Title

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ISBN

9798189270710

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Publication Date

2026-05-01

Peer reviewed|Thesis/dissertation

Expanding Ricci Solitons and the Expander Degree

by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Richard Bamler, Chair
Professor Vera Serganova
Professor Daniel Tataru

Spring 2026

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Abstract

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We consider the notion of **expander degree**, defined for certain smooth 4-orbifolds M with boundary, defined by Bamler and Chen. The expander degree is a topological invariant of M, and importantly, when the expander degree of a 4-manifold M is nonzero, there exists a gradient expanding soliton asymptotic to any nonnegatively curved cone metric. This fact is significant in the resolution of conical singularities in 4-dimensional Ricci flow by gradient expanding solitons.

We begin by reviewing Ricci flow and properties of Ricci solitons. Then, we give an introduction to the theory of the expander degree. Subsequently, we define a version of the expander degree in a cohomogeneity one setting and calculate it for certain topologies. In doing so, we describe a relation between the two notions of expander degree and compare the general degree theory with the cohomogeneity one setting.

To Amma, for being such a great mom.

Contents

Acknowledgments

First and foremost, I would like to express my deepest gratitude to my advisor, **Richard Bamler**. He has been incredibly generous with his time and has consistently provided me with invaluable advice. I am especially grateful for his unwavering belief in my abilities, even—and perhaps most importantly—during the times when I did not believe in myself. I also want to thank him for his practical support through Graduate Student Researcher positions funded by his grants, which made this work possible.

I am sincerely grateful to **Daniel Tataru** and **Vera Serganova** for serving on my committee and for their time in evaluating this dissertation.

A special thank you goes to **Eric Chen**, who was a postdoc at UC Berkeley and is now an assistant professor at UIUC. His advice and our many discussions throughout my PhD were incredibly helpful to my development as a researcher. I also owe a great deal to the geometry community at UC Berkeley. Thank you to **Garrett Brown** and **Yueqing Feng** for their camaraderie, and a specific thanks to **Carlos Esparza**, who was always available and willing to help me navigate difficult technical discussions.

Getting through a PhD requires a strong support system, and I am lucky to have had amazing friends by my side. Thank you to **Lewis Pan** for always letting me hang out in his office, and for keeping my spirits high with his incredible cooking and baking. Thank you to **Yuanpu (Galen) Liang**, **Zack Stier**, and **Catherine Lee** for being wonderfully supportive, friendly, and welcoming officemates. A specific thanks to Catherine for being such a good and reliable friend throughout the entirety of my PhD. I am also deeply grateful to **Ansuman Bardalai** for his friendship, for being great to hang out with, and for always knowing how to motivate me.

I want to extend a warm thank you to my neighbors, **Prakruthi Burra** and **Krishnakant Saboo**, who were always so welcoming in hosting us at their house and who steadfastly stuck by me during the toughest of times. Thank you as well to **Sukanya Kudva** and **Samvardhini Sridharan** for the countless walks and ice cream runs; you have both been wonderful friends and provided consistent, uplifting support throughout this entire journey.

Even from afar, my friends from my time in India—**Vinay Vikramaditya**, **Chinmaya Kausik**, **Rohit Kumar**, **Piush Jena**, and **Sanjeev Kumar**—have been a constant source of encouragement. Thank you for always being just a text away; I only wish I could visit you all more often.

Finally, my most profound gratitude belongs to my mom, **Gayatri Ramasami**. Thank you for being the only person who has stood with me my entire life. I could not have reached this milestone without your constant presence and support.

Chapter 1

Background and Introduction

This thesis is based on work of the author in [Abi25] entitled "Cohomogeneity One Expanding Ricci Solitons and the Expander Degree" and upcoming work. Broadly, the goal of the thesis is to review the literature on the **expander degree**, a topological invariant of smooth 4-orbifolds with boundary defined in [BC23], and calculate it in certain specialized settings.

In this chapter, we first present an overview of Ricci flow and Ricci solitons. In the next chapter, we review the literature on the expander degree. In Chapter 3, we define and calculate the **cohomogeneity one expander degree** as in [Abi25] in the cases of two specific smooth 4-manifolds with boundary. In Chapter 4, we compare the cohomogeneity one setting with the general setting and answer a version of an open question from [BC23] based on results from Chapter 3. In Chapter 5, we present some results on the structure of steady solitons that are asymptotically 2-cylindrical, which is relevant for the general question of expander degree (beyond cohomogeneity one).

1.1 Ricci Flow

The central object of study in this thesis is the Ricci flow equation, a geometric evolution equation introduced by Richard Hamilton in 1982 in [Ham82]. Let M be a smooth manifold. A one-parameter family of Riemannian metrics $g(t)$ on M is a solution to the Ricci flow equation if it satisfies the nonlinear partial differential equation:

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)} \tag{1.1}$$

where $\mathrm{Ric}_{g(t)}$ denotes the Ricci curvature tensor of the metric $g(t)$.

Ricci flow is essentially a non-linear heat equation for the Riemannian metric. Similar to how the standard heat equation smooths out temperature distributions, Ricci flow attempts to smooth out curvature irregularities and evolve the metric into one of constant curvature. In regions where the Ricci curvature is positive (where the local geometry is similar to that of the round sphere), the metric contracts; where it is negative (locally similar to hyperbolic space), the metric expands.

However, due to the nonlinearity of Equation (1.1), it does not always globally improve the geometry of the metric. More precisely, there exist Riemannian manifolds (M, g) for which the evolution of g by the Ricci flow leads to the development of a singularity in the metric in finite time. An example of this is the non-degenerate neckpinch, a metric on S^3 which develops a finite-time singularity modeled on the round cylinder $S^2 \times \mathbb{R}$. A thorough understanding of Ricci flow necessarily involves careful analysis of singularities.

In spite of the difficulties mentioned above, Ricci flow has had a significant impact on geometry and topology. A large body of applications are in dimension 3, perhaps most notably the Poincaré conjecture, in which the singularity models of Ricci flow were classified by Perelman in [Per02, Per03]. Perelman developed a surgery process to continue the flow beyond each of the possible singularities in 3-dimensions. In the rest of this chapter, we will explore solutions to the Ricci flow equation called **Ricci solitons** that allow us to understand singularity formation.

1.2 Self-Similar Solutions to the Ricci Flow Equation

Consider a Riemannian manifold (M, g) of any dimension with $\mathrm{Ric}_{g_0} \equiv 0$. The family of metrics $g(t) = g_0$ is a trivial solution to the Ricci flow.

Slightly more nontrivial examples are given by Einstein metrics. Consider a Riemannian manifold (M, g) that satisfies $\mathrm{Ric}_{g_0} = \lambda g$ for some constant $\lambda \in \mathbb{R}$. Then, the family of metrics

$$g(t) = (1 - 2\lambda t) g_0$$

is a solution to Equation (1.1). We note here that the behavior of the solution depends on the sign of λ. For $\lambda > 0$, the metric shrinks with time and collapses to a point $T = \frac{1}{2\lambda}$. The interval of existence of the solution is $(-\infty, T]$. An example of this case is the round shrinking sphere $(S^n, g_{\mathrm{round}})$ for $n \geqslant 2$, for which the associated value of λ is $n - 1$.

On the other hand, for $\lambda < 0$, the metric expands with time, and the interval of existence is $[T, \infty)$. An example of this case is hyperbolic space $(H^n, g_{\mathrm{standard}})$ for $n \geqslant 2$, for which the associated value of λ is $-(n - 1)$.

In all of the cases above, the metrics evolve purely by rescaling, and the metric at any time slice t is homothetic to the metric at any other time slice. However, there is an even more general version of a self-similar solution, called a **Ricci soliton**, in which the metric evolves by rescaling modulo diffeomorphisms. Ricci solitons, which can be seen as a generalization of Einstein metrics, arise due to the diffeomorphism invariance of Equation (1.1). More precisely, we have the following:

Definition 1.2.1. *A Riemannian manifold (M, g) is a* ***Ricci soliton*** *if there exists a smooth vector field X and a constant $\lambda \in \mathbb{R}$ such that:*

$$\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_X g = \lambda g$$

where Ric *is the Ricci tensor and $\mathcal{L}_X$ denotes the Lie derivative in the direction of X.*
The soliton is called a ***gradient Ricci soliton*** *if $X = \nabla f$ for some smooth function $f : M \to \mathbb{R}$, leading to:*

$$\mathrm{Ric}_g + \nabla^2 f = \lambda g$$

Ricci solitons are classified based on the sign of λ:

- **Shrinking** if $\lambda > 0$: These are generalizations of positively curved Einstein manifolds, such as the round shrinking sphere mentioned earlier. The interval of existence is of the form $(-\infty, T]$ for some $T \in \mathbb{R}$. Other examples of shrinking solitons include the round shrinking cylinder $(S^m \times \mathbb{R}^n, g_{\text{round}} \oplus g_{\text{Euc}})$ and the Gaussian shrinking soliton on $(\mathbb{R}^n, g_{\text{Euc}})$ with potential function $\frac{\lambda}{2}|x|^2$.

- **Steady** if $\lambda = 0$: When $\lambda = 0$, the gradient Ricci soliton equation simplifies to $\mathrm{Ric}_g + \nabla^2 f = 0$, and the interval of existence is $(-\infty, \infty)$. Steady solitons are a class of **eternal solutions**. Along with shrinking solitons, they are a class of **ancient flows**, since in both cases, the solution exists for all arbitrarily negative times. The most common example is the **Bryant soliton**, which is an asymptotically cylindrical $\mathrm{SO}(n)$ invariant metric on $\mathbb{R}^n$ with positive curvature that decays to 0 at infinity.

- **Expanding** if $\lambda < 0$: These are generalizations of negatively curved Einstein manifolds like hyperbolic space. Since the interval of existence is of the form $[T, \infty)$ for some $T > 0$, these are **immortal** flows. Many examples of expanding solitons are asymptotic to cones at infinity. The correspondence between expanding solitons and their asymptotic cones is the focus of this thesis.

Next, we will see how each kind of soliton is related to modeling singularities in Ricci flow.

1.3 Ricci Solitons and Singularities

As mentioned earlier, solutions to Equation (1.1) generally become singular in finite time. The singularities are studied by the method of blow-up analysis, a common technique in geometric flows. If the metric becomes singular at the point x at the time T, we can choose a sequence of times $t_i \to T$ and points $x_i \to x$ where the curvature $|\mathrm{Rm}|_{g_i(t)}$ blows up, we define a sequence of dilated solutions $g_i(t) = Q_i g(t_i + Q_i^{-1} t)$, where $Q_i = |\mathrm{Rm}(x_i, t_i)|$.

By a result of Hamilton, under appropriate conditions on the metric (such as uniform local curvature bounds and non-collapsing), the pointed sequence $(M, g_i(t), x_i)$ of Ricci flows converges to a singularity model $(M_\infty, g_\infty(t))$ which is an ancient flow.

Singularities are categorized into two types based on the asymptotic behavior of the curvature tensor Rm:

- **Type I Singularities:** These occur when the curvature blow-up rate is bounded by the reciprocal of the remaining time. Specifically, there exists a constant $C > 0$ such that:
$$\sup_M |\mathrm{Rm}| \leq \frac{C}{\sqrt{T-t}}$$
Type I singularities generally rescale to gradient shrinking solitons, such as the round shrinking sphere S^n or the round shrinking cylinder $S^n \times \mathbb{R}^m$.

- **Type II Singularities:** These represent a faster, more degenerate blow-up where no such constant C exists:
$$\limsup_{t\to T} \left(\sqrt{T-t} \sup_M |\mathrm{Rm}| \right) = \infty$$
Type II singularities often rescale to gradient steady solitons, such as the Bryant soliton.

In 3 dimensions, the singularity models have been classified as follows:

- **Compact Shrinkers:** The round shrinking sphere S^3 and its quotients (e.g., S^3/Γ).
- **Non-compact Shrinkers:** The round shrinking cylinder $S^2 \times \mathbb{R}$ and its $\mathbb{Z}_2$ quotient.
- **Steady Solitons:** The Bryant soliton, which is the unique 3D rotationally symmetric, noncollapsed, steady gradient soliton.

Thus, we see that in 3-dimensional Ricci flow, all singularity models are either gradient shrinking or steady Ricci solitons. However, in dimensions 4 dimensions and higher, the structures of singularity models are far more complicated. In [Bam20a, Bam20b], it has been shown that singularity models themselves may have singularities comprising a singular set of codimension 4. Additionally, conical singularity models appear in dimension 4.

Analogous to Perelman's work in 3 dimensions, it is important to understand how to continue the flow beyond singularities in 4-dimensions. Of particular interest are the conical singularities, since they do not have lower dimensional analogues.

Thus, one natural question that arises is whether there exists a gradient expanding soliton (of appropriate topology) that is asymptotic to a given cone. Work on this problem has been carried out by [BC23], in which a topological invariant called the **expander degree** of a smooth 4-orbifold with boundary M is defined. Given a cone of nonnegative scalar curvature, the expander degree roughly counts, with sign, the number of gradient expanding solitons defined on the interior of M that are asymptotic to the cone metric, and this invariant is independent of the choice of the geometry of the cone. We will introduce the expander degree in the next chapter and discuss variants of it and how to calculate them in subsequent chapters.

Chapter 2

Expander Degree

In the previous chapter, we introduced Ricci flow and understood how Ricci solitons play an important role in modeling singularities in Ricci flow. In this chapter, we will specialize to the case of conical singularities in 4-dimensional Ricci flow and their resolution by gradient expanding solitons. In most of this chapter, we will review work of Bamler and Chen in [BC23].

Work of Bamler in [Bam20a, Bam20b, Bam20c] shows a marked difference in the behavior of singularities in Ricci flow in 4 dimensions as compared to 3; as mentioned in the previous paragraph, singularities may be described by conical singularity models. Recent progress by [AK22] and [GS18] has suggested that conical singularities can be resolved by gradient expanding solitons which are asymptotically conical, and in particular asymptotic to the cone that models the singularity. This naturally motivates the following question from [BC23]:

Question: *Given a 4-dimensional Riemannian cone with non-negative scalar curvature, when is it possible to find a gradient expanding soliton with non-negative scalar curvature that is asymptotic to this cone?*

To resolve this question, Bamler and Chen construct an invariant called the **expander degree** of a smooth 4-dimensional orbifold $(X, \partial X)$ with boundary, denoted $\deg_{\exp}(X)$. Roughly, $\deg_{\exp}(X)$ is a signed count of the number of gradient expanding solitons on the interior $\mathrm{Int}(X)$ which are asymptotic near ∂X to a fixed cone metric with non-negative scalar curvature. Despite the definition involving cone metrics and gradient expanding soliton structures, the expander degree depends only on the smooth structure of X.

In the remainder of this section, we will outline the degree theory constructed in [BC23] and explain the proofs of some important results that will be particularly useful for us in the next chapter in the setting of the cohomogeneity one expander degree.

2.1 Overview of Degree Theory

Fix a smooth 4-dimensional orbifold with boundary $(X, \partial X)$ with isolated singularities. Suppose that X satisfies the following technical conditions:

1. The boundary ∂X, which is an orientable 3-manifold (or has such a double cover), admits a metric of positive scalar curvature. As a consequence of many results in topology, including the Geometrization Conjecture, ∂X is diffeomorphic to a connected sum:
$$\partial X \cong M_1 \# M_2 \# \dots \# M_k \# (S^2 \times S^1)_1 \# \dots \# (S^2 \times S^1)_l$$
where each M_i is a spherical space form.

2. X has a possibly infinite-sheeted orbifold cover $\hat{X}$ such that $\hat{X}$ is a manifold satisfying the following topological conditions:
$$H_2(X;\mathbb{Z}) = 0, \qquad H_1(X;\mathbb{Z}) \text{ is torsion free}$$

With the two conditions above, we can begin to define the space of asymptotically conical gradient expanding solitons over an orbifold with boundary $(X, \partial X)$. In what follows, k^* is either a large integer or infinity.

Definition 2.1.1. *An asymptotically conical gradient expanding soliton structure on Int(X) is given by a triple $(g, \nabla f, \gamma)$, where g denotes a C^{k^*}-regular Riemannian metric on Int(X) and ∇f is a C^{k^*}-regular gradient vector field such that the gradient expanding soliton equation holds:*

$$Ric_g + \nabla^2 f + g = 0$$

In addition, γ represents a C^{k^*}-regular cone metric on $\mathbb{R}^+ \times \partial X$ of the form $\gamma = dr^2 + r^2 h$, where h is a metric on ∂X such that there exists a tubular neighborhood $\partial X \subset U \subset X$ of the boundary, which, upon identification $U \equiv [1, \infty) \times \partial X$, the following hold on $(r_0, \infty) \times \partial X$:

$$g = \gamma + O(r^{-2}) \qquad \nabla^g f = -r\partial_r$$

where r is the first coordinate on $U \subset X$ and ∂_r is the associated coordinate vector field.

Definition 2.1.2. *Define an equivalence relation on triples $(g, \nabla f, \gamma)$ by $(g_1, \nabla f_1, \gamma_1)$ $(g_2, \nabla f_2, \gamma_2)$ if there exists a diffeomorphism $\phi : X \to X$ that fixes ∂X such that $\phi^* g_2 = g_1$ and $\phi^*(\nabla f_2) = \nabla f_1$.*

Denote by $\mathcal{M}^{k^*}_{grad}(X)$ the set of equivalence classes (under the relation defined above of triples) $(g, \nabla f, \gamma)$. $\mathcal{M}^{k^*}_{grad}(X)$ thus represents the space of asymptotically conical gradient expanding soliton structures on $(X, \partial X)$ up to isometry.

Denote by $\mathrm{Cone}^{k^*}(\partial X)$ *the space of C^{k^*}-regular cone metrics $\gamma = dr^2 + r^2 h$ with link $(\partial X, h)$.*

Having defined the relevant spaces, we can now consider the projection

$$\Pi : \mathcal{M}^{k^*}_{\mathrm{grad}}(X) \to \mathrm{Cone}^{k^*}(\partial X) \tag{2.1}$$

which is defined by $\Pi([g, \nabla f, \gamma]) = \gamma$. Note that this is well-defined by the definition of the equivalence relation, since the isometries fix the boundary and thus preserve the cone metric.

For technical reasons, it becomes important to restrict Π to the set of equivalence classes of solitons with nonnegative scalar curvature as follows:

$$\Pi : \mathcal{M}^{k^*}_{\mathrm{grad},R\geqslant 0}(X) \to \mathrm{Cone}^{k^*}_{R\geqslant 0}(\partial X) \tag{2.2}$$

It is a nontrivial result that restricting to solitons of nonnegative scalar curvature results in cones of nonnegative scalar curvature.

One of the main results of [BC23] is the fact that Π from Equation (2.2) is a proper map and that it has a well-defined degree in a generalized sense, denoted $\deg_{\mathrm{exp}}(X)$.

Note that since both of these spaces are infinite-dimensional, a major difficulty in defining the degree of $\Pi|_{\mathcal{M}^{k^*}_{\mathrm{grad},R\geqslant 0}(X)}$ is that since the space of gradient expanding solitons may not be a (Banach) manifold. This complicates the question of existence of a regular value for this map; as mentioned in Section 1.2 of [BC23], the derivative of $\Pi|_{\mathcal{M}^{k^*}_{\mathrm{grad},R\geqslant 0}(X)}$ may be degenerate everywhere along the differential.

To partially overcome this difficulty, we develop a theory of expander degree in the cohomogeneity one setting in the following chapter. Additionally, we explain a relation between this cohomogeneity one expander degree and the original invariant from [BC23]; this indicates that the cohomogeneity one setting may provide useful information about the general case.

2.2 Comparison with Cohomogeneity One Degree

We point out two main differences between the cohomogeneity one setting and the work of Bamler and Chen; firstly, we are able to avoid the assumption of nonnegative scalar curvature. In Lemma 4.4 of [BC23], a non-collapsing property is proven for gradient expanding solitons $(M, g, \nabla f)$ assuming the inequality $R_g \geqslant 0$. This is done by establishing a lower bound on Perelman's ν functional of the form

$$\nu[g] \geqslant -C(n, A)$$

where A is a constant such that the asymptotic cone $\gamma = dr^2 + r^2 h$ of (M, g) satisfies $|\mathrm{Rm}|_h \leqslant A$ and $\mathrm{inj}(h) \geqslant A^{-1}$.

This non-collapsing property is crucial in proving certain properties of gradient expanding solitons in terms of their asymptotic cones; one particularly important example is Lemma 4.32(a) of [BC23], in which it is shown that bounds on the geometry $|\mathrm{Rm}|_h \leqslant A$ and $\mathrm{inj}(h) \geqslant$

A^{-1} of the asymptotic cone imply a bound on the curvature of the soliton of the form $|\mathrm{Rm}|_g \leqslant C(A)$. This argument proceeds by assuming that a sequence exists which perpetually violates this bound, and producing a Cheeger-Gromov limit that is non-collapsed at all scales to reach a contradiction.

In the cohomogeneity one setting, as explained in the Appendix C in the following Chapter, the warped product structure of the solitons implies that geometric convergence of metrics is equivalent to convergence of sequences of solutions to a system of ODEs. Thus, we are in an entirely analytic situation in which the hypothesis of nonnegative scalar curvature is not required.

Additionally, unlike [BC23], our specialized cohomogeneity one expander degree theory is developed only for the smooth 4-manifolds with boundary $S^1 \times \mathbb{D}^3$ and $S^2 \times \mathbb{D}^2$ whose respective interiors are $S^1 \times \mathbb{R}^3$ and $S^2 \times \mathbb{R}^2$.

Notice that the manifold $M = S^2 \times \mathbb{R}^2$ does not satisfy $H_2(M, \mathbb{Z}) = \{0\}$; in [BC23], this assumption is necessary to prove an "instanton exclusion principle" in Lemma 4.35, which rules out the possibility of Cheeger-Gromov convergence to a Ricci-flat metric. In fact, even in the cohomgeneity one setting, there exists a (unique) nontrivial Ricci-flat metric on $S^2 \times \mathbb{R}^2$, known as the Euclidean Schwarzschild metric. However, again by using specialized ODE arguments, it is possible to prove that the Euclidean Schwarzschild metric does not arise as a limit and thus produces no complications.

Chapter 3

Cohomogeneity One Expander Degree

In the previous chapter, we described work of Bamler and Chen [BC23] in which a degree theory is constructed for asymptotically conical gradient expanding solitons. In this chapter, we define an analogous quantity which we call the **cohomogeneity one expander degree**, denoted $\deg_{\exp}^{\mathrm{sym}}$. Making this definition involves a certain properness result, whose proof takes up a bulk of this work.

Our main results are the following:

Theorem 3.0.1. $\deg_{\exp}^{\mathrm{sym}}(S^1 \times \mathbb{D}^3) = 1$

Theorem 3.0.2. $\deg_{\exp}^{\mathrm{sym}}(S^2 \times \mathbb{D}^2) = 0$

To prove this theorem, we construct a 2-parameter family of gradient expanding Ricci solitons each with an isometric action of $\mathrm{SO}(3) \times \mathrm{SO}(2)$ over the topologies $S^1 \times \mathbb{R}^3$ and $S^2 \times \mathbb{R}^2$. We note that the same solitons were originally constructed and analyzed in [BDGW15], [Win21], and [NW24] using a different coordinate system. In the case of each topology, the metric can be written as a doubly warped product

$$g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2} \tag{3.1}$$

for smooth functions a and b with a soliton potential function f which is also invariant under the group action. The high degree of symmetry possessed by these solitons implies that the expanding soliton equation $\mathrm{Ric}_g + \nabla^2 f + g = 0$ reduces to a system of 3 ordinary differential equations in a, b and f. It is possible to ensure that a soliton has the required topology by setting the initial conditions to the ODEs (at $r = 0$) appropriately. In both cases, one of the initial conditions is the value of $f''(0)$; as the soliton equations are degenerate at $r = 0$, $f''(0)$ must be specified in order to obtain a unique solution.

To prove Theorem 3.0.1 and Theorem 3.0.2, we first reconstruct the solitons from [BDGW15], [Win21], and [NW24] using the coordinate system of [A17]. Along the way, we prove the following theorem, which is a special case of the aforementioned work. We wish to point out that while this theorem is already known, the estimates we use in our alternate proof will be of further use when defining and calculating the expander degree.

Theorem 3.0.3. *Suppose M is diffeomorphic to either $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$. There exists a two-parameter family of complete gradient expanding Ricci solitons on M, each invariant under the standard action of* $\mathrm{SO}(3) \times \mathrm{SO}(2)$*. Further, these solitons are asymptotic to cones over the link $S^2 \times S^1$.*

In each case, the 2 parameters are the initial conditions of the soliton equations. In the $S^1 \times \mathbb{R}^3$ case, the pair of initial conditions is $(a(0), f''(0))$, where $a(0) \equiv a_0$ is the size of the S^1 orbit at $r = 0$ in (1.1) and $f''(0) \equiv f_0$ is as described above, while in the $S^2 \times \mathbb{R}^2$ case, the pair of initial conditions is $(b(0), f''(0))$ where $b(0) \equiv b_0$ is the size of the S^2 orbit at $r = 0$. In both cases, as the constructed solitons are asymptotic to cones, the functions $a(r)$ and $b(r)$ are asymptotic to linear functions, whose slopes we denote a'_∞ and b'_∞, respectively. The corresponding cone metric is $\gamma = ds^2 + (a'_\infty s)^2 g_{S^1} + (b'_\infty s)^2 g_{S^2}$.

We will show that for solitons of bounded curvature, the condition $f_0 < 0$ (along with either $a_0 > 0$ or $b_0 > 0$) is necessary and sufficient for a complete solution, in which case a and b are asymptotically linear. Note that as our goal is to analyze asymptotically conical solitons, we do not lose anything by assuming bounded curvature.

Thus, in the case of either topology, we can consider the map $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ which takes the initial conditions $(a_0, -f_0)$ (in the $S^1 \times \mathbb{R}^3$ case) or $(b_0, -f_0)$ (in the $S^2 \times \mathbb{R}^2$ case) to the slopes (a'_∞, b'_∞).

We further show that the asymptotic cone of the soliton varies continuously in the initial conditions, which amounts to showing that the map F is continuous. Further, we show that F is a proper map. This allows us to define the degree of the map F.

In [BC23], an invariant called the **expander degree**, denoted $\deg_{\exp}$, was defined for a certain class of compact smooth 4-orbifolds with boundary. The expander degree of such an orbifold roughly counts (with sign), for any fixed cone metric γ, the number of gradient expanding solitons defined on the interior of the orbifold with non-negative scalar curvature which are asymptotic to γ.

Analogous to [BC23], in this paper, we define the **cohomogeneity one expander degree**, denoted $\deg_{\exp}^{\mathrm{sym}}$, of the orbifolds $S^1 \times \mathbb{D}^3$ and $S^2 \times \mathbb{D}^2$, as the degree of the map F in the $S^1 \times \mathbb{R}^3$ and the $S^2 \times \mathbb{R}^2$ cases, respectively. We note that we do not require the hypothesis of nonnegative scalar curvature as in [BC23], although we reiterate that our results are only valid for cohomogeneity one solitons. Similar to the general case, $\deg_{\exp}^{\mathrm{sym}}$ represents the number (counted with sign) of cohomogeneity one gradient expanding solitions defined on the interior of the orbifold which are asymptotic to a fixed cone metric.

The proof of Theorem 3.0.3 is based on understanding the behaviors of the profile functions a and b in Equation (3.1); namely, that these functions are increasing. Additionally, the soliton potential f is non-positive with non-positive first and second derivatives. Putting these inequalities together, we show that the functions a, b and f can be extended to the interval $[0, \infty)$, showing that the corresponding solitons are complete. We note that these solitons were constructed previously and analyzed as particular cases of a more general method in [NW24, Win21, BDGW15]. However, the methods we use are more amenable to proving certain curvature estimates and proving a properness result used to define the cohomogeneity one expander degree.

