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Author
Craine, Roger

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Roger Craine

University of California at Berkeley

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The neoclassical rule and marginal q can be obtained by manipulating the firm's first-order conditions. Marginal q expresses the neoclassical flow condition in a present value form. Tobin's investment rule comes from combining Tobin's definition of q with the condition that no arbitrage profit opportunities exist in equilibrium.

In Section 3 I illustrate and compare the rules in a deterministic environment. Since the original derivations used deterministic models, the rules in Section 3 look familiar. Section 3 is self-contained.

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Introduction[1]

There are three popular "theories" of investment--the neoclassical model, Tobin's q, and marginal q. The neoclassical model and marginal q were derived in partial equilibrium models of optimal firm behavior. Tobin presented his q theory in a macroeconomic general equilibrium model. All the models are determininstic.

Economists frequently view these three "theories" as alternatives. I show that in a simple general equilibrium model with risk, the three investment rules yield the same Pareto-efficient decision. The rules derived in this paper are more general than the original, and most subsequent, presentations, since I explicitly model risk, and the asset prices and rates of return are endogenously determined. The investment rules are alternative, but equivalent, representations of the theory of optimal investment. I also illustrate the rules, and show the algebraic relationship between them, in a deterministic environment.

In 1963 Jorgenson, with important contributions by Hall (1967) and others, developed the neoclassical model of investment. The neoclassical theory is derived from a model of optimal firm behavior. In 1969 Tobin proposed his q theory of investment in a general equilibrium macroeconomic

1 I thank Jim Pierce for comments.

model. Tobin's q theory of investment comes from the market equilibrium condition that no arbitrage profit opportunities exist. Tobin does not explicitly model firm behavior. In 1977 Hall (p83) called q theory "the major competitor to Jorgenson's theoretical framework for investment". In the early '80s Abel(1980), Hayashi(1982), and Yoshikawa (1980) proposed a marginal q theory of investment. Sargent's (1979) text also contains a version of marginal q. Marginal q, like the neoclassical theory, is based on an explicit model of optimization by the firm. Abel and Hayashi show that marginal q and "average q", which they interpret as Tobin's q, only yield to the same investment decision under very special conditions. In general, average q leads to a suboptimal decision.[2]

With the exception of Tobin, the investment literature cited relies on a partial equilibrium approach focusing on optimal firm behavior.[3] Like Tobin, I choose a general equilibrium model, but I use the Arrow-Debreu framework which explicitly models firm behavior. The general equilibrium framework has several advantages. Most important is that Tobin's investment rule is not easy to interpret in a partial

2 The interpretation that average q is Tobin's q seems to be generally accepted (eg see Summers(1981), and Abel and Blanchard (1985)), although probably not by Tobin (see Tobin and Brainard (1977)).

3 Most of the work concentrated on the effect of taxes on the optimal investment decision so that a model of the firm was appropriate. I ignore taxes, and some other realistic complicating factors, because I want to establish equivalence between the rules. Making the environment more complex will not change the equivalence.

equilibrium analysis. In addition, we know that the Arrow-Debreu equilibrium allocation of resources is Pareto-efficient, so that any investment rule that we compare with the derived optimal rule must give the same decision or the rule is suboptimal. We also know that the vector of general equilibrium prices provides sufficient information for optimal decentralized decisionmaking. Therefore, if investment only depends on a subset of the vector of equilibrium prices (eg the rental rate of capital, or stock price), then the Pareto-efficient solution allows us to isolate the intertemporal prices that would be sufficient statistics for investment decisions if complete markets existed. Finally, although it makes no substantive difference for the main equivalence results established in this paper, the Arrow-Debreu complete-markets framework makes it easy to explicitly model risk, and the asset prices and rates of return are endogenously determined. As a consequence, the rules derived in this paper are more general than the original, and most subsequent, presentations.

The paper is organized as follows: Section 1 sets up a simple complete-markets economy and gives the equilibrium conditions. This section presents the household and firm objective functions and constraints, and the household decision rules. Section 2 focuses on the investment rules. The neoclassical rule and marginal q can be obtained by manipulating the first-order conditions that characterize optimal firm behavior. Marginal q expresses the neoclassical flow condition in a present value form. Tobin's investment rule comes from combining Tobin's definition of q with the condition that no arbitrage profit opportunities exist in equilibrium. All three rules give the same investment decision.

Section 3 illustrates and compares the rules in a deterministic environment. I show the algebraic relationship between them. Since the original derivations used deterministic models, the rules in Section 3 look familiar. Section 3 is self-contained and can be read without reading Sections 1 and 2.

