Commodity Money Equilibrium in a Convex Trading Post Economy with Transaction Costs

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"[An] important and difficult question...[is] not answered by the approach taken here: the integration of money in the theory of value..."
—— Gerard Debreu, *Theory of Value* (1959)

Abstract

Existence and efficiency of general equilibrium with commodity money is investigated in an economy where N commodities are traded at $\frac{N(N-1)}{2}$ commodity-pairwise trading posts. Trade is a resource-using activity recovering transaction costs through the spread between bid (wholesale) and ask (retail) prices. Budget constraints, enforced at each trading post separately, imply demand for a carrier of value between trading posts. Existence of general equilibrium is established under conventional convexity and continuity conditions while structuring the price space to account for distinct bid and ask price ratios. Commodity money flows are identified as the difference between gross and net inter-post trades.

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1 Introduction

It is well-known that the Arrow-Debreu model of Walrasian general equilibrium cannot account for money. Professor Hahn (1982) writes

"The most serious challenge that the existence of money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. A first, and...difficult...task is to find an alternative construction without...sacrificing the clarity and logical coherence ... of Arrow-Debreu."

This paper pursues development of foundations for a theory of money based on elaborating the detail structure of an Arrow-Debreu model. The elementary first step is to create a general equilibrium where there is a well defined demand for a medium of exchange — a carrier of value between transactions. This is arranged by replacing the single budget constraint of the Arrow-Debreu model with the requirement that the typical household or firm pays for its purchases directly at each of many separate transactions. Transactions take place at commodity-pairwise trading posts. Then a well-defined demand for media of exchange (commodity monies, not necessarily unique) arises endogenously as an outcome of the market equilibrium. The use of media of exchange is particularly evident when the structure of demands is characterized by an absence of double coincidence of wants, Jevons (1875). Media of exchange are characterized as the carrier of value between transactions (not fulfilling final demands or input requirements themselves), the difference between gross and net trades [1]. Related general equilibrium models with transaction cost include Foley (1970), Hahn (1971, 1973), Kurz(1974), Starrett (1973), Starr (2003C).

The trading post model is intended to provide a parsimonious [2] addition to the Arrow-Debreu model sufficient to generate a theory of money. The monetary structure of trade is shown to be a consequence of the price theory general equilibrium, not a separate assumption.

[1] The present model is an alternative to the fiat money models of overlapping generations, Wallace (1980) et al., and of search, Kiyotaki and Wright (1989) et al. There a unique unbacked fiat money of positive value is typically assumed and presented as a bubble. The models allow, as well, a non-monetary no-trade equilibrium where the fiat money has a value of zero. In the present model, the existence of media of exchange and their values are endogenously determined.

It is possible to accommodate in an Arrow-Debreu setting an intrinsically worthless paper money trading at a positive value and used as a common medium of exchange. The rationale is that taxes payable in paper money provide backing for a positive value, and low transaction cost ensures use as medium of exchange, Goldberg (2005), Smith (1776), Starr (2003A, 2003B).

[2] Consistent with Ockham's razor.

1.1 Structure of the Trading Post Model

In the trading post model, transactions take place at commodity pairwise trading posts (Shapley and Shubik (1977), Walras (1874), Wicksell (1936)) with budget constraints (you pay for what you get in commodity terms) enforced at each post. Prices — bid (wholesale) and ask (retail) — are quoted as commodity rates of exchange. Trade is arranged by firms, typically buying at bid prices and selling at ask prices, incurring costs (resources used up in the transaction process) and recouping them through the bid/ask spread. Market equilibrium occurs when bid and ask prices at each trading post have adjusted so that all trading posts clear.

1.2 Structure of the Proof

The structure of the proof of existence of general equilibrium follows the approach of Arrow and Debreu (1954), Debreu (1959), and Starr (1997). The usual assumptions of continuity, convexity (traditional but by no means innocuous in this context), and no free lunch/irreversibility are used. The price space at a trading post for exchange of one good at bid price for another at ask price is the unit 1-simplex, allowing any possible nonnegative relative price ratio. The price space for the economy as a whole then is a Cartesian product of unit 1-simplices. The attainable set of trading post transactions is compact. As in Arrow and Debreu (1954), the model considers transaction plans of firms and households artificially bounded in a compact set including the attainable set as a proper subset. Price adjustment to a fixed point with market clearing leads to equilibrium of the artificially bounded economy. But the artificial bounds are not a binding constraint in equilibrium. The equilibrium of the artificially bounded economy is as well an equilibrium of the original economy.

1.3 Conclusion: The medium(a) of exchange

The general equilibrium specifies each household and firm's trading plan. At the conclusion of trade, each has achieved a net trade. Gross trades include trading activity that goes to paying for acquisitions and accepting payment for sales rather than directly implementing desired net trades. It's easy to calculate gross trades and net trades at equilibrium. For households, the difference — gross trades minus net trades — represents trading activity in carriers of value between trades, media of exchange (perhaps including some arbitrage). Since firms perform a market-making function within trading posts, identification of media of exchange used by firms is not so straightforward. After netting out intra-post trades, the remaining difference between inter-post gross and net trades represents the firms' trade flows of media of exchange. In some examples (see Starr (2003A, 2003B, 2008)) the medium of exchange may be a single specialized commodity (the common medium of exchange). The approach of the present model is intended to provide a Walrasian general equilibrium theory of (commodity) money as a medium of exchange. It is sufficiently general to include both a single common medium of exchange and many goods simultaneously acting as media of exchange.

When will media of exchange actually be used in the trading post economy? Two conditions seem to be sufficient: desirability of trade, net of transaction costs; absence of double coincidence of wants. The logic is simple. If trade is desirable at prevailing equilibrium prices (net of transaction costs including the transaction cost of media of exchange) and there is no double coincidence of wants, then in order for trade to proceed fulfilling the budget constraint at each trading post separately, media of exchange will be used as carriers of value between trading posts. However, the absence of double coincidence of wants depends on prevailing prices as well as endowments and technology. It is problematic to characterize necessary and sufficient initial conditions so that absence of double coincidence is fulfilled. Hence the reliance on simple illustrative examples below. Nevertheless, the examples are intended to be robust. The parametric examples should be contained in an open subset of parameter space where the results of the example remain valid.

Conversely, there are two cases where trading post equilibria will have no use of media of exchange: full double coincidence of wants (subject to direct trade experiencing no higher transaction costs than indirect trade); and a no-trade equilibrium. Again, necessary and sufficient conditions, a priori, to fulfill these characteristics are not immediately evident.

2 Trading Posts

There are N tradeable goods denoted 1, 2, ..., N. They are traded for one another pairwise at trading posts. {i, j} (or equivalently {j,i}) denotes the trading post where goods i and j are traded for one another. There are $\frac{N(N-1)}{2}$ distinct trading posts.

3 Prices

Goods are traded directly for one another without distinguishing any single good as 'money'.

Let Δ represent the unit 1-simplex. At trading post {i, j}, the (relative) ask price of good i and (relative) bid price of good j are represented as $p^{\{i,j\}} \equiv (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \in$

Δ. In a (minor) abuse of notation, the ordering of i and j in the superscript on p will matter. The relative ask price of good j and bid price of i are represented as $p^{\{j,i\}} \equiv (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \in \Delta$. Thus there are two operative price 1-simplices at each trading post. The full price space then is $\Delta^{N(N-1)}$, the $N(N-1)$-fold Cartesian product of Δ with itself; its typical element is $p \in \Delta^{N(N-1)}$. Then the ask price of i at {i,j} in units of j is $\frac{a_i^{\{i,j\}}}{b_j^{\{i,j\}}}$ and the bid price of i is $\frac{b_i^{\{i,j\}}}{a_j^{\{i,j\}}}$.

