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Equilibrium of Incomplete European Option Markets

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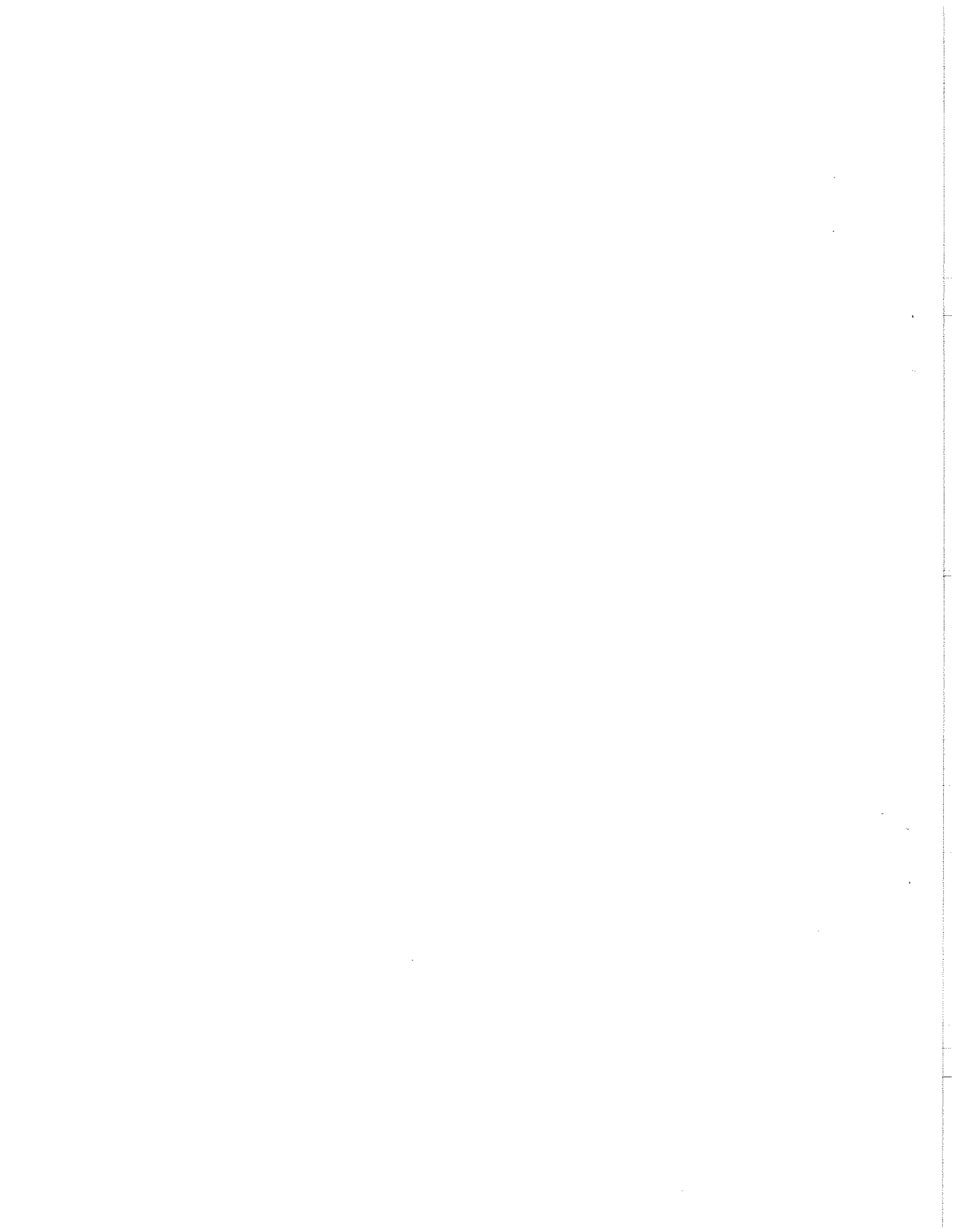
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Key words: incomplete European option markets, at the money strike prices, generic existence of competitive equilibria.

Abstract

We study a two period economy with commodity futures contracts and European call and put options written on one of the futures contracts. The strike prices of these options are set endogenously by an options exchange to be at the money, meaning the option strike prices are chosen to equal the price of the underlying commodity futures contract. We prove that, in the space of initial endowments, competitive equilibria generically exist.

