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Los Angeles

Integrability and Hidden Symmetries of Root Kerr

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy

in Physics

by

Christopher de Firmian

2026

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ABSTRACT OF THE DISSERTATION

Integrability and Hidden Symmetries of Root Kerr

by

Christopher de Firmian

Doctor of Philosophy in Physics

University of California, Los Angeles, 2026

Professor Zvi Bern, Chair

This dissertation investigates the dynamics of a charged, spinning probe governed by the Mathisson-Papapetrou-Dixon equations in the Root Kerr background: The electromagnetic spinning-disk solution obtained from the $G \to 0$ limit of Kerr-Newman. Root Kerr is related by the classical double copy to the Kerr spacetime, providing an electromagnetic analogue of this gravitational solution. I find generalizations of the "hidden" Carter and Rüdiger constants, showing the probe motion to be Liouville integrable through quadratic order in the probe spin. Their existence uniquely constrains the probe's multipole moments to those of the stationary Root Kerr field and requires its leading-order dynamical moments to vanish. These results further determine the corresponding classical Compton amplitude through spin squared, exhibiting spin exponentiation and agreeing with the classical limit of a minimally coupled Compton, derived through little group scaling and amplitude factorization. The failure of integrability beyond quadratic spin order indicates that hidden symmetries cannot provide a general principle for determining higher-order dynamical multipole moments or the associated Compton amplitude.

The dissertation of Christopher Steen de Firmian is approved.

Michael Gutperle

Mikhail Solon

Per Kraus

Zvi Bern, Committee Chair

University of California, Los Angeles

2026

DEDICATION

Throughout my Ph.D. I've felt beyond grateful for my supportive parents, Christine & Nick, as well as my family Bruce, Julie, Annick, John, and many others. I've been incredibly fortunate to meet and befriend mentors Trevor, Justin, Lukas, and Rachel, whose understanding, patience, and compassion I will aspire to for life. To Trevor and Justin, you are incredible and words cannot express how much I appreciate having met you. To Sandra, Madox, Joe, Aly, JuliAnne, and Trinity: Thank you; knowing that many of the struggles of the Ph.D. stage of life were not just my own, and the kinship that you offered me, helped me push through. To Stephanie, Christina, Brenda, Karina: Thank you for helping me navigate such a challenging part of life.

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Part of chapter 2 follows unpublished notes, *Introduction to the Mathisson-Papapetrou-Dixon Equations of Motion for Extended Objects in General Relativity*, unpublished, Scheopner, T. and Lichtner, F., 2024.

Chapter 5.2 contains a breakdown of a custom code, privately shared with me by Justin Vines and Jan Steinhoff.

Parts of chapters 4 and 6 follow the work of *Generalized Carter & Rüdiger Constants of* $\sqrt{\text{Kerr}}$, JHEP **05** (2026): 055, de Firmian, C. and Vines, J., 2026, for which the dissertation author was the primary investigator and author.

I am also grateful for the Mani L. Bhaumik Institute for Theoretical Physics and their support over these many years, and the U.S. Department of Energy's support, under award number DE-SC0009937.

VITA

Christopher de Firmian worked as a graduate student instructor and researcher at the University of California, Los Angeles, from 2020 through 2026. He initially worked in the field of celestial amplitudes, and pivoted to studying Root Kerr using the worldline formalism in fall of 2023. At the time of writing, he holds the following degrees:

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PUBLICATIONS

Christopher de Firmian and Justin Vines. "Generalized Carter & Rüdiger constants of $\sqrt{\text{Kerr}}$". In: *JHEP* 05 (2026), p. 055. DOI: 10.1007/JHEP05(2026)055. arXiv: 2602.18790 [gr-qc]

1 Introduction

With the groundbreaking detection of gravitational waves marking a new era of observational astrophysics [2, 3], high precision measurements of compact astrophysical objects, such as black holes and neutron stars, have compelled significant theoretical progress in our understanding of classical two-body dynamics. In an effort to extend predictions beyond geodesic motion, amplitude- and worldline-based approaches have pushed the envelope in modeling real-world spin effects, which will play an increasingly important role as gravitational wave detectors become more sensitive [4–6]. The study of generic spinning bodies in general relativity traces back to the 1950s [7–13], with many successful field theoretic and worldline-based approaches having since been developed to study spinning bodies in both the post-Newtonian (PN) approximation [14–67] and the post-Minkowskian (PM) approximation [68–108].

To model binary systems with spin, we employ the worldline formalism [11–13] to describe one body as a probe or *test body* in the background field of the other. Without spin, the probe's motion would be geodesic, however we can introduce deviation therefrom by modifying the geodesic action to include spin degrees of freedom. In the action, translational and rotational degrees of freedom, collectively referred to as *minimal*, can be coupled to the background field in order to construct generic higher dimensional operators. As when working in effective field theories, one hierarchically organizes such operators, whether in powers of spin, coupling, or other perturbative parameters, to select only the *relevant* interactions. By weighing each operator in the action by Wilson coefficients, one parameterizes families of probes, and the variation thereof yields effective equations of motion of a generic, minimal probe. The equations of motion resulting from varying said action are called the Mathisson-Papapetrou-Dixon (MPD) equations[7, 8, 10–13].

Complementary to the worldline formalism, are effective quantum field theories (QFTs), whose scattering amplitudes have been extensively studied and applied to gauge theories and gravitational systems. Central to this story is the tree-level electromagnetic Compton amplitude, governing the leading order behavior of a massive, charged object coupled to two external photons. A minimally coupled such Compton was derived from the amplitude principles of little group scaling and factorization in [109], and was later taken to its classical limit in [94]. The minimally coupled Compton exhibits an spurious pole, a singularity of the amplitude that does not correspond to any of the amplitude's factorization channels, for the massive particle's spin being greater than 1. However, upon taking its classical limit, the resulting classical minimal Compton exhibits the same spurious pole starting at $O(s^3)$, where s is the body's spin. With this classical limit giving rise to an inevitable spurious pole at $O(s^3)$ it is natural to wonder whether a sensible continuation of such an amplitude exists.

In contrast, using the worldline formalism to describe an effective, generic body coupled to electromagnetism, its classical Compton amplitudes were computed for generic Wilson coefficients in [110] through cubic order in spin. In the minimal Compton amplitude, such coefficients were fixed by factorization and little group scaling, however, in the latter case, despite it having no spurious poles, we need additional input in order to fix the Wilson coefficients. A number of these Wilson coefficients can be matched to the stationary field's multipole expansion of the body in question. Another set of Wilson coefficients, corresponding to so-called dynamical multipole moments, characterizing the body's response to external electromagnetic perturbations, are also necessary for fixing the classical Compton. Determination of the dynamical moments is, naturally predicated upon studying a system's response to an external perturbation, something which has been a subject of study in black hole perturbation theory, vis a vis Teukolsky-based methods [111]. The central object of study in this dissertation, called Root Kerr or $\sqrt{\text{Kerr}}$, is an electromagnetic charge configuration that can best be understood as the $G \to 0$ limit of the Kerr-Newman spacetime — the solution to Einstein's equations of a charged and spinning black hole. While turning off gravity by setting Newton's constant to zero returns this spacetime to flat, the residual gauge field from the black hole's charge remains. This field, and its associated compact charge and current distribution, is what's referred to as $\sqrt{\text{Kerr}}$, though we will mainly be referring to the gauge field and, equivalently, it's multipole moments. Though this may seem like a strange object to study, its dynamics prove analogous in many ways to that of the Kerr solution, in the sense of double copy.

Double copy, a term originally coined in the context of scattering amplitudes, usually refers to a prescription for combining amplitudes of, typically, simpler quantum field theories to produce those of complicated theories. As these relationships between asymptotic observables in quantum field theories emerged, it invited the question as to whether similar relationships between classical field configurations in the spacetime bulk exist. These so-called classical double copy relationships generally relate special solutions to Einstein's equations, those whose metrics can be written in Kerr-Schild form, to bulk gauge field configurations [112]. The classical double copy nomenclature follows from that the "double copy" metrics can be written in terms of dyads of the "single copy" gauge fields. It is due to these similarities that the single copies often serve as illuminating toy models for their double copy counterparts.

With our desire to study $\sqrt{\text{Kerr}}$ as a proxy for Kerr, we would like to determine the multipole moments which govern its dynamics. Beyond its stationary moments, however, studying its dynamical moments would require separating the Teukolsky equation in the Kerr-Newman spacetime, which is currently beyond the state of the art. Instead, we sought a different principle for determining the $\sqrt{\text{Kerr}}$ dynamical multipole moments. In [113] the authors studied the behavior of a generic, spinning, gravitating body in a background Kerr spacetime, and found that the probe's motion is only Liouville integrable, through $O(s^2)$, if the probe's multipole moments are those of a Kerr black hole. Thus, using the principle of Liouville integrability, this dissertation outlines the work done in finding conserved quantities for a generic, spinning probe coupled electromagnetically to a $\sqrt{\text{Kerr}}$ background. Analogously, we find that the motion of the probe is integrable only if its stationary multipole moments are those of $\sqrt{\text{Kerr}}$, and predict that the $\sqrt{\text{Kerr}}$, and in turn Kerr-Newman solution's, leading order response to an external electromagnetic perturbation vanishes. We also find that this integrability fails to persist beyond quadratic order in the probe's spin, a result verified by the asymptotic integrability community [114], hence the dynamical moments at cubic order in spin remain undetermined. Using the multipole moments from integrability through quadratic order in spin, we make a classical Compton amplitude for the $\sqrt{\text{Kerr}}$ tree-level scattering. This result agrees with the classical limit of the minimal Compton of [109]. While classical integrability does not serve as a correct guiding principle for higher order multipole moments, its failure at quadratic order in electromagnetism and quartic order in the gravitational double copy [115] hint at some deeper physical underpinnings that have yet to be understood about binary systems.

The structure of this dissertation is as follows. Chapter 2 is an overview of the worldline formalism, including derivations of the action, properties, and equations of motion[1, 116]. This chapter also includes the electromagnetic dynamical mass function used in finding the central result, summarized in chapter 6, of this work.

Chapter 3 is an overview of standard techniques used in constructing tree-level scattering amplitudes[117–119], with an emphasis on spinor helicity variables, their properties and a discussion of the modern tools of their computation. We also discuss the minimal Compton amplitude of [109] along with its classical limit from [94].

Chapter 4 discusses the central character of this dissertation: The Root Kerr electromagnetic field [1]. It outlines the role of Root Kerr as a single copy of Kerr and the Kerr-Schild form[120, 121], its derivation from both the Janis-Newman shift[122] and as a $G \to 0$ limit of the Kerr-Newman solution. All the necessary tensors and their relations pertaining to Root Kerr as a background field are defined herein.

Chapter 5 is an in-depth instruction manual for the Wolfram Mathematica code that underpins the Carter and Rüdiger conservation computation. It includes brief summaries of the xTensor, xPerm and xCoba packages, and their specific uses and custom code. Chapter 5 also details a number of the technical challenges pertaining to the automation of the computation. Section 5.7 importantly includes both the $O(s^2)$ and $O(s^3)$ ansätze.

Chapter 6 includes the generalized Carter and Rüdiger constants through $O(s^2)$, their implications for the classical Compton amplitude, and discusses the lack of integrability beyond quadratic order in spin of $\sqrt{\text{Kerr}}$ [1].

2 Worldline Formalism

2.1 Worldline Formalism from Action Principle

We seek to describe the dynamics of a body under the influence of gravity. For a generic, compact object evolving gravitationally, given a compactly supported, symmetric, and covariantly conserved energy-momentum[1] tensor $T^{\mu\nu}$, it was shown in [11–13]:

- It is possible to assign a worldline $z^{\mu}(\lambda)$ to the body, such that $z^{\mu}(\lambda)$ lies within the body, for all λ.
- It is possible to uniquely define the total linear momentum $p^{\mu}(\lambda)$ and total angular momentum $S^{\mu\nu}(\lambda)$ for any given worldline.
- The linear momentum $p^{\mu}(\lambda)$ satisfies $p^{\mu}p_{\mu} < 0$ and $p^{\mu}\dot{z} < 0$.
- While the choice of $z^{\mu}(\lambda)$ is not unique, there exists a unique choice of worldline for which $S^{\mu\nu}p_{\nu} = 0$. This choice corresponds to $z^{\mu}(\lambda)$ tracing the object's center of mass, and the constraint is referred to as the "spin supplementary condition".
- Along a particular worldline, p^{μ} and $S^{\mu\nu}$ satisfy the Mathisson-Papapetrou-Dixon (MPD) equations.
- The MPD equations depend on higher multipole moments of $T^{\mu\nu}$, and their evolution is fully determined by $\nabla_{\nu}T^{\mu\nu} = 0$.

In contrast to the work of Dixon, this chapter will follow the derivations of [116] in detailing how the worldline formalism and its equations of motion can, in quicker and arguably more familiar fashion, be derived from an action principle.

2.1.1 Degrees of Freedom

We begin by describing a compact body's trajectory in space by the degrees of freedom $z^{i}(t)$, for $i = 1, 2, 3$, where t is a time-like coordinate in whichever coordinate system one chooses.

[1] $T\mu\nu$ is subject to specific conditions.

Notably, this 3-component spatial construction is not relativistically covariant as is, so we, in order to make covariance manifest, introduce a zeroth component, $z^0(\lambda) = t$, in addition to the spatial $z^i(t)$, such that z^μ for $\mu = 0, 1, 2, 3$ repackages 3 degrees of freedom for the purpose of covariance. The catch, however, is that such repackaging requires the worldline to exhibit reparameterization invariance — the introduction of an additional degree of freedom is counterbalanced by the introduction of a gauge redundancy.

Additionally, a body may have rotational degrees of freedom, which we can covariantly keep track of using a tetrad. For each parameter value λ we assign 4 orthonormal vectors $\Lambda^\mu{}_A$, where μ is the vector index and A labels the vector in question. By orthonormality, we require that

$$g_{\mu\nu}\Lambda^\mu{}_A\Lambda^\nu{}_B = \eta_{AB}. \tag{1}$$

We define this tetrad such that the first vector, $\Lambda^\mu{}_0$ is a timelike, future-oriented vector, and the remaining three, $\Lambda^\mu{}_i$ as spatial vectors keeping track of the body's orientation. While $\Lambda^\mu{}_A$ has 16 components, 10 are fixed by orthonormality conditions, 3 by the choice of $\Lambda^\mu{}_0$, and the remaining 3 are the body's rotational degrees of freedom.

2.1.2 The Action and Reparameterization Invariance

Before writing an action, it is easier to, temporarily, repackage our spin degrees of freedom into a 1 index object, ϕ^a for $a = 1, 2, 3$, to shortcut some technicalities regarding the tetrad $\Lambda^\mu{}_A$. One can think of the ϕ^a as keeping track of the body's orientation using, say, 3 Euler angles or any other parameterization of choice. Now, with our 6 physical degrees of freedom, we define a worldline action

$$S[z, \phi] := \int d\lambda \mathcal{L}(z, \dot{z}, \phi, \dot{\phi}) \tag{2}$$

with

$$\dot{z}^\mu := \frac{dz^\mu}{d\lambda}, \qquad \dot{\phi}^a := \frac{d\phi^a}{d\lambda}. \tag{3}$$

The worldline parameter λ can be arbitrarily redefined to a new parameter λ' as some monotonically increasing function of λ, $\lambda'(\lambda)$, while keeping the worldline parameters unchanged

$$z'^{\mu}(\lambda') = z^{\mu}(\lambda), \qquad \phi'(\lambda') = \phi(\lambda). \tag{4}$$

This merely amounts to changing the labels of the events along the worldline, while, importantly, preserving their order. The caveat is that the variables $\dot{z}$ and $\dot{\Lambda}$, along with the measure $d\lambda$ incur Jacobians under re-parameterization, $\lambda \to \lambda'(\lambda)$, as

$$\dot{z}' = \frac{d\lambda}{d\lambda'}\dot{z}, \qquad \dot{\phi}' = \frac{d\lambda}{d\lambda'}\dot{\phi}, \qquad d\lambda = \frac{d\lambda}{d\lambda'}d\lambda'. \tag{5}$$

As shown in [116], one can enforce the reparameterization invariance of the action by introducing an einbein, $\alpha(\lambda)$ — a scalar worldline quantity defined for all λ — such that our action becomes

$$S[\alpha, z, \phi] := \int \mathcal{L}\left(z, \frac{\dot{z}}{\alpha}, \phi, \frac{\dot{\phi}}{\alpha}\right) \alpha d\lambda. \tag{6}$$

Its presence in the denominators of $\dot{z}$ and $\dot{\Lambda}$, as well as multiplying $d\lambda$, is to utilize its transformation property

$$\alpha(\lambda) = \frac{d\lambda'}{d\lambda}\alpha'(\lambda') \tag{7}$$

to cancel the Jacobians incurred by $\dot{z}$, $\dot{\Lambda}$, and $d\lambda$, leaving the action manifestly re-parameterization invariant.

Introducing conjugate variables as

$$p_{\mu} = \frac{\partial \mathcal{L}\left(z, \dot{z}/\alpha, \phi, \dot{\phi}/\alpha\right)}{\partial\left(\dot{z}^{\mu}/\alpha\right)}, \qquad \pi_{a} = \frac{\partial \mathcal{L}\left(z, \dot{z}/\alpha, \phi, \dot{\phi}/\alpha\right)}{\partial\left(\dot{\phi}^{a}/\alpha\right)}, \tag{8}$$

which are manifestly re-parameterization invariant since they're built from re-parameterization invariant quantities, we can define the Dirac Hamiltonian through the Legendre transformation

$$H(z, p, \phi, \pi) = \frac{\pi_a \dot{\phi}^a}{\alpha} + \frac{p_{\mu}\dot{z}^{\mu}}{\alpha} - \mathcal{L}\left(z, \frac{\dot{z}}{\alpha}, \phi, \frac{\dot{\phi}}{\alpha}\right). \tag{9}$$

Thus our action becomes

$$S[\alpha, z, p, \phi, \pi] = \int \left(\phi_a \dot{\pi}^a + p_\mu \dot{z}^\mu - \alpha H\left(z, p, \phi, \pi\right)\right) d\lambda, \tag{10}$$

where α's variation imposes reparameterization invariance and the so-called on-shell constraint:

$$\frac{\delta S}{\delta \alpha} = -H(z, p, \phi, \pi) = 0. \tag{11}$$

2.1.3 The Dynamical Mass Function

For a free particle, its Dirac Hamiltonian is given by

$$H_{\text{free}} = m - |p|, \tag{12}$$

however we can introduce additional degrees of freedom by considering the difference, δH, between the free Dirac Hamiltonian, H_{free} and the full Dirac Hamiltonian, H:

$$\delta H := H - H_{\text{free}} \Longrightarrow H(z, p, \phi, \pi) = -|p| + m + \delta H(z, p, \phi, \pi). \tag{13}$$

The on-shell constraint thus becomes

$$|p| = m + \delta H(z, p, \phi, \pi) = m + \delta H(z, u|p|, \phi, \pi), \tag{14}$$

where $u^\mu = p^\mu/|p|$. The solution for $|p|$ from equation 14 can be expressed $|p|$ in terms of z, u, ϕ, and π

$$|p| = \mathcal{M}(z, u, \phi, \pi) \tag{15}$$

where we refer to $\mathcal{M}$ as the dynamical mass function. Thus, we can write the full Dirac Hamiltonian as

$$H(z, p, \phi, \pi) = -|p| + \mathcal{M}(z, u, \phi, \pi), \tag{16}$$

which in turn allows us to write the corresponding action as

$$S[\alpha, z, p, \phi, \pi] = \int \left(p_\mu \dot{z}^\mu + \pi_a \dot{\phi}^a + \alpha\left(|p| - \mathcal{M}(z, u, \phi, \pi)\right)\right) d\lambda. \tag{17}$$

The ethos of trying to re-write our action in terms of the Dirac Hamiltonian is because it is automatically re-parameterization invariant and it is easier to construct from physical principles for the purposes of effectively describing a body. With an emphasis on constructing physical quantities, we would like for the Dirac Hamiltonian, and by extension the dynamical mass function, to be gauge invariant. As an example, suppose we wanted to couple the system to an electromagnetic gauge field, A_μ: The momentum truly conjugate to our worldline z would be the combination $p_\mu - qA_\mu$, which is not gauge invariant. With this momentum entering arguments of H and $\mathcal{M}$, this undermines our efforts to make these constructs physical. As a fix, we can shift our conjugate momentum by qA_μ, such that H and M depend only on the kinetic momentum p, thereby making them gauge invariant, albeit at the expense of shifting the $p_\mu \dot{z}^\mu$ term to $\left(p_\mu + qA_\mu\right)\dot{z}^\mu$. The presence of this gauge field outside $\mathcal{M}$ is merely a minor inconvenience for our purposes, and, in fact, we'll refer little to the gauge field A_μ that henceforth the term *momentum* shall refer to the kinetic momentum p^μ, rather than the full conjugate momentum.

2.1.4 Spin Degrees of Freedom and the Spin Supplementary Condition

As discussed, the rotational motion of the body are encoded in $\Lambda^\mu{}_A(\lambda)$, where we identify $\Lambda^0{}_A(\lambda)$ as the time direction. In particular, in the center of momentum frame, we can take $\Lambda^0{}_A(\lambda) = u^0$, an often convenient choice for computations. The set of rotations, $SO(3)$, that preserve this time-like direction can be written as

$$R^\mu{}_\nu = \exp\left[\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\phi_\mu(\lambda)u_\nu(\lambda)J_{\rho\sigma}\right]^\mu{}_\nu \tag{18}$$

where $J_{\rho\sigma}$ are the Lorentz generators

$$\left(J_{\alpha\beta}\right)^\mu{}_\nu = \delta^\mu{}_\nu\eta_{\alpha\nu} - \delta^\mu{}_\alpha\eta_{\beta\nu} \tag{19}$$

and $\phi_\rho(\lambda)$ are the physical degrees of freedom which parameterize $\Lambda^\mu{}_A(\lambda)$. Moreover, since the rotation parameterized by ϕ_μ leaves u invariant, we can take $\phi \cdot u = 0$. Given this, under a Lorentz boost

$$L^\mu{}_\nu = \exp\left[\theta^{\alpha\beta} J_{\alpha\beta}\right]^\mu{}_\mu \tag{20}$$

parameterized by $\theta^{\mu\nu}$, the Lorentz boost's action on $u^\mu \longrightarrow L^\mu{}_\nu u^\nu$ must duly be compensated by the induced transformation $\phi_\mu \longrightarrow \phi'_\mu$ such that the orthogonality condition $u \cdot \phi = 0$ is preserved in both frames. One can show that this implies the constraint

$$u^\mu \frac{d\phi_\rho}{d\theta^{\mu\nu}}\Big|_{\theta=0} = 0, \tag{21}$$

where the evaluation at $\theta = 0$ is a technical detail pertaining to the fact that the parameters $\theta^{\mu\nu}$ are considered in an infinitesimal neighborhood about the body's default orientation.

If we now take the worldline derivative of ϕ, and employ the covariant λ-derivative,

$$\frac{D}{D\lambda} = \frac{d}{d\lambda} + \dot{z}^\mu \nabla_\mu, \tag{22}$$

we can compute

$$\frac{d\phi_\rho}{d\lambda} = \frac{d\phi_\rho}{d\theta^{\mu\nu}} \frac{d\theta^{\mu\nu}}{d\lambda}. \tag{23}$$

Noting that the angular velocity tensor is given by

$$\Omega^{\mu\nu} = \frac{d\theta^{\mu\nu}}{d\lambda} = \eta^{AB} \Lambda^\mu{}_A \frac{D\Lambda^\nu{}_B}{D\lambda} \tag{24}$$

we have

$$\frac{d\phi_\rho}{d\lambda} = \frac{1}{2} \Omega^{\mu\nu} \frac{d\phi_\rho}{d\theta^{\mu\nu}}\Big|_{\theta=0}, \tag{25}$$

where the $\frac{1}{2}$ factor comes from the double-counting as a result of the anti-symmetric tensor contraction with another antisymmetric tensor.

Noether's Theorem tells us that the conserved charge generating an infinitesimal transformation is given by the canonical momentum, π_a, times the variation of the coordinate ϕ^a with respect to the transformation parameter:

$$S_{\mu\nu} = \pi_a \frac{d\phi^a}{d\theta^{\mu\nu}}\bigg|_{\theta=0}. \tag{26}$$

With the definition of the angular momentum tensor, $S_{\mu\nu}$, we can, using equation 21, arrive at the so-called spin supplementary condition (SSC):

$$u^\nu S_{\mu\nu} = \pi^a u^\nu \frac{d\phi_a}{d\theta^{\mu\nu}}\bigg|_{\theta=0} = 0, \tag{27}$$

which can be equivalently recast in terms of momentum $p^\mu S_{\mu\nu} = 0$.

Lastly, using our newly defined $S_{\mu\nu}$ and $\Omega^{\mu\nu}$, we seek to replace instances of ϕ and π in favor thereof. Since we have yet to build and specify a particular form of the dynamical mass function $\mathcal{M}$, the only remaining reference to π and ϕ is in the term $\pi_a \dot{\phi}_a$. Observe that this term

$$\pi_a \dot{\phi}_a = \pi_a \frac{d\phi^a}{d\lambda} = \pi_a \frac{d\phi^a}{d\theta^{\mu\nu}} \frac{d\theta^{\mu\nu}}{d\lambda} = \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu}, \tag{28}$$

thus our final action becomes

$$S[\alpha, z, p, S, \Omega] = \int \left((p_\mu + qA_\mu)\dot{z}^\mu + \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + \alpha\left(|p| - \mathcal{M}(z, u, \phi, \pi)\right) + \beta_\mu S^{\mu\nu} p_\nu \right) d\lambda, \tag{29}$$

where we explicitly include the spin supplementary condition $S^{\mu\nu} p_\nu = 0$ enforced by the Lagrange multiplier β_μ. Ordinarily, one ought also include the constraint of $\Lambda^\mu{}_0 = u^\mu$ with its own Lagrange multiplier, but it happens that, subject to the resulting equations of motion, that should $\Lambda^\mu{}_0 = u^\mu$ hold for the initial conditions then it necessarily holds for all times — a, perhaps, physically unsurprising result since the torques experienced by the body should only mix spatial components.

2.2 The Electromagnetic MPD Equations

So, in summary of the previous section, we arrive at the following: Specializing to flat spacetime, we describe a charged, extended body, with translational, $z^\mu(\tau)$, and rotational, $\Lambda^\mu{}_A(\tau)$, degrees of freedom, coupled to electromagnetism, $A_\mu(x)$ with the following action [13, 49, 77, 123–127]:

$$\mathcal{S}[z,p,e,S,A] = \int d\tau \left[(p_\mu + qA_\mu)\dot{z}^\mu + \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} - \frac{\alpha}{2}(p^2 + \mathcal{M}^2) - \beta^\mu S_{\mu\nu}p^\nu \right] \\ - \frac{1}{16\pi}\int d^4x F_{\mu\nu}F^{\mu\nu}. \tag{30}$$

Here $\Omega^{\mu\nu} = \Lambda^{\mu A}\dot{\Lambda}^\nu{}_A$ is the angular velocity tensor, $S_{\mu\nu}$ is the angular momentum spin tensor, and p^μ the momentum. The electromagnetic field strength tensor is the usual $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, and

$$\mathcal{M}^2 = \mathcal{M}^2(p,S,z)[F] = \mathcal{M}^2\left(p_\mu, S_{\mu\nu}, \{F_{\mu\nu,N}(z)\}_{N=0}^\infty\right) \tag{31}$$

is the so-called dynamical mass function squared. $F_{\mu\nu,N}(z) = F_{\mu\nu,\rho_1\ldots\rho_N} = \partial_{\rho_1}\ldots\partial_{\rho_N}F_{\mu\nu}$ denotes repeated partial differentiation of the field strength tensor, and α and β^μ serve as Lagrange multipliers to enforce the on shell constraint $p^2 = -\mathcal{M}^2$ and the spin supplementary condition $S_{\mu\nu}p^\nu = 0$, respectively. The total variation, neglecting boundary terms, is

$$\delta\mathcal{S} = \int d\tau \Bigg\{ \left[-\frac{d}{d\tau}(p_\mu + qA_\mu) + q\dot{z}^\nu\partial_\mu A_\nu - \frac{\alpha}{2}\frac{\partial\mathcal{M}^2}{\partial z^\mu} \right]\delta z^\mu + \left[\dot{z}^\mu - \alpha p^\mu - \frac{\alpha}{2}\frac{\partial\mathcal{M}^2}{\partial p_\mu} - \beta^\nu S_\nu{}^\mu \right]\delta p_\mu \\ + \frac{1}{2}\left[\dot{S}_{\mu\nu} - 2S_{\sigma[\mu}\Omega_{\nu]}{}^\sigma \right]\Lambda^{[\mu}{}_A\delta\Lambda^{\nu]A} + \frac{1}{2}\left[\Omega^{\mu\nu} - \alpha\frac{\partial\mathcal{M}^2}{\partial S_{\mu\nu}} - 2\beta^{[\mu}p^{\nu]} \right]\delta S_{\mu\nu} \Bigg\} \\ + \int d^4x \left[\frac{1}{4\pi}\partial_\nu F^{\nu\mu} + \int d\tau \left(q\dot{z}^\mu\delta^4(x-z) - \frac{\alpha}{2}\frac{\delta\mathcal{M}^2}{\delta A_\mu(x)} \right) \right]\delta A_\mu, \tag{32}$$

where the variation of the frame is orthogonal: $\delta\Lambda^\mu{}_A = \Lambda^\alpha{}_A\delta\theta_\alpha{}^\mu$, with $\delta\theta^{\mu\nu} = -\delta\theta^{\nu\mu}$ to ensure orthogonality of the resulting rotations. We can further eliminate reference to $\Omega^{\mu\nu}$, $\Lambda^\mu{}_A$, and

β^μ, along with solving algebraically for

$$\alpha = p \cdot \dot{z} \left(p^2 + \frac{p^\mu}{2} \frac{\partial \mathcal{M}^2}{\partial p^\mu} \right)^{-1}, \tag{33}$$

such that the transport equations for p_μ and $S_{\mu\nu}$ become

$$\dot{p}_\mu - qF_{\mu\nu}\dot{z}^\nu = \mathcal{F}_\mu = -\frac{\alpha}{2} \frac{\partial}{\partial z^\mu} \mathcal{M}^2 \tag{34}$$

$$\dot{S}_{\mu\nu} - 2p_{[\mu}\dot{z}_{\nu]} = \mathcal{N}_{\mu\nu} = -\alpha \left(p_{[\mu} \frac{\partial}{\partial p^{\nu]}} + 2S_{[\mu}{}^{\rho} \frac{\partial}{\partial S^{\nu]\rho}} \right) \mathcal{M}^2, \tag{35}$$

subject to the spin supplementary condition $p_\mu S^{\mu\nu} = 0$. We can further perform a change of variables by defining the momentum direction u^μ and the spin vector s^μ as

$$u^\mu = \frac{p^\mu}{\sqrt{-p^2}}, \quad s^\mu = -\frac{1}{2} \epsilon^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma}, \tag{36}$$

which in turn implies

$$S^{\mu\nu} = \epsilon^{\mu\nu}{}_{\rho\sigma} u^\rho s^\sigma. \tag{37}$$

Using our new variables, we take $\mathcal{M}^2(p, S, z) \to \mathcal{M}^2(u, s, z)$, i.e., we trade the spin tensor S for the spin vector s:

$$\left(\frac{\partial}{\partial p_\mu} \right)_S \mathcal{M}^2(p, s(p, S)) = \left[\left(\frac{\partial}{\partial p_\mu} \right)_s + \left(\frac{\partial s^\nu}{\partial p_\mu} \right)_S \left(\frac{\partial}{\partial s^\nu} \right)_p \right] \mathcal{M}^2(p, s), \tag{38}$$

$$\left(\frac{\partial}{\partial S_{\mu\nu}} \right)_p \mathcal{M}^2(p, s(p, S)) = \left(\frac{\partial s^\rho}{S_{\mu\nu}} \right)_p \left(\frac{\partial}{\partial s^\rho} \right)_p \mathcal{M}^2(p, s). \tag{39}$$