Then, we understand how the curvature of the soliton decays at infinity. We show that $|\mathrm{Rm}_g| \leqslant C/r^2$, where C is a constant that depends continuously on the initial conditions for either topology. From this, we show that the slopes $a'(r)$ and $b'(r)$ respectively converge to finite positive limiting values a'_∞ and b'_∞ as $r \to \infty$, indicating that the soliton metric g is asymptotic to the cone metric $\gamma = ds^2 + (a'_\infty s)^2 g_{S^1} + (b'_\infty s)^2 g_{S^2}$. We then extend this result to show that the asymptotic cone of an expanding soliton varies continuously as the soliton varies; this translates into the fact that the slopes a'_∞ and b'_∞ are continuous functions of the initial conditions.

Having done the above, we verify that in each case, the respective map F is proper using tools from [BC23] and proceed to calculate the cohomogeneity one expander degree individually in both cases.

In Sections 2,3 and 4, we derive the soliton equations and prove the monotonicity of the functions a, b and f. In Section 5, we show that the solitons in each case are complete and form a 2-parameter family. In Section 6 and 7, we show that the solitons are asymptotically conical and that the slopes vary continuously in the parameters. Then, we define the map F and show that it is continuous. Additionally, we quantify how close an asymptotic cone metric is to the corresponding expanding soliton in terms of the geometry of the cone. This relies on a technical non-existence result of certain two-ended expanding solitons. In Section 8, we describe how the asymptotic cones vary as the initial conditions approach their extreme values. Finally, in Section 9, we prove that F is proper using the results of Section 8. Then, we define the cohomogeneity one expander degree and prove Theorem 3.0.1 and Theorem 3.0.2.

In Appendix A, we derive the soliton equations under the assumption of the given symmetries. In Appendix B, we explain why the soliton equations have a local solution. In Appendix C, we explain the relationship between Cheeger-Gromov convergence of warped product metrics and convergence of the respective warping functions.

Acknowledgements: The author would like to thank his PhD advisor, Prof. Richard Bamler, for suggesting the problem and for many useful discussions and several pieces of useful advice. We also thank Eric Chen for helpful discussions.

3.1 Soliton Equations and Boundary Conditions

Our goal is to understand gradient expanding Ricci solitons (M, g, f) in 4 dimensions with an effective isometric action of $\mathrm{SO}(3) \times \mathrm{SO}(2)$. Thus, we look for metrics of the form

$$g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2}$$

with the normalization

$$\mathrm{Ric}_g + \nabla^2 f + g = 0$$

where $g_{S^1} = d\theta^2$ is the standard metric on the circle S^1 and g_{S^2} is the round metric on the sphere S^2. We note that both expanding and steady gradient symmetric Ricci solitons on various topologies have been studied in the past, such as in [BDGW15] and [BDW15], as well as [NW24] in which large families of expanding solitons were constructed, generalizing the $S^2 \times \mathbb{R}^2$ and $S^1 \times \mathbb{R}^3$ solitons constructed in this paper. Using a setup similar to that of [A17], in Appendix A, we explain how the expanding soliton condition is equivalent to the following equations for $r > 0$

$$f'' = \frac{a''}{a} + 2\frac{b''}{b} - 1 \tag{3.2}$$

$$a'' = -2\frac{a'b'}{b} + a'f' + a \tag{3.3}$$

$$b'' = \frac{1-(b')^2}{b} - \frac{a'b'}{a} + b'f' + b \tag{3.4}$$

where the smooth functions $f, a, b : [0, \infty) \to \mathbb{R}$ depend only on the coordinate r. Considering the soliton equations in this coordinate system as opposed to methods in the aforementioned papers simplifies the (analytic) proofs of completeness and allows us to prove (geometric) curvature estimates in Section 6. These estimates not only aid in proving that each soliton is asymptotically conical, but also gives us a notion of "uniform ϵ-conicality" (defined in Section 7), which allows us to define the cohomogeneity one expander degree later.

As g must be a smooth metric at $r = 0$, the boundary conditions must be chosen appropriately. It can be seen that some of the boundary conditions are determined by the topology of M. We will be particularly interested in the cases of M being diffeomorphic to $S^1 \times \mathbb{R}^3$ and $S^2 \times \mathbb{R}^2$.

Lemma 3.1.1. *Suppose* (M, g) *is a smooth Riemannian manifold with isometric effective action of* $\mathrm{SO}(3) \times \mathrm{SO}(2)$*, where* g *is as defined earlier in this section.*

For M *to be diffeomorphic to* $S^1 \times \mathbb{R}^3$*, it is necessary and sufficient that*

$$\begin{aligned} a(0) &> 0 & a^{odd}(0) &= 0 \\ b^{even}(0) &= 0 & b'(0) &= 1 \end{aligned}$$

Thus, the boundary conditions in this case are

$$a(0) = a_0, \qquad a'(0) = 0 \tag{3.5}$$

$$b(0) = 0, \qquad b'(0) = 1 \tag{3.6}$$

For M *to be diffeomorphic to* $S^2 \times \mathbb{R}^2$*, it is necessary and sufficient that*

$$\begin{aligned} a^{even}(0) &= 0 & a'(0) &= 1 \\ b(0) &> 0 & b^{odd}(0) &= 0 \end{aligned}$$

Thus, the boundary conditions in this case are

$$a(0) = 0, \qquad a'(0) = 1 \tag{3.7}$$

$$b(0) = b_0, \qquad b'(0) = 0 \tag{3.8}$$

where in each case, the corresponding parameter (a_0 *or* b_0) *is positive. Additionally, every cohomogeneity one gradient expanding Ricci soliton invariant under the standard* $\mathrm{SO}(2) \times \mathrm{SO}(3)$ *action over either of these topologies arises in this way.*

Proof. The lemma follows from Proposition 1 in the section "Doubly Warped Products" in Chapter 1 of [Pet], where it is proven how the boundary conditions above ensure that the topology is as required and that the metric is smooth. □

Next, we impose boundary conditions on f as in [A17]; since the soliton potential is determined only up to a constant, we can choose $f(0) = 0$. Additionally, we must have $\lim_{r \to 0} f'(r) = 0$ for Equations (3.3) and (3.4) to be satisfied in the limit $r \to 0$, giving the boundary condition $f'(0) = 0$.

In both cases, putting together the boundary conditions on a, b with those on f, Equation (3.2) is degenerate at $r = 0$, and a solution can be specified uniquely by imposing a value of $f''(0) \equiv f_0$. In Appendix B, using methods from [A17] and [Buz11], we show the degeneracy of the equations at $r = 0$ and how the boundary conditions above along with a value of f_0 ensure the existence of a unique local solution to Equations (3.2) to (3.4)

3.2 Soliton Identities

In this section, we collect some well-known soliton identities which we will combine with equations Equations (3.2) to (3.4) in later sections. Importantly, we show that the scalar curvature at $r = 0$ is determined by $f''(0) \equiv f_0$.

Suppose $(M, g, \nabla f)$ is a cohomogeneity one gradient expanding soliton as considered in Section 2. The following identities will be useful in the analysis of the soliton equations.

$$R + \Delta f + 4 = 0 \tag{3.9}$$

$$R + |\nabla f|^2 + 2f = \text{constant} = R(0) \tag{3.10}$$

where $\Delta \equiv \Delta_M$ is the Laplacian of (M, g). Equation (3.9) follows from taking the trace of the soliton equation while Equation (3.10) is an application of the second contracted Bianchi identity and the initial conditions $f(0) = f'(0) = 0$.

Lemma 3.2.1. *The soliton potential f satisfies $\Delta f - |\nabla f|^2 - 2f = 3f_0$ in the $S^1 \times \mathbb{R}^3$ case and $\Delta f - |\nabla f|^2 - 2f = 2f_0$ in the $S^2 \times \mathbb{R}^2$ case.*

Proof. Combining Equation (3.9) and Equation (3.10) gives $\Delta f - |\nabla f|^2 + 4 - 2f = -R(0)$. Using Equation (3.41) from Appendix A, rewrite Equation (3.9) as

$$R = -4 - f'' - \left(\frac{a'f'}{a} + 2\frac{b'f'}{b}\right)$$

In the $S^1 \times \mathbb{R}^3$ case, the boundary conditions Equations (3.5) and (3.6) imply that $\frac{a'f'}{a}(0) = 0$ while $2\frac{b'f'}{b} \to 2f''(0)$ as $r \to 0$ by L'Hôpital's rule. In the $S^2 \times \mathbb{R}^2$ case, Equations (3.7) and (3.8) imply that $\frac{a'f'}{a}(0) = f''(0)$ while $2\frac{b'f'}{b} \to 0$ as $r \to 0$. This shows that

$$R(0) = -3f_0 - 4 \text{ for } S^1 \times \mathbb{R}^3 \qquad\qquad R(0) = -2f_0 - 4 \text{ for } S^2 \times \mathbb{R}^2 \tag{3.11}$$

which gives us the result. $\square$

Lemma 3.2.2. *In the case of either topology, for a complete solution of bounded curvature to Equations* (3.2) *to* (3.4) *with the appropriate boundary conditions, we must have $f_0 \leqslant 0$.*

Proof. Under the assumption of bounded curvature, maximum principle methods (as in, for example, Theorem 2.3 of [BC23]; note the difference in normalizations) allow us to conclude that expanding solitons with the chosen normalization satisfy $R \geqslant -4$. Combining this inequality with Equation (3.11) gives us the result in each case. $\square$

Given a complete solution to Equations (3.2) to (3.4), the corresponding metric g as in Equation (3.1) is complete gradient expanding Ricci soliton metric.

A useful quantity is the ratio $P := \frac{b}{a}$. By calculating using the soliton equations, we see the following:

$$P'' = \left(f' - \frac{b'}{b} - 2\frac{a'}{a}\right)P' + \frac{1}{b^2}P \tag{3.12}$$

3.3 Monotonicity Properties

From now on, we will assume that $f''(0)$ is non-positive as described in the previous section. In this section, we will deduce the appropriate monotonicity properties of a, b and f. We will observe, similar to [A17] (which considered steady solitons as opposed to expanders) that a and b are monotonically increasing and that $f, f', f'' \leqslant 0$. Note that the results in this section do not assume completeness. Denote by $I \subseteq [0, \infty)$ the maximal interval of existence of the solutions to Equations (3.2) to (3.4); we know that I contains 0.

Lemma 3.3.1. *The functions a', b' are positive on $I - \{0\}$*

Proof. First, we look at the $S^1 \times \mathbb{R}^3$ case:

Using L'Hôpital's rule on Equation (3.3), we see that $a''(0) = \frac{a_0}{3}$. As $a'(0) = 0$, we see that $a > 0$ and $a' > 0$ on a small interval of the form $(0, \epsilon)$. Consider the first $r_0 > 0$ (if it exists) with $a'(r_0) = 0$; then Equation (3.3) becomes $a'' = a > 0$, implying $a' > 0$ for a small distance beyond r_0.

As $b'(0) = 1$, we see that $b > 0$ and $b' > 0$ on a small interval of the form $(0, \epsilon)$. Consider the first $r_0 > 0$ (if it exists) with $b'(r_0) = 0$, then Equation (3.4) becomes $b'' = \frac{1}{b} + b > 0$, implying $b' > 0$ for a small distance beyond r_0.

Now, we look at the $S^2 \times \mathbb{R}^2$ case:

As $a'(0) = 1$, we see that $a > 0$ on a small interval of the form $(0, \epsilon)$. As in the previous case, consider the first $r_0 > 0$ (if it exists) with $a'(r_0) = 0$, then Equation (3.3) becomes $a'' = a > 0$, implying $a' > 0$ for a small distance beyond r_0.

Using L'Hôpital's rule on Equation (3.4), we see that $b''(0) = \frac{b_0}{2}$. As $b'(0) = 0$, we see that $b > 0$ and $b' > 0$ on a small interval of the form $(0, \epsilon)$. Consider the first $r_0 > 0$ (if it exists) with $b'(r_0) = 0$, then Equation (3.4) becomes $b'' = \frac{1}{b} + b > 0$, implying $b' > 0$ for a small distance beyond r_0. □

To understand the behavior of f, we consider the cases $f''(0) < 0$ and $f''(0) = 0$ separately as follows.

Lemma 3.3.2. *If $f_0 < 0$, the functions f, f', f'' are negative on $I - \{0\}$ and hence f and f' are monotonically decreasing.*

Proof. By Lemma 3.2.1, we get

$$f'' + \left(\frac{a'}{a} + 2\frac{b'}{b}\right) f' - (f')^2 - 2f = 3f''(0) \tag{3.13}$$

in the $S^1 \times \mathbb{R}^3$ case (the other case is almost exactly the same, with $3f''(0)$ replaced by $2f''(0)$). As $f''(0) < 0$ and $f'(0) = 0$, we have $f, f' < 0$ on a small interval of the form $(0, \epsilon)$. If there is a point where $f' = 0$, let r_0 be the first such point. Then, we must have $f(r_0) < 0$. Then, at r_0, Equation (3.13) simplifies to

$$f''(r_0) = 3f''(0) + 2f(r_0) < 0$$

This shows that f is monotonically decreasing and $f' < 0$ for $r > 0$.

As $f_0 < 0$, we know that $f'' < 0$ on an interval of the form $(0, \epsilon)$. Differentiating Equation (3.13), we get

$$f''' = 2f' + 2f'f'' - \left(\frac{a'}{a} + 2\frac{b'}{b}\right)' f' - \left(\frac{a'}{a} + 2\frac{b'}{b}\right) f''$$

Applying equations Equations (3.2) to (3.4), we see that

$$\left(\frac{a'}{a} + 2\frac{b'}{b}\right)' = f'' + 1 - \left(\left(\frac{a'}{a}\right)^2 + 2\left(\frac{b'}{b}\right)^2\right)$$

Thus, we see that

$$f''' = f' + f'f'' + \left(\left(\frac{a'}{a}\right)^2 + 2\left(\frac{b'}{b}\right)^2\right) f' - \left(\frac{a'}{a} + 2\frac{b'}{b}\right) f''$$

This implies that at a point where $f'' = 0$, we have

$$f''' = \left(1 + \left(\frac{a'}{a}\right)^2 + 2\left(\frac{b'}{b}\right)^2\right) f' < 0 \tag{3.14}$$

which shows that $f'' < 0$.

□

Lemma 3.3.3. *If $f_0 = 0$ then $f \equiv 0$.*

Proof. Since the solution to equations Equations (3.2) to (3.4) depends continuously on the parameter $f''(0)$ (by the results of Appendix B), we know by Lemma 3.3.2 that $f, f', f'' \leqslant 0$ when $f''(0) = 0$. Equation Equation (3.13) becomes

$$f'' = -\left(\frac{a'}{a} + 2\frac{b'}{b}\right) f' + (f')^2 + 2f \tag{3.15}$$

Using Equation (3.15), by standard theory of ordinary differential equations, it is sufficient to show that f is identically zero in a neighborhood of 0 to conclude that $f \equiv 0$ on $\mathbb{R}$. Suppose by way of contradiction that this is not the case – then, there is an interval $(0, \epsilon)$ on which $f, f' < 0$. Then, we can rewrite Equation (3.15) as

$$f'' = -\left(\left(\frac{a'}{a} + 2\frac{b'}{b}\right) + f' + 2\frac{f}{f'}\right) f' \tag{3.16}$$

We notice that $f' \to 0$ as $r \to 0$ and that $\left(\frac{a'}{a} + 2\frac{b'}{b}\right) \to \infty$ as $r \to 0$ (since $b(0) = 0$ and $b'(0) = 1$ in the $S^1 \times \mathbb{R}^3$ case, and $a(0) = 0$ and $a'(0) = 1$ in the $S^2 \times \mathbb{R}^2$ case). Additionally,

$$\left|\frac{f(r)}{f'(r)}\right| = \left|\frac{\int_0^r f'(s)ds}{f'(r)}\right| \leqslant \left|\frac{rf'(r)}{f'(r)}\right| = r$$

where the inequality follows since $|f'(s)| \leqslant |f'(r)|$ for $s < r$, as $f'' \leqslant 0$. Thus, $\frac{f'}{f} \to 0$ as $r \to 0$.

This implies that the quantity in the parentheses in Equation (3.16) is positive at a point in $(0, \epsilon)$; this is a contradiction, as this would imply that $f'' > 0$ at that point. □

We have one more monotonicity result, for the quantity $P = \frac{b}{a}$.

Lemma 3.3.4. *Suppose there exists an $r_0 > 0$ with $P'(r_0) \geqslant 0$. Then, $P'(r) \geqslant 0$ for all $r \geqslant r_0$.*

Proof. Suppose r_0 exists and $r_1 > r_0$ is the first point beyond r_0 with $P'(r_1) = 0$. Then, by equation Equation (3.12) we see that $P''(r_1) = \frac{P}{b^2} > 0$, implying that P' remains nonnegative. As a consequence, if P is ever increasing, it remains increasing. □

3.4 Completeness of Solitons

The main results of this section are the following theorem and its corollary which show that there exists a 2-parameter family of complete cohomogeneity one gradient expanding Ricci solitons in the case of each topology. Note that our assumption of the initial condition $f''(0) \equiv f_0$ being negative is sufficient for a complete solution (and necessary for bounded curvature, as explained in Section 3).

Theorem 3.4.1. *For either set of boundary conditions Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8)*, if $f_0 < 0$ and $a_0, b_0 > 0$, there exists a unique complete solution $f, a, b : [0, \infty) \to \mathbb{R}$ to the soliton equations Equations* (3.2) *to* (3.4)*.*

We note that Theorem 3.4.1 is a special case of more general theorems in [NW24], [Win21], and [BDGW15]. For the sake of completeness, we provide an alternate proof in this special case. We also remark that we use some of the estimates in the proof from this section in future sections.

Corollary 3.4.2. *Let M be a smooth manifold diffeomorphic to $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$. Then there exists a two-parameter family of complete gradient expanding Ricci solitons on M which are cohomogeneity one and invariant under the standard action of* $\mathrm{SO}(3) \times \mathrm{SO}(2)$.

Proof. Completeness follows immediately from the theorem. In the $S^1 \times \mathbb{R}^3$ case, the parameters are a_0 and f_0, while in the $S^2 \times \mathbb{R}^2$ case, the parameters are b_0 and f_0. □

Proof of Theorem 3.4.1. We know by Appendix B that Equations (3.2) to (3.4) have a local solution. Then, we exhibit growth bounds on a and b that allow us to extend them indefinitely. Then, we repeat the process for f.

Lemma 3.4.3. *Suppose $(M, g, \nabla f)$ is a cohomogeneity one gradient expanding Ricci soliton diffeomorphic to $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$. Under the hypothesis of Theorem 3.4.1, a, b and a', b' remain bounded on the maximal interval of existence, which is $[0, \infty)$.*

Proof. By the existence of the local solution, the maximal interval of existence of the solutions to Equations (3.2) to (3.4) contains an interval of the form $(0, 2\epsilon)$ for some $\epsilon > 0$ and by Lemmas 3.3.1 and 3.3.2, we know that $a', b' > 0$ and $f' < 0$. Thus, a and b are positive on $(0, 2\epsilon]$. Then, we can rewrite Equation (3.3) and Equation (3.4) as

$$a'' < a \tag{3.17}$$

$$b'' < \frac{1}{b} + b \tag{3.18}$$

Multiplying Equation (3.17) by a' on both sides and integrating gives us

$$(a')^2 \leqslant a^2 - a(0)^2 + (a'(0))^2 \tag{3.19}$$

In the $S^1 \times \mathbb{R}^3$ case, using Equation (3.5) and Equation (3.6), Equation (3.19) implies that $a' \leqslant a$ on $[\epsilon, \infty)$. Integrating this shows that a is bounded by an exponential function.

In the $S^2 \times \mathbb{R}^2$ case, using Equation (3.7) and Equation (3.8), Equation (3.19) implies that $(a')^2 \leqslant a^2 + 1$ on $[\epsilon, \infty)$. If a is globally bounded by 1, then we are done. If not, we have $a(r_0) > 1$ for some $r_0 > \epsilon$; thus, by the monotonicity of a, we have $1 \leqslant a^2$ on $[r_0, \infty)$, giving us the bound $(a')^2 \leqslant 2a^2$. Integrating this shows that a is bounded by an exponential function.

For b, as $b' > 0$, we know that for $r > r_0$, $b > b(\epsilon) = C$ for some constant $C > 0$ by Lemma 3.3.1. This allows us to rewrite Equation (3.18) as

$$b'' < C + b \tag{3.20}$$

on the interval $[\epsilon, \infty)$. Now, in both cases, similar analysis shows that b is bounded by an exponential function whose value and derivative at $r = \epsilon$ match those of b. Thus, a and b do not blow up at finite r and these functions can be extended to $[0, \infty)$ by standard ODE theory. □

Lemma 3.4.4. *Under the hypothesis of Theorem 3.4.1, f and f' remain bounded on* $[0,\infty)$

Proof. We prove this in the $S^1 \times \mathbb{R}^3$ case; the $S^2 \times \mathbb{R}^2$ case is identical, except for changing the $3f''(0)$ term to $2f''(0)$. Applying Lemmas 3.3.1 and 3.3.2 to equation Equation (3.13), we see that

$$\begin{aligned} 0 <&(f')^2 \\ =& -(3f''(0)+2f) + \left(\frac{a'}{a} + 2\frac{b'}{b}\right) f' + f'' \\ <& -(3f''(0)+2f) \end{aligned}$$

This yields the inequality $f' > -\sqrt{-(3f''(0)+2f)}$. Solving the inequality shows that $|f|$ is bounded by a quadratic function.

More explicitly, suppose that $\tilde{f}$ satisfies the ODE corresponding to the differential inequality with $\tilde{f}(0) = f(0)$. Then, $\tilde{f}(r) = -\frac{r}{2}(2\sqrt{-3f''(0)} + r)$, and $\tilde{f}$ is a lower bound for f. Hence, by standard ODE theory as usual, f can be extended smoothly to $[0,\infty)$. □

Uniqueness follows from the uniqueness of the local solution in Appendix B and standard ODE theory. This concludes the proof of Theorem 3.4.1. □

3.5 Asymptotics

Suppose a, b, f satisfy the soliton equations Equations (3.2) to (3.4). The main result of this section is that the complete expanding Ricci solitons constructed in Section 5 are asymptotic to cones over the link $S^2 \times S^1$. Several technical lemmas analyzing the soliton equations will be needed before we conclude the result. We continue to assume that $f''(0) \equiv f_0 < 0$ to ensure that our solitons are complete.

In this section, the proofs will be carried out for the soliton equations in the $S^1 \times \mathbb{R}^3$ case; the other case is nearly identical, with the difference being the term $2f''(0)$ appearing instead of $3f''(0)$.

As we expect the expanding solitons to be asymptotic to cones, the quantities $\frac{a'}{a}$ and $\frac{b'}{b}$ should decay like $\frac{1}{r}$ to 0. We first show that these quantities are bounded by constants in a region near infinity.

Lemma 3.5.1. *Suppose a, b, f satisfy equations Equations* (3.2) *to* (3.4)*, with boundary conditions either Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8)*, depending on the topology. Then, there exists a $C > 0$ such that $a' < Ca$ and $b' < Cb$ on $[1,\infty)$, where C depends continuously on the initial conditions (so $C \equiv C(a_0, f_0)$ in the $S^1 \times \mathbb{R}^3$ case and $C \equiv C(b_0, f_0)$ in the $S^2 \times \mathbb{R}^2$ case)*

Proof. We will carry out the proof for b; the proof for a is almost identical to Equation (3.19). By Equation (3.18), we know that

$$\frac{b''}{b} < \frac{1}{b^2} + 1$$

which implies that

$$\begin{aligned}\left(\frac{b'}{b}\right)' &= \frac{b''}{b} - \left(\frac{b'}{b}\right)^2 \\ &< \frac{1}{b^2} + 1 - \left(\frac{b'}{b}\right)^2 \\ &\leqslant \frac{1}{b(1)^2} + 1 - \left(\frac{b'}{b}\right)^2\end{aligned}$$

on the interval $[1, \infty)$ (where the last step follows since $b > b(1)$, as b is increasing), giving us the inequality

$$\left(\frac{b'}{b}\right)' < C' - \left(\frac{b'}{b}\right)^2 \tag{3.21}$$

for the constant $C' = \frac{1}{b(1)^2} + 1$. Thus, the quantity $\frac{b'}{b}$ satisfies the inequality $u' + u^2 < C'$. Thus, $\frac{b'}{b}$ is bounded by the solution to the initial value problem

$$\begin{gathered} u'(x) + u(x)^2 = C' \\ u(1) = \tfrac{b'}{b}(1) \end{gathered}$$

This IVP can be solved exactly and the solution u is asymptotic to $\sqrt{C'}$. If $u(1) < \sqrt{C'}$, then u increases and becomes asymptotic to $\sqrt{C'}$, and if $u(1) > \sqrt{C'}$, then u decreases to $\sqrt{C'}$. As $\frac{b'(r)}{b(r)} \leqslant u(r)$ on $[1, \infty)$, any C greater than $\max(\sqrt{C'}, \frac{b'}{b}(1))$ makes the lemma true. Clearly, such a C can be chosen continuously in the initial conditions, since $b(1)$ and $b'(1)$ vary smoothly in the initial conditions as in the hypothesis of the lemma. $\square$

The next lemma is an improvement of Lemma 3.3.2. It shows that f'' is bounded from above by a negative constant.

Lemma 3.5.2. *Suppose a, b, f satisfy equations Equations* (3.2) *to* (3.4)*, with boundary conditions either Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8)*, depending on the topology. Then, there exists $\epsilon > 0$, depending continuously on the initial conditions (so $\epsilon \equiv \epsilon(a_0, f_0)$ or $\epsilon \equiv \epsilon(b_0, f_0)$ depending on the topology), such that for $r > 1$, we have*

$$f'' \leqslant -\epsilon$$

Proof. Recall from the proof of Lemma 3.3.2 that we have

$$f''' = f' + f'f'' + \left(\left(\frac{a'}{a}\right)^2 + 2\left(\frac{b'}{b}\right)^2\right) f' - \left(\frac{a'}{a} + 2\frac{b'}{b}\right) f''$$

For $r > 1$, we see that

$$f'''(r) < f'(r)(1 + f''(r)) - 3Cf''(r)$$

by the monotonicity properties and Lemma 3.5.1. Now, choose $0 < \epsilon < 1$ so that the quantity $f'(1)(1-\epsilon) + 3C\epsilon$ is negative. This is possible as this quantity is equal to $f'(1)$ (which is negative) for $\epsilon = 0$, so there must exist a positive ϵ satisfying the condition. Further adjust ϵ if needed so that $f''(1) < -\epsilon$. Then, for r in a small interval $[1, 1+\delta)$, we have $f'' < -\epsilon$.

Then, we see that if there exists a point $r > 1$ with $f''(r) = -\epsilon$, then

$$\begin{aligned} f'''(r) &< f'(r)(1 + f''(r)) - 3Cf''(r) \\ &= f'(r)(1-\epsilon) + 3C\epsilon \\ &< f'(1)(1-\epsilon) + 3C\epsilon \\ &< 0 \end{aligned}$$

where the third line follows by the monotonicity of f' and the last line by the choice of ϵ. Thus, $f'''(r) < 0$ at any point where $f''(r) = -\epsilon$, implying that $f'' < -\epsilon$ is a preserved condition and thus holds on $[1, \infty)$, implying the statement of the lemma.

Note that the choice of ϵ depends on C from Lemma 3.5.1 and $f'(1)$ and $f''(1)$, which together depend continuously on the initial conditions $f''(0)$ and a_0 or b_0. □

Now, we prove a lower bound on f'. Note that the geometric meaning of this bound is that $|\nabla f|$ has at most linear growth.