JEL Classification: 021, 026



I. INTRODUCTION

The canonical Arrow-Debreu model of complete contingent commodity markets is not meant to be descriptive of actual economies. Neither do Arrow securities [1], that pay out a dollar in just one state of nature, exist in practice. It is, therefore, important to study more realistic asset market structures. Such an inquiry has motivated the recent study of General Equilibrium of Incomplete asset markets (GEI) models. For an introduction to this literature, see Duffie [3]; Duffie, Shafer, Cass, Magill, Quinzii, and Geanakoplos [5]; Geanakoplos [7]; Geanakoplos, Magill, and Shafer [8]; Geanakoplos and Polemarchakis [9]; or Marimon [18]. GEI models study the behavior of an economy with an incomplete set of primitive assets. Primitive assets have payoffs that are not functions of other asset prices in an economy. In light of a counterexample to the existence of competitive equilibria due to Hart [12], current models (of real as opposed to nominal assets) focus on demonstrating GEI exist for all but an exceptional set of economies. Such genericity theorems are statements about most economies. These models can be easily generalized to include derivative assets whose payoffs depend on the price of a primitive asset, so long as that dependence is linear. An example of a linear derivative asset is a (bond) futures contract. There have been few studies extending these results to nonlinear derivative assets, which are those assets with

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payoffs which are nonlinear functions of an underlying asset's price. The most well-known example of a nonlinear derivative asset is a European option.¹ Adding European options to an economy without completing asset markets causes problems for even generic existence. Polemarchakis and Ku [19] have constructed a counterexample to generic existence of GEI in the case of exogenously determined strike prices for European call and put options. The reason for the failure of generic existence in their example is that an open neighborhood exists in the space of initial endowments and exogenously given strike price for which either the call or put option pays out zero in every state.

In this paper, we study the simplest possible two period general equilibrium model in order to demonstrate our main point. The asset structure consists of these assets traded in the initial period: two commodity futures contracts and two European options written on one of those commodity futures. The futures pay out commodities in the later period. We overcome the above difficulty by proving that GEI exist for generic endowments when the strike prices, which are sufficient to describe the asset structure for European options are not exogenous parameters, but instead are chosen endogenously by an option exchange to be at the money. This means the strike price of an option equals the initial period's endogenous price of the commodity futures contract. A central feature of actual practice by option exchanges is that option strike prices are set to be at (actually, close to) the money. This feature of our model permits us to prove the existence of competitive equilibria for generic initial endowments. In our model, in the space of initial endowments, generically an option is never out of the money for all states of nature.

Technically, by a no-arbitrage condition, appropriately normalized futures prices will satisfy a martingale property, for a set of (pseudo) probabilities or an equivalent martingale measure. (For the martingale

characterization of asset prices that admit no arbitrage, see Duta and Polemarchakis [6], Harrison and Kreps [11], and Huang and Litzenberger [14].) This means the initial price of a futures contract is a weighted average of its payoffs across states. When an option strike price is at the money, it will lie inside an interval, that is bounded by the highest and lowest payoffs of that futures contract across states. This interval will be nondegenerate provided those payoffs are not the same across states. We show that for generic initial endowments, this is true. This guarantees option payoffs are not zero for all states. Thus, in a model with at the money European options, we prove generically the asset return matrix will not drop rank and this implies the existence of competitive equilibria.

This paper offers a different resolution to Polemarchakis and Ku's [19] counterexample than Krasa [16], who shows the probability of an economy having an equilibrium converges to one with increasing variation in the aggregate initial endowment, while maintaining the exogenously given strike price assumption. Our model only requires option strike prices to be set at the money. We will prove that generically, there is enough variation in underlying commodity prices to ensure the existence of competitive equilibria. Our model also differs from another model of European options by Krasa and Werner [17] as their options are written on nominal assets and the option strike prices are given exogenously, and varied by renormalizing prices. While, our model concerns real numeraire options as payoffs are in terms of a numeraire good, their model concerns nominal options.

The rest of this paper is organized as follows. Section two presents our model and defines a competitive equilibrium. Section three defines two related concepts: a pseudo-equilibrium and a regular pseudo-equilibrium. Section four presents the main results. Section five offers concluding remarks. An appendix provides the proofs of all results.

