealth equals initial liquid wealth:

$$E^{\mathbb{Q}^*}\left[\frac{w_T^*(\lambda^*, \mathbb{Q}^*) - h_T}{1+R_f}\right] = l_0$$

4. The value that the investor assigns to the illiquid asset:

$$\widehat{h}_0 = E^{\mathbb{Q}^*}\left[\frac{h_T}{1+R_f}\right]$$

which is different from the level of the state variable h_0, depends on the investor preferences and is determined by the optimal valuation measure $\mathbb{Q}^*$. The investor values his total wealth at time 0 according to the optimal pricing measure and therefore he values his current wealth as:

$$w_0 = l_0 + \widehat{h}_0 \neq l_0 + h_0$$

2.2 Determination of the optimal allocation strategy

Once the optimal final wealth for each state is known, the determination of the optimal allocation strategy becomes the solution to a linear system of equations. From (9), the optimal liquid final wealth in each state is given by:

$$\left(w_i^T\right)^* - h_i^T = \left(\frac{\lambda^*}{(1+R_f)}\frac{q_i^*}{p_i}\right)^{-1/\gamma} - h_i^T$$

To obtain the optimal allocation to the liquid assets we consider separately the states where the risky liquid asset moves up from the cases where it moves down. To do this we take the conditional expectation of the liquid wealth over the possible states of h_T, controlling for the liquid risky asset s_T. Then the following linear system determines π_0^s, π_0^b :

$$\begin{aligned} l_u^* &\triangleq \sum_{i=1}^{2} f_i^* \left\{\frac{\left(w_i^T\right)^* - h_i^T}{(1+R_f)}\right\} = \pi_0^b + \pi_0^s \frac{u_s}{(1+R_f)} \\ l_d^* &\triangleq \sum_{i=3}^{4} f_i^* \left\{\frac{\left(w_i^T\right)^* - h_i^T}{(1+R_f)}\right\} = \pi_0^b + \pi_0^s \frac{d_s}{(1+R_f)} \end{aligned}$$

The solution of this system is given by:

$$\begin{aligned}\pi_0^b &= l_0 - \pi_0^s \\ \pi_0^s &= \frac{l_u^* - l_d^*}{u_s - d_s}(1 + R_f)\end{aligned}$$

The duality approach, by determining first the optimally allocated terminal wealth for each state, transforms the problem of determining the optimal allocations into the problem of replicating a derivative, trading only in the liquid assets, whose pay-off in each state is the optimal liquid wealth. An additional characteristic of this approach is that the analysis also provides for the stochastic discount factor which the agent can use to value any contingent claim on the liquid and the illiquid asset. The general ideas presented in this section carry over to the continuous time case.

3 The continuous time case

Now consider a continuous time economy where prices evolve stochastically in a filtered probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ supporting a two dimensional Brownian motion (W_t^1, W_t^2) where $\mathcal{F} = \{\mathcal{F}_t\}_{t \leq T}$ and $\mathcal{F}_t$ represents the augmented filtration generated by all the information reflected in the market up to time t and $\mathbb{P}$ is the objective probability measure. We fix a final time horizon T, the epoch at which the non traded (illiquid) asset becomes tradable and can be consumed.

The market is composed of three assets:

- The risk free bond B_t, whose dynamics is:

$$dB_t = rB_t dt \quad t \leq T$$

  where r is the continuously compounded risk free interest rate which, for simplicity, we assume to be constant.

- A traded liquid risky asset S_t, whose dynamics is:

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t^1 \quad t \leq T$$

where α ($> r$)is the continuously compounded expected rate of return on the risky liquid asset, and σ is the continuous standard deviation of the rate of return. [4]

- An illiquid risky asset H_t (no trading is allowed until time T, when it can be consumed), whose dynamics is:

$$\frac{dH_t}{H_t} = (\mu - \delta)dt + \eta \left(\rho dW^1 + \sqrt{1-\rho^2} dW^2 \right) \quad t \leq T$$

  where μ is the continuously compounded total expected rate of return on the risky illiquid asset, δ is the liquid continuous rate of dividend paid by the illiquid asset, η is the continuous standard deviation of the rate of return, and ρ is the correlation coefficient between the dynamics of the liquid and the illiquid risky asset. [5]

Since the illiquid asset H_t cannot be traded at any time $t < T$, it represents the level of a state variable associated with the illiquid asset. At time T, the state variable, H_T, becomes equal to the price of the illiquid asset.

The intertemporal optimization problem of the agent is given by:

$$\begin{aligned} & \sup_{(\pi,c)\in A(w_0,h_0)} E^{\mathbb{P}} \left[\int_0^T \beta_c e^{-\kappa u} U_\gamma (c_t)\, du + \beta_W e^{-\kappa T} U_\gamma \left(W_T^{\pi,c}\right) \right] \\ = & \sup_{(\pi,c)\in A(w_0,h_0)} V\left(c, W_T^{\pi,c}\right) \\ with \quad U_\gamma (x) = & \frac{x^{1-\gamma}}{1-\gamma} \quad for\ x > 0, \quad U_\gamma (x) = -\infty \quad for\ x \leq 0, \end{aligned}$$

where $W_T^{\pi,c}$ indicates the total wealth of the investor at time T which includes the liquid wealth, L_T, and the illiquid wealth, H_T; c_t indicates consumption at time t, κ indicates the subjective discount rate and finally β_c, β_W the relative weights of final utility of wealth with respect to the utility of intertemporal

[4]For simplicity we assume that the liquid risky asset pays no dividends, but the analysis would be the same if the asset paid a continuous dividend

[5]Since human wealth will enter in the state equation for liquid wealth equation only through dividends, the dividend plays exactly the same role of a stochastic income for the agent. For this reason, our allocation problem can be considered as the finite horizon counterpart of an allocation problem in the presence of an uninsurable stochastic income and H_t can be considered as the present value of all future wages.

consumption. We denote the dollar amounts invested in the liquid assets by the vector $\pi = \left(\pi_t^S, \pi_t^B\right)_{t \leq T}$.

We define a consumption plan as a triple $(c, \pi, W_T^{c,\pi})$. Then the set of admissible plans with initial liquid wealth l_0, and level of the illiquid state variable h_0, $A\left(l_0, h_0\right)$, is defined by the following restrictions:

- The consumption stream c and the final total wealth $(W_T^{\pi,c} = H_T + L_T^{\pi,c})$ must obey the following standard technical restrictions [6]:

$$(c, W) \in \mathcal{C} = \left\{ \begin{array}{c} c_t \;\geq 0 \; a.s. \; \mathcal{F}_t\text{-adapted}, \; E^{\mathbb{P}}\left[\int_0^T c_t^2 dt\right] < +\infty, \; t \leq T \\ W_T^{\pi,c} = H_T + L_T^{\pi,c} \geq 0 \; a.s. \; W_T^{\pi,c} \in \mathcal{L}^2\left(\mathcal{F}_T\right) \end{array} \right\}$$

- The strategy π finances a consumption stream c, given an initial liquid wealth $l_0 > 0$, and the initial level of the state variable $h_0 \geq 0$, if there exists a strictly positive liquid wealth process:

$$\begin{aligned} L_0 &= l_0 \\ L_t &= \pi_t^S + \pi_t^B \end{aligned} \tag{10}$$

  Any admissible plan must be self financing and therefore the dynamics of total liquid wealth is

$$dL_t^{\pi,c} = \delta H_t dt + \pi_t^S \frac{dS_t}{S_t} + \left(\pi_t^B r - c_t\right) dt$$

  We require that the process π_t^S is $\mathcal{F}-$adapted $S-$ integrable in order to avoid doubling strategies (see e.g. Duffie (1996)).

To find the optimal solution we follow the duality approach described in the binomial example. We denote with a $\sim$ all the discounted quantities (i.e. $\widetilde{S}_t = \exp\left(-rt\right) S_t$, $\widetilde{c}_t = \exp\left(-rt\right) c_t$, etc.), such that $d\widetilde{S}_t/\widetilde{S}_t = \left(\alpha - r\right) dt + \sigma dW_t^1$.

[6] Notice that these requirements, jointly with the parametric restrictions (34) and (35) reported in Appendix, are sufficient to guarantee that, starting with strictly positive liquid wealth, it will never be optimal to reach negative liquid wealth. This is suggested by the following informal argument: suppose on the contrary that a negative position in liquid wealth is possible, then there would be a small but non vanishing probability for the final total wealth to be negative. A formal proof should rely on the same arguments of proposition 1 in Duffie et al. (1997).

Then, the dynamics of discounted liquid wealth becomes (we drop the dependence on π and c for notational simplicity):

$$d\widetilde{L}_t = \delta \widetilde{H}_t dt + \widetilde{\pi}_t^S \frac{d\widetilde{S}_t}{\widetilde{S}_t} - \widetilde{c}_t dt$$

and rearranging terms

$$d\widetilde{L}_t - \delta \widetilde{H}_t dt + \widetilde{c}_t dt = \widetilde{\pi}_t^S \frac{d\widetilde{S}_t}{\widetilde{S}_t}$$

The allocation strategy is determined solely by $\widetilde{\pi}_t^S$. The bond amount in the portfolio can be recovered given total liquid wealth and the stock amount.