Lemma 3.5.3. *The inequality $f'(r) > -(r + C_1)$ holds on $[1, \infty)$, where $C_1 \in \mathbb{R}$ is a real constant depending continuously on the value of f_0. In fact, we can choose $C_1 = \sqrt{-3f_0}$*

Proof. We prove this in the $S^1 \times \mathbb{R}^3$ case; the other case is nearly identical. By the proof of Lemma 3.4.4, we have the inequality $f'(r) > -\sqrt{-(3f''(0) + 2f(r))}$. Consider the solution $\tilde{f}$ to the corresponding ODE $\tilde{f}'(r) = -\sqrt{-(3f''(0) + 2\tilde{f}(r))}$ with $f(1) = \tilde{f}(1) < 0$. Then, we have $f \geqslant \tilde{f}$ on $[1, \infty)$. This leads to the chain of inequalities

$$\begin{aligned} f'(r) &> -\sqrt{-(3f''(0) + 2f(r))} \\ &> -\sqrt{-(3f''(0) + 2\tilde{f}(r))} \\ &= \tilde{f}'(r) \end{aligned}$$

The ODE for $\tilde{f}$ can be solved explicitly as in Lemma 3.4.4, with

$$\tilde{f}(r) = -\frac{r}{2}\left(2\sqrt{-3f''(0)} + r\right) \qquad \tilde{f}'(r) = -(r + \sqrt{-3f''(0)})$$

and by substituting $\tilde{f}(r)$ into the inequality above, we see that $f'(r) > -(r + C_1)$ for some real constant C_1, as required. □

We can put the previous two lemmas together to control the growth rate of f in the following way:

Theorem 3.5.4. *Suppose a, b, f satisfy equations Equations (3.2) to (3.4), with boundary conditions Equations (3.5) and (3.6) or Equations (3.7) and (3.8). Then, the soliton vector field $f'(r)$ satisfies bounds of the following form:*

$$-(r + C_1) \leqslant f'(r) \leqslant -\epsilon(r - 1)$$

where C_1 is a constant depending on f_0 and $\epsilon \equiv \epsilon(a_0, f_0)$ or $\epsilon \equiv \epsilon(b_0, f_0)$ is a positive constant depending continuously on the initial conditions.

Proof. The lower bound is Lemma 3.5.3, while the upper bound follows by integrating the bound in Lemma 3.5.2 on $[1, r]$ and using the fact that $f'(1) < 0$ by Lemma 3.3.2. □

We will use Theorem 6.4 and further bounds on $f'(r)$ to provide growth bounds on a and b. For this, we will need the following ODE comparison result:

Lemma 3.5.5. *Suppose there exist functions v_1, $v_2 : \mathbb{R}^+ \to \mathbb{R}$ satisfying the differential inequalities*

$$v_1''(r) \leqslant -(r + C)v_1'(r) + v_1(r), \qquad v_2''(r) \geqslant -(r + C)v_2'(r) + v_2(r)$$

for $r \geq r_0$, with initial conditions $v_1(r_0) = v_2(r_0) = \alpha$ and $v_1'(r_0) = v_2'(r_0) = \alpha'$ and C is a positive real number. Then the following hold:

1. *$v_1(r) \leqslant c_1(r - r_0) + \alpha$ for all $r \in [r_0, \infty)$, for some positive constant c_1 which can be chosen uniformly in the constants C, α, and α'. In addition, $v_1' \leqslant c_1$ on this interval.*
2. *$v_2(r) \geqslant C_1(r - r_0) + \alpha$ for all $r \in [r_0, \infty)$, for some positive constant C_1 which can be chosen uniformly in the constants C, α, and α'. In addition, $v_2' \geqslant C_1$ on this interval.*

Proof. 1. For any $c_1 > 0$, consider the function $w_1 : [r_0, \infty) \to \mathbb{R}$ given by $w_1(r) = c_1(r - r_0) + \alpha$. The value of c_1 will be chosen below. Then, we see that

$$\begin{aligned}(v_1 - w_1)'' &= v_1'' \\ &\leqslant -(r + C)v_1' + v_1 \\ &\leqslant -(r + C)v_1' + v_1 + c_1(C + r_0) - \alpha \\ &= -(r + C)(v_1 - w_1)' + (v_1 - w_1)\end{aligned}$$

where the last inequality is true for $c_1 \geqslant \alpha/(C+r_0)$. Additionally, note that $\alpha = v_1(r_0) = w_1(r_0)$, and that $(v_1 - w_1)'(r_0) = \alpha' - c_1$.

Now, choose $c_1 > \max\{\alpha/(C+r_0), \alpha'\}$. Then, on $[r_0, \infty)$ we have the inequality

$$(v_1 - w_1)'' \leqslant -(r+C)(v_1 - w_1)' + (v_1 - w_1)$$

with $(v_1 - w_1)(r_0) = 0$ and $(v_1 - w_1)'(r_0) < 0$. Thus, $v_1 \leqslant w_1$ and $v_1' \leqslant w_1' = c_1$ on an interval of the form $[r_0, r_0 + \epsilon]$ for some $\epsilon > 0$. Let $r^* > r_0$ be the first point with $v_1'(r^*) = w_1'(r^*)$, if any such points exist; then, by the computation above, we have $(v_1 - w_1)''(r^*) \leqslant 0$. Thus, we see that $v_1 \leqslant w_1$ and $v_1' \leqslant w_1'$ on $[r_0, \infty)$. Clearly, the value of c_1 can be chosen uniformly in α, α' and C; for example, we may choose $c_1 = \max\{\alpha/(C+r_0), \alpha'\} + 1$

2. For any $C_1 > 0$, consider the function $w_2 : [r_0, \infty) \to \mathbb{R}$ given by $w_2(r) = C_1(r - r_0) + \alpha$. As in Part 1, the value of C_1 will be chosen below. Then, we see that

$$\begin{aligned}
(v_2 - w_2)'' &= v_2'' \\
&\geqslant -(r+C)v_2' + v_2 \\
&\geqslant -(r+C)v_2' + v_2 + C_1(C + r_0) - \alpha \\
&= -(r+C)(v_2 - w_2)' + (v_2 - w_2)
\end{aligned}$$

where the last inequality is true for $C_1 \leqslant \alpha/(C+r_0)$. Additionally, note that $\alpha = v_2(r_0) = w_2(r_0)$, and that $(v_2 - w_2)'(r_0) = \alpha' - C_1$.

Now, choose $C_1 < \min\{\alpha/(C+r_0), \alpha'\}$. Then, on $[r_0, \infty)$ we have the inequality

$$(v_2 - w_2)'' \geqslant -(r+C)(v_2 - w_2)' + (v_2 - w_2)$$

with $(v_2 - w_2)(r_0) = 0$ and $(v_2 - w_2)'(r_0) > 0$. Thus, as before, we see that $v_2 \geqslant w_2$ and $v_2' \geqslant w_2'$ on $[r_0, \infty)$. Clearly, the value of C_1 can be chosen uniformly in α, α' and C; for example, set $C_1 = \dfrac{\min\{\alpha/(C+r_0), \alpha'\}}{2}$ □

Lemma 3.5.6. *Consider Equations* (3.2) *to* (3.4)*, with initial conditions Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8)*. Then, on* $[1, \infty)$*, we have bounds of the form* $a(r), b(r) \geqslant c(r-1)$*, and the constant* $c \equiv c(a_0, f_0)$ *or* $c \equiv c(b_0, f_0)$ *can be chosen uniformly in the initial conditions.*

Proof. We will prove the result for a in the $S^1 \times \mathbb{R}^3$ case; the proofs for the other case are similar. Suppose we have an initial condition (a_0, f_0) and a compact set F containing it such that F avoids the boundary of the space of initial conditions (that is, both coordinates are nonzero for any element of F).

Step 1: Consider the soliton equation Equation (3.3) rewritten as $a'' = a + a'(-2\frac{b'}{b} + f')$. Using Lemmas 3.5.1 and 3.5.3, that $\frac{b'}{b} < C$ and $f' > -(r + C_1)$, and that $a' > 0$ by Lemma 3.3.1, we can extract the following inequality on $[1, \infty)$:

$$a''(r) \geqslant a(r) - (r + \bar{C})a'(r)$$

where $\bar{C} := C_1 + 2C$ is a positive constant, and a has initial conditions $a(1)$ and $a'(1)$.

Step 2: By continuous dependence in the initial conditions of solutions to Equations (3.2) to (3.4), we notice that the values $a(1)$ and $a'(1)$ are bounded from above and below (with nonzero lower bounds) by our choice of compact set F. Additionally, $\bar{C}$ can be chosen uniformly over F as well by the uniformity of C_1 and C. Then, by Part 1 of Lemma 6.5, we have a bound $a(r) \geqslant c_1(r-1) + a(1)$ for some $c_1 > 0$ which can be chosen uniformly over F and the constant $\bar{C}$. Thus, all a with initial conditions in K are bounded from below by a linear function. The proof for b is similar, following from equation Equation (3.4). □

Now that we understand how f' behaves for large r, we can calculate the asymptotics of a and b. We begin with lemmas describing the boundedness of the curvature and the rate of decay of the curvature near infinity. Before we do this, we prove a point-picking lemma that we will use multiple times in this paper. The lemma and proof are taken from [RFLN].

Lemma 3.5.7 (Point Picking)**.** *Let M be a complete manifold (with or without boundary) with $f : M \to (0, \infty)$ continuous, $x \in M$, and $d > 0$. Then, there is a $y \in B(x, 2df(x)^{-1/2})$ such that $f(y) \geqslant f(x)$ and $f \leqslant 4f(y)$ on $B(y, df(y)^{-1/2})$*

Proof. Set $y_0 = x$. If $y = y_0$ satisfies the required conditions, the proof concludes here. Else there exists $y_1 \in B(y_0, d/\sqrt{f(y_0)})$ such that $f(y_1) > 4f(y_0)$. If y_1 satisfies the required conditions, the proof concludes here. Otherwise, repeat this process to produce a sequence $\{y_j\}$. By repeatedly applying the triangle inequality, we obtain

$$d(y_j, x) = d(y_j, y_0) \leqslant \frac{d}{\sqrt{f(y_0)}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{j-1}}\right) < \frac{2d}{\sqrt{f(x)}}$$

Since the closure of $B(x, 2df(x)^{-1/2})$ is compact, we get an upper bound on f on this ball, so this process has to terminate. Thus, there exists a sufficiently large $j \in \mathbb{N}$ so that $y = y_j$ satisfies the required conditions. □

In the following lemma, we use geometric methods to prove estimates on the curvature. In using geometric convergence methods, we use the fact that since our metrics are warped products, the limit metrics are also warped products. The precise statement is proven in Appendix C.

Lemma 3.5.8. *Consider a complete cohomogeneity one gradient expanding Ricci soliton $(M, g, \nabla f)$ over either $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$, as in the setup of this paper. Then, there exists a constant $C > 0$ such that $|\mathrm{Rm}|_g \leqslant C$. Moreover, C can be chosen uniformly in the initial conditions (so $C \equiv C(a_0, f_0)$ or $C \equiv C(b_0, f_0)$, depending on the topology)*

Proof. We will carry out the proof in the $S^1 \times \mathbb{R}^3$ case; the other case is almost identical. Suppose that the statement of the lemma is not true. Then, there exists a sequence of solitons $(M, g_i, \nabla f_i)$ with initial conditions (a_0^i, f_0^i) lying in a compact set F of $\mathbb{R}^+ \times \mathbb{R}^-$ and points $p_i \in (M, g_i)$ with $r(p_i) = r_i$ and $|\mathrm{Rm}|_{g_i}(p_i) := Q_i \to \infty$. Additionally, for any sequence $\{D_i\} \to \infty$, by the previous lemma, we can assume that the p_i are chosen so that $|\mathrm{Rm}|_{g_i} \leqslant 4Q_i$ on $B_{g_i}(p_i, D_i/\sqrt{Q_i})$.

First, we claim that $\{r_i\}$ must be unbounded. Since the initial conditions (a_0^i, f_0^i) are bounded for every soliton in the sequence, we know that a_i and b_i and their derivatives remain uniformly bounded on any interval of the form $[0, R]$ for any $R > 0$, by the smooth dependence of the solutions of ODEs on initial conditions. As the sectional curvatures of (M, g_i) are smooth functions of a_i and b_i and their derivatives, we see that they must also remain bounded on points whose distance from the singular orbit lies in $[0, R]$. This implies that since $Q_i \to \infty$, the sequence r_i must be unbounded.

Now, rescale g_i by Q_i to get $\tilde{g}_i = Q_i g_i$, and consider the pointed sequence of manifolds $(M, \tilde{g}_i, \nabla f_i, p_i)$. The rescaled manifolds satisfy the equation

$$\mathrm{Ric}_{\tilde{g}_i} + \nabla^2 f_i + \frac{1}{Q_i}\tilde{g}_i = 0$$

where we now have the bound $|\mathrm{Rm}|_{\tilde{g}_i} \leqslant 4$ on $B_{\tilde{g}_i}(p_i, D_i)$. Now, since $D_i \to \infty$, for any $D > 0$, we have the bound $|\mathrm{Rm}|_{\tilde{g}_i} \leqslant C(D)$ on $B_{\tilde{g}_i}(p_i, D)$ (for all i) for some constant $C(D)$ depending on D.

By the equation above, this implies that $|\nabla^2 f_i|_{\tilde{g}_i} \leqslant C(D)$ on $B_{\tilde{g}_i}(p_i, D)$. We also have volume lower bounds of small r-balls at p_i, since the functions a_i and b_i are increasing. By Shi's estimates applied to the associated Ricci flows, we also have bounds on $B_{\tilde{g}_i}(p_i, D)$ on derivatives of the curvature of the form $|\nabla^k \mathrm{Rm}|_{\tilde{g}_i} \leqslant C_k(D)$ for $k \geqslant 1$ and for all i; these bounds provide bounds on higher derivatives of f_i as well.

Now, first we consider the case where (up to a subsequence) $|\nabla f|_{\tilde{g}_i}(p_i)$ becomes unbounded. Set

$$\tilde{f}_i := \frac{f_i - f_i(p_i)}{|\nabla f|_{\tilde{g}_i}(p_i)}.$$

Then, $|\nabla^2 \tilde{f}_i|_{\tilde{g}_i}$ converges to 0, so we have smooth pointed Cheeger-Gromov convergence of a subsequence of $(M, g_i, \nabla \tilde{f}_i, p_i)$ to a smooth non-flat Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ with a smooth function f_∞ satisfying $\nabla^2 f_\infty = 0$ and $|\nabla f_\infty|(p_\infty) = 1$. Thus, ∇f_∞ is a parallel vector field, which implies the splitting of (M_∞, g_∞). In addition, since $a_i(r_i)$ and $b_i(r_i)$ tend to infinity at least linearly in r_i by Lemma 3.5.6, and $Q_i > 1$ for large i, we see that M_∞ is diffeomorphic to $\mathbb{R}^4$ and carries a doubly warped product metric $g_\infty = dr^2 + a_\infty(r)^2 g_{\mathbb{R}} + b_\infty(r)^2 g_{\mathbb{R}^2}$ over $\mathbb{R} \times \mathbb{R}^2$, by Lemma 3.3.1 in Appendix C. As $\nabla^2 f_\infty = 0$, we see that the functions a_∞ and b_∞ are constant, implying that the limit is isometric to Euclidean space. However, $|\mathrm{Rm}|(p_\infty)$ is nonzero, which is a contradiction.

Now, consider the case where $|\nabla f|_{\tilde{g}_i}(p_i)$ remains bounded. Then, we have smooth pointed Cheeger-Gromov convergence of a subsequence of $(M, g_i, \nabla\tilde{f}_i, p_i)$ to a smooth non-flat steady Ricci soliton $(M_\infty, g_\infty, \nabla f_\infty, p_\infty)$. As before, we see that M_∞ is diffeomorphic to $\mathbb{R}^4$ and carries a doubly warped product metric over $\mathbb{R} \times \mathbb{R}^2$. Taking the quotient by $\mathbb{Z}$ and $\mathbb{Z}^2$ so that the orbits are compact, we have a steady Ricci soliton with 2 ends (since $r_i \to \infty$), which, according to the results of [MW11] must split as the product of $\mathbb{R}$ with a compact Ricci-flat 3-manifold N, which must be flat. This implies that M_∞ is flat, which is a contradiction.

Thus, such a sequence $\{r_i\}$ cannot exist, so the curvatures of the solitons must be uniformly bounded in terms of the initial conditions. □

Now, we improve the curvature bound from the previous lemma to quadratic curvature decay.

Lemma 3.5.9. *Suppose $(M, g, \nabla f)$ is a complete cohomogeneity one gradient expanding Ricci soliton over $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$, as considered in the setup of this paper. Then, the curvature satisfies the following bound for some $C > 0$. Moreover, C can be chosen uniformly in the initial conditions (so $C \equiv C(a_0, f_0)$ or $C \equiv C(b_0, f_0)$, depending on the topology)*

$$|\mathrm{Rm}|_g(r) \leqslant \frac{C}{r^2} \tag{3.22}$$

Proof. Suppose $(M, g, \nabla f)$ is an expanding soliton as in the hypothesis in the case of either topology. Consider the quantities $A := a'/a$ and $B := b'/b$. Then, we can rewrite Equation (3.3) and Equation (3.4) as

$$A' + A^2 = -2AB + Af' + 1$$

$$B' + B^2 = -B^2 - AB + Bf' + 1 + \frac{1}{b^2}$$

We rewrite the equations as

$$A' + A(A + 2B - f') = 1$$

$$B' + B(A + 2B - f') = 1 + \frac{1}{b^2}$$

Define $F : [1, \infty) \to \mathbb{R}$ to be the function satisfying $F'(r) = (A+2B)(r) - f'(r)$ and $F(1) = 0$. Note that $F'(r) = O(r)$ by Theorem 3.5.4 and Lemma 3.5.1. Using the definitions of A and B and soliton equation Equation (3.2), we see that $F'' = A' + 2B' - f'' = 1 - (\frac{a'}{a})^2 - 2(\frac{b'}{b})^2$, which is bounded uniformly in the initial conditions on $[1, \infty)$ by the result of Lemma 3.5.1. Then, we have the equations

$$A' + AF' = 1 \qquad B' + BF' = 1 + \frac{1}{b^2}$$

which we can solve to get

$$A(r) = e^{-F(r)} \int_1^r e^{F(u)} du + C_A e^{-F(r)}$$

$$B(r) = e^{-F(r)} \int_1^r e^{F(u)} du + e^{-F(r)} \int_1^r e^{F(u)} \frac{1}{b^2(u)} du + C_B e^{-F(r)}$$

Note that C_A and C_B vary continuously in the initial conditions, since $F(1)$, $A(1)$ and $B(1)$ vary continuously as well. Then, using the upper bound on $f'(r)$ from Theorem 3.5.4 and the bounds on A and B from Lemma 3.5.1, we see that the last term in both equations decays faster than $e^{-\epsilon r^2/2}$, for some $\epsilon > 0$ uniform in the initial conditions.

Claim: $A(r) = \dfrac{1}{F'(r)} + O\left(\dfrac{1}{F'(r)^3}\right)$ and $B(r) = \dfrac{1}{F'(r)} + O\left(\dfrac{1}{F'(r)^3}\right)$

Proof of Claim: We first prove the claim for A. By the sentence preceding the claim, the term $C_A e^{-F(r)}$ decays faster than any polynomial in $F'(r)^{-1}$, so it is enough to analyze the integral term. First, consider the quantity

$$\frac{\displaystyle\int_0^r e^{F(u)}\, du}{\dfrac{e^{F(r)}}{F'(r)}}.$$

Applying LHpitals rule, we get

$$L = \lim_{r\to\infty} \frac{\int_0^r e^{F(u)}\, du}{\frac{e^{F(r)}}{F'(r)}} = \lim_{r\to\infty} \frac{e^{F(r)}}{e^{F(r)}\left(1 - \frac{F''(r)}{(F'(r))^2}\right)} = \lim_{r\to\infty} \frac{1}{1 - \frac{F''(r)}{(F'(r))^2}} = 1.$$

where the last step follows since F'' is bounded and $F'(r) = O(r)$. This implies the asymptotic equivalence

$$e^{-F(r)} \int_0^r e^{F(u)}\, du = \frac{1}{F'(r)} + o\left(\frac{1}{F'(r)}\right) \quad \text{as } r \to \infty.$$

Now, we consider the limit

$$L = \limsup_{r\to\infty} \frac{e^{-F(r)} \displaystyle\int_0^r e^{F(u)}\, du - \dfrac{1}{F'(r)}}{\dfrac{1}{(F'(r))^3}}.$$

Multiplying the numerator and denominator by $e^{F(r)}$, this becomes:

$$L = \limsup_{r\to\infty} \frac{\displaystyle\int_0^r e^{F(u)}\,du - \frac{e^{F(r)}}{F'(r)}}{\dfrac{e^{F(r)}}{(F'(r))^3}}.$$

Applying LHpitals Rule (note that this becomes an inequality for the lim sup), we have

$$L \leqslant \limsup_{r\to\infty} \frac{e^{F(r)}\cdot \frac{F''(r)}{(F'(r))^2}}{e^{F(r)}\left(\frac{1}{(F'(r))^2} - \frac{3F''(r)}{(F'(r))^4}\right)} = \limsup_{r\to\infty} \frac{F''(r)}{1 - \frac{3F''(r)}{(F'(r))^2}}.$$

Since $F'(r)\to\infty$ and $F''(r)$ is bounded, this is $O(1)$. Thus,

$$e^{-F(r)}\int_0^r e^{F(u)}\,du = \frac{1}{F'(r)} + O\left(\frac{1}{(F'(r))^3}\right) \quad \text{as } r\to\infty.$$

For B, we see that the first term is identical to that of A, providing the leading order term $\frac{1}{F'(r)}$. In the second term, define $\tilde{F} := F - 2\ln(b)$. Then, we can rewrite this term as

$$e^{-F(r)}\int_1^r e^{F(u)}\frac{1}{b^2(u)}du = \frac{1}{b^2(r)}\left(e^{-\tilde{F}(r)}\int_1^r e^{\tilde{F}(u)}du\right)$$

Now, the term in parentheses above can be shown by a similar argument to decay with leading term $\dfrac{1}{\tilde{F}'(r)}$ (Note that $\tilde{F}'' = 1 - (\frac{a'}{a})^2 - 2\frac{b''}{b}$. This is bounded uniformly in the initial conditions on $[1,\infty)$ by Lemmas 3.5.1 and 3.5.8 since $-b''/b$ is a sectional curvature, allowing the argument for the first term for B to be applied to the second term as well). Thus, the second term for B is $O\left(\dfrac{1}{b(r)^2\tilde{F}'(r)}\right)$. Since $b(r)$ is known to grow at least linearly by Lemma 3.5.6, we can combine the decay rate of all 3 terms for B to see that the leading term is $1/F'(r)$ and that all other terms decay at least as fast as r^{-3}, proving the claim for B as well. ■

Using the claim and the fact that $F'(r)$ is of linear growth (uniform in the initial conditions), we see that

$$\frac{a''}{a}(r) = A'(r) + A(r)^2 = 1 - A(r)F'(r) + A(r)^2 = O\left(\frac{1}{F'(r)^2}\right) + O\left(\frac{1}{r^2}\right) = O\left(\frac{1}{r^2}\right)$$

$$\frac{b''}{b}(r) = B'(r) + B(r)^2 = 1 + \frac{1}{b^2(r)} - B(r)F'(r) + B(r)^2 = O\left(\frac{1}{F'(r)^2}\right) + O\left(\frac{1}{r^2}\right) = O\left(\frac{1}{r^2}\right)$$

$$\frac{a'b'}{ab}(r) = AB = O\left(\frac{1}{F'(r)^2}\right) = O\left(\frac{1}{r^2}\right)$$
$$\frac{1-(b')^2}{b^2}(r) = \frac{1}{b^2} - B^2 = O\left(\frac{1}{r^2}\right) + O\left(\frac{1}{F'(r)^2}\right) = O\left(\frac{1}{r^2}\right)$$

Thus, we have shown that all sectional curvatures (refer Appendix A) decay as r^{-2}. $\square$

Now that we know that the curvature tensor decays as r^{-2}, we can make a more precise statement about the asymptotics of f.

Lemma 3.5.10. *Suppose (a, b, f) satisfy equations Equations* (3.2) *to* (3.4)*, with boundary conditions either Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8)*. Then, the quantity $f'(r)+r$ has a finite limit at infinity, denoted K. In addition, we have the following inequality for $r > 0$:*

$$0 < |f'(r) + r - K| < \frac{C}{r}$$

Moreover, $C \equiv C(a_0, f_0)$ or $C \equiv C(b_0, f_0)$ can be chosen uniformly in the initial conditions and K is continuous in the initial conditions.

Proof. Equation Equation (3.2) can be written as

$$f'' + 1 = \frac{a''}{a} + 2\frac{b''}{b}$$

As $-\frac{a''}{a}$ and $-\frac{b''}{b}$ are components of the curvature tensor (components Rm_{1221} and Rm_{1331} respectively; refer Appendix A for details), they decay as r^{-2}. Thus, for some constant $C > 0$ (which can be chosen uniformly in the initial condtions) by Lemma 3.5.9, for $r > 0$, we have

$$0 \leqslant |f'' + 1| \leqslant \frac{C}{r^2}$$

As $f'(r) + r$ has derivative equal to $f''(r) + 1$, and since $f''(r) + 1$ decays like r^{-2}, we see that $f'(r) + r$ has a finite limit as $r \to \infty$, denoted by K. The uniformity of C allows us to see that K is continuous in the initial conditions.

Fix $r, s > 0$ and integrate the inequality above to get

$$0 \leqslant \left|\int_r^s (f''(t) + 1)dt\right| \leqslant \int_r^s |f''(t) + 1|dt \leqslant \frac{C}{r} - \frac{C}{s}$$

We thus see that

$$0 \leqslant |f'(r) + r - f'(s) - s| \leqslant \frac{C}{r} - \frac{C}{s}$$

Taking the limit in the above inequality as $s \to \infty$ gives the result. $\square$

Now, we can understand the asymptotic behavior of a and b.

Lemma 3.5.11. *Suppose a, b, f satisfy equations Equations (3.2) to (3.4) with initial conditions either Equations (3.5) and (3.6) or Equations (3.7) and (3.8), depending on the topology. Then we have the following inequalities for $r > 1$, which can be chosen uniformly in the initial conditions:*

$$0 \leqslant \left|1 + \frac{a'f'}{a}\right| \leqslant \frac{C}{r^2}$$
$$0 \leqslant \left|1 + \frac{b'f'}{b}\right| \leqslant \frac{C}{r^2}$$

where $C \equiv C(a_0, f_0)$ or $C \equiv C(b_0, f_0)$ can be chosen uniformly in the initial conditions.

Proof. From Equation (3.3), we see that

$$1 + \frac{a'f'}{a} = \frac{a''}{a} + \frac{2a'b'}{ab}$$

The terms on the right hand side comprise a component of the Ricci curvature (refer Appendix A for details), so the RHS must decay as r^{-2}, proving the first half of the lemma. The result for b follows analogously using Equation (3.4). □

Now, we will show that a and b are bounded above and below by linear functions.

Lemma 3.5.12. *Consider Equations (3.2) to (3.4), with initial conditions Equations (3.5) and (3.6) (in the $S^1 \times \mathbb{R}^3$ case) or Equations (3.7) and (3.8) (in the $S^2 \times \mathbb{R}^2$ case). Then, there exist constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and $C, C'D, D' \in \mathbb{R}$ so that we have $\alpha_1(r + C) < a(r) < \alpha_2(r + C')$ and $\beta_1(r + D') < b(r) < \beta_2(r + D')$. Additionally, $\alpha_1, \alpha_2, \beta_1, \beta_2$ depend continuously on the initial conditions.*

Proof. We will carry out the proof for a in the $S^1 \times \mathbb{R}^3$ case; similar arguments hold for b (the $\frac{1}{b}$ term can be bounded above by a constant C_b, since b is increasing; consider the quantity $b + C_b$). Note that Lemma 3.5.6 provides lower bounds; we show the upper bound. Suppose we have an initial condition inside a compact set of the form $(a_0, f_0) \in F \subset \mathbb{R}^+ \times \mathbb{R}^-$.

Step 1: Using Lemmas 3.3.1, 3.3.2, and 3.5.10 in Equation (3.3), for a large $r_0 > 0$, we can write the following inequality on the interval $[r_0, \infty)$:

$$a'' \leqslant a'(-r + C) + a$$

where C depends on the constant K from Lemma 3.5.10, and a has initial conditions $a(r_0)$ and $a'(r_0)$.

Step 2: We notice that the values $a(r_0)$ and $a'(r_0)$ are bounded from above and below (with nonzero lower bounds) by our choice of compact set F. Additionally, C can be chosen uniformly over F as well. Then, by Part 1 of Lemma 3.5.5, we have a bound $a(r) \leqslant C_1(r - r_0) + a(1)$ for some $C_1 > 0$ which can be chosen uniformly over F and the constant C. Thus, all a with initial conditions in K are bounded from below on $[r_0, \infty)$ by a linear function. The proof for b is similar, following from equation Equation (3.4). $\square$

Now, we can show that a and b are asymptotically linear.