Prices can then be read as rates of exchange between goods, distinguishing between bid (selling or wholesale) prices and ask (buying or retail) prices. Thus the ask price of a hamburger might be 5.0 chocolate bars and the bid price 3.0 chocolate bars. Note that the ask price of a chocolate bar then is the inverse of the bid price of a hamburger. That is, the ask price of a chocolate bar is 0.333 hamburger and the bid price of a chocolate bar is 0.2 hamburger.

4 Budget Constraints and Trading Opportunities

The budget constraint is simply that at each pairwise trading post, at prevailing prices, in each transaction, payment is given for goods received. That is, at trading post {i, j}, an ask/bid price pair is quoted $p^{\{i,j\}} \equiv (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \in \Delta$ expressing the ask price of i in terms of j and a bid price of j in terms of i. A firm or household's trading plan $(y, x) \in R^{2N(N-1)}$ specifies the following transactions at trading post {i,j}: $y_i^{\{i,j\}}$ (at ask prices — retail) in i, $y_j^{\{i,j\}}$ (at ask prices — retail) in j, $x_i^{\{i,j\}}$ (at bid prices — wholesale) in i, $x_j^{\{i,j\}}$ (at bid prices — wholesale) in j. Positive values of these transactions are purchases. Negative values are sales. At each trading post (of two goods) there are four quantities to specify in a trading plan. Then the budget constraint facing firms and households at each trading post is that value delivered must equal value received. That is

$$0 = (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \cdot (y_i^{\{i,j\}}, x_j^{\{i,j\}}) \ , \qquad 0 = (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \cdot (y_j^{\{i,j\}}, x_i^{\{i,j\}}) \qquad \textbf{(B)}$$

(B) says that purchases of i at the bid price are repaid by sales of j at the ask price, purchases of i at the ask price are repaid by sales of j at the bid price.

Given a price vector $p \in \Delta^{N(N-1)}$ the array of trades fulfilling **(B)** is the set of trades fulfilling the $N(N-1)$ local budget constraints at the trading posts. Denote this set

$$\mathbf{M}(p) \equiv \{(y, x) \in R^{2N(N-1)} | (y, x) \text{ fulfills } \textbf{(B)} \text{ at } p \text{ for all } i, j = 1, ..., N, i \neq j\}$$

5 Firms

The heavy lifting in this model is done by firms. They perform the market-making function, incurring transaction costs. The population of firms is a finite set denoted F, with typical element $f \in F$. Thus, firm f's technology set may specify that f's purchase of labor (retail) in exchange for i on the {i, labor} market and purchase of i and j wholesale on the {i,j} market allows f to sell i and j (retail) on the {i, j} market. That's how f can become a market maker. If there is a sufficient difference between bid and ask prices so that f can cover the cost of its inputs with a surplus left over, that surplus becomes f's profits, to be rebated to f's shareholders.

5.1 Transaction and Production Technology

Firm f's technology set is Y^f. We assume

P.0 $Y^f \subset R^{2N(N-1)}$

The typical element of Y^f is (y^f, x^f), a pair of $N(N-1)$-dimensional vectors. The $N(N-1)$-dimensional vector y^f represents f's transactions at ask (retail) prices; the $N(N-1)$-dimensional vector x^f represents f's transactions at bid (wholesale) prices. The 2-dimensional vector $y^{f\{i,j\}}$ represents f's transactions at ask (retail) prices at trading post {i,j}; the 2-dimensional vector $x^{f\{i,j\}}$ represents f's transactions at bid (wholesale) prices at trading post {i,j}. The typical co-ordinates $y_i^{f\{i,j\}}, x_i^{f\{i,j\}}$ are f's action with respect to good i at the {i,j} trading post. Since f may act as a wholesaler/retailer/market maker, entries anywhere in $(y^{f\{i,j\}}, x^{f\{i,j\}})$ may be positive or negative — subject of course to constraints of technology Y^f and prices $\mathbf{M}$(p). This distinguishes the firm from the typical household. The typical household can only sell at bid prices and buy at ask prices.

The entry $y_i^{f\{i,j\}}$, represents f's actions at ask prices with regard to good i at trading post {i,j}. $y_i^{f\{i,j\}} > 0$ represents a purchase of i at the {i,j} trading post (at the ask price). $y_i^{f\{i,j\}} < 0$ represents a sale of i at the ask price.

The entry $x_i^{f\{i,j\}}$, represents f's actions at bid prices with regard to good i at trading post {i,j}. $x_i^{f\{i,j\}} > 0$ represents a purchase of i at the trading post (at the bid price). $x_i^{f\{i,j\}} < 0$ represents a sale of i at the bid price.

A firm that is an active market-maker at {i,j} will typically buy at the bid price and sell at the ask price. A firm that is not a market-maker may have to pay retail — like the rest of us — selling at the bid price and buying at the ask price.

In addition to indicating the transaction possibilities, Y^f includes the usual production possibilities. The usual assumptions on production technology apply. For each $f \in F$, assume

P.I Y^f is convex.

P.II $0 \in Y^f$, where 0 indicates the zero vector in $R^{2N(N-1)}$.

P.III Y^f is closed.

The aggregate technology set is the sum of individual firm technology sets. $Y \equiv \sum_{f\in F} Y^f$. It fulfills the familiar *no free lunch* and *irreversibility* conditions.

P.IV [(a)] if $(y,x) \in Y$ and $(y,x) \neq 0$, then $y_i^{\{i,j\}} + x_i^{\{i,j\}} > 0$ for some i, j.

[(b)] if $(y,x) \in Y$ and $(y,x) \neq 0$, then $-(y,x) \notin Y$.

Denote the initial resource endowment of the economy as $r \in R_+^N$. Then we define the attainable production plans of the economy as

$$\hat{Y} \equiv \{(y,x) \in Y | r_i \geq \textstyle\sum_j (y_i^{\{i,j\}} + x_i^{\{i,j\}}) \text{ all } i = 1,2,...,N\}$$

Attainable production plans for firm f can then be described as

$$\hat{Y}^f \equiv \{(y^f, x^f) \in Y^f | \text{ there is } (y^k, x^k) \in Y^k \text{ for each } k \in F, k \neq f \text{ , so that }$$
$$[\sum_{k\in F, k\neq f} (y^k, x^k) + (y^f, x^f)] \in \hat{Y}\}.$$

Lemma 5.1: Assume P.0 - P.IV. Then $\hat{Y}$ and $\hat{Y}^f$ are closed, convex, and bounded. Proof: Starr (1997), Theorem 8.1, 8.2.

5.2 Firm Maximand and Transactions Function

The firm formulates a production plan and a trading plan. The firm's opportunity set for net yields after transactions fulfilling budget is $E^f(p) \equiv [\mathbf{M}(p) - Y^f] \cap R_+^{2N(N-1)}$. That is, consider the firm's production, purchase, and sale possibilities, net after paying for them, and what's left is the net yield. Using the sign conventions we've adopted — purchases are positive co-ordinates, sales are negative co-ordinates — the net yield is then the negative co-ordinates (supplies) in a trading plan that are not absorbed by payments due and the net purchases not required as inputs to the firm. The supplies are subtracted out, so the surpluses enter $E^f(p)$ as positive co-ordinates.

A typical element of these surplus supplies is denoted $(y', x') \in E^f(p)$. In this notation y' and x' are dummies, not actual marketed supplies and demands.

Now consider $(y', x') \in E^f(p)$. In each good i, the net surplus available in good i is $w_i^f \equiv \sum_{j=1}^N (y_i'^{\{i,j\}} + x_i'^{\{i,j\}})$ and firm f's surplus is the vector w^f of these co-ordinates. To give this notion a functional notation, let $W(y', x') \equiv w^f$ described here.

