Title
P-Recursive Integer Sequences and Automata Theory

Author
Garrabrant, Scott Michael

Publication Date
2015

Peer reviewed|Thesis/dissertation

University of California

Los Angeles

P-Recursive Integer Sequences and Automata Theory

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Scott Michael Garrabrant

2015

Abstract of the Dissertation

P-Recursive Integer Sequences and Automata Theory

by

Scott Michael Garrabrant

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2015

Professor Igor Pak, Chair

An integer sequence $\{a_n\}$ is called *polynomially recursive*, or *P-recursive*, if it satisfies a nontrivial linear recurrence relation of the form

$$q_0(n)\,a_n \,+\, q_1(n)\,a_{n-1} \,+\, \ldots \,+\, q_k(n)\,a_{n-k} \,=\, 0\,,$$

for some $q_i(x) \in \mathbb{Z}[x]$, $0 \le i \le k$. The study of P-recursive sequences plays a major role in modern Enumerative and Asymptotic Combinatorics. P-recursive sequences have *D-finite* (also called *holonomic*) generating functions.

This dissertation is on the application of automata theory to the analysis of P-recursive integer sequences, and is broken into three self-contained chapters. Chapters 1 and 2 contain negative results, simulating Turing machines within combinatorial structures to show these structures are not counted by a P-recursive sequence. In Chapter 1, we answer a question of Maxim Kontsevich by showing $[1]u^n$ is not always P-recursive when $u \in \mathbb{Z}[\mathrm{GL}(k,\mathbb{Z})]$. In Chapter 2, we disprove the celebrated Noonan-Zeilberger conjecture by showing that pattern avoidance is not P-recursive. These two chapters give the first results that disprove P-recursiveness using automata theory. Historically, results of this form have been mostly proven using analysis of the asymptotics.

Chapter 3 gives a full analysis of the class of integer sequences counting irrational tilings of a constant height strip. This class is a subset of P-recursive sequences, and we prove the equivalence of three definitions of this class: counting functions of irrational tilings, binomial multisums, and diagonals of $\mathbb{N}$-rational generating functions. In this analysis, we count paths through a labelled graph in a way that is equivalent to counting strings accepted by a given deterministic finite automaton where each letter appears n times.

The dissertation of Scott Michael Garrabrant is approved.

Rafail Ostrovsky

Bruce L. Rothschild

Alexander Sherstov

Igor Pak, Committee Chair

University of California, Los Angeles

2015

Table of Contents

Acknowledgments

All of the work in this dissertation is based on papers currently in preparation, co-authored with my advisor, Igor Pak. An alternate version of Chapter 2 has been accepted for the ACM-SIAM Symposium on Discrete Mathematics 2016.

We are very grateful to Michael Albert, Jean-Paul Allouche, Matthias Aschenbrenner, Cyril Banderier, Miklós Bóna, Alin Bostan, Mireille Bousquet-Mélou, Alexander Burstein, Ted Dokos, Michael Drmota, Misha Ershov, Joanna Garrabrant, Ira Gessel, Martin Kassabov, Nick Katz, Maxim Kontsevich, Andrew Marks, Sam Miner, Marni Mishna, Cris Moore, Alejandro Morales, Greta Panova, Robin Pemantle, Christophe Pittet, Christophe Reutenauer, Laurent Saloff-Coste, Bruno Salvy, Andy Soffer, Richard Stanley, Vince Vatter, Jed Yang and Doron Zeilberger for helpful conversations and useful comments at different stages of the project.

VITA

2011	B.A. in Mathematics (Honors) and Computer Science Pitzer College
2011-2015	Eugene Cota-Robles Fellow University of California, Los Angeles
2011-2013	Chancellor's Prize University of California, Los Angeles
2014	C.Phil in Mathematics University of California, Los Angeles
2015	Dissertation Year Fellow University of California, Los Angeles

CHAPTER 1

Words in Linear Groups, Random Walks, and P-Recursiveness

1.1 Introduction

An integer sequence $\{a_n\}$ is called *polynomially recursive*, or *P-recursive*, if it satisfies a nontrivial linear recurrence relation of the form

$$(*) \qquad q_0(n)\,a_n \,+\, q_1(n)\,a_{n-1} \,+\, \ldots \,+\, q_k(n)\,a_{n-k} \,=\, 0\,,$$

for some $q_i(x) \in \mathbb{Z}[x]$, $0 \le i \le k$. The study of P-recursive sequences plays a major role in modern Enumerative and Asymptotic Combinatorics, see e.g. [FS, Ges2, Odl, Sta1]. They have *D-finite* (also called *holonomic*) generating series

$$\mathcal{A}(t) \,=\, \sum_{n=0}^{\infty} a_n t^n\,,$$

and various asymptotic properties (see Section 1.5 below).

Let G be a group and $\mathbb{Z}[G]$ denote its group ring. For every $g \in G$ and $u \in \mathbb{Z}[G]$, denote by $[g]\,u$ the value of u on g. Let $a_n = [1]u^n$, which denotes the value of u^n at the identity element. When $G = \mathbb{Z}^k$ or $G = F_k$, the sequence $\{a_n\}$ is known to be P-recursive for all $u \in \mathbb{Z}[G]$, see [Hai]. Maxim Kontsevich asked whether $\{a_n\}$ is always P-recursive when $G \subseteq \mathrm{GL}(k, \mathbb{Z})$, see [S2]. We give a negative answer to this question:

Theorem 1.1.1. *There exists an element* $u \in \mathbb{Z}[SL(4, \mathbb{Z})]$*, such that the sequence* $\{\,[1]\,u^n\}$ *is not P-recursive.*

We give two proofs of the theorem. The first proof is completely self-contained and based on ideas from computability. Roughly, we give an explicit construction of a finite state automaton with two stacks and a non-P-recursive sequence of accepting path lengths (see Section 1.3). We then convert this automaton into a generating set $S \subset \mathrm{SL}(4, \mathbb{Z})$, see Section 1.4. The key part of the proof is a new combinatorial lemma giving an obstruction to P-recursiveness (see Section 1.2).

Our second proof of Theorem 3.1.2 is analytic in nature, and is the opposite of being self-contained. We interpret the problem in a probabilistic language, and use a number of advanced and technical results in Analysis, Number Theory, Probability and Group Theory to derive the theorem. Let us briefly outline the connection.

Let S be a generating set of the group G. Denote by $p(n) = p_{G,S}(n)$ the probability of return after n steps of a random walk on the corresponding Cayley graph $\mathrm{Cay}(G, S)$. Finding the asymptotics of $p(n)$ as $n \to \infty$ is a fundamental problem in probability, with a number of both classical and recent results (see e.g. [Pete, Woe]). In the notation above, we have:

$$p(n) = \frac{a_n}{|S|^n}, \quad \text{where} \quad a_n = [1]u^n \quad \text{and} \quad u = \sum_{s \in S} s.$$

Since P-recursiveness of $\{a_n\}$ implies P-recursiveness of $\{p(n)\}$, and much is known about the asymptotic of both $p(n)$ and P-recursive sequences, this connection can be exploited to obtain non-P-recursive examples (see Section 1.5). See also Section 3.11 for final remarks and historical background behind the two proofs.

1.2 Parity of P-Recursive Sequences

In this section, we give a simple obstruction to P-recursiveness.

Lemma 1.2.1. *Let $\{a_n\}$ be a P-recursive integer sequence. Consider an infinite binary word $\mathbf{w} = w_1 w_2 \ldots$ defined by $w_n = a_n$ textrmmod 2. Then, there exists a finite binary word v which is not a subword of w.*

Proof. Let $\eta(n)$ denote the largest integer r such that $2^r | n$. By definition, there exist polynomials $q_0, \ldots, q_k \in \mathbb{Z}[n]$, such that

$$a_n = \frac{1}{q_0(n)} \big(a_{n-1} q_1(n) + \ldots + a_{n-k} q_k(n) \big), \quad \text{for all } n > k.$$

Let ℓ be any integer such that $q_i(\ell) \neq 0$ for all i. Similarly, let m be the smallest integer such that $2^m > k$, and $m > \eta(q_i(\ell))$ for all i. Finally, let $d > 0$ be such that $\eta(q_d(\ell)) \leq \eta(q_i(\ell))$ for all $i > 0$.

Consider all n such that:

$$n = \ell \bmod 2^m, \quad w_{n-d} = 1 \quad \text{and} \quad w_{n-i} = 0 \quad \text{for all } \ i \neq 0, d. \tag{$\star$}$$

Note that $\eta(q_i(n)) = \eta(q_i(\ell))$ for all i, since $q_i(n) = q_i(\ell) \bmod 2^m$ and $\eta(q_i(\ell)) < m$. We have

$$\eta(a_n) = \eta\Big(a_{n-1} q_1(\ell) + \ldots + a_{n-k} q_k(\ell) \Big) - \eta(q_0(\ell)).$$

Since $\eta(a_{n-d} q_d(\ell)) < \eta(a_{n-i} q_i(\ell))$ for all $i \neq d$, this implies that

$$\eta(a_n) = \eta(a_{n-d} q_d(\ell)) - \eta(q_0(\ell)) = \eta(q_d(\ell)) - \eta(q_0(\ell)).$$

Therefore, $w_n = 1$ if and only if $\eta(q_d(\ell)) = \eta(q_0(\ell))$. This implies that w_n is independent of n, and must be the same for all n satisfying ($\star$). In particular, this means that at least one of the words $0^{k-d}10^{d-1}1$ and $0^{k-d}10^d$ cannot appear in $\mathbf{w}$ ending at a location congruent to ℓ modulo 2^m.

Consider the word $v = (0^{k-d}10^k10^{d-1})^{2^m}$. Note that $0^{k-d}10^k10^{d-1}$ has odd length, and contains both $0^{k-d}10^{d-1}1$ and $0^{k-d}10^d$ as subwords. Therefore, the word v contains both $0^{k-d}10^{d-1}1$ and $0^{k-d}10^d$ in every possible starting location modulo 2^m. This implies that v cannot appear as a subword of $\mathbf{w}$. $\square$

1.3 Building an Automaton

In this section we give an explicit construction of a finite state automaton with the number of accepting paths given by a binary sequence which does not satisfy conditions of Lemma 2.2.1.

Let $X \simeq F_3$ be the free group generated by x, 1_x, and 0_x. Similarly, let $Y \simeq F_3$ be the free group generated by y, 1_y, and 0_y. We assume that X and Y commute.