Lemma 3.5.13. *1. With K as in Lemma 3.5.10, the quantities $\frac{a(r)}{r-K}$ and $\frac{b(r)}{r-K}$ tend to finite limits as $r \to \infty$. Moreover, the quantities $a'(r)$ and $b'(r)$ tend to the same limits, respectively denoted a'_∞ and b'_∞.*

2. The quantities a'_∞ and b'_∞ are continuous in the initial conditions.

Proof. First, we prove the first statement. We will carry out the proof for a; the proof for b is analogous. We consider the quantity $\left|\dfrac{a(r)}{r-K} - a'(r)\right|$ for $r \geqslant \max(0, K)$

$$\begin{aligned}
\left|\frac{a(r)}{r-K} - a'(r)\right| &= \left|\frac{a(r)}{r-K} + \frac{a(r)}{f'(r)} - \frac{a(r)}{f'(r)} - a'(r)\right| \\
&\leqslant \left|\frac{a(r)}{r-K} + \frac{a(r)}{f'(r)}\right| + \left|\frac{a(r)}{f'(r)} + a'(r)\right| \\
&= a(r)\left|\frac{f'(r) + r - K}{f'(r)(r-K)}\right| + \left|\frac{a(r) + a'(r)f'(r)}{f'(r)}\right|
\end{aligned}$$

By Lemmas 3.5.10, 3.5.11, and 3.5.12 we see that the first term on the RHS is bounded by $\frac{C}{r^2}$ for some constant $C > 0$, while the second term is also bounded by $\frac{C}{r^2}$, both for $r > \max(0, K) + 1$.

Thus, we have for $r > \max(0, K) + 1$

$$\left|\frac{a(r)}{r-K} - a'(r)\right| < \frac{C}{r^2}. \tag{3.23}$$

This can be rewritten as

$$\left|(r-K)\frac{d}{dr}\left(\frac{a(r)}{r-K}\right)\right| < \frac{C}{r^2} \tag{3.24}$$

This shows that the derivative of $\frac{a(r)}{r-K}$ decays as r^{-3}. Thus, we see that $\frac{a(r)}{r-K}$ and thus $\frac{a(r)}{r}$ tends to a finite limit as $r \to \infty$. By Lemma 3.5.6, this limit is at least $c > 0$, so it must be positive.

Now, we see that a' and b' reach finite limits as $r \to \infty$. We denote these limits by a'_∞ and b'_∞. This concludes the proof of (1).

Now, for (2), observe that the constant C in equation Equation (3.24) depends uniformly on the initial conditions, as it only depends on similarly behaved constants from Lemmas 3.5.10, 3.5.11, and 3.5.12. Thus, the function $\frac{a(r)}{r-K}$ depends continuously on the initial conditions and its derivative is bounded by C/r^3 for a uniform constant C (on an interval of the form $[r_0, \infty)$, by the continuity of K). Thus, $\lim_{r\to\infty} \frac{a(r)}{r-K}$ is continuous in the initial conditions. Since we know from (1) that this constant is equal to a'_∞, we see that the slope a'_∞ is continuous in the initial conditions. The analogous argument shows the continuity of b'_∞ as well. □

Geometrically, this suggests that the gradient expanding solitons over $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$ are asymptotically conical. We formalize this as follows:

Theorem 3.5.14. *In either of the $S^1 \times \mathbb{R}^3$ and $S^2 \times \mathbb{R}^2$ cases, fix a cohomogeneity one gradient expanding soliton $(M, g, \nabla f)$ and a point p with $r(p) = 0$. Consider any sequence $\lambda_i \to 0$. Then, we have Gromov-Hausdorff convergence of $(M, \lambda_i^2 g, p)$ to a cone over the link $S^2 \times S^1$.*

Proof. Consider the sequence $\nu_i := \frac{1}{\lambda_i}$. Then, the rescaled metric $g_i := \lambda_i^2 g$ is a warped product given by

$$g_i = dr^2 + a_i(r)^2 g_{S^1} + b_i(r)^2 g_{S^2}$$

where $a_i(r) = \frac{a(\nu_i r)}{\nu_i}$ and $b_i(r) = \frac{b(\nu_i r)}{\nu_i}$. By (6.4), since we know that since $\lim_{r\to\infty} \frac{a(r)}{r-K} = a'_\infty$ and that the derivative of $\frac{a(r)}{r-K}$ decays as r^{-3}, we have the inequality for r greater than some large r_0 which is greater than K:

$$\left| \frac{a(r)}{r-K} - a'_\infty \right| \leqslant \frac{C}{r^2}$$

We also have

$$\left| \frac{a(r)}{r} - a'_\infty \right| \leqslant \left| \frac{a(r)}{r} - \frac{a(r)}{r-K} \right| + \left| \frac{a(r)}{r-K} - a'_\infty \right| \leqslant \frac{C}{r}$$

by using Lemma 3.5.12 on the first term. Thus, we have

$$|a(r) - a'_\infty r| \leqslant C$$

for $r > r_0$. We can immediately extend this bound to the interval $[0, \infty)$ (for a different C) by compactness of $[0, r_0]$. From this, we have

$$\left| \frac{a(\nu_i r)}{\nu_i} - a'_\infty r \right| \leqslant \frac{C}{\nu_i}$$

which implies that the functions $a_i(r)$ converge uniformly to $a'_\infty r$. A similar result holds for b_i. From this, it is easy to see that since the metrics on the rescaled solitons converge uniformly to the metric of the asymptotic cone, we have the required Gromov-Hausdorff convergence. $\square$

Based on the results of this discussion, we make the following definition.

Definition 3.5.15. *A cohomogeneity one gradient expanding soliton $(M, g, \nabla f)$ as considered in this paper is called **asymptotically conical** if it satisfies the following conditions:*

1. $|\mathrm{Rm}|_g(r) \leqslant \frac{C}{r^2}$ *for some* $C > 0$.
2. $|\nabla f|(r) \leqslant r + C$ *for some* $C \in \mathbb{R}$.
3. *There exists a sequence $\lambda_i \to 0$ so that $(M, \lambda_i^2 g, p)$ Gromov-Hausdorff converges to a cone metric γ with link $(S^2 \times S^1, h)$, where the metric h admits an isometric action of* $\mathrm{SO}(3) \times \mathrm{SO}(2)$.

From Lemmas 3.5.9, 3.5.10 and Theorem 3.5.14, we see that in the case of each topology, the solitons in the 2-parameter family from Theorem 3.4.1 are asymptotically conical as in Definition 3.5.15. We note that these solitons were shown to be asymptotically conical in previous work of [NW24], [Win21], and [BDGW15]. We have provided an alternate proof (using our slightly different definition of asymptotically conical) both for the sake of completeness of our work and additionally to use certain estimates from this section in future sections.

3.6 Relating Expanding Solitons to Asymptotic Cones

As we know that our solitons (M, g) are asymptotic to cones $(\mathbb{R}^+ \times S^2 \times S^1, \gamma)$, it will be important to understand how close g is to γ as well as the value of the distance from the singular orbit of M at which this happens. In this section, we show that the assignment of the asymptotic cone to a soliton is a continuous map by using results from the previous section, and also provide the setup to show that this assignment is a proper map. To do this, we introduce a notion called uniform ϵ-conicality and show that the closeness of an expanding soliton considered in this paper to its asymptotic cone is determined by the geometry of the cone. This is the main result of this section, used to prove the properness result of Section 9.

First, we make use of the following map, constructed in [BC23] which allows us to embed into a soliton its asymptotic cone.

Lemma 3.6.1. *Consider a cohomogeneity one gradient expanding Ricci soliton $(M, g, \nabla f)$ asymptotic to a cone $\gamma = ds^2 + s^2 h$ over a link $(S^2 \times S^1, h)$ as in Definition 3.5.15. Then, there exists a smooth embedding $\iota : \mathbb{R}_+ \times S^2 \times S^1 \to M$ satisfying the following*

1. $M - \iota((s,\infty) \times S^2 \times S^1)$ is compact for all $s \geqslant 0$.

2. $\iota^*(\nabla f) = -s\partial_s$, where s is the coordinate on $\mathbb{R}_+$ and ∂_s is the corresponding vector field.

3. The pullback metric $\iota^* g$ is smooth

Further, suppose that we have the following bounds, where the constants α and A_i are positive, and $m \geqslant 0$

$$\mathrm{inj}_\gamma \geqslant \alpha s \qquad\qquad |\nabla^{m,\gamma}\mathrm{Rm}_\gamma| \leqslant \frac{A_m}{s^{2+m}} \tag{3.25}$$

Then, there exists $S_0 \equiv S_0(\alpha, A_0)$ such that we have the following quantitative asymptotics of $\iota^* g$ to γ on $(S_0,\infty) \times S^2 \times S^1$

$$|\nabla^{m,\gamma}(\iota^* g - \gamma)| \leqslant \frac{C_m(\alpha, A_0, \ldots, A_m)}{s^{2+m}} \tag{3.26}$$

The map ι is unique in the following sense: suppose that ι' is another such smooth map satisfying conditions (1)-(3). Then, we must have $\iota' = \iota \circ (\mathrm{Id}_{\mathbb{R}^+}, \psi)$, where $\psi : (S^2 \times S^1, h) \to (S^2 \times S^1, h)$ is an isometry.

The conclusion of the lemma is essentially unchanged from Lemma 2.9 of [BC23]. The hypotheses that the solitons in the cohomogeneity one 2-parameter families from Theorem 3.4.1 are asymptotically conical was verified in the discussion following Definition 3.5.15.

In our warped product setting, it is clear that $\alpha = \mathrm{inj}_{S^2\times S^1} h = \pi \min\{a'_\infty, b'_\infty\}$. Since (M, g) is cohomogeneity one, we know that for any $p \in M$, that $\nabla f(p) \equiv f'(r)\partial_r|_p$ depends only on the coordinate r and not on the coordinates of $S^2 \times S^1$. By Part 2 of Lemma 3.6.1, since the trajectories of ι are the integral curves of this vector field, the coordinates on $S^2 \times S^1$ are unchanged along ι.

Thus, we may write $\iota(s, z) = (d(s), z)$, for some smooth function $d : \mathbb{R}^+ \to \mathbb{R}^+$ (here, we precompose ι with the required isometry of $S^2 \times S^1$ if necessary to ensure that the map ι leaves the coordinate on every $S^2 \times S^1$ orbit unchanged).

This lemma allows us to quantify the closeness between the soliton metric and the asymptotic cone via the following definition.

Definition 3.6.2. *Fix $\epsilon > 0$ and consider a cohomogeneity one gradient expanding soliton $(M, g, \nabla f)$ with topology $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$ asymptotic to the cone $(\mathbb{R}^+ \times S^2 \times S^1, \gamma = ds^2 + s^2 h)$. Suppose that there exists $S_0 > 0$ such that on $[S_0, \infty) \times S^2 \times S^1$, we have the following bound for all $m \leqslant 10$:*

$$|\nabla^{\gamma,m}(\iota^* g - \gamma)| \leqslant \epsilon \tag{3.27}$$

Then, $(M, g, \nabla f)$ is said to be ϵ-conical at distance $d(S_0)$ from the tip $r = 0$, where $d : \mathbb{R}^+ \to \mathbb{R}^+$ is the map defined as above with $\iota(s, z) = (d(s), z)$.

Using Definition 3.6.2, we will show that curvature and injectivity radius bounds on the links of the asymptotic cone γ of a soliton (M, g) are sufficient to control the distance from the singular orbit at which the solitons are ϵ-conical.

Lemma 3.6.3. *Suppose $(M, g_i, \nabla f_i)$ is a collection of cohomogeneity one gradient expanding Ricci solitons asymptotic to cone metrics γ_i with links $(S^2 \times S^1, h_i)$, where the link metrics are $h_i = (a'_{\infty,i})^2 g_{S^1} + (b'_{\infty,i})^2 g_{S^2}$. Suppose we have the following bound for all i:*

$$\min\{a'_{\infty,i}, b'_{\infty,i}\} \geqslant c$$

Then, for any fixed $\epsilon > 0$, there is a fixed constant $S_0 \equiv S_0(\epsilon, c) > 0$ so that each $(M, g_i, \nabla f_i)$ is ϵ-conical at $d_i(S_0)$. Additionally, we have the following bounds:

$$a'_{\infty,i}(1-\epsilon)^{1/2} S_0 \leqslant a_i(d_i(S_0)) \leqslant a'_{\infty,i}(1+\epsilon)^{1/2} S_0 \tag{3.28}$$

$$b'_{\infty,i}(1-\epsilon)^{1/2} S_0 \leqslant b_i(d_i(S_0)) \leqslant b'_{\infty,i}(1+\epsilon)^{1/2} S_0 \tag{3.29}$$

Proof. For a given expander $(M, g, \nabla f)$ asymptotic to the cone γ with asymptotic slopes a'_∞, b'_∞, the nonzero sectional curvatures (calculated using the coordinate system in Appendix A) of γ are

$$\mathrm{Rm}_{\gamma,2332} = \mathrm{Rm}_{\gamma,2442} = -\frac{1}{s^2} \qquad \mathrm{Rm}_{\gamma,3443} = \frac{1-(b'_\infty)^2}{b'^2_\infty}\frac{1}{s^2}$$

The bound in the hypothesis implies that $\mathrm{inj}(h_i)$ is bounded below, or that $\mathrm{inj}_{\gamma_i}(s) \geqslant \alpha s$ for some constant $\alpha > 0$. Additionally, since $b'_{\infty,i}$ is bounded from below, we can differentiate the curvature terms to see that we have the required bounds on $|\nabla^{m,\gamma}\mathrm{Rm}_\gamma|$ in the second part of Equation (3.25). Thus, by Lemma 3.6.1, there exists $R_0 \equiv R_0(c)$ such that the following bound holds for all i and $m \leqslant 10$ on $(R_0, \infty) \times S^2 \times S^1$:

$$|\nabla^{m,\gamma_i}(\iota_i^* g_i - \gamma_i)| \leqslant \frac{C_m(\alpha, A_0, \ldots, A_m)}{s^{2+m}}$$

Now, choose $S_0 > R_0$ so that the bound on the right hand side (for each $m \leqslant 10$) is less than ϵ for $s \geqslant S_0$. This proves the first statement of the lemma.

For the second statement, for $m = 0$, we have the following inequality on $(S_0, \infty) \times S^2 \times S^1$ for all i:

$$(1-\epsilon)\gamma_i \leqslant \iota_i^* g_i \leqslant (1+\epsilon)\gamma_i$$

By plugging in the appropriate unit vectors on $(\mathbb{R} \times S^1 \times S^2, \gamma_i)$ based at S_0 tangent to the S^1 and S^2 directions into the above inequalities, we obtain inequalities Equation (3.28) and Equation (3.29), respectively. □

With an additional diameter assumption on the links h_i, we will further show that the sequence $d_i(S_0)$ is bounded. In the remainder of this section, we will show this by contradiction; more specifically, assuming that $\{d_i(S_0)\}$ is unbounded, we take a geometric limit to produce a certain expanding soliton with two ends and identify a contradiction. For this, we will need some technical lemmas that allow us to take this limit and to show that two-ended expanding solitons satisfying certain conditions do not exist. The first lemma below shows that curvature bounds on a soliton provide lower bounds on the size of its S^2-orbit.

Lemma 3.6.4. *Suppose (M, g) is a cohomogeneity one gradient expanding Ricci soliton as considered in this paper. If (M, g) satisfies the bound $|Rm|_g \leqslant C$ for some constant $C > 0$ on a ball $B_g(p, D)$ (with $D > 1$) where $r(p) = r^*$, then we have the following bound:*

$$b(r^*) \geqslant \min\left\{\frac{1}{2}, \frac{1}{2C^{1/2}}\right\}$$

Proof. The curvature bound on the ball provides the following inequality on Rm_{3443} (refer Appendix A for the calculation of the sectional curvatures)

$$|\mathrm{Rm}_{3443}| = \left|\frac{1-(b')^2}{b^2}\right| \leqslant C$$

From this, we deduce the lower bound in the statement in the lemma. If the bound does not hold, the inequality $(b')^2 \geqslant (1 - Cb^2)$ implies that $b'(r^*) \geqslant \sqrt{3}/2$. Using the monotonicity of b from Lemma 3.3.1, $b'(r) \geqslant \sqrt{3}/2$ for all $r \leqslant r^*$, but this implies that b becomes negative (in finite distance from r^*) inside $B_g(p, D)$. $\square$

Next, we will show that certain kinds of cohomogeneity one gradient expanding solitons with 2 ends cannot arise as limits of solitons in the 2-parameter families we consider in this paper. First, we consider certain classes of cohomogeneity one Einstein metrics.

Lemma 3.6.5. *Suppose $(M, g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2})$ is a cohomogeneity one Einstein manifold satisfying $\mathrm{Ric}_g + g = 0$ on $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$. Then, we have the equalities $b' = Ca$ and $b'' = Ca'$ for some constant $C > 0$.*

Proof. Setting $f \equiv 0$ in Equations (3.2) to (3.4), we see that the Einstein equations are

$$\frac{a''}{a} = -\frac{2a'b'}{ab} + 1$$

$$\frac{b''}{b} = \frac{1-(b')^2}{b^2} - \frac{a'b'}{ab} + 1$$

$$\frac{a''}{a} + \frac{2b''}{b} - 1 = 0$$

From the first and third equations, we see that

$$\frac{b''}{b} = \frac{a'b'}{ab}$$

Then, the function $\frac{b'}{a}$ satisfies

$$\left(\frac{b'}{a}\right)' = \frac{b''}{a} - \frac{a'b'}{a^2} = \frac{b}{a}\left(\frac{b''}{b} - \frac{a'b'}{ab}\right) = 0$$

Thus, $\frac{b'}{a}$ must be a constant, implying the conclusion of the lemma. $\square$

Now, we rule out certain Einstein metrics on $\mathbb{R} \times S^2 \times S^1$. The key intuition underlying this proof is that the second equation (analogous to Equation (3.3)) for Einstein metrics cannot be satisfied near $-\infty$ since the curvature of the S^2 would become too large.

Lemma 3.6.6. *There are no cohomogeneity one Einstein metrics of the form* $\mathrm{Ric}_g + g = 0$ *on the space* $\mathbb{R} \times S^2 \times S^1$ *arising as Cheeger-Gromov limits of doubly warped product expanding solitons* $(M, g_i = dr^2 + a_i(r)^2 g_{S^1} + b_i(r)^2 g_{S^2})$*, where* a_i *and* b_i *are monotonically increasing functions.*

Proof. Suppose that such an Einstein metric exists. Then, the Einstein equations would have a solution (a, b) on the interval $(-\infty, \infty)$. As we are assuming that such an Einstein metric is a limit of doubly warped product expanding solitons, the monotonicity properties for such solitons carry over to give us the inequalities $a', b' \geqslant 0$.

Thus, we see that both a and b approach finite limits as r tends to $-\infty$ by monotonicity. From Lemma 3.6.5, we see that $b' = Ca$, so b' approaches a limit as r tends to $-\infty$. As b itself approaches a finite limit, b' must tend to 0 as r tends to $-\infty$. Then, we can rewrite the second Einstein equation as

$$1 + \frac{1 - (b')^2}{b^2} = 2C\frac{a'}{b} \tag{3.30}$$

which implies that

$$b^2 + 1 - (b')^2 = 2Ca'b \tag{3.31}$$

From Equation (3.30), we see that a' approaches a limit as r tends to $-\infty$, and this limit must be 0 as $a \geqslant 0$ everywhere. This leads to a contradiction, as the right hand side of Equation (3.31) approaches 0 as r tends to $-\infty$, while the left hand side approaches a nonzero value. Thus, there are no such Einstein metrics with two ends arising as such limits. $\square$

Now, we show that two-ended gradient expanding solitons satisfying the monotonicity properties (established for one-ended solitons in Section 4) cannot exist. Note that we do not rule out two-ended expanding solitons in general; in fact, in [Ram12], the existence of a 3-dimensional gradient expanding soliton on $\mathbb{R} \times S^1 \times S^1$ is established. In our situation, the topology and monotonicity properties will be key to the proof.

Lemma 3.6.7. *There does not exist a cohomogeneity one gradient expanding Ricci soliton on* $\mathbb{R} \times S^2 \times S^1$ *with the monotonicity properties* $a', b' \geqslant 0$ *and* $f', f'' \leqslant 0$.

Proof. Suppose $(M, g, \nabla f)$ is a cohomogeneity one gradient expanding Ricci soliton on $\mathbb{R} \times S^2 \times S^1$ with the given monotonicity properties. Recalling soliton identities Equation (3.9) and Equation (3.10), we have the following identities for some constant $C \in \mathbb{R}$:

$$R + \Delta f + 4 = 0,$$
$$R + |\nabla f|^2 + 2f = C.$$

Combining these identities, we see that

$$f = \frac{1}{2}(\Delta f - |\nabla f|^2 + C).$$

Using the monotonicity properties, we have $\Delta f = f'' + f'\left(\frac{a'}{a} + \frac{2b'}{b}\right) \leqslant 0$, so f is bounded from above.

Now, we make the following claim:

Claim: (M, g) must have bounded sectional curvature on the end where $r \to -\infty$.

Proof of Claim: Suppose this is not the case; then we can find a sequence of points $p_i \in (M, g)$ where $r(p_i) = r_i \to -\infty$ and $|\mathrm{Rm}|_g(p_i) \to \infty$. For any $D_i > 0$, using Lemma 3.5.7, we can find a sequence of points $q_i \in B_g(p_i, 2D_i/\sqrt{|\mathrm{Rm}|_g(p_i)})$ where

$$r(q_i) \leqslant r(p_i) + \frac{d(p_i, q_i)}{\sqrt{|\mathrm{Rm}|_g(p_i)}} \leqslant r_i + \frac{2D_i}{\sqrt{|\mathrm{Rm}|_g(p_i)}}$$

along with the bounds $|\mathrm{Rm}|_g(q_i) := Q_i \to \infty$ and $|\mathrm{Rm}_g| \leqslant 4Q_i$ on $B_g(q_i, D_i/\sqrt{Q_i})$. Now choose a sequence $D_i \to \infty$ such that $r(q_i) \to -\infty$. Rescale to get $\tilde{g}_i = Q_i g$, where the new inequality is $|\mathrm{Rm}_{\tilde{g}_i}| \leqslant 4$ on $B_{\tilde{g}_i}(q_i, D_i)$. Then, for any $D > 0$, we have $D < D_i$ for large i, so $|\mathrm{Rm}|_{\tilde{g}_i} \leqslant C(D)$ on $B_{\tilde{g}_i}(q_i, D)$ for a constant $C(D)$ depending on D.

By the monotonicity of f', since $r(q_i)$ is bounded above, we see that $|\nabla f|(q_i)$ is bounded. By Shi's estimates for the associated Ricci flows, we can derive bounds on the derivatives of the curvature on $B_{\tilde{g}_i}(q_i, D)$ as well as higher derivatives of f using the soliton equation. By Lemma 3.6.4, we have a uniform lower bound on $b(r(q_i))$. By multiplying a by a constant factor (note that this does not affect the soliton equations), we can assume that $a(r(q_i))$ is bounded from above and below.

Thus, by the previous paragraphs, we have the required volume and curvature bounds to take (up to a subsequence) a Cheeger-Gromov limit of $(M, g_i, \nabla \tilde{f}_i, q_i)$ to get a warped product metric $(M_\infty, g_\infty, \nabla f_\infty, q_\infty)$, which satisfies the steady soliton equation. If the size of the S^2 orbit remains bounded, M_∞ has topology $\mathbb{R} \times S^2 \times S^1$ and thus has two ends. If it becomes unbounded, then M_∞ has topology $\mathbb{R} \times \mathbb{R}^2 \times S^1$, but $M_\infty/\mathbb{Z}^2 \equiv \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ has two ends.

Thus, we have a steady soliton with two ends, which must split as the product of $\mathbb{R}$ and a compact 3-dimensional Ricci-flat manifold N by [MW11]. As N must be flat, we see that

(M_∞, g_∞) is isometric to a quotient of Euclidean space, but this contradicts $|\mathrm{Rm}_{g_\infty}(q_\infty)| \neq 0$. Thus, the claim must be true. □

Now, take a sequence of points p_i with $r(p_i) = r_i \to -\infty$ along this end. By the claim and its proof, we have lower volume bounds of small r-balls at p_i and curvature bounds on M (which imply bounds on the derivative of the curvature by Shi's estimates) along this end. Thus, we can take (up to a subsequence) a Cheeger-Gromov limit of $(M, g, p_i, \nabla f)$ to get a cohomogeneity one gradient expanding soliton $(M_\infty, g_\infty, p_\infty, \nabla f_\infty)$ with topology $\mathbb{R} \times S^2 \times S^1$ (by the monotonicity of a and b, and by rescaling a by a constant if necessary, the orbits stay bounded in diameter).

By the monotonicity of f_∞, since we chose a sequence $r_i \to -\infty$, we see that f_∞ must be bounded from below. From the beginning of the proof, we also know that f_∞ is bounded from above. This implies that $f'_\infty = f''_\infty = 0$, and that f_∞ is thus constant. This means that $(M_\infty, g_\infty, p_\infty, f_\infty)$ is an Einstein manifold on $\mathbb{R} \times S^2 \times S^1$ with two ends, which we have ruled out by Lemma 3.6.6. Thus, there are no such two ended expanding solitons. □

Using the technical lemmas above, we can prove the following improved version of Lemma 3.6.3.

Lemma 3.6.8. *Fix an $\epsilon \geqslant 0$ and suppose that $(M, g_i, \nabla f_i)$ is a sequence of gradient expanding Ricci solitons respectively asymptotic to cones over links $(S^2 \times S^1, h_i = (a'_{\infty,i})^2 g_{S^1} + (b'_{\infty,i})^2 g_{S^2})$. Suppose we have the following bounds for all i:*

$$\min\{a'_{\infty,i}, b'_{\infty,i}\} \geqslant c_1 \qquad b'_{\infty,i} \leqslant c_2$$

for some constants $c_1, c_2 > 0$. Then, we have the following:

1. *The scalar curvature satisfies $R_{g_i} \leqslant C(c_1)$ on M for some constant $C(c_1) > 0$.*
2. *There exists $r_0 > 0$ so that $(M, g_i, \nabla f_i)$ is uniformly ϵ-conical at a distance $r_i \leqslant r_0$ from the tip.*

Proof. Part 1 follows from the proof of Proposition 4.32 of [BC23]. To summarize briefly, first note that the bound $\min\{a'_{\infty,i}, b'_{\infty,i}\} \geqslant c_1$ in the hypothesis provides bounds on the injectivity radii and curvatures of the links in the form $|\mathrm{Rm}|_{h_i} \leqslant Q \equiv Q(c_1)$ and $\mathrm{inj}(h_i) \geqslant \alpha \equiv \alpha(c_1)$. In the first several paragraphs of the proof of Proposition 4.32 in [BC23], it is shown via the maximum principle that the quantity $R_{g_i} + |\nabla f_i|^2 = -f_i$ is bounded from above in the form $-f_i \leqslant C(\alpha, Q)$. This provides the uniform scalar curvature upper bound on (M, g_i).

We remark that we make no assumptions about the sign of of the scalar curvature, unlike in [BC23], in which it is used to prove a non-collapsing result. In a later section, we show that the assignment of the asymptotic cone to a cohomogeneity one soliton is a proper map irrespective of whether the solitons have nonnegative scalar curvature.

2. From Lemma 3.6.3, we know that we have a fixed $S_0 > 0$ such that $(M, g_i, \nabla f_i)$ is ϵ-conical at $d_i(S_0) := r_i$.