There are N-1 trading posts where each good i is traded, at N-1 rates of exchange. The notion of 'profit' is not well defined. In the absence of a single family of well-defined prices, it is difficult to characterize optimizing behavior for the firm. *Fautes de mieux* we'll give the firm a scalar maximand with argument p, y', x'. Firm f is assumed to have a real-valued, continuous maximand $v^f(p; y', x')$. We take v^f to be monotone and concave in (y', x'). This description of v^f includes as a special case the usual firm profit function (when p is sufficiently uniform across trading posts that the usual notion of profit is well defined).

The firm's optimizing choice (which may not be well defined) then is

$G^f(p) \equiv \{\text{argmax } v^f(p; y', x') \in E^f(p)\}$.

This results in the firm's market behavior (without any constraint requiring actions to stay in a bounded range) described by

$H^f(p) \equiv \{(y, x) \in \mathbf{M}(p) | [(y, x) + (y', x')] \in Y^f, (y', x') \in G^f(p)\}$. This marketed plan then results in the market and dividend plan

$S^f(p) \equiv \{(y, x; w) | (y, x) \in H^f(p), [(y, x) + (y', x')] \in Y^f, (y', x') \in G^f(p); w = W(y', x')\}$

The logic of this definition is that $(y', x') \geq 0$ is the surplus left over after the firm f has performed according to its technology and subject to prevailing prices.

It is possible that $S^f(p)$ is not well defined, since the opportunity set may be unbounded. In the light of Lemma 5.1, there is a constant $c > 0$ sufficiently large so that for all $f \in F$, $\hat{Y}^f$ is strictly contained in a closed ball, denoted B_c of radius c centered at the origin of $R^{2N(N-1)}$. Following the technique of Arrow and Debreu (1954), constrained market behavior for the firm will consist of limiting its production choices to $Y^f \cap B_c$. This leads to the constrained surplus

$\tilde{E}^f(p) \equiv [[\mathbf{M}(p) \cap B_c] - [Y^f \cap B_c]] \cap R_+^{2N(N-1)}$.

$\tilde{G}^f(p) \equiv \{\text{argmax } v^f(p; y', x') \in \tilde{E}^f(p)\}$.

$\tilde{H}^f(p) \equiv \{(y, x) \in \mathbf{M}(p) | [(y, x) + (y', x')] \in Y^f \cap B_c, (y', x') \in \tilde{G}^f(p)\}$.

The firm's constrained (to B_c) market behavior then is defined as

$\tilde{S}^f(p) \equiv \{(y, x; w) | (y, x) \in \tilde{H}^f(p), [(y, x) + (y', x')] \in Y^f \cap B_c, (y', x') \in \tilde{G}^f(p); w = W(y', x')\}$.

Lemma 5.2: Assume P.0 - P.IV. Then $\tilde{E}^f(p)$ is convex-valued, nonempty, upper and lower hemicontinuous.

Proof: (Note to the reader: The notation x^o appears in two distinct unrelated forms in this proof. Usually as part of firm f's planned transactions, but later — completely distinctly — in the quotation from Green and Heller (1981).) Upper hemicontinuity and convexity follow from closedness and convexity of the underlying sets. $0 \in \tilde{E}^f(p)$ always, so nonemptiness is fulfilled. Lower hemicontinuity requires some work.

Let $p^\nu \to p^o$, $(y^o, x^o) \in \tilde{E}^f(p^o)$. We seek $(y^\nu, x^\nu) \in \tilde{E}^f(p^\nu)$ so that $(y^\nu, x^\nu) \to (y^o, x^o)$. If $(y^o, x^o) = 0$, existence of $(y^\nu, x^\nu) \to (y^o, x^o)$ is trivially satisfied. Suppose instead $(y^o, x^o) \geq 0$ (the inequality applies co-ordinatewise). Then in an ϵ-neighborhood of (y^o, x^o), for ν sufficiently large, we seek to show that there is $(y^\nu, x^\nu) \in \tilde{E}(p^\nu)$. (y^ν, x^ν) is the required sequence. To demonstrate this, note that $\tilde{E}(p^\nu)$ is defined as the intersection of a convex-valued correspondence lower hemicontinuous in p with a constant convex set. When $(y^o, x^o) \geq 0$ and $(y^o, x^o) \in \tilde{E}^f(p^o)$ it follows that the relative interior of $\tilde{E}^f(p^o)$ is nonempty. It is sufficient then to apply Green and Heller (1981), p. 48, (8, lower), "If $\gamma_i, i = 1, 2$, are two l.h.c. convex-valued correspondences such that $int\gamma_1(x^o) \cap int\gamma_2(x^o) \neq \emptyset$, then $\gamma_1 \cap \gamma_2$ is l.h.c. at x^o. "

Lemma 5.3: Assume P.0 - P.IV. Then $\tilde{G}^f(p), \tilde{H}^f(p), \tilde{S}^f(p)$ are well defined, nonempty, upper hemicontinuous, and convex-valued for all $p \in \Delta^{N(N-1)}$.

Proof: Note compactness of B_c. Apply Theorem of the Maximum, continuity and concavity of v^f.

Lemma 5.4: Assume P.0 - P.IV. Let $[\tilde{G}^f(p) + \tilde{H}^f(p)] \cap \hat{Y}^f \neq \emptyset$. Then $[\tilde{G}^f(p) + \tilde{H}^f(p)] \subseteq [G^f(p) + H^f(p)]$.

Proof: Recall that B_c strictly includes $\hat{Y}^f$. Then the result follows from convexity of Y^f and $\hat{Y}^f$ and concavity of $v^f(p; y', x')$. The proof follows the model of Starr (1997) Theorem 8.3. Let $(y^{*\prime}, x^{*\prime}) \in \tilde{G}^f(p), (y^*, x^*) \in \tilde{H}^f(p), [(y^{*\prime}, x^{*\prime}) + (y^*, x^*)] \in \hat{Y}^f \subset B_c$. Use a proof by contradiction. Suppose not. Then there is $(y, x) \in Y^f$ so that $(y, x) - (y^o, x^o) = (y', x')$, where $v^f(p; y', x') > v^f(p; y^{*\prime}, x^{*\prime})$, $(y', x') \in E^f(p)$, and $(y^o, x^o) \in \mathbf{M}(p)$. But convexity of Y^f and concavity of v^f imply that on the chord between (y^*, x^*) and (y, x) there is $[\alpha(y^*, x^*) + (1 - \alpha)(y, x)] \in B_c$ for $1 \geq \alpha > 0$ where $v^f(p; [\alpha(y^{*\prime}, x^{*\prime}) + (1 - \alpha)(y', x')]) > v^f(p; y^{*\prime}, x^{*\prime})$. This is a contradiction.

5.3 Inclusion of constrained supply in unconstrained supply

$(y, x; w) \in \tilde{S}^f(p)$ implies $(y, x) \in B_c$, a bounded set. $w \in R^N_+$ is f's profits. By construction there is $K > 0$ so that w is contained in the nonnegative quadrant of a ball of radius K centered at the origin, denoted $B_K \subset R^N_+$.

Lemma 5.5: Let $p \in \Delta^{N(N-1)}$ such that $\tilde{S}^f(p) \cap [\hat{Y}^f \times B_K] \neq \emptyset$. Then $S^f(p)$ is well defined and nonempty. Further $\tilde{S}^f(p) \subseteq S^f(p)$.

Proof: Lemma 5.4.

6 Households

There is a finite set of households, H, with typical element h.