Section 4 adds a cost to adjusting capital in the stochastic general equilibrium environment. The original derivations of marginal q emphasize the importance of a cost to adjusting capital. Section 4 shows that when there are costs to adjusting capital, all the rules still lead to the same Pareto-efficient investment decision. Of course, the optimal decision is not the same decision as when there are no adjustment costs, and the rules are more complicated. Section 5 presents a summary and conclusions.

Section 1: A Simple Complete-Markets Economy

This section sets up a simple complete-markets economy with a representative household and firm. There is a single commodity that can be consumed or added to the capital stock. The firm produces the commodity with a technology that depends on the beginning-of-period capital and the realization of an exogenous random variable, the so-called "state of nature". The household purchases claims (options) for consumption contingent on a sequence of realizations of the stochastic shocks. The firm sells contingent-claims. All claims trading takes place in period one. Equilibrium in this simple economy is Pareto-efficient. Exogenous random shocks create uncertainty about future outcomes, but the complete contingent-claims market allows agents to make consistent plans that give a Pareto-efficient allocation of resources when executed.

The Household

The household owns all the resources in this economy, but it delegates production and investment decisions to the firm. The household buys options (contingent-claims) on consumption. The options are contingent on the realization of exogenous stochastic shocks ($s_t \in S$) which buffet the economy. Let $s(t) = (s_1, s_2, \ldots, s_t)$ denote a particular sequence of shocks, where $s(t) \in S(t)$. Let $P(s(t))$ denote the probability of that sequence.[1] An option contingent on the sequence s(t), entitles the owner to one unit of the consumption good in period t if the sequence of shocks s(t) is realized-- otherwise the option has no value. Let, p(s(t)) denote the price of an option conditional on the sequence s(t), and let c(s(t)) denote the number of these options purchased by the household.

The objective of the household is to maximize the expected value of,

$$1.1 \quad \sum_{t=1} D^{-(t-1)} \sum_{s(t)\in S(t)} P(s(t))U(c(s(t)))$$

a strictly concave, time additive, utility function; where $D=1+d$ is one plus the household rate of time preference. The budget constraint limits the value of household purchases,

$$1.2 \quad \sum_{t=1} \sum_{s(t)} p(s(t))c(s(t)) = V(s(1))$$

to the value of the firm, where V(s(1)) denotes the market value of firm.

Given the prices the household buys options until the present value of the marginal utility of consumption in each state weighted by the probability that the options get exercised,

$$1.3 \quad P(s(t))D^{-(t-1)}U_c(c(s(t))) = p(s(t))$$

1 In any period t the sequence of actual realizations (s(t)) and the probability of any potential sequence $P(s(t+T))$ are public knowledge.

is proportional to the option price, where the constant of proportionality is a LaGrangian multiplier which I normalize at one. Notice that the option prices contain a discount for time preference (the amount paid today for delivery in the future) and for uncertainty (the probability that the option will be exercised). The first-order condition 1.3 holds for all potential sequences s(t).

The Firm

The firm sells options. The objective of the firm is to choose a feasible set of option sales that maximizes,

$$1.4 \quad \sum_{t=1} \sum_{s(t)} p(s(t))c(s(t))$$

The options c(s(t)) obligate the firm to deliver c units of the consumption good at date t if the sequence of shocks, s(t), is realized. Since the option prices contain a discount for time-preference and risk, the objective function measures the present market value of revenue from current and future production. Call the objective function evaluated at the feasible maximum, V(s(1)).

Production technology constrains the firm's feasible option sales. Output in t depends on the (predetermined) beginning-of-period capital stock, k(s(t-1)), and the realization of the state of nature, s_t.[2] Output can be used to cover option obligations, or added to capital[3],

1.5 $c(s(t)) = f(k(s(t-1)),s_t)-\{k(s(t))-k(s(t-1))\}$

I assume the production function is strictly concave, separable in k and s, and that capital does not depreciate.

Equilibrium

Equilibrium is an Arrow-Debreu competitive equilibrium. The options market meets in period one after the realization of the stochastic shock s_1, and agents trade claims for all time periods and all potential sequences of stochastic stocks. Equilibrium is a set of prices so that the demand for options equals the supply of options for every potential sequence of shocks.