Suppose the sequence $\{r_i\}$ is unbounded. Let $p_i \in (M, g_i)$ be a sequence of points with $r(p_i) = r_i$. Then, by Definition 3.6.2, for sufficiently small ϵ, we know by ϵ-conicality that $|\mathrm{Rm}|_{g_i}(p_i)$ is close to $|\mathrm{Rm}|_{\gamma_i}(S_0, z)$ for all i. By the hypotheses that $c_2 \geqslant b'_{\infty,i} \geqslant c_1$, $|\mathrm{Rm}|_{\gamma_i}(S_0, z)$ is bounded from above and below for all i, implying the same conclusion for $|\mathrm{Rm}|_{g_i}(p_i)$. Additionally, using Equation (3.28) and Equation (3.29), we see that

$$
\begin{aligned}
&a'_{\infty,i}(1-\epsilon)^{1/2} S_0 \leqslant a_i(r_i) \\
&b'_{\infty,i}(1-\epsilon)^{1/2} S_0 \leqslant b_i(r_i) \leqslant b'_{\infty,i}(1+\epsilon)^{1/2} S_0
\end{aligned}
$$

From the above inequalities, and the hypotheses, we see that $a_i(r_i)$ is bounded from below while $b_i(r_i)$ is bounded from both above and below.

Now, define

$$\bar{M}_i := \{x \in (M, g_i) \mid r(x) \leqslant r_i\}$$

For any $D_i > 0$, by Lemma 3.5.7 (applied to the manifold with boundary $\bar{M}_i$), we can choose a point q_i satisfying

$$d(p_i, q_i) \leqslant \frac{2D_i}{\sqrt{|\mathrm{Rm}|_{g_i}(p_i)}} \quad \text{with} \quad r(q_i) \leqslant r_i,$$

so that for $Q_i := |\mathrm{Rm}|_{g_i}(q_i)$, we have

$$Q_i \geqslant |\mathrm{Rm}|_{g_i}(p_i) \quad \text{and} \quad |\mathrm{Rm}|_{g_i} \leqslant 4Q_i \quad \text{on} \quad B_{g_i}(q_i, D_i/\sqrt{Q_i}).$$

Note: a priori, this bound only holds on the intersection of this ball with the complete metric space $\bar{M}_i$; however, by ϵ-conicality, for sufficiently small $\epsilon > 0$ and for $r > r_i$, the curvature cannot be much larger than $|\mathrm{Rm}|(p_i)$, which is less than Q_i. Thus, the bound $\mathrm{Rm}|_{g_i} \leqslant 4Q_i$ holds on the entire ball $B_{g_i}(q_i, D_i/\sqrt{Q_i})$.

Now choose a sequence $\{D_i\} \to \infty$ such that $r(q_i) \to \infty$. Note that

$$r(q_i) \geqslant r_i - \frac{2D_i}{\sqrt{\mathrm{Rm}_{g_i}(p_i)}},$$

by the result of point-picking, so it is possible to choose such a sequence D_i. Then, rescale to get $\tilde{g}_i = Q_i g_i$ satisfying $|\mathrm{Rm}|_{\tilde{g}_i} \leqslant 4$ on $B_{\tilde{g}_i}(q_i, D_i)$. Then, for any $D > 0$, we have $D < D_i$ for large i, so $|\mathrm{Rm}|_{\tilde{g}_i} \leqslant C(D)$ on $B_{\tilde{g}_i}(q_i, D)$. By Shi's estimates (which can be applied, since $\{Q_i\}$ is bounded from below), we have bounds on the derivatives of the curvature of $\tilde{g}_i$ as well on the ball. By Lemma 3.6.4, we can ensure lower bounds on the sizes of the S^2 orbits at q_i. We may also rescale a by a constant if necessary to ensure that the sizes of the S^1 orbits remain bounded as well (note that this does not affect the soliton equations except at $r = 0$).

Now, consider the case where (a subsequence of) Q_i is unbounded from above. By Part 1, we have a uniform scalar curvature bound on (M, g_i) in terms of the curvature bound of the links. Thus, by the previous paragraph, we can consider (up to a subsequence) the Cheeger-Gromov limit of $(M, \tilde{g}_i, q_i)$ to get $(M_\infty, g_\infty, q_\infty)$. If the sizes of the S^1 and S^2 orbits containing q_i in (M, g_i) remain bounded, we have $M_\infty \equiv \mathbb{R} \times S^2 \times S^1$ topologically, which has two ends, since we assumed that D_i was chosen so that $r(q_i) \to \infty$. If the S^1 (S^2) orbits become unbounded, we may replace M_∞ by the quotient space $M_\infty/\mathbb{Z}$ ($M_\infty/\mathbb{Z}^2$), which has two ends.

The uniform scalar curvature bound implies that (M_∞, g_∞) is Ricci-flat. Then, (M_∞, g_∞) splits as a product of a line and a 3-dimensional Ricci-flat manifold, implying that M_∞ is flat. However, this is a contradiction to $|\mathrm{Rm}|_{g_\infty}(q_\infty) \neq 0$.

Now, consider the case where Q_i is bounded. By the proof of Lemma 4.32(b) of [BC23], $|\nabla f|$ is uniformly bounded on (M, g_i) for $r \leqslant r_i$. Using the soliton equation, the curvature bounds provide higher derivative bounds on f_i (we may add an appropriate constant to each f_i to ensure that $f_i(q_i) = 0$). Thus, we can take (up to a subsequence) the Cheeger-Gromov limit of the solitons $(M, g_i, \nabla f_i, q_i)$ to get a certain cohomogeneity one gradient expanding soliton $(M_\infty, g_\infty, \nabla f_\infty, q_\infty)$. As in the previous case, we can quotient M_∞ by translations if necessary to ensure that it has two ends.

Thus, we have a cohomogeneity one gradient expanding soliton with two ends. Note that this gradient expanding soliton satisfies the monotonicity properties $a', b' \geqslant 0$ and $f', f'' \leqslant 0$. By Lemma 3.6.7, we know that such solitons do not exist, leading us to a contradiction. Thus, $\{r_k\}$ is bounded, so choosing $r_0 = \max r_k$, we see that $(M, g_k, \nabla f_k)$ is uniformly ϵ-conical at $r_k \leqslant r_0$. □

Thus, if we know that a'_∞ and b'_∞ are uniformly bounded above and below, we see that any cohomogeneity one gradient expanding solitons asymptotic to such cones must be ϵ-conical within some fixed distance of the tip. Thus, given an $\epsilon > 0$, we can think about each expanding soliton $(M, g, \nabla f)$ as a union of two regions; an ϵ-conical region consisting of points where $r > r_0$ (these points are sufficiently far from the singular orbit at $r = 0$), and a compact set of points with $r \leqslant r_0$.

Now that we understand how the distance at which a soliton looks asymptotic to a cone depends on the geometry of the link of the cone, we define a map that essentially assigns to a soliton its asymptotic cone.

Definition 3.6.9. *Suppose (a, b, f) is a solution to equations Equations* (3.2) *to* (3.4)*, with boundary conditions either Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8)*. Let the corresponding soliton metric $g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2}$ be asymptotic to the cone metric $\gamma = ds^2 + (a'_\infty s)^2 g_{S^1} + (b'_\infty s)^2 g_{S^2}$. Then, we define the map $F : (0, \infty) \times (0, \infty) \to (0, \infty) \times (0, \infty)$ as*

$$F(a_0, -f_0) = (a'_\infty, b'_\infty)$$

in the case of $S^1 \times \mathbb{R}^3$ topology (boundary conditions Equations (3.5) *and* (3.6)*), and*

$$F(b_0, -f_0) = (a'_\infty, b'_\infty)$$

in the case of $S^2 \times \mathbb{R}^2$ topology (boundary conditions Equations (3.7) *and* (3.8)*).*

The definition above makes sense, as the soliton metrics in the 2-parameter families were verified to be asymptotic to cone metrics in Section 6.

Having established certain facts about ϵ-conicality of solitons so far in this section, we will analyze the map F and show that it has good properties in the rest of the paper.

Theorem 3.6.10. *In the case of either topology, the map F as in Definition 3.6.9 above is continuous.*

Proof. This is a restatement of the second statement of Lemma 3.5.13. □

Now that we know that F is a continuous function, it is natural to attempt to count, with sign, the number of expanding solitons which are asymptotic to a given cone. To do this, first we need to prove that F is a proper map. In Section 8, we will study the behavior of F as the initial conditions approach their extreme values, and use this to prove in Section 9 that F is proper in the case of each topology.

3.7 Asymptotic Behavior Near Extreme Values

We have established in the previous sections that each of the expanding solitons in the 2-parameter family over $S^2 \times \mathbb{R}^2$ or $S^1 \times \mathbb{R}^3$ are asymptotic to cones, with the cones defined by their limiting slopes a'_∞, b'_∞. Additionally, we know that these slopes are continuous functions of the initial conditions. Now, we will investigate what happens to the solitons as well as their asymptotic cones as the initial conditions tend to their extreme values. This will be used in the following section to show that the map F defined in the previous section is proper.

We will begin with the slightly simpler $S^1 \times \mathbb{R}^3$ case first. In this case, the two parameters are $a(0) \in (0, \infty)$ and $f''(0) \in (-\infty, 0)$. The next lemma shows that the dependence on $a(0)$ is very simple.

Lemma 3.7.1. *Suppose that the functions a, b and f satisfy the soliton equations with parameters $a(0) = a_0$ and $f''(0) = f_0$. Then, for any constant $c > 0$ the solution to the soliton equations with parameters $a(0) = ca_0$ and $f''(0) = f_0$, the solution to the soliton equations is given by the triple (ca, b, f).*

Proof. This follows from the simple observation that the transformation $a \to ca$ leaves Equations (3.2) to (3.4) unchanged. □

Thus, the dependence of the slopes on the parameter a_0 in the $\mathbb{S}^1 \times \mathbb{R}^3$ case is essentially trivial. With F as in Definition 3.6.9, in this case, we see that if $F(a_0, -f_0) = (a'_\infty, b'_\infty)$, then $F(ca_0, -f_0) = (ca'_\infty, b'_\infty)$. In Section 9, we reduce this to one variable function $F_1(-f_0) := F(1, -f_0)$

Now, we investigate what happens when the parameter f_0 tends towards the extreme of $f_0 = 0$.

Intuitively, as f_0 gets closer and closer to 0, the soliton potential f grows more slowly near 0, taking a longer region until it becomes asymptotic to $-\frac{1}{2}r^2$. By Lemma 3.3.3, we see that when $f''(0) = 0$, the corresponding (M, g) is an Einstein manifold. In the $S^1 \times \mathbb{R}^3$ case, we can describe this Einstein manifold in even more detail.

Lemma 3.7.2. *Consider the soliton equations Equations* (3.2) *to* (3.4) *with* $f''(0) = 0$ *and boundary conditions Equations* (3.5) *and* (3.6) *so that the manifold is diffeomorphic to* $S^1 \times \mathbb{R}^3$. *Then, the corresponding Riemannian manifold* $(M, dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2})$ *is a quotient of the hyperbolic model space with constant negative sectional curvature equal to* $-1/3$.

Proof. From Lemma 3.3.3, we know that $f \equiv 0$ on $[0, \infty)$. This implies that $\mathrm{Ric}_g + g = 0$, or that (M, g) is an Einstein manifold with negative scalar curvature. Then, the soliton equations reduce to the following:

$$\frac{a''}{a} + 2\frac{b''}{b} = 1$$

$$a'' = -2\frac{a'b'}{b} + a$$

$$b'' = \frac{1 - (b')^2}{b} - \frac{a'b'}{a} + b$$

Now, it is easy to verify that the functions $a(r) = a_0 \cosh\left(\frac{r}{\sqrt{3}}\right)$ and $b(r) = \sqrt{3}\sinh\left(\frac{r}{\sqrt{3}}\right)$ satisfy the given equations as well as boundary conditions Equations (3.5) and (3.6). Thus, they must be the unique solutions to the soliton equation in the case $f''(0) = 0$. Up to rescaling, this metric describes the hyperbolic model space $(\mathbb{R}^4, g_H)$ in cylindrical coordinates. □

Next, we will show that as $f_0 \to 0$, the slopes a'_∞ and b'_∞ tend to ∞ in both the $S^1 \times \mathbb{R}^3$ and $S^2 \times \mathbb{R}^2$ cases.

Recall the quantities $A := \frac{a'}{a}$ and $B := \frac{b'}{b}$ from the proof of Lemma 3.5.9. Using these quantities, we will rewrite the soliton equations in a slightly more convenient form for this analysis, beginning with the following lemma.

Lemma 3.7.3. *Consider a gradient expanding soliton* (M, g) *with* (a, b, f) *satisfying equations Equations* (3.2) *to* (3.4) *and boundary conditions either Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8). *For* A *and* B *as defined above, the following equations hold on* $[1, \infty)$*:*

$$A' = -A^2 - 2AB + Af' + 1 \tag{3.32}$$

$$B' = \frac{1}{b^2} - 2B^2 - AB + Bf' + 1 \tag{3.33}$$

Proof. This follows directly from the definitions of A and B and equations Equations (3.2) to (3.4). □

Lemma 3.7.4. *Consider a cohomogeneity one gradient expanding Ricci soliton* (M, g) *with* (a, b, f) *satisfying equations Equations* (3.2) *to* (3.4) *and boundary conditions either Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8). *Then, there exists a unique* $r_0 \equiv r_0(a_0, f_0)$ *or* $r_0(b_0, f_0)$ *such that* $f'(r_0) = -1$. *Suppose that the other initial condition* (a_0 *or* b_0) *lies in a compact interval of the form* $[\delta, 1/\delta]$ *for some* $\delta \in (0, 1)$. *As* f_0 *approaches* $-\infty$, r_0 *approaches* ∞.

Proof. We know from Lemma 3.3.2 and Theorem 3.5.4 that f' is monotonically decreasing and unbounded. Thus, for every $f_0 < 0$ and a_0 or b_0 lying in $[\delta, 1/\delta]$, there exists a unique $r_0 \equiv r_0(a_0, f_0)$ or $r_0(b_0, f_0)$ such that $f'(r_0) = -1$. By the continuity of solutions to Equations (3.2) to (3.4) in the initial conditions, and by the fact (Lemma 3.3.3) that $f_0 = 0$ implies that $f \equiv 0$ on $\mathbb{R}^+$, we know that as f_0 approaches 0 while the other initial condition (a_0 or b_0) lies in a compact set, r_0 tends to ∞. □

From now on, we will confine the initial condition (a_0 or b_0) to lie in a compact set, and consider values of f_0 sufficiently close to 0 so that the corresponding r_0 as defined above is greater than 1.

Lemma 3.7.5. *Consider a cohomogeneity one gradient expanding Ricci soliton* (M, g) *with* (a, b, f) *satisfying equations Equations* (3.2) *to* (3.4) *and boundary conditions either Equations* (3.5) *and* (3.6) *or Equations* (3.7) *and* (3.8). *Suppose the initial condition* (a_0 *or* b_0) *lies in a compact set* $[\delta, 1/\delta]$ *for some* $0 < \delta < 1$. *Fix a value* $f_0^* > 0$ *so that for* $f_0 \in (-f_0^*, 0)$, *and for* a_0 *or* b_0 *in* $[\delta, 1/\delta]$, *the corresponding* r_0 *from Lemma 3.7.4 is greater than* 1. *Then, the following inequalities hold:*

$$0 < \alpha_1 \leqslant A(1) \leqslant \alpha_2 \tag{3.34}$$

$$0 < \beta_1 \leqslant B(1) \leqslant \beta_2 \tag{3.35}$$

$$0 < a_1 \leqslant a(1) \leqslant a_2 \tag{3.36}$$

$$0 < b_1 \leqslant b(1) \leqslant b_2 \tag{3.37}$$

$$0 \leqslant 3 + \frac{2}{b(1)^2} \leqslant C_0$$

for positive constants $a_1, b_1, a_2, b_2, \alpha_1, \beta_1, \alpha_2, \beta_2, C_0$ *whose values depend only on* f_0^* *and* δ.

Proof. As explained in Section 2, equations Equations (3.2) to (3.4) are continuous in the initial conditions, including when $f_0 = 0$. By Lemma 3.3.3, $f_0 = 0$ implies that $f \equiv 0$. In this case, replicating the proof of Lemma 3.3.1, we see that the inequalities $a', b' > 0$ on $(0, \infty)$ continue to hold. Thus, we have $A(1), B(1) > 0$ for each $f_0 \in [-f_0^*, 0]$, proving Equation (3.34) and Equation (3.35) by compactness of this interval. Similarly, we have $a(1), b(1) > 0$, for each $f_0 \in [-f_0^*, 0]$, which proves Equation (3.36) and Equation (3.37) again by compactness. The last inequality follows from (8.6). $\square$

Now, we proceed to analyze Equation (3.32) and Equation (3.33). On the interval $[1, r_0]$, we have the following:

$$1 - A^2 - 2AB \geqslant A' \geqslant 1 - A^2 - 2AB - A$$
$$1 + \frac{1}{b(1)^2} - 2B^2 - AB \geqslant B' \geqslant 1 - AB - 2B^2 - B$$

From this, we can see that the following inequalities hold:

$$C_0 - (A + 2B)^2 \geqslant (A + 2B)' \geqslant 3 - (A + 2B)^2 - (A + 2B) \tag{3.38}$$

Then, we can show that $(A + 2B)$ is bounded in the following manner:

Claim 3.7.6. *Consider the setup of Lemma 3.7.5. Then, there exist constants $c, C > 0$ (depending only on f_0^* and δ) such that the following bounds hold on $[1, r_0]$:*

$$c \leqslant (A + 2B) \leqslant C$$

Proof. **Step 1:** By Equation (3.34) and Equation (3.35), we have lower and upper bounds at $r = 1$ of the form

$$\alpha_1 + 2\beta_1 \leqslant (A + 2B)(1) \leqslant \alpha_2 + 2\beta_2$$

where the bounds depend only on f_0^* and δ.

Step 2: Now, consider the equalities corresponding to inequalities Equation (3.38):

$$u' + u^2 = C_0 \qquad\qquad v' + v^2 + v = 3$$
$$(A + 2B)' + (A + 2B)^2 \leqslant C_0 \qquad\qquad (A + 2B)' + (A + 2B)^2 + (A + 2B) \geqslant 3$$

where u and v have initial condition $u(1) = v(1) = (A + 2B)(1)$. From these two equations, we see that if $u(s) = (A+2B)(s)$ for any $s \in [1, r_0]$, then $u'(s) \geqslant (A+2B)'(s)$, implying that $u \geqslant (A + 2B)$ on $[1, r_0]$. By analyzing the equation for v as well, we have $v \leqslant (A + 2B) \leqslant u$ on $[1, r_0]$.

First, consider the equation for u. This IVP can be solved exactly and the solution u is asymptotic to $\sqrt{C_0}$. If $u(1) < \sqrt{C_0}$, then u increases and becomes asymptotic to $\sqrt{C_0}$, and if $u(1) > \sqrt{C_0}$, then u decreases to $\sqrt{C_0}$. The equation for v can be analyzed to show that v exhibits similar behavior (for a different asymptotic constant value v_∞). Both equations have solutions asymptotic to positive constants u_∞ and v_∞ respectively, and the constant u_∞ depends on C_0, which itself depends on f_0^* and δ. Thus, we have $v \geqslant \min\{v(1), v_\infty\}$ and $u \leqslant \max\{u(1), u_\infty\}$ on $[1, r_0]$.

Conclusion: On $[1, r_0]$, since we have

$$\min\{v(1), v_\infty\} \leqslant v \leqslant A + 2B \leqslant u \leqslant \max\{u(1), u_\infty\}$$

the bounds from Step 1 show that we can set $c = \min\{\alpha_1 + 2\beta_1, v_\infty\}$ and $C = \max\{\alpha_2 + 2\beta_2, u_\infty\}$, proving the lemma. □

The following lemma establishes lower bounds on A and B on $[1, r_0]$.

Lemma 3.7.7. *Consider the setup of the Lemma 3.7.5. Then, there exist constants $\alpha, \beta > 0$ (depending on f_0^* and δ) such that the bounds $A \geqslant \alpha$, $B \geqslant \beta$ hold on $[1, r_0]$.*

Proof. Consider the following inequalities (derived from Equation (3.32) and Equation (3.33)) on $[1, r_0]$:

$$A' \geqslant 1 - A(A + 2B + 1)$$
$$B' \geqslant 1 - B(A + 2B + 1)$$

Using the upper bound on $(A + 2B)$ from Claim 3.7.6, we have the inequality on $[1, r_0]$:

$$A' \geqslant 1 - (C + 1)A$$

The method of analysis is similar to that of Claim 3.7.6. Consider the associated ODE

$$w'(r) = 1 - (C + 1)w(r)$$

with $w(1) = A(1)$. We see that $A \geqslant w$ on $[1, r_0]$.

Solving the ODE, we see that

$$w(r) = \frac{(A(1)(C + 1) - 1)e^{(C+1)(1-r)} + 1}{C + 1}$$

so w is asymptotic to a constant $w_\infty = \frac{1}{C+1}$. Thus, we see that $A \geqslant \min\{A(1), w_\infty\}$. Similarly, we see that $B \geqslant \min\{B(1), w_\infty\}$. Note that w_∞ depends only on C from Claim 3.7.6 (which depends only on f_0^* and δ), while the bounds $A(1) \geqslant \alpha_1$ and $B(1) \geqslant \beta_1$ from Equation (3.34) and Equation (3.35), respectively, are also dependent only on f_0^* and δ. Thus, we can set $\alpha = \min\{\alpha_1, w_\infty\}$ and $\beta = \min\{\beta_1, w_\infty\}$, making the lemma true. □

Using Lemma 3.7.7, for any value of f_0, we can show that the slopes a' and b' grow exponentially on the corresponding interval $[1, r_0]$.

Lemma 3.7.8. *Suppose we are in the setup of Lemma 3.7.5. Then, on the interval* $[1, r_0]$*, we have the following bounds:*

$$a(r) \geqslant a_1 e^{\alpha(r-1)} \qquad a'(r) \geqslant a_1 \alpha e^{\alpha(r-1)}$$

$$b(r) \geqslant b_1 e^{\beta(r-1)} \qquad b'(r) \geqslant b_1 \beta e^{\beta(r-1)}$$

where α *and* β *are as in Claim 3.7.6 and* a_1, a_2, b_1, b_2 *are as in Equation* (3.36) *and Equation* (3.37)*.*

Proof. Consider the inequality $A > \alpha$ on $[1, r_0]$. This is equivalent to the inequality

$$\frac{a'(r)}{a(r)} - \alpha > 0$$

Integrating this inequality gives

$$a(r) \geqslant a(1)e^{\alpha(r-1)}$$

Now, since $a(1) \geqslant a_1$ from Equation (3.36), by applying the inequality $a'(r) \geqslant \alpha a(r)$, we get the result for a. The same procedure for b gives the corresponding result. $\square$

The importance of the previous lemma is in the fact that a and b tend to grow exponentially at least until the point r_0, where $f' = -1$. This shows that the soliton mimics the behavior of an Einstein manifold (corresponding to $f''(0) = 0$) in a region close to the tip. As f_0 gets closer to 0, r_0 tends to ∞, suggesting that the slopes a' and b' grow exponentially on a larger and larger interval.

From this, we will show that a'_∞ and b'_∞ approach ∞ as f_0 approaches 0 (with the other initial condition, a_0 or b_0, lying in $[\delta, 1/\delta]$, as specified earlier in this section).

Theorem 3.7.9. *Fix* $\delta \in (0, 1)$ *and choose* $f_0^* > 0$ *so that the corresponding* r_0 *from Lemma 8.4 is greater than* 1 *for all* $f_0 \in (-f_0^*, 0)$*.*

1. *Consider cohomogeneity one gradient expanding solitons with topology* $S^1 \times \mathbb{R}^3$*. Suppose* $a_0^i \in [\delta, 1/\delta]$ *and* $f_0^i \in (-f_0^*, 0)$ *such that* f_0^i *converges to* 0*. Let* $F(a_0^i, f_0^i) = (a'_{\infty,i}, b'_{\infty,i})$*. Then, we have*

$$a'_{\infty,i} \text{ and } b'_{\infty,i} \text{ are unbounded from below } i \to \infty.$$

2. *Consider cohomogeneity one gradient expanding solitons with topology* $S^2 \times \mathbb{R}^2$*. Suppose* $b_0^i \in [\delta, 1/\delta]$ *and* $f_0^i \in (-f_0^*, 0)$ *such that* f_0^i *converges to* 0*. Let* $F(a_0^i, f_0^i) = (a'_{\infty,i}, b'_{\infty,i})$*. Then, we have*

$$a'_{\infty,i} \text{ and } b'_{\infty,i} \text{ are unbounded from below } i \to \infty.$$

Proof. The proof is essentially identical in both cases. In either case, fix a value $f_0 \in (-f_0^*, 0)$ and, depending on the topology, pick a_0 or b_0 lying in $[\delta, 1/\delta]$. Suppose (a, b, f) is the solution to Equations (3.2) to (3.4) with these initial conditions (as well as the initial conditions required to ensure the correct topology)

Step 1: Consider the quantities $\overline{a} = \frac{a}{a(r_0)}$ and $\overline{b} = \frac{b}{b(r_0)}$, where r_0 is the unique point where $f'(r_0) = -1$ as defined earlier in this section. Then, the equations for these quantities become

$$\overline{a}'' = -2\frac{\overline{a}'\overline{b}'}{\overline{b}} + \overline{a}'f' + \overline{a} \tag{3.39}$$

$$\overline{b}'' = \frac{\frac{1}{b(r_0)^2} - (\overline{b}')^2}{\overline{b}} - \frac{\overline{a}'\overline{b}'}{\overline{a}} + \overline{b}'f' + \overline{b} \tag{3.40}$$

with initial conditions $\overline{a}(r_0) = \overline{b}(r_0) = 1$ and $\overline{a}'(r_0)$ and $\overline{b}'(r_0)$ which are bounded from below by Lemma 3.7.7, since

$$\overline{a}'(r_0) = \frac{a'(r_0)}{a(r_0)} = A(r_0) > \alpha \qquad \text{(and similarly for } \overline{b}\text{)}$$

and from above by Claim 8.6, since

$$\overline{a}'(r_0) = A(r_0) < (A + 2B)(r_0) < C \qquad \text{(and similarly for } \overline{b}\text{)}$$

where all constants involved depend only on f_0^* and δ.

Step 2: By Claim 3.7.6, we know that $0 \leqslant B \leqslant C$ on $[1, r_0]$, where C only depends on f_0^* and δ. By an argument almost identical to that in the proof of Lemma 3.5.1, we can show that $B \leqslant C'$ on $[r_0, \infty)$ for some constant C' which also only depends on f_0^* and δ. Additionally, by Lemma 6.3, we see that the constant C_1 in the bound $f'(r) \geqslant -(r + C_1)$ can also be chosen uniformly in $f_0 \in (-f_0^*, 0)$, since $C_1 = \sqrt{-3f_0}$

Step 3: Applying these strengthened inequalities to equation Equation (3.39), we see that

$$\overline{a}'' = \overline{a}'\left(f' - 2\frac{\overline{b}'}{\overline{b}}\right) + \overline{a} \geqslant -\overline{a}'\left(r + C_1 + 2C'\right) + \overline{a}$$

where the constants C_1 and C' depend only on f_0^* and δ. The initial conditions are $\overline{a}(r_0)$ and $\overline{a}'(r_0)$ which are bounded from above and below by Step 1.

Step 4: Thus, setting $\bar{C} = C_1 + 2C'$, we have the following inequality on $\bar{a}$ on $[r_0, \infty)$:

$$\bar{a}''(r) \geqslant -(r + \bar{C})\bar{a}'(r) + \bar{a}(r)$$

with initial conditions $\bar{a}(r_0) = 1$ and $\bar{a}'(r_0) = A(r_0) > \alpha$. Now, by Part 2 of Lemma 3.5.5, we see that $\bar{a}'(r) \geqslant C_1$ on $[r_0, \infty)$ for some constant C_1 depending uniformly on the initial conditions of $\bar{a}(r_0)$ and $\bar{C}$. In fact, by the proof of this Lemma, we may choose

$$C_1 = \frac{\min\left\{\dfrac{\bar{C}}{1+r_0}, \alpha\right\}}{2}$$

Thus, we see that $\overline{a}'(r) \geqslant C_1 \equiv C_1(r_0)$ on $[r_0, \infty)$, which implies that $a'(r) \geqslant a(r_0)C_1$ on $[r_0, \infty)$, which implies that $a'_\infty \geqslant a(r_0)C_1 \geqslant a_1 e^{\alpha(r_0-1)}C_1$ by Lemma 3.7.8. Thus, we have a lower bound on a'_∞ (depending only on f_0^* and δ) over all values of $f_0 \in (-f_0^*, 0)$ and over all values of the other initial condition in $[\delta, 1/\delta]$.

