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Authors
Ekeland, Ivar
Karp, Larry
Sumaila, Rashid

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Equilibrium management of fisheries with altruistic overlapping generations

Ivar Ekeland, Larry Karp
and Rashid Sumaila

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Ivar Ekeland† Larry Karp‡ Rashid Sumaila§

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Abstract

We imbed a classic fishery model, where the optimal policy follows a Most Rapid Approach Path to a steady state, into an overlapping generations setting. The current generation discounts future generations' utility flows at a rate possibly different from the pure rate of time preference used to discount their own utility flows. The resulting model has non-constant discount rates, leading to time inconsistency. The unique Markov Perfect equilibrium to this model has a striking feature: provided that the current generation has *some* concern for the not-yet born, the equilibrium policy does not depend on the degree of that concern.

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Keywords: fisheries management, sustainable development, renewable resources, time inconsistency, hyperbolic discounting.

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†CEREMADE, Université Paris-Dauphine, France; ekeland@math.ubc.ca.

‡Department of Agricultural and Resource Economics, University of California, Berkeley, and the Ragnar Frisch Center for Economic Research; karp@berkeley.edu

§Fisheries Centre, University of Brititish Columbia, Vancouver, Canada; r.sumaila@fisheries.ubc.ca

1 Introduction

The Millennium Ecosystem Assessment identifies fisheries as one of the world's critical environmental stocks (United Nations 2005). Due to a combination of overfishing, loss of habitat, and climatic change, at least 30% of the world's fisheries are at risk of population collapse (Sumaila et al. 2011). Fisheries support the well-being of nations through direct employment in fishing, processing, and ancillary services amounting to over US$ 220 billion annually (Dyck and Sumailila 2010). Fish provide nearly 3.0 billion people with 15 percent of their animal protein needs; including post-catch activities and workers' dependants, marine fisheries support nearly 8% of the world's population (FAO, 2011). Fishery economists have used a particular optimal control model as a basis for recommending stock levels. We imbed this model in an overlapping generations (OLG) setting, thereby contributing to the field of fishery economics. Our paper also contributes to the broader field of environmental economics, in view of the importance of discounting in this field. More generally, the paper contributes to the study of OLG models and to the literature on non-constant (e.g., hyperbolic) discounting.

The actual problem of fishery management is intergenerational: those currently alive have to decide how much of the stock to retain for generations that will be alive in the distant future. Agents currently alive have a standard optimization problem in two cases: if they have no concern for those who have not yet been born, or if they discount the future utility flows of the not-yet-born at the same rate as they discount their own future utility. In all other cases, their implied discount rate is non-constant, either decreasing, as with hyperbolic discounting, or increasing. In these cases, the policy trajectory that is optimal for the current generation is not time consistent.

It is not reasonable for the current generation to act as if it could choose decisions for subsequent generations. We therefore consider a particular class of time consistent equilibria, in which the harvest decision at a point in time is conditioned on the stock of biomass – the state variable – at that point in time. We obtain a Markov equilibrium to the dynamic game amongst the sequence of policymakers. Each policymaker in this sequence is the representative agent at a point in time. The Markov Perfect Equilibrium is a subgame perfect Nash equilibrium to this sequential game: the policy rule chosen by each representative agent is optimal, given her beliefs about the policy rule that will be used in the future.

Perhaps the model most widely used to propose target stocks for fishery management, and certainly the model most widely used to explain the management problem, is linear in the harvest rate (Clark and Munro 1975), (Clark 2005). With this model, the benefit per unit of harvest is constant and the cost per unit of harvest is a decreasing convex function of the biomass of fish; harvest costs increase as the stock falls. The equation of motion equals the natural growth rate of biomass minus the harvest. The limitation of this model is its assumption that the flow of benefit minus cost is linear in the harvest. Fishery economists use this model because it provides a plausible, elegant, and easily interpreted recommendation: the stock should be driven as rapidly as possible to a steady state. The solution is "bang-bang", i.e. it involves a Most Rapid

Approach Path (Spence and Starrett 1975). The steady state depends on the per unit benefit of harvest, the growth equation, the harvest cost function, and importantly, on the discount rate used to evaluate future benefits.

We imbed this linear-in-control model into the sequential game described above, and obtain a striking conclusion: provided that the current generation has *some* concern for the net-yet born, the equilibrium policy rule, and thus the stock trajectory and the steady state, is independent of the *degree* of concern for future generations. The steady state stock here equals the level of a planner who has a constant discount rate equal to the pure rate of time preference minus the growth rate of the population that benefits from future harvests. If the current generation has zero concern for future generations, the steady state stock equals the level of a planner that has a constant discount rate, equal to a generation's pure rate of time preference plus their mortality rate, i.e. equal to their risk-adjusted discount rate. The difference between these two discount rates is twice the mortality rate minus the birth rate. For this model, the degree of the current generation's concern for the not-yet born is irrelevant to the equilibrium policy, provided that they have some concern for the not-yet born. There is a discontinuity in the equilibrium decision rule, in the limit as the current generation's concern for the not-yet born vanishes.

Our results depend on the linear-in-control framework, but not on other functional (e.g. the Gordon-Schaefer) assumptions. Our focus is the fishery problem, but the analysis contributes more generally to the literature on non-constant (including hyperbolic) discounting and to the OLG literature. OLG models used to study policy sometimes assume that the policy maker discounts the utility of the generation currently alive *back to the time of their birth*; see for example Calvo and Obstfeld (1988) page 414. Calvo and Obstfeld recognize that this assumption is "unnatural ...[because]...the planner is concerned with their welfare from the present time onward". Ignoring this fact and assuming instead that their utility is discounted back to the time of their birth, eliminates the time-inconsistency problem that is an integral part of this setting. Our sequential game model, and the focus on Markov perfection, provides an alternative that deals squarely with the time consistency issue. This class of OLG problems is sufficiently complicated that it makes sense to develop our understanding by dealing with special cases, such as the linear-in-control model.

2 The model

We first state the primitives of the model and then define the equilibrium.

2.1 Primitives

An agent's lifetime is exponentially distributed, with hazard rate ω. At time t the population size is $N(t)$. The memoryless feature of the exponential distribution means that all agents alive at a point in time have the same probability of dying over any

future interval, regardless of their current age. Therefore, agents alive at a point in time are indistinguishable from each other. There is literally a representative agent, so in this model there is no issue of aggregation of preferences over the agents currently alive. Agents are born at rate α; the growth rate of the population is $g = \alpha - \omega$.