6.1 Endowment and Consumption Set

$h \in H$ has a possible consumption set, taken for simplicity to be the nonnegative quadrant of R^N, R^N_+. $h \in H$ is endowed with $r^h >> 0$ assumed to be strictly positive to avoid boundary problems. $h \in H$ has a share $\alpha^{hf} \geq 0$ of firm f, so that $\sum_{h \in H} \alpha^{hf} = 1$.

6.2 Trades and Payment Constraint

$h \in H$ chooses $(y^h, x^h) \in R^{2N(N-1)}$ subject to the following restrictions. A household always balances its budget, sells wholesale and buys retail:

(i) $0 \geq x_i^{h\{i,j\}}$ for all i, j.

(ii) $y_i^{h\{i,j\}} \geq 0$ for all i, j.

(iii) $(y^h, x^h) \in \mathbf{M}(p)$

6.3 Maximand and Demand

Household h's share of profits from firm f is part of h's endowment and enters directly into consumption. When the profits of all firms $f \in F$, w^f in $(y^f, x^f; w^f)$, are well defined, f distributes to shareholders w^f, and h's consumption of good i is

(iv) $c_i^h \equiv r_i^h + [\sum_{f \in F} \alpha^{hf} w^f]_i + \sum_{j=1}^N x_i^{h\{i,j\}} + \sum_{j=1}^N y_i^{h\{i,j\}}$

However, prices p may be such that $S^f(p)$ is not well defined for some f. Then we may wish to discuss the constrained version of (iv), where $\tilde{w}^f$ comes from $(y^f, x^f; \tilde{w}^f) \in \tilde{S}^f(p)$.

(iv′) $c_i^h \equiv r_i^h + [\sum_{f \in F} \alpha^{hf} \tilde{w}^f]_i + \sum_{j=1}^N x_i^{h\{i,j\}} + \sum_{j=1}^N y_i^{h\{i,j\}}$

In addition, h's consumption must be nonnegative.

(v) $c^h \geq 0$. The inequality applies co-ordinatewise.

C.I For all $h \in H$, h's maximand is the continuous, quasi-concave, real-valued, strictly monotone, utility function $u^h(c^h)$. $u^h : \mathbf{R}_+^\mathbf{N} \rightarrow \mathbf{R}$.

h's planned transactions function is defined as D^h:$\Delta^{N(N-1)} \times R^{N\#F} \rightarrow R^{2N(N-1)}$. Let w denote $(w^1, w^2, w^3, ..., w^f, ..., w^{\#F})$.

$D^h(p, w) \equiv \{(y^h, x^h) \in R^{2N(N-1)} | (y^h, x^h)$ maximizes $u^h(c^h)$, subject to (i), (ii), (iii), (iv) and (v) $\}$. However, $D^h(p, w)$ may not be well defined when opportunity sets are unbounded (when ask prices of some goods are zero) and w may not be well defined for p such that $S^f(p)$ is not well defined for some f. To treat this issue, let $B_K^{\#F}$ be the $\#F$-fold Cartesian product of B_K, and define $\tilde{D}^h$:$\Delta^{N(N-1)} \times B_K^{\#F} \rightarrow B_c$.

$\tilde{D}^h(p, w) \equiv \{(y^h, x^h) | (y^h, x^h)$ maximizes $u^h(c^h)$, subject to (i), (ii), (iii), (iv′), (v), and $(y^h, x^h) \in B_c\}$. The restriction to B_c in this definition assures that $\tilde{D}^h(p)$ represents the result of optimization on a bounded set, and is well-defined.

Lemma 6.1: Assume P.0 - P.IV, C.I. Then $\tilde{D}^h(p, w)$ is nonempty, upper hemi-continuous and convex-valued, for all $p \in \Delta^{N(N-1)}$, $w \in B_K^{\#F}$. The range of $\tilde{D}^h(p, w)$

is compact. For (p,w) such that $|(y^h,x^h)| < c$ for (some) $(y^h,x^h) \in \tilde{D}^h(p,w)$, it follows that $\tilde{D}^h(p,w) \subseteq D^h(p,w)$.

Proof: (Note to the reader: This proof includes an unfortunate confusion of notation. c without superscript denotes a large real number indicating the radius of B_c, a ball strictly containing all attainable transactions of the typical firm. c^h and c^* (with superscript) denote consumption vectors.) Apply Theorem of the Maximum, noting continuity and quasi-concavity of u^h, convexity of constraint sets defined by (i)-(v) or by (i),(ii),(iii), (iv′), (v). Inclusion of $\tilde{D}^h(p,w)$ in $D^h(p,w)$ follows the pattern of Starr (1997) Theorem 9.1(b). Proof by contradiction. Suppose not. Then there is $(y^*,x^*) \in D^h(p,w)$ with associated c^* so that $u^h(c^*) > u^h(c^h)$. But recall $|(y^h,x^h)| < c$. On the chord between (y^h,x^h) and (y^*,x^*) there is $[\alpha(y^*,x^*)+(1-\alpha)(y^h,x^h)], 1 > \alpha > 0$, fulfilling (i), (ii), (iii), (iv′), (v), and $|[\alpha(y^*,x^*)+(1-\alpha)(y^h,x^h)]| = c$ so that $u(\alpha c^* + (1-\alpha)c^h) > u(c^h)$. This is a contradiction.

7 Excess Demand

Let $(p,w') \in \Delta^{N(N-1)} \times B_K^{\#F}$. Constrained excess demand and dividends at (p,w') is defined as

$\tilde{Z} : \Delta^{N(N-1)} \times B_K^{\#F} \to R^{2N(N-1)} \times B_K^{\#F}$.

$$\tilde{Z}(p,w') \equiv \{(\sum_{f\in F}(y^f,x^f)+\sum_{h\in H}\tilde{D}^h(p,w'),\ w^1,w^2,...,w^f,...,w^{\#F})|(y^f,x^f,w^f) \in \tilde{S}^f(p)\}.$$

Lemma 7.1: Assume P.0 - P.IV, and C.I. The range of $\tilde{Z}$ is bounded. $\tilde{Z}$ is upper hemi-continuous and convex-valued for all $(p,w') \in \Delta^{N(N-1)} \times B_K^{\#F}$.

Lemma 7.2 (Walras' Law): Let $(p,w') \in \Delta^{N(N-1)} \times B_K^{\#F}$. Let $(y,x,w) \in \tilde{Z}(p,w')$. Then for each $i,j = 1,...,N, i \neq j$, we have

$$0 = (a_i^{\{i,j\}}, b_j^{\{i,j\}}) \cdot (y_i^{\{i,j\}}, x_j^{\{i,j\}})\ , \qquad 0 = (a_j^{\{i,j\}}, b_i^{\{i,j\}}) \cdot (y_j^{\{i,j\}}, x_i^{\{i,j\}}) \qquad \textbf{(W)}.$$

Proof : The element (y,x) of $(y,x,w) \in \tilde{Z}(p,w')$ is the sum of elements (y^f,x^f) of $\tilde{S}^f(p)$ and (y^h,x^h) of $\tilde{D}^h(p,w')$ each of which is subject to **(B)**.

8 Equilibrium

Let Ξ denote a compact convex subset of $R^{2N(N-1)}$ so that $\Xi \times B_K^{\#F}$ includes the range of $\tilde{Z}$. Let $z \in \Xi$, $z \equiv ((y_1^{\{1,2\}}, x_2^{\{1,2\}}),...,(y_i^{\{i,j\}}, x_j^{\{i,j\}}),...,(y_{N-1}^{\{N-1,N\}}, x_N^{\{N-1,N\}}))$. Define $\rho : \Xi \to \Delta^{N(N-1)}$

$\rho(z) \equiv \{p^o \in \Delta^{N(N-1)} |$ For each $i, j = 1, 2, ..., N$, $i \neq j$, $p^{o\{i,j\}} \in \Delta$ maximizes $p^{\{i,j\}} \cdot (y_i^{\{i,j\}}, x_j^{\{i,j\}})$ subject to $p^{\{i,j\}} \in \Delta\}$.