A similar argument, analyzing equation 8.9, provides the analogous lower bound on b'_∞. Note that the only difference is from the term $\dfrac{1}{b(r_0)^2\overline{b}}$; however, this term is bounded from above by a constant C_b. We can apply the same procedure as for a to the quantity $b + C_b$, as we did in the proof of Lemma 3.5.13.

Step 5: Now, we prove the final step of the theorem in the $S^1 \times \mathbb{R}^3$ case; the other case is nearly identical. Suppose (a_0^i, f_0^i) is a sequence of initial conditions satisfying the hypotheses. Then, by the results of Step 4, we know that $a'_{\infty,i} \geqslant a_1 e^{\alpha(r_{0,i}-1)}C_1(r_{0,i})$, where $r_{0,i}$ is the unique value satisfying $f'(r_{0,i}) = -1$. Using Lemma 8.4, we know that $r_{0,i}$ approaches ∞. Thus, for sufficiently large i, $C_1(r_{0,i}) = \frac{\bar{C}}{1+r_{0,i}}$. Thus, we have

$$a'_{\infty,i} \geqslant \frac{a_1 \bar{C} e^{\alpha(r_{0,i}-1)}}{1+r_{0,i}}$$

Thus, we know that $r_{0,i}$ approaches ∞, $a'_{\infty,i}$ becomes unbounded as well. A similar argument works for $b'_{\infty,i}$, which concludes the proof of the theorem. □

3.8 Calculation of Expander Degree

In this section, we will study the relation between the initial conditions (a_0, f_0) or (b_0, f_0) and the slopes of the cone (a'_∞, b'_∞). Recall from Definition 3.6.9 that $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ was defined as

$$\begin{aligned} F(a_0, -f_0) &= (a'_\infty, b'_\infty) \qquad &\text{(in the } S^1 \times \mathbb{R}^3 \text{ case)} \\ F(b_0, -f_0) &= (a'_\infty, b'_\infty) \qquad &\text{(in the } S^2 \times \mathbb{R}^2 \text{ case)} \end{aligned}$$

where $(a'_\infty, b'_\infty) = \lim_{r\to\infty}(a'(r), b'(r))$, for the warping functions a and b in the soliton metric $g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2}$ and the soliton potential f satisfying Equations (3.2) to (3.4), with initial conditions Equations (3.5) and (3.6) for $S^1 \times \mathbb{R}^3$ and Equations (3.7) and (3.8) for $S^2 \times \mathbb{R}^2$. We will show that F is proper in the case of each topology.

Theorem 3.8.1. *1. The map F is proper in the case of $S^1 \times \mathbb{R}^3$ topology.*

2. The map F is proper in the case of $S^2 \times \mathbb{R}^2$ topology.

Proof. 1. In the $S^1\times\mathbb{R}^3$ case, consider a sequence $(a'_{\infty,i}, b'_{\infty,i})$ in the range of F that converges to the pair of positive real numbers (a'_∞, b'_∞). Suppose that $(a'_{\infty,i}, b'_{\infty,i}) = F(a_0^i, -f_0^i)$ for initial conditions a_0^i and f_0^i. Suppose that the corresponding expanding solitons $(M, g_i = dr^2 + a_i(r)^2 g_{S^1} + b_i(r)^2 g_{S^2}, \nabla f_i)$ are asymptotic to the cones γ_i. Recall from Definition 3.5.15 and Theorem 3.5.14 that this implies that for a point p with $r(p) = 0$ and any sequence $\lambda_i \to 0$, the sequence of pointed manifolds $(M, \lambda_i^2 g_i, p)$ converges in the Gromov-Hausdorff sense to a cone over the link $(S^2 \times S^1, h)$, where h admits an isometric action of $\mathrm{SO}(3) \times \mathrm{SO}(2)$. As a consequence, the functions a_i and b_i are asymptotically linear with $(a'_{\infty,i}, b'_{\infty,i}) = \lim_{r\to\infty}(a'_i(r), b'_i(r))$.

Fix a small $\epsilon > 0$. As $(a'_{\infty,i}, b'_{\infty,i})$ converges, we have uniform lower bounds $a'_\infty, b'_\infty \geqslant c$ for some $c > 0$, and $b'_{\infty,i} \leqslant C$ for some $C > 0$. Thus, by Lemma 3.6.3, we know that there is a uniform $S_0 > 0$ so that each $(M, g_i, \nabla f_i)$ is ϵ-conical at distance $d_i(S_0)$ from the tip, providing the following inequality from Equation (3.28):

$$a'_{\infty,i} S_0 (1-\epsilon)^{1/2} \leqslant a_i(d_i(S_0)) \leqslant a'_{\infty,i} S_0 (1+\epsilon)^{1/2}$$

By Part 2 of Lemma 3.6.8, we know that the sequence $d_i(S_0)$ is bounded from above by an $r_0 > 0$. Additionally, by the monotonicity of a from Lemma 3.3.1, we know that $a_i(r_0) \geqslant a_i(d_i(S_0)) \geqslant a_i(0)$.

From equation Equation (3.19), we know that $a'_i \leqslant a_i$ on $\mathbb{R}^+$. Integrating this inequality, we see that $a_i(r) \leqslant a_0^i e^r$, which gives us the inequality $a_i(r_0) \leqslant a_0^i e^{r_0}$. Combining this with the previous inequalities, we get the following for all i:

$$cS_0(1-\epsilon)^{1/2} \leqslant a'_{\infty,i} S_0 (1-\epsilon)^{1/2} \leqslant a_i(d_i(S_0)) \leqslant a_i(r_0) \leqslant a_0^i e^{r_0}$$

which shows that $a_0^i \geqslant cS_0(1-\epsilon)^{1/2} e^{-r_0}$, providing a lower bound on a_0^i.

For the upper bound on a_0^i, we know that for all i, we have

$$a_0^i \leqslant a_i(d_i(S_0)) \leqslant a'_{\infty,i} S_0 (1+\epsilon)^{1/2} \leqslant CS_0(1+\epsilon)^{1/2}$$

where the last inequality follows as convergent sequences are bounded. Thus, a_0^i is bounded both from above and below, so it lies in an interval of the form $[\delta, 1/\delta]$ for $\delta < 1$.

By Part 1 of Lemma 3.6.8, we have a uniform scalar curvature upper bound on $(M, g_i, \nabla f_i)$, and by equation Equation (3.11), we know that $R_i(0) = -4 - 3f_0^i$, so we must have that f_0^i is bounded from below. By the continuity of F (Theorem 3.6.10) and Theorem 3.7.9, and the fact that a_0 lies in a compact set by the previous paragraph, we know that since $a'_{\infty,i}$ and $b'_{\infty,i}$ are bounded from above, that f_0^i must be bounded from above by a constant $C < 0$. Thus, in this case, we know that the initial conditions (a_0^i, f_0^i) lie in a compact set, and thus a subsequence of the initial conditions converges. This shows that F is proper in this case.

2. In the $S^2\times\mathbb{R}^2$ case, suppose that $(a'_{\infty,i}, b'_{\infty,i}) = F(b_0^i, -f_0^i)$ for initial conditions b_0^i and f_0^i, with $(a'_{\infty,i}, b'_{\infty,i})$ converging to $(a'_\infty, b'_\infty) \in \mathbb{R}^+ \times \mathbb{R}^+$, and denote the corresponding expanding

solitons by $(M, g_i = dr^2 + a_i(r)^2 g_{S^1} + b_i(r)^2 g_{S^2}, \nabla f_i)$. First, the proof of the lower bound on f_0^i from the $S^1 \times \mathbb{R}^3$ case carries over (with the only change being that $R_i(0) = -4 - 2f_0^i$ in the $S^2 \times \mathbb{R}^2$ case). Additionally, as in the proof of the $S^1 \times \mathbb{R}^3$ case, we can prove the upper bound on b_0^i using uniform ϵ-conicality.

Now, we claim that b_0^i is bounded from below. Assume that this is false, and up to a subsequence, $b_0^i \equiv \lambda_i \to 0$. Then, consider the new quantities

$$\tilde{a}_i(r) = \frac{1}{\lambda_i} a_i(\lambda_i r) \qquad \tilde{b}_i(r) = \frac{1}{\lambda_i} b_i(\lambda_i r) \qquad \tilde{f}_i(r) = f_i(\lambda_i r)$$

which satisfy the following equations on $[0, \infty)$

$$\tilde{f}_i{}'' = \frac{\tilde{a}_i''}{\tilde{a}_i} + 2\frac{\tilde{b}_i''}{\tilde{b}_i} - \lambda_i^2$$

$$\tilde{a}_i'' = -2\frac{\tilde{a}_i'\tilde{b}_i'}{\tilde{b}_i} + \tilde{a}_i'\tilde{f}_i' + \lambda_i^2 \tilde{a}_i$$

$$\tilde{b}_i'' = \frac{1 - (\tilde{b}_i')^2}{\tilde{b}_i} - \frac{\tilde{a}_i'\tilde{b}_i'}{\tilde{a}_i} + \tilde{b}_i'\tilde{f}_i' + \lambda_i^2 \tilde{b}_i$$

with the initial conditions

$$\begin{array}{lll}
\tilde{a}_i(0) = 0 & \tilde{a}_i'(0) = 1 & \\
\tilde{b}_i(0) = 1 & \tilde{b}_i'(0) = 0 & \\
\tilde{f}_i(0) = 0 & \tilde{f}_i'(0) = 0 & \tilde{f}_i''(0) = (\lambda_i)^2 f_0^i
\end{array}$$

Then, in the limit as $i \to \infty$, we have the equations

$$f'' = \frac{a''}{a} + 2\frac{b''}{b}$$

$$a'' = -2\frac{a'b'}{b} + a'f'$$

$$b'' = \frac{1 - (b')^2}{b} - \frac{a'b'}{a} + b'f'$$

with initial conditions

$$\begin{aligned} a(0) &= 0 & a'(0) &= 1 & & \\ b(0) &= 1 & b'(0) &= 0 & & \\ f(0) &= 0 & f'(0) &= 0 & f''(0) &= 0 \end{aligned}$$

since f_0^i is negative and bounded from below. By an argument nearly identical to Part 2 of Lemma 4.2 of [A17], we see that this implies that $f \equiv 0$ on $\mathbb{R}^+$, thus implying that $g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2}$ is a Ricci-flat metric on $\mathbb{R}^2 \times S^2$. From Chapter 2 of [Pet], we know that this metric is asymptotic to the flat metric on $S^1 \times \mathbb{R}^3$ and that $a(r) \approx C$ and $b(r) \approx r$ for large r, where $C > 0$ is a constant.

Then, for any large $L > 0$, we can choose $r_0 > 0$ so that the following hold:

$$\frac{b(r_0)}{a(r_0)} \geqslant L \qquad \frac{d}{dr}\left(\frac{b(r)}{a(r)}\right)(r_0) > 0$$

Then, for sufficiently large i (depending on δ), we have

$$\frac{b_i(r_0)}{a_i(r_0)} \geqslant L - 1 \qquad \frac{d}{dr}\left(\frac{b_i(r)}{a_i(r)}\right)(r_0) > 0$$

This implies that for $P_i = \frac{b_i}{a_i}$ as defined in Section 3, we have that $P_i(\lambda_i r_0) \geqslant L - 1$ and $P_i'(\lambda_i r_0) \geqslant 0$. By Lemma 3.3.4, we know that $P_i(r) \geqslant L - 1$ for all $r \geqslant \lambda_i r_0$.

We also know that $\lim_{r\to\infty} P_i(r) = b'_{\infty,i}/a'_{\infty,i} < C$ for some constant $C > 0$ independent of i, by the convergence of $(a'_{\infty,i}, b'_{\infty,i})$ by hypothesis. However, this contradicts the conclusion of the previous paragraph by choosing L to be arbitrarily large. Thus, our assumption was false, and it must be the case that b_0^i is bounded from below.

Then, by the continuity of F and the fact that b_0 lies in a compact set and Theorem 3.7.9, we get that f_0^i must be bounded from above as well, just as in the $S^1 \times \mathbb{R}^3$ case.

Thus, we have shown that the initial conditions lie within a compact set in this case as well, so a subsequence of the initial conditions must converge, implying that the map F is proper in the $S^2 \times \mathbb{R}^2$ case. □

As a consequence of the previous theorem, we have the following:

Corollary 3.8.2. *The degrees of the maps F in the $S^1 \times \mathbb{R}^3$ case and in the $S^2 \times \mathbb{R}^2$ case are well defined.*

The importance of the properness of F is in concluding the corollary above; in [BC23], the expander degree of an orbifold was defined on the space of gradient expanding solitons on the interior of the orbifold with positive scalar curvature, albeit in a more general and non-symmetric setting. In our cohomogeneity one setting, we do not need this assumption.

Analogously, we can define a cohomogeneity one version of this quantity.

Definition 3.8.3. *The* ***cohomogeneity one expander degree****, denoted* $\deg_{\exp}^{\mathrm{sym}}$*, of the orbifolds* $S^1 \times \mathbb{D}^3$ *and* $S^2 \times \mathbb{D}^2$ *are defined as the topological degree of the corresponding maps* F *in the cases of the topologies* $S^1 \times \mathbb{R}^3$ *and* $S^2 \times \mathbb{R}^2$*, respectively.*

Now, we can calculate the cohomogeneity one expander degree in the case of each topology.

First, we consider the $S^1 \times \mathbb{R}^3$ case. We will calculate the limit of b'_∞ as $f_0 \to -\infty$.

Lemma 3.8.4. *In the case of* $S^1 \times \mathbb{R}^3$ *topology, consider a sequence of initial conditions* (a_0^i, f_0^i)*. Let* $(M, g_i, \nabla f_i)$ *be the corresponding cohomogeneity one gradient expanding solitons asymptotic to cones* $\gamma_i = dr^2 + r^2 h_i$ *over the link* $(S^2 \times S^1, h_i = (a'_{\infty,i})^2 g_{S^1} + (b'_{\infty,i})^2 g_{S^2})$*. Suppose* $f_0^i \to -\infty$*, and set*

$$(a'_{\infty,i}, b'_{\infty,i}) = F(a_0^i, f_0^i)$$

Then, $b'_{\infty,i}$ *converges to* 0*.*

Proof. Suppose the conclusion of the lemma is not true. Then, there would exist a sequence of expanding solitons $(M, g_i, \nabla f_i)$ with initial conditions (a_0^i, f_0^i) and asymptotic cone metrics γ_i such that $f_0^i \to -\infty$ but $b'_{\infty,i} \geqslant C$ for some $C > 0$.

Denote $\mathrm{inj}_{h_i} = \alpha_i$. Note that since $b'_{\infty,i}$ is bounded below, the sequence $\{\alpha_i\}$ is bounded from below iff $a'_{\infty,i}$ is bounded below. Suppose that $\alpha_i \to 0$. Then, we can consider the new sequence of expanding solitons $(M, \tilde{g}_i, \nabla f_i)$ with initial conditions $(a_0^i/\alpha_i, f_0^i)$. Using Lemma 3.7.1, we see that $(\tilde{a}_{\infty,i}, \tilde{b}_{\infty,i}) := F(a_0^i/\alpha_i, f_0^i) = (a'_{\infty,i}/\alpha_i, b'_{\infty,i})$. These solitons are respectively asymptotic to the cone metrics $\tilde{\gamma}_i = dr^2 + r^2 \tilde{h}_i$, in which the injectivity radii of $\tilde{h}_i$ are uniformly bounded from below. Thus, we have $\min\{\tilde{a}_{\infty,i}, \tilde{b}_{\infty,i}\} \geqslant c$, for some constant $c > 0$.

Then, by Part 1 of Lemma 3.6.8, the sequence $(M, \tilde{g}_i, \nabla f_i)$ would have uniformly bounded scalar curvature. However, we calculated in equation Equation (3.11) that $\tilde{R}_i(0) = -3f_0^i - 4$, which is clearly unbounded as $f_0^i \to -\infty$, which is a contradiction. Thus, we must have that $b'_{\infty,i} \to 0$ as $f_0^i \to -\infty$. □

Theorem 3.8.5. $\deg_{\exp}^{\mathrm{sym}}(S^1 \times \mathbb{D}^3) = 1$ *(up to sign)*

Proof. First, by Lemma 3.7.1, we see that changing a_0 simply scales a'_∞ and leaves b'_∞ invariant. Thus, we can consider the maps

$$F_1 : \mathbb{R}^+ \to \mathbb{R}^+$$

defined in the following way: suppose $F(1, -f_0) = (a'_\infty, b'_\infty)$. Then, set $F_1(-f_0) := p_2(F(1, -f_0)) = b'_\infty$, where p_2 is the projection onto the second component. We have the following lemma:

Lemma 3.8.6. *The degree of* F_1 *coincides with the degree of* F*.*

Proof of Lemma 3.8.6. Since a_0 merely scales a'_∞ and does not affect b'_∞, we can write $F(a_0, -f_0) \equiv (a_0 F_0(-f_0), F_1(-f_0))$ where F_1 is as above and $F_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. By Lemma 3.8.4, we have that $F_1(-f_0) \to 0$ as $-f_0 \to \infty$, and by Theorem 3.7.9, we know that $F_1(-f_0) \to \infty$ as $-f_0 \to 0$. Thus, F_1 is a proper map.

Now, consider the map $H : [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ given by $H(t,x,y) = ((1-t)x + txF_0(y), F_1(y))$. We will show that H is a proper homotopy between F and the map $(x,y) \mapsto (x, F_1(y))$. Suppose that (t_n, x_n, y_n) is a sequence in $[0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$ such that $H(t_n, x_n, y_n)$ converges. Then, since F_1 is a proper map and $F_1(y_n)$ converges, we may assume that y_n is contained in a compact set $[\delta, \frac{1}{\delta}]$ of $\mathbb{R}^+$ for some $\delta \in (0,1]$. Then, as F_0 is continuous, we see that $\{F_0(y_n)\} \subseteq F_0([\delta, \frac{1}{\delta}]) \subseteq \mathbb{R}^+$, so $F_0(y_n)$ is bounded from above and below independently of n. From this, it is easy to see that $(1-t_n) + t_n F_0(y_n)$ lies in a compact subset $[c, C]$ of $\mathbb{R}^+$. Then, since $(1-t_n)x_n + t_n x_n F_0(y_n)$ converges, we see that x_n also lies in a compact subset of $\mathbb{R}^+$. Finally, t_n lies in a compact set by compactness of $[0,1]$. This shows that H is a proper homotopy.

The map $(x,y) \mapsto (x, F_1(y))$ is a product map, so its degree is $\deg(x \mapsto x)\deg(F_1) = \deg(F_1)$. As proper homotopies preserve the degrees of continuous proper maps, we have that $\deg(F) = \deg(F_1)$. $\square$

Proof of Theorem 3.8.5 cont. By Lemma 3.8.6, it is enough to compute $\deg(F_1)$. Consider the map $H : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$, given by $H(s,x) = (1-s)F_1(x) + \frac{s}{x}$. It is straightforward to check that H is a proper homotopy between F_1 and the map $x \to \frac{1}{x}$ on $\mathbb{R}^+$. Thus, as proper homotopies preserve degree, it is clear that $\deg(F_1) = \deg(x \to \frac{1}{x}) = -1$, which is the same as 1 up to sign. $\square$

Remark: Similar methods are employed in [NW24] in their construction of expanders on $\mathbb{R}^3 \times S^1$. In the notation of that paper, a proof that σ_2 is continuous along with a properness result would constitute a proof of the previous theorem.

Next, we consider the $S^2 \times \mathbb{R}^2$ case.

Theorem 3.8.7. $\deg^{\mathrm{sym}}_{\mathrm{exp}}(S^2 \times \mathbb{D}^2) = 0$

Proof. We will show that F is not surjective; this is sufficient to prove that the degree is 0. Consider the set S defined as

$$S = \left\{(b_0, -f_0) \subseteq \mathbb{R}^+ \times \mathbb{R}^+ \mid b'_\infty = 1, \text{ where } (a'_\infty, b'_\infty) = F(b_0, -f_0)\right\}.$$

We claim that over all initial conditions in S, the value of a'_∞ is bounded from above. Suppose that this is not true. Then, there would exist a sequence of initial conditions (b_0^i, f_0^i) with $F(b_0^i, -f_0^i) = (a'_{\infty,i}, b'_{\infty,i})$ satisfying $b'_{\infty,i} = 1$ and $a'_{\infty,i} \to \infty$. Then, from equation Equation (3.3), we have the inequality $a''_i \leqslant a_i$, with the initial conditions $a_i(0) = 0, a'_i(0) = 1$. Integrating this inequality (using equation Equation (3.19) and the monotonicity of a), we see that $a_i(r) \leqslant \sinh(r)$ for all i.

Now, we clearly have bounds of the form

$$\min\{a'_{\infty,i}, b'_{\infty,i}\} \geqslant c_1 \qquad b'_{\infty,i} \leqslant c_2$$

for some constants $c_1, c_2 > 0$. Thus, from Lemma 3.6.3, we know that there exists an $S_0 > 0$ so that each soliton (M, g_i) is uniformly ϵ-conical at a distance $\{d_i(S_0)\}$ from the tip. Additionally, by Part 2 of Lemma 3.6.8, the sequence $d_i(S_0)$ is bounded above by a positive r_0. By inequality Equation (3.28) in the statement of Lemma 3.6.3 and the monotonicity of a in Lemma 3.3.1, this implies that

$$a'_{\infty,i} S_0 (1-\epsilon)^{1/2} \leqslant a_i(d_i(S_0)) \leqslant a_i(r_0)$$

Then, as $i \to \infty$, we have that $a'_{\infty,i}$ becomes unbounded from above by hypothesis, implying the same conclusion for $a_i(r_0)$ as well. However, the bound $a_i(r_0) \leqslant \sinh(r_0)$ indicates that $a_i(r_0)$ is bounded above, which is a contradiction. Thus, no such sequence of solitons (M, g_i) can exist, which implies that the value of a'_∞ is bounded over all solitons in S. Thus, there exist pairs (a'_∞, b'_∞) which are not in the image of F, so it is not surjective, and thus has degree 0. □

3.9 Appendix A: Derivation of Soliton Equations

Consider a metric g on a 4-manifold M of the form

$$g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2}$$

as in Section 2.

Choose a local orthonormal frame $e^1 = dr$, $e^2 = ad\theta$, and $e^i = b\hat{e}^i$ where the $\hat{e}^i$ form an orthonormal basis for S^2 for $i = 3, 4$. Denote the dual vector fields by E_i. In this frame, one can compute the nonzero components of the curvature to be:

$$\mathrm{Rm}_{1221} = -\frac{a''}{a} \qquad \mathrm{Rm}_{1331} = -\frac{b''}{b} \qquad \mathrm{Rm}_{1441} = -\frac{b''}{b}$$
$$\mathrm{Rm}_{2332} = -\frac{a'b'}{ab} \qquad \mathrm{Rm}_{2442} = -\frac{a'b'}{ab} \qquad \mathrm{Rm}_{3443} = \frac{1-(b')^2}{b^2}$$

Thus, the nonzero components of the Ricci tensor are

$$\mathrm{Ric}_{11} = -\frac{a''}{a} - 2\frac{b''}{b}$$
$$\mathrm{Ric}_{22} = -\frac{a''}{a} - 2\frac{a'b'}{ab}$$
$$\mathrm{Ric}_{33} = -\frac{b''}{b} - \frac{a'b'}{ab} + \frac{1-(b')^2}{b^2}$$

Consider a smooth function $f : M \to \mathbb{R}$ which is constant along the S^1 and S^2 directions – in other words, f depends only on r. The nonzero components of the Hessian $\nabla^2 f$ are

$$\nabla^2 f(E_1, E_1) = f''$$
$$\nabla^2 f(E_2, E_2) = \frac{a'f'}{a}$$
$$\nabla^2 f(E_3, E_3) = \nabla^2 f(E_4, E_4) = \frac{b'f'}{b}$$

Now, suppose that (M, g) is an expanding Ricci soliton with soliton potential f satisfying the equation

$$\mathrm{Ric}_g + \nabla^2 f + g = 0$$

In the frame chosen above, the soliton equations take the form

$$-\frac{a''}{a} - 2\frac{b''}{b} + f'' + 1 = 0$$
$$-\frac{a''}{a} - 2\frac{a'b'}{ab} + \frac{a'f'}{a} + 1 = 0$$
$$-\frac{b''}{b} - \frac{a'b'}{ab} + \frac{1-(b')^2}{b^2} + \frac{b'f'}{b} + 1 = 0$$

Rearranging the three equations above gives us the soliton equations Equations (3.2) to (3.4)

From the above, we also notice a relation between Δf and f'' as follows

$$\Delta f = f'' + \left(\frac{a'}{a} + 2\frac{b'}{b}\right) f' \tag{3.41}$$

3.10 Appendix B: Existence of Local Solutions

It is not immediately clear why equations Equations (3.2) to (3.4) have a unique solution given a value of $f''(0) \leqslant 0$. The following theorem, proven in [A17] explains why this is the case:

Theorem 3.11.1. *Let $n \in \mathbb{N}$, $c \in \mathbb{R}$ and U an open subset of $\mathbb{R}^n$ containing the origin. Let*

$$P : U \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n \qquad (u, r, \lambda) \to P(u, r, \lambda)$$

be a vector valued analytic function around $(\vec{0}, 0, c)$ such that $P(0, 0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$. If there is an open interval I containing c such that for all $\lambda \in I$, the matrix $\frac{\partial P}{\partial u}(\vec{0}, 0, \lambda)$ has no positive integer eigenvalues and

$$sup_{\lambda \in I, m \in \mathbb{N}} \left\| \left(mI_n - \frac{\partial P}{\partial u} \right)^{-1} \right\| = B < \infty.$$

then there exists an $\epsilon > 0$ and a one-parameter family of analytic vector-valued functions $u(\cdot, \lambda) : (-\epsilon, \epsilon) \to \mathbb{R}^n$ solving the ODE system

$$r \frac{du(r, \lambda)}{dr} = P(u(r, \lambda), r, \lambda) \tag{3.42}$$

$$u(0, \lambda) = 0$$

for $\lambda \in (c - \epsilon, c + \epsilon)$. Furthermore, u depends analytically on λ.

The proof, given in [A17], involves the construction of a formal power series for u which satisfies the system. It is shown that the series has a positive radius of convergence which establishes the existence of a local solution. Following Appleton's methods, we will transform the soliton equations into a form suitable to apply this theorem. The following is very similar to the proof of Theorem 2.1 in [A17].

First, we consider the $S^2 \times \mathbb{R}^2$ case. Let s denote the independent variable of the soliton equations. Note that $a'(0) = 1 \neq 0$ and $a' > 0$ for $s \in (0, \infty)$, so a can be chosen as the independent variable of the soliton equations under the coordinate change corresponding to

$$g = \frac{da^2}{h(a^2)} + g_{a,b(a)}$$

Setting $r = a^2$, we see as in the proof of Theorem 2.1 in [A17] that

$$\frac{dr}{ds} = 2\sqrt{rh(r)}$$

Thus, in this case, Equations (3.2) to (3.4) are transformed into the following (where we use $\dot{f}$ to denote $\frac{\partial f}{\partial r}$, etc.)

$$\ddot{f} = \frac{1}{4r}\frac{\dot{h}}{h} + 2\frac{\ddot{b}}{b} + \frac{1}{r}\frac{\dot{b}}{b} + \frac{\dot{b}\dot{h}}{bh} - \frac{1}{4rh} - \frac{1}{2r}\dot{f} - \frac{1}{2}\frac{\dot{h}}{h}\dot{f} \tag{3.43}$$

$$\dot{h} = -4h\frac{\dot{b}}{b} + 2h\dot{f} + 1 \tag{3.44}$$

$$\ddot{b} = \frac{1}{4rhb} - \frac{\dot{b}}{r} - \frac{1}{2}\frac{\dot{h}}{h}\dot{b} - \frac{(\dot{b})^2}{b} + \dot{f}\dot{b} + \frac{b}{4rh} \tag{3.45}$$

with boundary conditions

$$
\begin{aligned}
b(0) &= b_0 \\
\dot{b}(0) &= \frac{1}{4}\left(b_0 + \frac{1}{b_0}\right) \\
h(0) &= 1 \\
f(0) &= 0 \\
\dot{f}(0) &= \frac{f''(0)}{2} \equiv c
\end{aligned}
$$

The boundary condition $\dot{f}(0)$ was derived by applying l'Hôpital's rule to the quantity $\dot{f}(r) = \frac{f'(r)}{2a(r)a'(r)}$ and noting that $a'(0) = 1$. $\dot{b}(0)$ was derived similarly, noting that $b''(0) = \frac{1}{2}(b_0 + \frac{1}{b_0})$. Note that $\dot{f}(0)$ can take on any real number value.