The aggregate utility (profit) flow, shared by all agents alive at time t, regardless of their age, equals $u(t)$. In our setting, $u(t)$ is the aggregate flow of profits from the fishing sector. Agents' pure rate of time preference for their own future utility is δ, and they discount the future utility of the not-yet-born at the rate σ. Provided that $\sigma < \infty$, agents are "paternalistically" or "impurely" altruistic, because they care about the "direct", or "selfish", utility flows of agents who will be born in the future; they do not care how intervening generations value the utility of the agents born in the future. If they cared about their successors' valuation of their mutual successors' utility, they would be said to be purely altruistic (Ray 1987), (Andreoni 1989).[1]

The value that an agent today places on a flow at a future time t depends on the number of agents alive at that time, and on how the agent today values those agents' utility. For example, if the agent today puts the same weight on the utility of all agents alive at time t, regardless of their date of birth, then she would consider a flow twice as valuable if the population were $2N(t)$ rather than $N(t)$. She may, however, place more weight on the future utility of agents currently alive (members of her generation) than on the utility of agents who will be born between now and time t. In that case, the comparison of the two situations, where the time t population is either $N(t)$ or $2N(t)$, depends on the composition of the two populations, not just on their size. The discount factor reflects this kind of consideration, and therefore depends on the birth and death rates, α and ω, on the agent's pure rate of time preference, δ, and on her discount rate for the utility of the not-yet born, σ.

There are two components to the welfare of agents alive today. The selfish welfare component equals the present discounted stream of the agents' own expected future utility. All agents, regardless of their date of birth, use the risk adjusted discount rate $\delta + \omega$ to compute the selfish component of their welfare; this rate equals the pure rate of time preference plus the hazard rate for death. An equivalent interpretation of this component is that of the $N(t)$ agents alive at time t, only $e^{-\omega s}N(t)$ of them will be alive at time $t + s$, with $s \geq 0$. The present value of the time $t + s$ utility flow, for the agents alive at time t, is therefore $e^{-(\delta+\omega)s}N(t)$ times the $t + s$ utility flow.

The altruistic component of the welfare of those currently alive equals the selfish component of welfare for all agents who will be born in the future, discounted at the altruistic rate σ. A positive value of σ means that agents alive today place more weight on the selfish welfare of those who are born sooner rather than later: they

[1] Purely altruistic preferences require that agents apply non-negative weights to their successors' welfare. Saez-Marti and Weibull (2005) find conditions under which an arbitrary discount factor is consistent with purely altruistic preferences. However, their setting involves a succession of agents who each live a single period, and therefore it cannot be directly applied to our OLG setting. Work in progress examines the relation between pure and paternalistic altruism in an OLG model; details available on request.

care more about the utility of their children than they do about the utility of their (grand)^{m}children. For $\sigma = \infty$, the altruistic component vanishes, and we have a model with purely selfish agents who discount future utility flows at the risk adjusted rate $\delta + \omega$.

Table 1 collects the definitions of the parameters entering the discount factor.

ω	α	g	δ	σ
death rate	birth rate	growth rate $\alpha - \omega$	selfish pure rate of time preference	altruistic discount rate applied to unborn

Table 1: Parameters entering Discount Factor

We follow Ekeland and Lazrak (2010), who use a proposal in Sumaila and Walters (2005), to calculate the discount factor, $D(t)$:

$$D(t) = e^{-(\delta+\omega)t}\frac{\sigma-\delta}{\sigma-\alpha-\delta} + \left(\frac{\alpha}{\alpha+\delta-\sigma}\right)e^{-(\sigma-g)t}. \tag{1}$$

Appendix A contains the details. The discount factor exists provided that $\sigma \neq \alpha + \delta$, as we hereafter assume. In our model the flow payoff, $u(t)$, is positive. To ensure that the value of the program is finite, we also assume that $\sigma > g$.

The discount factor is a weighted sum (but not necessarily a convex combination) of exponential discount factors. The first involves the risk-adjusted discounting of those currently alive, $\delta + \omega$, and the second involves the difference between the discount rate applied to the unborn and the population growth rate. The discount rate, $r(t)$, and its time derivative, corresponding to the discount factor $D(t)$ are

$$r(t) = -\frac{\frac{dD}{dt}}{D} = \frac{-(\omega+\delta)(\sigma-\delta)+\alpha(\sigma-g)e^{-t(\sigma-\delta-\alpha)}}{(\delta-\sigma)+\alpha e^{-t(\sigma-\delta-\alpha)}}$$

$$\frac{dr}{dt} = \alpha e^{-t(\sigma-\delta-\alpha)}\frac{\sigma-\delta}{\left(\delta-\sigma+\alpha e^{-t(\sigma-\delta-\alpha)}\right)^2}\left(\sigma-\delta-\alpha\right)^2.$$

This discount rate is constant if $\sigma = \delta$ or if $\sigma = \infty$. (Recall that we assume throughout that $\sigma \neq \delta + \alpha$). We have:

$$\text{for } \sigma = \delta,\ r = \delta - g;\ \text{ for } \sigma = \infty,\ r = \delta + \omega.$$

For $\sigma \notin \{\delta, \delta + \alpha, \infty\}$, the discount rate is decreasing if $\sigma < \delta$ and increasing if $\sigma > \delta$. The initial value of the discount rate is $r(0) = \delta - g = \delta + \omega - \alpha$ for all finite $\sigma \neq \delta + \alpha$. Define $r_\infty = \lim_{t\to\infty} r(t)$. The signs of the derivatives and the asymptotic values of the discount rate are

$$\begin{aligned}
&\text{for } \delta+\alpha < \sigma < \infty\text{: } \tfrac{dr}{dt} > 0;\quad r_\infty = \delta + \omega \\
&\text{for } \delta < \sigma < \delta+\alpha\text{: } \tfrac{dr}{dt} > 0;\quad r_\infty = \sigma - g < \delta + \omega \\
&\text{for } \sigma < \delta\text{: } \tfrac{dr}{dt} < 0;\quad r_\infty = \sigma - g < \delta + \omega - \alpha.
\end{aligned} \tag{2}$$

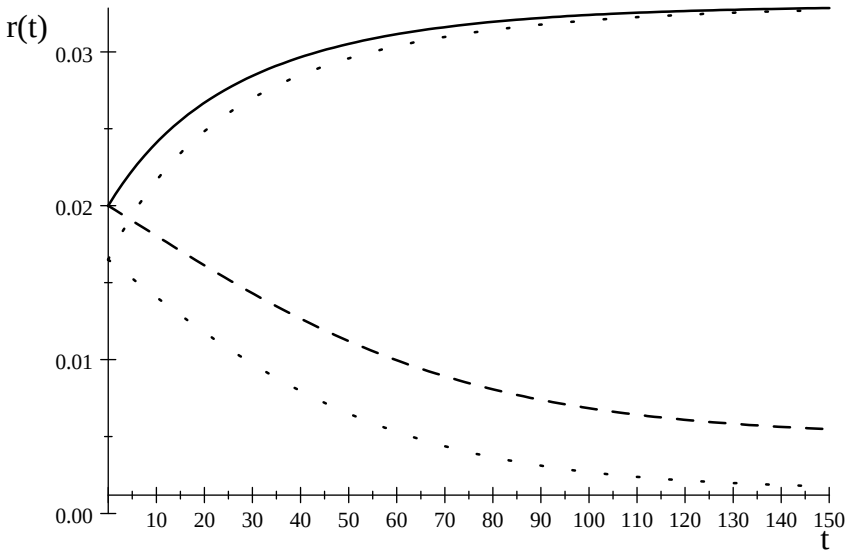

Figure 1: $\delta = 0.02$, $n = 0.013$. Solid curve shows $r(t)$ for $\sigma = 0.06$ and $g = 0$. Dashed curve shows $r(t)$ for $\sigma = 0.005$ and $g = 0$. The two dotted curves increase g to 0.0035

The case $\sigma < \delta$ corresponds to hyperbolic discounting, with the discount rate converging to $\sigma - g$. The case $\sigma > \delta$ corresponds to an increasing discount rate, converging to $\delta + \omega$. In both cases, however, the trajectory of discount rates lies below the selfish rate, $\delta + \omega$.