Lemma 8.1: ρ is upper hemi-continuous and convex-valued for all $z \in \Xi$.

Define $\Gamma : \Delta^{N(N-1)} \times \Xi \times B_K^{\#F} \to \Delta^{N(N-1)} \times \Xi \times B_K^{\#F}$.

$\Gamma(p, z, w') \equiv \rho(z) \times \tilde{Z}(p, w')$.

Lemma 8.2: Assume P.0 - P.IV, and C.I. Then Γ is upper hemi-continuous and convex-valued on $\Delta^{N(N-1)} \times \Xi \times B_K^{\#F}$. Γ has a fixed point (p^*, z^*, w^*) and $0 = z^*$.

Proof: Upper hemicontinuity and convexity are established in lemmas 7.1 and 8.1. Existence of the fixed point (p^*, z^*) then follows from the Kakutani fixed point theorem. To demonstrate that $z^* = 0$, note lemma 7.2 and strict monotonicity of u^h.

Definition: $(p^*, w^*) \in \Delta^{N(N-1)} \times B_K^{\#F}$ is said to be an equilibrium if

$$(0, w^*) \in \{(\sum_{f \in F}(y^f, x^f) + \sum_{h \in H} D^h(p^*, w^*),\ w^1, w^2, ..., w^f, ..., w^{\#F}) | (y^f, x^f, w^f) \in S^f(p^*)\}$$

where 0 is the origin in $R^{2N(N-1)}$.

Theorem 8.1: Assume P.0 - P.IV, C.I . Then there is an equilibrium $(p^*, w^*) \in \Delta^{N(N-1)} \times B_K^{\#F}$.

Proof: Apply Lemmas 5.5, 6.1, 8.2. Lemma 8.2 provides $(p^*, z^*, w^*) \in \Delta^{N(N-1)} \times \Xi \times B_K^{\#F}$ so that $0 = z^*$, where

$(z^*, w^*) \in \{(\sum_{f \in F}(y^f, x^f) + \sum_{h \in H} \tilde{D}^h(p^*, w^*),\ w^1, w^2, ..., w^f, ..., w^{\#F}) | (y^f, x^f, w^f) \in \tilde{S}^f(p^*)\}$. Then $\tilde{S}^f(p^*) \cap [\hat{Y}^f \times B_K] \neq \emptyset$, so by Lemma 5.5, $\tilde{S}^f(p^*) \subseteq S^f(p^*)$. $0 = z^*$, implies that $|(y^{*h}, x^{*h})| < c$, so by lemma 6.1, $\tilde{D}^h(p^*, w^*) \subseteq D^h(p^*, w^*)$. But then $(0, w^*) \in \{(\sum_{f \in F}(y^f, x^f) + \sum_{h \in H} D^h(p^*, w^*),\ w^1, w^2, ..., w^f, ..., w^{\#F}) | (y^f, x^f, w^f) \in S^f(p^*)\}$. Then (p^*, w^*) is an equilibrium.

8.1 No-arbitrage condition in Trading Post Equilibrium

At trading post equilibrium profitable arbitrage by households should not be possible at prevailing equilibrium prices. Otherwise, arbitrarily large trading profits would seem possible to the household. For simplicity, consider arbitrage among only two commodities, without loss of generality denoted 1 and 2. There is only one trading post {1, 2 } under consideration so the superscript designating the trading post can be omitted to simplify notation. The price vector is

$$((a_1, b_2), (a_2, b_1)) \in \Delta \times \Delta$$

where Δ is the unit 1-simplex. Recall that households sell at bid prices, b_1, b_2 and buy at ask prices a_1, a_2. Then from the household side the No Arbitrage Condition can be stated as

$$\frac{\mathrm{b}_1}{\mathrm{a}_2} \leq \frac{\mathrm{a}_1}{\mathrm{b}_2}.$$

This is demonstrated in the following way. Consider a single household, omitting the household superscript for simplicity. We have the following relations from the structure of the model:

$$x_1 \leq 0, x_2 \leq 0, y_1 \geq 0, y_2 \geq 0$$

$$-b_2 x_2 = a_1 y_1,\ -b_1 x_1 = a_2 y_2,\ x_2 = -\tfrac{\mathrm{a}_1}{\mathrm{b}_2} y_1,\ y_2 = -\tfrac{\mathrm{b}_1}{\mathrm{a}_2} x_1,\ x_1 = -\tfrac{\mathrm{a}_2}{\mathrm{b}_1} y_2,\ y_1 = -\tfrac{\mathrm{b}_2}{\mathrm{a}_1} x_2\ .$$

Consider household arbitrage in good 1, to accumulate large profits in good 2. Set $-x_1 = y_1 = \xi > 0$. Then $x_2 = -\frac{\mathrm{a}_1}{\mathrm{b}_2}\xi$ and $y_2 = -\frac{\mathrm{b}_1}{\mathrm{a}_2}(-\xi)$ or $y_2 + x_2 = \xi[\frac{\mathrm{b}_1}{\mathrm{a}_2} - \frac{\mathrm{a}_1}{\mathrm{b}_2}] =$ arbitrage profit. Hence the sufficient condition for arbitrage profit to be nonpositive is $\frac{\mathrm{b}_1}{\mathrm{a}_2} \leq \frac{\mathrm{a}_1}{\mathrm{b}_2}$.

Similarly consider household arbitrage in good 2 to accumulate large profits in good 1. Set $-x_2 = y_2 = \xi > 0$. Then $x_1 = -\frac{\mathrm{a}_2}{\mathrm{b}_1}\xi$ and $y_1 = -\frac{\mathrm{b}_2}{\mathrm{a}_1}(-\xi)$ or $y_1 + x_1 = \xi[\frac{\mathrm{b}_2}{\mathrm{a}_1} - \frac{\mathrm{a}_2}{\mathrm{b}_1}] =$ arbitrage profit. Hence a sufficient condition for arbitrage profit to be nonpositive is $\frac{\mathrm{b}_2}{\mathrm{a}_1} \leq \frac{\mathrm{a}_2}{\mathrm{b}_1}$ or equivalently

$$\frac{\mathrm{b}_1}{\mathrm{a}_2} \leq \frac{\mathrm{a}_1}{\mathrm{b}_2}.$$

9 Media of Exchange, Commodity Monies

Let $(y^h, x^h) \in D^h(p, w')$ be household h's $2N(N-1)$-dimensional transaction vector. The x co-ordinates are typically sales (negative sign) at bid prices; the y co-ordinates are typically purchases (positive sign) at ask prices. Then we can characterize h's gross transactions in good i as

$\sum_j y_i^{h\{i,j\}} - \sum_j x_i^{h\{i,j\}} \equiv \gamma_i^h$.

Further, the absolute value of h's net transactions in good i, is

$|\sum_j y_i^{h\{i,j\}} + \sum_j x_i^{h\{i,j\}}| \equiv \nu_i^h$.

The N-dimensional vector γ^h with typical element γ_i^h is h's gross trade. The N-dimensional vector ν^h with typical element ν_i^h is h's net trade vector (in absolute value). $\mu^h \equiv \gamma^h - \nu^h$ is h's flow of goods as media of exchange, gross trades minus net trades.

Since firms perform a market-making function, buying and selling the same good at a single trading post, a more complex view of their transactions is needed to sort out trading flows used as media of exchange. In particular, for firms, we should net out offsetting transactions within a single trading post. Thus for $f \in F$, f's gross transactions in i, netting out intra-post transactions is

$\sum_j |[y_i^{f\{i,j\}} + x_i^{f\{i,j\}}]| \equiv \gamma_i^f$.