Although all option trading occurs in period one, we can look at quantity realizations and the evolution of option values as if they were quantities and prices in a sequence economy. The household's (and firm's, which appear in Section 2) first-order conditions specify a set of contingent plans for quantities. In period t the household consumes, c(s(t)), ie the consumption goods its option

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2 In Kydland and Prescott's terminology this is a one-period "time to build" model. I also ignore other factor inputs (like labor), and taxes, which for my purpose only add "nuisance parameters" to the capital decision rule.
3 If production is less than option obligations the firm delivers capital to meet its obligations, ie it disinvests.

holdings entitle it to, see equation 1.3. Actual consumption is a random variable; the quantity of option contracts is not. The option contracts specify the sample space of consumption; the random variable s(t) determines actual consumption.

Option values also evolve randomly. All option prices, p(s(t)), for all potential sequences, s(t), are determined in period one. But, as time passes, only options conditional on the sequence of actual realizations retain value. Let $s(T) = (s(t), s(t+\tau))$; $T = t + \tau$. Now the equilibrium relative price of an option at time t equals the present value of the marginal rate of substitution weighted by the <u>conditional</u> probability that the sequence s(T), given s(t), occurs,

$$1.6 \quad \frac{p(s(t),s(t+\tau))}{p(s(t))} = \frac{P(s(t),s(t+\tau))}{P(s(t))} D^{-(t+\tau)} \left[\frac{U(c(s(t),s(t+\tau)))}{U(c(s(t)))}\right]$$

Options conditional on any other sequence, $p(s'(t),s(t+\tau))$ have no value. Since the relative prices of options that retain value remain the same in any period, agents have no wish to recontract and plans are executed.

To simplify the notation and to make comparisons with prices from a sequence economy more natural, I will use the notation p(s(t+T)) for the period t price of an option p(s(T)) given the sequence s(t).[4]

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4 At this abstract level of analysis we could set up an analogous sequence economy with spot markets, where spot prices, say ps(s(t)), are defined as,

$$ps(s(t)) = \{D^{(t-1)}/P(s(t))\}p(s(t))$$

and posit that agents know the probability distribution of spot prices, ie, they have rational expectations. The sequence economy has the same equilibrium allocation of resources as the complete-markets economy. To verify the allocation you can check that the spot prices satisfy the household and firm first-order conditions, equations 1.3 and 2.1.

Section 2: Investment Rules

This section compares the theoretical properties of the three popular investment rules--Jorgenson's neoclassical formulation, marginal q, and Tobin's q--using the general equilibrium model specified in Section 1. Manipulating the necessary condition to maximize the firm's objective function gives Jorgenson's rental rate formulation and marginal q. Tobin's investment rule comes from combining Tobin's definition of q with the condition that no arbitrage profit opportunities exist in equilibrium. All the rules give the same Pareto-efficient investment decision, so the rules are three alternative, but equivalent, representations of the theory of investment.

The objective of the firm is to choose a feasible set of options sales that maximizes the market value given in equation 1.4. The technology, specified in equation 1.5, constrains the feasible solution. The firm can rearrange sales temporally by transferring production between periods with investment, but total sales are limited by the production function. Thus the firm's decision comes down to the choice of a capital accumulation plan that maximizes the market value. The necessary condition equates the marginal loss in revenue from an option sale at time t in state s(t),

$$2.1 \quad p(s(t)) = \sum_{s(t+1)} p(s(t+1))\{f_k(k(s(t)))+1\}$$

with the marginal gain in revenue from transferring production forward one period through an additional unit of capital and <u>selling</u> the additional unit of capital next period. Equation 2.1 implies the set of optimal investment plans contingent on the realizations of the stochastic shocks. Recall that the firm collects the revenue from option sales now, but the option contract specifies delivery at a later date contingent on the state of nature. So, loosely interpreted, equation 2.1 says the current loss in marginal revenue equals the expected present value of the increase in future marginal revenue.[1]

The Neoclassical Model

Jorgenson's desired capital stock, k(s(t)), in the neoclassical formulation is the solution to equation 2.1,

$$2.2 \quad k(s(t)) = f_k^{-1}(\{p(s(t))/\sum_{s(t+1)} p(s(t+1))\}-1)$$

He labels $p(s(t))/\sum p(s(t+1)) -1$, the one-period return on the portfolio of options, the rental rate of capital.[2] The neoclassical formulation essentially equates the marginal

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1Substituting the spot prices(defined in footnote 4, Section 1) for the option prices makes this interpretation exact.
2 The return is the return on a one-period "bond". The portfolio $\sum p(s(t+1))$ entitles the holder to one unit of consumption next period whatever state of nature occurs, thus $p(s(t))/\sum p(s(t+1)) -1$ is the return to transferring a unit of consumption one-period into the future. I assumed the production function was separable in capital and shocks to simplify the notation for the "rental rate". For a nonseparable function the condition for a maximum,

$$p(s(t)) = \sum_{s(t+1)} p(s(t+1))\{f_k(k(s(t)),s_{t+1})+1\}$$

depends on the covariation of marginal productivity and prices.

flow return from physical capital to a flow return on a financial instrument.