Let the unit of time be a year. Figure 1 shows the graphs of r for $\delta = 0.02$ and $\omega = 0.013$, corresponding to an expected lifetime of 77 years. The increasing solid curve shows $r(t)$ for $\sigma = 0.06$ and the decreasing dashed curve shows $r(t)$ for $\sigma = 0.005$; both curves are for a constant population, $g = 0$. The dotted curves correspond to the same values, except with $g = 0.0035$; for this value, the population doubles in approximately 200 years.

The flow payoff, denoted $u(t)$ above, is $u(t) = (p - c(x_t))\, h_t$, where the state variable x_t is the biomass of fish, the decreasing convex function $c(x)$ is the unit cost of harvest, p is the price, and h_t is the harvest. The welfare of the agents alive at time t is the present discounted value of their selfish and altruistic flow of payoff,

$$\int_0^\infty D(s)\,(p - c(x_{t+s}))\, h_{t+s} ds. \tag{3}$$

The stock of fish evolves according to

$$\frac{dx(t)}{dt} = \dot{x}_t = f(x_t) - h_t. \tag{4}$$

In order to avoid uninteresting technical issues, we assume that harvest is bounded below by 0 and bounded above by $\bar{h} < \infty$.

Our discount function incorporates the assumption that an arbitrary future flow of profits, $u(t)$, is more valuable to the current generation when the population at t is larger. This assumption captures the idea that future profits are more important when more people depend on them. This interpretation is non-controversial if the flow is non-rival, e.g. if it is literally a public good, or used to finance a public good. For a rival good, a larger future population likely diminishes the value, to those currently alive, of the future flow, because they know that they will have to share the profits with more people. This interpretation gives rise to a model that is isomorphic to the one that we discuss in the text. We can use the results in the text to determine the MPE to this alternative model. The two models of discounting are equivalent if the population is constant. With a public good, the aggregate utility flow at t is $N(t)u(t)$ and with a private good where each individual obtains the same share, the individual flow is $\frac{u(t)}{N(t)}$. If N is constant, the flow payoffs differ by a constant factor, so the equilibrium is the same regardless of whether $u(t)$ is a public or a private good. (Appendix B).

2.2 The equilibrium

For $\sigma \notin \{\delta, \delta + \alpha, \infty\}$ the discount rate is non-constant, so a program that maximizes expression (3) subject to equation (4) is time inconsistent. We obtain a time consistent equilibrium by modelling the decision problem as a sequential game amongst agents who make decisions at different points in time. The agent at time t chooses the current harvest rate, taking as given the current state variable, x_t, under the belief that decisions at time $t + s$, $s > 0$, are given by a function $\chi(x_{t+s})$. We look for a symmetric, stationary, pure strategy Nash equilibrium to this game, a function $\chi(x)$ such that $h_t = \chi(x_t)$ is the optimal action for the agent at time t given the state variable x_t, when this agent believes that future actions will be $h_{t+s} = \chi(x_{t+s})$. These beliefs are confirmed in equilibrium for any possible subgame (any realization of x_{t+s}). That is, we obtain a Markov Perfect Equilibrium (MPE).

Karp (2007) studies the MPE for a more general class of games by taking the limit of a discrete stage infinite horizon game. In that game, each stage lasts for ε units of time, and the discount rate for the first S periods can take arbitrary values, but is constant for period $S+1$, $S+2...\infty$. The integral in expression (3) is replaced by an infinite sum, and the differential equation (4) is replaced by a difference equation. Harris and Laibson (2001) obtain the generalized Hamilton-Jacobi-Bellman (HJB) Equation for the case $S = 2$, which corresponds to the "β, δ" model of quasi-hyperbolic discounting (Laibson 1997). Their methods are easily extended to obtain the generalized HJB equation for the case of arbitrary finite S. Let $T = S\varepsilon$, the amount of time (as distinct from the number of periods) during which the discount rate may be nonconstant. Taking the formal limit of the discrete time generalized HJB equation as $\varepsilon \to 0$, holding T constant, gives the generalized continuous time HJB equation when the discount rate is allowed to be any function of time for $0 \leq t \leq T$, and is constant after T. One then takes the formal limit of that equation as $T \to \infty$.

Ekeland and Lazrak (2010) take a different route to studying this problem. They begin with the continuous time problem with arbitrary discounting function $r(s)$. At any time t, the agent is allowed to choose a policy over $(t, t+\varepsilon)$, taking as given the decision rule that will be used after $t+\varepsilon$. They obtain the necessary and sufficient condition for this agent's problem and then take the limit as $\varepsilon \to 0$. The two approaches lead to the same generalized HJB equation. Karp (2007) interprets this equation as the standard HJB equation for a "fictitious" optimal control problem: solving one is equivalent to solving the other. In the case at hand, solving the fictitious control problem turns out to be easier and more transparent than solving the generalized HJB equation, and we proceed to do so in the next section.

3 Results

We first explain the methods used to obtain a MPE and then characterize the unique equilibrium.

3.1 Obtaining the MPE

Using Proposition 1 and Remark 1 of Karp (2007), we obtain the MPE to our problem by solving the necessary conditions to the optimal control problem

$$\begin{gathered} J(x_t) = \max \int_0^\infty e^{-r_\infty \tau} \left[\left(p - c(x_{t+\tau}) \right) h_{t+\tau} - K\left(x_{t+\tau} \right) \right] d\tau \\ \text{subject to } \dot{x}_s = f(x_s) - h_s,\ x_t \text{ given.} \end{gathered} \tag{5}$$

Denote $\chi(x)$ as a (not necessarily unique) MPE decision rule, and define $U(x) := (p - c(x))\chi(x)$ as the flow of payoff under this decision rule, given the state variable x. The function $K(x)$ is

$$K(x_t) = \int_0^\infty D(\tau) \left(r(\tau) - r_\infty \right) U\left(x_{t+\tau} \right)) d\tau, \tag{6}$$

where $x_{t+\tau}$ is the solution to equation (4) given initial condition x_t and given $h_{t+s} = \chi(x_{t+s})$ for $s \geq 0$. We refer to the optimization problem (5) and the definition (6) as the "fictitious control problem". We use the necessary conditions to this problem to obtain a MPE to the game.