Moreover, defining the orthogonal projector to u, $\mathcal{P}$,

$$\eta^{\mu\nu} = \frac{\partial p^\mu}{\partial p_\nu} = \frac{\partial}{\partial p_\nu} (\mathcal{M} u^\mu) = \mathcal{M} \frac{\partial u^\mu}{\partial p_\nu} + u^\mu \frac{\partial \mathcal{M}}{\partial p_\nu} \tag{40}$$

$$\Longrightarrow \mathcal{M} \frac{\partial u^\mu}{\partial p_\nu} = \eta^{\mu\nu} - u^\mu \frac{\partial \sqrt{-p^2}}{\partial p_\nu} = \eta^{\mu\nu} - u^\mu \frac{-1}{2\mathcal{M}} \frac{\partial p^2}{\partial p_\nu} = \eta^{\mu\nu} + u^\mu u^\nu =: \mathcal{P}^{\mu\nu}, \tag{41}$$

we can take $\mathcal{M}^2$ to depend on p only through its direction u,

$$\frac{\partial \mathcal{M}^2}{\partial p^\mu} = \frac{\partial u^\nu}{\partial p^\mu}\frac{\partial \mathcal{M}^2}{\partial u^\nu} = \mathcal{P}^\nu{}_\mu \frac{1}{\mathcal{M}}\frac{\partial \mathcal{M}^2}{\partial u^\nu} = \mathcal{P}^\nu{}_\mu 2\frac{\partial \mathcal{M}}{\partial u^\nu} \tag{42}$$

in which case

$$p^\mu \frac{\partial \mathcal{M}^2}{\partial p^\mu} = p^\mu \mathcal{P}^\nu{}_\mu 2\frac{\partial \mathcal{M}}{\partial u^\nu} = (p^\nu - p^\nu)\, 2\frac{\partial \mathcal{M}}{\partial u^\nu} = 0 \Longrightarrow \alpha = \frac{-p^\mu \dot{z}_\mu}{-p^2} = -\frac{u^\mu \dot{z}_\mu}{\mathcal{M}}. \tag{43}$$

We can fix the worldline parameterization using Dixon's condition, $u\cdot\dot{z} = -1$ such that $\alpha = 1/\mathcal{M}$, and the resulting equations of motion become

$$\dot{p}_\mu - qF_{\mu\nu}\dot{z}^\nu = \mathcal{F}_\mu = -\frac{\alpha}{2}\frac{\partial}{\partial z^\mu}\mathcal{M}^2, \tag{44}$$

$$\dot{S}_{\mu\nu} - 2p_{[\mu}\dot{z}_{\nu]} = \mathcal{N}_{\mu\nu} = -\alpha\left(p_{[\mu}\frac{\partial}{\partial p^{\nu]}} + s_{[\mu}\frac{\partial}{\partial s^{\nu]}}\right)\mathcal{M}^2, \tag{45}$$

$$\dot{s}^\mu = \frac{\alpha}{2}\left(\epsilon^{\mu\nu}{}_{\rho\sigma}u^\rho s^\sigma \frac{\partial}{\partial s^\nu} - \frac{u^\mu}{\mathcal{M}}s^\nu\frac{\partial}{\partial z^\nu}\right)\mathcal{M}^2 + \frac{u^\mu}{\mathcal{M}}qF_{\nu\rho}s^\nu\dot{z}^\rho, \tag{46}$$

$$\dot{z}^\mu = u^\mu - \frac{\mathcal{N}^{\mu\nu}u_\nu}{\mathcal{M}} - S^{\mu\nu}\frac{\mathcal{F}_\nu + qF_{\nu\rho}(u^\rho - \mathcal{N}^{\rho\sigma}u_\sigma/\mathcal{M})}{\mathcal{M}^2 - \frac{1}{2}qF_{\alpha\beta}S^{\alpha\beta}}. \tag{47}$$

2.2.1 A General Dynamical Mass Function and Classical Compton

In [110] the authors constructed the most general electromagnetic dynamical mass function through third order in the probe spin:

$$\begin{aligned}
\mathcal{M}^2 = {} & m^2 + 2qC_1{}^\star F_{\mu\nu}s^\mu u^\nu + \frac{qC_2}{m}(s\cdot\partial)F_{\mu\nu}s^\mu u^\nu \\
& - \frac{qC_3}{3m^2}(s\cdot\partial)^{2\star}F_{\mu\nu}s^\mu u^\nu + \frac{qE_3}{3m^2}s^2(u\cdot\partial)^{2\star}F_{\mu\nu}s^\mu u^\nu \\
& + \frac{q^2D_{2a}}{m^2}\left(F_{\mu\nu}s^\mu u^\nu\right)^2 + \frac{q^2D_{2b}}{m^2}\left({}^\star F_{\mu\nu}s^\mu u^\nu\right)^2 + \frac{q^2D_{2c}}{m^2}s^2F^{\mu\nu}F_{\mu\nu} + \frac{q^2D_{2d}}{m^2}s^2u^\mu F_{\mu\nu}F^{\nu\rho}u_\rho \\
& + \frac{q^2D_{3a}}{m^3}{}_{\mu\nu}s^\mu u^\nu(s\cdot\partial)F_{\rho\sigma}s^\rho u^\sigma + \frac{q^2D_{3b}}{m^3}s^{2\star}F^{\mu\nu}u_\mu\partial_\nu F_{\rho\sigma}s^\rho u^\sigma \\
& + \frac{q^2D_{3c}}{m^3}s^{2\star}F^{\mu\nu}u_\mu(s\cdot\partial)F_{\nu\rho}u^\rho + \frac{q^2D_{3d}}{m^3}s^{2\star}F^{\mu\nu}s_\mu(u\cdot\partial)F_{\nu\rho}u^\rho \\
& + \frac{q^2D_{3e}}{m^3}{}^\star F^{\mu\nu}s_\mu(s\cdot\partial)F_{\nu\rho}s^\rho + \frac{q^2D_{3f}}{m^3}s^{2\star}F^{\mu\nu}(s\cdot\partial)F_{\mu\nu} + O\left(s^4\right),
\end{aligned} \tag{48}$$

where m is the non-dynamical mass of the probe, ${}^{\star}F$ denotes the dual of F, and C_i, E_i, & D_i are generic Wilson coefficients. While my work revolved around employing this dynamical mass function, along with the aforementioned electromagnetic MPD equations, to find dynamically conserved quantities, in [110] the authors use equation 48 and the MPD equations to compute the classical Compton amplitudes for these generic Wilson coefficients in $\mathcal{M}$ through third order in spin. The particular forms are rather lengthy, and will be presented in the Integrability Results chapter, once the coefficients have been fixed.

3 Helicity Amplitudes

Traditional approaches to calculating scattering amplitudes, involving the use of Feynman diagrams, quickly become computationally intractable due to the factorial growth of the number of diagrams. Additionally, much of the underlying physics was obscured by the diagram's lack of gauge invariance. The need for high precision scattering predictions from both collider experiments and, more recently, gravitational wave physics, necessitated the development of the so-called on-shell formalism. This formalism manifestly put the external momenta on-shell, satisfying their equation of motion, by expressing them in terms of spinor helicity variables. This chapter is a review of spinor helicity variables along with basic properties of tree-level scattering amplitudes. In this dissertation, this chapter serves the purpose of introducing the concepts of little-group scaling and factorization in the context of scattering amplitudes, as these are used in defining the minimal Compton amplitude.

3.1 On-Shell Formalism and Spinor Helicity Variables

To describe on-shell degrees of freedom, we elect to describe our $1 \oplus 0$ 4-Lorentz-vectors as $\frac{1}{2} \otimes \frac{1}{2}$ bi-$SL(2,\mathbb{C})$-spinors:

$$p^{\mu} \to p^{\dot{\alpha}\alpha} := \bar{\sigma}_{\mu}^{\dot{\alpha}\alpha} p^{\mu}, \tag{49}$$

where we define $(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} = (\mathbf{1}, -\sigma)$, with indices lowered as $\sigma^{\mu}_{\alpha\dot{\alpha}} := \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}$. Note the reversal of the order of the lower versus upper indices in this definition. The raising and lowering of $\mathrm{SL}(2,\mathbb{C})$ indices using this Levi-Civita convention corresponds to taking the inverse of the matrix components. Thus in this convention we have

$$\begin{gathered}
\sigma^{\mu}_{\alpha\dot{\alpha}} = (\mathbf{1}_{\alpha\dot{\alpha}}, \sigma_{\alpha\dot{\alpha}}) \qquad \sigma_{\mu\alpha\dot{\alpha}} = (\mathbf{1}_{\alpha\dot{\alpha}}, -\sigma_{\alpha\dot{\alpha}}) \\
\chi^{\alpha} = \epsilon^{\alpha\beta}\chi_{\beta} \qquad \bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}} \\
\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = \epsilon^{21} = \epsilon^{\dot{2}\dot{1}} = 1 \quad \epsilon_{21} = \epsilon_{\dot{2}\dot{1}} = \epsilon^{12} = \epsilon^{\dot{1}\dot{2}} = -1
\end{gathered}$$

It is convenient to note that

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_{\alpha}^{\ \gamma} \quad \epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\ \gamma}, \tag{50}$$

are SL$(2,\mathbb{C})$ invariant, since

$$\epsilon_{\alpha\beta} \longrightarrow M_\alpha{}^\gamma M_\beta{}^\delta \epsilon_{\gamma\delta} = \det(M)\,\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta} \tag{51}$$

$$\delta_\alpha{}^\beta \longrightarrow M_\alpha{}^\gamma M^\beta{}_\delta \delta_\gamma{}^\delta = M_\alpha{}^\gamma M^\beta{}_\gamma = M_\alpha{}^\gamma \left(M^{-1}\right)_\gamma{}^\beta = \delta_\alpha{}^\beta \tag{52}$$

Moreover we have the relations

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta} \quad \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} = 2\eta^{\mu\nu}, \tag{53}$$

from which we can express

$$p^{\dot\alpha\alpha}q^{\dot\beta\beta} = \bar\sigma_\mu^{\dot\alpha\alpha}p^\mu\bar\sigma_\nu^{\dot\beta\beta}q^\nu \tag{54}$$

$$\epsilon_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}p^{\dot\alpha\alpha}q^{\dot\beta\beta} = \underbrace{\epsilon_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}\bar\sigma_\mu^{\dot\alpha\alpha}\bar\sigma_\nu^{\dot\beta\beta}}_{2\eta_{\mu\nu}}p^\mu q^\nu = 2p\cdot q \tag{55}$$

The latter allows us to re-cast the on-shell condition as

$$m^2 = p^2 = \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}p^{\dot\alpha\alpha}p^{\dot\beta\beta} = \det(p^{\dot\alpha\alpha}). \tag{56}$$

Depending on the mass, we can express $p^{\dot\alpha\alpha}$ as

$$p^{\dot\alpha\alpha} = \sum_{I=1,2}\tilde\lambda_I^{\dot\alpha}\lambda^{\alpha I} \quad \text{for } m^2 > 0\text{, and} \tag{57}$$

$$p^{\dot\alpha\alpha} = \tilde\lambda^{\dot\alpha}\lambda^\alpha \quad \text{for } m^2 = 0, \tag{58}$$

where $\tilde\lambda_I \in (0,1/2)$ and $\lambda_I \in (1/2,0)$. Note that for $m^2 = 0$, $p^{\dot\alpha\alpha}$ is invariant under the transformation $\lambda \to z\lambda$ and $\tilde\lambda \to z^{-1}\tilde\lambda$, for $z \in \mathbb{C}\setminus\{0\}$, called the $U(1)$ *little group* transformation. For massive spinors, the little group action $\lambda^I \to \lambda^J L_J{}^I$, $\tilde\lambda_I \to \left(L^{-1}\right)_I{}^J\tilde\lambda_J$ is that of $SU(2)$.
We can invert these expressions

$$p^{\dot\alpha\alpha} := \bar\sigma_\mu^{\dot\alpha\alpha}p^\mu \tag{59}$$

$$\bar\sigma_\nu^{\dot\beta\beta}p^{\dot\alpha\alpha} = \bar\sigma_\nu^{\dot\beta\beta}\bar\sigma_\mu^{\dot\alpha\alpha}p^\mu \tag{60}$$

$$\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\bar{\sigma}_{\nu}^{\dot{\beta}\beta}p^{\dot{\alpha}\alpha} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\bar{\sigma}_{\nu}^{\dot{\beta}\beta}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}p^{\mu} \tag{61}$$

$$\bar{\sigma}_{\nu}^{\dot{\alpha}\alpha}p_{\dot{\alpha}\alpha} = 2\eta_{\mu\nu}p^{\mu} \quad \Longrightarrow p_{\mu} = \frac{1}{2}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}p_{\dot{\alpha}\alpha}, \tag{62}$$

and derive a reality condition on our bi-$SL(2,\mathbb{C})$ momentum:

$$p_{\mu} = (p_{\mu})^{*} \tag{63}$$

$$\frac{1}{2}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}p_{\dot{\alpha}\alpha} = \frac{1}{2}(\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha})^{*}(p_{\dot{\alpha}\alpha})^{*} \tag{64}$$

$$\frac{1}{2}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}p_{\dot{\alpha}\alpha} = \frac{1}{2}(\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha})^{*}(p_{\dot{\alpha}\alpha})^{*} \tag{65}$$

$$= \frac{1}{2}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}p^{*}_{\alpha\dot{\alpha}} \tag{66}$$

$$\Longrightarrow \frac{1}{2}\bar{\sigma}^{\mu\dot{\beta}\beta}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}p_{\dot{\alpha}\alpha} = \frac{1}{2}\bar{\sigma}^{\mu\dot{\beta}\beta}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha}p^{*}_{\alpha\dot{\alpha}} \tag{67}$$

$$\epsilon^{\dot{\beta}\dot{\alpha}}\epsilon^{\beta\alpha}p_{\dot{\alpha}\alpha} = \epsilon^{\dot{\beta}\dot{\alpha}}\epsilon^{\beta\alpha}p^{*}_{\alpha\dot{\alpha}} \tag{68}$$

$$p^{\dot{\beta}\beta} = p^{*\beta\dot{\beta}}. \tag{69}$$

For massless helicity spinors, this becomes

$$(\tilde{\lambda})^{\dot{\alpha}}(\lambda)^{\alpha} = (\tilde{\lambda}^{\alpha})^{*}(\lambda^{\dot{\alpha}})^{*} = (\lambda^{*})^{\dot{\alpha}}(\tilde{\lambda}^{*})^{\alpha}. \tag{70}$$

In order for each representation to transform irreducibly, we require that $\tilde{\lambda} = \pm\lambda^{*}$ and $\lambda = \pm\tilde{\lambda}^{*}$, where the $\pm$ sign is determined by the sign of p_0. In more standard notation, emphasizing the transformation properties, one writes $(\lambda^{\alpha})^{*} = \pm\tilde{\lambda}^{\dot{\alpha}}$ and $\lambda^{\alpha} = \pm(\tilde{\lambda}^{\dot{\alpha}})^{*}$, noting the conjugation switching between dotted and undotted indices. Subject to this reality condition, the little group transformation $\lambda \to z\lambda$ and $\tilde{\lambda} \to z^{-1}\tilde{\lambda}$ now restricts $|z| = 1$.

For real momenta, an explicit realization of the spinors is

$$\lambda^{\alpha} = \frac{1}{\sqrt{p^0 + p^3}}\begin{pmatrix} p^0 + p^3 \\ p^1 + ip^2 \end{pmatrix}^{\alpha} \quad \tilde{\lambda}^{\dot{\alpha}} = \frac{1}{\sqrt{p^0 + p^3}}\begin{pmatrix} p^0 + p^3 \\ p^1 - ip^2 \end{pmatrix}^{\dot{\alpha}}. \tag{71}$$

Moreover, utilizing the Levi-Civita symbol to raise and lower indices, $\lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta$ and $\tilde{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}$, one can construct the kinematic invariants

$$\left\langle \lambda_i \lambda_j \right\rangle := \lambda_i^\alpha \lambda_{j\alpha} = \epsilon_{\alpha\beta}\lambda_i^\alpha \lambda_j^\beta = -\left\langle \lambda_j \lambda_i \right\rangle =: \langle ij \rangle \tag{72}$$

$$[\tilde{\lambda}_i \tilde{\lambda}_j] := \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\beta}} \tilde{\lambda}_j^{\dot{\alpha}} = -[\tilde{\lambda}_j \tilde{\lambda}_i] =: [ij]. \tag{73}$$

Note the opposite index contraction between the left and right handed spinors.

These $SL(2,\mathbb{C})$ invariants allow for a convenient recasting of the massless Mandelstam invariant

$$s_{ij}^2 = (p_i + p_j)^2 = 2p_i \cdot p_j = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} p_i^{\dot{\alpha}\alpha} p_j^{\dot{\beta}\beta} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha \tilde{\lambda}_j^{\dot{\beta}} \lambda_j^\beta = \langle ij \rangle \, [ji]. \tag{74}$$

From our explicit forms of λ and $\tilde{\lambda}$ we have

$$\langle ij \rangle = \sqrt{|s_{ij}|} e^{i\phi_{ij}} \quad [ij] = -\sqrt{|s_{ij}|} e^{-i\phi_{ij}} \quad \cos\phi_{ij} = \frac{p_i^1 p_j^+ - p_j^1 p_i^+}{\sqrt{|s_{ij}|}\sqrt{p_i^+ p_j^+}} \quad \sin\phi_{ij} = \frac{p_i^2 p_j^+ - p_j^2 p_i^+}{\sqrt{|s_{ij}|}\sqrt{p_i^+ p_j^+}}, \tag{75}$$

where $p_i^\pm = p_i^0 \pm p_i^3$. Hence one can view the $SL(2,\mathbb{C})$ invariant angle and square brackets as the "square root" of the Lorentz invariant inner products. It is also useful to define the helicity operator

$$h_i = \frac{1}{2}\left[-\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \right], \tag{76}$$

which assigns helicity $-\frac{1}{2}$ to λ and helicity $+\frac{1}{2}$ to $\tilde{\lambda}$.

Consider the massless Dirac equation in momentum space, $\not{p} u(p) = 0$:

$$\begin{pmatrix} 0 & p_{\alpha\dot{\alpha}} \\ p^{\dot{\alpha}\alpha} & 0 \end{pmatrix} \begin{pmatrix} (u_L)_\alpha \\ (u_R)^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} p_{\alpha\dot{\alpha}} (u_R)^{\dot{\alpha}} \\ p^{\dot{\alpha}\alpha} (u_L)_\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{77}$$

and note that taking $u_R = \tilde{\lambda}$ and $u_L = \lambda$ solves the massless Dirac equation, since

$$p_{\alpha\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}} = \lambda_\alpha [pp] = 0 \tag{78}$$

$$p^{\dot{\alpha}\alpha}\lambda_\alpha = \tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha\lambda_\alpha = \tilde{\lambda}^{\dot{\alpha}}\langle pp\rangle = 0, \tag{79}$$

and we can thus write

$$u_+(p) = v_-(p) = \begin{pmatrix}\lambda_\alpha \\ 0\end{pmatrix} =: |p\rangle \quad u_-(p) = v_+(p) = \begin{pmatrix}0 \\ \tilde{\lambda}^{\dot{\alpha}}\end{pmatrix} =: |p]. \tag{80}$$

The Dirac equation can succinctly be written as $\not{p}\,|p\rangle = \not{p}|p] = 0$. With the definition $\bar{\psi} = \psi^\dagger\gamma^0$ we have

$$\bar{u}_+(p) = \bar{v}_-(p) = \begin{pmatrix}(\lambda_\alpha)^* & 0\end{pmatrix}\begin{pmatrix}0 & \mathbf{1} \\ \mathbf{1} & 0\end{pmatrix} = \begin{pmatrix}0 & \tilde{\lambda}_{\dot{\alpha}}\end{pmatrix} =: [p| \tag{81}$$

$$\bar{u}_-(p) = \bar{v}_+(p) = \begin{pmatrix}0 & (\tilde{\lambda}^{\dot{\alpha}})^*\end{pmatrix}\begin{pmatrix}0 & \mathbf{1} \\ \mathbf{1} & 0\end{pmatrix} = \begin{pmatrix}\lambda^\alpha & 0\end{pmatrix} =: \langle p|, \tag{82}$$

where we have assumed the momenta to be real. This allows us to write

$$\not{p} = \begin{pmatrix}0 & p_{\alpha\dot{\alpha}} \\ p^{\dot{\alpha}\alpha} & 0\end{pmatrix} = \begin{pmatrix}0 & \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} \\ \tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha & 0\end{pmatrix} = \begin{pmatrix}0 \\ \tilde{\lambda}^{\dot{\alpha}}\end{pmatrix}\begin{pmatrix}\lambda^\alpha & 0\end{pmatrix} + \begin{pmatrix}\lambda_\alpha \\ 0\end{pmatrix}\begin{pmatrix}0 & \tilde{\lambda}_{\dot{\alpha}}\end{pmatrix} = |p]\,\langle p| + |p\rangle\,[p|, \tag{83}$$

Moreover, we have $|-p\rangle := i\,|p\rangle$ and $|-p] := i|p]$, and thus $\langle -p| := i\,\langle p|$ and $[-p| := i[p|$. Spinor helicity variables also provide a convenient recasting of the spin 1 polarization basis vectors as bi-spinors via $\epsilon^{(\pm)}_{\alpha\dot{\alpha}}(p) = \sigma^\mu_{\alpha\dot{\alpha}}\epsilon^{(\pm)}_\mu(p)$ with

$$\epsilon^{(-)}_{\alpha\dot{\alpha}} = \sqrt{2}\frac{\lambda_\alpha\tilde{\xi}_{\dot{\alpha}}}{[\lambda\xi]} \quad \epsilon^{(+)}_{\alpha\dot{\alpha}} = \sqrt{2}\frac{\xi_\alpha\tilde{\lambda}_{\dot{\alpha}}}{\langle\xi\lambda\rangle}, \tag{84}$$

where the spinors ξ and $\tilde{\xi}$ are so-called "reference spinors" that can be chosen as long as they're not parallel to λ and $\tilde{\lambda}$. Note that choice of either polarization vector fixes the other via $\left(\epsilon^\mu_\pm\right)^* = \epsilon^\mu_\mp$, since $(\lambda_\alpha)^* = \tilde{\lambda}_{\dot{\alpha}}$ and $\langle ij\rangle^* = [ji]$. Graviton polarization vectors follow from

$$\epsilon^{(++)}_{\mu\nu}(p) = \epsilon^{(+)}_\mu(p)\epsilon^{(+)}_\nu(p), \quad \epsilon^{(--)}_{\mu\nu}(p) = \epsilon^{(-)}_\mu(p)\epsilon^{(-)}_\nu(p), \tag{85}$$

and hence satisfy $p^\mu \epsilon^{(\pm\pm)}_{\mu\nu}(p) = 0$. One can also choose them to be traceless: $\epsilon^{(\pm\pm)\mu}{}_{\mu} = 0$. We can verify that our momenta are still orthogonal to our polarization vectors, since

$$\begin{aligned}
p^\mu \epsilon^{(\pm)}_\mu(p) &= \frac{1}{2}\bar{\sigma}^{\mu\dot{\alpha}\alpha} p_{\dot{\alpha}\alpha} \frac{1}{2}\bar{\sigma}^{\dot{\beta}\beta}_\mu \epsilon^{(\pm)}_{\dot{\beta}\beta}(p) && (86)\\
&= \frac{1}{4}\underbrace{\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{\sigma}^{\dot{\beta}\beta}_\mu}_{2\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}} p_{\dot{\alpha}\alpha}\epsilon^{(\pm)}_{\dot{\beta}\beta}(p) && (87)\\
&= \frac{1}{2}p^{\dot{\beta}\beta}\epsilon^{(\pm)}_{\dot{\beta}\beta}(p) && (88)\\
&= \frac{1}{2}\tilde{\lambda}^{\dot{\beta}}\lambda^\beta \epsilon^{(\pm)}_{\dot{\beta}\beta}(p) = 0, && (89)
\end{aligned}$$

because

$$\begin{aligned}
\tilde{\lambda}^{\dot{\beta}}\lambda^\beta \epsilon^{(-)}_{\beta\dot{\beta}}(p) &= \tilde{\lambda}^{\dot{\beta}}\lambda^\beta \sqrt{2}\frac{\lambda_\beta \tilde{\xi}_{\dot{\beta}}}{[\lambda\xi]} = \langle\lambda\lambda\rangle \sqrt{2}\frac{[\xi\lambda]}{[\lambda\xi]} = 0 && (90)\\
\tilde{\lambda}^{\dot{\beta}}\lambda^\beta \epsilon^{(+)}_{\beta\dot{\beta}}(p) &= \tilde{\lambda}^{\dot{\beta}}\lambda^\beta \sqrt{2}\frac{\xi_\beta \tilde{\lambda}_{\dot{\beta}}}{\langle\xi\lambda\rangle} = [\lambda\lambda]\sqrt{2}\frac{\langle\lambda\xi\rangle}{\langle\xi\lambda\rangle} = 0. && (91)
\end{aligned}$$

The freedom of choice of reference spinors ξ and $\tilde{\xi}$ originates from the "gauge" freedom associated with the helicity spinor polarization vector. Consider the transformation $\xi \longrightarrow \xi + \delta\xi$ and its effect on the polarization spinor:

$$\epsilon^{(+)}_{\alpha\dot{\alpha}}(\xi+\delta\xi) = \sqrt{2}\frac{(\xi+\delta\xi)_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\langle(\xi+\delta\xi)\lambda\rangle} = \sqrt{2}\frac{\xi_\alpha\tilde{\lambda}_{\dot{\alpha}} + \delta\xi_\alpha\tilde{\lambda}_{\dot{\alpha}}}{\langle\xi\lambda\rangle + \langle\delta\xi\lambda\rangle}. \tag{92}$$

Treating $\delta\xi_\alpha$ as infinitesimal, we approximate

$$(\langle\xi\lambda\rangle + \langle\delta\xi\lambda\rangle)^{-1} = \langle\xi\lambda\rangle^{-1}\left(1 + \frac{\langle\delta\xi\lambda\rangle}{\xi\lambda}\right)^{-1} \approx \langle\xi\lambda\rangle^{-1}\left(1 - \frac{\langle\delta\xi\lambda\rangle}{\langle\xi\lambda\rangle}\right), \tag{93}$$

thus we have

$$\begin{aligned}
\epsilon^{(+)}_{\alpha\dot{\alpha}}(\xi+\delta\xi) &\approx \sqrt{2}\frac{\xi_\alpha\tilde{\lambda}_{\dot{\alpha}} + \delta\xi_\alpha\tilde{\lambda}_{\dot{\alpha}}}{\langle\xi\lambda\rangle}\left(1 - \frac{\langle\delta\xi\lambda\rangle}{\langle\xi\lambda\rangle}\right) && (94)\\
&\approx \epsilon^{(+)}_{\alpha\dot{\alpha}}(\xi) + \sqrt{2}\left(\frac{\delta\xi_\alpha\tilde{\lambda}_{\dot{\alpha}}\langle\xi\lambda\rangle}{\langle\xi\lambda\rangle^2} - \frac{\xi_\alpha\tilde{\lambda}_{\dot{\alpha}}\langle\delta\xi\lambda\rangle}{\langle\xi\lambda\rangle^2}\right) && (95)\\
&\approx \epsilon^{(+)}_{\alpha\dot{\alpha}}(\xi) + \sqrt{2}\frac{\tilde{\lambda}_{\dot{\alpha}}}{\langle\xi\lambda\rangle}\left(\delta\xi_\alpha\langle\xi\lambda\rangle - \xi_\alpha\langle\delta\xi\lambda\rangle\right). && (96)
\end{aligned}$$

As an intermediate result, it is useful to prove a statement of helicity spinor linear independence called the "Schouten identity":

$$\langle\lambda_1\lambda_2\rangle\,\lambda_3^\alpha + \langle\lambda_2\lambda_3\rangle\,\lambda_1^\alpha + \langle\lambda_3\lambda_1\rangle\,\lambda_2^\alpha = 0. \tag{97}$$

We start by noting that there are only two linearly independent $SL(2,\mathbb{C})$ spinors over $\mathbb{C}$, whose complex span is $\mathbb{C}^2$. Given that we have a two dimensional basis, we can take two of our three λ's to be linearly independent. Without loss of generality, we choose λ_1 and λ_2 and are thus guaranteed that $\langle\lambda_1\lambda_2\rangle \neq 0$. We can now express $\lambda_3 = a\lambda_1 + b\lambda_2$, for some $a, b \in \mathbb{C}$. To find a and b, it is easier to work in components

$$\lambda_3^\alpha = a\lambda_1^\alpha + b\lambda_2^\alpha \tag{98}$$

and take inner products with respect to the λ's:

$$\lambda_1 : \langle 31\rangle = a\langle 11\rangle + b\langle 21\rangle \Longrightarrow b = \frac{\langle 13\rangle}{\langle 12\rangle} \tag{99}$$

$$\lambda_2 : \langle 32\rangle = a\langle 12\rangle + b\langle 22\rangle \Longrightarrow a = \frac{\langle 32\rangle}{\langle 12\rangle}, \tag{100}$$

from which we find

$$\lambda_3 = \frac{\langle 32\rangle}{\langle 12\rangle}\lambda_1 + \frac{\langle 13\rangle}{\langle 12\rangle}\lambda_2 \Longrightarrow \langle 12\rangle\lambda_3 + \langle 31\rangle\lambda_2 + \langle 23\rangle\lambda_1 = 0. \tag{101}$$

We can now use the Schouten identity to simplify

$$\delta\xi_\alpha\,\langle\xi\lambda\rangle - \xi_\alpha\,\langle\delta\xi\lambda\rangle = \lambda_\alpha\,\langle\xi\delta\xi\rangle \tag{102}$$

and thus we have

$$\epsilon^{(+)}_{\alpha\dot\alpha}(\xi + \delta\xi) \approx \epsilon^{(+)}_{\alpha\dot\alpha}(\xi) + \sqrt{2}\frac{\tilde\lambda_{\dot\alpha}}{\langle\xi\lambda\rangle}\lambda_\alpha\,\langle\xi\delta\xi\rangle \tag{103}$$

$$\approx \epsilon^{(+)}_{\alpha\dot\alpha}(\xi) + p_{\dot\alpha\alpha}\sqrt{2}\frac{\langle\xi\delta\xi\rangle}{\langle\xi\lambda\rangle}. \tag{104}$$

Thus we see that changes in the reference spinor produce unphysical shifts of the polarization bi-spinors: namely shifts along the direction of propagation.

3.1.1 Properties of Partial Amplitudes

While we have, up until this point, focused on the kinematics entering amplitudes, external lines will, in general, have other quantum numbers which can determine selection rules for our scattering processes. While treatment of color is not essential to the contents of this dissertation, no introduction to amplitudes is complete without it, as we will later use properties of partial, or color-ordered, amplitudes. Thus, suppose we couple an $SU(N_c)$ gauge field, where N_c is the number of colors, to matter fields transforming in the fundamental representation of $SU(N_c)$. The gauge fields transforming under the adjoint, A^a_μ for $\mu = 0, 1, 2, 3$ and $a = 1, 2, ..., N_c^2 - 1$, incur factors of the structure constants f_{abc} at each three-point vertex, and a product of two f_{abc}'s for a 4-point vertex, in accordance with their Feynman rules. The presence of the 4-point vertex is necessarily present in the Yang-Mills Lagrangian because of $SU(N_c)$ gauge invariance, and thus is not independent of the three-point interaction. In fact, the 4-point interaction being proportional to two structure constants can be understood as two 3-point vertices being glued together.