3.1 Determination of optimal consumption and final wealth

In the continuous time model we are considering the no arbitrage condition implies the existence of a set of probability measures $\mathbb{Q}$ and correspondingly a set of stochastic discount factors $\xi_t^{\mathbb{Q}}$ [7] such that prices of tradeable assets are given by:

$$\begin{aligned} S_t &= e^{-r(T-t)} E_t^{\mathbb{Q}} [S_T] = \frac{E_t^{\mathbb{P}} \left[\xi_T^{\mathbb{Q}} S_T \right]}{\xi_t^{\mathbb{Q}}} \\ \forall \mathbb{Q} &\in \mathcal{D} \end{aligned}$$

where $E_t^{\mathbb{Q}}$ denotes the conditional expectation with respect $\mathcal{F}_t$ under the measure $\mathbb{Q}$. Moreover, in the illiquid market we are considering there is not an unique equivalent martingale measure. As we discuss in the Appendix, the presence of an illiquid asset requires an enlarged set of equivalent martingale measures $\mathcal{D}$. As usual in incomplete markets, the optimal equivalent martingale measure depends on the agent's preferences.

Following once again the duality approach suggested by Cox and Huang (1989), the optimal pair (c^*, W_T^*) can be determined solving the dual problem:

$$\sup_{(c, W_T) \in \mathcal{C}} V(c, W_T)$$

$$s.t. \quad E^{\mathbb{Q}} \left[\int_0^T e^{-rt} c_t dt + e^{-rT} (W_T - H_T) - \int_0^T e^{-rt} \delta H_t dt \right] \leq l_0 \quad \forall \mathbb{Q} \in \mathcal{D}$$

[7] See the Appendix for a rigorous definition of the relation between the stochastic discount factor and the optimal measure

We introduce the Lagrangian:

$$\mathcal{L}\left(c, W_T, H_T, \lambda, \mathbb{Q}\right) = V\left(c, W_T\right) - \lambda g^{\mathbb{Q}}\left(c, W_T, H_T\right), \quad \lambda > 0, \quad \mathbb{Q} \in \mathcal{D} \tag{11}$$

$$g^{\mathbb{Q}}\left(c, W_T, H_T\right) \triangleq E^{\mathbb{Q}}\left[\int_0^T e^{-rt} c_t dt + e^{-rT}\left(W_T - H_T\right) - \int_0^T e^{-rt}\delta H_t dt - l_0\right]$$

and impose the first order conditions to (11):

$$\begin{aligned} \beta_W e^{-\kappa T} U'_\gamma\left(W_T\left(\lambda, \xi_T^{\mathbb{Q}}\right)\right) &= \lambda \xi_T^{\mathbb{Q}} \\ \beta_c e^{-\kappa t} U'_\gamma\left(c\left(\lambda, \xi_T^{\mathbb{Q}}\right)\right) &= \lambda \xi_t^{\mathbb{Q}} \end{aligned}$$

Then, the optimal value for the Lagrangian parameter λ^* and the optimal martingale measure $\mathbb{Q}^*$ are determined by the minimization:

$$\inf_{\lambda>0, \mathbb{Q}\in\mathcal{D}} \mathcal{L}\left(c\left(\lambda, \xi_T^{\mathbb{Q}}\right), W_T\left(\lambda, \xi_T^{\mathbb{Q}}\right), H_T, \lambda, \mathbb{Q}\right) \tag{12}$$

Crucial to the determination of the explicit solution, under our assumptions about preferences and about the price dynamics, is the identification of the optimal valuation measure $\mathbb{Q}^*$ which is discussed in the Appendix. As in the binomial example, the optimal measure is determined in two stages. First, we impose the no arbitrage condition which applies to liquid assets. Then, optimizing over the set of martingale measures, we obtain the optimal solution to the dual minimization problem.

No arbitrage and optimality require that the stochastic discount factor, ξ^*, which determines the optimal consumption and the optimal wealth is the one used in a market composed only by liquid assets:

$$\begin{aligned} \frac{d\xi_t^*}{\xi_t^*} &= -rdt - \frac{(\alpha - r)}{\sigma} dW_t^1 \\ \xi_0^* &= 1 \end{aligned} \tag{13}$$

However, the optimal measure $\mathbb{Q}^*$ also determines the value to the investor for the illiquid asset H_t. The agent evaluates its illiquid wealth considering also the independent source of risk, W_t^2. As shown in the Appendix the change of measure $Z_T^2 = W_T^2 + \gamma\eta\left(1-\rho^2\right)^{\frac{1}{2}} T$ takes into account the shadow price of the illiquidity constraint.

Now, let $Z_T^1 = W_T^1 + (\alpha - r)\, T/\sigma$ and $Z_T^2 = W_T^2 + \gamma\eta\,(1-\rho^2)^{\frac{1}{2}}\, T$.

Then, under $\mathbb{Q}^*$ the dynamics of the (discounted) traded asset becomes a martingale:

$$\frac{d\widetilde{S}_t}{\widetilde{S}_t} = \sigma dZ_t^1 \tag{14}$$

While the dynamics of the (discounted) state variable associated with the non-traded asset becomes:

$$\begin{aligned} \frac{d\widetilde{H}_t}{\widetilde{H}_t} &= -\nu dt + \eta \left[\rho dZ_t^1 + \sqrt{1-\rho^2} dZ_T^2\right] &(15)\\ \nu &= -\left[(\mu - \delta - r) - \left(\frac{\alpha - r}{\sigma}\right)\eta\rho - \gamma\eta^2\left(1-\rho^2\right)\right] &(16)\end{aligned}$$

Note that the drift of the (discounted) process for the illiquid asset under the measure $\mathbb{Q}^*$ is not, in general, equal to zero.

Because of the transversality condition, in order to avoid bubbles, we require that $\nu - \delta > 0$.

At optimality the Lagrangian parameter λ^* is obtained by the minimization of (12). Its value is derived in the Appendix.

In conclusion, the final wealth and consumption from the optimal allocation which solve the original primal problem are given by the following expressions:

$$\begin{aligned} W_T\left(\lambda^*, \xi_T^*\right) &= \left(\lambda^* \frac{e^{\kappa T}}{\beta_W}\xi_T^*\right)^{-\frac{1}{\gamma}}, &(17)\\ c_t\left(\lambda^*, \xi_t^*\right) &= \left(\lambda^* \frac{e^{\kappa t}}{\beta_c}\xi_t^*\right)^{-\frac{1}{\gamma}} &(18)\end{aligned}$$

Remarkably, using results form Ocone and Karatzas (1991), it is possible to compute also the closed form expression for the optimal amount $\widetilde{\pi}_t^S$. Consider the (discounted) liquid wealth net of the dividends received and including the consumption withdrawals:

$$\begin{aligned} \widetilde{\Lambda}_t &= \widetilde{L}_t^* - \int_0^t \delta\widetilde{H_s} ds + \int_0^t \widetilde{c}_s^* ds &(19)\\ &= l_0 + \int_0^t \widetilde{\pi}_s^S \frac{d\widetilde{S}_s}{\widetilde{S}_s} \quad t \le T \end{aligned}$$

Since $\widetilde{\Lambda}_t$ is a martingale process under the optimal measure $\mathbb{Q}^*$, the Clark-Ocone formula [8] provides the expression of $\widetilde{\pi}_t^S$ as the conditional expectation of the Malliavin Derivative of $\widetilde{\Lambda}_T$ with respect to the process σZ_t^1:

$$\widetilde{\pi}_t^S = E_t^{\mathbb{Q}^*}\left[D_t \widetilde{\Lambda}_T\right]$$

By linearity of the derivative operator, $D_t\widetilde{\Lambda}_T$ can be decomposed as:

$$D_t\widetilde{\Lambda}_T = D_t\widetilde{W}_T^* - D_t\widetilde{H}_T - \int_0^T \delta D_t\widetilde{H}_s ds + \int_0^T D_t\widetilde{c}_s^* ds$$

Recall that expressions (17) and (18) provide explicit solutions for the optimal wealth and consumption:

$$\begin{aligned}
\widetilde{W}_T\left(\lambda^*, \xi_T^*\right) &= \left(\frac{\lambda^*}{\beta_W}\right)^{-\frac{1}{\gamma}} \exp\left[-\left(\kappa - r\right)\frac{T}{\gamma} - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 \frac{T}{\gamma} + \left(\frac{\alpha - r}{\sigma^2\gamma}\right)\sigma Z_T^1\right] \\
\widetilde{c}_t\left(\lambda^*, \xi_t^*\right) &= \left(\frac{\lambda^*}{\beta_c}\right)^{-\frac{1}{\gamma}} \exp\left[-\left(\kappa - r\right)\frac{t}{\gamma} - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 \frac{t}{\gamma} + \left(\frac{\alpha - r}{\sigma^2\gamma}\right)\sigma Z_t^1\right]
\end{aligned}$$