The corresponding net transaction is

$|\sum_j [y_i^{f\{i,j\}} + x_i^{f\{i,j\}}]| \equiv \nu_i^f$.

The N-dimensional vector γ^f with typical element γ_i^f is f's gross inter-post trade. The N-dimensional vector ν^f with typical element ν_i^f is h's net inter-post trade vector (in absolute value). $\mu^f \equiv \gamma^f - \nu^f$ is f's flow of goods as media of exchange, gross (inter-post) trades minus net trades.

The total (N-dimensional vector) flow of media of exchange among households and firms is then $\sum_{h\in H}\mu^h + \sum_{f\in F}\mu^f$. This expression, $\sum_{h\in H}\mu^h + \sum_{f\in F}\mu^f$, is the flow of commodity monies.

Thus the trading post equilibrium establishes a well-defined demand for media of exchange as an outcome of the market equilibrium. Media of exchange (commodity monies) are characterized as goods flows acting as the carrier of value between transactions (not fulfilling final demands or input requirements themselves), the difference between gross and net trades.

10 Walrasian Equilibrium, Trading Post Equilibrium, and Demand for Media of Exchange

10.1 Transaction Costs, Essential and Inessential Sequence Economies

The issues of general equilibrium with transaction cost, efficiency of allocation and the implications for the role of money appear in Foley (1970), Hahn (1971, 1973), and Starrett (1973). Foley (1970) considers a static equilibrium with (consistent with the Arrow-Debreu treatment) a single market meeting. All of the formal structure of the Arrow-Debreu economy is maintained while the transaction process is treated as a production activity. Each of N goods has a bid and ask (wholesale and retail) price with the resulting dimensionality of the price space at 2N. As in Debreu (1959) the count N includes futures markets for all of the relevant goods. Foley (1970)'s distinctive powerful insight is that this structure is mathematically equivalent to the Arrow-Debreu model. Assuming the usual continuity and convexity assumptions, a competitive equilibrium exists in the convex transaction cost economy, and the resulting allocation is Pareto efficient. The notion of Pareto efficiency here needs to take account of transaction costs: moving ownership from one firm or household to another is a resource using activity. Efficiency consists of efficient allocation net of the necessary resource cost of reassigning ownership.

Hahn (1973) treats the reopening of markets over time in a sequence economy, distinguishing between *essential* and *inessential* sequence economies. The issue treated is whether two otherwise identical economies have significantly different equilibrium prices and resource allocation depending on the character of the budget constraint: a single Arrow-Debreu budget for each household versus a time-dated sequence of budget constraints in a sequence economy. In this comparison it is necessary to take account of transaction costs, so the reference point is not the conventional Arrow-Debreu equilibrium without transaction costs, Debreu (1959). Rather, it is the allocation in an Arrow-Debreu economy with transaction costs, Foley (1970).

This paper adopts the same usage. The efficiency concept is subject to technically necessary transaction costs. A trading post equilibrium is 'inessential' if the resulting allocation is Walrasian, the same as in an Arrow-Debreu (Foley) economy with transaction costs. The equilibrium is inessential if the multi-faceted structure of the trading post budget constraint has no effect in itself on the resulting allocation of resources. Conversely, the trading post equilibrium will be described as 'essential' if the equilibrium resource allocation is non-Walrasian, differing because of the structure of budget constraints.

Then the resource allocation in an inessential trading post economy is a Walrasian equilibrium allocation and it is Pareto efficient by the First Fundamental Theorem of Welfare Economics. Conversely, a trading post economy is essential when the multi - faceted structure of budget constraints renders the equilibrium allocation of resources different from an Arrow-Debreu equilibrium (taking full account of the effect of transaction costs, with a complete array of futures markets). Then the equilibrium allocation will not be a Walrasian equilibrium and may be Pareto inefficient. The inefficiency arises in either of two ways: additional resources may be expended in fulfillment the multiplicity of budget constraints, or the allocation may be shifted (relative to Walrasian equilibrium) to fulfill the additional constraints. Since these circumstances represent real resource allocations to fulfill a purely administrative constraint, the reallocation is regarded as Pareto inefficient. This treatment is similar to Hahn (1973)'s treatment of sequence economies. A full development of efficiency conditions and detailed characterization of (in)essentiality is a significant topic, beyond the scope of this paper.

The array of economies subject to general equilibrium modeling includes essential and inessential trading post economies with resultant Walrasian and non-Walrasian allocations. Since the designation 'essential' or 'inessential' is based on the character of endogenous equilibrium pricing, it seems problematic to distinguish essential from inessential trading post economies a priori. The alternative is to review examples, several of which are presented below.

10.2 Economies actively Using Media of Exchange

The examples of sections 10.3.1 and 10.4.1 below illustrate the notion of trading post economies using media of exchange in equilibrium. They are characterized by economies where trade is mutually advantageous but direct trade between suppliers and final demanders at trading posts may be more costly in resources than indirect trade through a lower transaction cost instrument. This typically reflects two elements of the example: direct exchange is not fully mutually satisfactory because of absence of double coincidence of wants; transaction costs in some commodity may be lower than others, favoring its use as a carrier of value in exchange. For a particularly simple example, see Starr (2008). It is difficult fully to characterize the attributes of an economy, a priori, that will lead to these conditions, hence the reliance on examples. Nevertheless, the examples are intended to be robust. The parameters of the examples are intended to be elements of an be an open subset of parameter space where similar results hold.

10.3 Pareto Efficiency of Trading Post Equilibrium with Transaction Costless Media of Exchange

When there is a generally available zero-transaction cost medium of exchange, the trading post equilibrium will be inessential and the resulting allocation of resources Pareto efficient (taking into account transaction costs). The allocation will be a Walrasian equilibrium. Supposing that the transaction costs of media of exchange in advanced monetary economies are low (if not nil), the zero-cost case should be a significant limiting case.

However important, the result is not deep. The presence of a costless medium of exchange means that price ratios in a trading post economy will be the same as those of the corresponding Arrow-Debreu economy. The example of section 10.3.1 below illustrates the efficiency. The point of comparison is an economy with transaction costs, complete markets, efficient allocation in general equilibrium, a single budget constraint for each household and well-defined profit maximand for each firm, as in Foley (1970). Then apply the First Fundamental Theorem of Welfare Economics.

10.3.1 Example: A Natural Money absent Double Coincidence of Wants; Pareto Efficient Allocation in Trading Post Equilibrium

Let $H \equiv \{h = 1, 2, ..., N\}$ where $r^h_h = 100$ and where $u^h(c^h) = 20c^h_{h+1} + \sum_{n \neq h+1, n=1}^{N} c^h_n$ for $h = 1, ..., 99$, and for $h = N$, $u^h(c^h) = 20c^h_1 + \sum_{n \neq 1, n=2}^{N} c^h_n$. There are N households named h = 1, 2, ...,N; each endowed with 100 units of good h and strongly preferring good h+1 (mod N) to all others.

There are $N(N-1)/2$ firms denoted $\{i, j\}, j > i, i, j = 1, 2, ..., N$. The transaction technology of $\{i, j\}, i \neq 1$ is $Y^{\{i,j\}} \equiv \{(y, x) | \text{for } k = i, j, 0 \geq y_k \geq -0.8x_k; \text{for } k \neq i, j, y_k = x_k = 0\}$. For $\{i, j\}, i = 1, Y^{\{i,j\}} \equiv \{(y, x) | \text{for } k = 1, y_1 = -x_1, \text{for } j \neq 1, 0 \geq y_j \geq -0.8x_j; \text{for k } \neq i, j, y_k = x_k = 0\}$. That is, for each pair of goods there is a distinct trading post firm {i,j} and there is no arbitrage by firms between posts. Trade in all goods except good 1 experiences a 20% loss in the trading process.