As in [A17], we reduce Equations (3.43) to (3.45) to a system of first-order ODEs. As f does not appear in the equations, the system can be considered first-order in $\dot{f}$. Setting $F = \dot{f}$ and $B = \dot{b}$, we can rewrite the equations as a first-order system in (F, h, b, B) as:

$$
\begin{aligned}
r\dot{F} &= \frac{1 + b^2 - 4B^2hr - b^2F(1 + 2Fh)r + 4bBh(-1 + 2Fr)}{2b^2h} \\
r\dot{h} &= -4hr\frac{B}{b} + 2hrF + r \\
r\dot{b} &= Br \\
r\dot{B} &= \frac{1}{4hb} - B + \frac{rB^2}{b} - \frac{rB}{2h} + \frac{b}{4h}
\end{aligned}
$$

Defining $u(r,c) \equiv (u_1(r,c), u_2(r,c), u_3(r,c), u_4(r,c)) = (F(r)-c, h(r)-h(0), b(r)-b(0), B(r)-B(0))$, we have an ODE system of the following form with c as a real parameter:

$$
\begin{aligned}
r\frac{du_i}{dr} &= P_i(u, r, c) \\
u_i(0, c) &= 0 \quad \text{for } i = 1, 2, 3, 4,
\end{aligned}
\tag{3.46}
$$

where P is an analytic function in the neighborhood of the point $(\vec{0}, 0, c)$ in $\mathbb{C}^6$ and $P(\vec{0}, 0, c) = 0$. We compute $\frac{\partial P_i}{\partial u_j}$ at $(\vec{0}, 0, c)$ and obtain

$$
\begin{bmatrix}
0 & -\frac{1+b_0^2}{2b_0^2} & \frac{-1+b_0^2}{2b_0^3} & -\frac{2}{b_0} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -\frac{1}{4}(b_0 + \frac{1}{b_0}) & \frac{1}{4}(1 - \frac{1}{b_0^2}) & -1
\end{bmatrix}.
$$

To apply the theorem, we calculate

$$\det\left(mI - \frac{\partial P}{\partial u}\right) = m^3(m+1),$$

and this matrix has no positive integer roots, so its inverse exists for all $m \in \mathbb{N}$. We can check that there exists $B \in \mathbb{R}$ such that

$$\left(mI - \frac{\partial P}{\partial u}\right)^{-1} < B$$

for all $m \in \mathbb{N}$. Thus, by Theorem 3.11.1, there exists a local solution to the system with the given boundary conditions. This proves the local existence in the case of boundary conditions corresponding to $S^2 \times \mathbb{R}^2$.

For the $S^1 \times \mathbb{R}^3$ case, we cannot use a similar procedure, as considering b as the independent variable of the soliton equations leads to a system of equations in which the differential equation for the corresponding quantity $r\dot{F}$ retains a singularity at $r = 0$. Instead, note that any SO(3) $\times$ SO(2)-symmetric solitons on $S^1 \times \mathbb{R}^3$ are cohomogeneity one with a singular orbit at $r = 0$, falling into the framework of [NW24] in which expanding Ricci solitons of warped product type generalizing the case of $S^1 \times \mathbb{R}^3$ topology are constructed, with local existence of following from results in [Buz11]. Although the soliton equations in [Buz11] are written with respect to different quantities, the results imply the required local existence and uniqueness to Equations (3.2) to (3.4) with initial conditions Equations (3.5) and (3.6), as is also mentioned in Section 1 of [NW24].

Appendix C: Smooth Cheeger-Gromov Convergence

In this section, we prove certain facts about Cheeger-Gromov convergence of cohomogeneity one solitons that have been used extensively in this paper. The results have parallels to Lemma 3.5 of [BHZ22]; in our setup, we have 2 warping functions instead of one, which introduces slight differences.

Lemma 3.3.1. *Suppose (M, g, p_i) is a sequence of pointed Riemannian manifolds with topology either $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$, where the metrics are of the warped product form*

$$g_i = dr^2 + a_i(r)^2 g_{S^1} + b_i(r)^2 g_{S^2}$$

with $r(p_i) = 0$ for each i where the functions a_i and b_i have domain $[-L_i, \infty)$, where $L_i \to \infty$. Suppose the monotonicity bounds $a_i', b_i' \geqslant 0$ hold as well. Consider bounds of the following form for all i:

$$|\nabla^k \mathrm{Rm}_{g_i}| \leqslant C_k(D) \text{ on the interval } [-D, D] \text{ for all } i, \text{ for any } D > 0,\ k \geqslant 0$$

$$\alpha_1 \leqslant a_i(0) \leqslant \alpha_2 \qquad\qquad \beta_1 \leqslant b_i(0) \leqslant \beta_2$$

1. *Suppose we have the above bounds on g_i. Then, (M, g_i, p_i) converges in the Cheeger-Gromov sense to a smooth Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ with topology $\mathbb{R}\times S^2\times S^1$ and a cohomogeneity one metric g_∞.*

2. *Suppose we have the curvature bounds, but that either $a_i(0)$ or $b_i(0)$ or both approach infinity. Then (M, g_i, p_i) converges in the Cheeger-Gromov sense to a smooth Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ with topology $\mathbb{R}\times S^2\times\mathbb{R}^1$ or $\mathbb{R}\times\mathbb{R}^2\times S^1$ or $\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^1$ respectively, and a warped product metric g_∞ which is invariant under the appropriate isometry group (depending on the topology).*

Proof. 1. The sectional curvature bounds give us the following inequalities for all i:

$$\left|\frac{a_i''}{a_i}\right| \leqslant C(D) \qquad \left|\frac{b_i''}{b_i}\right| \leqslant C(D) \qquad \left|\frac{a_i'b_i'}{a_ib_i}\right| \leqslant C(D) \qquad \left|\frac{1-(b_i')^2}{b_i^2}\right| \leqslant C(D) \tag{3.47}$$

Step 1: We derive bounds on b. This step is similar to Step 1 in Lemma 3.5 of [BHZ22].

Consider the fourth equation of Equation (3.47). On the interval $[-(D+1), D+1]$, we have that $(b_i')^2 \geqslant 1 - b_i^2C(D+1)$. From this, we claim that $b_i(-D) \geqslant \min\{\frac{1}{2C(D+1)^{1/2}}, \frac{1}{2}\}$. If this were false, then we would have the inequality $b_i'(-D) \geqslant \frac{\sqrt{3}}{2}$. Then, we know that for $r \in -[(D+1), -D]$,

$$b_i'(r)^2 \geqslant 1 - C(D+1)b_i(r)^2 \geqslant 1 - C(D+1)b_i(-D)^2 \geqslant \tfrac{3}{4}$$

where we used the hypothesis of the monotonicity of b_i. Thus, $b_i'(r) \geqslant \frac{\sqrt{3}}{2}$ on $[-(D+1), -D]$. But then, $b_i(-(D+1))$ would become negative since $b_i(-D)$ is also bounded from above by $\frac{1}{2}$. Thus, we have a contradiction, so we must have the bound $b_i(-D) \geqslant \min\{\frac{1}{2C(D+1)^{1/2}}, \frac{1}{2}\}$. By the monotonicity of b_i, this bound holds on $[-D, D]$. Thus, we have

$$C(D) \leqslant b_i(r)$$

on $[-D, D]$ for all i.

Then, again by the fourth equation of Equation (3.47), we know that on $[-D, D]$

$$0 \leqslant \frac{(b_i')^2}{b_i^2} \leqslant C(D) + \frac{1}{b_i^2} \leqslant C(D) + C(D) \equiv C(D)$$

by using the lower bound on b_i in the above inequality. Thus, we have the following

$$0 \leqslant \frac{d}{dr}\log(b_i) \leqslant C(D) \qquad\qquad 0 \leqslant \frac{b_i'(0)}{b_i(0)} \leqslant C \tag{3.48}$$

By integrating the inequality above, we get a lower bound $c(D, \beta_1) \leqslant b_i(-D)$, which extends to a lower bound on $[-D, D]$ by monotonicity.

By Equation (3.48) and the hypothesis that $b_i(0) \leqslant \beta_2$, we have a bound of the form $b_i'(0) \leqslant C(\beta_2)$. Then, using the second equation of Equation (3.47), we know that since $b_i(0)$ and $b_i'(0)$ are bounded from above, we can integrate the bound $b_i'' \leqslant C(D)b_i$ to get an exponential growth upper bound for $b_i(D)$. By the monotonicity of b_i, this bound holds on $[-D, D]$. Thus, to summarize, we have the following bounds on $[-D, D]$ for b_i for all i:

$$c(D, \beta_1) \leqslant b_i(r) \leqslant C(D, \beta_2) \tag{3.49}$$

Step 2: By the first equation of Equation (3.47), we know that $|a_i''| \leqslant C(1)a_i$ on $[-1, 1]$. Applying this to the subinterval $[-1, 0]$, we know that by the monotonicity of a_i that $a_i \leqslant \alpha_2$ on $[-1, 0]$. Thus, we have $|a_i''| \leqslant C(1)\alpha_2$ on $[-1, 0]$. This implies that $a_i'(0)$ is bounded from above by a constant $C(\alpha_2)$; otherwise, a_i' would be very large on the interval $[-1, 0]$ and $a_i(-1)$ would be negative. Then, as in Step 1, we can get an exponential growth upper bound for $a_i(D)$ of the form $a_i(D) \leqslant C(D, \alpha_2)$ by integrating the inequality $|a_i''| \leqslant C(D)a_i$.

By monotonicity and the lower bounds $a_i(0) \geqslant \alpha_1$, and $b_i(0) \geqslant \beta_1$, we have lower volume bounds on small s-balls for any point q with $r = 0$. By the curvature bound, by the Bishop-Gromov inequality, we have lower volume bounds of s-balls (whose centers are at distance at most D to q) by constants $C(D, \alpha_1, \beta_1)$. From this, we have a lower bound of the form $a(-D) \geqslant C(D, \alpha_1, \beta_1)$. By monotonicity, this bound holds on $[-D, D]$. Thus, to summarize, we have the following bounds on $[-D, D]$ for a:

$$c(D, \alpha_1, \beta_1) \leqslant a_i(r) \leqslant C(D, \alpha_2) \tag{3.50}$$

Step 3: By Equation (3.49) and Equation (3.50), we have upper bounds on a_i and b_i by constants of the form $a_i \leqslant C(D, \alpha_2)$ and $b_i \leqslant C(D, \beta_2)$, respectively. Using the first and second equations of Equation (3.47), on $[-D, D]$ we have bounds of the form (for all i)

$$|a_i'| \leqslant C(D, \alpha_2) \qquad |b_i'| \leqslant C(D, \beta_2)$$

In addition, using the lower bounds on a and b on $[-D, D]$, we have for all i

$$\left|\frac{a_i'}{a_i}\right| \leqslant C(D, \alpha_1, \alpha_2, \beta_1) \qquad \left|\frac{b_i'}{b_i}\right| \leqslant C(D, \beta_1, \beta_2) \tag{3.51}$$

Step 4: Now, the curvature derivative bounds imply the following inequalities for all i:

$$\left|\frac{d^k}{dr^k}\left(\frac{a_i''}{a_i}\right)\right| \leqslant C_k(D) \qquad \left|\frac{d^k}{dr^k}\left(\frac{b_i''}{b_i}\right)\right| \leqslant C_k(D) \tag{3.52}$$

We use these bounds to prove bounds of the form on $[-D, D]$ for all i

$$|a_i^{(k)}| \leqslant C(C_0, \cdots, C_k, D, \alpha_1, \alpha_2, \beta_1) \qquad |b_i^{(k)}| \leqslant C(C_0, \cdots, C_k, D, \beta_1, \beta_2) \tag{3.53}$$

The $k = 0$ and $k = 1$ cases are taken care of by Steps 1 to 3. For $k \geqslant 2$, we use Equation (3.52) and the bounds on a_i'/a_i and b_i'/b_i along with the bounds on a_i and b_i derived above to prove Equation (3.53) by induction. Then, by Arzela-Ascoli, we have subsequential convergence in $C^\infty_{loc}(\mathbb{R})$ of a_i and b_i to smooth positive functions $a_\infty, b_\infty : \mathbb{R} \to \mathbb{R}^+$. Thus, we have a smooth metric $g_\infty = dr^2 + a_\infty(r)^2 g_{S^1} + b_\infty(r)^2 g_{S^2}$ on $\mathbb{R} \times S^2 \times S^1$.

By hypothesis, our manifolds have topology $S^1 \times \mathbb{R}^3$ or $S^2 \times \mathbb{R}^2$, with a singular orbit at $r = -L_i$. Now, with $U_i := (-L_i/2, L_i/2) \times S^2 \times S^1$, consider the maps $\phi_i : U_i \to M$, where ϕ_i maps the points with coordinates (r, z_2, z_1) to the point in M in the orbit at distance $r + L_i$ from the singular orbit and whose coordinates on S^2 and S^1 are z_2 and z_1, respectively. Note that we have chosen $p_i \in (M, g_i)$ to have $r(p_i) = 0$ for all i. Then, we have $\phi_i^* g_i - g_\infty \to 0$ in $C^\infty_{loc}(\mathbb{R} \times S^2 \times S^1)$. Passing to a further subsequence, we get the convergence $\phi_i^{-1}(p_i) \to p_\infty$.

For 2, for ease of notation we assume that both orbit sizes at 0 blow up; the proof is similar if only one of them does. We consider the functions $\tilde{a}_i := \frac{a_i}{a_i(0)}$, $\tilde{b}_i := \frac{b_i}{b_i(0)}$. Note that the same bounds as in Equation (3.48) to Equation (3.53) can be proven for these functions, except without any reference to the α_i and β_i constants. Thus, by Arzela-Ascoli, we have convergence in $C^\infty_{loc}(\mathbb{R})$ of $\tilde{a}_i$ and $\tilde{b}_i$ to smooth positive functions $a_\infty, b_\infty : \mathbb{R} \to \mathbb{R}^+$.

Now, consider the smooth metric $g_\infty = dr^2 + a_\infty(r)^2 g_{\mathbb{R}^1} + b_\infty(r)^2 g_{\mathbb{R}^2}$ on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^1$. Since $a_i(0)$ and $b_i(0)$ converge to ∞, it is easy to see that there are diffeomorphisms $\phi_{1,i} : (U_{1,i}, g_{E_1}) \to (S^1, g_{a_i(0)})$ and $\phi_{2,i} : (U_{2,i}, g_{E_2}) \to (S^2, g_{b_i(0)})$ (where $(U_{k,i}, g_{E_k})$ is a subset of $\mathbb{R}^k$ with the Euclidean metric and $g_{a_i(0)}$ and $g_{b_i(0)}$ are respectively the metrics on S^2 and S^1 of the sizes in the subscripts and the $U_{k,i}$ cover $\mathbb{R}^k$) such that $\phi_{1,i}^* g_{a_i(0)} \to g_{E_1}$ in $C^\infty_{loc}(\mathbb{R})$ and $\phi_{2,i}^* g_{b_i(0)} \to g_{E_2}$ in $C^\infty_{loc}(\mathbb{R}^2)$. Now, consider the diffeomorphisms $\phi_i : (-L_i/2, L_i/2) \times U_{1,i} \times U_{2,i} \to (M, g_i)$ which map (r, z_1, z_2) to the point in M at distance $r + L_i$ from the singular orbit and whose coordinates on S^2 and S^1 are respectively given by $\phi_{2,i}(z_2)$ and $\phi_{1,i}(z_1)$. Then, we have $\phi_i^* g_i - g_\infty \to 0$ in $C^\infty_{loc}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^1)$. Convergence of $\phi_i^{-1}(p_i)$ follows from the fact that $r(p_i) = 0$ and by the fact that the convergence of the spheres to Euclidean spaces can always be chosen to be pointed (by adjusting by translations if necessary). $\square$

Chapter 4

Comparison with the General Setting

In Chapter 2, we provided a brief overview of the theory of the expander degree from [BC23]. In Chapter 3, we defined a parallel version of this theory in the cohomogeneity one setting. In this chapter, we will leverage the results from the previous chapter to provide useful examples or intuition in the general setting.

4.1 Multiple Solitons Asymptotic to the Same Cone

We begin with the following question, asked as Question 1.18 in [BC23]:

Question: Is there an example of two non-isometric gradient expanding solitons with non-negative scalar curvature, defined on the same orbifold $\mathrm{Int}(X)$, that are asymptotic to the same conical metric $\gamma \in \mathrm{Cone}^{\infty}_{R\geqslant 0}(\partial X)$?

We answer the following version of the question, without assumptions on the scalar curvature:

Question (modified): Is there an example of two non-isometric gradient expanding solitons defined on the same orbifold $\mathrm{Int}(X)$, that are asymptotic to the same conical metric $\gamma \in \mathrm{Cone}^{\infty}(\partial X)$?

We provide a positive answer to the modified question, based on Theorem 3.8.7, from which we know that $\deg^{\mathrm{sym}}_{\exp}(S^2 \times \mathbb{D}^2) = 0$. Now, consider the space $\mathrm{Cone}^{\infty,\mathrm{sym}}_{R\geqslant 0}(S^2 \times S^1)$, which is the space of cones over the link $S^2 \times S^1$ with a metric of the form $h = (a'_\infty)^2 g_{S^1} + (b'_\infty)^2 g_{S^2}$.

Now, pick any cone such that (a'_∞, b'_∞) is in the image of the map F. Since $\deg^{\mathrm{sym}}_{\exp}(S^2 \times \mathbb{D}^2) = 0$, there must exist at least 2 preimages of (a'_∞, b'_∞) under F. Each preimage is a set of initial conditions of the form $(b_0, -f_0)$ which corresponds to a cohomogeneity one gradient expanding soliton on $S^2 \times \mathbb{R}^2$ which is asymptotic to the chosen cone. The two solitons are clearly not isometric, yet they are asymptotic to the same cone. Note that despite the fact that both the cone and the solitons in this example are cohomogeneity one, this still provides an answer to the modified question in the general setting.

We find it surprisingly hard to prove the existence of gradient expanding solitons on $S^2 \times \mathbb{R}^2$ with positive scalar curvature, despite the fact that numerical simulations strongly suggest their existence. This precludes us from providing a rigorous example for the original Question.

As mentioned in [BC23], a positive answer to this question would suggest that the process of resolution of singularities in Ricci flow in dimensions 4 and higher is non-unique. To confirm this belief, it needs to be further shown that the cone in the above argument actually arises as a conical singularity in 4-dimensional Ricci flow.

4.2 Relating the Notions of Expander Degree

In Chapter 2, we recalled the definition of $\deg_{\exp}(X)$ through the definition of the space $\mathcal{M}^{k^*}_{\mathrm{grad},R\geqslant 0}(X)$ of asymptotically conical gradient expanding solitons with nonnegative scalar curvature.

For our chosen orbifolds $X = S^1 \times \mathbb{D}^3$ or $X = S^2 \times \mathbb{D}^2$, if we define cohomogeneity one versions of these spaces, while removing the scalar curvature restriction, we get the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}^{\infty}_{\mathrm{grad}}(X) \arrow[r, "\Pi"] & \mathrm{Cone}^{\infty}(\partial X) \\
\mathcal{M}^{\infty,\mathrm{sym}}_{\mathrm{grad}}(X) \arrow[u, "i"] \arrow[r, "\Pi_{\mathrm{sym}}"] & \mathrm{Cone}^{\infty,\mathrm{sym}}(\partial X) \arrow[u, "i"'] \\
\mathbb{R}^+ \times \mathbb{R}^+ \arrow[u, "p"] \arrow[r, "F"'] & \mathbb{R}^+ \times \mathbb{R}^+ \arrow[u, "\cong"']
\end{tikzcd}$$

where the map F is as defined in Chapter 3 and Π is from Equation (2.1) for $k^* = \infty$, and Π_{sym} is the restriction of Π. The map p takes a pair of initial conditions, either $(a_0, -f_0)$ or $(b_0, -f_0)$, and associates to it the equivalence class of the soliton with metric $g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2}$ and soliton vector field $\nabla f = f'(r)\partial_r$, where a, b, f satisfy the soliton equations Equations (3.2) to (3.4). More precisely, we define

$$\begin{aligned}
p(a_0, -f_0) &= [(g, \nabla f, \gamma)] \qquad (S^1 \times \mathbb{D}^3) \\
p(b_0, -f_0) &= [(g, \nabla f, \gamma)] \qquad (S^2 \times \mathbb{D}^2)
\end{aligned}$$

We first make the following claim:

Claim: p is a continuous bijection.

Proof: First, by Section 3.1 and Lemma 3.1.1, every equivalence class in $M_{\text{grad}}^{\infty,\text{sym}}(X)$ contains a representative $(g, \nabla f, \gamma)$ where $g = dr^2 + a(r)^2 g_{S^1} + b(r)^2 g_{S^2}$. There is a one-to-one mapping between such metrics (up to isometry) and pairs of initial conditions in $\mathbb{R}^+ \times \mathbb{R}^+$, which shows that p is a bijection.

Now, recall that the metric on $M_{\text{grad}}^{\infty,\text{sym}}(X)$, inherited from $M_{\text{grad}}^{\infty}(X)$, is given by

$$d\big([(g_1, \nabla f_1, \gamma_1)], [(g_2, \nabla f_2, \gamma_2)]\big) := \sum_{j=1}^{\infty} 2^{-j} G\big(d'_j([(g_1, \nabla f_1, \gamma_1)], [(g_2, \nabla f_2, \gamma_2)])\big) \tag{4.1}$$

$$+ \big| \max |\mathrm{Rm}_{g_1}|_{g_1} - \max |\mathrm{Rm}_{g_2}|_{g_2} \big| + \|\gamma_1 - \gamma_2\|, \tag{4.2}$$

where $G(x) = \frac{x}{1+x}$ and the last norm is the Banach norm on $\mathrm{Cone}^{\infty,\text{sym}}(S^2 \times S^1)$ and

$$d'_j\big([(g_1, \nabla f_1, \gamma_1)], [(g_2, \nabla f_2, \gamma_2)]\big) := \sup_{x,y \in [j^{-1}, j] \times N} |d_{g_1}(\widetilde{\iota}_1(x), \widetilde{\iota}_1(y)) - d_{g_2}(\widetilde{\iota}_2(x), \widetilde{\iota}_2(y))|.$$

where $\tilde{\iota}$ is the unique map from Lemma 3.15(b) of [BC23]. $\tilde{\iota}$ has the property that $\tilde{\iota}^*(\nabla f) = -r\partial_r$.

For simplicity, we will carry out the proof in the case $X = S^1 \times \mathbb{D}^3$. Consider a sequence of initial conditions $(a_0^i, -f_0^i)$ that converge to a sequence $(a_0, -f_0)$, and let the associated metrics g_i in the classes $(g_i, \nabla f_i, \gamma_i)$ converge to a metric g (in the usual sense of convergence of metrics). By Theorem 3.6.10, the map F is continuous, implying that the asymptotic cone metrics γ_i converge to γ. Additionally, $\big| \max |\mathrm{Rm}_{g_i}|_{g_i} - \max |\mathrm{Rm}_g|_g \big|$ also approaches 0.

Now, fix $\epsilon > 0$. Note that $0 \leqslant G(x) \leqslant 1$ since the arguments of G are positive. Now, the summation can be broken up as

$$\begin{aligned}
\sum_{j=1}^{\infty} 2^{-j} G\big(d'_j([(g_i, \nabla f_i, \gamma_i)], [(g, \nabla f, \gamma)])\big) &= \sum_{j=1}^{j_0} 2^{-j} G\big(d'_j([(g_i, \nabla f_i, \gamma_i)], [(g, \nabla f, \gamma)])\big) \\
&\quad + \sum_{j=j_0+1}^{\infty} 2^{-j} G\big(d'_j([(g_i, \nabla f_i, \gamma_i)], [(g, \nabla f, \gamma)])\big) \\
&\leq G\big(d'_{j_0}([(g_i, \nabla f_i, \gamma_i)], [(g, \nabla f, \gamma)])\big) + \epsilon \\
&\leq d'_{j_0}([(g_i, \nabla f_i, \gamma_i)], [(g, \nabla f, \gamma)]) + \epsilon
\end{aligned}$$

by choosing j_0 sufficiently large.

Now, similar the discussion surrounding Lemma 3.6.1, it can be shown that $\tilde{\iota} : \mathbb{R}^+ \times S^2 \times S^1 \to \mathrm{Int}(X)$ is determined entirely by the similarly named map $\tilde{\iota} : \mathbb{R}^+ \to \mathbb{R}^+$. Additionally, since

$\tilde{\iota}^*(\nabla f) = -r\partial_r$, we see that $\tilde{\iota}$ is determined by ∇f. Since ∇f_i converges to ∇f uniformly on compact sets, we have that $\tilde{\iota}_i$ converges uniformly to $\tilde{\iota}$ on compact sets. Thus, by choosing i sufficiently large, we can ensure that $d'_{j_0}([(g_i, \nabla f_i, \gamma_i)], [(g, \nabla f, \gamma)]) \leqslant \epsilon$. This ensures that the summation term can be made arbitrarily small by choosing i sufficiently large, and thus that the equivalence classes $[(g_i, \nabla f_i, \gamma_i)]$ converge to $[(g, \nabla f, \gamma)]$ in the topology on $M^{\infty,\mathrm{sym}}_{\mathrm{grad}}(X)$ determined by the metric, proving that p is continuous.

Now, we show that p is a homeomorphism by proving that its inverse is continuous.

Claim: p^{-1} is continuous.

Proof: Consider a sequence of equivalence classes $[(g_i, \nabla f_i, \gamma_i)] = p(a_0^i, -f_0^i)$ such that $[(g_i, \nabla f_i, \gamma_i)]$ converge to $[(g, \nabla f, \gamma)]$. Then, since γ_i converges to γ, by the fact that F is proper (Theorem 3.8.1), we know that any subsequence of the sequence $(a_0^i, -f_0^i)$ must have at least one subsequential limit, say $(a_0, -f_0)$. Suppose that there exists a different subsequential limit $(a_0', -f_0')$. This leads to a contradiction, since $p(a_0, -f_0) = p(a_0', -f_0') = [(g, \nabla f, \gamma)]$, but we know that p is a bijection. Thus, since every subsequence of $(a_0^i, -f_0^i)$ has a further convergent subsequence (with a unique subsequential limit across all subsequences), the sequence $(a_0^i, -f_0^i)$ converges to $(a_0, -f_0)$, showing the continuity of p^{-1}.

Thus, as p is a homeomorphism, the map Π_{sym} is essentially the same as the map F, and thus it is possible to define the invariant degree $\deg(\Pi_{\mathrm{sym}}) = \deg(F)$. Note that this equality holds only up to sign without taking into consideration the relative orientations of the two spaces.

It is expected that $\deg_{\exp}(X) = \deg(\Pi_{\mathrm{sym}})$, as it is believed that the non-symmetric solitons do not contribute to the signed count in the expander degree. However, this is not rigorously known at this time.

Instead, we relate our notion of degree to that of [BC23] in the following lemma. The local theory of the expander degree can be redefined in the cohomogeneity one setting, with a notable simplification that all such solitons are gradient, since any smooth vector field on $\mathbb{R}^+$ is the gradient of a smooth function. As in [BC23], we denote by $\mathrm{index}(-L_p)$ the Morse index of the Einstein operator at $p \in M^{\infty,\mathrm{sym}}_{\mathrm{grad}}(X)$ for a representative $p = [(g, \nabla f, \gamma)]$.