As proved in the Appendix of Detemple, Garcia and Rindisbacher (2003), the Malliavin derivative (with respect to σZ_t^1) of a lognormal random variable can be computed according to the usual differentiation rule, thus we get:

$$\begin{aligned}
D_t\widetilde{W}_T\left(\lambda^*, \xi_T^*\right) &= D_t\left(\lambda^*\frac{e^{\kappa T}}{\beta_W}\xi_T^*\right)^{-\frac{1}{\gamma}} = \left(\frac{\alpha - r}{\gamma\sigma^2}\right)\widetilde{W}_T^*|_{t\leq T} \\
D_t\widetilde{c}_s\left(\lambda^*, \xi_t^*\right) &= e^{-rs}D\left(\lambda^*\frac{e^{\kappa s}}{\beta_c}\xi_s^*\right)^{-\frac{1}{\gamma}} = \left(\frac{\alpha - r}{\gamma\sigma^2}\right)\widetilde{c}_s^*|_{t\leq s}
\end{aligned}$$

[8]Clark Ocone formula is an extension of Ito representation theorem, it states that the random variable $\widetilde{\Lambda}_T$ can be decomposed as follows:

$$\widetilde{\Lambda}_T = E^{\mathbb{Q}}\left[\widetilde{\Lambda}_T\right] + \int_0^T E_t^{\mathbb{Q}}\left[D_t\widetilde{\Lambda}_T\right]\sigma dZ_t^1$$

where $D_t\widetilde{\Lambda}_T$ is the Malliavin derivative of the process $\widetilde{\Lambda}_T$. Its applications to dynamic portfolio allocation have been discussed in Detemple Garcia and Rindisbacher (2003). The Clark-Ocone formula loosely corresponds to the first order Taylor formula applied to stochastic processes. Comparing the expression for $\widetilde{\Lambda}_T$ with (19) it is immediate to determine the expression of the allocation strategy in terms of the Malliavin derivative.

Also $\widetilde{H}_T$ is lognormally distributed (with respect to σZ_t^1), therefore

$$\begin{aligned} D_t \widetilde{H}_T &= \frac{\eta\rho}{\sigma} \widetilde{H}_T \\ D_t \widetilde{H}_s &= \frac{\eta\rho}{\sigma} \widetilde{H}_s |_{t \leq s} \end{aligned}$$

Summing up the expressions for the derivatives we get the Malliavin derivative of $D_t \widetilde{\Lambda}_T$. The explicit allocation strategy at time 0 is then given by the expectation:

$$\begin{aligned} \pi_0^S &= \widetilde{\pi}_0^S = E^{\mathbb{Q}^*}\left[D_0 \widetilde{\Lambda}_T\right] = \\ &= \left(\frac{\alpha - r}{\gamma\sigma^2}\right) E^{\mathbb{Q}^*}\left[\widetilde{W}_T^* + \int_0^T \widetilde{c}_s^* ds - \widetilde{H}_T - \int_0^T \delta \widetilde{H_t} dt\right] \\ &\quad + \left(\frac{\alpha - r}{\gamma\sigma^2} - \frac{\eta\rho}{\sigma}\right) E^{\mathbb{Q}^*}\left[\widetilde{H}_T + \int_0^T \delta \widetilde{H_t} dt\right] \\ &= \left(\frac{\alpha - r}{\gamma\sigma^2}\right) l_0 \\ &\quad + \left(\frac{\alpha - r}{\gamma\sigma^2} - \frac{\eta\rho}{\sigma}\right) h_0 \left[e^{-\nu T} + \delta\nu^{-1}\left(1 - e^{-\nu T}\right)\right] \end{aligned} \tag{20}$$

Finally, the actual amounts invested are:

$$\begin{aligned} \pi_t^S &= e^{rt} \widetilde{\pi}_t^S \\ \pi_t^B &= L_t^* - \pi_t^S \end{aligned}$$

Note that when no illiquid asset exists ($h_0 = 0$) this result (20) is exactly Merton's (1969).

The value at time 0 of the illiquid asset for the investor is given by:

$$\widehat{h}_0 = E^{\mathbb{Q}^*}\left[\widetilde{H}_T + \int_0^T \delta \widetilde{H_t} dt\right] = h_0 \left[e^{-\nu T} + \delta\nu^{-1}\left(1 - e^{-\nu T}\right)\right]$$

From this point of view h_0 represents the notional (or implied, or accounting) value of the illiquid asset; but the price at which the investor would be willing to sell it at time 0 is $\widehat{h}_0$; this value is then the value to the investor of the illiquid asset. Illiquidity affects the value of the asset by the factor

$\widehat{h}_0/h_0 \leq 1$. This cost depends on all the parameters of the two stochastic process, in addition to the preference parameter γ. Similarly, the value at time 0 of the total wealth of the investor is

$$\widehat{w}_0 = l_0 + \widehat{h}_0 \tag{21}$$

The amount invested in the liquid risky asset can then be written as a function of $\widehat{h}_0$ and $\widehat{w}_0$:

$$\pi_0^S = \left(\frac{\alpha - r}{\gamma\sigma^2}\right)\widehat{w}_0 - \frac{\eta\rho}{\sigma}\widehat{h}_0 \tag{22}$$

Then, the fraction of wealth invested in the liquid risky asset is given by:

$$\frac{\pi_0^S}{\widehat{w}_0} = \left(\frac{\alpha - r}{\gamma\sigma^2}\right) - \frac{\eta\rho}{\sigma}\frac{\widehat{h}_0}{\widehat{w}_0} \tag{23}$$

Note that the first term of this expression represents the exact Merton's proportions of total wealth, but in this case total wealth is the sum of the liquid wealth and the value to the agent of the illiquid wealth. The second term takes into account the diversification effect that the degree of correlation between the risky assets has on the optimal allocation to the risky liquid asset.

3.2 Optimal consumption stream

The optimal consumption flow at time 0 can be easily computed from equation (18) given expression (13) and the optimal value for the Lagrangian parameter λ^*, computed in the Appendix:

$$\begin{aligned}
c_0 &= \left(\frac{\lambda^*}{\beta_c}\right)^{-\frac{1}{\gamma}} (\xi_0^*)^{-\frac{1}{\gamma}} = \left(\frac{\lambda^*}{\beta_c}\right)^{-\frac{1}{\gamma}} \\
\lambda^* &= \left\{\frac{l_0 + h_0\left[e^{-\nu T} + \delta\nu^{-1}\left(1 - e^{-\nu T}\right)\right]}{\beta_c^{1/\gamma} m^{-1}\left(e^{mT} - 1\right) + \beta_W^{1/\gamma} e^{mT}}\right\}^{-\gamma} \\
m &= \frac{\kappa}{\gamma} - r\left(1 - \frac{1}{\gamma}\right) - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2\left(\frac{1}{\gamma} - \frac{1}{\gamma^2}\right)
\end{aligned}$$

hence:

$$
\begin{aligned}
c_0 &= \left\{ \frac{l_0 + h_0 \left[e^{-\nu T} + \delta \nu^{-1} \left(1 - e^{-\nu T} \right) \right]}{m^{-1} \left(e^{mT} - 1 \right) + \left(\beta_c^{-1} \beta_W \right)^{1/\gamma} e^{mT}} \right\} = \frac{\widehat{w}_0}{K^C} \\
K^C &= \left(m^{-1} \left(e^{mT} - 1 \right) + \left(\beta_c^{-1} \beta_W \right)^{1/\gamma} e^{mT} \right)
\end{aligned}
$$

As in Merton (1969), the agent will consume an amount proportional to his total wealth, but where wealth is defined as the sum of his liquid wealth plus the value of his illiquid wealth (which is adjusted for illiquidity). The proportionality constant, however, is smaller that in the Merton two-asset case because the Sharpe ratio for the two asset case is always larger that the Sharpe ratio of the one asset case. As a consequence, illiquidity induces the agent to reduce consumption both because the value to the investor of the illiquid asset is smaller and also because the proportionality constant is reduced.

3.3 The utility cost of illiquidity

Illiquidity is a constraint on possible allocations. In this Section we evaluate the utility cost to the investor of these restrictions. For an unconstrained investor his welfare depends only on the total amount of his initial endowment w_0, while for a constrained investor, it will depend on the level of liquid wealth l_0 and on the level of the state variable associated with the illiquid asset h_0. We compute the amount of liquid wealth l_0^{CE} required by a constrained investor to reach the same level of utility of an unconstrained investor with total wealth $w_0 = l_0 + h_0$, for a fixed level of illiquid asset h_0.