II. MODEL

Consider a pure exchange economy in which there are G ($g = 1, \dots, G$) physical goods; two periods: 0 and 1; and S states of nature in period one. There are $N = G(1 + S)$ spot commodity markets, G spot ones in period zero, and G spot ones in each of the S states of nature in period one. There are H ($h = 1, \dots, H$) households, each with a (column) vector of initial commodity endowments, $e^h = (e^h(0), e^h[1]) \in \mathbb{R}_{++}^N$, where $e^h[1]$ has components $e^h(s)$. Let $e = (e^1, \dots, e^H)$. We denote households' demands for goods by a (column) vector $x^h = (x^h(0), x^h[1])$, where $x^h[1]$ has components $x^h(s)$. We denote the (row) vector of spot commodity prices by $p = (p(0), p[1]) \in \mathbb{R}^N$, where $p[1]$ has components $p(s)$. We assume that households possess conditional perfect foresight (or rational expectations) in the sense of knowing the entire function $p(s)$, but not which s will prevail. Households are also assumed to have strictly monotone and strictly convex differentiable preferences satisfying the usual boundary condition (Debreu's [2] smooth preference assumptions).

The asset structure of this economy is as follows. There are two commodity futures contracts, written on, without loss of generality, the first two commodities. Each commodity futures contract pays out a unit of that commodity in each state of nature in period one. Specific examples of commodities that futures contracts are traded on in practice include gold, orange juice, soybean, and wheat. Futures contracts are traded in period zero. We define the liquidation value of the j th commodity futures contract f in state s to be $v_{sj} = p_j(s)$, the price of commodity j in that state. The S by 2 matrix $V_f(p)$ has typical element v_{sj} . We denote commodity futures prices in period zero by a row vector $q_f \in \mathbb{R}_{++}^2$. Also, there are two European

options written on, without loss of generality, the first commodity futures contract. The common strike price of both options is denoted by k .

All prices (of goods and assets) in period zero are relative to the price of the numeraire asset: commodity futures two, so $q_{f2} = 1$. In period one, since commodity futures two expires, we use as numeraire, the liquidation value of that contract, namely commodity two's state-dependent price; so for all s , $p_2(s) = 1$. This means that for all s , $v_2(s) = 1$. For example, we can think of commodity two as being gold and period one commodity prices and option payoffs being expressed in terms of gold. The option strike price in period one is also relative to the liquidation value of the gold futures contract, namely the state-dependent price of gold. All period zero prices, be they of commodities or other assets, are expressed in terms of period zero gold futures prices. Thus, we define European option payoffs in terms of the liquidation value of the first commodity futures contract:

DEFINITION. A European call option in state s , for $s = 1, \dots, S$ pays off

$$v_c(p_1(s)) = \max [0, p_1(s) - k].$$

DEFINITION. A European put option in state s , for $s = 1, \dots, S$ pays off

$$v_p(p_1(s)) = \max [0, k - p_1(s)].$$

We assume incompleteness of asset markets, meaning that we assume sufficiently many states exist so that $S > 4$ (actually only $S > 3$ is required since by the put-call parity theorem, any one of our four assets can be replicated by a portfolio of the remaining three). We define a vector of European call and put option prices by $q_E = (q_c, q_p)$. We define the vector of asset prices to be $q = (q_E, q_f)$. A household's portfolio choice is given by θ^h

$= (\theta^h_E, \theta^h_f) \in \mathbb{R}^4$, with $\theta^h_E = (\theta^h_c, \theta^h_p)$ and $\theta^h_f = (\theta^h_{f1}, \theta^h_{f2})$. An allocation for the economy is given by (x, θ) , where x has components x^h and θ has components θ^h .

In our model, an economy is parameterised only by initial endowments, $e \in \mathbb{R}^{HN}_{++}$. This differs from either the model of Polemarchakis and Ku [19] or that of Krasa [16], in which an economy is parameterised not only by initial endowments, but also by exogenously given strike prices. In our model, the strike price in an economy is not exogenously given, but instead will endogenously be set at the money. Notice that we also hold utility functions u^h fixed.

DEFINITION. A competitive equilibrium for the economy given by $e \in \mathbb{R}^{HN}_{++}$ is a 4-tuple, $(x, \theta, p, q) \in \mathbb{R}^{HN}_+ \times \mathbb{R}^{4H} \times \mathbb{R}^N_+ \times \mathbb{R}^4_+$, consisting of an allocation and prices for commodities and assets, where $(x^h, \theta^h) = \arg \max u^h(x^h)$ subject to:

$$\sum_h x^h = \sum_h e^h, \quad (2.1)$$

$$\sum_h \theta^h = 0, \quad (2.2)$$

$$p(0)[x^h(0) - e^h(0)] + q_c \theta^h_c + q_p \theta^h_p + q_{f1} \theta^h_{f1} + \theta^h_{f2} = 0; \quad (2.3)$$

and for all $s = 1, \dots, S$;

$$p(s)[x^h(s) - e^h(s)] = v_c(p_1(s)) \theta^h_c + v_p(p_1(s)) \theta^h_p + p_1(s) \theta^h_{f1} + \theta^h_{f2}. \quad (2.4)$$