In contrast, the Feynman rules governing the vertices of fundamental and anti-fundamental matter introduce $SU(N_c)$ Lie algebra generators $(T^a)^i{}_j$, where $a = 1, 2, ..., N_c^2 - 1$ as is the dimension of the Lie algebra, and $i, j \in \{1, ..., N_c\}$, since each T^a is an N_c by N_c matrix, that can be taken to be traceless and Hermitian. While one can, and does, separate all these color-dependent structures from the kinematical piece of the amplitude by factoring them out front, this leads to somewhat of T's and f's with varying indices, some of which commute while others don't. Fortunately, in order to systematize the treatment of color dependence of a given amplitude, we can substitute all instances of the structure constants with generators using

$$f^{abc} = -\frac{i}{\sqrt{2}} \operatorname{Tr}\left(T^a \left[T^b, T^c\right]\right), \tag{105}$$

where the $-i/\sqrt{2}$ pre-factor is consistent with the conventions of appendix A. Thus we can eliminate all the structure constants and transforms the color dependence of the amplitude to products of T^a's. In this expression, un-contracted indices correspond to the color quantum numbers of external lines, with fundamental and anti-fundamental indices correspond to Fermions and anti-Fermions, respectively, and the adjoint indices corresponding to $SU(N_c)$ gauge bosons. Contracted indices, corresponding to interior lines of our Feynman graph, are all that remain. Given that we're left with the three aforementioned types, fundamental, anti-fundamental, and adjoint, we can make progress by eliminating the adjoint indices in favor of fundamental and anti-fundamental ones. Any pair of contracted adjoint indices can be replaced using

$$(T^a)_{i_1}^{j_1}\,(T^a)_{i_2}^{j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N_c}\delta_{i_1}^{j_1}\delta_{i_2}^{\bar{j}_2}, \tag{106}$$

a relation deriving from the completeness of the $U(N_C)$ gauge group. Thus the color dependence of any diagram reduces to an expression involving either closed traces of generators, or strings of generators with open indices, corresponding to external color indices.

For pure gluon amplitudes, things are even simpler due to the absence of open fundamental indices and the fact that, as far as the interaction vertices are concerned, the $U(N_c)$ and $SU(N_c)$ gauge groups are indistinguishable. To see this, note that the generator of the $U(1) < U(N_c)$ subgroup, called T^0, can be taken proportional to the identity, as is unsurprising given the $U(1)$ is Abelian.

The associated interaction vertex for this generator is identically zero, which takes a couple of lines to prove. First note that the structure constants are cyclic in their indices:

$$f^{abc} = -\frac{i}{\sqrt{2}}\,\mathrm{Tr}\big[T^a[T^b,T^c]\big] = -\frac{i}{\sqrt{2}}\Big(\mathrm{Tr}\big[T^aT^bT^c\big] - \mathrm{Tr}\big[T^aT^cT^b\big]\Big) \tag{107}$$

$$= -\frac{i}{\sqrt{2}}\Big(\mathrm{Tr}\big[T^bT^cT^a\big] - \mathrm{Tr}\big[T^bT^aT^c\big]\Big) \tag{108}$$

$$= -\frac{i}{\sqrt{2}}\,\mathrm{Tr}\big[T^b[T^c,T^a]\big] = f^{bca}. \tag{109}$$

Because the $U(1)$ generator, T^0, is proportional to the identity, it commutes with all the other $SU(N_c)$ generators. Consequently, at any vertex where the $U(1)$ gauge field couples to the

$SU(N_c)$ gauge field, the incurred structure constant is zero:

$$f^{ab0} = -\frac{i}{\sqrt{2}}\mathrm{Tr}[T^a \underbrace{[T^b, T^0]}_{0}] = 0. \tag{110}$$

By cyclicity of f^{abc} this is true regardless of where the zero index appears.

With no external fundamental or anti-fundamental indices to speak of, the color dependence in tree-level all-gluon amplitudes reduces to a single trace of generators times the kinematic-dependent partial amplitude:

$$\mathcal{A}_n^{a_1 a_2 \ldots a_n}\left(\{h_i, p_i\}\right) = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}\left(T^{a_{\sigma_1}} T^{a_{\sigma_2}} \cdots T^{a_{\sigma_n}}\right) \cdot$$
$$A_n\left(\{p_{\sigma_1}, h_{\sigma_1}\}, \{p_{\sigma_2}, h_{\sigma_2}\}, ..., \{p_{\sigma_n}, h_{\sigma_n}\}\right). \tag{111}$$

Here a_i, h_i, and p_i denote the i^{th} external gluon's color, helicity, and momentum, respectively, and $S_n/\mathbb{Z}_n \cong S_{n-1}$ is the set of all non-cyclic permutations of n elements. Evident from that the colored amplitude, $\mathcal{A}_n^{a_1 a_2 \ldots a_n}$ carries color indices, it is apparent that it is gauge-dependent. In contrast, the partial amplitude A_n, carrying no indices, is an $SU(N_c)$ invariant object.

As a summary of general properties of tree-level partial amplitudes, not just gluons, color ordered amplitudes are gauge invariant and obey the properties:

- Complex conjugation $*$ acts to flip the helicities of the external legs:

$$[A(1, ..., n)]^* = A(\bar{1}, ..., \bar{n}), \tag{112}$$

- Charge conjugating the entire amplitude is equivalent to multiplying by a negative:

$$A(1_q, 2_{\bar{q}}, 3, ..., n) = -A(1_{\bar{q}}, 2_q, 3, ..., n), \tag{113}$$

- Cyclicity in its arguments:

$$A(1, ..., n) = A(2, ..., n, 1) \tag{114}$$

- Reversing the order of n external legs is equivalent to multiplying by $(-1)^n$:

$$A(1, ..., n) = (-1)^n A(n, n-1, ..., 1). \tag{115}$$

- Summing over cyclic permutations of all legs but the n^{th} yields zero:

$$\sum_{\sigma\in\mathbb{Z}_{n-1}} A\left(\sigma_1, \ldots, \sigma_{n-1}, n\right) = 0, \tag{116}$$

The important take-away, for the purposes of this dissertation, is that we are able to turn out attention away from the preceding color factors and focus our attention on the color-ordered amplitudes. This allows us to focus on the kinematical problem of building amplitudes, without having to worry about and carry around other quantum numbers.

3.1.2 Three Point Amplitudes from Symmetry

While scattering amplitudes are often described in introductory quantum field theory textbooks as being Lorentz invariant, this misconception is not quite true. Amplitudes transform under the so-called little groups of their momenta: The group of Lorentz transformations that leave a momentum invariant, though not all of spacetime. For a massive momentum, one can boost to a frame wherein its three spatial components are zero, wherefrom any rotation in 3 dimensions leaves its momentum unchanged. For a massless momentum, no such boost exists and one can, at most, rotate about the axis defined by the massless momentum's direction. Consequently, the massive and massless momenta have little groups $SU(2)$ and $U(1)$, respectively. The quantum numbers associated with these little groups, spins and helicities, are what we use to characterize massive and massless particle states, respectively.

The principle of little group scaling — how amplitudes transform under the little group of the external momenta — is an essential ingredient for determining the minimal Compton amplitude of [109], which is a central object in this dissertation. For the purposes of this chapter, we will restrict our scope to massless momenta and their helicities. To this end, we begin by noting the action of the helicity operator on an amplitude

$$h_i A(\{\lambda_1, \tilde{\lambda}_1\}, ..., \{\lambda_n, \tilde{\lambda}_n\}) = \frac{1}{2}\left[-\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}\right] A(\{\lambda_1, \tilde{\lambda}_1\}, ..., \{\lambda_n, \tilde{\lambda}_n\}), \tag{117}$$

where h_i is the helicity of the i^{th} particle. This requirement, along with only using Lorentz invariant ingredients in building our amplitudes, is sufficient to affix the 3-point amplitudes using kinematics alone.

Given that each of the external legs is massless, $p_i^2 = 0$, momentum conservation yields

$$\sum_{i=1}^{3} p_i^\mu = 0 \Longrightarrow \sum_{\substack{i=1 \\ i\neq j}}^{3} p_i^\mu = -p_j^\mu \Longrightarrow \sum_{\substack{i=1 \\ i\neq j}}^{3} \underbrace{p_i^2}_{0} + 2\sum_{\substack{a=1 \\ a\neq j \\ a>b}}^{3} \sum_{\substack{b=1 \\ b\neq j}}^{3} p_a \cdot p_b = 0. \tag{118}$$

Note that the double sum only runs over one pair of indices (a, b), namely those that satisfy $a \neq b$, $b \neq j$, and $j \neq a$. Thus, without loss of generality, since we can re-label the momenta at will, we have

$$p_a \cdot p_b = 0 \quad \forall \quad a, b \in \{1, 2, 3\}. \tag{119}$$

Noting that we can express $p \cdot q = \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}p^{\dot{\alpha}\alpha}q^{\dot{\beta}\beta}$, we have

$$0 = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}p_i^{\dot{\alpha}\alpha}p_j^{\dot{\beta}\beta} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta}\tilde{\lambda}_i^{\dot{\alpha}}\lambda_i^\alpha\tilde{\lambda}_j^{\dot{\beta}}\lambda_j^\beta = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_j^{\dot{\beta}}\epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta = [ji]\,\langle ij\rangle\,, \tag{120}$$

and hence we have two possible non-trivial solutions:

$$\langle ij\rangle = 0 \text{ and } [ij] \neq 0, \quad \text{ or } \quad \langle ij\rangle \neq 0 \text{ and } [ij] = 0 \quad \forall i \neq j, \quad i, j \in \{1, 2, 3\}. \tag{121}$$

We have, to this point, shown that either angle or square brackets for each *distinct* pair (i, j) is necessarily zero. We can derive a stronger constraint using the Schouten identity

$$\langle ij\rangle\,\lambda_k^\alpha + \langle jk\rangle\,\lambda_i^\alpha + \langle ki\rangle\,\lambda_j^\alpha = 0 \tag{122}$$

$$[ij]\tilde{\lambda}_k^{\dot{\alpha}} + [jk]\tilde{\lambda}_i^{\dot{\alpha}} + [ki]\tilde{\lambda}_j^{\dot{\alpha}} = 0. \tag{123}$$

Momentum conservation for the three point amplitudes dictates that for each pair i and j, either their angle or their square bracket is zero. Thus at least two of the terms in either equation 122 or 123 must be zero. This necessarily implies that the outstanding term must also be zero, hence either all the angle brackets are zero or all the square brackets are zero.
For real momenta we have $(\lambda^{\alpha})^* = \pm\tilde{\lambda}^{\dot{\alpha}}$ and $\lambda^{\alpha} = \pm(\tilde{\lambda}^{\dot{\alpha}})^*$, hence

$$\langle ij\rangle = \epsilon_{\alpha\beta}\lambda_i^{\alpha}\lambda_j^{\beta} = \epsilon_{\alpha\beta}(\tilde{\lambda}_i^{\dot{\alpha}})^*(\tilde{\lambda}_j^{\dot{\beta}})^* = [ji]^*, \tag{124}$$

and therefore

$$0 = [ji]\,\langle ij\rangle = |[ji]|^2 = |\langle ij\rangle|^2 = 0 \Longrightarrow [ji] = \langle ij\rangle = 0. \tag{125}$$

Thus we see that there is no scattering of massless particles with real momenta at three point. To avoid this, we consider the scattering kinematics in complexified spacetime, wherein $p_{\dot{\alpha}\alpha}$ need no longer be Hermitian, allowing λ and $\bar{\lambda}$ to remain independent.
Now, since either all of the angle or all of the square brackets, though not both, need to be zero, we have heavily constrained the possible forms of the amplitudes. The last piece of the puzzle is little group scaling. First note the action of the helicity operator on a power of angle and square brackets:

$$h_i\langle ij\rangle^m = -\frac{1}{2}m\langle ij\rangle^m \quad h_i[ij]^m = \frac{1}{2}m[ij]^m. \tag{126}$$

Now consider a three point amplitude with the helicity assignments 1^{-s}, 2^{-s}, 3^{+s}. We have established that we must construct this object from a rational function of either angle or square brackets. Let's consider the former:

$$A\left(1^{-s}, 2^{-s}, 3^{+s}\right) \sim \langle 12\rangle^{n_{12}}\langle 23\rangle^{n_{23}}\langle 31\rangle^{n_{31}} \tag{127}$$

We can now use the helicity operator, knowing the helicities of each of the associated legs:

$$h_1\langle 12\rangle^{n_{12}}\langle 23\rangle^{n_{23}}\langle 31\rangle^{n_{31}} = \left(-\frac{n_{12}}{2}-\frac{n_{31}}{2}\right)\langle 12\rangle^{n_{12}}\langle 23\rangle^{n_{23}}\langle 31\rangle^{n_{31}} \implies \left(-\frac{n_{12}}{2}-\frac{n_{31}}{2}\right) = -s \tag{128}$$

$$h_2\langle 12\rangle^{n_{12}}\langle 23\rangle^{n_{23}}\langle 31\rangle^{n_{31}} = \left(-\frac{n_{12}}{2}-\frac{n_{23}}{2}\right)\langle 12\rangle^{n_{12}}\langle 23\rangle^{n_{23}}\langle 31\rangle^{n_{31}} \implies \left(-\frac{n_{12}}{2}-\frac{n_{23}}{2}\right) = -s \tag{129}$$

$$h_3\langle 12\rangle^{n_{12}}\langle 23\rangle^{n_{23}}\langle 31\rangle^{n_{31}} = \left(-\frac{n_{23}}{2}-\frac{n_{31}}{2}\right)\langle 12\rangle^{n_{12}}\langle 23\rangle^{n_{23}}\langle 31\rangle^{n_{31}} \implies \left(-\frac{n_{23}}{2}-\frac{n_{31}}{2}\right) = s. \tag{130}$$

Solving this system of equations we have that

$$(n_{12}, n_{23}, n_{31}) = (3s, -s, -s), \tag{131}$$

hence our three point amplitude takes the form:

$$A^{\text{MHV}}\left(1^{-s}, 2^{-s}, 3^{+s}\right) \sim \left[\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right]^s, \tag{132}$$

where the proportionality factor, ig, can be found through matching to Feynman rules. The abbreviation "MHV" stands for maximally helicity violating amplitude. Helicity violation seems to suggest some sort of conservation thereof, which is not the case. The violation merely refers to the mismatch between the number of external legs with positive and negative helicities. In a sense, a tree-level amplitude with all positive or negative helicities would be maximally violating, in this sense, but amplitudes with all positive or negative helicities are zero. In fact, tree-level amplitudes with all legs of one helicity but one are also zero. Thus we reserve the descriptor "maximally violating" for the first non-zero helicity configurations, namely those where all but two external legs are of the same helicity. The three point amplitude, whose existence seems to clash with this nomenclature, is the exception. Since the massless 3 point amplitude vanishes for real momenta, regardless of its helicity configuration, this doesn't pose a contradiction. Thus, we can pose the definition that an MHV amplitude has all positive external helicities, except in two legs, with its parity conjugate, $\overline{\text{MHV}}$, having the opposite.

For the 3-point $\overline{MHV}$ amplitude, one could do a similar analysis with little group scaling, but it is more straightforward to relate the two using an explicit parity transformation. Note that under parity

$$p^\mu = (p_0, p_1, p_2, p_3)^\mu \to p'^\mu = (p_0, -p_1, -p_2, -p_3)^\mu \tag{133}$$

$$\begin{aligned} p^{\dot\alpha\alpha} = \bar\sigma_\mu^{\dot\alpha\alpha} p^\mu \to p'^{\dot\alpha\alpha} &= \sigma_\mu^{\dot\alpha\alpha} p^\mu = \sigma_\mu^{\dot\alpha\alpha} \frac{1}{2}\bar\sigma^{\mu\dot\beta\beta} p_{\dot\beta\beta} \\ &= \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\alpha\gamma}\bar\sigma_{\mu\gamma\dot\gamma}\frac{1}{2}\bar\sigma^{\mu\dot\beta\beta} p_{\dot\beta\beta} = \epsilon^{\dot\alpha\gamma}\epsilon^{\alpha\dot\gamma}\epsilon_{\dot\gamma}{}^{\dot\beta}\epsilon_\gamma{}^\beta p_{\dot\beta\beta} = p^{\alpha\dot\alpha}, \end{aligned} \tag{134}$$

which at the level of helicity ($m^2 = 0$) spinors becomes

$$(\tilde\lambda)^{\dot\alpha}(\lambda)^\alpha \to (\tilde\lambda)^\alpha(\lambda)^{\dot\alpha} = (\lambda)^{\dot\alpha}(\tilde\lambda)^\alpha, \tag{135}$$

that is we merely exchange left and right handed spinors under parity. From this we can compute the transformations of the angle and square brackets

$$\langle ij\rangle = \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta \to \epsilon_{\alpha\beta}(\tilde\lambda_i)^\alpha(\tilde\lambda_j)^\beta = [ji] \tag{136}$$

$$[ij] = \epsilon_{\dot\alpha\dot\beta}\tilde\lambda_i^{\dot\beta}\tilde\lambda_j^{\dot\alpha} \to \epsilon_{\dot\alpha\dot\beta}(\lambda_i)^{\dot\beta}(\lambda_j)^{\dot\alpha} = \langle ji\rangle. \tag{137}$$

Thus we have

$$A_3^{\overline{\text{MHV}}}\left(1^+, 2^+, 3^-\right) = -ig\frac{[12]^3}{[23][31]}. \tag{138}$$

We can verify that little group scaling, indeed, produces the correct amplitude by starting from our color-ordered Feynman rules for the 3-point vertex[118]:

$$\begin{aligned} A(1^-, 2^-, 3^+) = g\frac{i}{\sqrt{2}}\Big[&\varepsilon_1^{(-)}\cdot\varepsilon_2^{(-)}(p_1 - p_2)\cdot\varepsilon_3^{(+)} + \varepsilon_2^{(-)}\cdot\varepsilon_3^{(+)}(p_2 - p_3)\cdot\varepsilon_1^{(-)} \\ &+\varepsilon_3^{(+)}\cdot\varepsilon_1^{(-)}(p_3 - p_1)\cdot\varepsilon_2^{(-)}\Big], \end{aligned} \tag{139}$$

wherein the polarization vectors are

$$\varepsilon_{\alpha\dot\alpha}^{(-)} = \sqrt{2}\frac{\lambda_\alpha\tilde\xi_{\dot\alpha}}{[\lambda\xi]} \quad \varepsilon_{\alpha\dot\alpha}^{(+)} = \sqrt{2}\frac{\xi_\alpha\tilde\lambda_{\dot\alpha}}{\langle\xi\lambda\rangle}. \tag{140}$$

The dot products between polarizations and the momenta can be computed as

$$p_i \cdot \varepsilon_j^{(h)} = \frac{1}{2}\bar{\sigma}^{\mu\dot{\alpha}\alpha} p_{\dot{\alpha}\alpha} \frac{1}{2}\bar{\sigma}_\mu^{\dot{\beta}\beta} \varepsilon_{\dot{\beta}\beta}^{(h)}(p_j) = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\tilde{\lambda}_i)_{\dot{\alpha}}(\lambda_i)_\alpha \varepsilon_{\dot{\beta}\beta}^{(h)}(p_j) = \begin{cases} \frac{[ij]\langle\xi i\rangle}{\sqrt{2}\langle\xi j\rangle} & \text{for } h = + \\ \frac{\langle ji\rangle[i\xi]}{\sqrt{2}[j\xi]} & \text{for } h = -, \end{cases} \tag{141}$$

and similarly the dot products between different polarization vectors is given by

$$\varepsilon_i^{(h)} \cdot \varepsilon_j^{(h')} = \frac{1}{2}(\varepsilon^{(h)})^{\dot{\alpha}\alpha}(\varepsilon^{(h')})_{\dot{\alpha}\alpha} = \begin{cases} +\frac{\langle\xi\xi'\rangle[ji]}{\langle\xi i\rangle\langle\xi' j\rangle} & (h,h') = (+,+) \\ -\frac{\langle\xi j\rangle[\xi' i]}{\langle\xi i\rangle[\xi' j]} & (h,h') = (+,-) \\ -\frac{\langle i\xi'\rangle[j\xi]}{\langle j\xi'\rangle[i\xi]} & (h,h') = (-,+) \\ +\frac{\langle ij\rangle[\xi'\xi]}{[i\xi][j\xi']} & (h,h') = (-,-). \end{cases} \tag{142}$$

As mentioned when we defined the polarization vectors in 84, the spinors ξ and ξ' are reference spinors that can be chosen, so long as they're not collinear with any of the momenta's helicity spinors. Choosing $\xi = \xi'$, we can take $\varepsilon_i^{(\pm)} \cdot \varepsilon_j^{(\pm)} = 0$. Moreover, using momentum conservation, $p_1 + p_2 + p_3 = 0$, and transversality, $p_i \cdot \epsilon_i = 0$, we can write

$$A(1^-,2^-,3^+) = g\frac{i}{\sqrt{2}}\left[\varepsilon_2^{(-)} \cdot \varepsilon_3^{(+)}(2p_2 + p_1) \cdot \varepsilon_1^{(-)} + \varepsilon_3^{(+)} \cdot \varepsilon_1^{(-)}(-2p_1 - p_2) \cdot \varepsilon_2^{(-)}\right] \tag{143}$$

$$= ig\sqrt{2}\left[\varepsilon_2^{(-)} \cdot \varepsilon_3^{(+)} p_2 \cdot \varepsilon_1^{(-)} - \varepsilon_3^{(+)} \cdot \varepsilon_1^{(-)} p_1 \cdot \varepsilon_2^{(-)}\right] \tag{144}$$

$$= ig\sqrt{2}\left[-\frac{\langle 2\xi\rangle[3\xi]}{\langle 3\xi\rangle[2\xi]}\frac{\langle 12\rangle[2\xi]}{\sqrt{2}[1\xi]} + \frac{\langle\xi 1\rangle[\xi 3]}{\langle\xi 3\rangle[\xi 1]}\frac{\langle 21\rangle[1\xi]}{\sqrt{2}[2\xi]}\right] \tag{145}$$

$$= ig\left[-\frac{\langle 2\xi\rangle[3\xi]}{\langle 3\xi\rangle}\frac{\langle 12\rangle}{[1\xi]} - \frac{\langle\xi 1\rangle[\xi 3]}{\langle\xi 3\rangle}\frac{\langle 21\rangle}{[2\xi]}\right] \tag{146}$$

$$= -ig\frac{\langle 12\rangle[3\xi]}{\langle 3\xi\rangle}\left[\frac{\langle 2\xi\rangle}{[1\xi]} + \frac{\langle 1\xi\rangle}{[2\xi]}\right] \tag{147}$$

$$= ig\frac{\langle 12\rangle[3\xi]}{\langle 3\xi\rangle}\left(\frac{\langle\xi 1\rangle[1\xi] + \langle\xi 2\rangle[2\xi]}{[1\xi][2\xi]}\right). \tag{148}$$

Note that we can write

$$\langle\xi\lambda\rangle[\lambda\xi] = \xi^\alpha\lambda_\alpha\tilde{\lambda}_{\dot{\beta}}\tilde{\xi}^{\dot{\beta}} = \xi^\alpha p_{\alpha\dot{\beta}}\tilde{\xi}^{\dot{\beta}} =: \langle\xi|p|\xi], \tag{149}$$

and therefore

$$\langle\xi 1\rangle[1\xi] + \langle\xi 2\rangle[2\xi] = \langle\xi|p_1 + p_2|\xi] = \langle\xi| - p_3|\xi] = -\langle\xi 3\rangle[3\xi]. \tag{150}$$

Plugging this into our amplitude expression yields

$$A(1^-, 2^-, 3^+) = ig\frac{\langle 12\rangle[3\xi]}{\langle 3\xi\rangle}\left(\frac{-\langle\xi 3\rangle[3\xi]}{[1\xi][2\xi]}\right) = ig\frac{\langle 12\rangle[3\xi]^3}{[1\xi][2\xi]}. \tag{151}$$

Lastly we can re-write, again using momentum conservation to derive

$$\frac{[3\xi]}{[1\xi]} = \frac{\langle 13\rangle[3\xi]}{\langle 13\rangle[1\xi]} = -\frac{\langle 1|p_3|\xi]}{\langle 3|p_1|\xi]} = -\frac{\langle 1|p_1 + p_2|\xi]}{\langle 3|p_2 + p_3|\xi]} = -\frac{\langle 1|p_2|\xi]}{\langle 3|p_2|\xi]} = -\frac{\langle 12\rangle[2\xi]}{\langle 32\rangle[2\xi]} = \frac{\langle 12\rangle}{\langle 23\rangle} \tag{152}$$

$$\frac{[3\xi]}{[2\xi]} = \frac{\langle 23\rangle[3\xi]}{\langle 23\rangle[2\xi]} = -\frac{\langle 2|p_3|\xi]}{\langle 3|p_2|\xi]} = -\frac{\langle 2|p_1 + p_2|\xi]}{\langle 3|p_1 + p_3|\xi]} = -\frac{\langle 2|p_1|\xi]}{\langle 3|p_1|\xi]} = -\frac{\langle 21\rangle[1\xi]}{\langle 31\rangle[1\xi]} = \frac{\langle 12\rangle}{\langle 31\rangle}, \tag{153}$$

we can finally simplify our amplitude to

$$A(1^-, 2^-, 3^+) = ig\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \tag{154}$$

in agreement with the little group scaling derivation.

3.2 Recursion Relations for Amplitudes

On-shell recursion relations, most famously the Britto-Cachazo-Feng-Witten (BCFW) relation, provide a constructive framework for building higher point amplitudes from lower point amplitudes. They involve gluing together lower point amplitudes along possible factorization channels in order to build higher point amplitudes with the correct factorization properties.

To derive the BCFW recursion relation, we begin by considering an n-point tree level amplitude $A(p_1, ..., p_n)$ and introduce the following deformation of the spinors of two adjacent particles 1 and n (often indicated as $[n1\rangle$):

$$\lambda_1 \to \hat{\lambda}_1(z) = \lambda_1 - z\lambda_n, \qquad \tilde{\lambda}_1 \to \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1, \tag{155}$$

$$\lambda_n \to \hat{\lambda}_n = \lambda_n, \qquad \tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n(z) = \tilde{\lambda}_n + z\tilde{\lambda}_1, \tag{156}$$

where z is some complex number. These transformations of the spinor helicity variables induce the following transformations of the bi-$SL(2, C)$ momenta:

$$p_1^{\dot{\alpha}\alpha} \to \hat{p}_1^{\dot{\alpha}\alpha} = \hat{\tilde{\lambda}}_1^{\dot{\alpha}} \hat{\lambda}_1^{\alpha} = \tilde{\lambda}_1^{\dot{\alpha}} (\lambda_1 - z\lambda_n)^{\alpha} = p_1^{\dot{\alpha}\alpha} - z\tilde{\lambda}_1^{\dot{\alpha}} \lambda_n^{\alpha} \tag{157}$$

$$p_n^{\dot{\alpha}\alpha} \to \hat{p}_n^{\dot{\alpha}\alpha} = \hat{\tilde{\lambda}}_n^{\dot{\alpha}} \hat{\lambda}_n^{\alpha} = (\tilde{\lambda}_n + z\tilde{\lambda}_1)^{\dot{\alpha}} \lambda_n^{\alpha} = p_n^{\dot{\alpha}\alpha} + z\tilde{\lambda}_1^{\dot{\alpha}} \lambda_n^{\alpha}. \tag{158}$$

Importantly, these shifts preserve momentum conservation and on-shell conditions:

$$(\hat{p}_1)_\mu(z) + (\hat{p}_n)_\mu(z) = \frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha} \left((\hat{p}_1)_{\dot{\alpha}\alpha} + (\hat{p}_n)_{\dot{\alpha}\alpha}\right) \tag{159}$$

$$= \frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha} \left((p_1)_{\dot{\alpha}\alpha} + (p_n)_{\dot{\alpha}\alpha} + z\tilde{\lambda}_1^{\dot{\alpha}}\lambda_n^{\alpha} - z\tilde{\lambda}_1^{\dot{\alpha}}\lambda_n^{\alpha}\right) \tag{160}$$

$$= (p_1)_\mu + (p_n)_\mu, \tag{161}$$

and

$$\det(\hat{p}_1^{\dot{\alpha}\alpha}(z)) = \frac{1}{2}\hat{p}_1^{\dot{\alpha}\alpha}(z)\hat{p}_1^{\dot{\beta}\beta}(z)\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{162}$$

$$= \frac{1}{2}(\tilde{\lambda}_1^{\dot{\alpha}}(\lambda_1 - z\lambda_n)^{\alpha})(\tilde{\lambda}_1^{\dot{\beta}}(\lambda_1 - z\lambda_n)^{\beta})\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{163}$$

$$= \frac{1}{2}[11]\left(\langle 11\rangle + z^2\langle nn\rangle - z(\langle 1n\rangle + \langle n1\rangle)\right) = 0, \tag{164}$$

$$\det(\hat{p}_n^{\dot{\alpha}\alpha}(z)) = \frac{1}{2}\hat{p}_n^{\dot{\alpha}\alpha}(z)\hat{p}_n^{\dot{\beta}\beta}(z)\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{165}$$

$$= \frac{1}{2}((\tilde{\lambda}_n + z\tilde{\lambda}_1)^{\dot{\alpha}}\lambda_n^{\alpha})((\tilde{\lambda}_n + z\tilde{\lambda}_1)^{\dot{\beta}}\lambda_n^{\beta})\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} \tag{166}$$

$$= \frac{1}{2}\langle nn\rangle\left([nn] + z^2[11] + z([n1] + [1n])\right) = 0. \tag{167}$$

3.2.1 Analytic Properties of Tree Level Amplitudes

To motivate the idea of amplitude factorization, imagine that a scattering process occurs, but a subset of the external legs, say the subset $\{1, ..., i\}$, interact only among themselves in a small region of spacetime, with the complementary set of legs, $\{i+1, ..., n\}$ interacting among themselves in a separate region to the first. For this to be one single amplitude, these separate interactions must be connected by an intermediate propagator. Now consider taking these two regions to be infinitely far away from one another. Akin to how external particles can propagate out to infinity by satisfying the equations of motion, taking these two regions to be infinitely separated requires taking momentum propagating in-between to go on-shell. As the connecting momentum approaches the propagator pole the amplitude diverges causing the overall amplitude to diverge. This "pinching" of the overall amplitude — taking the internal momentum to go on shell — splits the amplitude into two sub-amplitudes that, mathematically, are the residue about the propagator connecting them. Because the connecting propagator's momentum is taken to go on shell, the sub-amplitudes see connecting momentum, in this pinching limit, as yet another on-shell external momentum coming from infinitely far away.