The optimal allocation for the unconstrained agent is given by the solution of Merton (1969), applied on a market where there are two liquid risky assets with risk and return characteristics corresponding to those of the two assets S_t and H_t. For a given initial wealth w_0, the unconstrained agent will allocate to the two risky assets the dollar amounts:

$$
\begin{pmatrix} \pi_U^S \\ \pi_U^H \end{pmatrix} = \frac{1}{\gamma} \left(\Sigma \Sigma^T \right)^{-1} \begin{pmatrix} \alpha - r \\ \mu - r \end{pmatrix} w_0 \tag{24}
$$

From Merton (1969), the expression for the indirect utility of wealth for the unconstrained investor is:

$$\begin{aligned}
V^U(w_0) &= \left(K^U\right)^\gamma \frac{(w_0)^{1-\gamma}}{1-\gamma} \\
K^U &= m_U^{-1}\left(e^{m_U T}-1\right)+\left(\beta_c^{-1}\beta_W\right)^{1/\gamma} e^{m_U T} \\
m_U &= \frac{\kappa}{\gamma} - r\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}M\left(\frac{1}{\gamma}-\frac{1}{\gamma^2}\right) \\
M &= \begin{pmatrix}\alpha - r \\ \mu - r\end{pmatrix}' \left(\Sigma\Sigma^T\right)^{-1} \begin{pmatrix}\alpha - r \\ \mu - r\end{pmatrix}
\end{aligned}$$

The indirect utility of wealth for the constrained investor is easily obtained replacing the optimal consumption and optimal wealth in the objective function, $V(c^*, W^*)$, and taking expectations:

$$\begin{aligned}
V^C(h_0, l_0) &= \left(K^C\right)^\gamma \frac{\left(\widehat{w}_0(l_0, h_0)\right)^{1-\gamma}}{1-\gamma} \\
\widehat{w}_0(l_0, h_0) &= l_0 + h_0\left[e^{-\nu T} + \delta\nu^{-1}\left(1-e^{-\nu T}\right)\right] \\
K^C &= m^{-1}\left(e^{mT}-1\right) + \left(\beta_c^{-1}\beta_W\right)^{1/\gamma} e^{mT} \\
m &= \frac{\kappa}{\gamma} - r\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2\left(\frac{1}{\gamma}-\frac{1}{\gamma^2}\right)
\end{aligned}$$

Then, l_0^{CE} , the certainty equivalent level of liquid wealth which provides to the constrained investor the same level of utility of an unconstrained investor with endowment $w_0 = l_0 + h_0$ is given by:

$$\begin{aligned}
V^C\left(l_0^{CE}, h_0\right) &= V^U(l_0 + h_0) \\
\frac{l_0^{CE}}{l_0} &= \left(\frac{K^C}{K^U}\right)^{\gamma/(\gamma-1)}\left(1+\frac{h_0}{l_0}\right) - \frac{h_0}{l_0}\left[e^{-\nu T} + \delta\nu^{-1}\left(1-e^{-\nu T}\right)\right]
\end{aligned}$$

Since $M \geq \left(\frac{\alpha - r}{\sigma}\right)^2$, it can be shown that $K^U \leq K^C$. Therefore $l_0^{CE} \geq l_0$. As expected, the illiquidity constraint decreases the utility of the agent. The additional liquid wealth required to compensate the investor for the illiquidity of the asset is $l_0^{CE} - l_0$.

4 Illustrative example

To gain more insight into the results obtained, in this section we illustrate our findings through a numerical example. Consider and investor with an horizon of 20 years and with a coefficient of risk aversion of $\gamma = 3$ who is allocating funds to a liquid risky asset with an expected rate of return of $\alpha = 0.08$ and volatility $\sigma = 0.15$. In addition he holds an illiquid risky asset with a drift $\mu = 0.07$ and volatility $\eta = 0.20$. Even though the illiquid asset can not be traded, it pays a liquid dividend yield of $\delta = 0.05$. Finally, there is a liquid riskless asset with a constant interest rate $r = 0.03$.

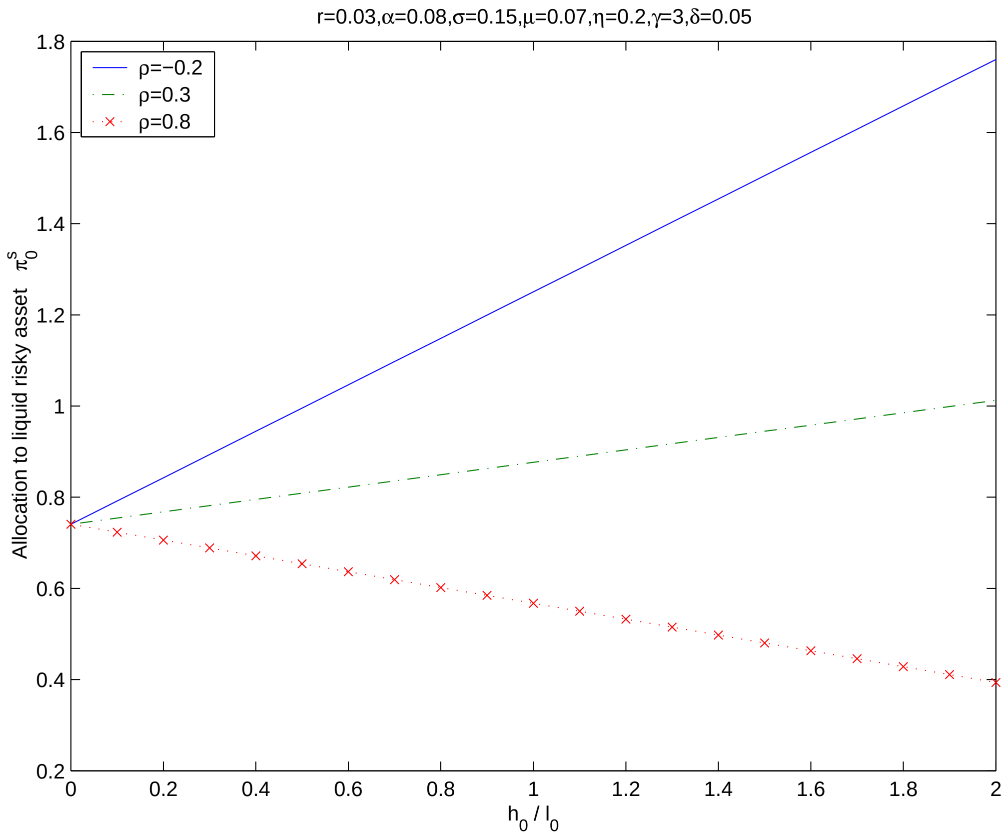

Figure 1: Allocation to the Risky Liquid Asset as a Function of h_0 fixing $l_0 = 1$.

Figure 1 shows the allocation to the risky liquid asset as a function of h_0 fixing $l_0 = 1$, for different values of the correlation between the liquid and the illiquid asset returns. From equation (20) this relation is linear. Note that when we increase h_0 the total wealth of the investor increases also. From the figure we see that the allocation to the liquid risky asset increases for negative correlation and also for moderately positive correlation as well, while for high correlation this allocation decreases. For sufficiently low correlation the diversification effect of having the illiquid asset (even if it cannot be traded) increases the optimal allocation to the liquid risky asset, even to the point of borrowing at the risk free asset to invest in the liquid risky asset. Only when the correlation is sufficiently high and the diversification effect of holding the illiquid risky asset diminishes, does the optimal allocation to the liquid risky asset decreases when h_0 increases.

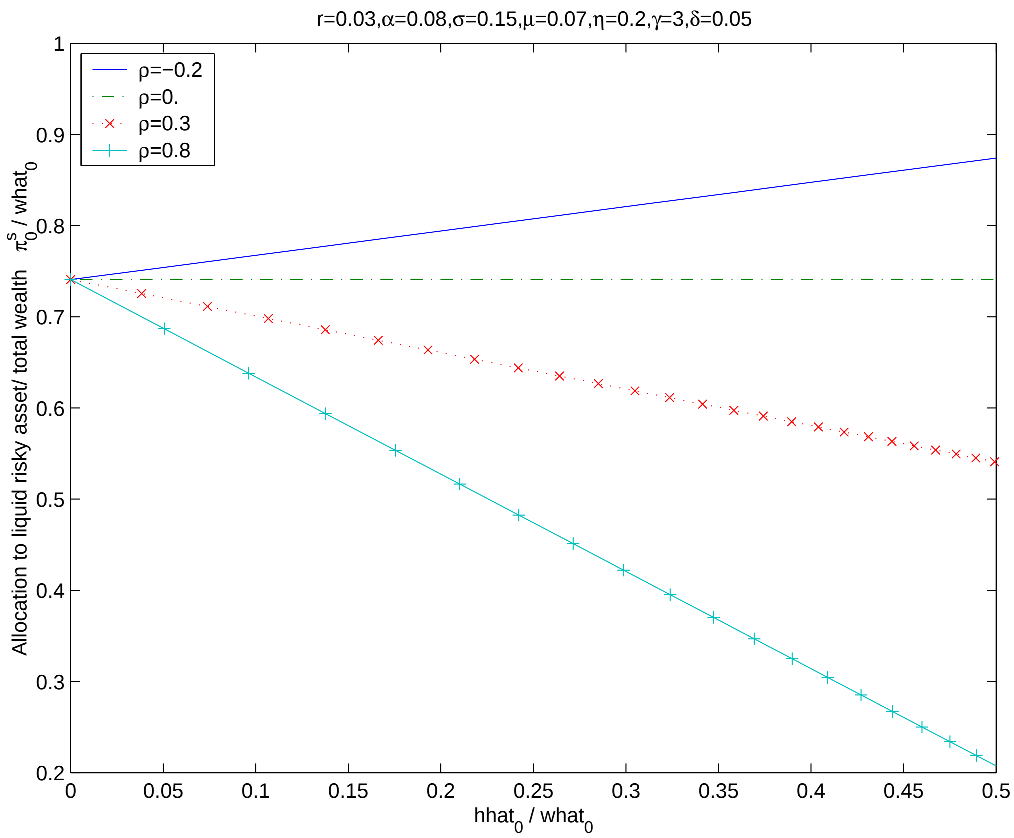

Figure 2: Fraction of Total Wealth Allocated to the Risky Liquid Asset as a Function of the Fraction of Illiquid Wealth.