We define $V(p)$ to be the S by 4 dimensional matrix of asset returns with typical element $(v_c(p_1(s)), v_p(p_1(s)), p_1(s), 1)$. Also, we define the $S+1$ by 4 dimensional matrix $W(p, q)$, with the first row of $W(p, q)$ being the vector $-q$; and the next S rows of $W(p, q)$ being $V(p)$. Then, we can rewrite

equations (2.3) - (2.4) as $p(x^h - e^h) = W(p, q)\theta^h$. Define the subspace of income transfers in \mathbb{R}^{S+1} generated by the columns of $W(p, q)$ as follows, $\langle W(p, q) \rangle = \{ \tau \in \mathbb{R}^{1+S} \mid \exists \theta \in \mathbb{R}^4 \text{ such that } \tau = W\theta \}$. Finally, define the (dual) orthogonal subspace of state prices: $\langle W(p, q) \rangle^\perp = \{ \alpha \in \mathbb{R}^{1+S} \mid \alpha W = 0 \}$.

We introduce the concept of no-arbitrage asset prices:²

DEFINITION. q is a no-arbitrage vector of commodity futures prices and commodity futures option prices if there does not exist a portfolio $\theta \in \mathbb{R}^M$ that yields a semipositive return $W(p, q)\theta \geq 0$.

Clearly, q is an equilibrium price for assets only if q is a no-arbitrage vector of asset prices. The following lemma characterizes a no-arbitrage asset price vector by showing the existence of positive state prices, β .

LEMMA. If q is a no-arbitrage asset price, then $\exists \beta = (\beta_0, \beta_1, \dots, \beta_S) \in \mathbb{R}_{++}^{1+S}$, such that:

$$q_E = \sum_{s=1}^S \beta_s v_E(p_1(s)) \text{ for } E = c \text{ or } p, \quad (2.5)$$

$$q_{fj} = \sum_{s=1}^S \beta_s p_j(s) \text{ for } j = 1 \text{ or } 2. \quad (2.6)$$

Notice that $\sum_{s=1}^S \beta_s = 1$ from equation (2.6) because $q_{f2} = 1$ and for all s , $p_2(s) = 1$. This means the price of a futures contract is equal to the conditional expectation of its payoffs across states under the probability vector β . These β_s coefficients form a set of pseudo-probabilities or an equivalent martingale measure, as explained in Harrison and Kreps [11].

If the strike price of the call option is chosen by the options exchange

to be the endogenous price of the first commodity futures contract, then $k = q_{f1}$ and $v_c(p_1(s)) = \max [0, p_1(s) - k] = \max [0, p_1(s) - \sum_{s=1}^S \beta_s p_1(s)]$.

By the Put-Call Parity Theorem, one of the four assets in our model is redundant. Since the gold futures contract is a numeraire asset, its price is already determined to be one. We consider only these three assets: the first commodity futures contract, the European call and European put options. So, we redefine $V(p)$ to be the S by 3 matrix of nonredundant asset returns with typical element $(v_c(p_1(s)), v_p(p_1(s)), p_1(s))$.

III. RELATED EQUILIBRIUM CONCEPTS

We introduce two concepts of equilibria besides the competitive one, namely that of pseudo-equilibria and regular pseudo-equilibria, which are used in the proof of generic existence of competitive equilibria.

First, we eliminate the asset prices by rewriting the period zero budget constraint (2.3) by using the no-arbitrage conditions (2.5) - (2.6):

$$p(0)[x^h(0) - e^h(0)] + \sum_j [\sum_s \beta_s p_j(s)] \theta_{fj}^h + \sum_E \sum_s \beta_s v_E(p_1(s)) \theta_E^h = 0.$$

Using the period one budget constraint (2.4), this can be rewritten:

$$p(0)[x^h(0) - e^h(0)] + \sum_s \beta_s \{p(s)[x^h(s) - e^h(s)]\} = 0.$$

We follow Husseini, Lasry, and Magill [15] and work on the price simplex $\Delta_{+}^{N-1} = \{p \in \mathbb{R}_{+}^N \mid \sum_{i=1}^N p_i = 1\}$. Notice we can use the homogeneity (of degree zero) property of the period one budget constraint (in the vector of period one spot prices) to rescale spot prices so that $\beta_s p(s)$ can be replaced by $p(s)$ without affecting that budget constraint.³ So, we can rewrite the above equation:

$$p(0)[x^h(0) - e^h(0)] + \sum_s p(s)[x^h(s) - e^h(s)] = 0.$$

This can be more compactly expressed as $p(x^h - e^h) = 0$. Once a competitive equilibrium is found, we use the homogeneity property of the period one budget constraint (in the vector of period one spot prices) to rescale spot prices so that $p_2(s) = 1$ for all $s = 1, \dots, S$.