The validity of this approach requires that the value function $J(x)$ and the function $K(s)$ are differentiable. We verify differentiability in Lemma 1 below.[2] We obtain a MPE by solving the necessary conditions to a control problem with constant discount

[2]Karp (2007) assumes at the outset that the policy rule $\chi(x)$ is differentiable, but that assumption is needed only later in his paper, not for Proposition 1 and Remark 1, which are all that we rely on. However, differentiability of the functions $J(x)$ and $K(x)$ are required. Similarly, Ekeland and Lazrak (2010) assume that the policy rule is differentiable, but an extension of their argument shows that in the current problem, differentiability of $\chi(x)$ is not required.

rate r_∞. The integrand in this control problem equals the integrand in the original game, minus the function $K(x)$. That function depends on the MPE decision rule, $\chi(x)$. In general, replacing the original game by the fictitious control problem does not seem to have advanced matters much, because it appears that we need to know the function $K(x)$ to solve the control problem, and $K(x)$ depends on the unknown MPE decision rule. In addition, in general we can not give an intuitive meaning to the function $K(x)$. For the problem at hand, however, there is a simple solution to the problem, and an intuitive interpretation of $K(x)$.

The simplicity arises because the fictitious control problem is linear in the control variable, harvest. For *any* policy rule that results in a differentiable $J(x)$ and $K(x)$, the optimal decision must be on either boundary, $h = 0$ or $h = \bar{h}$, unless a particular function (the "switching function"), defined below, vanishes. The linearity makes this problem tractable.

The asymptotic discount rate, r_∞, takes two possible values, depending on whether $\delta + \alpha < \sigma < \infty$ or $\sigma < \delta + \alpha$. We consider these two cases separately, because the parameter r_∞ is used to discount the payoff in the fictitious control problem, and it also appears in the definition of $K(x)$.

For $\delta + \alpha < \sigma < \infty$, the asymptotic discount rate is $r_\infty = \delta + \omega$. Some calculations establish

$$D(t)\left(r(t) - r_\infty\right) = -\alpha e^{-t(\sigma - g)},$$

which implies

$$-K\left(x_t\right) = \alpha \int_0^\infty e^{-\tau(\sigma-g)} U\left(x_{t+\tau}\right)) d\tau. \tag{7}$$

Here, $-K$ is an annuity, which if received in perpetuity and discounted at the birth rate α, equals the present discounted stream of the future payoff, discounted at $\sigma - g$, the altruistic discount rate minus the growth rate. The fictitious control problem includes this annuity in the flow payoff.

For $\sigma < \delta + \alpha$, $r_\infty = \sigma - g$. In this case,

$$D\left(t\right)\left(r\left(t\right) - r_\infty\right) = -\left(\sigma - \delta\right) e^{-t(\omega + \delta)},$$

which implies

$$-K\left(x_t\right) = \left(\sigma - \delta\right) \int_0^\infty e^{-\tau(\delta+\omega)} U\left(x_{t+\tau}\right)) d\tau. \tag{8}$$

Here, $-K$ is an annuity, which if received in perpetuity and discounted at the rate $\sigma - \delta$, equals the present discounted stream of the future payoff, discounted at the risk adjusted rate $\delta + \omega$. Again, this annuity is part of the flow payoff in the fictitious control problem. For $\sigma < \delta$, this annuity is negative, a fact that we discuss below.

3.2 Equilibrium results

In the problem with a constant discount rate, r this fishery problem has a well-known solution. It is optimal to set the harvest level at its maximum or minimum value ($\bar{h}$

or 0) in order to drive the stock of fish as quickly as possible to its steady state level, the solution to

$$r = f'(x) - \frac{c'(x)f(x)}{p - c(x)}. \tag{9}$$

In the interest of simplicity, we adopt

Assumption 1: For $0 < \delta - g \leq r \leq \delta + \omega$ there exists a unique solution to equation (9), decreasing in r.

Assumption 2: The growth function $f(x)$ is concave with $f(0) = 0$ and $f'(0) > 0$.

Assumption 3: The value of x below which profits are negative, defined as $x_{\min}$, is positive and $f(x_{\min}) - \bar{h} < 0$.[3]

Assumption 1 implies that in the standard constant discounting problem, a larger discount rate lowers the steady state stock, thereby lowering the steady state flow of profit. Assumption 2 excludes the possibility of "critical depensation", the situation where for sufficiently small initial conditions, the resource is doomed to extinction even in the absence of harvest. Assumption 3 means that although it is feasible to drive the stock below $x_{\min}$, it is never part of an equilibrium strategy to do so. Therefore, the non-negativity constraint on the stock is not binding.

The current value Hamiltonian for the fictitious control problem is

$$H = (p - c(x) - \psi)\, h - K(x) + \psi f(x)$$

where ψ is the current value costate variable, and the function $(p - c(x) - \psi)$ is known as the switching function. The costate equation is

$$\dot{\psi} = (r_\infty - f'(x))\, \psi + c'(x)h^* + K'(x), \tag{10}$$

where h^* is the optimal control. ("Optimal" for the fictitious control problem, or "equilibrium" for the sequential game.) Due to the linearity in h of the Hamiltonian, an optimal harvest rate must be on the boundary unless the switching function is 0. The harvest rate can be at an interior value for an interval of time (with positive measure) if and only if the switching function is identically 0 during that interval. Differentiating this identity with respect to time and using equations (4) and (10) imply that the switching function is identically 0 if and only if x is a solution to

$$r_\infty = f'(x) - \frac{c'(x)f(x) + K'(x)}{p - c(x)}. \tag{11}$$

Equations (9) and (11) have the same form, apart from the presence of $K'(x)$ on the right side of the latter.

The following proposition summarizes our main result

[3] If $f'(x) - \frac{c'(x)f(x)}{(p-c(x))f(x)}$ is a decreasing function, and if there exists a carrying capacity x^c at which $f(x^c) = 0 > f'(x^c)$, then Assumption (iii) implies Assumption (i).

Proposition 1 *We maintain Assumptions 1 - 3 and require $\sigma \neq \delta + \alpha$. (i) Within the class of pure strategy equilibria that generate differentiable value functions, the unique MPE to the game amongst the sequence of representative agents, is to follow a most rapid approach path (MRAP) to drive the stock of fish to a level x^*, and thereafter to maintain that stock by harvesting at rate $f(x^*)$:*

$$h^* = \begin{cases} 0 \text{ for } x < x^* \\ f(x^*) \text{ for } x = x^* \\ \bar{h} \text{ for } x > x^* \end{cases} \tag{12}$$

(ii) For $\sigma < \infty$, the steady state is the solution to equation (9) with $r = \delta - g$. (iii) For $\sigma = \infty$, the steady state is the solution to equation (9) with $r = \delta + \omega$.

Analysis of the fictitious control problem requires establishing that $K'(x)$ exists. Under the policy in equation (12) it is obvious that $K'(x)$ exists for $x \neq x^*$. We need only show that it exists (i.e. that the left and right derivatives are equal) at $x = x^*$, the point at which there is a discontinuous change in the harvest rate. We also need to evaluate that derivative. We have

Lemma 1 *Under the policy given in equation (12), for $\sigma < \infty$, the derivative $K'(x^*)$ is*

$$K'(x^*) = \begin{cases} \alpha(p - c(x^*)) \text{ for } \sigma > \delta + \alpha \\ \\ (\sigma - \delta)(p - c(x^*)) \text{ for } \sigma < \delta + \alpha \end{cases} \tag{13}$$

The lemma states a "smooth pasting" condition, a phenomena that appears in many contexts, e.g. stochastic control. The proof involves routine calculations, available on request. Note that, evaluated at x^* the annuity $-K$ is an increasing function of x if $\sigma < \delta$ and a decreasing function of x if $\sigma > \delta$. We return to this observation in Section 4.