The resulting equilibrium prices, for $i, j \neq 1$ are $(a^{\{i,j\}}_i, b^{\{i,j\}}_j) = (\frac{5}{8}, \frac{3}{8})$. For $i = 1, j \neq 2$ we have, $(a^{\{1,j\}}_1, b^{\{1,j\}}_j) = (\frac{1}{2}, \frac{1}{2})$, $(a^{\{1,j\}}_j, b^{\{1,j\}}_1) = (\frac{5}{9}, \frac{4}{9})$. For $\{1, 2\}$ we have $(a^{\{1,2\}}_1, b^{\{1,2\}}_2) = (\frac{1}{2}, \frac{1}{2})$, $(a^{\{1,2\}}_2, b^{\{1,2\}}_1) = (\frac{5}{9}, \frac{4}{9})$.

The trade flows for $h = 2, 3, ..., N - 1$, are $(x^{h\{h,1\}}_h, y^{h\{h,1\}}_1) = (-1, 1)$, $(x^{h\{1,h+1\}}_1, y^{h\{1,h+1\}}_{h+1}) = (-1, 0.8)$. For $h = N, (x^{N\{1,N\}}_N, y^{N\{1,N\}}_1) = (-1, 0.8)$. For $h = 1, (x^{1\{1,2\}}_1, y^{1\{1,2\}}_2) = (-1, 0.8)$. That is, direct trade of most goods i for j is prohibitively expensive, losing 40% of the goods in the transaction process. Indirect trade, through good 1, is more attractive since good 1 itself is transaction costless. The typical pattern of trade then is that household h sells endowment, good h, for good 1, then sells good 1 for the desired good, h+1. In the process, only 20% of goods are lost to transaction costs. In this is example all trade goes through good 1, and for N-1 of N traders good 1 is a medium of exchange. The allocation is Pareto efficient.

Is the Trading Post equilibrium a Walrasian equilibrium? Individual agent trading behavior in the trading post model differs from Walrasian behavior (e.g.in Foley (1970)) since it includes active use of a medium of exchange, good 1. But those trades are costless and net out to zero. The resulting resource allocation is fully consistent with Walrasian equilibrium and in a Foley (1970) economy (Arrow-Debreu with transaction costs) the allocation could be supported by Walrasian equilibrium prices. The allocation is Pareto efficient. This trading post economy is inessential.

10.4 Pareto Inefficiency of Trading Post Equilibrium with Costly Media of Exchange; An essential trading post economy

As in Hahn (1973) and Starrett (1973)'s analysis of a sequence economy, when the multi-faceted structure of the budget constraint in the trading post economy significantly affects the real allocation of resources, the resulting allocation is Pareto inefficient. This occurs because real resources spent or reallocated in fulfillment of the administrative requirement of budget constraints represent a waste. The expenditure or reallocation is administratively required but technically unnecessary.

10.4.1 Example: An essential trading post economy; Pareto Inefficient Allocation in Trading Post Equilibrium

The following example simply follows the format of the previous example, except that there is no costless medium of exchange. The result is a non-Walrasian Pareto inefficient allocation. The mechanism of inefficiency is transparent. Transactions will use the medium of exchange and incur the cost of doing so. The cost is a wasted resource; it is administratively required but fulfills no technical function. Let the population H and H's endowments and preferences be as described in section 10.3.1. There are $N(N-1)/2$ firms denoted $\{i,j\}, j > i, i,j = 1,2,...,N$. The transaction technology of $\{i,j\}, i \neq 1$ is $Y^{\{i,j\}} \equiv \{(y,x)|\text{for } k = i,j, 0 \geq y_k \geq -0.8x_k; \text{for } k \neq i,j, y_k = x_k = 0\}$. For $\{i,j\}, i = 1, Y^{\{i,j\}} \equiv \{(y,x)|\text{for } k = 1, y_1 = -x_1, \text{for } j \neq 1, 0 \geq y_1 + y_j \geq -0.9x_1 - 0.8x_j; \text{for k } \neq i,j, y_k = x_k = 0\}$. That is, for each pair of goods there is a distinct trading post firm {i,j} and there is no arbitrage by firms between posts. Trade in all goods except good 1 experiences a 20% loss of each good in the trading process; trading two goods incurs two 20% losses, 20% of each. Trade in good 1 with any other good j experiences a 30% loss in good j (a 10% saving compared to using any good other than 1 as medium of exchange, hence the desirability of trading through good 1 if a medium of exchange is to be used).

The resulting equilibrium prices, for $i,j \neq 1$ are $(a_i^{\{i,j\}}, b_j^{\{i,j\}}) = (\frac{5}{8}, \frac{3}{8})$. For $i = 1, j \neq 2$ we have, $(a_1^{\{1,j\}}, b_j^{\{1,j\}}) = (\frac{1}{2}, \frac{1}{2})$, $(a_j^{\{1,j\}}, b_1^{\{1,j\}}) = (\frac{10}{17}, \frac{7}{17})$. For $\{1,2\}$ we have $(a_1^{\{1,2\}}, b_2^{\{1,j\}}) = (\frac{1}{2}, \frac{1}{2})$, $(a_2^{\{1,2\}}, b_1^{\{1,j\}}) = (\frac{10}{17}, \frac{7}{17})$.

The trade flows for $h = 2,3,...,N-1$, are $(x_h^{h\{h,1\}}, y_1^{h\{h,1\}}) = (-1,1)$, $(x_1^{h\{1,h+1\}}, y_{h+1}^{h\{1,h+1\}}) = (-1, 0.7)$. For $h = N, (x_N^{N\{1,N\}}, y_1^{N\{1,N\}}) = (-1, 0.7)$. For $h = 1, (x_1^{1\{1,2\}}, y_2^{1\{1,2\}}) = (-1, 0.7)$. That is, direct trade of most goods i for j is prohibitively expensive, losing 40% of the goods in the transaction process. This reflects the absence of double coincidence of wants. A typical household directly trading good h for good h+1 necessarily incurs transaction costs on both sides of the bargain. Indirect trade, through good 1, is more attractive since good 1 itself carries lower transaction costs. The typical pattern of trade then is that household h sells endowment, good h, for good 1, then sells good 1 for the desired good, h+1. In the process, only 30% of good h+1 is lost to transaction costs.

In this example all trade goes through good 1, and for N-1 out of N traders good 1 is a medium of exchange. The allocation is not however Pareto efficient. Some of the resources used in the transaction process, 20% of gross endowment, is technically necessary to the reallocation. It is not wasted. But the transaction costs associated merely with fulfilling the pairwise trading post budget constraint, 10% of total endowment, is administratively necessary but not technically necessary. It's a waste. The equilibrium allocation represents the outcome in an essential trading post economy. It is not Pareto efficient.

Is the Trading Post equilibrium a Walrasian equilibrium? Individual agent trading behavior in the trading post model differs from Walrasian behavior (e.g.in Foley (1970)) since it includes active use of a medium of exchange, good 1. Those trades net out to a loss. The resulting resource allocation is inconsistent with Walrasian equilibrium. In a Foley (1970) economy (Arrow-Debreu with transaction costs) the allocation cannot be supported by Walrasian equilibrium prices and it is Pareto inefficient. This trading post economy equilibrium is essential.

10.5 Economies not Using Media of Exchange: Double Coincidence of Wants and Inactive Trade

Economies with full double coincidence of wants will typically not use media of exchange in trading post equilibrium. Supplies are directly exchanged for demands [3].