Marginal q

Marginal q transforms the neoclassical flow condition into a present value. Marginal q says the firm should invest until the present value of the marginal revenue product of capital equals the cost of investment. Integrating 2.1 forward in time gives the present value of the marginal revenue product of capital. Marginal q is analogous to an asset evaluation equation. p(s(t)) is the current asset price, $\sum p(s(t+1))f_k$ is the "expected" present value of the dividend, and $\sum p(s(t+1))$ is the "expected" present value of the asset next period. Recursive substitution for next period's asset value (integrating equation 2.2 forward in time and over all potential sequences s(t+T)) gives,

$$2.5 \quad p(s(t)) = \sum_{T=1} \sum_{s(t+T)} p(s(t+T)) f_k(k(s(t-1+T)))$$

an asset evaluation equation for the marginal unit of capital. The right-hand-side measures the present value of the stream of dividends from the investment. The-left-hand-side is the current price of the asset.

Dividing by the price (cost) of investment gives the expression for marginal q,

$$2.6 \quad mq(s(t)) = \sum_{T=1} \sum_{s(t+T)} \frac{p(s(t+T))}{p(s(t))} f_k(k(s(t-1+T)))$$

Evaluated at the optimum, marginal q equals one--the firm should invest (decreasing the marginal productivity of capital) until the present value of the dividend stream from investment equals the price. Recall that the option prices discount the future values of the physical product of capital for risk and time.[3] Marginal q is the stock (integral) representation of the neoclassical flow condition.

Tobins' q

A Tobin's q investment rule can be derived by combining Tobin's definition of q with the condition that no arbitrage profit opportunities exist in a competitive equilibrium. Tobin's q does not specify an explicit model of optimization; nevertheless, it gives the same investment decision as the neoclassical model or marginal q.

Tobin defines q as the ratio of the equity value of the firm, say E(s(t)), to the replacement cost of <u>existing</u> capital, p(s(t))k(s(t-1)).[4] He says when q is greater than one, the firm should invest because the equity market's evaluation of potential earnings exceeds the replacement cost of existing capital, and vice versa.

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3 Abel (1980), Hayashi (1982), and Summers (1981) emphasize the importance of adjustment costs for capital in deriving marginal q. I add adjustment costs in Section 4. Adjustment costs are not necessary if the firm's value function is bounded.

4 At least I interpret Tobin's definition of q as a function of the beginning of period capital stock, see Tobin (1982).

In the complete-markets economy the market value of the firm is the value of options sold. Evaluated at the equilibrium,

2.10 $V(s(t)) =$

$$\max \sum_{T=0} \sum_{s(t+T)} p(s(t+T))c(s(t+T));\ s(t),\ k(s(t-1))$$

the market value of the firm is the maximum of the present value of earnings from current and future production obligations, given the initial capital stock and the sequence of stochastic shock realizations. The equity value of the firm is,

2.11 $E(s(t)) = V(s(t))-p(s(t))c(s(t))$

$$= \sum_{T=1} \sum_{s(t+T)} p(s(t+T))c(s(t+T))$$

the market value minus the current dividend, $p(s(t)c(s(t))$,[5]

which equals the present value of option contracts for future delivery. Notice 2.11 can also be written as,

2.11′ $E(s(t)) = \sum_{s(t+1)} V(s(t+1))$

since,

2.12 $V(s(t+1)) =$

$$\max \sum_{T=1} \sum_{s(t+T)} p(s(t+T))c(s(t+T));\ s(t+1),\ k(t)$$

Equation 2.12 is the market value of the of the firm in t+1 given the realization of the stochastic shock, s_{t+1}, <u>and the capital stock in period t</u>, k(t). Summing over all possible realizations for the shock gives the present value of earnings from the current (optimal) capital stock.

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5 This follows the convention that stock prices are quoted ex-dividend.

The condition for no arbitrage profits is,

2.13 $p(s(t))k(s(t)) = E(s(t))$

The market (replacement) value of the <u>optimal</u> capital stock, $p(s(t))k(s(t))$, must equal the the equity value of the firm; otherwise, riskless arbitrage profit opportunities exist. For example, if the market (replacement) value of the current capital stock is less than the equity value of the firm, then an agent could sell options worth, $E(s(t))$, the equity value, buy the capital to meet these obligations for less, and make an arbitrage profit. On the other hand, if the market value of capital exceeds the equity value of the firm, then an agent could buy the firm for the equity value, sell the capital stock, and make an arbitrage profit.