An argument that parallels the proof of the lemma shows that the value function is differentiable. Therefore, the costate variable, ψ can be written as a continuous function of x, $\psi = \psi(x)$; the costate variable equals $J'(x)$.

3.3 Proof of the proposition

We first establish parts (i) and (ii). The Markov assumption means that at any value of the stock, the equilibrium harvest does not depend on whether the stock has approached this value from above or from below. In a model with a single state variable, pure strategy Markov equilibrium trajectories cannot cycle. In view of the linearity of the problem, harvest takes a boundary value unless x satisfies equation (11). We proceed under the hypothesis that the control rule is a MRAP of the form of equation (12), and we then verify this hypothesis.

For $\infty > \sigma > \delta + \alpha$, $r_\infty = \delta + \omega$. Using this equality and the first line of equation (13) implies that x^* is the solution to

$$\begin{aligned} \delta + \omega &= f'(x) - \frac{c'(x)f(x) + \alpha\left(p - c\left(x\right)\right)}{p - c(x)} \Longrightarrow \\ \delta + \omega - \alpha &= \delta - g = f'(x) - \frac{c'(x)f(x)}{p - c(x)}. \end{aligned} \tag{14}$$

For $\sigma < \delta + \alpha$, $r_\infty = \sigma - g$. Using this value of r_∞ and the second line of equation (13) implies that x^* is the solution to

$$\sigma - g = f'(x) - \frac{c'(x)f(x) + (\sigma - \delta)\left(p - c\left(x\right)\right)}{p - c(x)},$$

which reproduces equation (14). By Assumption 1, there is a unique solution to this equation.

We now confirm that the control rule must be the MRAP in equation (12) with x^* equal to the solution to equation (14). The hypothesis that the control rule is a MRAP is equivalent to the claim that the graph of the switching function, $(p - c(x) - \psi\left(x\right))$ is negative for $x < x^*$ and positive for $x > x^*$. A proof by contradiction establishes this claim.

First, in the neighborhood of the carrying capacity (defined as the value of $x > 0$ at which $f(x) = 0$), the switching function is not negative. If it were negative, then for fish stocks sufficiently close to the carrying capacity, it would be a MPE to extract nothing forever. That cannot be an equilibrium, because any deviation gives a higher payoff. Therefore, if for any $x > x^*$ the switching function is negative, it must cross the x axis again from below, at a point where $x > x^*$. That possibility violates Assumption 1, which imposes uniqueness.

The switching function cannot be positive in the neighborhood of $x = x_{\min}$, the largest stock value at which average profits are 0. If the switching function were positive in this neighborhood, then for values of x close to $x_{\min}$ the solution to the fictitious control problem would drive the stock of fish to $x_{\min}$ and then harvest at the rate that keeps it there, $f(x_{\min})$. This outcome results in a zero flow of profit at the steady state $x_{\min}$. Any deviation involving lower harvest for an interval of time results in positive profits. Therefore, the proposal to drive x to $x_{\min}$ can not solve the fictitious control problem, and hence is not a MPE. (An appendix, available on request, verifies this informal argument by showing that the trajectory that drives x to $x_{\min}$ violates a transversality condition.) Therefore, if the switching function is positive for any $x < x^*$ it must cross the x axis at some $x < x^*$, so that the switching function is negative in the neighborhood of $x_{\min}$. This multiple crossing violates Assumption 1.

If $\sigma = \delta$ then the original problem involves the constant discount rate $\delta - g$. The result is a standard problem, for which the solution is well-known: follow a MRAP until reaching x^*, the solution to equation (14). The only remaining case is where $\sigma = \infty$, where again we have a standard problem, but here the constant discount rate is $\delta + \omega$. The solution is to follow a MRAP to x^*, the solution to equation (9) with $r = \delta + \omega$.

This completes the proof.

4 Discussion

In optimal control models with constant discounting, the optimal policy depends, sometimes sensitively, on the discount rate. For example, in the case of climate change, the optimal carbon tax might change by a factor of 10 as the discount rate varies between levels that different economists consider reasonable (Stern 2007), (Nordhaus 2007). A large part of the debate about climate change, and about environmental policy more generally, turns on the degree to which we think it appropriate to value utility flows of distant generations.

A standard approach uses the social discount rate to value these flows. The social discount rate is the sum of the pure rate of time preference used to evaluate our own future utility flows, and a term that depends on future consumption growth (set equal to 0 in our setting). The problem with this approach is that our own future utility flows and those of the not-yet born belong to different categories; there is no reason that we should apply the same degree of impatience to discount them. An overlapping generations model, in which current generations might discount their own future utility flows and those of successive generations at different rates avoids this conflation of distinct categories, but typically leads to the problem of time inconsistency of optimal programs. Because it is unreasonable for people living today to believe that they can choose policies that will be in effect generations from now, we are led to replace the optimization problem usually used to study resource issues with a sequential game amongst successive overlapping generations.

If we happen to evaluate our own and future generations' utility flows using the same discount rate (i.e. if $\sigma = \delta$), then the discount rate for each of the infinite succession of planners in this game is constant, equal to $\delta - g$, the pure rate of time preference minus the population growth rate. In this case, the sequential game and the 'benevolent' social planner's problem are equivalent. At the other extreme, if we have no concern for future generations ($\sigma = \infty$), then the equilibrium to the sequential game is identical to the solution for the selfish social planner; it involves the risk adjusted discount rate $\delta + \omega$, the pure rate of time preference plus the mortality rate. These results hold in the case of a pure public good. The problem with a private (rival) good is isomorphic to the public good problem, and requires only trivial changes.

Neither of these two extremes ($\sigma = \delta$ or $\sigma = \infty$) is particularly compelling. It is hard to believe that we are completely indifferent to the utility of distant generations. However, the choice $\sigma = \delta$ is arbitrary, in view of fact that own-utility and other's-utility belong to different categories. Certainly the choice $\sigma = \delta$ does not have a convincing ethical basis. Why should the life-time stream of consumption of someone born in t years be worth only the fraction $e^{-\delta t}$ of the consumption stream of someone born today? The 'benevolent' social planner values the stream of consumption of the agent born today more than that of the agent born in t years. Because these are different

agents, not the same agent at different points in time, this 'benevolent' planner earns her inverted commas.

The striking result is that the unique equilibrium (within the class giving rise to differentiable value functions) to the sequential game replicates this 'benevolent' social planner. Provided that agents have some concern for the future, the degree of their concern is irrelevant. This outcome is better for future generations compared to the selfish outcome (where $\sigma = \infty$), but for the reason discussed above it can not be viewed as a particularly ethical outcome. Thus, from the standpoint of future generations, the news is mixed: even large changes in the extent to which current agents value future generations' utility flows have no effect on those flows.