Next, we eliminate portfolio trades by rewriting the period one budget constraint as $p[1] \cap ([x^h[1] - e^h[1]) \in \langle V(p) \rangle$, where $\langle V(p) \rangle$ is the subspace of R^S spanned by the 3 column vectors of $V(p)$ and the box notation means $p[1] \cap (x^h[1] - e^h[1]) = (p(s)(x^h(s) - e^h(s)))_{s=1}^S$. To define pseudo-equilibria and regular pseudo-equilibria, we replace $\langle V(p) \rangle$, the subspace of actual income transfers which can be achieved by trading in assets, with a trial subspace of feasible income transfers. So as to have a sufficiently rich family of subspaces from which to find an equilibrium one we require a convenient way to vary these subspaces. We follow Duffie and Shafer [4] and study the Grassmanian manifold, $G(S, 3)$, which consists of all 3-dimensional subspaces of R^S . This means replacing the subspace of income transfers $\langle V(p) \rangle$ attainable with asset trading by a trial subspace of income transfers, $L \in G(S, 3)$, in the budget constraints of households 2 through H .

Following the literature, there is no loss of generality if household one is not constrained by asset markets. We define the budget correspondence $B: \Delta^{N-1}_+ \times G(S, 3) \times R^{N}_{++} \rightarrow R^N_+$ for $h = 2, \dots, H$:

$$B(p, L; e^h) = \{x \in R^N_+ \mid p(x^h - e^h) = 0 \text{ and } p_1 \cap (x^h[1] - e^h[1]) \in L\} \quad (3.1)$$

DEFINITION. A pseudo-equilibrium over the Grassmanian manifold for the economy parameterised by $e \in R^{HN}_{++}$ is a three-tuple consisting of a commodity allocation, commodity prices, and a trial subspace of income

transfers, $(x, p, L) \in \mathbb{R}^{HN}_+ \times \Delta^{N-1}_+ \times G(S, 3)$ such that:

$$x^1 = \arg \max u^1(x^1) \text{ subject to } p(x^1 - e^1) = 0, \quad (3.2)$$

$$x^h = \arg \max u^h(x^h) \text{ subject to } x^h \in B(p, L; e^h), \text{ for } h = 2, \dots, H, \quad (3.3)$$

$$\sum_h x^h = \sum_h e^h, \quad (2.1)$$

$$L \succ \langle V(p) \rangle. \quad (3.4)$$

Finally, we define:⁴

DEFINITION. A pseudo-equilibrium (x, p, L) is regular if $\langle V(p) \rangle = L$.

Existence of pseudo-equilibrium is a consequence of the fixed point theorem of Husseini, Lasry, and Magill [15], that is also proved in Hirsch, Magill, and Mas-Colell [13]. We state this powerful theorem in our notation:

Husseini-Lasry-Magill Fixed Point Theorem: Let $\Phi: \Delta^{N-1}_+ \times G(S, M) \rightarrow \mathbb{R}^{N-1}$

and $\Psi: \Delta^{N-1}_+ \times G(S, M) \rightarrow \mathbb{R}^{SM}$ be continuous functions such that $\Delta^{N-1}_+ \supset$

$\Phi(\partial \Delta^{N-1}_+, L), \forall L \in G(S, M)$. Then, $\exists (p^*, L^*) \in (\text{int } \Delta^{N-1}_+) \times G(S, M)$ such that

$\Phi(p^*, L^*) = p^*$ and $L^* \succ \langle \Psi(p^*, L^*) \rangle$.

IV. RESULTS

Following Husseini, Lasry, and Magill [15], we have:

THEOREM 1. A pseudo-equilibrium exists for any economy parameterised by

$e \in \mathbb{R}^{HN}_{++}$.