Therefore, the condition that no arbitrage profit opportunities exist in equilibrium implies that the market value, or replacement cost, of the firm's <u>optimal</u> capital stock in the current period equals the current period equity value of the firm. Substituting the no arbitrage condition, $p(s(t))k(s(t))=E(s(t))$, for the equity value in Tobin's definition of q,

2.14 $q(s(t)) =$

$$\frac{p(s(t))k(s(t))}{p(s(t))k(s(t-1))} = \frac{k(s(t-1))}{k(s(t-1))} + \frac{k(s(t)) - k(s(t-1))}{k(s(t-1))}$$

gives the Tobin q investment rule. The firm should invest until the replacement cost of the initial capital stock plus current investment equals the equity value of the firm.

In this Section, I showed that all three rules lead to the same Pareto-efficient investment decision, so the rules are alternative representations of a single theory of investment. The neoclassical model and marginal q are equivalent representations of the firm's first-order condition for maximization. Tobin's investment rule is like a dynamic programming algorithm; the equity market reveals the value of the firm given the current and all future optimal decisions, so investment can be calculated by simply subtracting the market value of the beginning-of-period capital from the equity market solution.

Section 3: An Illustration

In section 2, I showed that the three investment rules all lead to the same Pareto-efficient decision in a simple complete-markets general equilibrium model with risk. The original derivations used deterministic models, and were not placed in an Arrow-Debreu general equilibrium framework. To illustrate the results and compare the models in a more familiar context, in this section I evaluate each of the rules in a deterministic environment, and in the steady-state.

In a deterministic economy there is only one possible state of nature, so the option (Arrow-Debreu contingent-claim) prices collapse to the current price of a discount bond that delivers one unit of consumption in the future. In the steady-state all quantities are constant so investment is zero. Each of the rules yields the optimal investment decision. Furthermore, the neoclassical rule and marginal q, since they are derived from the maximization condition for the firm, can be inverted in the steady-state giving an expression for the optimal capital stock as a function of the household rate of time preference. In the steady-state Tobin's q equals one since optimal investment is zero. A q of one, however, simply reflects the no-arbitrage-profit condition. Inverting the no-arbitrage-profit condition does not give the optimal capital stock. Instead, it gives the "average q" investment rule, which is, in general, suboptimal.

The Neoclassical Model

The neoclassical investment rule comes from the firm's first-order condition for maximization. In my stripped-down model,[6] to satisfy the first-order condition, the firm chooses a capital stock so that the current value of the marginal revenue product of capital next period plus the

6 Equation 1.4 gives the firm's objective function, and equation 1.5 gives the production constraint.

current value of a unit of capital next period equals the current price,

3.1 $p = p_{+1}\{f_k(k)+1\} = p_{+1}f_k(k)+p_{+1}$

The notation p_{+1} denotes the (discount) price of a one-period bond. Jorgenson labels $r=p/p_{+1}-1$ (the return on a one-period bond) the rental rate of capital. Rearranging 3.1 and using the Jorgenson definition of the rental rate of capital gives the neoclassical investment rule--the firm invests until the marginal product of capital equals the rental rate.

Combining the firm's and household's first-order conditions (equation 1.3) gives the general equilibrium condition that,

3.2 $f_k(k) = r = p/p_{+1}-1 = \dfrac{U_c(c)}{(1+d)^{-1}U_c(c_{+1})} -1$

the marginal product of capital equals the rental rate of capital, which in equilibrium equals the household's intertemporal marginal rate of substitution minus one. d is the household rate of time preference. Evaluating 3.2 at the steady-state ($c=c_{+1}$) gives the familiar condition that the marginal product of capital equals the household rate of time discount. Inverting the marginal productivity condition defines the steady-state optimal capital stock [7]

3.3 $k = f_k^{-1}(d)$

7 Since equation 3.2 holds in every period the neoclassical rule equating the marginal product of capital with the rental rate also defines an optimal path for capital (and investment) in the non-steady-state.

as a function of the household rate of time preference. In the steady-state the beginning-of-period capital equals the optimal capital stock, so investment is zero.