The fact that the current generation's problem is linear in its control *regardless of the Markovian policies that future generations use* means that each agent's action is always at the boundary of its feasible set, unless the state variable takes a particular value. In the one-state variable model, this value equals the steady state. Mild assumptions on the primitives of the model insure that this value is unique. Thus, the issue of multiplicity of Markov equilibria does not arise here.

We also used the fact that the MPE satisfies the necessary conditions to a fictitious control problem, one that includes an annuity $(-K)$ that is a function of the current state variable. The annuity depends on actions that future generations take, and in equilibrium those actions depend on the current state via its effect on future state variables. Evaluated at the steady state, this annuity is a positive and decreasing function of x if $\sigma > \delta$ and it is a negative and increasing function if $\sigma < \delta$. Thus, if agents are relatively selfish $(\sigma > \delta)$ this fictitious control problem assigns a positive but decreasing amenity value to the state, arising from the actions of subsequent agents. The reverse holds if agents are relatively altruistic $(\sigma < \delta)$. These contrasting effects just offset the differences in the constant discount rate used in this fictitious control problem. They cause the solution to be independent of σ.

We emphasized the importance of the fact that the problem is linear in the control. Our results do not hold for more general classes of problems. Given the difficulty of obtaining general results for this type of game, it is instructive of build up our intuition using special cases. This paper contributes to that ongoing effort.

A Derivation of discount function

The $N_0 e^{gt}\alpha dt$ agents born during the interval $(t, t+dt)$ die at rate ω and they discount their own utility at rate δ so their present discounted value of their selfish payoff from the program $u(s)$ is (with $N \equiv N_0$)

$$Ne^{gt}\alpha dt \int_t^\infty e^{-\delta(s-t)} e^{-\omega(s-t)} u\left(s\right) ds. \tag{15}$$

The representative agent alive at time 0 discounts the payoff of generations born in the future at rate σ, so this representative agent's altruistic value of the selfish utility received by the agents born during $(t, t+ds)$ is

$$Ne^{(g-\sigma)t}\alpha dt \int_t^\infty e^{-\delta(s-t)} e^{-\omega(s-t)} u\left(s\right) ds.$$

The current representative agent's value of the selfish utility received by all agents who will be born in the future is therefore

$$N\alpha \int_0^\infty e^{(g-\sigma)t} \left(\int_t^\infty e^{-\delta(s-t)} e^{-\omega(s-t)} u\left(s\right) ds \right) dt =$$
$$N\alpha \int_0^\infty e^{-(\omega+\delta)s} u\left(s\right) \left(\tfrac{e^{(\alpha-\sigma+\delta)s}-1}{\alpha-\sigma+\delta} \right) ds.$$

The equality follows from changing the order of integration and simplifying.

The current representative agent discounts the future utility of those currently alive at rate δ and knows that these agents die at rate ω, so her risk-adjusted discount rate for them is $\delta + \omega$. Her (selfish) valuation of their lifetime welfare is therefore

$$N \int_0^\infty e^{-(\delta+\omega)s} u(s) ds.$$

The representative agent's total welfare is the sum of welfare attributed to the utility of the agents who will be born in the future (the altruistic component), and of the agents who are currently alive (the selfish component):

$$N \int_0^\infty \left[e^{-(\omega+\delta)s} \alpha \left(\frac{e^{(\alpha-\sigma+\delta)s} - 1}{\alpha - \sigma + \delta} \right) + e^{-(\delta+\omega)s} \right] u\left(s\right) ds.$$

The discount factor for the time t utility flow is

$$D(t) := e^{-(\omega+\delta)t} \alpha \left(\frac{e^{(\alpha-\sigma+\delta)t} - 1}{\alpha - \sigma + \delta} \right) + e^{-(\delta+\omega)t}. \tag{16}$$

Simplifying this equation yields equation (1). Note that the right side of equation (16) is positive and it converges to 0 as $t \to \infty$, because $\omega + \delta - (\alpha - \sigma + \delta) = \sigma - (\alpha - \omega) = \sigma - g > 0$, where the inequality holds by assumption.

B Private versus public goods

Our model of discounting provides an "exact" description of paternalistic altruism in the case where the population size is constant, or if population is growing but the flow (here, fishing profits) is a non-rival public good. The model can be viewed only as an approximation if g is non-zero and the flow is not a pure public good. Clearly, profits are not a pure public good, unless they happen to be used to finance a public good. This appendix explains in greater detail the meaning of our discounting model in the text, when $g \neq 0$. It also provides results for an alternative model.

Our discounting function assumes that if agents have some concern for future generations ($\sigma < \infty$), the current generation values future profit flows more highly, the larger is the number of agents in the future, i.e., the larger is g. It also assumes that if agents have no concern for future generations ($\sigma = \infty$), then their evaluation of future profit flows is independent of g. These assumptions are non-controversial when the flow is a pure public good. But the division of a profit flow is a zero sum game. The text defends our modeling approach on the ground that it captures the idea that a particular flow of aggregate income should be more valuable if there are more people who need to use it.

Here we explore the opposite view. Our model assumes that utility is linear in profits; we need that assumption in order to have a model in which the payoff is linear in the control variable, harvest. Suppose that profits at a point in time are divided amongst all agents alive equally, and that aggregate utility equals the sum of utility of all agents. In that case, the value of an aggregate profit flow $u(t)$ given population $N(t)$ is $\frac{u(t)}{N(t)}N(t)$, i.e. it is independent of $N(t)$. However, the present value of that future flow, to an agent today depends on the composition of the population at time t, unless she discounts her own and future generations welfare at the same rate: $\sigma = \delta$.

Denote the population at time 0 as N, so the population at time t is Ne^{gt}. There are $Ne^{gt}\alpha dt$ agents born during the interval $(t, t+dt)$. At time s each of these obtains the flow $\frac{u(s)}{Ne^{g(t+s)}}$. The aggregate selfish life-time welfare of these agents is

$$Ne^{gt}\alpha dt \int_t^\infty e^{-\delta(s-t)} e^{-\omega(s-t)} \frac{u\,(s)}{Ne^{g(t+s)}} ds. \tag{17}$$

Expression (15) and (17) differ because the latter equation assumes that each agent gets the same share of the flow and that the sum of shares equals 1; that is, the flow is a private rather than a public good.

The representative agent (who aggregates the preferences of her generation) alive at time 0 discounts the selfish payoff of generations born in the future at rate σ, so this representative agent's altruistic value of the selfish utility received by the agents born

during $(t, t+ds)$ is[4]

$$e^{-\sigma t} N e^{gt} \alpha dt \int_t^{\infty} e^{-\delta(s-t)} e^{-\omega(s-t)} \frac{u(s)}{N e^{g(t+s)}} ds.$$

The current representative agent's altruistic value of the direct utility received by all agents who will be born in the future is therefore

$$\begin{array}{c} \int_0^{\infty} e^{-\sigma t} N e^{gt} \alpha dt \left(\int_t^{\infty} e^{-\delta(s-t)} e^{-\omega(s-t)} \frac{u(s)}{N e^{g(t+s)}} ds \right) \\ = \alpha \int_0^{\infty} e^{-(g+\omega+\delta)s} u(s) \left(\int_0^{s} e^{-(\sigma-\delta-\omega)t} dt \right) ds \\ = \int_0^{\infty} \alpha \frac{1-e^{-(\sigma-\omega-\delta)s}}{\sigma-\omega-\delta} e^{-(g+\omega+\delta)s} u(s) ds \end{array}$$

The first equality follows from changing the order of integration and simplifying. Note that $\frac{1-e^{-(\sigma-\omega-\delta)s}}{\sigma-\omega-\delta} > 0$ for $s > 0$, so the discount factor used to calculate the altruistic component of welfare is always positive (as required by a sensible model). This discount factor also converges to 0 as $t \to \infty$ if and only if both $g+\omega+\delta > 0$ and $\sigma - \omega - \delta + g + \omega + \delta = \sigma + g > 0$ are true. Thus, convergence requires that the size of the population is not falling too quickly.