To make this precise, let P be the sum of a subset of the external momenta and note that, for a color-ordered amplitude, the propagator will take the form

$$\frac{i}{(p_i + p_{i+1} + \ldots + p_j)^2}. \tag{168}$$

Thus the deformed amplitude will only have simple poles in z of the form

$$\frac{i}{\hat{P}_i^2(z)} := \frac{i}{(\hat{p}_1(z) + p_2 + \ldots + p_{i-1})^2} = \frac{i}{(p_i + \ldots + p_{n-1} + \hat{p}_n(z))^2}. \tag{169}$$

Note that any propagator involving both or neither of $\hat{p}_1(z)$ and $\hat{p}_n(z)$ would be independent of z by virtue of these momenta being oppositely shifted, and hence not contribute any poles in z. Moreover, since

$$(\hat{p}_1)_\mu(z) = (p_1)_\mu + \frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}(-z(\tilde{\lambda}_1)_{\dot{\alpha}}(\lambda_n)_\alpha) = (p_1)_\mu - z\langle n|\bar{\sigma}_\mu/2|1] \tag{170}$$

$$(\hat{p}_n)_\mu(z) = (p_n)_\mu + \frac{1}{2}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}(z(\tilde{\lambda}_n)_{\dot{\alpha}}(\lambda_n)_\alpha) = (p_n)_\mu + z\langle n|\bar{\sigma}_\mu/2|1], \tag{171}$$

we have that

$$\begin{aligned}(\hat{p}_1(z) + p_2 + \ldots + p_{i-1})^2 &= ((p_1)_\mu + (p_2)_\mu + \ldots + (p_{i-1})_\mu - z\langle n|\bar{\sigma}_\mu/2|1]) \\ &\qquad \cdot (p_1^\mu + p_2^\mu + \ldots + p_{i-1}^\mu - z\langle n|\bar{\sigma}^\mu/2|1]) \qquad (172) \\ &= (P_i)_\mu(P_i)^\mu - z\langle n|(P_i)_\mu\bar{\sigma}^\mu|1] + z^2\langle n|\bar{\sigma}_\mu/2|1]\langle n|\bar{\sigma}^\mu/2|1] \qquad (173) \\ &= P_i^2 - z\langle n|P_i|1] + \frac{z^2}{4}(\lambda_n)^\alpha\bar{\sigma}^\mu_{\alpha\dot{\alpha}}(\tilde{\lambda}_1)^{\dot{\alpha}}(\lambda_n)^\beta\bar{\sigma}_{\mu\beta\dot{\beta}}(\tilde{\lambda}_1)^{\dot{\beta}} \qquad (174)\end{aligned}$$

$$= P_i^2 - z\langle n|P_i|1] + \frac{z^2}{4}(\lambda_n)^\alpha(\tilde\lambda_1)^{\dot\alpha}(\lambda_n)^\beta(\tilde\lambda_1)^{\dot\beta}2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} \tag{175}$$

$$= P_i^2 - z\langle n|P_i|1] + \frac{z^2}{2}\langle nn\rangle[11] = P_i^2 - z\langle n|P_i|1] \tag{176}$$

where $P_i := \sum_{j=1}^{i-1} p_j$ and

$$\langle n|\bar\sigma|1] = (\lambda_n)^\alpha\bar\sigma_{\alpha\dot\beta}(\tilde\lambda_1)^{\dot\beta} = \epsilon^{\alpha\gamma}(\lambda_n)_\gamma\bar\sigma_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\delta}(\tilde\lambda_1)_{\dot\delta} = (\lambda_n)_\gamma\epsilon^{\gamma\alpha}\epsilon^{\dot\delta\dot\beta}\bar\sigma_{\alpha\dot\beta}(\tilde\lambda_1)_{\dot\delta} = (\lambda_n)_\alpha\bar\sigma^{\dot\beta\alpha}(\tilde\lambda_1)_{\dot\beta}. \tag{177}$$

Thus we have

$$\frac{i}{\hat P_i^2(z)} := \frac{i}{(\hat p_1(z) + p_2 + ... + p_{i-1})^2} = \frac{i}{(p_i + ... + p_{n-1} + \hat p_n(z))^2} \tag{178}$$

$$= \frac{i}{P_i^2 - z\langle n|P_i|1]}, \tag{179}$$

and we deduce that $A_n(z)$ is meromorphic in z with simple poles at

$$z_{P_i} = \frac{P_i^2}{\langle n|P_i|1]} \quad \forall i \in [3, n-1]. \tag{180}$$

The restriction on i comes from the fact that for $i = 1$ we have $P_1 = 0$, for $i = 2$ we have $P_2 = p_1$ and thus $\langle n|p_1|1] = \lambda_n^\alpha (p_1)_{\alpha\dot\alpha}\tilde\lambda_1^{\dot\alpha} = \lambda_n^\alpha(\lambda_1)_\alpha(\tilde\lambda_1)_{\dot\alpha}\tilde\lambda_1^{\dot\alpha} = \langle n1\rangle[11] = 0$. The case $i = n$ yields $P_n = p_1 + ... + p_{n-1} = -p_n$ by momentum conservation, and thus $\langle n|P_n|1] = -\langle n|p_n|1] = -\langle nn\rangle[n1] = 0$. Thus our deformed propagator, $i/\hat P_i(z)^2$ can be re-written explicitly in terms of the meromorphic structure of z:

$$\frac{i}{\hat P_i^2(z)} = \frac{i}{P_i^2(z) - z\langle n|P_i|1]} = -\frac{i}{\langle n|P_i|1]}\frac{1}{z - z_{P_i}}. \tag{181}$$

Consequently, as $z \to z_P$ and $\hat P_i \to 0$, the amplitude $A_n(z)$ factorizes as

$$\lim_{z\to z_{P_i}} A_n(z) = -i\frac{1}{z - z_{P_i}}\frac{1}{\langle n|P_i|1]}\sum_{h=\pm} A_L\left(\hat 1(z_{P_i}), 2, ..., i-1, -\hat P_i^{-h}(z_{P_i})\right) \times A_R\left(\hat P_i^h(z_{P_i}), i, ..., n-1, \hat n(z_{P_i})\right), \tag{182}$$

where, in general, one ought to sum over both the helicity configurations $h = \pm$ of the internal line. To recover $A_n(z = 0)$, we can use the Cauchy integral formula to construct it from our deformed $A_n(z)$:

$$A_n(z = 0) = \frac{1}{2\pi i} \oint_C \frac{dz}{z} A_n(z), \tag{183}$$

where the contour C encloses the origin, but not any other poles of $A_n(z)$. We have no intention of evaluating this by direct integration, but rather we intend to use the meromorphic nature of $A_n(z)$ along with the residue theorem to evaluate it. The idea is to "push outward" the integration contour, encircling any singularities of $A_n(z)$ that we might come across. Attention to detail will tell you that this reverses the direction of integration for the poles picked up in this outward sweep, and that one ought to consider the potentially divergent behavior of the contour at infinity. With this in mind one finds

$$\begin{aligned}
&\frac{1}{2\pi i} \oint_{C_0} \frac{dz}{z} A_n(z) = \\
&\sum_{i=3}^{n-1} \sum_{h=\pm} A_L \left(\hat{1}(z_{P_i}), 2, ..., i-1, -\hat{P}_i^{-h}(z_{P_i})\right) \frac{i}{P_i^2} A_R \left(\hat{P}_i^h(z_{P_i}), i, ..., n-1, \hat{n}(z_{P_i})\right) \\
&\qquad - \mathrm{Res}_{z\to\infty} \frac{A(z)}{z}.
\end{aligned} \tag{184}$$

In the case that the boundary term vanishes, one arrives at the BCFW recursion relation,

$$A_n = \sum_{i=3}^{n-1} \sum_{h=\pm} A_L \left(\hat{1}(z_{P_i}), 2, ..., i-1, -\hat{P}_i^{-h}(z_{P_i})\right) \frac{i}{P_i^2} A_R \left(\hat{P}_i^h(z_{P_i}), i, ..., n-1, \hat{n}(z_{P_i})\right), \tag{185}$$

where $z_{P_i} = \frac{P_i^2}{\langle n|P_i|1]}$, and $P_i = p_1 + ... + p_{i-1}$. The assumption that the residue at infinity vanishes requires that $A_n(z) \cong z^{-1}$, which depends on the helicities of the shifted legs. One can show that

$$A(\hat{1}^+, \hat{n}^-) \overset{z\to\infty}{\sim} \frac{1}{z} \qquad A\left(\hat{1}^+, \hat{n}^+\right) \overset{z\to\infty}{\sim} \frac{1}{z} \qquad A(\hat{1}^-, \hat{n}^-) \overset{z\to\infty}{\sim} \frac{1}{z} \tag{186}$$

$$A(\hat{1}^-, \hat{n}^+) \overset{z\to\infty}{\sim} z^3. \tag{187}$$

The $\hat{1}^-$ and $\hat{n}^+$ shift is thus not preferred for recursion purposes. Partial amplitude properties, such as cyclicity and parity, can aid in arranging the appropriate helicity configurations such that the BCFW shift does not induce a pole at infinity. Theories whose amplitudes' BCFW shifts incur unavoidable boundary terms are referred to as non-constructible.

3.2.2 Applications of Amplitude Recursion Relations

Starting from the 3 point amplitudes as the base case, we can use the BCFW recursion relation to build an expression for a generic MHV amplitude. Without loss of generality, we can consider the case where leg 1 and n have negative helicity. By the BCFW recursion relation, we can write an n-point MHV amplitude as

$$A_n^{\mathrm{MHV}} = \sum_{i=3}^{n-1} \sum_{h=\pm} A_L \left(\hat{1}^-(z_{P_i}), 2, ..., i-1, -\hat{P}_i^{-h}(z_{P_i}) \right) \frac{i}{P_i^2} A_R \left(\hat{P}_i^h(z_{P_i}), i, ..., n-1, \hat{n}(z_{P_i})^- \right). \tag{188}$$

Note that because A_n^{MHV} only has two external legs with negative helicity, each belonging to the A_L and A_R amplitudes, either A_L or A_R will only have one negative helicity gluon. Thus our sum restricts itself to having a 3-point amplitude on either side of the BCFW bridge, since higher point amplitudes with less than two negative helicity gluons are zero. Subject to these constraints the recursion relation becomes:

$$A_n^{\mathrm{MHV}} \tag{189}$$
$$= A_3^{\overline{\mathrm{MHV}}} \left(\hat{1}^-(z_{P_3}), 2^+, -\hat{P}_3^+(z_{P_3}) \right) \frac{i}{P_3^2} A_{n-1}^{\mathrm{MHV}} \left(\hat{P}_3^-(z_{P_3}), 3^+, ..., n-1^+, \hat{n}(z_{P_3})^- \right) \tag{190}$$
$$+ A_{n-1}^{\mathrm{MHV}} \left(\hat{1}^-(z_{P_{n-1}}), 2^+, ..., n-2^+, -\hat{P}_{n-1}^-(z_{P_{n-1}}) \right) \frac{i}{P_{n-1}^2} A_3^{\overline{\mathrm{MHV}}} \left(\hat{P}_{n-1}^+(z_{P_{n-1}}), n-1^+, \hat{n}(z_{P_{n-1}})^- \right).$$

We now turn out attention to

$$A_3^{\overline{\mathrm{MHV}}} \left(\hat{P}_{n-1}^+(z_{P_{n-1}}), n-1^+, \hat{n}(z_{P_{n-1}})^- \right) = A_3^{\overline{\mathrm{MHV}}} \left(P_{n-1} - z_{P_{n-1}} \tilde{\lambda}_1 \lambda_n{}^+, n-1^+, p_n + z_{P_{n-1}} \tilde{\lambda}_1 \lambda_n{}^- \right). \tag{191}$$

Using momentum conservation, we have

$$P_{n-1} = p_1 + ... + p_{n-2} = -p_{n-1} - p_n, \tag{192}$$

The requirement for 3 point $\overline{MHV}$ amplitudes to be non-zero is that all angle brackets are zero. This poses an issue because, for $\hat{p}_n^{\dot{\alpha}\alpha} = \hat{\tilde{\lambda}}_n^{\dot{\alpha}} \hat{\lambda}_n^{\alpha} = (\tilde{\lambda}_n + z\tilde{\lambda}_1)^{\dot{\alpha}} \lambda_n^{\alpha}$, we have

$$0 = \langle \hat{p}_n | p_{n-1} \rangle = \langle p_n | p_{n-1} \rangle, \tag{193}$$

and hence $A_3^{\overline{\text{MHV}}}$ only contributes when p_{n-1} and p_n are collinear. We thus discard this case and concentrate on the first term now:

$$A_n^{\text{MHV}} = A_3^{\overline{\text{MHV}}} \left(\hat{1}^-(z_{P_3}), 2^+, -\hat{P}_3^+(z_{P_3}) \right) \frac{i}{P_3^2} A_{n-1}^{\text{MHV}} \left(\hat{P}_3^-(z_{P_3}), 3^+, ..., n-1^+, \hat{n}(z_{P_3})^- \right) \tag{194}$$

with

$$z_{P_3} = \frac{(p_1 + p_2)^2}{\langle n | p_1 + p_2 | 1]} = \frac{\langle 12 \rangle [21]}{\langle n2 \rangle [21]} = \frac{\langle 12 \rangle}{\langle n2 \rangle}. \tag{195}$$

We use the previously derived 3 point amplitude

$$A_3^{\overline{\text{MHV}}} \left(\hat{1}^-(z_{P_3}), 2^+, -\hat{P}_3^+(z_{P_3}) \right) = -ig \frac{[2\ (-\hat{P}_3(z_{P_3}))]^3}{[\hat{1}(z_{P_3})\ 2][(-\hat{P}(z_{P_3}))\ \hat{1}(z_{P_3})]}, \tag{196}$$

and guess

$$A_{n-1}^{\text{MHV}} \left(\hat{P}_3^-(z_{P_3}), 3^+, ..., n-1^+, \hat{n}(z_{P_3})^- \right) = ig^{n-3} \frac{\langle \hat{n}(z_{P_3})\ \hat{P}_3(z_{P_3}) \rangle^3}{\langle \hat{P}_3(z_{P_3}) 3 \rangle \langle 34 \rangle ... \langle (n-1)\ \hat{n}(z_{P_3}) \rangle} \tag{197}$$

to show that the Parke Taylor formula holds recursively, noting the base case of the A_3^{MHV} amplitude. Noting that λ_n and $\tilde{\lambda}_1$ remain unshifted in our $[n1\rangle$ shift, as well as the following identities

$$\langle \hat{n} \hat{P}_3 \rangle [\hat{P}_3 2] = \lambda_n^{\alpha} \hat{P}_{3\alpha\dot{\alpha}} \tilde{\lambda}_2^{\dot{\alpha}} = \lambda_n^{\alpha} (p_{1\alpha\dot{\alpha}} + p_{2\alpha\dot{\alpha}} - z_{P_3} \tilde{\lambda}_{1\dot{\alpha}} \lambda_{n\alpha}) \tilde{\lambda}_2^{\dot{\alpha}} = \lambda_n^{\alpha} p_{1\alpha\dot{\alpha}} \tilde{\lambda}_2^{\dot{\alpha}} = \langle n1 \rangle [12] \tag{198}$$

$$\langle 3\hat{P}_3\rangle[\hat{P}_3 1] = \lambda_3^\alpha \hat{P}_{3\alpha\dot{\alpha}}\tilde{\lambda}_1^{\dot{\alpha}} = \lambda_3^\alpha(p_{1\alpha\dot{\alpha}} + p_{2\alpha\dot{\alpha}} - z_{P_3}\tilde{\lambda}_{1\dot{\alpha}}\lambda_{n\alpha})\tilde{\lambda}_1^{\dot{\alpha}} = \lambda_3^\alpha p_{2\alpha\dot{\alpha}}\tilde{\lambda}_1^{\dot{\alpha}} = \langle 32\rangle[21] \tag{199}$$

$$P_3^2 = (p_1 + p_2)^2 = \langle 12\rangle[21], \tag{200}$$

we can put things together

$$A_n^{\text{MHV}} = ig^{n-2}\frac{[2\;(-\hat{P}_3(z_{P_3}))]^3}{[\hat{1}(z_{P_3})\;2][(-\hat{P}(z_{P_3}))\;\hat{1}(z_{P_3})]}\frac{1}{\langle 12\rangle[21]}\frac{\langle \hat{n}(z_{P_3})\;\hat{P}_3(z_{P_3})\rangle^3}{\langle \hat{P}_3(z_{P_3})3\rangle\langle 34\rangle...\langle (n-1)\;\hat{n}(z_{P_3})\rangle} \tag{201}$$

$$= -ig^{n-2}\frac{[2\;\hat{P}_3(z_{P_3})]^3}{[\hat{1}(z_{P_3})\;2][\hat{P}_3(z_{P_3})\;\hat{1}(z_{P_3})]}\frac{1}{\langle 12\rangle[21]}\frac{\langle \hat{n}(z_{P_3})\;\hat{P}_3(z_{P_3})\rangle^3}{\langle \hat{P}_3(z_{P_3})3\rangle\langle 34\rangle...\langle (n-1)\;\hat{n}(z_{P_3})\rangle} \tag{202}$$

$$= ig^{n-2}\frac{\langle n1\rangle^3[12]^3}{[12][21]\langle 12\rangle\langle 23\rangle[21]\langle 34\rangle...\langle (n-1)n\rangle} \tag{203}$$

$$A_n^{\text{MHV}}(1^-, 2^+, 3^+, ..., n-1^+, n^-) = ig^{n-2}\frac{\langle n1\rangle^4}{\langle 12\rangle\langle 23\rangle...\langle (n-1)n\rangle\langle n1\rangle}. \tag{204}$$

Thus we see that, with the 3-point amplitude base case, we have shown by induction that MHV amplitudes follow the Parke-Taylor formula.

Note that the MHV amplitude does not have multi-particle poles because, by the BCFW recursion relation and the fact that $A_n(1^\pm, 2^+, ..., n^+) = 0$, whenever we construct one we find that we can at most have 3-point vertices on either side of the BCFW bridge. Anything more would lead to either the left or the right amplitude being zero, thus at most two adjacent momenta can occur the in BCFW propagator. Moreover, MHV amplitudes are holomorphic functions of the spinor variables λ.

For completeness, let's note that the anti-MHV amplitude, $A_n^{\overline{\text{MHV}}}(1^+, 2^-, 3^-, ..., n-1^-, n^+)$, can be found by the same trick as we used for the 3-point amplitudes: Parity flips helicities of external legs and exchanges angle and square brackets. Thus we have

$$A_n^{\overline{\text{MHV}}}(1^+, 2^-, 3^-, ..., n-1^-, n^+) = ig^{n-2}\frac{[n1]^4}{[12][23]...[(n-1)n][n1]}. \tag{205}$$

Interestingly, the MHV amplitude is indifferent to where the negative helicity legs occur in the order.

3.3 Factorization Properties of Tree-Level Amplitudes

From the BCFW recursion relations, it is apparent that the analytic structure of tree-level amplitudes is governed by propagator poles and their residues. The residues are given by products of lower-point tree-level amplitudes. Color ordered amplitudes can have poles when momenta $P_{ij} := p_i + p_{i_1} + ... + p_j$ go on shell, and the amplitudes factorize as

$$A_n^{\text{tree}}(1, \ldots, n) \sim \sum_h A_L \left(i, \ldots, j, -P^h\right) \frac{i}{P_{ij}^2} A_R \left(P^{-h}, j+1, \ldots, i-1\right). \tag{206}$$

P_{ij} creates a *two particle pole* if P_{ij} is only formed by two external momenta; otherwise it's called a *multi-particle pole*. With the BCFW recursion prescribing a method for constructing higher point amplitudes from lower point ones, the, in a sense, reverse process of factorizing amplitudes on internal BCFW bridge momenta proves incredibly powerful. Suppose one encounters a candidate amplitude that one wishes to verify the validity of. Checking that the candidate amplitude correctly factorizes into products of proper, lower-point amplitudes serves as a strong way of checking that the candidate amplitude is indeed correct.

Factorization, as a condition for healthy amplitudes, can also be used to constrain and bootstrap amplitudes, along with other principles such as little group scaling, unitarity, and crossing symmetry, to name a few examples. As was shown for the three-point amplitude, where it was constrained up to an overall constant by little group scaling and Lorentz invariance, generic amplitude principles can prove powerful in predicting scattering amplitudes, especially in cases where one is not bestowed with a Lagrangian.

3.4 The Minimal Compton Amplitude & Classical Limit

With the success of little group scaling for determining the three-point amplitudes, and amplitude factorization in dissecting higher point amplitudes, one might ask whether, at 4-point, little group scaling and factorization in the s, t, and u channels suffices for constraining the tree-level

Compton amplitude. In the landmark paper [109], dubbing their result a so-called "Minimal" Compton amplitude, the authors investigated exactly this, and found for a massive spin $S \leq 1$ particle

$$\mathcal{A}_{\text{QED}}\left(-\mathbf{1}^S, \mathbf{2}^S, 3^-, 4^+\right) = \frac{y^2}{t_{13}t_{14}}\left(\frac{\langle 3\mathbf{1}\rangle[4\mathbf{2}] - \langle 3\mathbf{2}\rangle[4\mathbf{1}]}{y}\right)^{2S} \tag{207}$$

where the boldfaced spinor helicity variables denote the $SU(2)$ multiplet of two helicity spinors corresponding to a massive external leg. In equation 207, $t_{1i} = (p_1 - q_i)^2 - m^2$, $s_{34} = (q_3 + q_4)^2$, and $y := [4|p_1|3\rangle$, where p_1 and p_2 are the massive legs. The bold-face notation suppresses the $SU(2)$ indices, $I = 1, 2$, but each term is, indeed, an $SU(2)$ invariant due to the presence of two bold-faced variables within each. The interesting observation, hence $S \leq 1$, is that because y can become zero when q_3 becomes collinear with p_1, equation 207 exhibits a spurious pole for $S > 1$. As mentioned, restricting $S \leq 1$ addresses this limitation, but taking the classical limit of 207, as was done for quantum electrodynamics (QED) in [94], yields

$$\mathcal{A}^s = \frac{y^2}{t_{13}t_{14}} \exp\left\{\left[q_4 - q_3 + \frac{(t_{14} - t_{13})}{y} w\right] \cdot \mathfrak{a}\right\}, \tag{208}$$

where $\mathfrak{a}$ becomes the classical rescaled spin $a^\mu = S^\mu/m$ in the classical limit. If one expands this classical Compton amplitude, one encounters a spurious pole when going beyond $O(a^2)$ — that is quadratic order in spin. Consequently, the classical limit of the minimal Compton of equation 207 exhibits unphysical poles when going beyond quadratic order in spin. This begs the question as to whether other physical principles can be used to find a minimal Compton amplitude to higher orders in spin.

4 Root Kerr

4.1 Classical Double Copy & The Root Kerr Field

The term double copy refers to, in the context of scattering amplitudes of quantum field theories, to a mapping between amplitudes of, typically, simpler theories to typically more complicated theories. The famous relation between gauge theory and gravitational amplitudes, often summarized as gravity = (gauge theory)^2, is a prominent example of a large web of relations between various quantum field theories' amplitudes [128]. Given the existence of such relations between various theories' asymptotic observables, it naturally begs the question as to whether similar relations exist for classical field configurations in the bulk. The study of classical double copy solutions, specifically between classical gauge theories and general relativity, generally revolves around the decomposition of certain spacetime metrics into a so-called Kerr-Schild form.

For a class of solutions to Einstein's equations, called Kerr-Schild solutions [120], one can decompose their metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi k_\mu k_\nu, \tag{209}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and k_μ is a vector field which is null with respect to both the Minkowski and the full metric: $g^{\mu\nu} k_\mu k_\nu = 0 = \eta^{\mu\nu} k_\mu k_\nu$. This decomposition is permitted for a number of highly symmetric solutions, and our attention will be turned to such a case: The Kerr spacetime. The associated "single copy" gauge field is of the form $A_\mu = \phi k_\mu$, where ϕ and φ differ by factors of the coupling constants of the theories in question. For our purposes, we turn our attention to the relationship between the Kerr spacetime solution and its single copy: The electromagnetic gauge field called "Root Kerr" or $\sqrt{\text{Kerr}}$. This chapter, specifically the Root Kerr overview, largely follows my paper [1].

4.1.1 The Janis-Newman Shift

In 1965, [129] showed that performing a complex coordinate translation $x^\mu \to x^\mu - ia^\mu$, for $a^\mu = (0, 0, 0, a)$, mapped the Schwarzschild solution to Einstein's equations to those of the Kerr metric with rescaled spin a. In 2002, [122] showed an analogous result, starting from the Coulomb potential for a charge q and arriving at a rigid rotating charge distribution defined by the potential

$$\Phi \to \frac{q}{r^2} \to \frac{q}{\sqrt{x^2 + y^2 + (z - ia)^2}}. \tag{210}$$

The resulting electric and magnetic fields are given by

$$\vec{E} + i\vec{B} = -\nabla\Phi = \frac{q(\vec{r} - ia\hat{z})}{|\vec{r} - ia\hat{z}|^{3/2}}, \tag{211}$$

for $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\vec{a} = a\hat{z}$. The associated charge distribution is a rotating disk, of angular velocity $1/a$, in the $z = 0$ plane with charge distribution

$$\sigma = -\frac{q}{2\pi} \frac{a}{(a^2 - r^2)^{3/2}} \qquad \text{for } r < a, \tag{212}$$

otherwise zero.

As figure 1 depicts, in the sense of Janis-Newman shifts and classical double copy, how $\sqrt{\text{Kerr}}$ is to Coulomb as Kerr is to Schwarzschild, and $\sqrt{Kerr}$ is to Kerr as Coulomb is to Schwarzschild. Because of this, even if only as a toy model, $\sqrt{\text{Kerr}}$ proves as a useful electromagnetic model for its gravitational double copy, Kerr.

4.1.2 G →0 Limit of Kerr-Newman

In the case of Kerr, k_μ exhibits a classical double copy relationship with the following electromagnetic gauge field [121]: Considering the Kerr-Newman solution, a charged and spinning black hole, one can take the $G \to 0$ limit, at fixed charge and fixed spin per mass, to recover a flat spacetime metric along with a nontrivial electromagnetic field, that of a charged spinning ring-disk singularity. Specifically, consider the Kerr-Newman metric and vector potential in

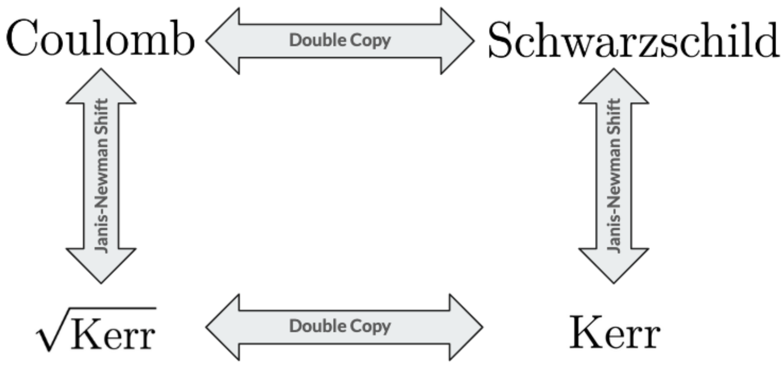

Figure 1: Janis-Newman Shift & Double Copy

Boyer-Lindquist coordinates (t, r, θ, ϕ),

$$\begin{aligned} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= -\frac{\Delta}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right)^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}\left(\left(r^2+a^2\right)\mathrm{d}\phi - a\,\mathrm{d}t\right)^2 \\ A_\mu \mathrm{d}x^\mu &= -\frac{Qr}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right), \end{aligned} \tag{213}$$

with

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 + GQ^2 - 2GMr, \tag{214}$$

outside a black hole with mass M, spin $S = Ma$, and electric charge Q. Taking the $G \to 0$ limit we have

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \frac{\Sigma}{r^2+a^2}\,\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \left(r^2+a^2\right)\sin^2\theta\,\mathrm{d}\phi^2, \tag{215}$$

$$A_\mu \mathrm{d}x^\mu = -\frac{Qr}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right), \tag{216}$$

namely a flat metric in oblate-spheroidal coordinates along with the $\sqrt{\text{Kerr}}$ gauge field. We can relate these to the Cartesian coordinates as

$$t = t, \quad x + iy = \sqrt{r^2+a^2}\sin\theta\, e^{i\phi}, \quad z = r\cos\theta, \tag{217}$$

implying

$$(r + ia\cos\theta)^2 = x^2 + y^2 + (z + ia)^2 = |\boldsymbol{x} + i\boldsymbol{a}|^2. \tag{218}$$

Defining, via Cartesian components, the time-like Killing vector $T^\mu := (1, 0, 0, 0)$, the rescaled spin vector $a^\mu := (0, 0, 0, a)$, and the spacetime displacement from the origin $x^\mu := (t, x, y, z)$, then $|\mathbf{x}| := \sqrt{x_\mu x^\mu + (T_\mu x^\mu)^2}$ is the magnitude of the projection of x^μ orthogonal to T^μ. With this and

$$\mathcal{R} := r + ia\cos\theta = |\boldsymbol{x} + i\boldsymbol{a}| = \sqrt{(x + ia)_\mu (x + ia)^\mu + \left[T_\mu (x + ia)^\mu\right]^2}, \tag{219}$$

it can be shown that

$$A_\mu = \left[T_\mu \cos(a\cdot\partial) + \epsilon_{\mu\nu\alpha\beta} T^\nu a^\alpha \partial^\beta \frac{\sin(a\cdot\partial)}{a\cdot\partial}\right] \frac{Q}{|\boldsymbol{x}|}, \tag{220}$$

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} = \frac{1}{2} G_{\mu\nu}{}^{\alpha\beta} T_\alpha (x + ia)_\beta \frac{Q}{\mathcal{R}^3} + c.c., \tag{221}$$

where $G_{\mu\nu}{}^{\alpha\beta} := 2\delta_{[\mu}{}^{\alpha}\delta_{\nu]}{}^{\beta} + i\epsilon_{\mu\nu}{}^{\alpha\beta}$ is the tensor which projects a 2-form onto 4 times its anti-self dual part. An important building block for working with the $\sqrt{\text{Kerr}}$ background is the anti-self dual 2-form

$$N_{\mu\nu} := iG_{\mu\nu}{}^{\alpha\beta} T_\alpha \partial_\beta \mathcal{R} = iG_{\mu\nu}{}^{\alpha\beta} T_\alpha (x + ia)_\beta \frac{1}{\mathcal{R}}, \tag{222}$$

as it allows us to re-write $F_{\mu\nu}$ as

$$F_{\mu\nu} = -\frac{iQ}{2\mathcal{R}^2} N_{\mu\nu} + \frac{iQ}{2\overline{\mathcal{R}}^2} \bar{N}_{\mu\nu}, \tag{223}$$

along with expressing the Killing-Yano tensor

$$Y_{\mu\nu} = \frac{1}{2}\left(\mathcal{R} N_{\mu\nu} + \overline{\mathcal{R}} \bar{N}_{\mu\nu}\right), \tag{224}$$

from which the Killing tensor $K_{\mu\nu} = Y_{\mu\rho}Y^{\rho}{}_{\nu}$ can be built. N and its complex conjugate $\bar{N}$ can be thought of as "square roots of the metric" in the sense

$$N_{\mu\rho}N_{\nu}{}^{\rho} = \eta_{\mu\nu} = \bar{N}_{\mu\rho}\bar{N}_{\nu}{}^{\rho}. \tag{225}$$

Moreover the tensor $\pi_{\mu\nu} := N_{\mu\rho}\bar{N}^{\rho}{}_{\nu}$ satisfies $\pi_{\mu\nu} = \pi_{\nu\mu} = \bar{\pi}_{\mu\nu}$ and $\pi_{\mu}{}^{\mu} = 0$.
Lastly, these building blocks have the following closed set of differential relations:

$$\partial_\alpha N_{\mu\nu} = \frac{-i}{\mathcal{R}}\left(G_{\mu\nu\alpha\beta} - N_{\mu\nu}N_{\alpha\beta}\right)T^{\beta}, \tag{226}$$

$$\partial_\mu \mathcal{R} = -iN_{\mu\nu}T^{\nu}, \tag{227}$$

$$\partial_\alpha Y_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}T^{\beta}, \tag{228}$$

given $\partial_\alpha G_{\mu\nu\rho\sigma} = 0$. These relations are used in reducing derivatives of tensors to fundamental variables, as discussed in section 5.6. These same constructs of $\mathcal{R}$, N, and Y exist analogously in the Kerr spacetime, and are straight-forwardly related to the Newman-Penrose tetrad associated with oblate-spheroidal coordinates.