Marginal q

Marginal q expresses the neoclassical flow condition in present value terms; marginal q says the firm should invest until the present value of the marginal revenue product of capital equals the current price of a unit of capital.[8]

Recursive substitution for p_{+T} (integrating equation 3.1 forward in time) in the firm's first-order condition,

$$3.4 \quad p = \sum_{T=1} p_T f_k(k)$$

equates the present value of the future marginal revenue stream from the last unit of investment to the price of investment. Dividing the right-hand-side of 3.4 by the current price, p, defines marginal q,

$$3.5 \quad mq(k) = \sum_{T=1} \frac{p_T f_k(k)}{p} = \sum_{T=1} (1+r_T)^{-1} f_k(k)$$

where r_T is the return on a T period bond. The marginal q investment rule states that the firm should invest (reducing

8 The authors who developed marginal q used models with a cost to adjusting capital. Section 4 adds a cost to adjusting capital and shows that, although the optimal investment decision changes, all the rules still give the same decision.

the marginal product of capital) until marginal q equals one.

Evaluating the marginal q rule at the steady-state gives,

$$3.6 \quad mq(k)-1 = \frac{f_k(k)}{d}-1 = \frac{f_k(k)}{r}-1$$

an equation that can be solved for the implied optimal capital stock, which is the capital stock given in equation 3.3.

Tobin's q

Interpreting Tobin's investment rule, especially in the steady-state, is somewhat trickier. Tobin's investment rule comes from combining his definition of q with the market equilibrium condition that no arbritrage profit opportunities exist. Tobin defines q as,[9]

$$3.7 \quad q(k,k_0) = E(k)/pk_0$$

the equity value of the firm (E(k)) relative to the replacement cost (market value) of the existing capital stock (pk_0). I use k_0 to denote the beginning-of-period, or existing, stock of capital.

9 At least I interpret Tobin's definition of q as equation 3.7,(see Tobin 1982).

The no-arbitrage-profit condition says that, in a competitive equilibrium, an investor must be indifferent between purchasing an equity which entitles him to the future earnings of the firm and purchasing the capital stock which produces that earnings stream. Owners of the equity capital receive dividend payments (div)

3.8 $div_T = p_T\{f(k_{T-1})-(k_T-k_{T-1})\}$

which are the net real earnings of the firm. I multiplied these by the discount price (p_T) so div is the present value of a dividend paid at *T*. The equity value of the firm is the maximum of present value of the dividend stream,[10]

3.9 $E(k) = \sum_{T=1} div_T$

where I follow the convention that the current equity is valued "ex dividend", ie the sum starts at one.

The condition for no arbitrage profit opportunities in a competitive equilibrium is,

3.10 $pk = E(k)$

that the market value of capital equals the equity value of the firm, or present value of production from capital. Equation 3.10 is the condition for no arbitrage profits. It is not Tobin's investment rule.

Combining Tobin's definition of q with the no-arbitrage-profit condition gives the Tobin q investment rule,

10 Tobin's definition presumes that the equity market evaluates firm earnings at the maximum, which it will in a complete markets equilibrium, see Section 2.

3.11 $q(k,k_0) = \frac{E(k)}{pk_0} = \frac{p(k_0+dk)}{pk_0} = 1 + \frac{dk}{k_0}$

where I substituted the definition that the current capital stock (k) equals the initial capital stock (k_0) plus the current investment (dk). Rearranging gives an investment rule that says the firm should invest until,

3.11' $q(k,k_0) - \frac{dk}{k_0} = 1$

q minus the ratio of current investment to the beginning-of-period capital equals one.

The Tobin q investment rule gives the same decision as the marginal q rule, and the neoclassical rule. To simplify exposition of the algerbraic relationship between the rules, consider the Taylor's series approximation of the equity equation 3.9 around the beginning-of-period capital stock, k_0,[11]

3.12 $E(k) = E(k_0)+E_k(k_0)dk$

or, using the definition of dividends in equation 3.8,

3.12' $E(k) = E(k_0)+\sum_{T=1} p_T f_k(k_0)dk$

Equation 3.12 says the current value of equities equals the present value of earnings evaluated at the suboptimal initial capital stock (k_0) plus the present value of the marginal increase in earnings from the optimal current investment. Substituting 3.12 in the investment rule gives,

11 In the non-steady-state capital is a path, $k=(k_1,k_2,.....)$, so define k_0 as that path minus the investment decision,ie $k_0=(k_1-dk,k_2-dk,.....)$.

$$3.13 \quad q(k,k_0) - \frac{dk}{k_0} = \left[\frac{E(k_0)}{pk_0} + \frac{E_k(k_0)}{p} - 1\right]\frac{dk}{k_0}$$

$$= \frac{E(k_0)}{pk_0} + \left[\sum_{\tau=1} \left\{\frac{p_\tau f_k(k_0)}{p}\right\} - 1\right]\frac{dk}{k_0} = 1$$