The current representative agent's aggregation of her generation's preferences attributed to their selfish welfare is

$$N \int_0^{\infty} e^{-(\delta+\omega)s} \frac{u(s)}{e^{gs} N} ds = \int_0^{\infty} e^{-(\alpha+\delta)s} u(s) ds.$$

Here also the discount factor is positive and converges to 0 because $\alpha + \delta > 0$. The total welfare is the sum of the altruistic and the selfish components:

$$\int_0^{\infty} \left(\alpha \frac{1-e^{-(\sigma-\omega-\delta)s}}{\sigma-\omega-\delta} e^{-(\delta+\omega+g)s} + e^{-(\alpha+\delta)s} \right) u(s) ds.$$

The discount factor is

$$\begin{array}{c} D(t) = \alpha \frac{1-e^{(-\sigma+\omega+\delta)t}}{\sigma-\omega-\delta} e^{-(\delta+\omega+g)t} + e^{-(\alpha+\delta)t} \\ \frac{-\alpha}{\sigma-\omega-\delta} e^{-(\sigma+g)t} + \left(1 + \frac{\alpha}{\sigma-\omega-\delta}\right) e^{-(\alpha+\delta)t} \end{array} \tag{18}$$

This discount factor is the sum of the discount factors applied to the altruistic and the selfish components of welfare, each of which is positive. This discount factor converges if $\sigma + g > 0$. Note that with a public good, convergence of payoffs requires that the growth rate not be too large ($g < \sigma$) whereas with a private good, convergence holds if $g > -\sigma$.

[4]This expression equals the *aggregate* value that *all* agents alive at time 0 attribute to the selfish welfare of agents born during $(0, t)$. It is not the value that a single agent alive at time 0 attributes to this welfare; if it were, we would have to multiply it by N to obtain the aggregate value.

In order to make this model of discounting easily comparable to the one that we study in the text, define

$$\begin{aligned} \tilde{\sigma} \equiv \sigma + 2g \Longrightarrow \sigma = \tilde{\sigma} - 2g \\ \tilde{\delta} \equiv \delta + g \Longrightarrow \delta = \tilde{\delta} - g. \end{aligned} \tag{19}$$

With these definitions, we can rewrite equation (18) as

$$\begin{aligned} D(t) &= \frac{-\alpha}{\tilde{\sigma}-2g-\omega-\left(\tilde{\delta}-g\right)} e^{-(\tilde{\sigma}-2g+g)t} + \left(1 + \frac{\alpha}{\tilde{\sigma}-2g-\omega-\left(\tilde{\delta}-g\right)}\right) e^{-\left(\tilde{\delta}-g+\alpha\right)t} \\ &= \frac{-\alpha}{\tilde{\sigma}-\alpha-\tilde{\delta}} e^{-(\tilde{\sigma}-g)t} + \frac{\tilde{\sigma}-\tilde{\delta}}{\tilde{\sigma}-\alpha-\tilde{\delta}} e^{-\left(\tilde{\delta}+\omega\right)t}. \end{aligned} \tag{20}$$

Comparing the last line of equation (20) with equation (1) shows that the two are equivalent, except that the latter involves the "adjusted" preference parameters $\tilde{\delta}$ and $\tilde{\sigma}$. This fact allows us to use our earlier results to write down the equilibrium in the new model of discounting. We summarize the results, for the alternative discounting model presented in this appendix:

Corollary 1 *(i) Given the preference parameters δ and σ and the growth parameters α and ω, if $\sigma < \infty$, the unique MPE (within the class that generate differentiable value functions) is the MRAP given by equation (12) with the steady state x^* given by equation (9), where $r = \tilde{\delta} - g = \delta$. If $\sigma = \infty$, the value of r that determines the steady state is $\tilde{\delta} + \omega = \delta + \alpha$. (ii) If $g > 0$, the discounting model in this appendix leads to a lower steady state, one associated with a higher discount rate, r, compared to the discounting model in the text. If $g < 0$, the new discounting model leads to a higher steady state. For $g = 0$, the two models are equivalent.*

References

Andreoni, J. (1989): "Giving with impure altruism: applications to charity and Ricardian equivalence," *Journal of Political Economy*, 97, 1447–1458.

Calvo, G., and M. Obstfeld (1988): "Optimal time-consistent fiscal policy with finite lifetimes," *Econometrica*, 56, 411–32.

Clark, C. (2005): *Mathematical Bioeconomics: the optimal management of renewable resources*. Wiley Interscience, 2'nd edition.

Clark, C. W., and G. Munro (1975): "Economies of fishing and modern capital theory: a simplified approach," *Journal of Environmental Economics and Management*, 2, 92–106.

Dyck, A., and U. R. Sumailila (2010): "Economic impact of ocean fish populations in the global fishery," *Journal of Bioeconomics*, 12, 227–43.

Ekeland, I., and A. Lazrak (2010): "The golden rule when preferences are time inconsistent," *Mathematical and Financial Economics*, 4(1).

Harris, C., and D. Laibson (2001): "Dynamic Choices of Hyperbolic Consumers," *Econometrica*, 69(5), 935–957.

Karp, L. (2007): "Non-constant discounting in continuous time," *Journal of Economic Theory*, 132, 557 – 568.

Laibson, D. (1997): "Golden eggs and hyperbolic discounting," *Quarterly Journal of Economics*, 62, 443–78.

Nordhaus, W. D. (2007): "A Review of the Stern Review on the Economics of Climate Change," *Journal of Economic Literature*, (3), 686 – 702.

Ray, D. (1987): "Nonpaternalistic intergenerational altruism," *Journal of Economic Theory*, (41), 112–132.

Saez-Marti, M., and J. Weibull (2005): "Discounting and altruism to future decision- makers," *Journal of Economic Theory*, (122), 254–66.

Spence, M., and D. Starrett (1975): "Most Rapid Approach Paths in Accumulation Problems," *International Economic Review*, 16, 388–403.

Stern, N. (2007): *The Economics of Climate Change*. Cambridge University Press.

Sumaila, U., and C. Walters (2005): "Intergenerational discounting: a new intuitive approach," *Ecological Economics*, 52, 135–142.

Sumailia, U., W. Cheung, V. Lam, D. Pauly, and S. Herrick (2011): "Climate change impacts on the biophysics and economics of world fisheries," *Nature Climate Change*, doi:10.1038/nclimate1301.