Figure 2 presents a rescaled plot of Figure 1. In order to keep constant total wealth (liquid wealth plus the value to the investor of the illiquid wealth) this figure shows the fraction of total wealth allocated to the liquid risky asset as a function of the proportion of illiquid wealth, as described by equation (23). Note that when the correlation is zero the proportion allocated to the liquid risky assets is constant and this constant is the same as in Merton, however, the definition of total wealth is different. When the correlation is negative (positive) the proportion is increasing (decreasing) in the fraction of the illiquid asset. In this graph the slope of the lines is (the negative of) the "beta coefficient" of the illiquid asset return with respect to the liquid asset return, and the intercept is the usual Merton proportion.

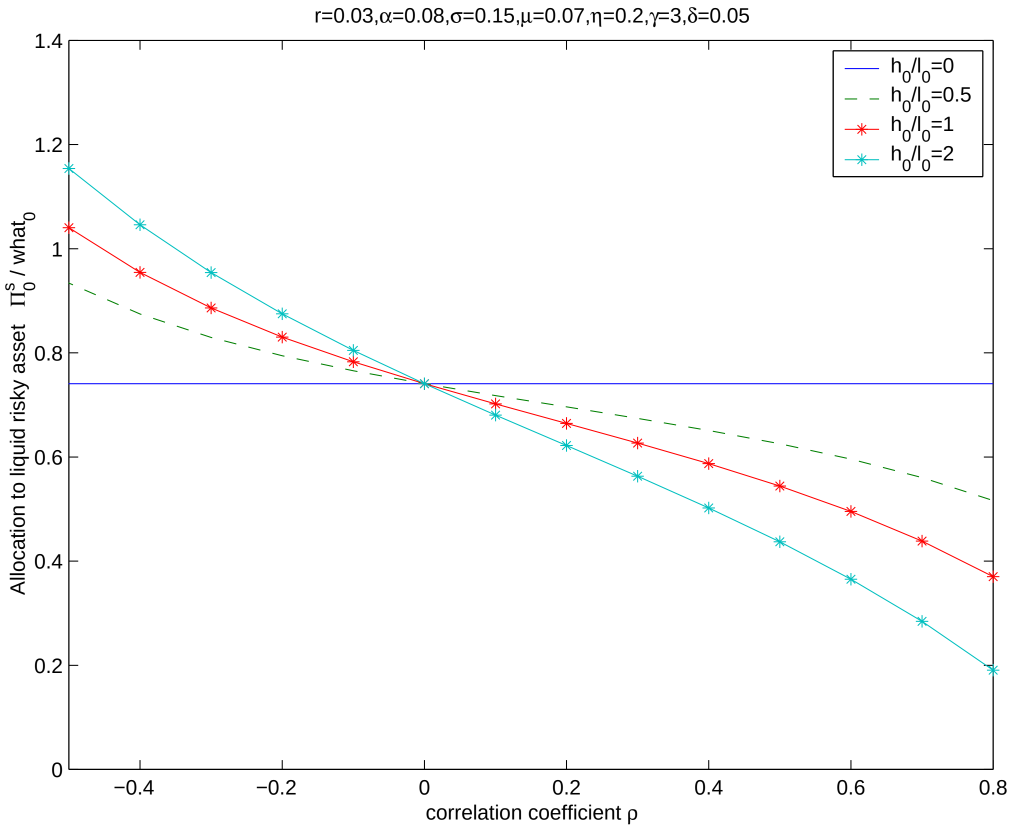

Figure 3: Allocation to the Risky Liquid Asset as a Function of the Correlation between the Returns of the Liquid and Illiquid Assets.

Figure 3 shows the allocation to the risky liquid asset (as a proportion of total wealth) as a function of the correlation, for different proportions of illiquid to liquid wealth. Naturally the correlation effect is stronger the higher is the proportion of illiquid wealth.

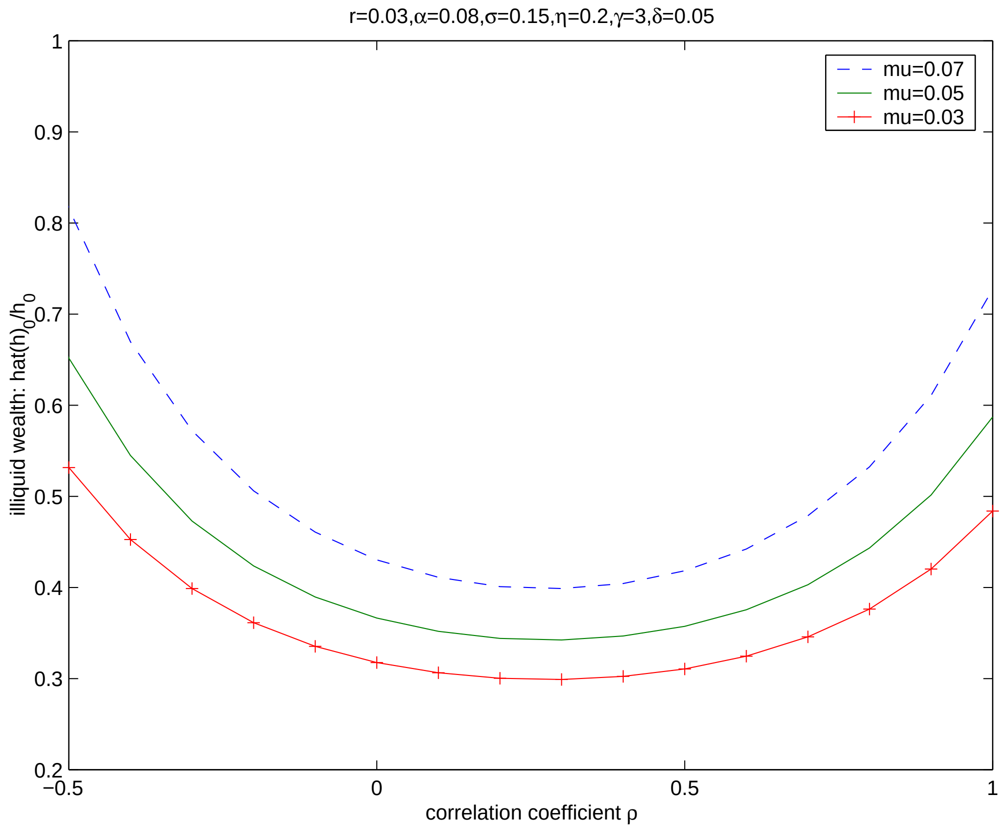

Figure 4: Shadow Price of the Illiquid Asset as Measured by $\widehat{h}_0/h_0$ as a Function of the Correlation for Different μ's.

Figure 4 provides an indication of the shadow price for the illiquidity constraint, measured by $\widehat{h}_0/h_0$, as a function of the correlation for different levels of the total expected return of the illiquid asset process. The value of the illiquid asset to the investor $(\widehat{h}_0)$ is always lower than its notional level (h_0). Interestingly, is non monotonic with respect to the correlation. For low and high correlation the value is greater than for intermediate values. For negative correlation the diversification potential of the illiquid asset makes it valuable to the investor even if the agent cannot trade on it. Then, after reaching a minimum value, the ratio starts increasing again; this is due to the fact that the higher is the correlation, the more the illiquid asset can be hedged with the liquid one. Naturally, this ratio is critically dependent on μ; the higher is μ the more valuable is the illiquid asset.