We show that in the space of initial endowments, a pseudo-equilibrium

is generically regular. A pseudo-equilibria is not regular if $V(p)$ drops rank. This could happen because for some $p(s)$ values, the columns of $V(p(s))$ are linearly dependent. This would occur if, for example, the call option pays off zero in all S states. This will not happen if the price of commodity one differs across (at least) two states. We show that, in the space of initial endowments, this is generically so.

In fact, we show that a stronger condition holds, namely that in the space of initial endowments, generically, $p_1(s)$ is distinct over all states. We actually prove that, in the space of initial endowments, generically, the negation of that statement does not hold. This is enough to guarantee that, in the space of initial endowments, generically $V(p)$ has maximal rank.

THEOREM 2. There is an open and dense set Ω , contained in $\mathbb{R}^{\text{HN}}_{++}$, such that for any economy parameterised by $e \in \Omega$, every pseudo-equilibrium of that economy is regular.

Following Duffie and Shafer [4] or Hussein, Lasry, and Magill [15] we now state our final result:

THEOREM 3. There is an open and dense set Ω , contained in $\mathbb{R}^{\text{HN}}_{++}$, such that for any economy parameterised by $e \in \Omega$, a competitive equilibrium exists.

V. CONCLUSIONS

We have shown that European options can be successfully incorporated into a GEI model once we treat strike prices as endogenously chosen and not exogenously given. We view our model as bridging two strands of the GEI

literature, namely those involving exogenous asset structures and those involving endogenous asset structures. In our model, we do not explain why there are only two futures contracts or two European options, but we do determine the strike prices of those options endogenously in equilibrium.

We observe that, in practice, options exchanges introduce call and put options with strike prices that are rounded to the nearest multiple of ten or five dollars above and below the current primitive asset price. This means that options might not be at the money, but only near the money. If an option is near enough the money, however, this guarantees that its payoffs are not zero for all states as long as there are two states, $s \neq s^*$, such that $q_{f1}(s) \neq q_{f1}(s^*)$. Thus, our model could easily be generalized to multiple strike prices as long as these lie in the generically nondegenerate interval formed by the lowest and highest underlying primitive asset prices across states.

APPENDIX

PROOF OF LEMMA: This is a direct consequence of Farkas' theorem, which implies that for any $(1+S)$ by 4 matrix W one and only one of these two conditions holds: either the intersection of $\langle W(p, q) \rangle$ and $R_+^{1+S} \setminus 0$ is non-empty or the intersection of $\langle W(p, q) \rangle^\perp$ and R_{++}^{1+S} is non-empty.

This means either $\exists \theta \in R^4$ with $W\theta \geq 0$ or $\exists \alpha = (\alpha_0, \alpha_1, \dots, \alpha_S) \in R_{++}^{1+S}$ such that $\alpha W = 0$. Since q is assumed to be a no-arbitrage asset price, there does not exist a $\theta \in R^4$ such that $W\theta \geq 0$. This means $\exists \alpha \in R_{++}^{1+S}$ such that $\alpha W = 0$, or that $0 = -\alpha_0 q_E + \sum_{s=1}^S \alpha_s v_E(p_1(s))$ for $E = c$ or p , and that for $j = 1, 2$; $0 = -\alpha_0 q_{fj} + \sum_{s=1}^S \alpha_s p_j(s)$. Notice that we can divide both sides of these equations by the scalar α_0 , since that is guaranteed to be positive. This results in the desired β 's and equations (2.5) - (2.6)

PROOF OF THEOREM 1: We apply the Husseini-Lasry-Magill fixed point theorem with $M = 3$, Φ equal to a price adjustment function that is a modification of the aggregate excess demand function as constructed in Husseini, Lasry, and Magill [15], and $\Psi(p, L) = V(p)$. It can be checked as in Husseini, Lasry, and Magill [15] that Φ is inward pointing, so that a fixed point (p^*, L^*) exists with $\Phi(p^*, L^*) = p^*$ and $L^* \supset \langle \Psi(p^*, L^*) \rangle$. This fixed point is a pseudo-equilibrium.