Alternatively, the economy may not use media of exchange simply because trade is unattractive. There are two obvious cases: a Pareto efficient endowment and prohibitive transaction costs.

[3] Exceptions to this generalization occur where multiple trades through a medium of exchange incur lower cost than a single direct trade. That reflects some cost associated with the interaction between the goods traded directly (e.g. gasoline and matches) or economies of scale in a high volume market with a common medium of exchange, Starr (2003B).

10.5.1 Full double coincidence of wants with linear transaction costs

Consider the following economy with full double coincidence of wants. Let $N \geq 2$ be an even integer. Let $H \equiv \{h = 1, 2, ..., N\}$ where $r^h_h = 100$ and where for h odd $u^h(c^h) = 20c^h_{h+1} + \sum_{n \neq h+1, n=1}^N c^h_n$, and for h even , $u^h(c^h) = 20c^h_{h-1} + \sum_{n \neq h-1, n=1}^N c^h_n$. There are N households named h = 1, 2, ...,N; each endowed with 100 units of good h and the odd numbered households strongly preferring good h+1, the even numbered households strongly preferring good h-1. Direct trade with the neighbor is the obvious policy. This will be true even if there is a low transaction cost instrument available, so long as direct trade is no more costly than indirect trade through the low transaction cost instrument.

Assume a population of firms and transaction technologies the same as in section 10.4.1.

The resulting equilibrium prices, for $i, j \neq 1$ are $(a^{\{i,j\}}_i, b^{\{i,j\}}_j) = (\frac{5}{9}, \frac{4}{9})$. For $\{1, 2\}$ we have $(a^{\{1,2\}}_1, b^{\{1,2\}}_2) = (\frac{10}{17}, \frac{7}{17})$, $(a^{\{1,2\}}_2, b^{\{1,2\}}_1) = (\frac{1}{2}, \frac{1}{2})$.

The trade flows for h odd, $h \neq 1, 2$ are $(x^{h\{h,h+1\}}_h, y^{h\{h,h+1\}}_{h+1}) = (-1, .8)$, $(x^{h\{h,h+1\}}_{h+1}, y^{h\{h,h+1\}}_h) = (0, 0)$. For h=even, $(x^{h\{h,h-1\}}_h, y^{h\{h,h-1\}}_{h-1}) = (-1, .8)$, $(x^{h\{h,h-1\}}_{h-1}, y^{h\{h,h-1\}}_h) = (0, 0)$. For $h = 1, 2$, $(x^{1\{1,2\}}_1, y^{1\{1,2\}}_2) = (-1, 0.7)$, $(x^{1\{1,2\}}_2, y^{1\{1,2\}}_1) = (0, 0)$, $(x^{2\{1,2\}}_1, y^{2\{1,2\}}_2) = (0, 0)$, $(y^{2\{1,2\}}_1, x^{2\{1,2\}}_2) = (1, -1)$.

All of the trade flows in this allocation are direct trade. There is no trade in media of exchange. This reflects the endowment, demand, and transaction cost structure: there is a double coincidence of wants, so there is little incentive to trade indirectly, and no transaction cost advantage to indirect trade. Thus, the example generates a trading post equilibrium without use of a medium of exchange. The trading structure and resulting allocation are Pareto efficient, and constitute a Walrasian equilibrium (allowing for transaction costs). The trading post economy is inessential. That is, the trade flows and resulting allocations would be the same — allowing for similar transaction technology — in a unified (Foley (1970)) trading setting.

10.5.2 Inactive trade: Pareto efficient endowment

In an economy where there is no need for trade, there is no use for media of exchange. If the endowment is Pareto efficient, there will be no use of media of exchange in a trading post equilibrium.

10.5.3 Inactive Trade: Prohibitive transaction costs

A far more interesting reason for a nil demand for media of exchange is overwhelming transaction costs. Costs high enough to discourage all trade will eliminate the demand for media of exchange as well.

Assume household population, tastes and endowments, the same as in section 10.3.1.

There are $N(N-1)/2$ firms denoted $\{i,j\}, j > i, i,j = 1,2,...,N$. The transaction technology of $\{i,j\}$, all i, j, is $Y^{\{i,j\}} \equiv \{(y,x) | \text{for } k = i,j, 0 \geq y_k \geq -0.1x_k; \text{for } k \neq i,j, y_k = x_k = 0\}$. That is, for each pair of goods there is a distinct trading post firm {i,j} and there is no arbitrage by firms between posts. Trade in all goods experiences a 90% loss in the trading process. A pair of trades, using an intermediary good compounds the loss: 99% loss in two successive trades.

The resulting equilibrium prices, for i,j are $(a_i^{\{i,j\}}, b_j^{\{i,j\}}) = (\frac{99}{100}, \frac{1}{100})$. The endowment is the equilibrium allocation. No household wishes to trade at a discount of 99% — but this is just break-even for the firms considering the oppressive transaction technology. The allocation is non-Walrasian and is far from Pareto efficient — one-step rearrangements for each good would be a grand Pareto improvement, even incurring 90% transaction costs. But that calculation ignores the 90% transaction cost on payment of quid pro quo, necessarily incurred in a trading post equilibrium. This calculation reflects the dual problems of transaction costs and absence of double coincidence of wants — if there were a better match of suppliers with demanders even 90% transaction costs could be borne and mutually beneficial trades undertaken. But the absence of double coincidence of wants means that each trade undertaken benefits directly only one side. Two trades and two sets of transaction costs must be incurred in the trading post economy, and transaction costs then swamp the gains from trade.

11 Conclusion

This essay creates a parsimonious model where a medium of exchange (commodity money) is an outcome of the (slightly augmented) Arrow-Debreu general equilibrium. The monetary structure of trade is a result of the price theory general equilibrium. Monetary trade is not a separate assumption; monetary exchange is an outcome, a direct implication of the general equilibrium when there are multiple distinct budget constraints facing each agent.

The trades of firms and households in a trading post economy may be characterized by many separate transactions, each fulfilling a separate budget constraint. In an economy of N commodities there are $N(N-1)/2$ trading posts, one for each pair of goods. The trading post model reformulates the budget so that each of many separate transactions fulfills its own budget constraint. This treatment generates a demand for carriers of value (media of exchange) moving among successive trades, Starr (2003A, 2003B). Virtually the same axiomatic structure, Arrow and Debreu (1954), that ensures the existence of general equilibrium in the model of a unified market without transaction costs yields existence of equilibrium and a well-defined demand for media of exchange in this disaggregated setting.

Trading post equilibria are Pareto efficient when they are simply the elaboration of an underlying Walrasian equilibrium, an inessential trading post economy; see also Hahn (1973). However, the multiplicity of separate budget constraints and the additional transaction costs incurred or avoided may skew the allocation and pricing (an essential trading post equilibrium). Then the equilibrium cannot be supported by a Walrasian price structure and the allocation will be Pareto inefficient; see also Starrett (1973).

The price system is informative not only about scarcity and desirability. It also prices liquidity. Transaction costs generate a spread between bid and ask prices at each trading post. The bid-ask spread tells firms and households which goods are liquid, easily traded without significant loss of value, and which are illiquid, unsuitable as carriers of value between trades, Menger (1892). The multiplicity of budget constraints creates the demand for liquidity; the bid-ask spreads signal its supply. When liquidity is too expensive (example 10.5.3), media of exchange will not be used. When liquidity is inexpensive and helpful in achieving a Pareto improving allocation (example 10.3.1), media of exchange will be actively traded in equilibrium. The trading post model endogenously generates a designation and a flow of commodity money(ies). The existence of (commodity) money and the monetary structure of trade is an outcome of the general economic equilibrium. Money is not a separate assumption; it is a result of the equilibrium allocation.

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