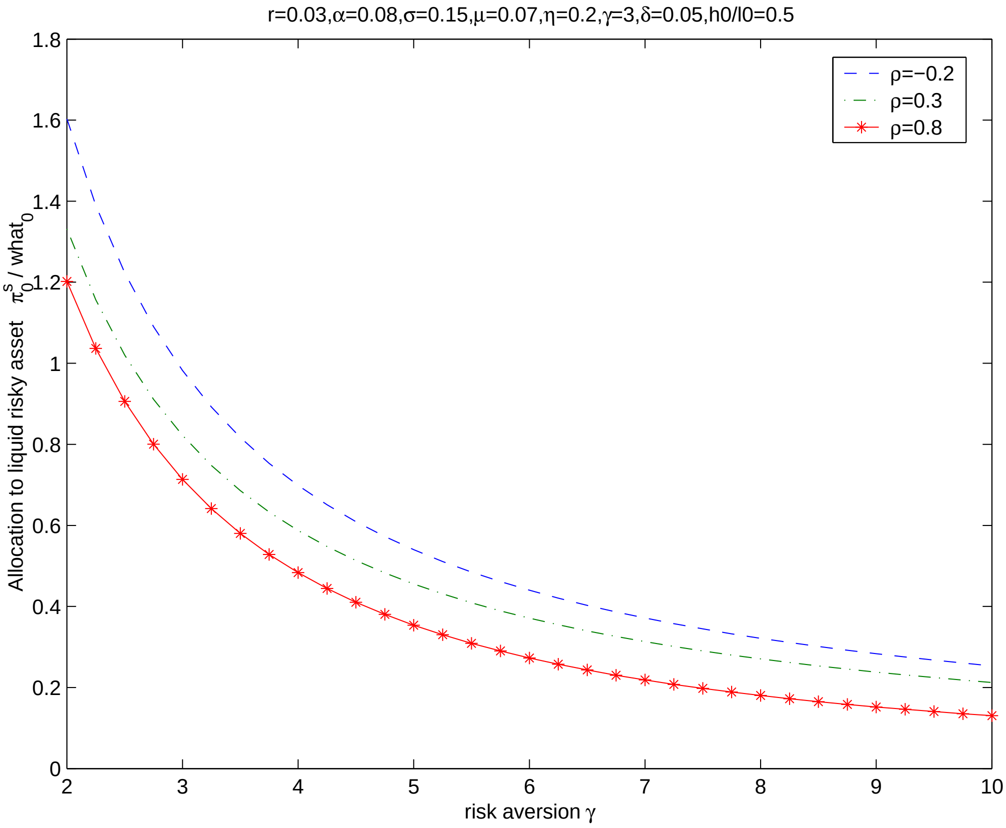

Figure 5: Proportional Allocation to the Liquid Risky Asset as a Function of the Risk Aversion Parameter.

Figure 5 presents the proportional allocation to the liquid risky asset as a function of the coefficient of risk aversion for different levels of the correlation. As expected the higher is the risk aversion the lower is the allocation to the risky asset, and the higher the correlation the lower is the proportional allocation.

Figure 6 shows the liquid wealth, l_0^{CE}, that would be required to compensate the investor for the illiquidity of the constrained asset in order to obtain the same utility as an equivalent investor that has no illiquidity constraints and invests a total amount $l_0 + h_0$. In this figure we plot l_0^{CE} as a function of h_0/l_0 for different correlations and normalizing $l_0 = 1$. First note

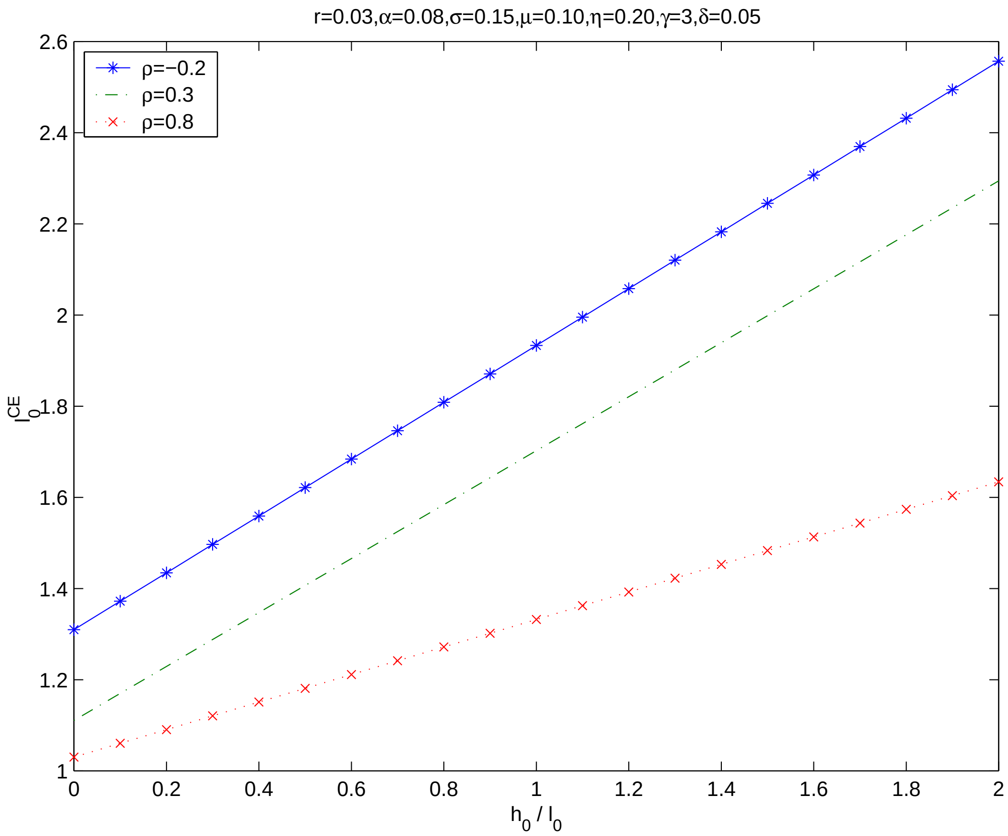

Figure 6: Total Liquid Wealth Required to Compensate for Illiquidity as a Function of h_0/l_0 for Different Correlations.

that even when the fraction of illiquid wealth is zero, l_0^{CE} is larger than 1 because in one case the agent is only investing in the liquid asset and in the other the agent in optimizing over the two assets. Thus, for the correlation smaller that 1 there is always an advantage to diversification. As expected, l_0^{CE} is an increasing function of the fraction of illiquid wealth and a decreasing function of the correlation. This last is due to the fact that the more negative is the correlation the larger is the advantage from diversification in the unconstrained case.

5 Summary and conclusions

We study the problem of optimal asset allocation in the presence of an illiquid asset. The illiquid asset cannot be traded, but it generates a liquid dividend that can be consumed or invested in liquid assets. This liquid dividend has many interpretations depending on the nature of the illiquid asset. An important application is when the illiquid asset is human wealth and the dividend is labor income. There is a vast literature in economics and finance trying to understand the effect of stochastic labor income on optimal consumption and asset allocation. We obtain closed form solution to this problem in the relevant case of time separable power utility of consumption and terminal wealth.

An important by-product of our analysis is that we derive a valuation procedure for liquid and illiquid assets. In particular, we are able to compute the value that the agent assigns to the illiquid asset, that is, the shadow price of illiquidity. The framework, however, allows, given the preferences of the investor, to value any contingent claim on the illiquid asset or on both the liquid and illiquid asset.

The approach we develop can also be used to solve the optimal asset allocation problem in the presence of borrowing and short selling constraints as discussed in general terms by He and Pages (1993). In particular, it would be interesting to study the effect that these constraints have on the value that the agent assigns to his illiquid asset.

Perhaps the most challenging extension of our analysis is market equilibrium. If the risky liquid asset is the market portfolio, and the illiquid asset of each agent in the economy is its human wealth, the aggregation problem involves heterogeneous valuations of human wealth holdings of all the agents in the economy. The possibility of asymmetric information effects raises the issue of the impact of moral hazard and adverse selection on such market equilibrium.

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6 Appendix

6.1 Determination of the optimal measure $\mathbb{Q}^*$ in the continuous time problem

The stochastic optimization problem we solve belongs to the class of utility maximization problems in the presence of a random endowment. The theoretical aspects of the duality approach have been analyzed and solved by Cuoco (1997), Cvitanic, Schachermayer and Wang (2001), Karatzas and Zitkovic (2003), and Hugonnier and Kramkov (2004). We refer to the above references for a rigorous discussion of the mathematical problem and for a proof of the existence of an optimal measure; here we identify the optimal pricing measure $\mathbb{Q}^*$ in the specific case of our dynamic allocation problem.

In particular the model described in Section 3 belongs to the class of "Ito process models" as defined in (Karatzas and Zitkovic (2003) Example 4.1).