Define a directed graph Γ on vertices $\{s_1, \ldots, s_8\}$, and with edges as shown in Figure 1.1. Some of the edges in Γ are labeled with elements of X, Y, or both. For a path γ in Γ, denote by $\omega_X(\gamma)$ the product of all elements of X in γ, and by $\omega_Y(\gamma)$ denote the product of all elements of Y in γ. By a slight abuse of notation, while traversing γ we will use ω_X and ω_Y to refer to the product of all elements of X and Y, respectively, on edges that have been traversed so far.

Finally, let b_n denote the number of paths in Γ from s_1 to s_8 of length n, such that $\omega_X(\gamma) = \omega_Y(\gamma) = 1$. For example, the path

$$\gamma : \quad s_1 \xrightarrow{xy} s_1 \to s_2 \xrightarrow{1_y x^{-1}} s_4 \xrightarrow{1_y^{-1} 1_x} s_4 \xrightarrow{y^{-1}} s_5 \to s_6 \xrightarrow{1_x^{-1}} s_8$$

is the unique such path of length 7, so $b_7 = 1$.

Lemma 1.3.1. *For every $n \geq 1$ we have $b_n \in \{0, 1\}$. Moreover, every finite binary word is a subword of $\mathbf{b} = b_1 b_2 \ldots$*

Proof. To simplify the presentation, we split the proof into two parts.

(a) The structure of paths. Let γ be a path from s_1 to s_8. Denote by k the number of times γ traverses the loop $s_1 \xrightarrow{xy} s_1$. The value of ω_X after traversing these k loops is x^k, and the value of ω_Y is y^k.

There must be k instances of the edge $s_4 \xrightarrow{y^{-1}} s_5$ in γ to cancel out the y^k. Further, any time the path traverses this edge, the product ω_Y must change from

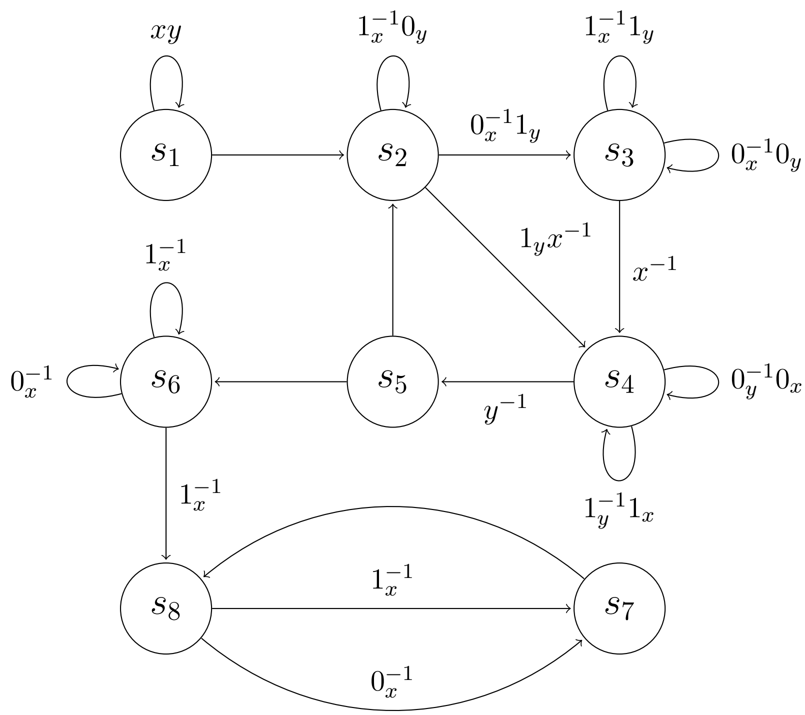


Figure 1.1: The graph Γ.

some y^j to y^{j-1}, with no 0_y or 1_y terms. Therefore, every time γ enters the vertex s_4, it must traverse the two loops $s_4 \xrightarrow{1_y^{-1}1_x} s_4$ and $s_4 \xrightarrow{0_y^{-1}0_x} s_4$ enough to replace any 0_y and 1_y terms in ω_Y with 0_x and 1_x terms in ω_X. This takes the binary word at the end of ω_Y, and moves it to the end of ω_X in the reverse order.

Similarly, any time γ traverses the edge $s_3 \xrightarrow{x^{-1}} s_4$ or $s_2 \xrightarrow{1_y x^{-1}} s_4$, the product ω_X must change from some x^j to x^{j-1}, with no 0_x or 1_x terms. Every time γ enters the vertex s_2, it must remove all 0_x and 1_x terms from ω_X before transitioning to s_4. The s_2 and s_3 vertices ensure that as this binary word is deleted from ω_X, another binary word is written at the end of ω_Y such that the reverse of the binary word written at the end of ω_Y is one greater as a binary integer than the word removed from the end of ω_X.

Every time γ traverses the edge $s_4 \xrightarrow{y^{-1}} s_5$, the number written in binary at the end of ω_X is incremented by one. Thus, after traversing this edge k times, the X word will consist of k written in binary, and ω_Y will be the identity. At this point, γ will traverse the edge $s_5 \xrightarrow{y^{-1}} s_6$.

After entering the vertex s_6, all of the 0_x and 1_x terms from ω_X will be removed. Each time a 1_x term is removed, γ can move to the vertex s_8. From s_8, the 0_x and 1_x terms will continue to be removed, but γ will traverse two edges for every term removed, thus moving at half speed. After all of these terms are removed, the products $\omega_X(\gamma)$ and $\omega_Y(\gamma)$ are equal to identity, as desired.

(b) The length of paths. Now that we know the structure of paths through Γ, we are ready to analyze the possible lengths of these paths. There are only two choices to make in specifying a path γ : first, the number $k = k(\gamma)$ of times the loop from s_1 to itself is traversed, and second, the number $j = j(\gamma)$ of digits still on $\omega_X(\gamma)$ immediately before traversing the edge from s_6 to s_8. The number j must be such that the j-th binary digit of k is a 1.

When γ reaches s_5 for the first time, it has traversed $k+4$ edges. In moving from the i-th instance of s_5 along γ to the $(i+1)$-st instance of s_5, the number of edges traversed is $3 + \lfloor 1 + \log_2(i) \rfloor + \lfloor 1 + \log_2(i+1) \rfloor$, three more than the sum of the number of binary digits in i and $i+1$. Therefore, the number of edges traversed by the time γ reaches s_6 is equal to

$$k + 5 + \sum_{i=1}^{k-1} (3 + \lfloor 1 + \log_2(i) \rfloor + \lfloor 1 + \log_2(i+1) \rfloor).$$

If $j = 1$, the edge from s_6 to s_8 is traversed at the last possible opportunity and $\lfloor 1 + \log_2(k) \rfloor$ more edges are traversed. However, if $j > 1$, there are an additional $j - 1$ edges traversed, since the s_7 and s_8 states do not remove ω_X terms as efficiently as s_6. In total, this gives $|\gamma| = L(k(\gamma), j(\gamma))$, where

$$L(k, j) \;=\; j - 1 + \lfloor 1 + \log_2(k) \rfloor + k + 5 + \sum_{i=1}^{k-1} \Big(3 + \lfloor 1 + \log_2(i) \rfloor + \lfloor 1 + \log_2(i+1) \rfloor\Big).$$

This simplifies to

$$L(k,j) \;=\; j + 6k + 2\sum_{i=1}^{k} \lfloor \log_2 i \rfloor \,.$$

Since $1 \le j \le \lfloor 1 + \log_2(k) \rfloor$, we have $L(k+1,1) > L(k,j)$ for all possible values of j. Thus, there are no two paths of the same length, which proves the first part of the lemma.

Furthermore, we have $b_n = 1$ if and only if $n = L(k,j)$ for some $k \ge 1$ and j such that the j-th binary digit of k is a 1. Thus, the binary subword of $\mathbf{b}$ at locations $L(k,1)$ through $L(k, \lfloor 1 + \log_2(k) \rfloor)$ is exactly the integer k written in binary. This is true for every positive integer k, so $\mathbf{b}$ contains every finite binary word as a subword. $\square$

Example 1.3.2. For $k = 3$ and $j = 2$, we have $L(k,j) = 24$. This corresponds to the unique path in Γ of length 24:

$$\begin{aligned} s_1 &\xrightarrow{xy} s_1 \xrightarrow{xy} s_1 \xrightarrow{xy} s_1 \to s_2 \xrightarrow{1_y x^{-1}} s_4 \xrightarrow{1_y^{-1} 1_x} s_4 \xrightarrow{y^{-1}} s_5 \to s_2 \\ &\xrightarrow{1_x^{-1} 0_y} s_2 \xrightarrow{1_y x^{-1}} s_4 \xrightarrow{1_y^{-1} 1_x} s_4 \xrightarrow{0_y^{-1} 0_x} s_4 \xrightarrow{y^{-1}} s_5 \to s_2 \xrightarrow{0_x^{-1} 1_y} s_3 \xrightarrow{1_x^{-1} 1_y} s_3 \\ &\xrightarrow{x^{-1}} s_4 \xrightarrow{1_y^{-1} 1_x} s_4 \xrightarrow{1_y^{-1} 1_x} s_4 \xrightarrow{y^{-1}} s_5 \to s_6 \xrightarrow{1_x^{-1}} s_8 \xrightarrow{1_x^{-1}} s_7 \to s_8 \,. \end{aligned}$$

1.4 Proof of Theorem 3.1.2

1.4.1 From automata to groups

We start with the following technical lemma.