5 Computation Overview

To evaluate ansätze and verify the conservation of the Carter and Rüdiger constants, we seek to replace functions differentiated with respect to the worldline parameter, say the proper time τ, with functions purely on configuration space. Performing this replacement of τ-dotted variables with the equations of motion, derived from the action principle, and then setting the resulting expression to zero, allows us to solve for generic Wilson coefficients in the action, as well as coefficients in the Carter and Rüdiger ansätze, that yield Carter and Rüdiger constant conservation. This chapter details, and serves as an instruction manual, for many of the intermediate steps in performing this computation using the Wolfram Mathematica packages xTensor, xAct, and xCoba. Section 5.1 gives an overview of the functionalities of the xTensor Mathematica package. Sections 5.2, 5.3, 5.4, 5.6, and 5.7 pertain to the core details of the computation, namely computing and spin-expanding the equations of motion, automating the tensor contraction algebra, reducing to an independent set of variables, building the ansätze, and solving for the coefficients. Chapter 5.8 details an auxiliary code, using the package xCoba, to numerically verify the algebraic relations, identities and results used in the main xTensor code. Section 5.5 covers aspects of expediting and automating the computation. As a matter of convention for this chapter, any commas or periods at the end of code environments should be excluded from the reading of the code itself, as these exist for sentence punctuation.

5.1 xTensor Basics

The xTensor Mathematica package is a component of the xAct suite — a collection of free, open source Mathematica packages capable of handling abstract tensor calculus, such as those required in special and general relativity. It enables the construction and manipulation of manifolds and tensor objects thereon, allowing the computerization of tensor algebra operations in a coordinate free context. Its cousin, the xCoba (coordinate-basis) package allows further specification of charts and tensor components, and will be discussed in section 5.8.

5.1.1 Manifolds, Tensors, and Indices

After loading the xTensor package into a kernel using

```
<< xAct`xTensor`
```

we are able to define a manifold using the function

```
DefManifold[ST,4,IndexRange[a,l]],
```

where *ST*, short of space-time, is the manifold's label, 4 is its dimension, and IndexRange[a,l] associates the list of index symbols a,b,...,l with said manifold. The package will throw errors should one attempt to use indices for tensors defined over *ST* that fall outside this list. The label *ST* will serve as the underpinning of all subsequent structures, and subsequent tensors, indices, covariant derivatives, connections, et cetera, are defined in relation thereto.
We can define tensors using the

```
DefTensor[F[a,b],{ST},Antisymmetric[{a,b}]]
```

command. Here F[a,b] is a tensor defined on the manifold *ST* with placeholder indices a and b, between which it exhibits antisymmetry. One could, analogously, define different tensors, with other sets of indices and symmetries. For example, one can define $\nabla_\mu \nabla_\nu F_{\rho\sigma}$ as

```
DefTensor[DDF[a,b,c,d],{ST},
    {Symmetric[{a,b},Antisymmetric[{c,d}]]}],
```

where the antisymmetry of F is given by Antisymmetric[], and the commutativity of covariant derivatives is encoded in Symmetric[], assuming we're working in a flat chart.
Moreover, one can define a Metric using the function

```
DefMetric[-1,g[-a,-b],CD,FlatMetric->True],
```

which holds a special place in the space of tensors on the manifold. For one, the presence of the metric immediately allows for the lowering and raising of indices within each tensor,

```
F[a,b]g[-b,-c]=F[a,-c],
```

and will be used for all raising and lowering manipulations of indices within the underlying manifold's range. In DefMetric, the second argument gives name and associated indices to the metric, while the first denotes its signature, with -1 being a Lorentzian manifold. xTensor makes no assumptions regarding the particular choice of metric, mostly minus versus mostly plus. In xTensor, the distinction between signatures generally does not show up when dealing with abstract tensor index manipulation, since this is a matter for components, though there is one notable exception: The Levi-Civita tensor. Upon defining a metric g, xTensor also immediately defines a totally anti-symmetric symbol, epsilong[a,b,c,d]. In 4 dimensions with Lorentzian signature, $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -4!$, where the -1 factor comes from the metric signature. Because of this, even though xTensor does not care for specific components, this particular argument *does* matter for chart-independent computational purposes.
The third argument of DefMetric specifies the name of the metric-compatible covariant derivative that also automatically gets defined. The fourth argument, FlatMetric-¿True, tells xTensor, in its chart-independent ways, that we're working in a flat spacetime and to assume that the covariant derivatives commute, i.e. $CD[a][CD[b][...]] = CD[b][CD[a][...]]$.

5.1.2 Parameters and Constant Symbols

xTensor also handles parameters, such as a worldline's proper time. We can introduce a new parameter τ with

```
DefParameter[\[Tau]],
```

and allow subsequently defined structures, such as a scalar dynamical mass function,

```
DefTensor[M[],{ST,\[Tau]}],
```

to be differentiated with respect to τ. The keen-eyed reader might notice that the dynamical mass function $\mathcal{M}$ is a function defined *only* along the worldline, not all of spacetime, and hence including ST as a dependence in its definition is technically improper. However, in order to cue xTensor's chain-rule to appropriately catch all dependencies, it is necessary to include both ST and τ. Were one to omit ST, then covariant derivatives, which should act on the structures such as $F_{\mu\nu}$ sitting inside $\mathcal{M}$, would pass over such instances of $F_{\mu\nu}$.

In order to take such worldline derivatives, one employs the function

```
ParamD[\[Tau]][...],
```

which symbolically distributes the τ-derivatives across the expression's interior in accordance with the Leibniz rule. Once distributed, one can then employ replacement rules, such as those for equations of motion

```
ParamD[\[Tau]][S[a_,b_]]:>...,
```

to suitably replace the derivatives.

To contrast, suppose one wishes to introduce variables which one wishes to keep explicitly constant with respect to all xTensor dependencies. To this end, one can use the function DefConstantSymbol as

```
DefConstantSymbol[c]
```

or

```
DefConstantSymbol[{c1,c2,c3,...,cn}]
```

to define such a variable, or a list of constant variables. While this may seem unnecessary, upon the application of xTensor's ToCanonical, along with other Head-sensitive operations, it is prudent to encode any constant variables using DefConstantSymbol to avoid xTensor canonicalization issues.

5.1.3 Modules, Dummy Indices, and Scalars

Beyond their definitions through DefTensor, we routinely employed replacement rules for tensors, and their derivatives, in terms of others. Often, the right-hand side of the replacement rule wound up involving index contractions, hence we often needed to use Module wrappers in order to define dummy indices. As an example, mapping F to its dual, FD with the function

```
FtoFD = {F[a_, b_] :> Module[{c, d},
-1/2*(epsilon\[Eta][a, b, -c, -d]*FD[c, d])
]}
```

warranted the use of a Module in order to treat the indices c and d as dummies, such that, along with the DelayedRule symbol `:>`, new indices are assigned to the index contraction at every instance of FtoFD being called.

One could achieve the same as Module, in this instance, by applying the ReplaceDummyIndices function to the right-hand side. This proves efficient in instances where the right-hand side has yet to be fully computed, as every time a DelayedRule is called it's fully re-evaluated. In cases where a replacement rule is inserting a large expression, it is prudent to pre-compute the expression once outside of a rule, such as for a triple covariantly differentiated field strength,

```
cdcdcdF = CD[l][CD[k][CD[j][F[h, i]/.Fton]]] // CC,
```

whereafter one can define a delayed replacement rule

```
DDDFton={DDDF[a_, b_, c_, d_, e_] :>
    ReplaceDummies[Evaluate[cdcdcdF //ReplaceIndex[#,
    {l -> a, k -> b, j -> c, h -> d, i ->e}] &
]]};,
```

without re-evaluating the derivative upon each calling of DDDFton. ReplaceIndex serves to swap the l,k,j,h,i for the pattern matching a,b,c,d,e, as defined in the left-hand side. The ReplaceDummies function, here, is superior to the Module wrapper since it's a priori unclear which dummy indices are introduced by the Mathematica kernel in computing cdcdcdF, for which Module would require an exact list but ReplaceDummies remains flexible in regard to.

Lastly, xTensor automatically defines the functions PutScalar and NoScalar. PutScalar identifies index-contracted tensors and collects them under a single Scalar head. NoScalar, when applied to an expression, does the opposite — removing every instance of the Scalar in favor of its constituents, along with applying ReplaceDummyIndices as to not overuse existing indices. A note of caution in using these, however, is that replacement rules seem inconsistent in their ability to match inside the Scalar head, as Mathematica often encounters errors when performing series expansions involving Scalar wrappers, such as in expanding the denominator of the $\dot{z}$ equation of motion.

5.2 Vector Derivative Code

In the electromagnetic MPD equations, the transport equations for the spin tensor S, vector s, and worldline z^μ, involve taking derivatives of a generic dynamical mass function $\mathcal{M}$ with respect to worldline vector quantities: Momentum p (or u) and spin s. This occurs through the computation of the torque tensor $\mathcal{N}$ and explicitly in the equation of motion of the spin vector s:

$$\dot{S}_{\mu\nu} - 2p_{[\mu}\dot{z}_{\nu]} = \mathcal{N}_{\mu\nu} = -\alpha\left(p_{[\mu}\frac{\partial}{\partial p^{\nu]}} + s_{[\mu}\frac{\partial}{\partial s^{\nu]}}\right)\mathcal{M}^2, \tag{229}$$

$$\dot{s}^\mu = \frac{\alpha}{2}\left(\epsilon^{\mu\nu}{}_{\rho\sigma}u^\rho s^\sigma\frac{\partial}{\partial s^\nu} - \frac{u^\mu}{\mathcal{M}}s^\nu\frac{\partial}{\partial z^\nu}\right)\mathcal{M}^2 + \frac{u^\mu}{\mathcal{M}}qF_{\nu\rho}s^\nu\dot{z}^\rho, \tag{230}$$

$$\dot{z}^\mu = u^\mu - \frac{\mathcal{N}^{\mu\nu}u_\nu}{\mathcal{M}} - S^{\mu\nu}\frac{\mathcal{F}_\nu + qF_{\nu\rho}(u^\rho - \mathcal{N}^{\rho\sigma}u_\sigma/\mathcal{M})}{\mathcal{M}^2 - \frac{1}{2}qF_{\alpha\beta}S^{\alpha\beta}}. \tag{231}$$

This particular form of the MPD equations came about as a consequence of the change of variables from $\mathcal{M}(p, S, z) \to \mathcal{M}(u, s, z)$. While derivative of $\mathcal{M}$ with respect to momentum p stands out, it can easily be re-cast in terms of u via the chain rule

$$\frac{\partial\mathcal{M}^2}{\partial p^\mu} = \mathcal{P}^\nu{}_\mu 2\frac{\partial\mathcal{M}}{\partial u^\nu}, \tag{232}$$

where $\mathcal{P}^\nu_\mu = u^\nu u_\mu + \eta^\nu{}_\mu$ is the projector onto u. As a result, it is prudent to explicitly write a dynamical mass function in terms of the variables s and u, in lieu of S and p, for easy differentiation.

Taking these derivatives for generic dynamical mass functions becomes arduous with an increasing number of effective terms in the action, warranting automation, and, unlike the built-in covariant differentiation machinery that comes from defining a manifold and metric in xTensor, this vector differentiation machinery is custom.

5.2.1 Differential Operators

First we begin with defining a function multiD as

```
multiD[der_, f_[args__]] := (
Derivative[Sequence @@ #][f][args]
& /@ IdentityMatrix[Length[{args}]]) .
(der /@ ReplaceDummies /@ {args});,
```

which serves, in essence, as multi-variable chain rule. The IdentityMatrix creates an $n \times n$ identity matrix, where n is the number of arguments of f. We then apply Derivative[Sequence @@][f][args], parses the identity matrix as n lists and feeds them to Mathematica's Derivative function. Each list is then converted to a Sequence, merely removing the curly-brackets of the list, and threaded into a list of derivatives,

```
{Derivative[1,0,0,...,0][f][args],
Derivative[0,1,0,...,0][f][args],
Derivative[0,0,1,...,0][f][args],
...,
Derivative[0,0,0,...,1][f][args],},
```

giving the gradient of f with respect to its arguments, args. The gradient is then dot-producted into the der, where der is a differential operator that we feed to multiD. Schematically the output looks something like

```
multiD[der,f[x_1,x_2,x_3,...,x_n]]=
Derivative[x_1][f]*der[x_1]+
Derivative[x_2][f]*der[x_2]+
...
Derivative[x_n][f]*der[x_n].
```

In the case that der is merely a list of numbers, an easy interpretation of this is as a directional derivative in the direction der. However, we leave the option for der to be an operator, which we reserve for the operator Dg.

The differential operator Dg manifestly satisfies the algebraic properties of a derivation through the lines

```
Dg[sum_Plus] := Map[Dg, sum];
```

```
Dg[x_ y_] := Dg[x] y + x Dg[y];,
```

which encode linearity over sums and the Leibniz product rule.

Dg also specifies objects which it treats as constant, for instance constants defined through xTensor's DefConstantSymbol along with pure numbers and mathematical constants:

```
Dg[_?ConstantQ] = 0;.
```

Additionally, Dg's independencies are explicitly listed for a number of tensors,

```
Dg[\[Eta][__]] = 0;
Dg[epsilon\[Eta][__]] = 0;
...,
```

because we intend to use it for differentiating instances of p^{μ}, u^{μ}, and s^{μ}, only, hence the omission of these three variables from the list of Dg's independencies.

Lastly, Dg occasionally must act on non-constant scalars, such as the dynamical mass function. If said scalar is wrapped in a Scalar head, then Dg removes the head and introduces dummy indices:

```
Dg[Scalar[expr_]]:=Dg[ReplaceDummies[expr]];.
```

If the expression is scalar but constructed using a set of variables, then

```
Dg[f_?ScalarFunctionQ[args___]] := multiD[Dg, f[args]];
```

serves to distribute Dg, in the multivariable chain rule sense of multiD, onto all the arguments entering the scalar's construction.

With this, we finally define our vector-derivative function,

```
Dvec[v_, i_][e_] := Dg[e] /. { Dg[v[j_]] :> delta[i, j] } //
    dropDvar;,
```

where dropDvar is defined as

```
dropDvar[ex_] := ex /. {
            Dg[X_[_]] /; MemberQ[{ u, s, p }, X] -> 0
    } /. {
```

```
                Dg[X_] :> Throw[Dg[X]]
    };.
```

In other words, all the derivative structure and properties are encapsulated by the function Dg. Dvec then employs Dg across an expression, and, once distributed across all fundamental variables, namely s, p, and u since they are not included in Dg's independences, then Dvec replaces instances of

$$\frac{\partial s^\mu}{\partial s^\nu}, \quad \frac{\partial u^\mu}{\partial u^\nu}, \quad \frac{\partial p^\mu}{\partial p^\nu} \tag{233}$$

and replaces them with $\delta^\mu{}_\nu$. However, since this replacement will not match instances of

$$\frac{\partial s^\mu}{\partial p^\mu}, \quad \frac{\partial s^\mu}{\partial p^\mu}, \quad \frac{\partial s^\mu}{\partial p^\mu}, \quad ... \tag{234}$$

despite these terms being caught by Dg, we need the additional function dropDvar which, once the terms of equation 233 have been suitably replaced, eliminates all instances of terms in equation 234.

5.3 Spin Expansion

In order to show the Carter and Rüdiger constant conservation to a particular order in spin, it is necessary to expand the MPD equations of motion to a particular order in spin. This section details many of the tricks and subleties of such a procedure in Wolfram Mathematica with xTensor.

5.3.1 Spin Counting Parameter os and Spin Weight

To manage these spin expansions in Wolfram Mathematica, we find it prudent to introduce a scalar series expansion parameter, referred to as os, for the purposes of employing Mathematica's Series[expr_,{os,0,1}] expansion in that variable, since Series encounters difficulty when dealing with the spin vector s[a_] or spin tensor objects S[a_,b_] directly. For most purposes, barring the caveats discussed in this appendix, it suffices to replace each instance of s[a_] or S[a_,b_] with itself times os. For example, the equations of motion, to all orders in spin, for $\dot{p}$, $\dot{u}$, and $\dot{z}$ face no such caveats, and simply introduce an os whenever either s[a_] or S[a_,b_] occur.

```
pEOM = {ParamD[\[Tau]][p[a_]] :>
     Module[{b}, q*F[a, b] zDot[-b]] + \[ScriptCapitalF][a]} //
   ReplaceDummies;
uEOM = {ParamD[\[Tau]][
     u[a_]] :> (ParamD[\[Tau]][p[a]/(-Module[{b}, p[b] p[-b]])^(
         1/2)] /. pEOM /. ptou // ReplaceDummies)};
zEOM = {zDot[a_] :>
     Module[{b, c, d},
      u[a] - (\[ScriptCapitalN][a, -b] u[b])/
       M[] - (os* S[a, -b] \[ScriptCapitalF][b] +
          os* S[a, b] q F[-b, -c] u[c] -
          os/M[]*S[a, b] q F[-b, -c] \[ScriptCapitalN][c,
            d] u[-d])/(M[]^2 + q *os*FDsu[])] // ReplaceDummies};
```

Figure 2: MPD Transport Equations Code for p, u, and $\dot{z}$.

In the 5.3.1 code, every *explicit* mention of spin, whether vector or tensor, has an os factor. In the equations of motion for $\dot{p}$ and $\dot{u}$, there is only *implicit* spin dependence through the force vector $\mathcal{F}$, or ScriptCapitalF, the torque tensor $\mathcal{N}$, or ScriptCapitalN, the dynamical mass function $\mathcal{M}$, or M[], and FDsu[], which is an atomic Mathematica object that placeholds for Scalar[FD[a,b]s[-a]u[-b]].

One exception pertains to whenever one substitutes for $\dot{s}$ or $\dot{S}$, where having an additional os-factor in front interferes with Wolfram's built in pattern-recognition. Since not every instance of s[a] or S[a,b] that occurs during a computation is immediately preceded by a os, due to the various orderings and factoring of scalars, the matching for a combination of terms can be significantly slower, if not outright unsuccessful. Instead, we divide the os factor from both sides of the replacement rule, making the replacement rule only for one entity. This can be seen explicitly in the equation of motion for $\dot{S}$,

```
SEOM = {ParamD[\[Tau]][S[a_, b_]] :>
     1/os (p[a] zDot[b] - p[b] zDot[a] + \[ScriptCapitalN][a, b])}
        //ReplaceDummies;
```

where this inverse os factor is made explicit. One might wonder if this creates any problems in regards to inverse powers of os, since not all terms in the right hand side, when expressed fully, contain their own powers of os. It is easy to verify, however, that any inverse os terms on the right hand side are identically zero due to anti-symmetry.

Similarly, in the equation of motion for $\dot{s}$,

```
sEOM = {ParamD[\[Tau]][s[a_]] :>
     Module[{b, c, d},
      1/(2 M[]) (epsilon\[Eta][a, b, c, d] DsMsq[-b] u[-c] s[-d] -
          u[a]/M[] s[b] DzMsq[-b]) +
       u[a]/M[] q*F[-b, -c] s[b] zDot[c]] // ReplaceDummies};
```

the os factors are implicit, through the dynamical mass function M and derivatives DsMsq[-b] and DzMsq[-b]. In this instance, the presence of s on both sides causes the os parameter to divide out and, naively, leads to mismatched os and s counting. As we saw in the S[a,b] equation of motion, however, the important thing is the spin *weight* matching on both sides of the replacement rule, which can be verified by checking if the substitution rule is invariant under taking os-¿1 and then rescaling all explicit spin structures by os, i.e. s[a_]-¿os*s[a], S[a_,b_]-¿os*S[a,b], FDsu[]-¿os*FDsu[], Fsu[]-¿os*Fsu[], et cetera.

The spin vector derivative of the dynamical mass function squared, $\frac{\partial \mathcal{M}^2}{\partial s^\mu}$, also involves an inverse power of os due to the explicit removal of a spin vector:

```
DsMsq[a_] := 1/os Dvec[s, a][MOS] // CC // ReplaceDummies;
```

Lastly, the variable transformations between S and s, the Schouten identities involving spin, and the replacement rules for the atomic scalars in the dynamical mass function all exhibit equal spin weight on both sides.

5.3.2 In-homogeneous Spin Weight and Series Expansion

As alluded to in the previous section, some constructions are of in-homogeneous spin weight, namely different terms within an expression carry differing powers of spin, or os. These warrant particular attention in the sense that they ought be substituted in *fully* before performing a Series expansion in os. While one can, in principle, delay substitution of some in-homogeneous spin weight quantities and perform subsequent expansions to account for this, one runs the risk of oversight and foregoing the full spin expansion of some terms. In particular, given the worldline derivative chain-rule-based approach of this computation, one often encounters the worldline derivatives acting on a tensor, say $\frac{d}{d\tau}F^{\mu\nu}(z(\tau))$, are not preemptively expanded into $\dot{z}^{\rho}\partial_{\rho}F^{\mu\nu}$ before the application of the $\dot{z}$ equation of motion. As a consequence, and debugging insight, worldline derivatives occurring in the computation *after* the Series expansion are an indication of the output not being fully evaluated, requiring more work to substitute for and re-expand the term in spin. This particular limitation can largely be addressed by the use of UpValues, which will be discussed more extensively in 5.4, to immediately replace instances of $\frac{d}{d\tau}$ with $\dot{z}^{\rho}\partial_{\rho}$ whenever acting on spacetime-defined tensors evaluated along the worldline.

```
ParamD /: ParamD[\[Tau]][CD[a_][expr_]] :=
 Module[{b}, zDot[-b] CD[b][CD[a][expr]]];
Y /: ParamD[\[Tau]][Y[a_, b_]] :=
  Module[{c}, zDot[-c] (CD[c][Y[a, b]])] // CC // ReplaceDummies;
F /: ParamD[\[Tau]][F[a_, b_]] :=
  Module[{c}, zDot[-c] (DF[c, a, b])] // CC // ReplaceDummies;
```

Consequently, we suggest the general prescription of applying all replacement rules with in-homogeneous spin weight *before* performing the Series expansion in os. In addition to the anticipated culprits of $\mathcal{M}$, its derivatives, aequations of motion, and others, this particularly applies to the substitution of the momentum p for its normalized counterpart u. In alignment with our choice, as briefly discussed in section 2.2, to parameterize our dynamical mass function $\mathcal{M}(p, S, z) \to \mathcal{M}(u, s, z)$ such that $\mathcal{M}$ depends only on p through its direction u, it is natural to reduce the independent, normalized variable u, unlike $p = \mathcal{M}u$ which exhibits s-dependence through its magnitude $\mathcal{M}$.

Turning to the idiosyncrasies of the Series operation itself, it does not permit the use of a xTensor Tensor -head as the expansion parameter argument, hence the need to introduce os as a counting and expansion parameter. Moreover, Series often encounters problems when performing series expansions of expressions with non-atomic objects in denominators, even when wrapped in a Scalar -head. The use of atomic variables is particularly apparent in Wolfram Language implementation of the $\dot{z}$ equation of motion, as can be seen in 5.3.1. Also in the denominator is M[]^2, $whichnecessitatestheconstructionofthedynamicalmassfunctionas$

```
MtoFus = {M[] :> (m^2 + 2 q*os*C1*FDsu[] + C2/m*q*os^2*sDFsu[] -
      C3/(3 m^2)*q*os^3*ssDDFDsu[] + E3/(3 m^2)*q*os^3*ssuuDDFDsu[]
         +
      D2a/m^2*q^2*os^2*Fsu[]^2 + D2b/m^2*q^2*os^2*FDsu[]^2 +
      D2c/m^2*q^2*os^2*ssFF[] + D2d/m^2*q^2*os^2*ssuFFu[] +
      D3a/m^3*q^2*os^3*FDsusDFsu[] + D3b/m^3*q^2*os^3*ssFDuDFsu[] +
      D3c/m^3*q^2*os^3*ssFDusDFu[] + D3d/m^3*q^2*os^3*ssFDsuDFu[] +
      D3e/m^3*q^2*os^3*FDssDFs[] + D3f/m^3*q^2*os^3*ssFDsDF[])^(1/2)
         };
```

where each of the atomic tensor objects DFsu[], sDFsu[], ssDDFDsu[], et cetera, require their own replacement rule for *after* the Series expansion in os has occurred. Also, after Series has been performed, it is important to apply //Normal before continuing the application of other rules or functions.

This "trick" of atom-izing the Scalar terms also proves necessary in the 5.6.3 section, due to Wolfram Mathematica's intricacies of the CoefficientRules -function when it comes to the particular variables it can take as arguments.

5.3.3 Preemptive Series Expansion of the Equations of Motion

Most of the replacement rules in the previous sections involved the use of DelayedRule (`:>`), as opposed to the regular Rule (-¿), along with a Module -environment on the right hand side. Module, through its first argument, specifically serves to create so-called dummy indices for tensor index contractions on the right hand side. Coupled with the use of DelayedRule, this serves to ensure that, whenever the DelayedRule in question is called, new, previously-unused dummy indices are introduced to facilitate the tensor contractions within the Module. This is important because, without new, unique dummy indices, the use of the replacement rule could lead to duplicate pairs of dummy indices, leading to ambiguous expressions in the computation, akin to seeing $s^\mu s_\mu T^\mu T_\mu$ — are the vectors contracting with themselves or each other? The use of both Module and `:>` avoids this, as it re-evaluates the second Module argument whenever called.

This poses another problem, however, namely one of efficiency: Every time the DelayedRule is called, it has to recompute the replacement, which greatly affects heavier computations. Although not problematic for Module arguments that require no additional evaluation, if, for example, the right hand side is lengthy in its InputForm and one wishes to avoid typographical errors from copy-pasting, we prescribe the following:

Define an immutable variable that you compute *once*, say the covariant derivative of the field strength $\nabla_\mu F_{\nu\rho}$,

```
cdF = CD[j][F[h, i]] /. Fton // CC;
```

wherefrom you define a DelayedRule that only replaces the free and dummy indices, e.g.

```
DFton = Alternatives @@ {CD[a_][F[b_, c_]], DF[a_, b_, c_]} :>
   ReplaceDummies[
    Evaluate[cdF // ReplaceIndex[#, {j -> a, h -> b, i -> c}] &]];
```

thereby preserving the advantages of both prescriptions. Applying this to spin is particularly important, since, to os^3, the equations of motion can take several minutes each to evaluate. Consequently, we compute them once,

```
SEOMOS =
```

```
  ParamD[\[Tau]][os*S[i, j]] /. SEOM /. zEOM /. ptou // CC //
              RA[MtoFus] // Series[#, {os, 0, maxOS}] & // Normal
              // RA[Stos] // CC // RA[DMFrules] // RA[Frules]
              // NoScalar //ExpandAll // CC
              // CleanUp[#, nRules] & // PutScalar
              // Simplification;
pEOMOS = ParamD[\[Tau]][p[i]] ...
uEOMOS = ParamD[\[Tau]][u[i]]...
sEOMOS = ParamD[\[Tau]][os*s[i]]...
zEOMOS = zDot[i]...
```

and save them to external Wolfram language .wl files, as to import rather than re-compute if the kernel needs restarting. With these variables pre-computed we define functions for each of the above variables, such as osSEOM for SEOMOS,

```
osSEOM[expr_, n_Integer /; n >= 1] :=
 Module[{rhs}, rhs = 1/os Series[SEOMOS, {os, 0, n}] // Normal;
  ReplaceAll[expr, ParamD[\[Tau]][S[a_, b_]] :>
    ReplaceDummies[Evaluate[rhs // ReplaceIndex[#, {i -> a, j -> b}]
        &]]]]
```

which produces a replacement rule that substitutes $\dot{S}^{\mu\nu}$ for its equation of motion, with the correct free and new dummy indices, but only up to the order specified by argument n. This also proves powerful for computational efficiency, as any terms of higher spin order than the maximally desired are entirely redundant and only serve to delay the computation unnecessarily. Furthermore, it is prudent to design this series truncation of the MPD equations to be variable in order of spin, since many Carter or Rüdiger terms are not homogeneous in spin weight. For example, suppose one seeks to show conservation to third order in spin, then any ansatz terms of order spin squared need only substituting using linear order of spin MPD equations of motion, as any higher orders would yield quartic spin terms.

Again, there's a slight exception in the construction of these functions due to $\dot{s}$ and $\dot{S}$ being of spin weight 1. Given our desire to avoid unnecessary factors of os on the left hand side due to pattern matching, SEOMOS is computed from $\dot{S}$ with a factor of os, and said factor is later divided on the right hand side in osSEOM. This is in contrast to the MPD equations which have spin weight zero on the left hand side, such as

```
pEOMOS =
  ParamD[\[Tau]][p[i]] /. pEOM /. zEOM /. ptou // CC // RA[MtoFus]
     // Series[#, {os, 0, maxOS}] & // Normal // RA[Stos] // CC
  // RA[DMFrules] // RA[Frules] // NoScalar // ExpandAll // CC
  // CleanUp[#, nRules] & // PutScalar // Simplification;
ospEOM[expr_, n_Integer /; n >= 0] :=
 Module[{rhs}, rhs = Series[pEOMOS, {os, 0, n}] // Normal;
  ReplaceAll[expr, ParamD[\[Tau]][p[a_]] :>
    ReplaceDummies[Evaluate[rhs // ReplaceIndex[#, {i -> a}] &]]]]
```

which proceeds without said subtlety.

5.4 Computer Algebra and Replacements

5.4.1 Manual Replacement Rules

In this section I discuss the necessary variable- and pattern-substitution machinery that I implemented in order to simplify contracted tensor expressions. These methods center around the use of manually applied replacement rules and the use of so-called UpValues. While manual replacement rules warrant calling either /. rule or //ReplaceAll[, rule]& on each expression on which they're applied, UpValues are automatic — in the sense that they test for and apply themselves whenever they encounter a matching Head. While UpValues seem to be inherently advantageous over manual replacement for this reason, it is not always prudent to apply replacements so liberally. In particular, if UpValues, associated with some Head, automatically expand a variable, that would otherwise cancel or simplify without the UpValue application, into a much larger expression that doesn't immediately allow Wolfram to recognize its simplicity. Consequently, it is most prudent to implement UpValues for smaller, albeit more frequent, algebraic operations, such as tensor contractions, than for the substitution of entire equations of motion or other lengthy replacement rules. For the equations of motion, as discussed in 5.3, the computationally economical choice of only inserting the necessary orders in spin is not something that UpValues can accommodate.

Barring the exceptions outlined in 5.3, replacement rules for tensor algebra simplifications generally ought to be of the DelayedRule -kind, due to the frequent necessity for mapping free indices from the matched, left hand side onto the right hand side, along with the introduction of dummy indices on the right. The inclusion of Module environments and ReplaceDummies on the right also serves to avoid potential issues regarding over-used indices within the same expression — something one can easily encounter without these practices. Such issues are, of course, even more likely with the use of non-delayed Rule, which must be avoided whenever the right hand side of the rule involves dummy indices, because even ReplaceDummies will only be called once, namely at the initial evaluation of the Rule. As an example, replacing F with its dual FD, we have

```
FtoFD = {F[a_, b_] :>
```

```
    Module[{c, d}, -1/2*(epsilon\[Eta][a, b, -c, -d]*FD[c, d])]} ;
```

in contrast to replacing Scalar-ized tensors with atomic placeholders

```
{Scalar[n[-a, -b] T[a] u[b]] -> nTu[],
  Scalar[n[-a, -b] s[a] u[b]] -> nsu[],
  ..., Scalar[T[a] u[-a]] -> Tu[]};
```

wherein the right hand sides are *always* unchanged. The one exception to this Rule versus DelayedRule prescription is in the argument of the custom-made Replacements function, what will be detailed in the 5.5 section.