The major technical difficulty in the formulation of the dual problem is the definition of a set of admissible measures $\mathbb{Q}$. The proper domain for the dual problem $\mathcal{D}$, which has been introduced in (Cvitanic, Schachermayer and Wang (2001) pg.263), is:

$$\mathcal{D} = \{\mathbb{Q} \in (\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}))^* \mid \|\mathbb{Q}\| = 1 \text{ and } \langle \mathbb{Q}, X \rangle \leq 0 \; \forall X \in \mathcal{C}\}$$

where $\mathcal{C}$ is the set of $\mathcal{F}_T$ measurable random variables in $\mathbb{L}^\infty$ dominated by a dynamic trading strategy admissible in the market and $\langle \mathbb{Q}, X \rangle$ denotes the expectation of X under the measure $\mathbb{Q}$.

In order to identify uniquely the optimal solution $\mathbb{Q}^*$, it is important to provide a characterization of the elements of $\mathcal{D}$ (see e.g. the Appendix of Cvitanic, Schachermayer and Wang (2001)). Since $\mathcal{D}$ is a subset of $(\mathbb{L}^\infty)^*$, each pricing measure $\mathbb{Q}$ is required to be only finitely additive, as opposed to the standard case where measures are required to be countably additive. While most of the properties of standard measures can be extended to the finitely additive case, some properties of standard stochastic calculus do not apply. In particular the Radon Nikodym theorem does not hold in its standard form and the corresponding derivative is not uniquely defined. For this reason the determination of the optimal measure is more elaborate than the standard situation.

First we formulate a conjecture about the optimal measure and then we provide evidence that this conjecture satisfies sufficient conditions for optimality.

We specify our conjecture using the following important property: a finitely additive measure $\mathbb{Q} \in \mathcal{D}$ defined on a domain (σ-algebra) $\mathcal{G}$ can be uniquely decomposed as a sum of two terms, a regular and a purely singular part:

$$\mathbb{Q} = \mathbb{Q}^r_{\mathcal{G}} + \mathbb{Q}^s_{\mathcal{G}}, \quad \mathbb{Q}^r_{\mathcal{G}} \geq 0, \mathbb{Q}^s_{\mathcal{G}} \geq 0$$

where the subscript indicates that such separation depends on the domain $\mathcal{G}$. As explained in (Karatzas and Zitkovic (2003) pg. 9), the separation between the regular part and the singular part of the measure depends on the information set (σ-algebra) where the measure is defined. Moreover the smaller information set the bigger the regular component.

Only the regular component admits a Radon-Nikodym derivative, while the singular part of the measure does not have a properly defined density.

In our problem the largest domain for $\mathbb{Q}$ is of course the full information set for the agent $\mathcal{F}_T$, which is the information set where the paths for both the Brownian motions are observed: $\mathcal{F}_T = \sigma(W^1_T, W^2_T)$.

The regular component in $\mathcal{F}_T$, $d\mathbb{Q}^r_{\mathcal{F}_T}/d\mathbb{P}$, provides the expression for the stochastic discount factor which is defined as:

$$\xi_t = e^{-r(T-t)} E^{\mathbb{P}}_t \left[\frac{d\mathbb{Q}^r_{\mathcal{F}_T}}{d\mathbb{P}} \right] \tag{25}$$

Therefore the regular component of the measure can be fixed, as in the usual case, specifying the evolution of the stochastic discount factor $\xi_t^{\mathbb{Q}}$.

Consider now the information set $\sigma\left(W_T^1\right)$ generated by the Brownian motion which drives the liquid assets $W_{t\leq T}^1$. Of course $\sigma(W_T^1)$ is properly contained in $\mathcal{F}_T$.

We conjecture that the optimal measure $\mathbb{Q}^*$ becomes completely regular, i.e. no singular component remains, when we restrict the domain of the measure to $\sigma(W_T^1)$. In addition, we conjecture that the Radon Nikodym derivative for the measure restricted to the smaller information set $\sigma(W_T^1)$ has an additional factor, $f^*\left(W_T^2\left(\omega\right)\right)$, not present when the domain is $\mathcal{F}_T$:

$$\frac{d\mathbb{Q}^r_{\sigma(W_T^1)}}{d\mathbb{P}\mid_{\sigma(W_T^1)}} = \left(\frac{d\mathbb{Q}^r_{\mathcal{F}_T}\mid_{\sigma(W_T^1)}}{d\mathbb{P}\mid_{\sigma(W_T^1)}}\right) f^*\left(W_T^2\left(\omega\right)\right)$$

In this way the singular component $\mathbb{Q}^s_{\mathcal{F}_T}$ is determined. It has a well defined Radon Nikodym derivative only when the measure is restricted on the information set $\sigma(W_T^1)$ and such derivative is:

$$\left(\frac{d\left(\mathbb{Q}^r_{\mathcal{F}_T}\mid_{\sigma(W_T^1)} - \mathbb{Q}^r_{\sigma(W_T^1)}\right)}{d\mathbb{P}\mid_{\sigma(W_T^1)}}\right) = \left(\frac{d\mathbb{Q}^r_{\mathcal{F}_T}\mid_{\sigma(W_T^1)}}{d\mathbb{P}\mid_{\sigma(W_T^1)}}\right)\left(f^*\left(W_T^2\left(\omega\right)\right) - 1\right)$$

Our conjecture is completed by the specification of the evolution for the stochastic discount factor ξ_t^*:

$$\begin{aligned} \frac{d\xi_t^*}{\xi_t^*} &= \mu^\xi dt + \sigma_1^\xi dW_t^1 + \sigma_2^\xi dW_t^2 \\ \mu^\xi &= -r \quad \sigma_1^\xi = -\frac{(\alpha - r)}{\sigma} \quad \sigma_2^\xi = 0 \end{aligned} \tag{26}$$

and by the specification of the expression for $f^*\left(W_T^2\left(\omega\right)\right)$ in the singular component $\xi_T^*\left(f^*\left(W_T^2\left(\omega\right)\right) - 1\right)$:

$$f^*\left(W_T^2\left(\omega\right)\right) = \exp\left(-\frac{1}{2}\gamma^2\eta^2\left(1-\rho^2\right)T + \gamma\eta\left(1-\rho^2\right)^{1/2} W_T^2\right)$$

Now we provide evidence that the above conjecture is optimal. We consider the special case where there is no intertemporal consumption and no dividends ($\beta_W = 1$, $\beta_c = 0$, $\delta = 0$). The extension to the case with consumption and dividends does not change the nature of the results (see Karatzas and Zitkovic (2003)).

Let ξ^* be the stochastic discount factor corresponding to the regular component of the optimal measure $\mathbb{Q}^*$. The condition that $\mathbb{Q}^*$ is an equivalent martingale measure implies that $\xi_t^* S_t$ and $\xi_t^* B_t$ have to follow a martingale process under $\mathbb{P}$ and this is equivalent to:

$$\mu^{\xi} = -r \tag{27}$$

and that:

$$\sigma_1^{\xi} = -\frac{(\alpha - r)}{\sigma} \tag{28}$$

Given the expression for $f^*\left(W_T^2(\omega)\right)$ which we determine below, the conditions for optimality given in (Cvitanic, Schachermayer and Wang (2001) equation (4.1) pg. 264) are verified and in particular the condition that the expectation with respect to the singular component of the measure of the optimal wealth (17):

$$E^{\mathbb{P}}\left[\left(f^*\left(W_T^2(\omega)\right) - 1\right)\xi_T^*\left(\lambda^*\frac{e^{\kappa T}}{\beta_W}\xi_T^*\right)^{-\frac{1}{\gamma}} \mid \sigma(W_T^1)\right] = 0$$

if and only if:

$$\sigma_2^{\xi} = 0$$

Finally we determine the expression for $f^*\left(W_T^2(\omega)\right)$ by solving a new optimization problem related with the original one. In this problem the only source of uncertainty is given by $W_T^2(\omega)$ while the specific path for the noise term which drives the liquid risky asset, $\left\{\overline{W_T^1(\omega)}\right\}_{t\leq T}$, is supposed to be known. Then the only asset whose evolution is uncertain is the illiquid asset:

$$H_t = h_0 \exp\left(\left(\mu - \frac{1}{2}\eta^2\right)t + \eta\rho\overline{W_t^1(\omega)} + \eta\left(1-\rho^2\right)^{1/2} W_t^2(\omega)\right)$$
$$for \quad W_t^1(\omega) = \overline{W_t^1(\omega)}$$

The risky asset price is the product of two components:

$$H_t = R_t\left(\overline{W_t^1(\omega)}\right) H_t^c,$$

one deterministic

$$R_t\left(\overline{W_t^1(\omega)}\right) = h_0 \exp\left(\left(\mu - r - \frac{1}{2}\eta^2\rho^2 + g\eta\left(1-\rho^2\right)^{1/2}\right)t + \eta\rho\overline{W_t^1(\omega)}\right)$$

and one stochastic component, H_t^c:

$$H_t^c = \exp\left\{\left(r - \frac{1}{2}\eta^2\left(1-\rho^2\right) - g\eta\left(1-\rho^2\right)^{1/2}\right)t + \eta\left(1-\rho^2\right)^{1/2} W_t^2(\omega)\right\}$$

which evolves according to the stochastic differential equation:

$$\begin{aligned} \frac{dH_t^c}{H_t^c} &= \left(r - g\eta\left(1-\rho^2\right)^{1/2}\right)dt + \eta\left(1-\rho^2\right)^{1/2} dW_t^2(\omega) \\ H_0^c &= 1 \end{aligned}$$

where, for convenience, the drift for the stochastic process H_t^c is parametrized by g.