Lemma 1.4.1. *Let $G = F_{11} \times F_3$. Then there exists an element $u \in \mathbb{Z}[G]$, such that $[1]u^{2n+1}$ is always even, and $\mathbf{w} = w_1 w_2 \ldots$ given by $w_n = \left(\frac{1}{2}\,[1]u^{2n+1}\right) \mathit{mod}\, 2$, is an infinite binary word that contains every finite binary word as a subword.*

Proof. We label the generators of F_{11} as $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, x, 0_x, 1_x\}$ and label the generators of F_3 as $\{y, 0_y, 1_y\}$. Consider the following set S of 19 elements of G:

(1) $z_1 = s_1^{-1}xys_1$,

(2) $z_2 = s_1^{-1}s_2$,

(3) $z_3 = s_2^{-1}1_x^{-1}0_ys_2$,

(4) $z_4 = s_2^{-1}0_x^{-1}1_ys_3$,

(5) $z_5 = s_3^{-1}1_x^{-1}1_ys_3$,

(6) $z_6 = s_3^{-1}0_x^{-1}0_ys_3$,

(7) $z_7 = s_3^{-1}x^{-1}s_4$,

(8) $z_8 = s_2^{-1}1_yx^{-1}s_4$,

(9) $z_9 = s_4^{-1}1_y^{-1}1_xs_4$,

(10) $z_{10} = s_4^{-1}0_y^{-1}0_xs_4$,

(11) $z_{11} = s_4^{-1}y^{-1}s_5$,

(12) $z_{12} = s_5^{-1}s_2$,

(13) $z_{13} = s_5^{-1}s_6$,

(14) $z_{14} = s_6^{-1}1_x^{-1}s_6$,

(15) $z_{15} = s_6^{-1}0_x^{-1}s_6$,

(16) $z_{16} = s_6^{-1}1_x^{-1}s_8$,

(17) $z_{17} = s_7^{-1}s_8$,

(18) $z_{18} = s_8^{-1}1_x^{-1}s_7$,

(19) $z_{19} = s_8^{-1}0_x^{-1}s_7$.

Let Γ be as defined in the previous section. For every edge from $s_i \xrightarrow{r} s_j$ in Γ, there is one element of S equal to $s_i^{-1}rs_j$. We show that the number of ways to multiply n terms from S to get $s_1^{-1}s_8$ is exactly b_n.

First, we show that there is no product of terms in S whose F_{11} component is the identity. Assume that such a product exists, and take one of minimal length. If there are two consecutive terms in this product such that s_i at the end of one term does not cancel the s_j^{-1} at the start of the following term, then either the s_i must cancel with a s_i^{-1} before it or the s_j^{-1} must cancel with a s_j after it. In both cases, this gives a smaller sequence of terms whose product must have F_{11} component equal to the identity. If the s_i at the end of each term cancels the s_j^{-1} at the beginning of the next term, then this product corresponds to a cycle $\gamma \in \Gamma$ such that $\omega_X(\gamma)$ is the identity. Straightforward analysis of Γ shows that no such cycle exists, so there is no product of terms in S whose product F_{11} component equal to the identity.

This also means that the s_i at the end of each term must cancel the s_j^{-1} at the start of the following term, since otherwise ether the s_i must cancel with a s_i^{-1}

before it or the s_j^{-1} must cancel with a s_j after it, forming a product of terms in S whose F_{11} component is equal to the identity.

Since each s_i cancels with an s_i^{-1} at the start of the following term, the product must correspond to a path $\gamma \in \Gamma$. If γ is from s_i to s_j, the product will evaluate to $s_i^{-1}\omega_X(\gamma)\omega_Y(\gamma)s_j$. Therefore, the number of ways to multiply n terms from S to get $s_1^{-1}s_8$ is equal to b_n.

We can now define $u \in \mathbb{Z}[G]$ as

$$u = 2s_8^{-1}s_1 + \sum_{z_i \in S} z_i .$$

We claim that $\frac{1}{2}[1]u^{2n+1} = b_{2n} \bmod 2$. We already showed that one cannot get 1 by multiplying only elements of S, so the $2s_8^{-1}s_1$ term must be used at least once. If this term is used more than once, then the contribution to $[1]u^{2n+1}$ will be 0 mod 4. Therefore, we need only consider the cases where this term is used exactly once, so $\frac{1}{2}[1]u^{2n+1}$ is equal modulo 2 to the number of products of the form

$$2 = z_{i_1} \dots z_{i_{k-1}}(2s_8^{-1}s_1)z_{i_{k+1}} \dots z_{i_{2n+1}}. \qquad (\star\star)$$

This condition holds if and only if

$$z_{i_{k+1}} \dots z_{i_{2n+1}} z_{i_1} \dots z_{i_{k-1}} = s_1^{-1}s_8,$$

which can be achieved in b_{2n} ways.

There are $2n+1$ choices for the location k of the $2s_8^{-1}s_1$ term, and for each such k, there are b_{2n} solutions to $(\star\star)$. This gives

$$\frac{1}{2}[1]u^{2n+1} = (2n+1)b_{2n} = b_{2n} \mod 2,$$

which implies $w_n = b_{2n}$. By Lemma 1.4.1, we conclude that $\mathbf{w}$ is an infinite binary word which contains every finite binary word as a subword. $\square$

1.4.2 Counting words mod 2

We first deduce the main result of this paper and then give a useful minor extension.

Proof of Theorem 3.1.2. The group $\mathrm{SL}(4,\mathbb{Z})$ contains $\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$ as a subgroup. The group $\mathrm{SL}(2,\mathbb{Z})$ contains Sanov's subgroup isomorphic to F_2, and thus every finitely generated free group F_ℓ as a subgroup (see e.g. [dlH]). Therefore, $F_{11}\times F_3$ is a subgroup of $\mathrm{SL}(4,\mathbb{Z})$, and the element $u\in\mathbb{Z}[F_{11}\times F_3]$ defined in Lemma 1.4.1 can be viewed as an element of $\mathbb{Z}[\mathrm{SL}(4,\mathbb{Z})]$.

Let $a_n=[1]u^n$. By Lemma 1.4.1, the number a_{2n+1} is always even, and the word $\mathbf{w}=w_1w_2\ldots$ given by $w_n=\frac{1}{2}\,a_{2n+1} \bmod 2$ is an infinite binary word which contains every finite binary word as a subword. Therefore, by Lemma 2.2.1, the sequence $\left\{\frac{1}{2}\,a_{2n+1}\right\}$ is not P-recursive. Since P-recursivity is closed under taking a subsequence consisting of every other term, the sequence $\{a_n\}$ is also not P-recursive. $\square$

Theorem 1.4.2. *There is a group $G\subset SL(4,\mathbb{Z})$ and two generating sets $\langle S_1\rangle=\langle S_2\rangle=G$, such that for the elements*

$$u_1=\sum_{s\in S_1}s\,,\qquad u_2=\sum_{s\in S_2}s\,,$$

we have the sequence $\{[1]u_1^n\}$ is P-recursive, while $\{[1]u_2^n\}$ is not P-recursive.

Proof. Let $G=F_{11}\times F_3$ be as above. Denote by X_1 and X_2 the standard generating sets of F_{11} and F_3, respectively. Finally, let $S_1=(X\times 1)\cup(1\times Y)$,

$$w_1=\sum_{x\in X_1}x,\quad w_2=\sum_{x\in X_2}x.$$

Recall that if $\{c_n\}$ is P-recursive, then so is $\{c_n/n!\}$ and $\{c_n \cdot n!\}$. Observe that

$$\sum_{n=0}^{\infty} [1]u_1^n \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} [1]w_1^n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} [1]w_2^n \frac{t^n}{n!}\right),$$

and that $\{[1]w_1^n\}$ and $\{[1]w_2^n\}$ are P-recursive by Haiman's theorem [Hai]. This implies that $\{[1]u_1^n\}$ is also P-recursive, as desired.

Now, let $S_2 = 2S_1 \cup S$, where S is the set constructed in the proof of Lemma 1.4.1, and $2S_1$ means that each element of S_1 is taken twice. Observe that $[1]u_2^n = [1]u^n \bmod 2$, where u is as in the proof of Theorem 3.1.2. This implies that $\{[1]u_1^n\}$ is not P-recursive, and finishes the proof. $\square$

1.5 Asymptotics of P-recursive sequences and the return probabilities

1.5.1 Asymptotics

The asymptotics of general P-recursive sequences is undersood to be a finite sum of the terms

$$A\,(n!)^s\,\lambda^n\,e^{Q(n^\gamma)}\,n^\alpha\,(\log n)^\beta,$$

where $s, \gamma \in \mathbb{Q}$, $\alpha, \lambda \in \overline{\mathbb{Q}}$, $\beta \in \mathbb{N}$, and $Q(\cdot)$ is a polynomial. This result goes back to Birkhoff and Trjitzinsky (1932), and also Turrittin (1960). Although there are several gaps in these proofs, they are closed now, notably in [Imm]. We refer to [FS, §VIII.7], [Odl, §9.2] and [Pak] for various formulations of general asymptotic estimates, an extensive discussion of priority issues and further references.

For the integer P-recursive sequences which grow at most exponentially, the asymptotics have further constraints summarized in the following theorem.

Theorem 1.5.1. *Let $\{a_n\}$ be an integer P-recursive sequence defined by $(*)$, and such that $a_n < C^n$ for some $C > 0$ and all $n \geq 1$. Then*

$$a_n \sim \sum_{i=1}^{m} A_i \, \lambda_i^n \, n^{\alpha_i} \, (\log n)^{\beta_i} \, ,$$

where $\alpha_i \in \mathbb{Q}$, $\lambda_i \in \overline{\mathbb{Q}}$ and $\beta_i \in \mathbb{N}$.

The theorem is a combination of several known results. Briefly, the generating series $\mathcal{A}(t)$ is a G-functions in a sense of Siegel (1929), which by the works of André, Bombieri, Chudnovsky, Dwork and Katz, must satisfy an ODE which has only regular singular points and rational exponents (see a discussion on [And, p. 719] and an overview in [Beu]). We then apply the Birkhoff–Trjitzinsky theorem, which in the regular case has a complete and self-contained proof (see Theorem VII.10 and subsequent comments in [FS]). We refer to [Pak] for further references and details.

1.5.2 Probability of return

Let G be a finitely generated group. A generating set S is called *symmetric* if $S = S^{-1}$. Let H be a subgroup of G of finite index. It was shown by Pittet and Saloff-Coste [PS2], that for two symmetric generating sets $\langle S \rangle = G$ and $\langle S' \rangle = H$ we have

$$(\diamond) \qquad C_1 p_{G,S}(\alpha_1 n) \, < \, p_{G,S'}(n) \, < \, C_2 p_{G,S}(\alpha_2 n),$$

for all $n > 0$ and fixed constants $C_1, C_2, \alpha_1, \alpha_2 > 0$. For $G = H$, this shows, qualitatively, that the asymptotic behavior of $p_{G,S}(n)$ is a property of a group. The following result gives a complete answer for a large class of groups.