United Nations (2005): *Millennium Ecosystem Assessment*. United Nations, http://www.maweb.org/en/index.aspx.

C Referees' appendix

This appendix is available on request, but not intended for publication

C.1 Proof of Lemma

We provide details for the case $\sigma > \delta + \alpha$, where $r_\infty = \delta + \omega$. Define $P(\epsilon)$ as the amount of time it takes the state variable to move from $x^* + \varepsilon$ to x^* using the control rule in equation (12); ε may be either positive or negative, but is small. With this definition, the control rule (12), and equation (7), we have

$$-K(x^* + \varepsilon) = \alpha \left(h^* \int_0^{P(\varepsilon)} e^{-\tau(\sigma - g)} \left(p - c\left(x_{t+\tau}\right)\right) d\tau + \left(p - c\left(x^*\right)\right) f\left(x^*\right) \int_{P(\varepsilon)}^{\infty} e^{-\tau(\sigma-g)} d\tau \right) . \quad (21)$$

The first integral on the right side is the contribution to $-K$ of the flow payoff during the approach to the steady state value x^*; the second integral equals the contribution due to the steady state flow payoff.

We want to show that the left and right derivatives are equal, i.e. $\lim_{\varepsilon \to 0} \frac{dK(x^* + \varepsilon)}{d\varepsilon}$ has the same value regardless of whether ε approaches 0 from above or below. Consider the case where $\varepsilon > 0$, so $h^* = \bar{h}$ over $[0, P)$. Integrating equation (4) we have $-\varepsilon = \int_t^{t+P} dx = \int_0^P \left(f\left(x_{t+\tau}\right) - \bar{h}\right) d\tau$. (The first term is $-\varepsilon$ because here $\varepsilon > 0$, so $x_{t+P} = x^* < x^* + \varepsilon = x_t$.) In this case,

$$\frac{dP}{d\varepsilon} = \frac{-1}{f(x^*) - \bar{h}}. \quad (22)$$

Using equations (21) and (22) we have

$$\lim_{\varepsilon \to 0^+} \frac{dK(x^* + \varepsilon)}{d\varepsilon} = \alpha \left(p - c\left(x^*\right)\right) \left(\bar{h} - f\left(x^*\right)\right) \frac{1}{f(x^*) - \bar{h}} = \alpha \left(p - c\left(x^*\right)\right).$$

Now consider the case where $\varepsilon < 0$, so $h^* = 0$ over $[0, P)$. Here, $\varepsilon = \int_t^{t+P} dx = \int_0^P f\left(x_{t+\tau}\right) d\tau$, and

$$\frac{dP}{d\varepsilon} = \frac{1}{f(x^*)}. \quad (23)$$

Using equation (21) and (23) we have

$$\lim_{\varepsilon \to 0^-} \frac{dK(x^* + \varepsilon)}{d\varepsilon} = -\alpha \left(p - c\left(x^*\right)\right) \left(0 - f\left(x^*\right)\right) \frac{1}{f(x^*)} = \alpha \left(p - c\left(x^*\right)\right).$$

Thus, the left and right derivatives are equal, as shown in the first line of equation (13).

The argument for $\sigma < \delta + \alpha$ parallels the above. In this case, using the control rule (12), and equation (8), we have

$$-K\left(x^* + \varepsilon\right) = (\sigma - \delta) \left(h^* \int_0^{P(\varepsilon)} e^{-\tau(\delta + \omega)} \left(p - c\left(x_{t+\tau}\right)\right) d\tau + \left(p - c\left(x^*\right)\right) f\left(x^*\right) \int_{P(\varepsilon)}^{\infty} e^{-\tau(\delta+\omega)} d\tau \right).$$

Equation (22) still applies for $\varepsilon > 0$ and equation (23) for $\varepsilon < 0$. We have

$$\lim_{\varepsilon \to 0^+} \frac{dK(x^*+\varepsilon)}{d\varepsilon} =$$
$$(\sigma - \delta)(p - c(x^*))\left(\bar{h} - f(x^*)\right)\frac{1}{f(x^*)-\bar{h}} = (\sigma - \delta)(p - c(x^*)).$$

A similar argument shows that the left derivative $\lim_{\varepsilon \to 0^-} \frac{dK(x^*+\varepsilon)}{d\varepsilon}$, has the same value, shown in the second line of equation (13).

C.2 The transversality condition

Here we confirm that a trajectory that drives the resource to the point where it is not economically viable and thereafter keeps it at that level, is not an equilibrium. Define T as the date at which the resource reaches $x_{\min}$ in this candidate. At T the fictitious control problem effectively ends; there is no scrap value, so the continuation payoff at T is 0. In addition, $K(x_{\min}) = 0$ from the definition of $K(S)$ and the fact that the equilibrium flow payoff for $t > T$ is identically 0 in this candidate. In the fictitious control problem, a necessary condition for a program that drives the stock to $x_{\min}$ is that the Hamiltonian vanish at T:

$$\begin{aligned} H(T) &= \left[(p - c(x) - \psi)h - K(x) + \psi f(x)\right]_{|x=x(T)=x_{\min}} \\ &= \left[(p - c(x) - \psi)\bar{h} - K(x) + \psi f(x)\right]_{|x=x(T)=x_{\min}} = 0 \Longrightarrow \\ \psi(T)\left(f(T) - \bar{h}\right) = 0 &\Longrightarrow \psi(T) = 0 \Longrightarrow p - c(x) - \psi(T) = 0. \end{aligned} \tag{24}$$

(An obvious abuse of notation replaces the arguments $x_T = x_{\min}$ by T.) The third line of equation (24) follows from the fact that $K(x_{\min}) = 0$ and Assumption (iii) that states $f(x_{\min}) - \bar{h} < 0$. Thus, the switching function is 0 at $x = x_{\min}$. In order for the hypothesized trajectory to be optimal, the switching function must be positive for larger values of x. That is, the switching function must approach 0 from above, as $t \to T$. Consequently, the time derivative of the switching function must be non-positive at $t = T$.

We need to know whether the switching function, $\pi(S) - \psi$, approaches 0 from above or below as $t \to T$. Consider the case where $\sigma > \delta + \alpha$ where $r_\infty = \delta + \omega$. Using equation (7),

$$K'(x) = \frac{(\sigma - g)K + \alpha(p - c(x)h)}{f(x) - h}.$$

Substitute this equation into the costate equation (10) to write the time derivative of the switching function, on the candidate equilibrium, in the neighborhood of T (where $h = \bar{h}$):

$$\frac{d(p-c(x)-\psi)}{dt} =$$
$$-c'(x)\left(f(x) - \bar{h}\right) - \left[(\delta + \omega - f'(x))\psi(x) + c'(x)\bar{h} + \frac{(\sigma-g)K+\alpha(p-c(x)h)}{f(x)-h}\right]$$

Evaluating the right side of this equation at $x = x_{\min}$, the right side simplifies to $-c'(x_{\min}) f(x_{\min}) > 0$. This inequality is our contradiction, because our hypothesis requires that the time derivative of the switching function is non-positive at $x = x_{\min}$. A parallel argument deals with the case where $\sigma < \delta + \alpha$.