PROOF OF THEOREM 2: By the smooth preference assumption, the solution to each agent's utility maximization problem exists, is unique, and yields a system of household demand functions of spot commodity prices. Consider

a modified aggregate excess demand function $z^*: \mathbb{R}^N_{++} \times G(S, 3) \times \mathbb{R}^{HN}_{++} \rightarrow$

\mathbb{R}^N defined by the formula:

$$z^*(p, L; e) = F^1(p; 1) - e^1 + \sum_{h=2}^m (F^h(p, L; e^h) - e^h), \quad (\text{A.1})$$

where F^h are individual household demand functions. Then, $z^*(p, L; e) = 0$ if and only if the aggregate excess demand is zero and $pe^1 = 1$. Consider the

projection of $V(p)$ onto the orthogonal subspace L^\perp of L , $\psi: \mathbb{R}^N_{++} \times G(S, 3) \rightarrow$

\mathbb{R}^{3S} defined by the formula:

$$\psi(p, L) = \Pi_{L^\perp} V(p), \quad (\text{A.2})$$

Notice that $L \supset \langle V(p) \rangle$ is equivalent to $\psi(p, L) = 0$. Finally, consider the

function $h: \mathbb{R}^N_{++} \times G(S, 3) \times \mathbb{R}^{HN}_{++} \rightarrow \mathbb{R}^N \times \mathbb{R}^{3S}$ defined by the formula:

$$h(p, L; e) = (z^*(p, L; e), \psi(p, L)). \quad (\text{A.3})$$

Define the space of economies to be $E = \mathbb{R}^{HN}_{++}$. We define the set M of

pseudo-equilibrium prices corresponding to an economy e , $\forall e \in E$:

$$M(e) = \{(p, L) \in \mathbb{R}^N_{++} \times G(S, 3) \mid h(p, L; e) = 0\}. \quad (\text{A.4})$$

Step 1: Generically, h is transverse to 0.

Notice that $D_e h(z^*) = -I$ has maximal rank N . By the Transversality theorem [10, p. 68], h_e is transverse to 0 for all e in E except for a closed

set of Lebesgue measure zero, where $h_e: \mathbb{R}^N_{++} \times G(S, 3) \rightarrow \mathbb{R}^N \times \mathbb{R}^{3S}$ is

defined by $h_e(p, L) = h(p, L; e)$. Let $E_0 = \{e \in E \mid h_e \text{ is transverse to } 0\}$.

Then, E_0 is an open set with null complement in E .

Step 2: Generically, $M(e) = h_e^{-1}(0)$ is a submanifold of dimension zero.

By the Preimage theorem [10], $M(e)$ is a submanifold $\mathbb{R}^N_{++} \times G(S, 3)$ and $\dim M(e) = \dim$ of the domain minus the \dim of the range. The domain is $\mathbb{R}^N_{++} \times G(S, 3)$ and so has $\dim = N + 3(S-3)$. As for the range $\mathbb{R}^N \times \mathbb{R}^{3S}$, its dimension must be calculated in light of the security market function being only dependent on the subspace that $V(p)$ spans and not on the particular matrix representation of $V(p)$. This defines an equivalence relation on the space of S by 3 matrices: A and B are equivalent if both A and B span the identical subspace. This means that from the apparent range we have to subtract as many degrees of freedom as equivalent matrix representations of $V(p)$. Thus, the dimension of the range is also $N + 3(S-3)$, the dimension of the domain.

Step 3: Generically, $\forall s \neq s^*, p_1(s) \neq p_1(s^*)$.

Generically, the negation of the above statement is that $\exists s, s^*$ such that $p_1(s) = p_1(s^*)$. If that were to be true, then we can use those states s and s^* to define the map $g(p, L; e) = (p_1(s) - p_1(s^*))$. Define two new maps: $f(p, L; e) = (h(p, L; e), g(p, L; e))$, and $f_e(p, L) = f(p, L; e)$. Then, $f_e^{-1}(0)$ is the subset of $M(e) = h_e^{-1}(0)$ for which the first period price of the first commodity is the same for two states of nature s and s^* . A natural question that arises is if $0 \in \mathbb{R}^N \times \mathbb{R}^{3S} \times \mathbb{R}$ is a regular value of f_e or if $f_e^{-1}(0)$ is a (sub)manifold. This means asking if $D_{(p, L; e)} f$ has maximal rank $\forall (p, L; e) \in f^{-1}(0)$. But, we know that $\forall (p, L; e) \in h^{-1}(0)$, the rank of $D_{(p, L; e)} h = N$. We show that $D_{(p, L; e)} g = 1$. Together, these two facts imply the rank $D_{(p, L; e)} f = N + 1$ whenever $(p, L; e) \in f^{-1}(0)$.