Theorem 1.5.2. *Let G be an amenable subgroup of $GL(k, \mathbb{Z})$ and S is a symmetric generating set. Then either G has polynomial growth and polynomial return* *probabilities:*

$$A_1 n^{-d} < p_{G,S}(2n) < A_2 n^{-d},$$

or G has exponential growth and mildly exponential return probabilities:

$$A_1 \rho_1^{\sqrt[3]{n}} < p_{G,S}(2n) < A_2 \rho_2^{\sqrt[3]{n}},$$

for some $A_1, A_2 > 0$, $0 < \rho_1, \rho_2 < 1$, and $d \in \mathbb{N}$.

The theorem is again a combination of several known results. Briefly, by the Tits alternative, group G must be virtually solvable, which implies that it either has a polynomial or exponential growth (see e.g. [dlH]). By the *quasi-isometry* ($\diamond$), we can assume that G is solvable. In the polynomial case, the lower bound follows from the CLT by Crépel and Raugi [CR], while the upper bound was proved by Varopoulos using the Nash inequality [V1] (see also [V3]). For the more relevant to us case of exponential growth, recall Mal'tsev's theorem, which says that all solvable subgroups of $\mathrm{SL}(n, \mathbb{Z})$ are polycyclic (see e.g. [Sup, Thm. 22.7]). For polycyclic groups of exponential growth, the upper bound is due to Varopoulos [V2] and the lower bound is due to Alexopoulos [Ale]. We refer to [PS3] and [Woe, §15] for proofs and further references, and to [PS1] for a generalization to discrete subgroups of groups of Lie type.

1.5.3 Applications to P-recursiveness

We can now show that non-P-recursiveness for amenable linear groups of exponential growth.

Theorem 1.5.3. *Let G be an amenable subgroup of $GL(k, \mathbb{Z})$ of exponential growth, and let S be a symmetric generating set. Then the probability of return sequence $\{p_{G,S}(n)\}$ is not P-recursive.*

Proof. It is easy to see that H has exponential growth, so Theorem 1.5.2 applies.

Let $a_n = |S|^n p_{G,S}(n) \in \mathbb{N}$ as in the introduction. If $\{p_{G,S}(n)\}$ is P-recursive, then so is $\{a_{2n}\}$. On the other hand, Theorem 1.5.1 forbids mildly exponential terms $\rho^{\sqrt[3]{n}}$ in the asymptotics of a_{2n}, giving a contradiction. $\square$

To obtain Theorem 3.1.2 from here, consider the following linear group $H \subset \mathrm{SL}(3,\mathbb{Z})$ of exponential growth:

$$H = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} & y_1 \\ x_{2,1} & x_{2,2} & y_2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{s.t.} \quad \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^k, \; k \in \mathbb{Z} \right\}$$

(see e.g. [Woe, §15.B]). Observe that $H \simeq \mathbb{Z} \ltimes \mathbb{Z}^2$, and therefore solvable. Thus, H has a natural symmetric generating set

$$E = \left\{ \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 0 & \pm 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

By Theorem 1.5.3, the probability of return sequence $\big\{p_{H,E}(n)\big\}$ is not P-recursive, as desired.

1.6 Final Remarks

1.6.1

Kontsevich's question was originally motivated by related questions on the "categorical entropy" [DHKK]. In response to the draft of this paper, Ludmil Katzarkov, Maxim Kontsevich and Richard Stanley asked us if the examples we construct satisfy *algebraic differential equations* (ADE), see e.g. [Sta1, Exc. 6.63]. We believe that the answer is No, and plan to explore this problem in the future.

1.6.2

The motivation behind the proof of Theorem 3.1.2 lies in the classical result of Mihaĭlova that $G = F_2 \times F_2$ has an undecidable group membership problem [Mih]. In fact, we conjecture that the problem whether $\{[1]\,u^n\}$ is P-recursive is undecidable. We refer to [Hal] for an extensive survey of decidable and undecidable matrix problems.

1.6.3

Following the approach of the previous section, Theorem 1.5.3 can be extended to all polycyclic groups of exponential growth and solvable groups of finite Prüfer rank [PS4]. It also applies to various other specific groups for which mildly exponential bounds on $p(n)$ are known, such as the *Baumslag–Solitar groups* $\mathrm{BS}_q \subset \mathrm{GL}(2, \mathbb{Q})$, $q \geq 2$, and the *lamplighter groups* $L_d = \mathbb{Z}_2 \wr \mathbb{Z}^d$, $d \geq 1$, see e.g. [Woe, §15]. Let us emphasize that P-recursiveness fails for *all* symmetric generating sets in these cases. In view of Theorem 1.4.2, the P-recursiveness fails for *some* generating sets of non-amenable groups containing $F_2 \times F_2$. This suggests that P-recursiveness of all generating sets is a rigid property which holds for very few classes of group. We conjecture that it holds for all nilpotent groups.

1.6.4

Lemma 2.2.1 can be rephrased to say that the *subword complexity function* $c_{\mathbf{w}}(n) < 2^n$ for some n large enough (see e.g. [AS, BLRS]). This is likely to be far from optimal. For example, for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, we have $\mathbf{w} = 101000100000001\ldots$ In this case, it is easy to see that the word complexity function $c_{\mathbf{w}}(n) = \Theta(n)$, cf. [DS]. It would be interesting to find sharper upper bounds on the maximal growth of $c_{\mathbf{w}}(n)$, when $\mathbf{w}$ is the infinite parity word of a P-recursive sequence. Note that $c_{\mathbf{w}}(n) = \Theta(n)$ for all automatic sequences [AS,

§10.2], and that the exponentially growing P-recursive sequences modulo almost all primes are automatic provided deep conjectures of Bombieri and Dwork, see [Chr].

1.6.5

The integrality assumption in Theorem 1.5.1 cannot be removed as the following example shows. Denote by a_n the number of *fragmented permutations*, defined as partitions of $\{1, \ldots, n\}$ into ordered lists of numbers (see sequence A000262 in [OEIS]). It is P-recursive since

$$a_n = (2n-1)\, a_{n-1} - (n-1)(n-2)\, a_{n-2} \quad \text{for all} \ \ n > 2.$$

The asymptotics is given in [FS, Prop. VIII.4]:

$$\frac{a_n}{n!} \sim \frac{1}{2\sqrt{e\pi}}\, e^{2\sqrt{n}}\, n^{-3/4}.$$

This implies that the theorem is false for the *rational*, at most exponential P-recursive sequence $\{a_n/n!\}$, since in this case we have mildly exponential terms. To understand this, note that $\sum_n a_n t^n/n!$ is not a G-function since the *lcm* of denominators of $a_n/n!$ grow superexponentially.

1.6.6

Proving that a combinatorial sequence is not P-recursive is often difficult even in the most classical cases. We refer to [B+, BRS, BP, FGS, Kla, MR] for various analytic arguments. As far as we know, this is the first proof by a computability argument.

References

[AS] J.-P. Allouche and J. Shallit, Automatic sequences, Cambridge U. Press, Cambridge, UK, 2003.

[Ale] G. Alexopoulos, A lower estimate for central probabilities on polycyclic groups, *Canad. J. Math.* **44** (1992), 897–910.

[And] Y. André, Séries Gevrey de type arithmétique. I. Théorèmes de pureté et de dualité (in French), *Ann. of Math.* **151** (2000), 705–740.

[B+] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, Generating functions for generating trees, *Discrete Math.* **246** (2002), 29–55.

[BLRS] J. Berstel, A. Lauve, C. Reutenauer and F. Saliola, *Combinatorics on Words: Christoffel Words and Repetitions in Words*, AMS, Providence, RI, 2009.

[Beu] F. Beukers, *E-functions and G-functions*, course notes (2008); available from the Southwest Center for Arithmetic Geometry website http://swc.math.arizona.edu/aws/2008/

[BRS] A. Bostan, K. Raschel and B. Salvy, Non-D-finite excursions in the quarter plane, *J. Combin. Theory, Ser. A* **121** (2014), 45–63.

[BP] M. Bousquet-Mélou and M. Petkovšek, Walks confined in a quadrant are not always D-finite, *Theoret. Comput. Sci.* **307** (2003), 257–276.

[Chr] G. Christol, Globally bounded solutions of differential equations, in *Lecture Notes in Math.* **1434**, Springer, Berlin, 1990, 45–64.

[CR] P. Crépel and A. Raugi, Théorème central limite sur les groupes nilpotents (in French), *Ann. Inst. H. Poincaré Sect. B* **14** (1978), 145–164.

[DHKK] G. Dimitrov, F. Haiden, L. Katzarkov and M. Kontsevich, Dynamical systems and categories; arXiv:1307.8418.

[dlH] P. de la Harpe, *Topics in Geometric Group Theory*, University of Chicago Press, Chicago, 2000.

[DS] E. Deutsch and B. E. Sagan, Congruences for Catalan and Motzkin numbers and related sequences, *J. Number Theory* **117** (2006), 191–215.

[FGS] P. Flajolet, S. Gerhold and B. Salvy, On the non-holonomic character of logarithms, powers, and the nth prime function, *Electron. J. Combin.* **11** (2004/06), A2, 16 pp.

[FS] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, Cambridge, 2009.

[Ges] I. Gessel, Symmetric Functions and P-Recursiveness, *J. Combin. Theory, Ser. A* **53** (1990), 257–285.

[Hai] M. Haiman, Noncommutative rational power series and algebraic generating functions, *European J. Combin.* **14** (1993), 335–339.

[Hal] V. Halava, Decidable and Undecidable Problems in Matrix Theory, *TUCS Tech. Report* **127** (âĂŐ1997), 62 pp.

[Imm] G. K. Immink, Reduction to canonical forms and the Stokes phenomenon in the theory of linear difference equations, *SIAM J. Math. Anal.* **22** (1991), 238–259.

[Kla] M. Klazar, Irreducible and connected permutations, *ITI Series Preprint* **122** (2003), 24 pp.; available at http://kam.mff.cuni.cz/~klazar/irre.pdf

[Mih] K. A. Mihaĭlova, The occurrence problem for direct products of groups, *Mat. Sb.* **70** (1966), 241–251.

[MR] M. Mishna and A. Rechnitzer, Two non-holonomic lattice walks in the quarter plane, *Theoret. Comput. Sci.* **410** (2009), 3616–3630.

[Odl] A. M. Odlyzko, Asymptotic enumeration methods, in *Handbook of Combinatorics*, Vol. 2, Elsevier, Amsterdam, 1995, 1063–1229.

[OEIS] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org.

[Pak] I. Pak, Asmptotics of combinatorial sequences, a survey in preparation.