5.4.2 UpValues

Turning our attention to UpValues, the syntax is a bit different. UpValues are relations that associate to a particular Head which, when encountered by the kernel, Wolfram runs through the Head's UpValues and applies them where applicable. For example, noting our normalization conditions of $u^2 = -1$ and $T^2 = -1$, we can associate the UpValues with their respective Heads as

```
u /: u[-a_] u[a_] := -1;
T /: T[-a_] T[a_] := -1;
```

such that, whenever Wolfram sees any instance of u or T it looks for places to simplify u[-a_] u[a_] := -1 and T[-a_] T[a_] := -1, respectively.

Similarly, one can enforce the spin-supplementary-condition, $S_{\mu\nu}p^{\nu} = 0$ as

```
S /: p[a_] S[b_, -a_] := 0;
```

and Dixon's worldline parameterization condition $u \cdot \dot{z} = -1$ as

```
zDot /: u[a_] zDot[-a_] := -1;
```

where zDot is the head associated with $\dot{z}$. In the last two examples, the UpValue could have been associated with either head, but it's generally prudent, whenever choosing which of multiple heads to associate an UpValue with, to pick the UpValue with the fewest UpValues already associated. The reason is, again, a matter of efficiency: Having a large number of UpValues associated to one symbol, at most one of which will match at a time, increases the resources wasted by Mathematica every time it encounters that particular Head. The UpValues associated to a symbol can be inspected and cleared using

```
UpValues[symbol]
UpValues[symbol] = {}
```

respectively.

As the previous examples suggest, UpValues are associated to a particular symbol using the syntax /:, where the Head of the symbol for which Mathematica is on the "lookout" is on the left thereof, and the substitution, in the form of SetDelayed as opposed to the DelayedRule used in the manual replacement rules, is on the right. It's still important to use appropriate pattern matching, such as underscores in the indices a_ and -a_ to pattern-match to generic indices in the contractions.

5.4.3 Pattern Matching and Mathematica's ToCanonical

Within the scope of the replacement rules and UpValues discussed so far, one shortcoming of Wolfram Mathematica's pattern recognition is that, though the underscores on index arguments are sufficient for matching to other index symbols, it does not catch instances of when an index pair is raised/lowered in comparison to the pattern specified in the replacement rule. For example, the UpValues

```
s /: s[a_] u[-a_] := 0;
s /: s[b_] S[a_, -b_] := 0;
```

will *not* replace instances of s[-a] u[a] or s[-b] S[a, b] with zero, due to the contracted index being mismatched, in the sense of raising and lowering, relative to the rule's pattern. The FullForm of the rule reveals why, since

```
In:=s[a_] u[-a_]//FullForm
Out=Times[s[Pattern[a,Blank[]]],u[Times[-1,Pattern[a,Blank[]]]]]
```

while

```
In:= s[-a] u[a] // FullForm
```

```
Out=Times[s[Times[-1,a]],u[a]]
```

meaning that, while the upper-index pattern Pattern[a,Blank[]] can match to a lower index without problem, the lower index pattern, Times[-1,Pattern[a,Blank[]]]], can only match to indices with Times[-1,...] Heads around them, i.e. only lowered indices. Consequently, every contracted index-pair that you wish to match to must have a higher-lower and a lower-higher pattern for general purpose contractions. Moreover, it is prudent to define the matching pattern for free indices as raised by default, since the pattern matching for lowered free indices would not match to their raised counterparts.

A similar hurdle applies whenever contracting tensors with more indices. Even if Wolfram Mathematica is provided with the relevant internal symmetries of the tensors, it will not automatically catch contractions unless they pedantically agree with the replacement rule or UpValue. As an example,

```
In:= n[a, b] P[-b, c] /. n[a_, b_] P[-b_, c_] -> 1
Out=1
```

because the expression exactly matches the pattern. If we, however, permute the indices of P such that the expression becomes

```
In:= n[a, b] P[c, -b] /. n[a_, b_] P[-b_, c_] -> 1
```

then the pattern does not match and returns n[a, b] P[c, -b]. Hence, for general purpose tensor algebra, one needs contraction rules for all the index permutations and the pairwise raising and lowering of each pair of dummies.

There is, however, one saving grace for most cases of index contraction trouble: xTensor's ToCanonical function. ToCanonical is, relative to Simplify, a lightweight simplification-like function that exploits the symmetries of the expression's tensors in order to find cancellations. In particular, it roughly does the following

1. Re-labels and raises & lowers dummy index pairs in a consistent fashion across terms.

2. Applies permutation symmetries of tensors to standardize (canonicalize) expressions and detect terms that vanish by symmetry.

3. Performs index raising and lower operations through contractions with the Manifold Metric.

4. Combines or cancels terms that become identical after canonicalization.

As an example,

```
In:= S[a, b] T[-a]//OutputForm
Out=S[a, b] T[-a]
```

leaves the tensor indices unchanged, but hitting the expressions with ToCanonical yields

```
In:= S[a, b] T[-a] //ToCanonical//OutputForm
Out=-(S[b, -a] T[a])
```

where the anti-symmetry of S has been used to incur an overall negative sign in order to put the contracted a indices next to each other, along with putting the raised dummy index on T as opposed to S. To the other points, ToCanonical will annihilate terms that a zero by index symmetry

```
In:= S[a, b] P[-a,-b] //ToCanonical
Out=0
```

if given the definite index permutation properties of tensors, say symmetric P[a,b]=P[b,a] and anti-symmetric $S[a, b] = -S[b, a]$. On the last note, ToCanonical also combines or cancels expressions that wouldn't explicitly match, unless canonicalized first:

```
In:= n[a, b] S[-a, -b] - S[a, -b] n[-a, b] //ToCanonical
Out=0
```

Equipped with ToCanonical, replacement rules and, in particular, UpValues are incredibly powerful *if* the pattern matching looks for the canonicalized form, since applications of ToCanonical map tensors and their contractions to the patterns which match to the UpValues. However, while a careful implementation of the canonicalized UpValues and replacement rules could, in principle, suffice for catching all the possible tensor contractions amenable to simplification, this approach incurs practical problems when dealing with larger tensor expressions.

5.4.4 Practical Limitations of ToCanonical

Once the spacetime Manifold and Metric have been defined through xTensor, we begin defining the relevant tensors for our computation using the DefTensor function. For the purposes of tensor manipulation, such as in ToCanonical, xTensor keeps a record of the tensors defined within the kernel and, importantly, keeps track of the *order* in which they're defined. This ordering is imposed on tensor expressions such that tensors that were defined first are left of tensors defined later, such as in

```
In:= P[a, b] nb[-b, -c] n[c, d] //OutputForm
Out=n[c, d] nb[-b, -c] P[a, b]
```

where the ordering is reversed since I defined these n, nb, and P as

```
DefTensor[n[a, b], {ST, \[Tau]}, Antisymmetric[{a,b}]];
DefTensor[nb[a, b], {ST, \[Tau]}, Antisymmetric[{a,b}]];
DefTensor[P[a, b], {ST, \[Tau]}, Symmetric[{a,b}]];
```

where ST is the spacetime Manifold and [Tau] are the proper time dependencies, respectively. Furthermore, this ordering has important implications for the use of replacement rules and UpValues because the ordering affects the particular way in which ToCanonical outputs an expression. If one inputs a number of replacement rules or UpValues and later changes the order in which tensors are defined, say by moving their definitions in the code, canonical form of expressions might be affected and the previously matching UpValues may now fail. In other words, best practice is to avoid changing around the order in which tensors are defined and to re-canonicalize any patterns in the event of such changes.

ToCanonical encounters further complications regarding replacements and UpValues when concerning tensors in large expressions. Even if appropriately canonicalized, pattern matching may fail due to tensors having different canonicalized appearances when strung together in more than one contraction. For example, in

```
In[1]:= S[a, b] T[-a] // ToCanonical // OutputForm
Out[1]=-(S[b, -a] T[a])
In[2]:= S[a, b] T[-a] u[-b] // ToCanonical // OutputForm
```

```
Out[2]= S[-a, -b] T[a] u[b]
```

the presence of a contraction with u[b] affects both the upper/lowered nature of the indices on S[a,b], but ToCanonical also prefers permuting the indices, even incurring a negative from antisymmetry of S, when only T[a] is contracted. With more complex expressions, pre-empting all the patterns of ToCanonical becomes cumbersome and often impractical, as it only takes one mismatch to prevent appropriate simplification and cancellation. To be clear: This is not a case of the contracted tensors being "separated" in Wolfram's FullForm, as the pattern matching can see across this gap, but rather that the canonicalized tensor index positions are sensitive to this ordering, as well as to the presence of additional tensors present in the contracted expression.

In order to address these limitations of ToCanonical's intricate nature, I implement a function called Replacements to generate replacement rules to catch all the patterns, the details of which are discussed in the 5.5 section.

5.5 xPerm Combinatorics

5.5.1 xPerm Basics

The xPerm package works with permutations using any of 4 different presentations, having the heads Perm, Images, Cycles, and Rules. For a particular permutation σ of n elements, sending $i \to \sigma_i$, we can present the permutation as

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix}, \tag{235}$$

with each column of 235 indicating which element is sent to which target. The Cycles notation yields the permutation in the form of disjoint cyclic permutations, and Rules yields a list of replacement rules from i to σ_i. The difference between the Perm and Images notation can be understood as the permutation acting on the image space versus the domain, or as "left" and "right" actions of σ, respectively. The second row of equation 235 is the image of $(1, 2, \dots, n)$ under the action of σ, thus xPerm would store it as $(\sigma_1, \sigma_2, \dots, \sigma_n)$ in the Images notation. In contrast, the Perm notation would store $(\sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_n^{-1})$, since the left action on the indices corresponds to the inverse as a right action on the slots. Writing this in two-line notation,

$$\begin{pmatrix} \sigma_1^{-1} & \sigma_2^{-1} & \dots & \sigma_n^{-1} \\ 1 & 2 & \dots & n \end{pmatrix}, \tag{236}$$

one can see that the permutation of elements is no different than in equation 235. This distinction of notations acting left- or right-ward is important since xPerm associates a tensor's symmetries with the permutations of its tensor *slots*, as opposed to its indices, the latter of which is more fungible, such as in index relabeling. As a last tidbit, xPerm, unlike default Mathematica, allows for permutations to be signed, in the even or odd sense, which allows for easier handling of tensors with anti-symmetry. Naturally, whenever composing two signed permutations, the resulting sign is the product of the two signs.

Turning out attention to tensors, we can access the symmetry data using the xPerm function SymmetryOf, which wraps the following data in the Symmetry Head: The number of indices of the tensor, the tensor and its raw slots, the slots to index rules, and a strong generating set of the tensor's symmetries. Taking the totally antisymmetry tensor, ϵ^{abcd}, as an example, SymmetryOf returns

```
In[]:=SymmetryOf[epsilon\[Eta][a,b,c,d]]
Out[]=Symmetry[4,
    epsilon\[Eta][slot[1], slot[2], slot[3], slot[4]],
    {"\[FilledCircle]1" -> a,"\[FilledCircle]2" -> b,
        "\[FilledCircle]3" -> c,"\[FilledCircle]4" -> d},
    StrongGenSet[{1, 2, 3, 4},
        GenSet[Cycles[{1, 2}], -Cycles[{2, 3}], -Cycles[{3, 4}]]]]
```

where I have suppressed instances of xAct‘xTensor‘Private‘slot and xAct‘xPerm‘Cycles to slot and Cycles, respectively, for readability. The second component from SymmetryOf is the skeleton of the tensor, with slot placeholders for re-insertion of indices. The third component provides a list of replacement rules that reinstitute indices into the slots of the second component. One can trivially reconstruct the input tensor from SymmetryOf:

```
In[]:=SymmetryOf[epsilon\[Eta][a,b,c,d]];
       %[[2]]/.%[[3]]
Out[]=epsilon\[Eta][a,b,c,d]
```

We, however, are not merely interested in recovering the trivial permutation. The objective of using xPerm is to create a function that, when given a particular replacement rule, will automatically generate all the patterns that we would want the left hand side to match to. This means considering all positions of the contracted indices in the expression, including swapping raised and lowered pairs. For the input replacement rule $n^{ab}\bar{n}^{c}{}_{b} \rightarrow -P^{ac}$, we would want our function to return all the following rules

```
{n[a_, b_] nb[c_, -(b_)] -> -P[a, c],
 n[b_, a_] nb[c_, -(b_)] -> P[a, c],
 n[a_, b_] nb[-(b_), c_] -> P[a, c],
 n[b_, a_] nb[-(b_), c_] -> -P[a, c],
 n[a_, -(b_)] nb[c_, b_] -> -P[a, c],
```

```
n[-(b_), a_] nb[c_, b_] -> P[a, c],
n[a_, -(b_)] nb[b_, c_] -> P[a, c],
n[-(b_), a_] nb[b_, c_] -> -P[a, c]}
```

where we have a total of 8 rules resulting from each of n and nb having a symmetry group order of 2, and the raising and lowering swap of one pair of dummy indices. A custom Mathematica function, Replacements, that does exactly this, and its implementation will be the discussion of the rest of this section.

5.5.2 Strong Generating Sets

To the uninitiated physicist, the phrase *strong* generating set might seem odd, as how can one generating set be stronger than another — doesn't every generating set, definitionally, generate the group in question. While this is, of course, true, the best choice of generators, as well as a strategy for combining them to efficiently generate the group, is a matter for computational group theory. While I won't give a detailed exposition on the Schreier-Sims algorithm, I will give the notion of a strong generating set, its *base*, and, hopefully, the reader some intuition for how this systematizes the construction of the entire group.

As an example to start, suppose you have a cube, defined by its vertices 1,...,8, with each vertex connected to three others by edges.

Insert Labeled Cube Figure and Cite https://haskellformaths.blogspot.com/2010/02/how-to-find-strong-generating-set.html

The group of symmetries of this cube can be generated by the strong generating set

```
{(1,2)(3,4)(5,6)(7,8),
(1,3,4,2)(5,7,8,6),
(1,5,7,8,4,2)(3,6),
(2,3)(6,7),
(2,5,3)(4,6,7),
(3,5)(4,6)}
```

where the () parentheses denote the permutation cycles notation. Of course, this set generates the whole set of symmetries of the group, but it also exhibits a particular structure. The first three elements move 1, the next two fix 1 but move 2, and the last fixes both 1 and 2, but moves 3. This stratification of the generators into "levels" 1, 2, and 3, gives rise to what we call the base, [1,2,3], of the strong generating set. So, suppose we discarded the elements of level 1, the remaining elements would generate the stabilizer of the element 1. Similarly, discarding both levels 1 and 2, the last level generates the stabilizer of points 1 and 2. Hence, the base of a strong generating set can be understood as defining a sequence of subgroups, each subgroup stabilizing more points than the last, called a stabilizer chain. In other words, for a group G, a base $B = [\beta_1, \beta_2, ..., \beta_k]$, associates to a stabilizer chain of subgroups

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq ... \geq G^{(k)}, \tag{237}$$

where each $G^{(i)}$ is the stabilizer of the first i base points.

The insight that proves this structure computationally useful for generating the group, is that one can build all the elements of G by taking *traversals* of this stabilizer chain. The i^{th} traversal $T_i := G^{(i-1)}/G^{(i)}$ is a set of representative permutations which act on elements i, i+1,...n, with the action of $G^{(i)}$ quotiented out. Now, of course, we don't have these groups on hand, since we're in fact trying to construct them. Instead one pursues the following algorithm, making use of the strong generating set structure, to construct these traversals.

One begins by constructing the first traversal T_1 by finding the orbit of 1 under all the generators of $G^{(0)}$, and collecting one representative for each element of the orbit of 1. The particular choice of representative is unimportant because we care only for the equivalence class in question, i.e. where 1 moved in its orbit. The subsequent traversals will affix the rest of the permutations' actions within each equivalence class. Then, to construct T_2, we consider the image of all elements of T_1 under the action of the generators of $G^{(2)}$, the generators that move 2 but keep 1 fixed. Since T_1 moves 1, this preliminarily moves 1, so we "pull back" the results using the representative T_1 elements in order to have T_2 elements only shift 2. As in the case of constructing T_1, we don't care about the particular permutation of subsequent elements because we're only interested in the equivalence class defined by the image of point 2 under T_2. Then to construct T_3, we do the same as in the previous step, but with T_2 in lieu of T_1. This process continues until all the traversals T_i have been computed, which we can use to explicitly construct the group by composing all the possible combinations of generators from each of the traversals T_i. Since any element of the group can be uniquely expressed as $g = t_1 t_2 t_3 ... t_n$, where $t_i \in T_i$, one can easily identify the order of the group as the product of the orders of the traversals.

5.5.3 Generating and Permuting Replacements

To build the relevant permutations, while avoiding the construction of redundant replacement rules, we seek to generate all the permutations of the tensor structure's *contracted* indices only, since any free indices will be matched with a Pattern and thus any particular representative ordering of free indices will match to all. To generate these, we will generate the full set of index permutations, and then quotient out those of the free indices, leaving those of the contracted.

When using the xPerm function SymmetryOf on a composite expression, built from a product of tensors, it returns the strong generating set of the entire expression. The SymmetryOf of a product of tensors *almost* yields a direct product group structure, since, while these symmetry groups are independent, the function also captures the symmetries associated with the interchangeability of identical tensors, something which is redundant for the purposes of our pattern matching. Thus, instead of trying to construct a larger symmetry group, and correspondingly a larger quotient group due to tensor interchangeability, we instead employ the following strategy.

Take a generic, tensorial product and break it into its individual tensors. Apply SymmetryOf to each individual tensor, wherefrom the individual tensors' symmetry groups can be constructed. By taking a direct product to reconstruct the overall symmetry group, we automatically quotient out the symmetries of the interchangeability of any identical tensors, and then take their direct product. This avoids the computational bloat and technical challenges of constructing a quotient group that includes more symmetries.

Specifically, we begin with our Replacements function taking an input Rule[lhs-¿rhs], and, if constructed from a tensor product, i.e. having the Head Times, lhs is converted to a list of tensors, whereafter SymmetryOf is individually applied to the elements:

```
If[Head[lhs] === Times,
```

```
  lhsSyms = Replace[SymmetryOf /@ lhs, x_Times :> List @@ x, {0}],
  lhsSyms = List@SymmetryOf@lhs];
```

Once the individual tensors' SymmetryOf have been listed, we combine them into an overall generating set using the custom code for taking direct products

```
ShiftSym[s_Symmetry, shiftNumber_Integer] :=
 Symmetry @@ Join[List @@ s[[1 ;; 3]] /.
    xAct`xTensor`Private`slot[n_] :>
     xAct`xTensor`Private`slot[n + shiftNumber],
     {s[[4]] /. m_List :> m + shiftNumber}]

CombineSym[s1_Symmetry, s2_Symmetry] :=
 Module[{shifts2 = ShiftSym[s2, s1[[1]]]},
  Symmetry[s1[[1]] + shifts2[[1]], Times[s1[[2]], shifts2[[2]]],
   Join[s1[[3]], shifts2[[3]]], JoinSGS[s1[[4]], shifts2[[4]]]]]
CombineSym[{s1_Symmetry, rest___}] := Fold[CombineSym, s1, {rest}]
CombineSym[s1_Symmetry] := s1
```

where ShiftSym translates the permutation σ by m, leaving the identity permutation on the first m elements:

$$ShiftSym : \sigma = (\sigma_1, \sigma_2, \sigma_3, ..., \sigma_n) \mapsto (1, 2, 3, ..., m, \sigma_1 + m, \sigma_2 + m, \sigma_3 + m, ..., \sigma_n + m). \tag{238}$$

This function is used in CombineSym to translate Symmetry s2, say σ, by the length of Symmetry s1, say ρ, and then compose them as

$$CombineSym : (\rho_1, ..., \rho_m) \times (\sigma_1, ..., \sigma_n) \mapsto (\rho_1, ..., \rho_m, \sigma_1 + m, ..., \sigma_n + m)\,, \tag{239}$$

exactly how a direct product would act. CombineSym also Folds to combine an entire list of permutations in the order that they appear. Using CombineSym on the list of tensor symmetries,

```
    lhsSym = CombineSym[lhsSyms];
```

we can use xPerm's Dimino function to generate the un-quotiented group

```
    allPerms = Dimino[lhsSym[[4]]];
```

This group, however, includes permutations of the free-index slots, which we seek to quotient out. We define

```
quotPerms =
  If[Length[allPerms] == 1, TranslatePerm[allPerms, Rules],
   TranslatePerm[InversePerm /@ TranslatePerm[
    CanonicalPerm[#, lhsSym[[1]], Stabilizer[Flatten[dummyIndices],
    lhsSym[[4]]], freeIndices, {DummySet[M, dummyIndices,1]}]
    & /@ allPerms,
    {Perm, lhsSym[[1]]}] // DeleteDuplicates, Rules]];
```

to find the coset representatives of each equivalence class of allPerms modulo the permutations of free indices. Specifically, the xPerm function CanonicalPerm yields, for the input #, the canonical representative of # modulo the symmetry group given by lhsSym. The rest of CanonicalPerm's inputs serve to affix which indices are moved by lhsSym.

With the quotient group, we lastly seek to apply these as replacement rules to our original expression, which is done in

```
repRules =
   Table[Rule[(lhsSym[[2]] /. VTT) /.
        List @@ nosign[quotPerms[[j]]] /. (lhsSym[[3]] /.
          binaryLetters[[i]] /. indicesToPattern),
          (PermutationSign[quotPerms[[j]]]*rhs //ReplaceDummies)],
          {i, 1, Length[binaryLetters]}, {j, 1, Length[quotPerms]}]
        // Flatten;
```

Regarding repRules, the thing worth mentioning is the inclusion of the binaryLetters replacement rule, which serves to take every pair of contracted, dummy indices and produce both a lowered-raised and a raised-lowered pair, as to include all the patterns that might match to contracted expressions. With this, we have a list of replacement rules for every possible up/down and index permuted contraction.

5.6 Multinomial Reduction and Schouten Identities

Since the Carter and Rüdiger constants, C and Q, and their worldline derivatives, $\dot{C} = \frac{dC}{d\tau}$ and $\dot{Q} = \frac{dQ}{d\tau}$, are scalars, it proves useful to reduce our expressions to a set of functionally independent variables in order to translate the condition $\dot{C} = \dot{Q} = 0$ into a requirement of all individual multinomial terms' coefficients being zero, i.e. we guarantee that said multinomial is zero *if and only if* its coefficients are. While a set of functionally *dependent* variables could be *sufficient* for identifying solutions, by setting coefficients to zero, it's not *necessary*, and thus one cannot exhaustively conclude that there are no solutions merely through zeroing coefficients. For the $O(s^2)$ Carter and Rüdiger computations, this distinction was unimportant because we found a solution without fully reducing to independent variables. However, to exhaustively conclude that there are no solutions at $O(s^3)$ we required an additional change of variables.

5.6.1 Variables and Rational Parameterizations

As a preliminary attempt to find such a set of variables, for $\dot{C}$ and $\dot{Q}$, we employ a spacetime basis spanned by u^μ, $N^{\mu\nu}u_\nu$, $\bar{N}^{\mu\nu}u_\nu$, and $\pi^{\mu\nu}u_\nu$, where $\Pi^{\mu\nu} = N^{\mu\rho}\bar{N}_\rho{}^\nu$, and consider all contractions possible between this spacetime basis and the vectors s^μ, u^μ, and T^μ, yielding the set

$$n_{su} \quad \bar{n}_{su} \quad P_{su} \quad n_{Tu} \quad \bar{n}_{Tu} \quad P_{Tu} \quad P_{uu} \quad T \cdot u. \tag{240}$$

For conciseness, we have used the shorthand notation to denote the vector contraction with a tensor as follows: $\pi_{ss} = \pi_{\mu\nu}s^\mu s^\nu$, $N_{su} = N_{\mu\nu}s^\mu u^\nu$, $\bar{N}_{Tu} = \bar{N}_{\mu\nu}T^\mu u^\nu$, et cetera.

We employ the following Schouten-like identities,

$$s \cdot T = \frac{N_{su}N_{Tu} + \bar{N}_{su}\bar{N}_{Tu} + \pi_{su}\pi_{Tu} + (T \cdot u)\pi_{su}\pi_{uu} - \bar{N}_{su}N_{Tu}\pi_{uu} - N_{su}\bar{N}_{Tu}\pi_{uu}}{-1 + (\pi_{uu})^2} \tag{241}$$

$$\pi_{sT} = \frac{(T \cdot u)\pi_{su} - \bar{N}_{su}N_{Tu} - N_{su}\bar{N}_{Tu} + N_{su}N_{Tu}\pi_{uu} + \bar{N}_{su}\bar{N}_{Tu}\pi_{uu} + \pi_{su}\pi_{Tu}\pi_{uu}}{-1 + (\pi_{uu})^2} \tag{242}$$

$$N_{sT} = \frac{(T \cdot u)N_{su} + \pi_{su}\bar{N}_{Tu} - \bar{N}_{su}\pi_{Tu} - (T \cdot u)\bar{N}_{su}\pi_{uu} - \pi_{su}N_{Tu}\pi_{uu} + N_{su}\pi_{Tu}\pi_{uu}}{-1 + (\pi_{uu})^2} \tag{243}$$

$$\bar{N}_{sT} = \frac{(T \cdot u)\bar{N}_{su} + \pi_{su}N_{Tu} - N_{su}\pi_{Tu} - (T \cdot u)N_{su}\pi_{uu} - \pi_{su}\bar{N}_{Tu}\pi_{uu} + \bar{N}_{su}\pi_{Tu}\pi_{uu}}{-1 + (\pi_{uu})^2} \tag{244}$$

$$s \cdot s = \frac{(N_{su})^2 + (\bar{N}_{su})^2 + (\pi_{su})^2 - 2N_{su}\bar{N}_{su}\pi_{uu}}{-1 + (\pi_{uu})^2} \tag{245}$$

$$\pi_{ss} = \frac{-2N_{su}\bar{N}_{su} + (N_{su})^2\pi_{uu} + (\bar{N}_{su})^2\pi_{uu} + (\pi_{su})^2\pi_{uu}}{-1 + (\pi_{uu})^2}, \tag{246}$$

in order to reduce our multinomial to what is almost an independent set. This "almost"-independent set was sufficient to identify the solutions to the $O(s^2)$ Carter and Rüdiger constant problem, but was insufficient in ruling out the ansatz space at $O(s^3)$ because of the non-trivial relation,

$$1 - (\pi_{uu})^2 - (T \cdot u)^2 - (N_{Tu})^2 - (\bar{N}_{Tu})^2 - (\pi_{Tu})^2 + 2N_{Tu}\bar{N}_{Tu}\pi_{uu} - 2(T \cdot u)\pi_{Tu}\pi_{uu} = 0, \tag{247}$$

between five elements $\pi_{uu}, T \cdot u, N_{Tu}, \bar{N}_{Tu}$, and π_{Tu}.

Because of the particular nature of equation 247, we could not find a polynomial parameterization of π_{uu}, $T \cdot u$, N_{Tu}, $\bar{N}_{Tu}$, and π_{Tu} in terms of 4 variables. However, as with the previous Schouten identities, we can employ rational parameterizations and, eventually, multiply out the denominator of the entire $\dot{C}$ and $\dot{Q}$ expressions. To this end, we found the following *four* variables s, t, u, v which rationally parameterize the intermediate variables x_1, x_2, x_3, x_4, and x_5 as

$$x_1 = \frac{2s(1 - u^2)}{(1 + u^2)(1 + s^2 + t^2)} + \frac{2u(1 - s^2 - t^2)(1 - v^2)}{(1 + u^2)(1 + s^2 + t^2)(1 + v^2)} \tag{248}$$

$$x_2 = \frac{2s}{1 + s^2 + t^2} \tag{249}$$

$$x_3 = \frac{2t(u^2 - 1)}{(1 + u^2)(1 + s^2 + t^2)} + \frac{4uv(1 - s^2 - t^2)}{(1 + u^2)(1 + s^2 + t^2)(1 + v^2)} \tag{250}$$

$$x_4 = \frac{2t}{1 + s^2 + t^2} \tag{251}$$

$$x_5 = \frac{1 - u^2}{1 + u^2}, \tag{252}$$

which in turn parameterize

$$\pi_{uu} = \frac{1 - x_1^2}{1 + x_1^2} \tag{253}$$

$$\bar{n}_{Tu} = \frac{2x_3}{1 + x_3^2 + x_4^2} \tag{254}$$

$$T \cdot u = \frac{2x_4}{1 + x_3^2 + x_4^2} \tag{255}$$

$$n_{Tu} = \frac{2(1 - x_1^2)x_3}{(1 + x_1^2)(1 + x_3^2 + x_4^2)} + \frac{2x_1(1 - x_2^2)(1 - x_3^2 - x_4^2)}{(1 + x_1^2)(1 + x_2^2)(1 + x_3^2 + x_4^2)} \tag{256}$$

$$\pi_{Tu} = \frac{2(-1 + x_1^2)x_4}{(1 + x_1^2)(1 + x_3^2 + x_4^2)} + \frac{4x_1x_2(1 - x_3^2 - x_4^2)}{(1 + x_1^2)(1 + x_2^2)(1 + x_3^2 + x_4^2)}, \tag{257}$$

such that equation 247 is always satisfied. This parameterization is, perhaps, unsurprising, given that 247 looks akin to some 4-dimensional hypersurface in a 5-dimensional parameter space. Additionally, π_{uu}, $T \cdot u$, N_{Tu}, $\bar{N}_{Tu}$, and π_{Tu} are all dimensionless, unlike the variables constructed from contractions with the spin vector, hence the parameterizing variables of s, t, u, and v need not have mass dimensions for themselves.

We believe the remaining set of scalars to be independent, in the sense of having no algebraic relations between them, by virtue of degree of freedom counting. Counting degrees of freedom of the problem, we identify a total of 10 parameters and dynamical degrees of freedom. Firstly, we have the probe mass scale m, coupling $q * Q$, and the background (rescaled) spin a. By azimuthal symmetry, our probe's worldline only has two degrees of freedom: radial r and polar angle θ. Moreover, the probe's spin 4-vector s^μ suggests 4 naive spin degrees of freedom, but the quadratic Casimir of s^2 and orthogonality relation $s \cdot u = 0$ brings this down to two. Then we have the translational momentum p^μ with 4 components, but subject to $p^2 = -\mathcal{M}^2$, giving another 3. Hence, in total, we have 3+2+2+3=10.

Comparing this to the variables on which we Collect, we have $q * Q$, m, R and $\bar{R}$, along with the 7 scalars π_{su}, N_{su}, $\bar{N}_{su}$, N_{Tu}, $\bar{N}_{Tu}$, and π_{uu}. Given the constraint of 247, however, there are only 6 independent scalars, which is in agreement with the parameter counting. Hence, once 247 has been used to eliminate all instances of π_{uu}, the remaining variables are algebraically independent and we can zero the resulting multinomial coefficients without worry.

5.6.2 Computational Bottlenecks

As mentioned, the repeated substitution of these rational parameterizations introduces a large number of factors in the terms' denominators. Since we only care about the parameters that zero the entire expression, one could, in principle, bring the entire expression over one denominator and continue therefrom. Practically, however, this proves to be a massive computational hurdle. The initial $\dot{C}$ and $\dot{Q}$ expressions involve millions of terms, and the resulting intermediate algebra causes both unreasonable scaling in compute time and memory usage. In fact, at this stage of the computation, any use of functions such as Simplify, Together, or any code that requires comparing of multinomial terms, beyond collecting or cancellation, should be used as a last resort.