Lemma 4.2.1. *Let $\Pi_{\mathrm{sym}} : M^{\infty,\mathrm{sym}}_{\mathrm{grad}}(X) \to \mathrm{Cone}^{\infty,\mathrm{sym}}(\partial X)$ be as defined earlier. Then the standard topological degree of Π_{sym} satisfies the following formula:*

$$\deg(\Pi_{\mathrm{sym}}) = \sum_{p \in (\Pi_{\mathrm{sym}})^{-1}(\gamma)} (-1)^{\mathrm{index}(-L_p)}$$

for any regular value $\gamma \in \mathrm{Cone}^{\infty,\mathrm{sym}}(\partial X)$.

Proof. Let $\gamma \in \mathrm{Cone}^{\infty,\mathrm{sym}}(\partial X)$ be a regular value of the proper map Π_{sym}. Note that Π_{sym} is the restriction of the map Π which is shown in Section 10 of [BC23] to be $C^{1,\alpha}$ for $\alpha \in (0,1)$. By the classical definition of the topological degree for maps between oriented manifolds of the same dimension, we have:

$$\deg(\Pi_{\mathrm{sym}}) = \sum_{p\in(\Pi_{\mathrm{sym}})^{-1}(\gamma)} \mathrm{sgn}(\det d(\Pi_{\mathrm{sym}})_p)$$

where $\mathrm{sgn}(\det d(\Pi_{\mathrm{sym}})_p) \in \{\pm 1\}$ is $+1$ if the differential

$$d(\Pi_{\mathrm{sym}})_p : T_p M^{\infty,\mathrm{sym}}_{\mathrm{grad}}(X) \to T_\gamma \mathrm{Cone}^{\infty,\mathrm{sym}}(\partial X)$$

is orientation-preserving, and -1 if it is orientation-reversing.

Fix an arbitrary preimage point $p \in (\Pi_{\mathrm{sym}})^{-1}(\gamma)$. As γ is a regular value, the differential $d(\Pi_{\mathrm{sym}})_p$ is a vector space isomorphism. Consequently, its kernel and cokernel are both trivial zero-dimensional vector spaces:

$$\begin{aligned}&K := \ker(d(\Pi_{\mathrm{sym}})_p) = \{0\}, \quad \text{and}\\ &C := \mathrm{coker}(d(\Pi_{\mathrm{sym}})_p) = T_\gamma \mathrm{Cone}^{\infty,\mathrm{sym}}(\partial X)/\mathrm{Im}(d(\Pi_{\mathrm{sym}})_p) = \{0\}.\end{aligned}$$

As in Proposition 10.1(j*) of [BC23], we now examine the two linear maps required to define the orientation on $T_p M^{\infty,\mathrm{sym}}_{\mathrm{grad}}(X)$:

1. **The Quotient Differential:** The map

$$\overline{d(\Pi_{\mathrm{sym}})_p} : T_p M^{\infty,\mathrm{sym}}_{\mathrm{grad}}(X)/K \to \mathrm{Im}(d(\Pi_{\mathrm{sym}})_p).$$

   Since $K = \{0\}$ and $\mathrm{Im}(d(\Pi_{\mathrm{sym}})_p) = T_\gamma \mathrm{Cone}^{\infty,\mathrm{sym}}(\partial X)$, this map is canonically identified with the full differential $d(\Pi_{\mathrm{sym}})_p$. Therefore, its orientation sign ϵ_1 is precisely the sign of the Jacobian determinant:

$$\epsilon_1 = \mathrm{sgn}(\det d(\Pi_{\mathrm{sym}})_p)$$

2. **The Shift Map:** The map $S_p : C \to K$. Since both C and K are the trivial zero-dimensional vector space $\{0\}$, S_p is the unique linear map $\{0\} \to \{0\}$. Thus, its orientation sign is strictly positive:

$$\epsilon_2 = +1$$

By hypothesis, the orientation of $T_p M^{\infty,\mathrm{sym}}_{\mathrm{grad}}(X)$ is globally chosen such that the relation $\epsilon_1 \cdot \epsilon_2 = (-1)^{\mathrm{index}(-L_p)}$ holds. Substituting $\epsilon_2 = +1$ and $\epsilon_1 = \mathrm{sgn}(\det d(\Pi_{\mathrm{sym}})_p)$ into this relation yields:

$$\mathrm{sgn}(\det d(\Pi_{\mathrm{sym}})_p) = (-1)^{\mathrm{index}(-L_p)}$$

Thus, we have

$$\begin{aligned}\deg(\Pi_{\text{sym}}) &= \sum_{p\in(\Pi_{\text{sym}})^{-1}(\gamma)} \operatorname{sgn}(\det d(\Pi_{\text{sym}})_p) \\ &= \sum_{p\in(\Pi_{\text{sym}})^{-1}(\gamma)} (-1)^{\text{index}(-L_p)}\end{aligned}$$

which proves the lemma. □

For our particular examples, when $X = S^2 \times \mathbb{D}^2$, all degrees above are 0. However, when $X = S^1 \times \mathbb{D}^3$, the degree is only known up to sign ± 1, indicating that the index index$(-L_p)$ is known to be either always odd or always even.

As an additional consequence, when $X = S^2 \times \mathbb{D}^2$, we have the following:

$$0 = \deg(\Pi_{\text{sym}}) = \sum_{p\in(\Pi_{\text{sym}})^{-1}(\gamma)} (-1)^{\text{index}(-L_p)}$$

Since the summation is equal to 0, for any regular value γ in the image of Π_{sym}, there must exist at least 2 preimages of γ, so there must exist at least one preimage $p = [(g, \nabla f, \gamma)]$ whose index index$(-L_p)$ is odd. This indicates that the operator $-L_p$ at the preimage p has at least one negative eigenvalue, representing an unstable direction. Thus, we have the existence of an unstable expanding soliton.

Chapter 5

Structure of Some 4-Dimensional Steady Solitons

In [BCDMZ22], the authors consider 4-dimensional gradient steady Ricci soliton singularity models, particularly those whose tangent flow at infinity is the shrinking soliton given by the product of $\mathbb{R}$ with a spherical space form S^3/Γ. In this section, we take an approach analogous to that of their paper to study the structure of 4-dimensional steady soliton singularity models whose tangent flow is given by the shrinking soliton $S^2 \times \mathbb{R}^2$. These solitons are otherwise referred to as asymptotically 2-cylindrical.

We prove a result about the structure of asymptotically 2-cylindrical singularity models in 4-dimensions that is analogous to Proposition 3.8 of [BCDMZ22]. This result is part of a method to prove the more general result that the expander degree of certain smooth manifolds with boundary, such as $S^2 \times \mathbb{R}^2$ is 0 even without the cohomogeneity one assumption of the previous chapter.

Analogous to [BCDMZ22], we proceed by defining ϵ-necks for $S^2 \times \mathbb{R}^2$.

Definition 5.0.2. *Let (M, g) be a Riemannian manifold and let $\varepsilon > 0$. An open subset $U \subset M$ is called an ε-neck if it is modeled on $S^2 \times \mathbb{R}^2$ in the following sense.*

U is an ε-neck for $S^2 \times \mathbb{R}^2$ if there exists a bijective ε-homothety

$$\Phi : S^2 \times \left(-\tfrac{1}{\varepsilon}, \tfrac{1}{\varepsilon}\right) \times \left(-\tfrac{1}{\varepsilon}, \tfrac{1}{\varepsilon}\right) \to U.$$

A point $x \in U$ is called a center *of the ε-neck if*

$$x \in \Phi\big(S^2 \times \{0\} \times \{0\}\big)$$

for such a map Φ.

Now, we prove a version of the [BCDMZ22] result for steady soliton singularity models with tangent flow at infinity $S^2 \times \mathbb{R}^2$. We will assume throughout that our steady soliton is normalized so that the identity $R_g + |\nabla f|^2 = 1$ is satisfied.

Lemma 5.0.3. *Consider a 4-dimensional steady gradient Ricci soliton singularity model $(M^4, g(t), f(t))$ whose tangent flow at infinity is $S^2 \times \mathbb{R}^2$. Then, for any $\epsilon > 0$, there exists an unbounded set N_ϵ such that every point in N_ϵ is the center of an ϵ-neck for $S^2 \times \mathbb{R}^2$.*

Proof. The proof of this lemma is similar to that of Proposition 3.8 of [BCDMZ22]. In Proposition 3.1 of that paper, all possible tangent flows at infinity of 4-dimensional steady gradient Ricci soliton singularity models are classified. By Proposition 3.2 of [BCDMZ22], we know that the tangent flow at infinity is unique. Thus, we know that any tangent flow at infinity of $(M, g(t))$ is $(S^2 \times \mathbb{R}^2, g_{\text{cyl}})$, where

$$g_{\text{cyl}} = g_{S^2} + g_{\mathbb{R}^2}$$

where g_{S^2} is the round metric on the sphere S^2 and $g_{\mathbb{R}^2}$ is the Euclidean metric.

Consider any point $(x, 0) \in M \times \mathbb{R}$, and let $\lambda > 0$. Let $(z_\lambda(x), -1/\lambda)$ be an H_4-center of $(x, 0)$ (refer [Bam20a] for the definition of H_n-centers), and define $g_\lambda(t) = \lambda g(t/\lambda)$. Then, we know that there exists an $\epsilon \equiv \epsilon(\lambda) > 0$ and a diffeomorphism $\Psi_\lambda : B^{\text{cyl}}_{1/\epsilon} \to B(z_\lambda(x), 1/\epsilon, g_\lambda(-1))$ such that $\lim_{\lambda \to 0} \epsilon(\lambda) = 0$ and

$$\|\Psi^* g_\lambda(-1) - g_{\text{cyl}}\|_{C^{[1/\epsilon]}(B^{\text{cyl}}_{1/\epsilon})} \leqslant \epsilon$$

where $B^{\text{cyl}}_{1/\epsilon}$ denotes any ball of radius $1/\epsilon$ in $(S^2 \times \mathbb{R}^2, g_{\text{cyl}})$. This implies that $z_\lambda(x)$ is the center of an ϵ-neck for $S^2 \times \mathbb{R}^2$ in $(M, g_\lambda(-1))$.

Set $g := g(0)$. From the Ricci soliton equation $\text{Ric}_{g(t)} + \nabla^2 f(t) = 0$, we see that the vector field $\nabla_g f$ generates the 1-parameter group of diffeomorphisms $\Phi_t : M \to M$. Then, as in Proposition 3.8 of [BCDMZ22], it is clear that $g_\lambda(-1) = \lambda \Phi^*_{-1/\lambda} g$. Thus, we have the following composition of diffeomorphisms:

$$B^{\text{cyl}}_{1/\epsilon} \xrightarrow{\Psi_\lambda} B(z_\lambda(x), 1/\epsilon\, ; g_\lambda(-1)) \xrightarrow{\Phi_{-1/\lambda}} B\big(w_\lambda(x), 1/(\sqrt{\lambda}\epsilon); g\big) =: \mathfrak{N}_\lambda(x),$$

where $w_\lambda(x) := \Phi_{-1/\lambda}(z_\lambda(x))$ for every $x \in M$. Thus, we have

$$\|\lambda(\Phi_{-1/\lambda} \circ \Psi_\lambda)^* g - g_{\text{cyl}}\|_{C^{[1/\epsilon]}(B^{\text{cyl}}_{1/\epsilon})} \leq \epsilon.$$

which implies that $|\text{Rm}_g|(y) \approx \lambda$ for all $y \in \mathfrak{N}_\lambda(x)$.

Now, choose $\overline{\lambda} \equiv \overline{\lambda}(x)$ so that for $\lambda \leqslant \overline{\lambda}$, we have $\epsilon(\lambda) \leqslant 10^{-6}$ and the set $V_\lambda(x)$, defined by

$$V_\lambda(x) := B(z_\lambda(x), 10\sqrt{H_4}; g_\lambda(-1)) = B(z_\lambda(x), 10\sqrt{H_4/\lambda}; g(-1/\lambda)) \tag{5.1}$$

is diffeomorphic to the corresponding ball in $S^2 \times \mathbb{R}^2$. Define $U_\lambda(x)$ by

$$U_\lambda(x) := B(w_\lambda(x), 10\sqrt{H_4/\lambda}; g) = \Phi_{-1/\lambda}(V_\lambda(x))$$

Following [BCDMZ22], we define $\alpha B(x,r;g) := B(x,\alpha r;g)$. Then, we write

$$N_\epsilon = \bigcup_{x\in M} N_\epsilon(x) \quad \text{where} \quad N_\epsilon(x) := \bigcup_{\bar{\lambda}>\lambda>0} 10U_\lambda(x)$$

As in [BCDMZ22], every point in N_ϵ is the center of an ϵ-neck for $S^2\times\mathbb{R}^2$ for the following reason: since $10(10\sqrt{H_4}) \ll 1/\epsilon \leqslant 10^6$, we see that $10U_\lambda(x)$ is in the middle of the neck region $\mathfrak{N}_\lambda(x) = \frac{1}{10\epsilon}U_\lambda(x)$, and so every point in $10U_\lambda(x)$ is the center of an ϵ-neck, and thus the same is true for N_ϵ.

It remains to show that N_ϵ has a connected unbounded component. To do this, we show that the sets $U_\lambda(x)$ are not disjoint for values of λ sufficiently close to a fixed λ_0 and y close to a fixed x. First, we make the following claim (essentially identical to the one in [BCDMZ22]; for convenience, we reproduce the proof here)

Claim 1: Fix $x\in M$. For any $\lambda_0>0$, there exists a $\delta\equiv\delta(\lambda_0,x)$ so that if $|\lambda-\lambda_0|\leqslant\delta$, then

$$U_\lambda(x)\cap U_{\lambda_0}(x)\neq\emptyset$$

Proof of Claim 1: For each $x\in M$, define

$$\begin{aligned} V'_\lambda(x) &:= \tfrac{4}{5}V_\lambda(x) := B\big(z_\lambda(x), 8\sqrt{H_4}\,;\, g_\lambda(-1)\big)\\ U'_\lambda(x) &:= \tfrac{4}{5}U_\lambda(x) := \Phi_{-1/\lambda}\big(V'_\lambda(x)\big) = B\big(w_\lambda(x), 8\sqrt{H_4/\lambda}\,;\, g\big)\end{aligned}$$

Suppose the claim is not true; then, there would exist a sequence $\lambda_j\to\lambda_0$ such that for all j, we have

$$U_{\lambda_j}(x)\cap U_{\lambda_0}(x)=\emptyset$$

By applying the diffeomorphism Φ_{1/λ_0} to this, we see that

$$V_{\lambda_0}(x)\cap\Phi_{\frac{1}{\lambda_0}-\frac{1}{\lambda_j}}(V_{\lambda_j}(x))=\emptyset. \tag{5.2}$$

For any sufficiently small $\beta>0$, there exists $\bar{j}=\bar{j}(\beta,\lambda_0)$ such that for $j\geq\bar{j}$,

$$\delta_j := \frac{1}{\lambda_0}-\frac{1}{\lambda_j}\in(-\beta,\beta).$$

For each $y\in V'_{\lambda_j}(x_0)$, by definition,

$$d(y, z_{\lambda_j}(x); g_{\lambda_j}(-1)) < 8\sqrt{H_4}.$$

Then

$$d(\Phi_{-\delta_j}(y), y; g_{\lambda_j}(-1)) \leqslant \left|\int_{-\delta_j}^0 |\nabla f|_{g_{\lambda_j}(-1)}(\Phi_s(y))\,ds\right| \leqslant |\delta_j|\sqrt{1/\lambda_j} \leqslant \beta\sqrt{\beta+1/\lambda_0} < 1$$

if $\beta < \bar{\beta}(\lambda_0)$, where the penultimate inequality follows from the definition of δ_j. Thus (5.2) yields

$$V'_{\lambda_j}(x) \subset \Phi_{\delta_j}(V_{\lambda_j}(x)), \quad \text{and hence} \quad V'_{\lambda_0}(x) \cap V'_{\lambda_j}(x) = \emptyset.$$

Now, we apply the Gaussian concentration estimate (Proposition 3.13) of [?] to the H_4-center $(z_\lambda(x), -1/\lambda)$ of the point $(x, 0)$ with $A = 64$ to get

$$\nu_{x,0;-1/\lambda_0}(V'_{\lambda_0}(x)) \geqslant 1 - \frac{1}{64} > 0.9$$

By Lemma 3.3 of [BCDMZ22] (assuming that $-1/\lambda_0 < -1/\lambda_j$; the proof is similar in the other case), we know that

$$\nu_{x,0;-1/\lambda_0}(V'_{\lambda_j}(x)) \geqslant \frac{1}{2}$$

Since both sets have $\nu_{x,0;-1/\lambda_0}$-measure at least $\frac{1}{2}$, this is a contradiction to the previously obtained conclusion that $V'_{\lambda_0}(x) \cap V'_{\lambda_j}(x) = \emptyset$. This completes the proof of the first claim.

Claim 2: Fix $(x_1, 0)$ and $(x_2, 0)$ in $M \times \{0\}$. Then, there exists $\lambda^* \equiv \lambda^*(x_1, x_2)$ such that for $\lambda \leqslant \lambda^*$, we have

$$U_\lambda(x_1) \cap U_\lambda(x_2) \neq \emptyset$$

Proof of Claim 2: Denote the distance $d_g(x_1, x_2)$ by D. Fix any $\lambda > 0$ and consider H_4-centers $(z_1, -1/\lambda)$ of $(x_1, 0)$ and $(z_2, -1/\lambda)$ of $(x_2, 0)$. Then, we see that

$$\begin{aligned}
d_{g(-1/\lambda)}(z_1, z_2) &= \mathrm{Var}(\delta_{z_1}, \delta_{z_2})^{1/2} \\
&\leqslant \mathrm{Var}(\delta_{z_1}, \nu_{x_1,0;-1/\lambda})^{1/2} + \mathrm{Var}(\nu_{x_1,0;-1/\lambda}, \nu_{x_2,0;-1/\lambda})^{1/2} + \mathrm{Var}(\nu_{x_2,0;-1/\lambda}, \delta_{z_2})^{1/2} \\
&\leqslant 2\sqrt{H_4/\lambda} + \mathrm{Var}(\nu_{x_1,0;-1/\lambda}, \nu_{x_2,0;-1/\lambda})^{1/2} \\
&\leqslant 2\sqrt{H_4/\lambda} + \sqrt{d_g^2(x_1, x_2) + H_4/\lambda} \\
&\leqslant 3\sqrt{H_4/\lambda} + D
\end{aligned}$$

where the second inequality follows by Lemma 3.2 of [Bam20a] and the third inequality follows by Corollary 3.8 of [Bam20a].

Thus, for sufficiently small $\lambda > 0$, the sets $V_\lambda(x_1)$ and $V_\lambda(x_2)$ defined in (5.1) must intersect, since the distance $d_{g(-1/\lambda)}(z_1, z_2)$ is at most $3\sqrt{H_4/\lambda} + D \ll (10\sqrt{H_4/\lambda} + 10\sqrt{H_4/\lambda})$. This is sufficient to prove Claim 2.

Now, it follows from Claim 1 that $N_\epsilon(x)$ is connected for all x. From Claim 2, it follows that N_ϵ is connected. Clearly N_ϵ is unbounded, completing the proof of the Lemma. $\square$

We also need the following lemma that tells us how the scalar curvature decays along integral curves of the soliton vector field.

Lemma 5.0.4. *Let $(M^4, g, \nabla f)$ be a smooth four-dimensional gradient steady Ricci soliton whose tangent flow at infinity is the shrinking cylinder $(S^2 \times \mathbb{R}^2, g_{\mathrm{cyl}})$. For any $p \in M$, let σ denote the integral curve of $-\nabla f$ with $\sigma(0) = p$. Then, for some constant $c > 0$, for $t \geqslant 0$, we have the following inequality:*

$$R(\sigma(t)) \leqslant \frac{R(p)}{1 + R(p)ct} \leqslant \frac{1}{ct} \tag{5.3}$$

Proof. We have the following on (M, g):

$$\Delta R = \mathrm{o}(R^2) \quad \text{and} \quad 2|\mathrm{Ric}|^2 = R^2 + \mathrm{o}(R^2). \tag{5.4}$$

These are proven similarly to (3.18) and (3.19) of [BCDMZ22]; if (5.4) were not true, there would exist a sequence of points converging to infinity whose rescalings converge to a soliton that is not $S^2 \times \mathbb{R}^2$, since the scaling-invariant identities $R_{\bar{g}}^{-2} \Delta_{\bar{g}} R_{\bar{g}} = 0$ and $2R_{\bar{g}}^{-2} |\operatorname{Ric}_{\bar{g}}|^2 = 1$ hold on $(S^2 \times \mathbb{R}^2, \bar{g})$.

Now, we have the following inequality on (M, g):

$$\begin{aligned}(\nabla f, \nabla R) = \partial_t R &= \Delta R + 2|\mathrm{Ric}_g|^2 \\ &= \mathrm{o}(R^2) + R^2 \\ &\geq cR^2.\end{aligned} \tag{5.5}$$

for some positive constant c, where the second line follows from equation (5.4). Now, let σ be the integral curve of $-\nabla f$ with $\sigma(0) = p$, and let $\phi(x) = 1/R(x)$. Then, we have

$$\begin{aligned}\frac{d}{ds}(\phi \circ \sigma)(s) &= \langle \nabla \phi, \dot{\sigma} \rangle (\sigma(s)) \\ &= \langle \nabla \phi, -\nabla f \rangle (\sigma(s)) \\ &= \left\langle \nabla \left(\frac{1}{R} \right), \nabla(-f) \right\rangle (\sigma(s)) \\ &\geqslant c\end{aligned}$$

where the last line follows by (5.5). Now, integrating the inequality $\frac{d}{ds}(\phi \circ \sigma)(s) \geqslant c$ from $s = 0$ to $s = t$ gives gives

$$\phi(\sigma(t)) \geqslant \phi(x) + ct$$

implying that for all $t > 0$, we have (by definition of ϕ)

$$R(\sigma(t)) \leqslant \frac{1}{\frac{1}{R(p)} + ct}$$

which proves the lemma. $\square$

Now, having understood the structure of $(M^4, g(t), f(t))$ at infinity, we proceed to understand the complementary region, where the soliton does not look close to its tangent flow at infinity. We show that no point is very far away from the region that looks close to $S^2 \times \mathbb{R}^2$.

Lemma 5.0.5. *Consider a smooth 4-dimensional steady gradient Ricci soliton singularity model $(M^4, g, \nabla f)$ whose tangent flow at infinity is isometric to $(S^2 \times \mathbb{R}^2, g_{cyl})$. Denote by N_ϵ the region in M which consists of points which are the centers of ϵ-necks for $S^2 \times \mathbb{R}^2$. Suppose further that $(M, g, \nabla f)$ satisfies the following technical assumptions:*

Assumption A: *The inequality on the scalar curvature $R(x) \geqslant c_0(\epsilon)$ holds for some constant $c_0(\epsilon)$ depending only on $\epsilon > 0$ for all $x \in \partial N_\epsilon$.*

Assumption B: *There is no smooth embedding of $RP^2 \times (-1, 1)^2$ into M.*

Then, there exists a constant $D \equiv D(\epsilon)$ such that the following holds:

$$x \in M - N_\epsilon \implies d_g(x, N_\epsilon) \leqslant D$$

Note: It can be shown that Assumption A holds for all steady soliton singularity models whose tangent flow at infinity is asymptotically 2-cylindrical as considered in this section. The proof is technical so for simplicity we include it as an assumption. Additionally, Assumption B can be done away with by a modification of the definition of the set N_ϵ to include ϵ-necks for the $\mathbb{Z}_2$ quotient of $S^2 \times \mathbb{R}^2$ as well. Again, for simplicity, it is kept as an assumption.

Proof. Suppose that the lemma is not true. Then, there would exist a sequence of points $\{x_j\} \subset (M, g)$ with $d_g(x_j, N_\epsilon) \to \infty$. Consider the sequence of pointed manifolds $(M, g, x_j, \nabla f_j)$, where $f_j = f - f(x_j)$ (so that $f(x_j) = 0$). Additionally, by the soliton identity $R_g + |\nabla f|_g^2 = 1$, we know that $|\nabla f_j|(x_j)_g$ is bounded. From the soliton equation $\mathrm{Ric}_g + \nabla^2 f_j = 0$ and the fact that (M, g) is smooth, we see that since the curvature Rm_g is bounded we have the bound $|\nabla_g^2 f_j|_g \leqslant C$ which holds on all of (M, g) for some constant $C > 0$. By Shi's estimates, we have similar bounds on higher derivatives of f_j which are uniform in j.

Thus, by the previous paragraph, we have Cheeger-Gromov convergence of a subsequence of $(M, g, x_j, \nabla f_j)$ to a smooth 4-dimensional steady soliton singularity model $(M_\infty, g_\infty, x_\infty, \nabla f_\infty)$. By Assumption B, the tangent flow at infinity of $(M_\infty, g_\infty, \nabla f_\infty)$ cannot be $((S^2 \times \mathbb{R})/\mathbb{Z}_2) \times \mathbb{R}$. Additionally, by the definition of the points x_j, this tangent flow cannot be $S^2 \times \mathbb{R}^2$ either. Thus, by Propositions 3.1 and 3.2 of [BCDMZ22], the unique tangent flow at infinity of (M_∞, g_∞) must be $(S^3/\Gamma) \times \mathbb{R}$, for some finite group Γ.

Thus, by definition of Cheeger-Gromov convergence, we have an exhaustion $\{U_j\}$ of (M_∞) and maps $\Phi_j : U_j \to M$ which are diffeomorphisms onto their images, with pointwise convergence $\Phi_j \to \mathrm{Id}_{M_\infty}$ and smooth convergence of the pullback metrics $\Phi_j^* g_j \to g_\infty$. Then, for any $\epsilon > 0$, there exists a $j \equiv j(\epsilon)$ such that on $B_{g_j}(x_j, 1/\epsilon)$, we have

$$\|(\Phi_j^{-1})^* g_\infty - g_j\|_{C^{[1/\epsilon]}} \leq \epsilon. \tag{5.6}$$

Now, since the tangent flow at infinity of (M, g) is $(S^2 \times \mathbb{R}^2, g_{\mathrm{cyl}})$, and since the flow of the vector field $-\nabla f$ generates the diffeomorphisms $\Psi_{-t} : M \to M$, for each x_j we can find a $t_j > 0$ such that $\Psi_{-t_j}(x_j)$ is in the region $N_\epsilon \subset M$ of points that are centers of ϵ-necks for $S^2 \times \mathbb{R}^2$. Let t'_j be the first $t > 0$ such that $\sigma_j(t'_j) = \Psi_{-t'_j}(x_j) := y_j$ is in the boundary region ∂N_ϵ. Then, by Assumption A, we have the inequality

$$R(y_j) \geqslant c_0(\epsilon) \tag{5.7}$$

Since σ_j is the integral curve of $-\nabla f$ starting at x_j, by Lemma 5.0.4, we know that for all j, we have

$$R(y_j) \leqslant \frac{1}{ct'_j} \tag{5.8}$$

We then have

$$d(x_j, N_\epsilon) \leqslant d(x_j, y_j) = \left| \int_0^{-t'_j} |\nabla f|_g \big(\Psi_s(x_j)\big)\, ds \right| \leqslant t'_j$$

where the last inequality follows as $|\nabla f|_g \leqslant 1$, by the soliton identity $R + |\nabla f|^2 = 1$. Thus, the sequence t'_j is unbounded, so there exists points $y_j \in \partial N_\epsilon$ of arbitrarily low scalar curvature by Equation (5.8), which contradicts Equation (5.7). Thus, such a sequence of points $\{x_j\}$ cannot exist, implying that the conclusion of the Lemma is true. $\square$

Thus, analogous to Proposition 3.8 [BCDMZ22], we have understood the structure of asymptotically 2-cylindrical steady soliton singularity models by constructing the analogous set N_ϵ. Unlike the 3-cylindrical case, the complement $M - N_\epsilon$ is not compact, however, in the previous lemma, we show that no point is very far from N_ϵ. This will be useful in proving the general expander degree result mentioned earlier in the section.

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