[Pete] G. Pete, *Probability and Geometry on Groups*, Lecture notes for a graduate course, 2013, 203 pp.; available at http://www.math.bme.hu/~gabor/PGG.pdf

[PS1] C. Pittet and L. Saloff-Coste, Random walk and isoperimetry on discrete subgroups of Lie groups, in *Sympos. Math.* **XXXIX**, Cambridge Univ. Press, Cambridge, 1999, 306–319.

[PS2] C. Pittet and L. Saloff-Coste, On the stability of the behavior of random walks on groups, *J. Geom. Anal.* **10** (2000), 713–737.

[PS3] C. Pittet and L. Saloff-Coste, A survey on the relationships between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples; preprint (2001), available at http://www.math.cornell.edu/~lsc/articles.html

[PS4] C. Pittet and L. Saloff-Coste, Random walks on finite rank solvable groups, *Jour. EMS* **5** (2003), 313–342.

[S1] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1 and 2, Cambridge Univ. Press, Cambridge, UK, 1997 and 1999.

[S2] R. P. Stanley, D-finiteness of certain series associated with group algebras, in *Oberwolfach Rep.* **11** (2014), 708; available at http://tinyurl.com/lza6v2e

[Sup] D. A. Suprunenko, *Matrix groups*, AMS, Providence, RI, 1976.

[V1] N. Th. Varopoulos, Théorie du potentiel sur des groupes et des variétés (in French), *C.R. Acad. Sci. Paris Sér. I Math.* **302** (1986), no. 6, 203–205.

[V2] N. Th. Varopoulos, Groups of superpolynomial growth, in *Harmonic analysis*, Springer, Tokyo, 1991, 194–200.

[V3] N. Th. Varopoulos, Analysis and geometry on groups, in *Proc. ICM Kyoto*, Math. Soc. Japan, Tokyo, 1991, 951–957.

[Woe] W. Woess, *Random walks on infinite graphs and groups*, Cambridge U. Press, Cambridge, 2000.

CHAPTER 2

Pattern avoidance is not P-recursive

2.1 Introduction

Combinatorial sequences have been studied for centuries, with results ranging from minute properties of individual sequences to broad results on large classes of sequences. Even just listing the tools and ideas can be exhausting, which range from algebraic to bijective, to probabilistic and number theoretic [Rio]. The existing technology is so strong, it is rare for an open problem to remain unresolved for more than a few years, which makes the surviving conjectures all the more interesting and exciting.

The celebrated *Noonan–Zeilberger Conjecture* is one such open problem. It is a central problem in the area of *pattern avoidance*, which has been very popular in the past few decades, see e.g. [Bóna, Kit]. The problem was first raised as a question by Gessel in 1990, see [Ges2, §10]. In 1996, it was upgraded to a conjecture and further investigated by Noonan and Zeilberger [NZ], see also §**??**. There are now hundreds of papers in the area with positive results for special sets of patterns. Here we use Computability Theory to disprove the conjecture.

Let $\sigma \in S_n$ and $\omega \in S_k$. Permutation σ is said to *contain* the *pattern* ω if there is a subset $X \subseteq \{1, \ldots, n\}$, $|X| = k$, such that $\sigma|_X$ has the same relative order as ω. Otherwise, σ is said to *avoid* ω. Fix a set of patterns $\mathcal{F} \subset S_k$. Denote by $C_n(\mathcal{F})$ the number of permutations $\sigma \in S_n$ *avoiding* the patterns $\omega \in \mathcal{F}$. The sequence $\{C_n(\mathcal{F})\}$ is the main object in the area, extensively analyzed from analytic, asymptotic and combinatorial points of view (see §2.9.1).

An integer sequence $\{a_n\}$ is called *polynomially recursive*, or *P-recursive*, if it satisfies a nontrivial linear recurrence relation of the form

$$q_0(n)\, a_n \, + \, q_1(n)\, a_{n-1} \, + \, \ldots \, + \, q_k(n)\, a_{n-k} \, = \, 0\,,$$

for some $q_i(x) \in \mathbb{Z}[x]$, $0 \le i \le k$. The study of P-recursive sequences plays a major role in modern Enumerative and Asymptotic Combinatorics (see e.g. [FS, Odl, Sta1]). They have *D-finite* (also called *holonomic*) generating series and various asymptotic properties (see §2.9.2).

Conjecture 2.1.1 (Noonan–Zeilberger)**.** *Let $\mathcal{F} \subset S_k$ be a fixed set of patterns. Then the sequence $\{C_n(\mathcal{F})\}$ is P-recursive.*

The following is the main result of the paper.

Theorem 2.1.2. *The Noonan–Zeilberger conjecture is false. More precisely, there exists a set of patterns $\mathcal{F} \subset S_{80}$, such that the sequence $\{C_n(\mathcal{F})\}$ is* not *P-recursive.*

We should mention that in contrast with most literature in the area which studies small sets of patterns, our set $\mathcal{F}$ is enormously large and we make no effort to decrease its size (cf. §2.9.5). To be precise, we construct two large sets $\mathcal{F}, \mathcal{F}' \subset S_{80}$ and show that at least one of them gives a counterexample to the conjecture. In fact, it is conceivable that a single permutation pattern $\omega = (1324)$ may be sufficient for non-P-recursiveness (see §2.9.4).

The proof of Theorem 2.1.2 is based on the following idea. Roughly, we show that every *two-stack automaton M* can be emulated by a finite set of permutation patterns. More precisely, we show that the number of accepted paths of M is equal to $C_n(\mathcal{F})$ mod 2, for a subset of integers n forming an arithmetic progression (see Main Lemma 2.3.2). This highly technical construction occupies much of the paper. The rest of the proof is based on our approach in Chapter 1, where we resolved *Kontsevich's problem* on the P-recursiveness of certain numbers of words in linear groups. The ability to emulate *any* two-stack automaton M in the weak sense described above is surprisingly powerful (see below).

We apply our results to the following decidability problem. Two sets of patterns $\mathcal{F}_1$ and $\mathcal{F}_2$ are called *Wilf–equivalent*, denoted $\mathcal{F}_1 \sim \mathcal{F}_2$, if $C_n(\mathcal{F}_1) = C_n(\mathcal{F}_2)$

for all n. In [V2], Vatter asked whether it is decidable when two patterns are Wilf–equivalent. Here we resolve a mod-2 version of this problem (cf. Section 2.8 and §2.9.7).

Theorem 2.1.3. *The problem whether $C_n(\mathcal{F}_1) = C_n(\mathcal{F}_2)$ mod 2 for all $n \in \mathbb{N}$, is undecidable.*

The rest of the paper is structured as follows. We begin with an explicit construction of a two-stack automata with a non-P-recursive number of accepted paths (Section 2.2). In Section 2.3, we reduce the proof of Theorem 2.1.2 to the Main Lemma 2.3.2 on embedding two-stack automata into pattern avoidance problems. The proof of the Main Lemma spans the next four sections. We first present the construction in Section 2.4. In Section 2.5, we prove the lemma modulo a number of technical results. We illustrate the construction in a lengthy example in Section 2.6, and prove the technical results in Section 2.7. We proceed to prove Theorem 2.1.3 in Section 2.8 We conclude with final remarks and open problems in Section 2.9.

2.2 Two-stack automata

In this section we construct an automaton with a non-P-recursive number of accepted paths. The construction is technical, but elementary. Although it is more natural if the reader is familiar with basic Automata Theory (see e.g. [HMU, Sip]), the construction is completely self-contained and is given in the language of elementary Graph Theory. The proof, however, is not self-contained and follows a similar proof in Chapter 1.

To be precise, we give an explicit construction of a graph, where the vertices have certain variables as weights. We then count the number a_n of paths of length n between two fixed vertices, where only certain weight sequences are allowed (we call these *balanced paths*). The non-P-recursiveness of a_n mod 2 is explained below.

2.2.1 The motivation

It is relatively easy to present a construction of an automaton which produces a non-P-recursive sequence $\{a_n\}$ of balanced paths. Our goal in this section is stronger – the sequence $\{a_n\}$ we get is not equal to *any* P-recursive sequence modulo 2. Somewhat informally, we call such automaton *non-P-recursive*. The advantages afforded by the modulo 2 property are technical and will become clear later in this paper.

Our main tool for building a non-P-recursive two-stack automaton is the following result.

Theorem 2.2.1 (Lemma 1.2.1 of Chapter 1)**.** *Let $\{a_n\}$ be a P-recursive integer sequence. Consider an infinite binary word $\overline{\alpha} = (\alpha_1\alpha_2\ldots)$, defined by $\alpha_n := a_n$ mod 2. Then, there exists a finite binary word which is not a subword of $\overline{\alpha}$.*

What follows is a construction of a two-stack automaton such that the corresponding binary sequence $\overline{\alpha}$ contains every finite binary subword by design. There are many such automata, in fact. We give a complete description of this one as we need both its notation and additional properties of the construction later on.

2.2.2 The setup

Let Γ be a finite directed graph with vertices $v_1, \ldots, v_m$. Let X denote the set of labels of the form x_i and let X^{-1} denote the set of labels of the form x_i^{-1}, where i is any integer. Define Y and Y^{-1} similarly. Label each vertex of Γ with an element of $X \cup X^{-1} \cup Y \cup Y^{-1} \cup \{\varepsilon\}$.

Let $\rho(v)$ denote the label on vertex v. We say that $w_1 \sim w_2$ if $w_1, w_2 \in X \cup X^{-1}$

or if $w_1, w_2 \in Y \cup Y^{-1}$. If Γ has an edge from v_i to v_j, we say that $v_i \to v_j$.

Contrary to standard notation, we refer to a path $\gamma = \gamma_1 \dots \gamma_n$, where each γ_i is a vertex, not an edge, and we say that such a path is of length n, even though it only has $n-1$ edges.

We further require that $\rho(v_1) = \rho(v_2) = \varepsilon$, and that there is no edge $v_i \to v_j$, with $\rho(v_i) \sim \rho(v_j)$. A graph Γ satisfying all of the above conditions is called a *two-stack automaton.*

As we traverse a path γ, we keep track of two words $w_X \in X^\star$ and $w_Y \in Y^\star$, which start out empty. Whenever we enter a vertex with label x_i, we append x_i to the end of w_X. When we enter a vertex with label x_i^{-1}, we remove x_i from the end of w_X. We modify w_Y similarly when entering vertices with label y_i or y_i^{-1}. When we enter a vertex with label ε, we do nothing. A path is called *balanced* if every step of this process is well defined and both w_X and w_Y are empty at the end of the path. Let $G(\Gamma, n)$ denote the number of balanced paths in Γ from v_1 to v_2 of length n.