The second term on the right-hand-side is marginal q, see equation 3.5, evaluated at the initial capital stock, $mq(k_0)$. If optimal investment is positive, the present value of the marginal revenue product of capital evaluated at the initial level of capital exceeds the cost of the investment goods, and vice-versa. Optimal investment is the value of dk that sets $q-dk/k_0$ equal to one. Notice that evaluating the right-hand-side of equation 3.13 at the optimal capital stock gives a value of one. The first term is the arbitrage equilibrium condition, $E(k)/pk=1$. The second term is marginal q minus one, $mq(k)-1$, which equals zero evaluated at the optimal capital stock.

Substituting the definition $mq(k_0)$ in equation 3.13 gives a compact expression for the Tobin q investment rule,

$$3.14 \quad q(k,k_0) - \frac{dk}{k_0} = \frac{E(k_0)}{pk_0} + [mq(k_0)-1]\frac{dk}{k_0} = 1$$

In the steady-state, since the beginning-of-period capital stock, k_0, is the optimal capital stock, Tobin's investment rule collapses to,

$$3.15 \quad q(k_0,k_0) = \frac{E(k_0)}{pk_0} = 1$$

the condition for no arbitrage profits, $E(k)=E(k_0)=pk_0$.

The no-arbitrage-profit condition cannot be inverted to determine the optimal capital stock. Substituting the present value of the steady-state dividend stream for the equity value gives,

$$3.16 \quad q(k_\infty, k_\infty) = \frac{f(k_\infty)}{d} = 1 \quad , \text{ or}$$

$$3.16' \quad \frac{f(k_\infty)}{k_\infty} = d$$

a no-arbitrage-profit condition that equates the dividend-price ratio (average product of capital) to the household rate of time preference.

The no-arbitrage-profit condition reflects the distribution of firm earnings--not the allocation decision. The firm's net revenue is always exhausted in payments to the factors. In the steady-state, the owners of the firm receive the average product of capital (average real earnings) times the number of shares they own in dividend payments, or $f(k)=(f(k)/k)k$. The optimal capital stock is determined by the marginal valuation of capital, equation 3.14.

Inverting the arbitrage equilibrium condition, 3.16′, gives the steady-state version of the so-called "average q",

$$3.17 \quad k = h^{-1}(d) \neq f_k^{-1}(d); \quad h = \frac{f(k)}{k}$$

investment (capital) rule. The average q rule is a suboptimal allocation rule except in the special case where the average product of capital equals the marginal product.

Section 4: Adjustment Costs

Many argue that the constraint in equation 1.5 allows one to define the optimal capital stock, but not investment which is the change in capital. Of course, given the optimal path for capital, in a discrete time model investment is defined by the identity, $I(s(t))=k(s(t))-k(s(t-1))$.[1] But in a continuous-time model discrete jumps in the stock of capital imply infinite investment, so investment is not well-defined. Adjustment costs smooth changes in capital, leading to a well defined investment equation.

Adding a cost to adjusting capital to the model in Section 2 changes the objective function for the firm and the equity value of the firm. Adjustment costs complicate the math and change the optimal investment decision, but the three rules give the same Pareto-efficient decision. So adjustment costs do not alter the basic result that the rules are alternative, but equivalent, representations of a single theory.

Adjustment costs reduce the gross flow from capital by more than the market cost of the new investment,

4.1 $$p(s(t))c(s(t)) = p(s(t))\{f(k(s(t-1)),s_t)-I(s(t))-g(I(s(t))\}$$

---

1 If the production function is homogeneous of degree one, then the firm's objective function, equation 1.4, is unbounded so the maximization problem is not well posed.

since some of the output is consumed by installing (or dismantling) the capital. Let the adjustment cost function, $g(I(s(t))$ be convex and equal to zero at $I(s(t))=0$.

The objective of the firm is to maximize the market value, given in equation 1.4, subject to the modified constraint 4.1, which includes the cost of adjusting capital. Consider the dynamic programming solution. Using the notation from Section 2, define,

4.2 $$V(s(t+1)) = \max \sum_{T=1} \sum_{s(t+T)} p(s(t+T))c(s(t+T));\ s(t+1)$$

as the maximum of the market value in t+1 given all future optimal (contingent) decisions, ie all $k(s(t+T))$, $T>0$. $V(s(t+1))$ is a function of the choice variable in period t, $k(s(t))$, and the realization of the state of nature, s_{t+1}. So, the firm's decision is to choose the current capital stock, subject to the constraint 4.1,