The original problem involves the utility maximization of final wealth of an investor with power utility function:

$$E^{\mathbb{P}}\left[\frac{(L_T + H_T)^{1-\gamma}}{1-\gamma}\right]$$

Now consider a related artificial problem for a fixed path for the noise $\overline{W_t^1(\omega)}$ [9]. In this artificial problem the investor has a utility of terminal wealth of the HARA class given by:

$$E^{\mathbb{P}}\left[\frac{(K_T + V_T)^{1-\gamma}}{1-\gamma}\right]$$

where K_T is a constant and the agent can invest in a complete market composed by one risky asset $H_t^c\left(W_T^2(\omega)\right)$ and one riskless asset with rate of return r. In this artificial market we are assuming that the risky asset $H_t^c\left(W_T^2(\omega)\right)$ is tradable. V_t is the wealth at time t and evolves as

$$\frac{dV_t}{V_t} = p\frac{dH_t^c}{H_t^c} + (1-p)\, r dt$$

[9] Below we consider the conditions under which the solution to this artificial problem is equivalent to the solution of the original problem.

where p is the proportion of wealth invested in the risky asset.

When g is constant the problem can be solved using the duality approach (see Merton 1992) and the optimal allocation in this artificial market is given by a constant proportion optimal strategy:

$$p = \frac{-g\eta\left(1-\rho^2\right)^{1/2}}{\gamma\eta^2\left(1-\rho^2\right)}$$

Observe that by construction the risk premium g in this artificial market parameterizes the changes of measure $W_t^2 \to Z_t^2 = W_t^2 - gT$. [10]

Our objective is to determine g such that the investor will optimally allocate all his wealth to the risky asset ($p = 1$). In that case:

$$V_t = H_t^c$$

and, assuming that

$$K_T = L_T\left(\overline{W_T^1(\omega)}\right) R_T^{-1}\left(\overline{W_T^1(\omega)}\right)$$

then at optimality the artificial problem would be equivalent to the original one. Following an argument introduced in Karatzas and Cvitanic (1996), we impose the condition that the optimal unconstrained allocation in the artificial market coincides exactly with the constrained solution in the original allocation problem. Thus we determine the parameter g such that the investor is indifferent between a continuous trading strategy on H_t^c and a buy and hold strategy.

$$\begin{aligned} 1 &= \frac{-g\eta\left(1-\rho^2\right)^{1/2}}{\gamma\eta^2\left(1-\rho^2\right)} \\ g &= -\gamma\eta^2\left(1-\rho^2\right) \end{aligned}$$

Since the term $-\gamma\eta^2\left(1-\rho^2\right)$ does not depend on the specific path followed by the risk source W_t^1 the above analysis holds for all paths $W_T^1(\omega)$ and,

[10] The corresponding Radon Nikodym derivative:

$$f\left(W_T^2(\omega)\right) = \exp\left(-\frac{1}{2}g^2T + gW_T^2(\omega)\right)$$

parameterizes the admissible singular components of the optimal measure in the original problem.

therefore, it uniquely defines the singular measure in the original problem. The corresponding optimal expression for $f^*\left(W_T^2(\omega)\right)$ is then given by:

$$f^*\left(W_T^2(\omega)\right) = \exp\left(-\frac{1}{2}\gamma^2\eta^2\left(1-\rho^2\right)T + \gamma\eta\left(1-\rho^2\right)^{1/2}W_T^2\right)$$

and the optimal measure in the original problem is completely determined.

An additional independent verification that the above solution to the dual problem is optimal could be carried out formulating the dual problem in terms of a dynamic programming problem. It can be verified that the above solution verifies the Hamilton Jacobi Bellman (HJB) equation for the dual problem and provides via convex duality also the solution for the HJB equation of the primal problem.

6.2 Computation of λ^*

In order to conclude the computation we need to compute the optimal λ^* such that at optimality:

$$E^{\mathbb{Q}^*}\left[\left(W_T\left(\lambda^*,\mathbb{Q}^*\right) - H_T - \int_0^T \delta H_t dt\right) + \int_0^T \xi_t^* c_t\left(\lambda^*,\mathbb{Q}^*\right)dt\right] = l_0$$

or equivalently the λ^* which minimizes the expression

$$\begin{aligned}\lambda^* \;=\; & \arg\inf_{\lambda\in\mathbb{R}_+}\left\{\frac{\gamma}{1-\gamma}\lambda^{(\gamma-1)/\gamma}(E^{\mathbb{P}}\left[\int_0^T \beta_c e^{-\kappa u}\left(\frac{e^{\kappa u}}{\beta_c}\xi_u^*\right)^{(\gamma-1)/\gamma}du\right.\right.\\ & \left.\left.+\beta_W e^{-\kappa T}\left(\frac{e^{\kappa T}}{\beta_W}\xi_T^*\right)^{(\gamma-1)/\gamma}\right) + \lambda\left(l_0 + E^{\mathbb{Q}^*}\left[\left(H_T + \int_0^T \delta H_t dt\right)\right]\right)\right\}\end{aligned}$$

then applying first order conditions with respect to λ we get:

$$\left(\lambda^*\right)^{-1/\gamma} = \frac{l_0 + E^{\mathbb{Q}^*}\left[\left(H_T + \int_0^T \delta H_t dt\right)\right]}{E^{\mathbb{P}}\left[\int_0^T \beta_c e^{-\kappa u}\left(\frac{e^{\kappa u}}{\beta_c}\xi_u^*\right)^{(\gamma-1)/\gamma}du + \beta_W e^{-\kappa T}\left(\frac{e^{\kappa T}}{\beta_W}\xi_T^*\right)^{(\gamma-1)/\gamma}\right]}$$

then:

$$\lambda^* = \left\{\frac{l_0 + E^{\mathbb{Q}^*}\left[H_T + \int_0^T \delta H_t dt\right]}{\int_0^T\left(\frac{e^{\kappa u}}{\beta_c}\right)^{-1/\gamma}E^{\mathbb{Q}^*}\left[\xi_u^{*(-1/\gamma)}\right]du + \left(\frac{e^{\kappa T}}{\beta_W}\right)^{-1/\gamma}E^{\mathbb{Q}^*}\left[\xi_T^{*(-1/\gamma)}\right]}\right\}^{-\gamma} \tag{29}$$

and finally we compute the explicit formula for the expectations:

$$E^{\mathbb{Q}^*}\left[H_T\right] = h_0 \exp\left[(\mu-\delta-r) - \left(\frac{\alpha-r}{\sigma}\right)\eta\rho - \gamma\eta^2\left(1-\rho^2\right)\right]T \tag{30}$$

$$\begin{aligned} E^{\mathbb{Q}^*}\left[\int_0^T \delta H_t dt\right] &= h_0\delta \int_0^T \exp\left\{\left[(\mu-\delta-r) - \left(\frac{\alpha-r}{\sigma}\right)\eta\rho\right]t\right\} dt \\ &= h_0\delta\frac{\exp\left(-\nu T\right)-1}{-\nu} \qquad (31) \\ -\nu &= (\mu-\delta-r) - \left(\frac{\alpha-r}{\sigma}\right)\eta\rho - \gamma\eta^2\left(1-\rho^2\right) \end{aligned}$$

$$E^{\mathbb{Q}^*}\left[\xi_u^{r(-1/\gamma)}\right] = \exp\left(mt\right) \tag{32}$$

$$m = \frac{\kappa}{\gamma} - r\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\left(\frac{\alpha-r}{\sigma}\right)^2\left(\frac{1}{\gamma}-\frac{1}{\gamma^2}\right) \tag{33}$$

thus we finally obtain the expression for λ^*:

$$\lambda^* = \left\{\frac{l_0 + h_0\left[e^{-\nu T} + \delta\nu^{-1}\left(1-e^{-\nu T}\right)\right]}{\beta_c^{1/\gamma}m^{-1}\left(e^{mT}-1\right) + \beta_W^{1/\gamma}e^{mT}}\right\}^{-\gamma}$$

Substituting (30,31), (32) into (29) we get expression for the dynamics for the stochastic discount factor given in the main part of the paper.

Throughout the paper the following restrictions are assumed on the parameters: the standard transversality conditions in order to avoid the possibility of growth of discounted utility (see e.g. Merton (1992, pg. 110)):

$$m = \frac{\kappa}{\gamma} - r\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\left(\frac{\alpha-r}{\sigma}\right)^2\left(\frac{1}{\gamma}-\frac{1}{\gamma^2}\right) \tag{34}$$

and an additional transversality condition on the evolution of the illiquid asset to avoid bubbles:

$$\nu > \delta. \tag{35}$$