Define an involution $\pi_\gamma \in S_n$ as follows. If the above process writes an instance of a label to w_X or w_Y at some time t_i, and removes the same instance of that label for the first time at time t_j, then $\pi(t_i) = t_j$ and $\pi(t_j) = t_i$. If the process does not write or remove anything at a time step t_k, then $\pi(t_k) = t_k$. For example, if $\rho(\gamma_1)\rho(\gamma_2)\dots\rho(\gamma_9) = \varepsilon x_1 y_1 x_1 y_1^{-1} x_1^{-1} \varepsilon x_1^{-1} \varepsilon$, then $\pi_\gamma = (2\ 8)(3\ 5)(4\ 6)$. This gives the following alternate characterization of balanced paths.

Proposition 2.2.2. *A path γ is balanced if and only if there exists an involution $\pi_\gamma \in S_n$ such that:*

(1) $\rho(\gamma_i) = \varepsilon$ *for all* $\pi_\gamma(i) = i$,

(2) $\rho(\gamma_i) \in X \cup Y$ *and* $\rho(\gamma_{\pi_\gamma(i)}) = \rho(\gamma_i)^{-1}$, *for all* $\pi_\gamma(i) > i$, *and*

(3) There are no i and j with $\rho(\gamma_i) \sim \rho(\gamma_j)$ such that $i < j < \pi_\gamma(i) < \pi_\gamma(j)$.

Further, this involution π_γ is uniquely defined for each balanced γ.

The proof is straightforward.

2.2.3 Non-P-recursive automaton

We are now ready to present a construction of such automaton Γ_1, which is given in Figure 2.1. The construction is based on a smaller automaton Γ_2 we introduced in Chapter 1.

Lemma 2.2.3. *There exists a two-stack automaton Γ_1, such that $\alpha_n := G(\Gamma_1, n) \in \{0, 1\}$ for all n, and such that the word $\overline{\alpha} = (\alpha_1\alpha_2\ldots)$ is an infinite binary word which contains every finite binary word as a subword.*

Proof. We give an explicit automaton Γ_1 in Figure 2.1. This automaton is formed by modifying the automaton Γ_2, given in Chapter 1. Here we use $\varepsilon_1, \ldots, \varepsilon_8$ to denote the same trivial label ε; we make this distinction only for the purpose of illustration. The vertex v_1 is the shaded vertex labelled ε_1, and the vertex v_2 is the shaded vertex labelled ε_8. The vertex labelled ε_i in Γ_1 corresponds to the vertex labelled s_i in Γ_2.

The primary difference between Γ_1 and Γ_2 is that Γ_1 has labels on vertices while Γ_2 has labels on edges. The labels were also changed by replacing 0_x, 1_x, and x with x_0, x_1, and x_2 respectively, and similarly for y. Since the lengths of the paths change slightly, we get a slightly different formula, but the analysis is similar.

In counting paths we follow the proof of Lemma 1.3.1 in Chapter 1. The valid paths through Γ_2 have

$$\mu \;=\; j + 6k + 2\sum_{i=1}^{k} \lfloor \log_2 i \rfloor \quad \text{edges,}$$

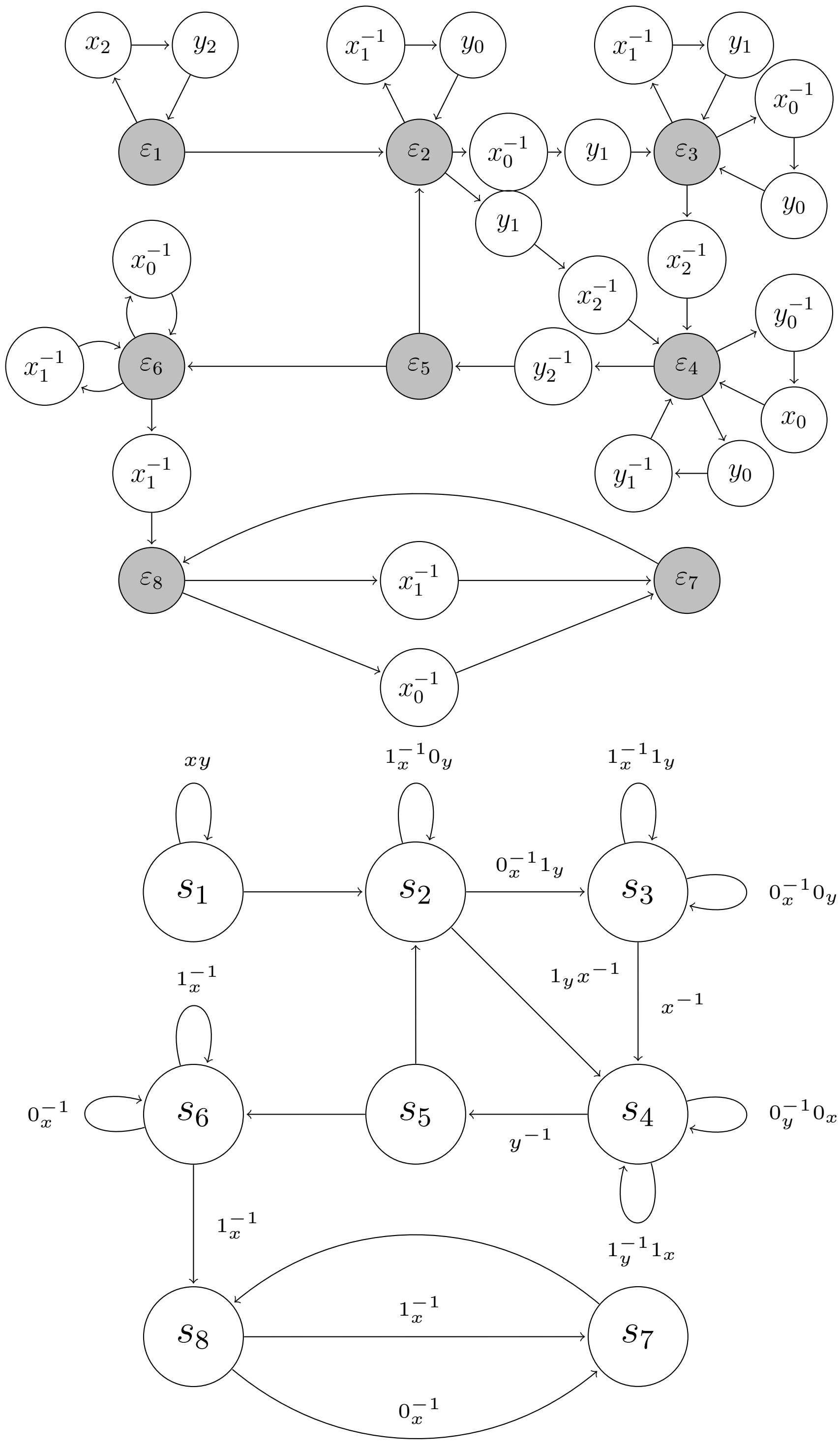


Figure 2.1: The automata Γ_1 (top), and Γ_2 (bottom).

for some positive integers j and k such that the j-th binary digit of k is a 1. Every path through Γ_1 will similarly have $(\mu + 1)$ vertices with label ε.

Such paths will also have a total of $4k$ vertices labelled x_2, x_2^{-1}, y_2 or y_2^{-1}, since k copies each of x_2 and y_2 are written and removed in the computation. Similarly, every binary integer from 1 to k is written and removed from both tapes, so the vertices with the remaining 8 labels are used

$$\nu \;=\; 4k \,+\, 4\sum_{i=1}^{k} \lfloor \log_2 i \rfloor \quad \text{times in total.}$$

In summary, we have $G(\Gamma_1, n) = 1$ for all $n = (\mu+1) + \nu + 4k$, where j and k are positive integers such that the j-th binary digit of k is a 1, and $G(\Gamma_1, n) = 0$ otherwise. The word $\overline{\alpha} = (\alpha_1\alpha_2\ldots)$ then contains the positive integer k written out in binary starting at location

$$n \;=\; 2 \,+\, 14k \,+\, 6\sum_{i=1}^{k} \lfloor \log_2 i \rfloor\,,$$

and will therefore contain every finite binary word as a subword.

Finally, note that in the notion of "valid path" from Chapter 1, it was possible for some instance of x^{-1} to cancel with a later instance of x. For our purposes, this difference is irrelevant since the words defined by paths in Γ_2 do not have such cancellations. $\square$

2.3 Main Lemma and the proof of Theorem 2.1.2

In this section we first change our setting from pattern avoidance to slightly more general but equivalent notion of *partial pattern avoidance*. We state the Main Lemma 2.3.2 and show that it implies Theorem 2.1.2.

2.3.1 Partial patterns

A 0-1 matrix is called a *partial pattern* if every row and column contains at most one 1. Clearly, every permutation pattern is also a partial pattern. We say that a permutation matrix M *contains* a partial pattern L, if L can be obtained from M by deleting some rows and columns; we say that M *avoids* L otherwise. Given a set $\mathcal{F}$ of partial patterns, let $\mathcal{C}_n(\mathcal{F})$ denote the set of $n \times n$ matrices M which avoid all partial patterns in $\mathcal{F}$. By analogy with the usual permutation patterns, let $C_n(\mathcal{F}) = |\mathcal{C}_n(\mathcal{F})|$.

Proposition 2.3.1. *Let $\mathcal{F}_1$ be a finite set of partial patterns. Then there exists a finite set of the usual permutation patterns $\mathcal{F}_2$, such that $C_n(\mathcal{F}_1) = C_n(\mathcal{F}_2)$ for all $n \in \mathbb{N}$.*

Proof. First, let us prove the result for a single partial pattern. Let L be a partial pattern of size $p \times q$, and let $k = p + q$. Denote by $\mathcal{P}_n(L)$ be the set of $n \times n$ permutation matrices containing L. Let us show by induction that for all $n \geq k$, every permutation matrix $M \in \mathcal{P}_n(L)$ contains a matrix in $\mathcal{P}_k(L)$. Indeed, the claim is trivially true for $n = k$. For larger n, observe that every $n \times n$ permutation matrix M which contains L must also contain some i-th row and j-th column, such that $M_{p,q} = 1$, and neither i-th row nor j-th column intersect L. This follows from the fact that otherwise $\operatorname{rank}(M) \leq i + j < n$. Deleting these row and column gives a smaller permutation matrix which contains L, proving the induction claim.