4.3 $$\max_{k(s(t))} \{(p(s(t))c(s(t)) + \sum_{s(t+1)} V(s(t+1)))\};$$

$$s(t),\ k(s(t-1))$$

that maximizes the current market value of the firm. The necessary condition is,

4.4 $$[p(s(t)) \frac{\partial c(s(t))}{\partial k(s(t))} + \sum_{s(t+1)} \frac{\partial V(s(t+1))}{\partial k(s(t))}]\ dk(s(t)) = 0$$

or,

4.4′ $$p(s(t))\{1+g_I(I(s(t)))\} = \sum_{s(t+1)} p(s(t+1))[f_k(k(s(t)))+\{1+g_I(I(s(t+1)))\}]$$

The neoclassical formulation is the dynamic programming solution, ie the value of capital, k(s(t)), that satisfies equation 4.4 given all future optimal decisions.

Marginal q comes from using the maximum principle to solve the problem, eq see Abel(1980) or Hayashi. Integrating 4.4 forward in time (recursively substituting for $p(s(t+T))\{1+g_I(I(s(t+T)))\}$ gives,

4.5 $$p(s(t))\{1+g_I(I(s(t)))\} = \sum_{T=1} \sum_{s(t+T)} p(s(t+T))f_k(k(s(t-1+T)))$$

Define marginal q, as I did in Section 2,

4.6 $$mq(s(t)) = \sum_{T=1} \sum_{s(t+T)} \frac{p(s(t+T))}{p(s(t))} f_k(k(s(t-1+T)))$$

as the present value of investment.[2] This definition gives the optimal investment decision as a function,

4.7 $$I(s(t)) = g_I^{-1}(mq(s(t))-1)$$

that sets marginal q to one.

Tobin's investment rule can no longer be interpreted as the consequence of a market arbitrage equilibrium condition since setting up a new firm costs more than adding capital to an existing firm. However, in equilibrium, the equity value of the firm is still,

4.8 $$E(s(t)) = V(s(t))-p(s(t))c(s(t))$$

---

2 Equation 4.6 is analogous Abel's (1984, equation (6)) and Hayashi's (1982, equation 11) definition of marginal q.

the market value of the firm--which takes into account the cost of adjusting capital--minus the current dividend--which includes the current adjustment cost. So Tobin's q gives the investment rule,

$$4.9 \quad p(s(t))\{k(s(t-1))+I(s(t))\} = E(s(t)) \quad , \text{ or}$$

$$4.9' \quad q(s(t)) - \frac{I(s(t))}{k(s(t-1))} = \frac{E(s(t))}{p(s(t))k(s(t-1))} \frac{-I(s(t))}{k(s(t-1))} = 1$$

that the firm should invest until the market value (replacement value ignoring adjustment costs) of the current capital equals the equity value of the firm.

Section 5: Summary and Conclusions

In this paper I showed that in a simple general equilibrium model with complete-markets and risk, the three popular "theories" of investment--the neoclassical model, Tobin's q, and marginal q--give the same Pareto-efficient decision. The investment rules are alternative, but equivalent, representations of the theory of optimal investment. The rules derived in this paper are actually more general than the original presentations, since I explicitly model risk, and the asset prices and rates of return are endogenously determined.

The neoclassical rule and marginal q come from manipulating the firm's first-order conditions for maximization. Marginal q expresses the neoclassical flow condition in a present value form. Abel and Hayashi derive marginal q using the maximum principle.

Tobin's q investment rule comes from combining Tobin's definition of q with the condition that no arbitrage profit opportunities exist in equilibrium. Tobin's q rule is like a dynamic programming solution. In equilibrium, when markets are complete, the equity value of the firm is the <u>maximum</u> of the present value of future earnings. The equity value reveals the future value of the firm, so optimal investment is the difference between the beginning-of-period capital and the optimal capital stock implied by the equity evaluation.

If the rules differ in real world applications it is because markets are incomplete, or (unspecified) disequilibrium constraints affect the rules differently. In empirical applications the rules yield different parameter estimates, but none of them works very well. Clark (1979) compared the post-sample predictions from several rules, including the neoclassical model and Tobin's q. He found a simple accelerator model performed best. Abel and Blanchard (1985) estimated a marginal q model, and found that it leaves a large serially correlated portion of investment unexplained. I conjecture (see Craine (1986) for a detailed explanation) that in real world applications, where markets are incomplete, observable market prices--eg equity values or the rental rates of capital-- are noisy signals of the relevant complete-markets intertemporal prices, so investment is not highly correlated with the observable market prices.

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