As g does not depend on e or L , we have $D_e g = D_L g = 0$. Note $D_p g$ is an N -dimensional vector having zeroes for all entries that are partial derivatives of g with respect to $p_j(s)$, for $j \neq 1$ and for all s ; 1 for the entry that is the partial derivative of g with respect to $p_1(s)$; and -1 for the entry that is the partial derivative of g with respect to $p_1(s^*)$; and zeroes for all those entries that are partial derivatives of g with respect to $p_j(s')$ if $s' \neq s$ or s^* . So, the rank of $D_p g$ is 1. So is the rank of $D_{(p, L; e)} g$. Thus, $0 \in \mathbb{R}^N \times \mathbb{R}^{3S} \times \mathbb{R}$ is a regular value of f . Define $E_1 = \{e \in E \mid f_e \text{ is transverse to } 0\}$. We know E_1 is open with null complement in E , by the Transversality theorem. We know $\forall e \in E_1$, that $f_e^{-1}(0)$ is a submanifold. In fact, $\forall e \in E_1$, we already know by the Preimage theorem $\dim f_e^{-1}(0) = -1$. Thus, $\forall e \in E_1$, no pseudo-equilibrium of the economy e satisfies the additional property of having the same first period price of the first commodity for two states.

Step 4: Generically, $\langle V(p) \rangle$ is a 3-dimensional subspace of \mathbb{R}^S .

Let Ω be the intersection of E_0 and E_1 . Since both of these are open sets with null complements in E , so is Ω . If $e \in \Omega$, then $\forall (p, L) \in M(e)$, the nonredundant asset matrix $V(p)$ will now be shown to have maximal rank, namely three. We know that the first column has at least one zero element (as the put option has at least one zero element). But, we also know the first column has at least one non-zero element, namely $p_1(s) - k$, for some s , by the martingale characterization of $k = q_{f_1}$ and step 3. Similarly, the second column is also going to have at least one zero element and one

non-zero element, namely $k - p_1(s^*)$, for some s^* . The third column is just a vector having entries $p_1(s)$. We show that these column vectors are linearly independent. Let square brackets around a scalar denote the vector with those components. If

$$\lambda_1 [v_c(p_1(s))] + \lambda_2 [v_p(p_1(s))] + \lambda_3 [p_1(s)] = [0]. \quad (\text{A.5})$$

Then, with $S > 3$, either there are several s^* for which $v_c(p_1(s^*)) = 0$ or $v_p(p_1(s^*)) = 0$. Without loss of generality, assume the former. Then for those s^* , where square brackets around a scalar now denote the subvector consisting of those nonzero components, we have:

$$\lambda_2 [k - p_1(s^*)] + \lambda_3 [p_1(s^*)] = [0], \text{ or} \quad (\text{A.6})$$

$$(\lambda_3 - k\lambda_2) [p_1(s^*)] + \lambda_2 [k] = 0 \quad (\text{A.7})$$

Since for generic e , $[p_1(s^*)]$ is not a constant vector, $[p_1(s^*)]$ and $[k]$ are linearly independent. Thus, $\lambda_2 = 0$ and $(\lambda_3 - k\lambda_2) = 0$, so $\lambda_3 = 0$ and $\lambda_1 = 0$.

Step 5: Generically, $\langle V(p) \rangle = L$.

By the above, $\forall (p, L) \in M(e)$ with $e \in \Omega$, the subspace $\langle V(p) \rangle$ has the same dimension as L . But, we already know $L \supset \langle V(p) \rangle$, from the definition of a pseudo-equilibrium. By a well-known theorem from linear algebra, it follows that $L = \langle V(p) \rangle$. So, every pseudo-equilibrium of $e \in \Omega$ is regular.

PROOF OF THEOREM 3: It is well-known that a competitive equilibrium can be constructed from a regular pseudo-equilibrium. (See Duffie and Shafer [4] or Hussein, Lasry, and Magill [15] for details.) Therefore, by Theorem 2, for $e \in \Omega$, the set of competitive equilibria is non-empty.

FOOTNOTES

¹ A European call option gives its holder the right, as opposed to an obligation, to buy a specified amount of a primary asset or commodity for a specified price, known as the strike or exercise price, on a certain date, known as the date of expiration. A European put option is analogously defined with sell replacing buy.

² We are using the notational convention that $x \geq 0$ means $x_i \geq 0$ for all i and $x_i > 0$ for some i .

³ We point out although the options' payoffs are nonlinear functions when viewed as functions of just the underlying price of commodity one, since they and the strike price are with respect to a numeraire, namely the price of commodity two, they are homogenous of degree zero as functions of all commodity prices and the strike price.

⁴ Duffie and Shafer [4] refer to a regular pseudo-equilibrium as an effective equilibrium.

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