We conclude that $\mathcal{F}_2(L) := \cup_{\ell \leq k} \mathcal{P}_\ell(L)$ is the desired set of matrices for $\mathcal{F} = \{L\}$. In full generality, take $\mathcal{F}_2 = \cup_{L \in \mathcal{F}} \mathcal{F}_2(L)$. The details are straightforward. □

It therefore suffices to disprove the Noonan–Zeilberger Conjecture 2.1.1 for partial patterns.

Lemma 2.3.2 (Main Lemma)**.** *Let Γ be a two-stack automaton. Then there exist* *sets $\mathcal{F}$ and $\mathcal{F}'$ of partial patterns, and some integers $c, d \geq 1$, such that*

$$C_{cn+d}(\mathcal{F}) - C_{cn+d}(\mathcal{F}') \;=\; G(\Gamma, n) \;\, mod\; 2\,, \quad for\ all \;\;\, n \in \mathbb{N}.$$

The proof of the Main Lemma is given in Section 2.5.

2.3.2 Proof of Theorem 2.1.2

By Lemma 2.2.3, there exists a two-stack automaton Γ_1, such that the infinite binary word $\overline{\alpha} = (\alpha_1\alpha_2 \ldots)$ given by $\alpha_n = G(\Gamma_1, n)$, contains every finite binary word as a subword. By Lemma 2.3.2, there exist integers c and d and two sets $\mathcal{F}$ and $\mathcal{F}'$ of partial patterns such that $C_{cn+d}(\mathcal{F}) - C_{cn+d}(\mathcal{F}') = G(\Gamma_1, n)$ mod 2, for all n.

If Conjecture 2.1.1 is true, then both $\{C_n(\mathcal{F})\}$ and $\{C_n(\mathcal{F}')\}$ are P-recursive sequences. Since P-recursive sequences are closed under taking the differences and subsequences with indices in arithmetic progressions (see e.g. [Sta1, §6.4]), this means that the sequence $\{a_n\}$, defined as $a_n = \{C_{cn+d}(\mathcal{F}) - C_{cn+d}(\mathcal{F}')\}$, is also P-recursive. On the other hand, from above, we have $\alpha_n = a_n$ mod 2. This gives a contradiction with Theorem 2.2.1.

The second part of the theorem requires a quantitative form of the Main Lemma and is given as Corollary 2.4.4. □

2.4 The construction of an automaton in the Main Lemma

2.4.1 Notation

The construction of sets of matrices $\mathcal{F}, \mathcal{F}'$ has two layers and is quite involved, so we try to simplify it by choosing a clear notation. We use $\mathcal{A}_g$ to denote a certain subset of $g \times g$ matrices, which we call an *alphabet* and use as building blocks. We use English capitals with various decorations, notably A, A', B, B', E, L, P, Q, R, S, T_k and Z_p, to denote 0-1 matrices of size at most $g \times g$. We use script capital letters $\mathcal{F}_i, \mathcal{F}'_i, \mathcal{W}_i, \mathcal{W}'_i$, to denote the sets of larger matrices (partial patterns) which form sets $\mathcal{F}, \mathcal{F}'$. Each is of size at most $8g \times 8g$, and some of the matrices are denoted W_i and W'_i.

On a bigger scale, we use $M = M(*,*)$ to denote large block matrices, with individual blocks $M^{i,j}$ being either zero or matrices in the alphabet $\mathcal{A}_g$. For the proof of Theorem 2.1.2 we take $g = 10$, but for Theorems 2.1.3 we need larger g. When writing matrices, we use a dot $(\cdot)$ within a matrix to represent a single 0 entry, and a circle $(\circ)$ to represent a $g \times g$ submatrix of zeros.

2.4.2 The alphabet

A permutation matrix is called *simple* if it contains no permutation matrix as a proper submatrix consisting of consecutive rows and columns, other than the trivial 1×1 permutation matrix.

Define an alphabet $\mathcal{A}_g$ of all $g \times g$ simple permutation matrices which contain the following matrix as a submatrix:

$$L = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Proposition 2.4.1. *We have:* $|\mathcal{A}_g| \to g!/e^2$ *as* $g \to \infty$.

Proof. It was shown in [AAK] that the probability that a random $g \times g$ permutation matrix M is simple tends to $1/e^2$ as $g \to \infty$ (see also [OEIS, A111111]). On the other hand, the probability that M avoids L tends to 0 as $g \to \infty$. Thus, the probability that $M \in \mathcal{A}_g$ tends to $1/e^2$, as desired. $\square$

By the proposition, we can fix an integer g large enough that $|\mathcal{A}_g| > 5+m+r$, where m is the number of vertices in Γ and r is the number of distinct labels in $X \cup Y$ on vertices of Γ.

We build our forbidden partial patterns out of elements of $\mathcal{A}_g$ as follows. Choose five special matrices $P, Q, B, B', E \in \mathcal{A}_g$, as well as two classes of matrices, $T_1, \ldots, T_m \in \mathcal{A}_g$, and $Z_1, \ldots, Z_r \in \mathcal{A}_g$. Here the matrices $T_1, \ldots, T_m$ represent the m vertices in Γ. Let T_i denote the matrix corresponding to v_i. The matrices $Z_1, \ldots, Z_r$ represent the r labels in $X \cup Y$. Let $s(Z_p)$ denote the label which corresponds to Z_p. Let us emphasize that these choices are arbitrary as the only important properties of these $(5+m+r)$ matrices is that they are all in $\mathcal{A}_g$ and distinct.

2.4.3 Forbidden matrices

Let $\mathcal{F}_1$ denote the set of all $g \times (g+1)$ or $(g+1) \times g$ partial patterns formed by taking a matrix A in $\mathcal{A}_g$, and inserting a row or column of all zeros somewhere in the middle of A.

Let $\mathcal{F}_2$ denote the set of all $(2g+1) \times (5g+1)$ or $(5g+1) \times (2g+1)$ partial patterns whose bottom left $g \times g$ consecutive submatrix is a B or B' and whose top right $g \times g$ consecutive submatrix is T_j for some j.

Let $\mathcal{F}_3$ denote the four element set consisting of the $(2g+1) \times g$ and $g \times (2g+1)$ partial pattern formed by inserting $g+1$ rows of zeros below Q, inserting $g+1$ rows of zeros above P, inserting $g+1$ columns of zeros to the right of Q, or inserting $g+1$ columns of zeros to the left of P.

Let $\mathcal{F}_4 = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \cup \mathcal{W}_4 \cup \mathcal{W}_5$, where $\mathcal{W}_i$ are defined as follows. Let $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ denote the sets of matrices of the form W_1, W_2 and W_3, respectively:

$$W_1 = \begin{pmatrix} \circ & \circ & T_i & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_j & \circ & \circ \\ L & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & Z_p \\ \circ & \circ & \circ & \circ & \circ & \circ & T_k & \circ \\ \circ & B' & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & R & \circ & \circ & \circ \\ \circ & \circ & \circ & Z_p & \circ & \circ & \circ & \circ \end{pmatrix}, \ W_2 = \begin{pmatrix} \circ & \circ & \circ & \circ & Z_p & \circ & \circ & \circ \\ \circ & \circ & \circ & T_i & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & T_j & \circ \\ \circ & L & \circ & \circ & \circ & \circ & \circ & \circ \\ Z_p & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & T_k \\ \circ & \circ & B' & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & R & \circ & \circ \end{pmatrix},$$

$$W_3 = \begin{pmatrix} \circ & \circ & T_i & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_j & \circ \\ L & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & E & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & T_k \\ \circ & B' & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & R & \circ & \circ \end{pmatrix}.$$

In all three cases, we require $L, R \in \{B, B'\}$ and $v_i \to v_j \to v_k$. In W_1, we require $\rho(v_j) = s(Z_p)$. In W_2, we require $\rho(v_j) = (s(Z_p))^{-1}$. In W_3, we require $\rho(v_j) = \varepsilon$.

Similarly, let $\mathcal{W}_4$ and $\mathcal{W}_5$ denote the set of all matrices of the form W_4 and W_5 respectively:

$$W_4 = \begin{pmatrix} \circ & \circ & \circ & \circ & T_1 & \circ \\ \circ & P & \circ & \circ & \circ & \circ \\ \circ & \circ & E & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_k \\ B' & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & R & \circ & \circ \end{pmatrix}, \ W_5 = \begin{pmatrix} \circ & \circ & T_i & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_2 \\ L & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & E & \circ & \circ \\ \circ & \circ & \circ & \circ & Q & \circ \\ \circ & B' & \circ & \circ & \circ & \circ \end{pmatrix}.$$

Here, in W_4, we require $R \in \{B, B'\}$ and $v_1 \to v_k$, and in W_5, we require $L \in$

$\{B, B'\}$ and $v_i \to v_2$. Finally, let $\mathcal{F}_5$ denote the set of all patterns of the form

$$\begin{pmatrix} \circ & Z_p & \circ \\ \circ & \circ & Z_q \\ B & \circ & \circ \end{pmatrix}, \quad \begin{pmatrix} \circ & Z_p & \circ \\ \circ & \circ & Z_q \\ B' & \circ & \circ \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \circ & \circ & T_j \\ Z_p & \circ & \circ \\ \circ & Z_q & \circ \end{pmatrix}, \quad \text{where } s(Z_p) \sim s(Z_q).$$

Lemma 2.4.2 (Explicit Construction)**.** *Given a two-stack automaton* Γ*, let*

$$\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \quad \textit{and} \quad \mathcal{F}' := \mathcal{F} \cup \{B, B'\},$$

where $\mathcal{F}_1, \ldots, \mathcal{F}_5, B, B'$ *are defined as above. Then, for all* n*, we have:*

$$C_m(\mathcal{F}) - C_m(\mathcal{F}') = G(\Gamma, n) \ \textit{mod}\ 2, \quad \textit{where } m = (3n+2)g\,.$$

The Main Lemma 2.3.2 follows immediately from this result.

2.4.4 Counting Partial Patterns

We will now analyze the above construction in the specific case of Γ_1.

Theorem 2.4.3. *There exists a set* $\mathcal{F}$ *of at most* 6854 *partial patterns of size at most* 80×80 *such that* $\{C_n(\mathcal{F})\}$ *is not P-recursive.*

Converting these partial patterns into the usual permutation patterns would require many more patterns. However all the patterns avoided would still be size at most 80×80.