Specifically, regarding the cancellation of denominators in the $\dot{C}$ and $\dot{Q}$ expressions, I prescribe the use of Wolfram's Cancel function applied *individually* to each term of the expression. In other words, take the entire expression, multiply it by sufficiently large polynomial such that all the factors in denominators can cancel out, and then apply the Cancel function to each term individually:

```
relevantNumerator=Total[Cancel /@ List @@ Expand[Cdot*denom]].
```

This does the same as bringing the entire expression over a common denominator with Together and Numerator, but avoids all the intermediate work that Together performs in order to collect and simplify terms in the intermediate stages.

5.6.3 Solving for Coefficients

Once reduced to multinomials in terms of $s, t, u, v, n_{su}, \bar{n}_{su}$, and π_{su}, we are ready to employ Wolfram Mathematica's CoefficientRules function. I stress the importance of having fully reduced the expression to a multinomial, without any denominators, since the CoefficientRules function does not work as intended for rational functions or power series with negative exponents. Assuming the correct input form, we use this CoefficientRules to give us a list of all the individual terms' coefficients,

```
CoefficientRules[poly,{x1,x2,...}]
```

```
gives the list {{e11,e12,...}→c1,{e21,e22,...}→c2,...} of exponent
   vectors and coefficients for the monomials in poly with respect to
   the xi,
```

where we only care about the right hand side of the resulting replacement rules. Since the output of CoefficientRules has the structure of an Association, we can easily access these coefficients using the Wolfram Values function. The resulting Values, when set to zero, yield a system of equations over all undetermined parameters in the ansatz and Wilson coefficients in the action, that we can apply Wolfram's Solve function to. When this is performed using the coefficients of an algebraically independent set of variables, this necessarily catches all the solutions, and hence one can rule out parameter spaces, namely those detailed in the ansätze chapter.

Lastly, on the note of computational efficiency, the Solve function worked through a system of hundreds of thousands of equations to be solved over around 250 variables, total. While the system was linear in most of these variables, namely the generic coefficients of the ansatz terms — since the worldline parameter differentiation acts on each linearly, some non-linearities in the worldline's action's Wilson coefficients are incurred. Consequently, the system is significantly more intensive to solve with the inclusion of the Wilson coefficients, and we recommend that, order by order in spin, that the previous order's results are immediately used to fix the Wilson coefficients before evaluating the subsequent order's ansätze. However, given the scaling of both the ansätze and the number of terms in the action at higher orders in spin, one might consider using Wolfram's NSolve as a preliminary tool, in hopes of finding these coefficients by decimal approximation. We tried applying the NSolve approach at $O(s^3)$, however we encountered division by zero errors, which we were unable to resolve. Upon returning to the regular Solve, we were also able solve the $O(s^3)$ case with less than 32 gigabytes of memory on a single Wolfram kernel.

5.7 Ansatz Building Blocks and Principles

In order to identify hidden conserved quantities, we employ generic ansätze in terms of the physical degrees of freedom. To systematize this search, we organize our terms order-by-order in spin, s. Additionally, we require each term be time-reversal and parity symmetric, as well as gauge invariance. Our building blocks are T^μ, u^μ, $\epsilon^{\mu\nu\rho\sigma}$, $S^{\mu\nu}$, $Y^{\mu\nu}$, $F^{\mu\nu}$, and covariant derivatives of $F^{\mu\nu}$, as well as scalars q and m for the purposes of scaling dimensions. T^μ is odd under time-reversal, and $Y^{\mu\nu}$ is odd under parity, with $\epsilon^{\mu\nu\rho\sigma}$ being odd under both. Due to having several tensors of multiple indices, we decided to choose a different set of tensors to build our ansätze. We decided to construct terms from the physical E and B fields, as well as derivatives thereof known as *electromagnetic tidal tensors*, in order to avoid overcounting as best as possible.

5.7.1 Electromagnetic Fields and Tidal Tensors

In the $(-,+,+,+)$ metric signature, and with $\epsilon_{0ijk} = \epsilon_{ijk}$, and thus $\epsilon^{0ijk} = -\epsilon^{ijk}$, the gauge field $A^a = (\phi, A^i)$, $A_a = (-\phi, A_i)$, and the field strength components $F_{ab} = 2\partial_{[a}A_{b]} = \partial_a A_b - \partial_b A_a$ decompose into the electric and magnetic fields, E and B, as

$$E_i = -F_{0i} = -\dot{A}_i - \partial_i\phi \tag{258}$$

$$B_i = {}^\star F_{0i} = \frac{1}{2}\epsilon_i{}^{jk}F_{jk} = \epsilon_i{}^{jk}\partial_j A_k \tag{259}$$

where ${}^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$. One can, of course, dually relate ${}^\star F_{ij} = \epsilon_{ij}{}^k E_k$. Taking the time direction to be that of a unit timelike vector u^μ ($u^2 = -1$), we covariantly define the 4-vector fields

$$E_\mu = -u^\nu F_{\nu\mu}, \qquad B_\mu = u^{\nu\star}F_{\nu\mu}, \tag{260}$$

which are orthogonal to u due the antisymmetry of F and ${}^\star F$. The inverse mapping, the field strength tensor and its dual in terms of E and B, can be expressed as

$$F_{\mu\nu} = 2u_{[\mu}E_{\nu]} + \epsilon_{\mu\nu}{}^{\rho\sigma}u_\rho B_\sigma, \qquad {}^\star F_{\mu\nu} = -2u_{[\mu}B_{\nu]} + \epsilon_{\mu\nu}{}^{\rho\sigma}u_\rho E_\sigma. \tag{261}$$

Now consider the first derivative $\partial_\rho F_{\mu\nu}$ of a vacuum field strength. Its independent irreducible components with respect to a time direction u^μ are the symmetric, trace-free, orthogonal-to-u tensors

$$\dot{E}_\mu = -u^\nu u^\rho \partial_\rho F_{\nu\mu}, \quad \dot{B}_\mu = u^\nu u^\rho \partial_\rho {}^\star F_{\nu\mu}, \tag{262}$$

along with the symmetric, also known as *tidal*, tensors

$$\mathcal{E}_{\mu\nu} = -u^\sigma \partial_\rho F_{\sigma(\mu} \mathcal{P}_{\nu)}{}^\rho, \quad \mathcal{B}_{\mu\nu} = u^\sigma \partial_\rho {}^\star F_{\sigma(\mu} \mathcal{P}_{\nu)}{}^\rho, \tag{263}$$

where $\mathcal{P}^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$ is the perpendicular projector onto u. Analogous to constructing E and B from F in the u-frame, we build the vectors

$$X_\mu = -u^{\nu\star} Y_{\nu\mu}, \quad W_\mu = -u^\nu Y_{\nu\mu}, \tag{264}$$

which have corresponding inverses

$${}^\star Y_{\mu\nu} = 2u_{[\mu} x_{\nu]} + \epsilon_{\mu\nu}{}^{\rho\sigma} u_\rho w_\sigma, \qquad -Y_{\mu\nu} = -2u_{[\rho} w_{\sigma]} + \epsilon_{\mu\nu}{}^{\rho\sigma} u_\rho x_\sigma. \tag{265}$$

Along with these changes, we also employ the use of the spin vector $s^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu S_{\rho\sigma}$ in lieu of its tensorial counterpart $S^{\mu\nu}$ in our pursuit of minimizing all the various index contractions. The cost of these choices is the need to include explicit $\epsilon^{\mu\nu\rho\sigma}$'s in our ansätze, however we found this to be the easier approach as far as organizing and tallying the various index contraction combinations.

5.7.2 $O(s^2)$ Ansatz

Organized in terms of powers of s, q and T, we took our ansatz to be a real linear combination of the following terms:

$O(s^0)\,O(q^0)\,O(T^0)$ $\quad p^2 W^2 \oplus p^2 X^2$

$O(s^1)\,O(q^0)\,O(T^0)$ $\quad m\; s \cdot W$

$$\mathcal{O}(T^1) \qquad p\cdot T\ s\cdot W \oplus \epsilon[TXps]$$

$$\mathcal{O}(q^1)\,\mathcal{O}(T^0)$$

$$q\begin{pmatrix} \begin{matrix} X^2\ B\cdot s \\ X\cdot B\ X\cdot s \\ W^2\ B\cdot s \\ W\cdot B\ W\cdot s \end{matrix} & \oplus & \begin{matrix} X\cdot W\ E\cdot s \\ W\cdot E\ X\cdot s \\ X\cdot E\ W\cdot s \end{matrix} & \oplus & \epsilon[XWBs] \end{pmatrix}$$

$$\mathcal{O}(s^2)\,\mathcal{O}(q^0)\,\mathcal{O}(T^1) \qquad s^2\ T\cdot u$$

$$\mathcal{O}(T^2) \qquad (s\cdot T)^2 \oplus s^2(T\cdot u)^2$$

$$\mathcal{O}(q^1)\,\mathcal{O}(T^0)$$

$$\frac{q}{m}\begin{pmatrix} \begin{matrix} s^2\ E\cdot X \\ E\cdot s\ s\cdot X \\ s^2\ B\cdot W \\ B\cdot s\ s\cdot W \end{matrix} & \oplus & \begin{matrix} X\cdot\mathcal{E}\cdot X\ s^2 \\ s\cdot\mathcal{E}\cdot s\ X^2 \\ X\cdot\mathcal{E}\cdot s\ X\cdot s \\ W\cdot\mathcal{E}\cdot W\ s^2 \\ s\cdot\mathcal{E}\cdot s\ W^2 \\ W\cdot\mathcal{E}\cdot s\ W\cdot s \end{matrix} & \oplus & \begin{matrix} X\cdot\mathcal{B}\cdot W\ s^2 \\ X\cdot\mathcal{B}\cdot s\ W\cdot s \\ W\cdot\mathcal{B}\cdot s\ X\cdot s \\ s\cdot\mathcal{B}\cdot s\ X\cdot W \end{matrix} & \oplus & \begin{matrix} \epsilon[u\dot{E}XW]\ s^2 \\ \epsilon[su\dot{E}X]\ W\cdot s \\ \epsilon[Wsu\dot{E}]X\cdot s \\ \epsilon[XWsu]\ \dot{E}\cdot s \\ \epsilon[u\dot{B}Xs]\ X\cdot s \\ \epsilon[u\dot{B}Ws]\ W\cdot s \end{matrix} \end{pmatrix}$$

$$\mathcal{O}(T^1)$$

$$\frac{q}{m}\begin{pmatrix} \begin{matrix} \epsilon[XBsu]\ T\cdot s \\ \epsilon[BsuT]\ X\cdot s \\ \epsilon[suTX]\ B\cdot s \\ \epsilon[uTXB]\ s^2 \end{matrix} & \oplus & \begin{matrix} \epsilon[WEsu]\ T\cdot s \\ \epsilon[EsuT]\ W\cdot s \\ \epsilon[suTW]\ E\cdot s \\ \epsilon[uTWE]\ s^2 \end{matrix} & \oplus & (T\cdot u)\begin{bmatrix} X\cdot E\ s^2 \\ X\cdot s\ E\cdot s \\ W\cdot B\ s^2 \\ W\cdot s\ B\cdot s \end{bmatrix} \end{pmatrix}$$

$O(q^2)\ O(T^0)$

$$\frac{q^2}{m^2}\begin{pmatrix}\begin{array}{c} E\cdot s\ E\cdot X\ X\cdot s \\ E^2\ (X\cdot s)^2 \\ (E\cdot s)^2\ X^2 \\ s^2\ (E\cdot X)^2 \\ s^2\ E^2\ X^2 \\ E\cdot s\ E\cdot W\ W\cdot s \\ E^2\ (W\cdot s)^2 \\ (E\cdot s)^2\ W^2 \\ s^2\ (E\cdot W)^2 \\ s^2\ E^2\ W^2 \end{array} & \oplus & \begin{array}{c} B\cdot s\ B\cdot X\ X\cdot s \\ B^2\ (X\cdot s)^2 \\ (B\cdot s)^2\ X^2 \\ s^2\ (B\cdot X)^2 \\ s^2\ B^2\ X^2 \\ B\cdot s\ B\cdot W\ W\cdot s \\ B^2\ (W\cdot s)^2 \\ (B\cdot s)^2\ W^2 \\ s^2\ (B\cdot W)^2 \\ s^2\ B^2\ W^2 \end{array} & \oplus & \begin{array}{c} s\cdot E\ E\cdot X\ X\cdot s \\ s\cdot E\ B\cdot W\ X\cdot s \\ s\cdot X\ B\cdot E\ W\cdot s \\ s\cdot B\ E\cdot X\ W\cdot s \\ s\cdot B\ E\cdot W\ X\cdot s \\ s\cdot E\ X\cdot W\ B\cdot s \\ s^2\ E\cdot B\ X\cdot W \\ s^2\ E\cdot X\ B\cdot W \\ s^2\ E\cdot W\ X\cdot B \end{array}\end{pmatrix}$$

In the above we have used the shorthand $\cdot$ to denote Lorentz index contraction, squaring to denote 4-vector norm, and $\epsilon[wxyz] = \epsilon^{\mu\nu\rho\sigma} w_\mu x_\nu y_\rho z_\sigma$. Each term listed, whether included in a column or separated by a direct sum symbol $\oplus$, is included as part of the ansatz, with each term being weighed by a dimensionless coefficient.

In the above ansatz, one notable inclusion is the presence of valid terms as well as said same terms but multiplied by a factor of $T^\mu u_\mu$ to some power. Because $T^\mu u_\mu$ is dimensionless and respects our parity and time reversal, since the time-direction component of u^0 is also odd under time reversal, we can take arbitrarily high powers of $T\cdot u$ and multiply existing ansatz terms to create new, valid ones, without incurring higher orders of spin or field strength. Because of u's equation of motion, these contribute and mix nontrivially with other ansatz terms, as evident by the presence of a $s^2(T\cdot u)^2$ term in the resulting $O(s^2)$ Carter constant. While their presence may not be *necessary*, the inclusion of $s^2(T\cdot u)^2$ in the $O(s^2)$ ansatz did yield significant simplification of the final result, from dozens of terms down to the eventual 7 terms of $O(s^2)$. It is, as of now, an open question whether such $T\cdot u$ inclusions are warranted in higher powers for future ansätze, in order to properly rule out the entire parameter spaces. Starting from an ansatz without explicit $T\cdot u$ factors, one could multiply a power series function,

$$P(T\cdot u) = 1 + g_1(T\cdot u) + g_2(T\cdot u)^2 + \ldots = 1 + \sum_{n=1} g_n(T\cdot u)^n \tag{266}$$

with generic coefficients g_i, to help control the dependence incurred by these higher powers. Specifically, since the dynamical behavior of $P(T \cdot u)$ is given by

$$\dot{P}(T \cdot u) = T \cdot \dot{u} \sum_{n=1} g_n n (T \cdot u)^{n-1}, \tag{267}$$

studying the asymptotic limit and how $\dot{P}$ falls off as $r \to \infty$, one could find the maximum power of $T \cdot u$ required such that mixing with the (explicitly) $T \cdot u$-independent ansatz occurs.

As pointed out to me by Mikhail Solon, one can equivalently view this series as a series expansion in $T \cdot p/m$, which parallels a non-relativistic limit if one assumes that the expansion parameter p/m is small.

5.7.3 $O(s^3)$ Ansatz

Going beyond $O(s^2)$, we followed the schematic outline

$$\begin{gathered} YYpp \\ {}^{\star}YTSp \quad \oplus \quad YYFS \\ TTSS \quad \oplus \quad {}^{\star}YTFpss \quad \oplus \quad YY\partial Fpss \quad \oplus \quad YYFFss \\ TTFSss \quad \oplus \quad {}^{\star}YT\partial FSss \quad \oplus \quad YY\partial\partial FSss \quad \oplus \quad YTFFsss \quad \oplus \quad {}^{\star}YYF\partial Fsss \end{gathered}$$

for the Carter constant ansatz, where we wish to consider all possible contractions between these objects. The mass dimensions are fixed by including q's whenever an instance of F occurs, and suitably multiplying by inverse powers of the scalar mass m. The p^μ's can be replaced by mu^μ, since $p^\mu - mu^\mu = O(s)$ and any difference incurred is parameterized by the generic ansatz at the next order in spin. While it is conceivable that such schematic does not capture all necessary terms, we have found this to be successful in the computation of the $\sqrt{\text{Kerr}}$ Carter and Rüdiger constants, as well as in Justin Vines', and collaborators', analogous result for a spinning probe coupled to the Kerr spacetime. It should also be mentioned that the above structure is ad-hoc modified by introducing extra terms proportional to $T \cdot u$, as discussed in section 5.7.2. Given the presence of a mass scale m in the problem, we are able to combine both the Carter and

Rüdiger ansätze under a single ansatz, which proves useful as it's, a priori, not obvious which of the ansatz terms will ultimately belong to which conserved quantity.

This particular presentation is merely for the sake of readability and identification of the lowest order result with that of the Carter constant presented in standard literature. As discussed for the parameterization of the $\mathcal{O}(s^2)$ ansatz, we employed the aforementioned X, W, s, E, B, $\dot{E}$, $\dot{B}$, and tidal tensors $\mathcal{E}$, $\mathcal{B}$ in lieu of the schematic building blocks. In addition to these from $\mathcal{O}(s^2)$, for $\mathcal{O}(s^3)$ we need to span the space of double-differentiated field strength tensors, requiring the introduction of tensors

$$\ddot{E}_\mu := -u^\nu u^\rho u^\sigma \partial_\sigma \partial_\rho F_{\nu\mu}, \qquad \dot{\mathcal{E}}_{\mu\nu} := -u^\sigma u^\kappa \partial_\kappa \partial_\rho F_{\sigma(\mu} \mathcal{P}_{\nu)}{}^\rho, \qquad \mathfrak{E}_{\mu\nu\rho} := -u^\sigma \partial_\kappa \partial_\lambda F_{\sigma(\mu} \mathcal{P}_\nu{}^\kappa \mathcal{P}_{\rho)}{}^\lambda, \tag{268}$$

$$\ddot{B}_\mu := u^\nu u^\rho u^\sigma \partial_\sigma \partial_\rho {}^\star F_{\nu\mu}, \qquad \dot{\mathcal{B}}_{\mu\nu} := u^\sigma u^\kappa \partial_\kappa \partial_\rho {}^\star F_{\sigma(\mu} \mathcal{P}_{\nu)}{}^\rho, \qquad \mathfrak{B}_{\mu\nu\rho} := u^\sigma \partial_\kappa \partial_\lambda {}^\star F_{\sigma(\mu} \mathcal{P}_\nu{}^\kappa \mathcal{P}_{\rho)}{}^\lambda. \tag{269}$$

Using the same short-hand conventions as in the $\mathcal{O}(s^2)$ case, we investigated the space of functions at $\mathcal{O}(s^3)$:

$$\frac{q}{m^2}\left(\begin{matrix} B \cdot ss^2 \\ B \cdot s(T \cdot s)^2 \\ B \cdot TT \cdot ss^2 \end{matrix} \quad \oplus \quad \epsilon_{TuEs} T \cdot us^2 \quad \oplus \quad \begin{matrix} X \cdot s\mathcal{B}_{ss} \\ s^2 \mathcal{B}_{Xs} \\ W \cdot s\mathcal{E}_{ss} \\ s^2 \mathcal{E}_{Ws} \end{matrix} \quad \oplus \quad (T \cdot u) \begin{bmatrix} X \cdot s\mathcal{B}_{ss} \\ s^2 \mathcal{B}_{Xs} \\ W \cdot s\mathcal{E}_{ss} \\ s^2 \mathcal{E}_{Ws} \end{bmatrix} \right)$$

$$\frac{q}{m^2}\begin{pmatrix} \epsilon_{uTXs}\mathcal{E}_{ss} & & \epsilon_{uTWs}\mathcal{B}_{ss} \\ \epsilon_{uTX}{}^\mu \mathcal{E}_{\mu s} s^2 & & \epsilon_{uTW}{}^\mu \mathcal{B}_{\mu s} s^2 \\ \epsilon_{uTs}{}^\mu \mathcal{E}_{\mu X} s^2 & \oplus & \epsilon_{uTs}{}^\mu \mathcal{B}_{\mu W} s^2 \\ \epsilon_{uTs}{}^\mu \mathcal{E}_{\mu s} X \cdot s & & \epsilon_{uTs}{}^\mu \mathcal{B}_{\mu s} W \cdot s \\ \epsilon_{uXs}{}^\mu \mathcal{E}_{\mu T} s^2 & & \epsilon_{uWs}{}^\mu \mathcal{B}_{\mu T} s^2 \\ \epsilon_{uXs}{}^\mu \mathcal{E}_{\mu s} T \cdot s & & \epsilon_{uWs}{}^\mu \mathcal{B}_{\mu s} T \cdot s \end{pmatrix}$$

$$
\frac{q}{m^2}\begin{pmatrix}
\begin{matrix} T\cdot sX\cdot s\dot{B}\cdot s \\ T\cdot X\dot{B}\cdot ss^2 \\ T\cdot \dot{B}X\cdot ss^2 \\ X\cdot \dot{B}T\cdot ss^2 \end{matrix}
\oplus
\begin{matrix} T\cdot sW\cdot s\dot{E}\cdot s \\ T\cdot W\dot{E}\cdot ss^2 \\ T\cdot \dot{E}W\cdot ss^2 \\ W\cdot \dot{E}T\cdot ss^2 \end{matrix}
\oplus
\begin{matrix} \epsilon_{TX\dot{E}s}s^2 \\ \epsilon_{TW\dot{B}s}s^2 \end{matrix}
\oplus
(T\cdot u)\begin{bmatrix} \epsilon_{TX\dot{E}s}s^2 \\ \epsilon_{TW\dot{B}s}s^2 \end{bmatrix}
\end{pmatrix}
$$

$$
\frac{q}{m^2}\begin{pmatrix}
\begin{matrix} \mathfrak{B}_{sss}X^2 \\ \mathfrak{B}_{Xss}s\cdot X \\ \mathfrak{B}_{XXs}s^2 \end{matrix}
\oplus
\begin{matrix} \mathfrak{B}_{sss}W^2 \\ \mathfrak{B}_{Wss}s\cdot W \\ \mathfrak{B}_{WWs}s^2 \end{matrix}
\oplus
\begin{matrix} \mathfrak{E}_{sss}X\cdot W \\ \mathfrak{E}_{Xss}s\cdot W \\ \mathfrak{E}_{Wss}X\cdot s \\ \mathfrak{E}_{sXW}s^2 \\ \epsilon_{XWs}{}^{\mu}\mathfrak{B}_{\mu ss} \end{matrix}
\oplus
\begin{matrix} \epsilon_{uXWs}\dot{\mathcal{B}}_{ss} \\ \epsilon_{uXW}{}^{\mu}\dot{\mathcal{B}}_{\mu s}s^2 \\ \epsilon_{uXs}{}^{\mu}\dot{\mathcal{B}}_{\mu W}s^2 \\ \epsilon_{uXs}{}^{\mu}\dot{\mathcal{B}}_{\mu s}W\cdot s \\ \epsilon_{uWs}{}^{\mu}\dot{\mathcal{B}}_{\mu s}X\cdot s \end{matrix}
\end{pmatrix}
$$

$$
\frac{q}{m^2}\begin{pmatrix}
\begin{matrix} \epsilon_{uXs}{}^{\mu}\dot{\mathcal{E}}_{\mu X}s^2 \\ \epsilon_{uXs}{}^{\mu}\dot{\mathcal{E}}_{\mu s}X\cdot s \\ \epsilon_{uWs}{}^{\mu}\dot{\mathcal{E}}_{\mu W}s^2 \\ \epsilon_{uWs}{}^{\mu}\dot{\mathcal{E}}_{\mu s}W\cdot s \end{matrix}
\oplus
\begin{matrix} \ddot{B}\cdot sX^2s^2 \\ \ddot{B}\cdot s(X\cdot s)^2 \\ \ddot{B}\cdot XX\cdot ss^2 \\ \ddot{B}\cdot sW^2s^2 \\ \ddot{B}\cdot s(W\cdot s)^2 \\ \ddot{B}\cdot WW\cdot ss^2 \end{matrix}
\oplus
\begin{matrix} \ddot{E}\cdot sX\cdot sW\cdot s \\ \ddot{E}\cdot sX\cdot Ws^2 \\ \ddot{E}\cdot XW\cdot ss^2 \\ \ddot{E}\cdot WX\cdot ss^2 \end{matrix}
\oplus
\epsilon_{XWs\ddot{B}}s^2
\end{pmatrix}
$$

$$
\frac{q^2}{m^3}\begin{pmatrix}
\begin{matrix} W\cdot EE\cdot ss^2 \\ W\cdot s(E\cdot s)^2 \\ W\cdot sE^2s^2 \\ W\cdot BB\cdot ss^2 \\ W\cdot s(B\cdot s)^2 \\ W\cdot sB^2s^2 \end{matrix}
\oplus
(T\cdot u)\begin{bmatrix} W\cdot EE\cdot ss^2 \\ W\cdot s(E\cdot s)^2 \\ W\cdot sE^2s^2 \\ W\cdot BB\cdot ss^2 \\ W\cdot s(B\cdot s)^2 \\ W\cdot sB^2s^2 \end{bmatrix}
\oplus
\begin{matrix} X\cdot sE\cdot sB\cdot s \\ X\cdot EB\cdot ss^2 \\ X\cdot BE\cdot ss^2 \\ E\cdot BX\cdot ss^2 \end{matrix}
\oplus
(T\cdot u)\begin{bmatrix} X\cdot sE\cdot sB\cdot s \\ X\cdot EB\cdot ss^2 \\ X\cdot BE\cdot ss^2 \\ E\cdot BX\cdot ss^2 \end{bmatrix}
\end{pmatrix}
$$

$$
\frac{q^2}{m^3}\begin{pmatrix}
\begin{matrix} \epsilon_{uTXE}E\cdot ss^2 \\ \epsilon_{uTXs}(E\cdot s)^2 \\ \epsilon_{uTXs}E^2s^2 \\ \epsilon_{uTEs}E\cdot Xs^2 \\ \epsilon_{uTEs}E\cdot sX\cdot s \\ \epsilon_{uXEs}E\cdot sT\cdot s \\ \epsilon_{uTEs}E\cdot Ts^2 \end{matrix}
\oplus
\begin{matrix} \epsilon_{uTXB}B\cdot ss^2 \\ \epsilon_{uTXs}(B\cdot s)^2 \\ \epsilon_{uTXs}B^2s^2 \\ \epsilon_{uTBs}B\cdot Xs^2 \\ \epsilon_{uTBs}B\cdot sX\cdot s \\ \epsilon_{uXBs}B\cdot sT\cdot s \\ \epsilon_{uTBs}B\cdot Ts^2 \end{matrix}
\oplus
\begin{matrix} \epsilon_{uTWE}B\cdot ss^2 \\ \epsilon_{uTWB}E\cdot ss^2 \\ \epsilon_{uTEB}W\cdot ss^2 \\ \epsilon_{uWEB}T\cdot ss^2 \end{matrix}
\end{pmatrix}
$$

$$\frac{q^2}{m^3}\begin{pmatrix} \epsilon_{usTW}E\cdot Bs^2 \\ \epsilon_{usTE}W\cdot Bs^2 \\ \epsilon_{usTB}W\cdot Es^2 \\ \epsilon_{usWE}T\cdot Bs^2 \\ \epsilon_{usWB}T\cdot Es^2 \\ \epsilon_{usEB}T\cdot Ws^2 \end{pmatrix.} \oplus \begin{matrix} \epsilon_{usTW}E\cdot sB\cdot s \\ \epsilon_{usTE}W\cdot sB\cdot s \\ \epsilon_{usTB}W\cdot sE\cdot s \\ \epsilon_{usWE}T\cdot sB\cdot s \\ \epsilon_{usWB}T\cdot sE\cdot s \\ \epsilon_{usEB}T\cdot sW\cdot s \end{matrix}\Bigg)$$

$$\frac{q^2}{m^3}\left(\begin{matrix} \mathcal{B}_{ss}E\cdot XX\cdot s \\ \mathcal{B}_{ss}E\cdot sX^2 \\ \mathcal{B}_{sX}E\cdot sX\cdot s \\ \mathcal{B}_{sX}E\cdot Xs^2 \end{matrix} \oplus \begin{matrix} \mathcal{B}_{sE}(X\cdot s)^2 \\ \mathcal{B}_{sE}X^2s^2 \\ \mathcal{B}_{XE}X\cdot ss^2 \\ \mathcal{B}_{XX}E\cdot ss^2 \end{matrix} \oplus \begin{matrix} \mathcal{B}_{ss}E\cdot WW\cdot s \\ \mathcal{B}_{ss}E\cdot sW^2 \\ \mathcal{B}_{sW}E\cdot sW\cdot s \\ \mathcal{B}_{sW}E\cdot Ws^2 \end{matrix} \oplus \begin{matrix} \mathcal{B}_{sE}(W\cdot s)^2 \\ \mathcal{B}_{sE}W^2s^2 \\ \mathcal{B}_{WE}W\cdot ss^2 \\ \mathcal{B}_{WW}E\cdot ss^2 \end{matrix}\right)$$

$$\frac{q^2}{m^3}\left(\begin{matrix} \mathcal{E}_{ss}B\cdot XX\cdot s \\ \mathcal{E}_{ss}B\cdot sX^2 \\ \mathcal{E}_{sX}B\cdot sX\cdot s \\ \mathcal{E}_{sX}B\cdot Xs^2 \end{matrix} \oplus \begin{matrix} \mathcal{E}_{sB}(X\cdot s)^2 \\ \mathcal{E}_{sB}X^2s^2 \\ \mathcal{E}_{XB}X\cdot ss^2 \\ \mathcal{E}_{XX}B\cdot ss^2 \end{matrix} \oplus \begin{matrix} \mathcal{E}_{ss}B\cdot WW\cdot s \\ \mathcal{E}_{ss}B\cdot sW^2 \\ \mathcal{E}_{sW}B\cdot sW\cdot s \\ \mathcal{E}_{sW}B\cdot Ws^2 \end{matrix} \oplus \begin{matrix} \mathcal{E}_{sB}(W\cdot s)^2 \\ \mathcal{E}_{sB}W^2s^2 \\ \mathcal{E}_{WB}W\cdot ss^2 \\ \mathcal{E}_{WW}B\cdot ss^2 \end{matrix}\right)$$

$$\frac{q^2}{m^3}\left(\begin{matrix} \mathcal{E}_{ss}X\cdot WE\cdot s \\ \mathcal{E}_{ss}W\cdot EX\cdot s \\ \mathcal{E}_{ss}E\cdot XW\cdot s \end{matrix} \oplus \begin{matrix} \mathcal{E}_{sX}W\cdot sE\cdot s \\ \mathcal{E}_{sW}E\cdot sX\cdot s \\ \mathcal{E}_{sE}X\cdot sW\cdot s \end{matrix} \oplus \begin{matrix} \mathcal{E}_{sX}W\cdot Es^2 \\ \mathcal{E}_{sW}E\cdot Xs^2 \\ \mathcal{E}_{sE}X\cdot Ws^2 \end{matrix} \oplus \begin{matrix} \mathcal{E}_{XW}E\cdot ss^2 \\ \mathcal{E}_{WE}X\cdot ss^2 \\ \mathcal{E}_{EX}W\cdot ss^2 \end{matrix}\right)$$

$$\frac{q^2}{m^3}\left(\begin{matrix} \mathcal{B}_{ss}X\cdot WB\cdot s \\ \mathcal{B}_{ss}W\cdot BX\cdot s \\ \mathcal{B}_{ss}B\cdot XW\cdot s \end{matrix} \oplus \begin{matrix} \mathcal{B}_{sX}W\cdot sB\cdot s \\ \mathcal{B}_{sW}B\cdot sX\cdot s \\ \mathcal{B}_{sB}X\cdot sW\cdot s \end{matrix} \oplus \begin{matrix} \mathcal{B}_{sX}W\cdot Bs^2 \\ \mathcal{B}_{sW}B\cdot Xs^2 \\ \mathcal{B}_{sB}X\cdot Ws^2 \end{matrix} \oplus \begin{matrix} \mathcal{B}_{XW}B\cdot ss^2 \\ \mathcal{B}_{WB}X\cdot ss^2 \\ \mathcal{B}_{BX}W\cdot ss^2 \end{matrix}\right)$$

$$\frac{q^2}{m^3}\left(\begin{matrix} \epsilon_{XEs}{}^{\mu}\mathcal{E}_{\mu s}X\cdot s \\ \epsilon_{XEs}{}^{\mu}\mathcal{E}_{\mu X}s^2 \end{matrix} \oplus \begin{matrix} \epsilon_{WEs}{}^{\mu}\mathcal{E}_{\mu s}W\cdot s \\ \epsilon_{WEs}{}^{\mu}\mathcal{E}_{\mu W}s^2 \end{matrix} \oplus \begin{matrix} \epsilon_{XBs}{}^{\mu}\mathcal{B}_{\mu s}X\cdot s \\ \epsilon_{XBs}{}^{\mu}\mathcal{B}_{\mu X}s^2 \end{matrix} \oplus \begin{matrix} \epsilon_{WBs}{}^{\mu}\mathcal{B}_{\mu s}W\cdot s \\ \epsilon_{WBs}{}^{\mu}\mathcal{B}_{\mu W}s^2 \end{matrix}\right)$$

$$\frac{q^2}{m^3}\left(\begin{matrix} \epsilon_{XWEs}\mathcal{B}_{ss} \\ \epsilon_{XWE}{}^{\mu}\mathcal{B}_{\mu s}s^2 \end{matrix} \oplus \begin{matrix} \epsilon_{sXW}{}^{\mu}\mathcal{B}_{\mu s}E\cdot s \\ \epsilon_{sWE}{}^{\mu}\mathcal{B}_{\mu s}X\cdot s \\ \epsilon_{sEX}{}^{\mu}\mathcal{B}_{\mu s}W\cdot s \end{matrix} \oplus \begin{matrix} \epsilon_{sXW}{}^{\mu}\mathcal{B}_{\mu E}s^2 \\ \epsilon_{sWE}{}^{\mu}\mathcal{B}_{\mu X}s^2 \\ \epsilon_{sEX}{}^{\mu}\mathcal{B}_{\mu W}s^2 \end{matrix}\right)$$

$$\frac{q^2}{m^3}\left(\begin{matrix} \epsilon_{XWBs}\mathcal{E}_{ss} \\ \epsilon_{XWB}{}^{\mu}\mathcal{E}_{\mu s}s^2 \end{matrix} \oplus \begin{matrix} \epsilon_{sXW}{}^{\mu}\mathcal{E}_{\mu s}B\cdot s \\ \epsilon_{sWB}{}^{\mu}\mathcal{E}_{\mu s}X\cdot s \\ \epsilon_{sBX}{}^{\mu}\mathcal{E}_{\mu s}W\cdot s \end{matrix} \oplus \begin{matrix} \epsilon_{sXW}{}^{\mu}\mathcal{E}_{\mu B}s^2 \\ \epsilon_{sWB}{}^{\mu}\mathcal{E}_{\mu X}s^2 \\ \epsilon_{sBX}{}^{\mu}\mathcal{E}_{\mu W}s^2 \end{matrix}\right)$$

$$
\frac{q^2}{m^3}\begin{pmatrix}
\epsilon_{uXE\dot{E}}X\cdot ss^2 & & \epsilon_{uWE\dot{E}}W\cdot ss^2 & & \epsilon_{uXB\dot{B}}X\cdot ss^2 & & \epsilon_{uWB\dot{B}}W\cdot ss^2 \\
\epsilon_{usE\dot{E}}(X\cdot s)^2 & & \epsilon_{usE\dot{E}}(W\cdot s)^2 & & \epsilon_{usB\dot{B}}(X\cdot s)^2 & & \epsilon_{usB\dot{B}}(W\cdot s)^2 \\
\epsilon_{usE\dot{E}}X^2s^2 & & \epsilon_{usE\dot{E}}W^2s^2 & & \epsilon_{usB\dot{B}}X^2s^2 & & \epsilon_{usB\dot{B}}W^2s^2 \\
\epsilon_{usX\dot{E}}E\cdot sX\cdot s & \oplus & \epsilon_{usW\dot{E}}E\cdot sW\cdot s & \oplus & \epsilon_{usX\dot{B}}B\cdot sX\cdot s & \oplus & \epsilon_{usW\dot{B}}B\cdot sW\cdot s \\
\epsilon_{usX\dot{E}}E\cdot Xs^2 & & \epsilon_{usW\dot{E}}E\cdot Ws^2 & & \epsilon_{usX\dot{B}}B\cdot Xs^2 & & \epsilon_{usW\dot{B}}B\cdot Ws^2 \\
\epsilon_{usXE}\dot{E}\cdot sX\cdot s & & \epsilon_{usWE}\dot{E}\cdot sW\cdot s & & \epsilon_{usXB}\dot{B}\cdot sX\cdot s & & \epsilon_{usWB}\dot{B}\cdot sW\cdot s \\
\epsilon_{usXE}\dot{E}\cdot Xs^2 & & \epsilon_{usWE}\dot{E}\cdot Ws^2 & & \epsilon_{usXB}\dot{B}\cdot Xs^2 & & \epsilon_{usWB}\dot{B}\cdot Ws^2
\end{pmatrix}
$$

$$
\frac{q^2}{m^3}\begin{pmatrix}
\epsilon_{usXW}E\cdot\dot{B}s^2 & & \epsilon_{usXW}E\cdot s\dot{B}\cdot s & & \\
\epsilon_{usXE}W\cdot\dot{B}s^2 & & \epsilon_{usXE}W\cdot s\dot{B}\cdot s & & \epsilon_{uXWE}\dot{B}\cdot ss^2 \\
\epsilon_{usX\dot{B}}W\cdot Es^2 & \oplus & \epsilon_{usX\dot{B}}W\cdot sE\cdot s & \oplus & \epsilon_{uWE\dot{B}}X\cdot ss^2 \\
\epsilon_{usWE}X\cdot\dot{B}s^2 & & \epsilon_{usWE}X\cdot s\dot{B}\cdot s & & \epsilon_{uE\dot{B}X}W\cdot ss^2 \\
\epsilon_{usW\dot{B}}X\cdot Es^2 & & \epsilon_{usW\dot{B}}X\cdot sE\cdot s & & \epsilon_{u\dot{B}XW}E\cdot ss^2 \\
\epsilon_{usE\dot{B}}X\cdot Ws^2 & & \epsilon_{usE\dot{B}}X\cdot sW\cdot s & &
\end{pmatrix}
$$

$$
\frac{q^2}{m^3}\begin{pmatrix}
\epsilon_{usXW}B\cdot\dot{E}s^2 & & \epsilon_{usXW}B\cdot s\dot{E}\cdot s & & \\
\epsilon_{usXB}W\cdot\dot{E}s^2 & & \epsilon_{usXB}W\cdot s\dot{E}\cdot s & & \epsilon_{uXWB}\dot{E}\cdot ss^2 \\
\epsilon_{usX\dot{E}}W\cdot Bs^2 & \oplus & \epsilon_{usX\dot{E}}W\cdot sB\cdot s & \oplus & \epsilon_{uWB\dot{E}}X\cdot ss^2 \\
\epsilon_{usWB}X\cdot\dot{E}s^2 & & \epsilon_{usWB}X\cdot s\dot{E}\cdot s & & \epsilon_{uB\dot{E}X}W\cdot ss^2 \\
\epsilon_{usW\dot{E}}X\cdot Bs^2 & & \epsilon_{usW\dot{E}}X\cdot sB\cdot s & & \epsilon_{u\dot{E}XW}B\cdot ss^2 \\
\epsilon_{usB\dot{E}}X\cdot Ws^2 & & \epsilon_{usB\dot{E}}X\cdot sW\cdot s & &
\end{pmatrix}
$$

Since E, B, X, and W are just linear combinations of the vectors $N^{\mu\nu}u_\nu$ and $\bar{N}^{\mu\nu}u_\nu$, $\epsilon_{\mu\nu\rho\sigma}$ contracted with 3 or more of these automatically yield zero, but have been included in the above ansatz for counting purposes.

The above $O(s^3)$ parameter space was searched, and no generalizations of the Carter and Rüdiger constants were found beyond $O(s^2)$. Given the nature of our heuristic approach in ansatz-ing, we cannot rule out the existence Carter and Rüdiger constants beyond $O(s^2)$, however the lack of their existence within a large parameter space is in agreement with the asymptotic analysis done in [114], wherein the authors claim that no such $O(s^3)$ extensions exist.

5.8 xCoba Numerical Testing

As a check on the computational results found using the xTensor package, xCoba, Coba referring to coordinate-basis, provides method for checking the abstract and symbolic tensor index expressions of xTensor in a particular choice of coordinates. xCoba facilitates the constructions of bases, charts, and specific tensor components with respect thereto. We, specifically, employed xCoba for the purposes of verifying identities as well as checking results during our debugging of the xTensor code. This enabled an independent check on the code's correctness and, hence, the validity of our results. We were also able to numerically discover the algebraic relation amongst several of our variables, without which we could not have appropriately ruled out the $O(s^3)$ ansatz space.

5.8.1 Charts, Metrics, and Levi-Civitas in xCoba

To implement xCoba into a notebook, we called the functions

```
<<xAct`xCoba`;
```

xCoba depends on structures of xTensor, which are prerequisitely loaded with xCoba. Just like in our xTensor code, we initialize the same manifold, parameters and tensors

```
DefManifold[ST,4,IndexRange[a,l]]
DefParameter[\[Tau]]
DefTensor[R[]]
...
```

with the notable exception of *not* calling DefMetric. While the inclusion of the other tensors makes the porting of xTensor code into our xCoba notebook easy, as the kernel understands our existing objects, the definition of a metric, in the xTensor sense, will conflict with the definition of a metric in xCoba, causing issues with raising, lowering, and contractions of indices when working with components.

Our first step in xCoba is to define the coordinate chart in which we wish to work:

```
CartesianCoords = {t[], x[], y[], z[]}
DefChart[Carte, ST, {0, 1, 2, 3}, CartesianCoords]
```

In DefChart, we specify the shorthand name for Cartesian coordinates, Carte, the underlying spacetime manifold, ST, and the list of coordinates, CarteCoords, that make up the chart. The list of numbers $\{0, 1, 2, 3\}$ tells DefChart to associate t[] with the zeroth direction, x[] with the first, and so on.

With a chart in hand, we can now define a CTensor, xCoba's head for component-wise defined tensors,

```
gC=CTensor[DiagonalMatrix[{-1, 1, 1, 1}], {-Carte, -Carte}, 0]
```

representing the metric

$$gC = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{270}$$

where the inclusion of {-Carte, -Carte} denotes that the metric is defined in Cartesian coordinates, both of which are lower indices. With this CTensor in hand, we can now define a metric, in the xCoba way of explicitly referencing a coordinate-basis,

```
SetCMetric[gC, Carte, SignatureOfMetric -> {3, 1, 0}]
```

where the {3,1,0} list denotes the number of $+1$, -1, and 0 diagonal entries of the metric.

We now have a CMetric, which I stress not to define concurrently with a regular xTensor metric, and xCoba will automatically use it when adjusting the components through raising, lowering, and contracting CTensor indices. Unlike when defining the xTensor metric, this construction does not automatically include a CTensor equivalent of the totally antisymmetric ϵ tensor. Constructing such a tensor requires doing so by hand, thus we start with defining a placeholder Levi-Civita tensor

```
DefTensor[epsilon[a,b,c,d],ST,{Antisymmetric[{a, b, c, d}]}]
```

to which xCoba can assign the value $\epsilon^{0123} \rightarrow 1$ using

```
ComponentValue[
epsilon[{0, Carte}, {1, Carte}, {2, Carte}, {3, Carte}], 1]
```

We then, using the defined total antisymmetry of epsilon, we proceed to generate all the components

```
epsilon[{a, Carte}, {b, Carte}, {c, Carte}, {d,
Carte}] // ComponentArray
```

and assign component values using

```
ComponentValue/@%
```

From this we can define a totally antisymmetric CTensor with the command

```
epsilongC=CTensor[
(epsilon[{a, Carte}, {b, Carte}, {c, Carte}, {d, Carte}] //
   ComponentArray) /. TensorValues[epsilon], {Carte, Carte,
   Carte, Carte}]
```

5.8.2 CTensors for Root Kerr

In order to construct the necessary building blocks of the Root Kerr field in CTensor, we first begin with the scalars $\mathcal{R} = \sqrt{(x+ia)_\mu (x+ia)^\mu + \left[T_\mu (x+ia)^\mu\right]^2}$ and its complex conjugate, $\bar{\mathcal{R}}$. Here $T^\mu = (1,0,0,0)$ and $a^\mu = (0,0,0,a)$ are the timelike killing- and rescaled spin-vectors of the background. For the ease of taking complex conjugates, as well as for later simplification, we introduce the assumptions

```
DefConstantSymbol[A]
assumptions={a > 0,
     Im[t[]] == 0,Im[x[]] == 0,Im[y[]] == 0,Im[z[]] == 0}]
```

such that a and the spacetime coordinates are treated as real by the kernel. Thus we can define T^μ and a^μ as CTensors

```
TC = CTensor[{1, 0, 0, 0}, {Carte}]
```

```
aC = CTensor[{0, 0, 0, A}, {Carte}]
```

from which we can define the intermediate CTensor

```
In[]:= tempC = CTensor[CarteCoords + I*Table[AvecC[{a, Carte}],
   {a, 0, 3}], {Carte}]
Out[]= CTensor[{t[], x[], y[], I A + z[]}, {Carte}, 0]
```

such that $\mathcal{R}$ can be constructed as

```
In[]:= RC = Scalar[(tempC[a]TempC[-a]+(TC[a]tempC[-a]^2))^(1/2)]
Out:=(x^2+y^2+(I*A+z)^2)^(1/2)
```

and analogously for the complex conjugate $\bar{\mathcal{R}}$.

The rest of the $\sqrt{\text{Kerr}}$ structure can rather easily be constructed from the self and anti-self dual projectors

$$G_{\mu\nu}{}^{\alpha\beta} := 2\delta_{[\mu}{}^{\alpha}\delta_{\nu]}{}^{\beta} + i\epsilon_{\mu\nu}{}^{\alpha\beta} \qquad \bar{G}_{\mu\nu}{}^{\alpha\beta} := 2\delta_{[\mu}{}^{\alpha}\delta_{\nu]}{}^{\beta} - i\epsilon_{\mu\nu}{}^{\alpha\beta}, \tag{271}$$

which, in turn, define

$$N_{\mu\nu} := iG_{\mu\nu}{}^{\alpha\beta}T_{\alpha}(x+ia)_{\beta}\frac{1}{\mathcal{R}} \quad N_{\mu\nu} := -i\bar{G}_{\mu\nu}{}^{\alpha\beta}T_{\alpha}(x-ia)_{\beta}\frac{1}{\bar{\mathcal{R}}}. \tag{272}$$

The CTensor constructions of these objects are simply constructed:

```
GC = CTensor[Table[
    gC[{a, Carte}, {c, Carte}]*gC[{b, Carte}, {d, Carte}]
    - gC[{b, Carte}, {c, Carte}]*gC[{a, Carte}, {d, Carte}]
    + I*epsilongC[{a, Carte}, {b, Carte}, {c, Carte}, {d, Carte}],
{a, 0, 3}, {b, 0, 3}, {c, 0, 3}, {d, 0, 3}],
{Carte, Carte, Carte, Carte}]
nC = Head[I/RC GC[a, b, c, d] TC[-c] tempC[-d]]
```

and analogously for their complex conjugates. The remaining tensors, $F^{\mu\nu}$, $Y^{\mu\nu}$, and $\pi^{\mu\nu}$ are defined in their usual senses from N and $\bar{N}$.

With the background tensors suitably presented in terms of components and coordinates, what remains to parameterize are the worldline-defined tensor objects. Since we have chosen the probe spin vector s^μ and rescaled momentum u^μ as our independent variables for parameterizing the probe's motion, we seek to construct CTensor versions thereof. We seek to do so randomly, such that re-running a numerical check can reveal if an accidental coincidence occurred. The only two subleties in defining CTensors for s and u randomly are that these might not be orthogonal and u might not be appropriately normalized. Hence we want to construct two random CTensors, s and u, satisfying $s^\mu u_\mu = 0$ and $u^\mu u_\mu = -1$.

To accomplish this, we begin by generating preliminary versions of s and u

```
preuC = CTensor[Flatten[{0,RandomInteger[{0, 10}, 3]}], {Carte}]
presC = CTensor[RandomInteger[{0, 10}, 4], {Carte}]
```

where each entry is an integer between 0 and 10. In the case of the preliminary u vector, we have taken the first entry to be zero. We then define the full u CTensor with a suitable time-component as to normalize it:

```
uC = Head[(1 + preuC[a] preuC[-a])^(1/2) TC[b] + preuC[b]]
```

For the s CTensor, we merely project out any u-direction from the preliminary s tensor

```
sC = Head[presC[b] + (presC[a] uC[-a]) uC[b] // Simplification]
```

as to satisfy the orthogonality to u.

From these, one can construct the spin tensor S as

```
SC = Head[epsilongC[a, b, c, d] uC[-c] sC[-d] // Simplification]
```

along with any other worldline objects.

5.8.3 Debugging xTensor Results

Having defined all the xTensor tensors and xCoba CTensors, all that remains to numerically evaluate an xTensor expression is to replace the xTensor versions with their xCoba counterparts. For this reason, we have largely employed the nomenclature of concatenating a "C" to the end of our xTensors when naming our CTensors. Thus, we can write a list of rules

```
XtoC={}
AppendTo[XtoC,S[a_,b_] :> SC[a,b]]
AppendTo[XtoC,R[] :> RC]
AppendTo[XtoC,g[a_,b_] :> gC[a,b]]
AppendTo[XtoC,epsilong[a_,b_,c_,d_] :> epsilongC[a,b,c,d]]
...
```

such that we can apply XtoC to any xTensor expression for a numerical result. The only remaining step is to assign numerical values to the various coordinates and other parameters, such as t[], x[], y[], z[], q, Q, m.

Equipped with this code, we extensively tested various aspects of our code. For every replacement rule, with the exception of the equations of motion, written as lhs-¿rhs, we numerically verified that lhs-rhs==0. In order to verify the equations of motion, we similarly numerically verified that the worldline time derivatives of $s \cdot p$, $p_\mu S^{\mu\nu}$, $u_\mu S^{\mu\nu}$, $p^2 + \mathcal{M}^2$, s^2, and $\dot{z} \cdot u + 1$ were identically zero. These, we are confident that our replacement rules and equations of motion were correctly implemented in the xTensor Mathematica code.

6 Integrability Results

The resulting Mathisson-Papapetrou-Dixon (MPD) equations of motion [7, 8, 10–13], governing the worldline and spin evolution, are said to be Liouville integrable if a sufficient number of conserved quantities exist. For a *spinless*, gravitating probe in a Kerr spacetime background, the geodesic equation, $p^\nu \nabla_\nu p^\mu = 0$, requires 4 independent conserved quantities to integrate on an 8-dimensional phase space. The Kerr spacetime background has 2 independent Killing vectors, associated with time-translation and azimuthal-rotation invariance, from which we have the conservation of energy and one component of angular momentum. Another constraint comes from worldline re-parameterization invariance in the form of the "on shell" constraint $-m^2 = g_{\mu\nu} p^\mu p^\nu$. Lastly, and unexpectedly, Carter [130] found an additional, independently conserved constant $K_{\mu\nu} p^\mu p^\nu$, where $K_{\mu\nu} = K_{\nu\mu}$ is a Killing tensor satisfying $\nabla_{(\mu} K_{\nu)\rho} = 0$ — a generalization of the Killing vector condition $\nabla_{(\mu} \xi_{\nu)} = 0$. Thus geodesic motion in a Kerr spacetime is integrable, enabling the separation and reduction of the problem to one dimensional integrals.

Going beyond geodesic motion, we can generalize to a body with spin tensor $S_{\mu\nu} = -S_{\nu\mu}$, where we now have an additional 6 degrees of freedom. Via a Lagrange multiplier in the worldline action, one can enforce 3 additional constraints, $S_{\mu\nu} p^\nu = 0$, called the spin supplementary condition (SSC), which can be understood as fixing the worldline along the object's center of mass. Additionally, the quadratic Casimir $\frac{1}{2} S_{\mu\nu} S^{\mu\nu}$ is independently conserved, leaving 2 unconstrained spin degrees of freedom. With suitable modifications, the on-shell constraint and two Killing vectors yield 3 additional conserved quantities. Lastly, for a black-hole-like probe, generalizations of Carter's constant along with another constant found by Rüdiger [131, 132] were found to $O(S^2)$ [113]. Consequently, with 8+2 degrees of freedom and 5 conserved quantities, the motion of a test black hole in a Kerr spacetime is integrable up to second order in spin. The focus of this chapter is the analogous result for the $\sqrt{\text{Kerr}}$ probe in a $\sqrt{\text{Kerr}}$ background, through second order in the probe spin. Moreover, we discuss the lack of generalizations of the electromagnetic Carter and Rüdgier constants beyond $O(s^2)$, the implications for the minimal electromagnetic Compton amplitude, and ties to asymptotic integrability.

6.1 $O(s^2)$ Carter, Rüdiger, and Compton Results

In our action, the most general dynamical mass function to $O(s^3)$ was given by [110] and quoted in equation 48. The variables used in this chapter are defined in Worldline Formalism and Root Kerr chapters.

In addition to the conserved quantities resulting from our two Killing vectors $\xi \in \{t, e_\phi\}$,

$$\mathcal{P}_\xi = \left(p_\mu + qA_\mu\right)\xi^\mu + \frac{1}{2}S^{\mu\nu}\partial_\mu\xi_\nu, \tag{273}$$

we find that, to $O(S^2)$, the following generalizations of Carter and Rüdiger's constants, C and Q, are conserved:

$$\begin{aligned} C = {} & Y_{\mu\rho}Y_\nu{}^\rho p^\mu p^\nu - qF^\mu{}_\sigma S^{\nu\sigma}Y_{\mu\rho}Y_\nu{}^\rho + 4T^\lambda \epsilon_{\lambda\rho\sigma[\mu}Y_{\nu]}{}^\sigma p^\mu S^{\nu\rho} \\ & - (s^\mu T_\mu)^2 + s^\mu s_\mu (T^\nu u_\nu)^2 - \frac{q}{m}s^\mu s_\mu T^\nu u_\nu F^{\rho\sigma *}Y_{\rho\sigma} - \frac{2q}{m}s^\mu T_\mu F^{\rho\sigma}s^\nu u_\rho{}^* Y_{\nu\sigma} \\ & + \frac{3q}{m}T^\mu u_\mu F_\nu{}^\rho s^\nu s^{\sigma *}Y_{\sigma\rho} + \frac{q}{m}s^\mu s_\mu F_\nu{}^\rho T^\nu u^{\sigma *}Y_{\sigma\rho} + \frac{q}{m}s^\mu s^\nu u^{\rho *}Y_\rho{}^{\sigma *}Y_\sigma{}^\gamma \nabla_\nu F_{\mu\gamma}, \end{aligned} \tag{274}$$

and

$$Q = \frac{1}{2}S^{\mu\nu *}Y_{\mu\nu} + \frac{3q}{2m^2}F_{\mu\nu}s^\mu u^\nu s^\rho u^{\sigma *}Y_{\rho\sigma} + \frac{q}{2m^2}F_{\mu\nu}s^\mu s^{\rho *}Y_\rho{}^\nu - \frac{q}{m^2}s^\mu s_\mu F_\alpha{}^\gamma u^\alpha u^{\beta *}Y_{\beta\gamma}, \tag{275}$$

only for a unique choice of the Wilson coefficients in the action. As an important note, the conservation of 274 and 275 is compatible with the background field charge of $Q \to -Q$, or equivalently $F^{\mu\nu} \to -F^{\mu\nu}$, everywhere in the expressions for C, Q, and MPD equations.

The conservation of C and Q fixes the multipole coefficients $C_1 = C_2 = 1$ in 48, in agreement with the $\sqrt{\text{Kerr}}$ stationary multipole moments from [110]. Moreover, the so-called dynamical multipole moments, D_1,D_2,D_3, and D_4 are fixed to be zero, implying that the leading order $\sqrt{\text{Kerr}}$ electromagnetic tidal deformation of the $\sqrt{\text{Kerr}}$ solution vanishes. $D_1 = D_2 = D_3 = D_4 = 0$ also implies that the resulting Compton helicity amplitudes from [110] exhibit the "spin-exponentiation" property up to second order. Physically, a stand-alone $\sqrt{\text{Kerr}}$ charge configuration would be unstable under its own field, hence it seems implausible that it should be indifferent to leading order electromagnetic perturbations. This conundrum is resolved in understanding $\sqrt{\text{Kerr}}$ as the $G \to 0$ *limit* of the Kerr-Newman spacetime: The electromagnetic charge distribution is affixed by the gravitational singularity, not by its own field, and the vanishing dynamical multipole moments are to be understood as the Kerr-Newman spacetime's response to a perturbation, not the standalone $\sqrt{\text{Kerr}}$ charge configuration. Investigating the Kerr-Newman spacetime's response to such perturbations is currently beyond the state of the art of black hole perturbation theory, such as Teukolsky-based approaches in [111], hence the desire for other physical principles, such as Liouville integrability, in fixing undetermined Wilson coefficients.

As for the implications of $\sqrt{\text{Kerr}}$ integrability for its Compton scattering, consider a particular Lorentz frame, associated to inertial Cartesian coordinates $x^\mu = (t, x, y, z)$, where $\sqrt{\text{Kerr}}$ has initial velocity $v^\mu = (1, 0, 0, 0)$ and the incoming and outgoing photons have wavevectors $k_1^\mu = \omega(1, 0, 0, 1)$ and $k_2^\mu = \omega(1, \sin\theta, 0, \cos\theta)$, respectively. In terms of the squared momentum transfer $q^2 = (k_2 - k_1)^2 = 4\omega^2 \sin^2 \frac{\theta}{2}$, the helicity preserving A_{++} and reversing A_{+-} Compton amplitudes from [110] become

$$\mathcal{A}_{++} = \frac{q^2 - 4\omega^2}{2\omega^2} e^{(k_1 + k_2 - 2w)\cdot a} + O(S^3) \tag{276}$$

$$\mathcal{A}_{+-} = \frac{q^2}{2\omega^2} e^{(k_1 - k_2)\cdot a} + O(S^3), \tag{277}$$

where $\omega = -v \cdot k_1 = -v \cdot k_2$, and $w^\mu = \omega\left(1, \tan\frac{\theta}{2}, i\tan\frac{\theta}{2}, 1\right)$. Hence with our identification of Carter and Rüdiger's constants, and thereby integrability through $O(s^2)$, the associated $\sqrt{\text{Kerr}}$ Compton amplitudes are fixed, in agreement with the minimal Compton from equation 208, along with exhibiting the conjectured spin-exponentiation pattern. It is, therefore, natural to wonder if such integrability persists at higher orders in spin.

6.1.1 Lack of Integrability Beyond $O(s^2)$

The generalization of our $O(s^2)$ ansätze for Carter and Rüdiger is given in the ansatz building sub-chapter. We searched a large parameter space through $O(s^3)$ and did not identify any Carter or Rüdiger constants, for any choices of Wilson coefficients in the dynamical mass function. Consequently, we believe that no such generalizations exist. Concurrently, a parallel result to our investigation was published in [114], where the authors perform an asymptotic investigation, using the Dirac-bracket formalism of [115, 133]. The authors found that, by studying the leading order asymptotic behavior of the ansatz space that we investigated in the bulk, they ruled out the existence of integrability for a $\sqrt{\text{Kerr}}$ probe in a $\sqrt{\text{Kerr}}$ background — a result that agrees with our bulk computation. Interestingly, while the gravitational Carter and Rüdiger constants were found for a Kerr probe in a Kerr background in [113], the asymptotic integrability formalism showed an analogous result in [115], wherein the same asymptotic analysis ruled out the integrability in the gravitational case beyond quartic order in the probe spin. The quadratic versus quartic order integrability in electromagnetism versus gravity is yet another testament to the classical double-copy relationship between the $\sqrt{\text{Kerr}}$ and Kerr solutions.

Despite our attempts to attribute physical significance to the breakdown of integrability, it is presently unclear as to why integrability should fail at orders quadratic and quartic in $\sqrt{\text{Kerr}}$ and Kerr, respectively. One might postulate that, beyond a certain order, each theory exhibits mixing of conservative and radiative effects – perhaps better to be understood from the associated scattering amplitudes. For black holes, for instance, their spin is upper-bounded by the extremal black hole case, hence the powers in spin can loosely be compared to powers of the coupling constant, G. Hence, sufficiently high orders in spin may mix with loop amplitudes that introduce non-conservative dynamics beyond $G^{\triangle}$.

Regardless of the particular reason for the breakdown of integrability, there still remains the task of finding the generalizations of the gravitational Carter and Rüdiger constants beyond $O(s^2)$, up to, at least, $O(s^4)$. This will be my, and my co-author Justin Vines', next project.

A Amplitudes Conventions

$$\left[T^a, T^b\right] = i\sqrt{2} f^{abc} T^c$$

$$\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab}$$

$$f^{abc} = -\frac{i}{\sqrt{2}} \mathrm{Tr}\left(T^a \left[T^b, T^c\right]\right)$$

$$(T^a)_{i_1}^{j_1} (T^a)_{i_2}^{j_2} = \delta_{i_1}^{j_2} \delta_{i_2}^{j_1} - \frac{1}{N} \delta_{i_1}^{j_1} \delta_{i_2}^{j_2}$$

$$\left(T_A^a\right)^{bc} := -i\sqrt{2} f^{abc}$$

Ingoing gluon	$\epsilon_{s\mu}(p)$	$p \cdot \epsilon_s(p) = 0$ and $p^2 = 0$
Outgoing gluon	$\epsilon^*_{s\mu}(p)$	$p \cdot \epsilon^*_s(p) = 0$ and $p^2 = 0$

Incoming fermion	$u_s(p)$	$p^2 = m^2$
Incoming anti-fermion	$\bar{v}_s(p)$	$p^2 = m^2$

Outgoing fermion	$\bar{u}_s(p)$	$p^2 = m^2$
Outgoing anti-fermion	$v_s(p)$	$p^2 = m^2$

where $(\not{p} - m)u(p) = 0$, and $(\not{p} + m)v(p) = 0$. Moreover, as $m \to 0$, one has the identification

$$\underbrace{u_{\pm}(p)}_{:=\frac{1}{2}(1\pm\gamma_5)u(p)} \quad \leftrightarrow \quad \underbrace{v_{\mp}(p)}_{:=\frac{1}{2}(1\mp\gamma_5)v(p)}$$

for spin 1 gauge fields with the associated polarization vectors $\epsilon^{\mu}_{\pm}(p)$, one has

$$p \cdot \epsilon_{\pm}(p) = 0$$

$$\epsilon_{\pm} \cdot \epsilon_{\mp} = -1$$

$$\epsilon_{\pm} \cdot \epsilon_{\pm} = 0$$

$$\left(\epsilon^{\mu}_{\pm}\right)^* = \epsilon^{\mu}_{\mp}$$

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