Corollary 2.4.4. *There exists a set* $\mathcal{F}$ *of* 80×80 *permutation matrices such that the sequence* $\{C_n(\mathcal{F})\}$ *is not P-recursive. In particular,* $|\mathcal{F}| < 80! < 10^{119}$.

Proof of Theorem 2.4.3. Observe that Γ_1 has 31 vertices and uses 6 labels in $X \cup Y$. Therefore we need $5 + 31 + 6 = 42$ matrices in $\mathcal{A}_g$. Let $g = 10$. Consider the following simple 9×9 pattern

$$L' = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Note that there are 60 ways to insert 1 into L' to form a simple 10×10 pattern. Indeed, the 1 may inserted anywhere other than the 4 corners or the 36 locations that would form a 2×2 consecutive submatrix. All 60 of these 10×10 are distinct and in $\mathcal{A}_g$.

For $\mathcal{F}_1$, we actually only need to include the 42 matrices in $\mathcal{A}_{10}$ which are actually used, so $|\mathcal{F}_1| = 42 \cdot 9 \cdot 2 = 756$. Similarly, for $\mathcal{F}_2$, there are 2 choices for the bottom left 10×10 consecutive submatrix and 31 choices for the top right 10×10 consecutive submatrix. There are 41 entries in the middle and at most one of them can be a 1, which can be satisfied in 42 ways. Therefore, $|\mathcal{F}_2| = 2 \cdot 31 \cdot 2 \cdot 42 = 5208$. Clearly, $|\mathcal{F}_3| = 4$.

Let us show that $|\mathcal{F}_4| = 292$. Indeed, a matrix in $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$ is defined by the path $v_i \to v_j \to v_k$ and the choices for L and R. There are 71 paths in Γ of length 3, so we have $|\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3| = 71 \cdot 4 = 284$. A matrix in $\mathcal{W}_4$ is defined by the vertex v_k and the choices for R, so we have $|\mathcal{W}_4| = 4$. Similarly, a matrix in $\mathcal{W}_5$ is defined by the vertex v_i and the choices for L, so $|\mathcal{W}_5| = 4$.

Finally, for $\mathcal{F}_5$, there are 6 choices for Z_p, 3 choices for Z_q and $31 + 2$ choices for the B, B' or T_j. Therefore, $|\mathcal{F}_5| = 6 \cdot 3 \cdot 33 = 594$. In total, $\mathcal{F}$ consists of $|\mathcal{F}| = 756 + 5208 + 4 + 292 + 594 = 6854$ partial patterns of dimensions at most 80×80. The set $\mathcal{F}'$ has two extra matrices, but can be made smaller than $\mathcal{F}$ since avoiding B' makes all matrices $W_i \in \mathcal{F}_4$ redundant. $\square$

2.5 Proof of the Explicit Construction Lemma 2.4.2

In this section we give a proof of Lemma 2.4.2 by reducing it to three technical lemmas which are proved in Section 2.7. Briefly, since $\mathcal{F} \subset \mathcal{F}'$, we have $\mathcal{C}_n(\mathcal{F}') \subseteq \mathcal{C}_n(\mathcal{F})$ for all n. Denote $\mathcal{D}_n = \mathcal{C}_n(\mathcal{F}') \smallsetminus \mathcal{C}_n(\mathcal{F})$. We construct an explicit involution ϕ on $\mathcal{D}_n$ and analyze the set of fixed points $\mathcal{D}'_n$. We show that the set $\mathcal{D}'_n$ has a very rigid structure emulating the working of a given two-stack automaton Γ.

2.5.1 Preliminaries

The key idea of an involution ϕ defined below is a switch $B \leftrightarrow B'$ between submatrices B and B', in such a way that the fixed points $\mathcal{D}'_n$ of ϕ avoid B'. The remaining copies of B create a general diagonal structure of the matrices in $\mathcal{D}'_n$, and enforce the location of all other submatrices from the alphabet. We invite the reader to consult the example in the next section to have a visual understanding of our approach.

We also need a convenient notion of a *marked submatrix*. Such marked submatrix will always be a B, and is located at a specific position in forbidden matrices W_i. This is best illustrated in the matrix formulas below, where marked submatrix B is boxed.

Let $\mathcal{F}'_4 = \mathcal{W}'_1 \cup \mathcal{W}'_2 \cup \mathcal{W}'_3 \cup \mathcal{W}'_4 \cup \mathcal{W}'_5$, where $\mathcal{W}'_i \subseteq \mathcal{W}_i$ are defined to have no submatrices B' (so $L = R = B$ in the notation above), and where the elements of $\mathcal{W}'_4$ have a unique marked submatrix B. Precisely, let $\mathcal{W}'_1$, $\mathcal{W}'_2$ and $\mathcal{W}'_3$ denote the set of matrices of the form W_1', W_2' and W_3', respectively:

$$W_1' = \begin{pmatrix} \circ & \circ & T_i & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_j & \circ & \circ \\ B & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & Z_p \\ \circ & \circ & \circ & \circ & \circ & \circ & T_k & \circ \\ \circ & \boxed{B} & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & B & \circ & \circ & \circ \\ \circ & \circ & \circ & Z_p & \circ & \circ & \circ & \circ \end{pmatrix}, W_2' = \begin{pmatrix} \circ & \circ & \circ & \circ & Z_p & \circ & \circ & \circ \\ \circ & \circ & \circ & T_i & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & T_j & \circ \\ \circ & B & \circ & \circ & \circ & \circ & \circ & \circ \\ Z_p & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & T_k \\ \circ & \circ & \boxed{B} & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & B & \circ & \circ \end{pmatrix},$$

$$W_3' = \begin{pmatrix} \circ & \circ & T_i & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_j & \circ \\ B & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & E & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & T_k \\ \circ & \boxed{B} & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & B & \circ & \circ \end{pmatrix}.$$

Similarly, let $\mathcal{W}_4'$, and $\mathcal{W}_5'$ denote the set of matrices of the form W_4' and W_5', respectively:

$$W_4' = \begin{pmatrix} \circ & \circ & \circ & \circ & T_1 & \circ \\ \circ & P & \circ & \circ & \circ & \circ \\ \circ & \circ & E & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_k \\ \boxed{B} & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & B & \circ & \circ \end{pmatrix}, \quad W_5' = \begin{pmatrix} \circ & \circ & T_i & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_2 \\ B & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & E & \circ & \circ \\ \circ & \circ & \circ & \circ & Q & \circ \\ \circ & \boxed{B} & \circ & \circ & \circ & \circ \end{pmatrix}.$$

Of course, all these W_i' satisfy the same conditions as W_i in the previous section.

2.5.2 Construction of the involution ϕ

From this point on, let $m = (3n+2)g$. Given a $m \times m$ permutation matrix M, let $M^{i,j}$ refer to the $g \times g$ submatrix in rows $g(i-1)+1$ through gi, and columns $g(j-1)+1$ through gj.

First, observe that $\mathcal{D}_n := \mathcal{C}_m(\mathcal{F}) \smallsetminus \mathcal{C}_m(\mathcal{F}')$ is the set of all $m \times m$ matrices avoiding $\mathcal{F}$ with at least one submatrix B or B'. Since $\mathcal{D}_n$ avoids $\mathcal{F}_1$, every submatrix B or B' in a matrix in $\mathcal{D}_n$ must be a consecutive $g \times g$ block. A submatrix B or B' in a matrix M of $\mathcal{D}_n$ is called *blocked* if replacing it with B' or B, respectively, would result in a matrix not in $\mathcal{D}_n$.

Lemma 2.5.1. *Consider the map ϕ on $\mathcal{D}_n$ which takes the leftmost unblocked submatrix B or B' and replaces it with B' or B respectively. The map ϕ is an involution on $\mathcal{D}_n$. Furthermore, the fixed points of ϕ are the $m \times m$ matrices M such that:*

(1) matrix M avoids $\mathcal{F}$,

(2) matrix M avoids B',

(3) matrix B is a submatrix of M, and

(4) every submatrix B inside M is a marked submatrix of a matrix in $\mathcal{F}'_4$.

Let $\mathcal{D}'_n = \mathrm{Fix}(\phi)$ denote the set of all fixed points of ϕ. Since ϕ is an involution, we conclude that $|\mathcal{D}_n| = |\mathcal{D}'_n| \bmod 2$, so it suffices to show that $|\mathcal{D}'_n| = G(\Gamma, n) \bmod 2$.

2.5.3 The structure of $\mathcal{D}'_n$

Let γ be a path from v_1 to v_2 which is not necessarily balanced, and let $\pi \in S_n$. Denote by $M := M(\gamma, \pi)$ the $m \times m$ permutation matrix given by:

(1) $M^{2,2} = P$,

(2) $M^{3n+1,3n+1} = Q$,

(3) $M^{3i+2,3i-2} = B$ for all i,

(4) $M^{3i-2,3i+2} = T_j$ for all i, where $\gamma_i = v_j$,

(5) $M^{3i,3j} = E$ whenever $\rho(\gamma_i) = \varepsilon$ and $\pi(i) = j$,

(6) $M^{3i,3j} = Z_p$ whenever $\rho(\gamma_i) = s(Z_p)$ and $\pi(i) = j$,

(7) $M^{3i,3j} = Z_p$ whenever $\rho(\gamma_i) = (s(Z_p))^{-1}$ and $\pi(i) = j$,

(8) $M^{i,j} = 0$ is a zero matrix otherwise.

Lemma 2.5.2. *Every matrix in $\mathcal{D}'_n$ is of the form $M(\gamma, \pi)$, where γ is a path of length n in Γ, and $\pi \in S_n$.*

Note, however, not every matrix $M(\gamma, \pi)$ is in $\mathcal{D}'_n$. The following lemma gives a complete characterization. Recall that given a balanced path γ, there is a unique permutation π_γ associated with γ given in Proposition 2.2.2.

Lemma 2.5.3. *Let γ be a path in Γ of length n, and let $\pi \in S_n$. Then, $M(\gamma, \pi) \in \mathcal{D}'_n$ if and only if γ is balanced and $\pi = \pi_\gamma$.*

Lemmas 2.5.1, 2.5.2 and 2.5.3 easily imply the Main Lemma.

2.5.4 Proof of Lemma 2.4.2

We have $C_m(\mathcal{F}) - C_m(\mathcal{F}') = |\mathcal{D}_n|$ by definition. Lemma 2.5.1 shows that ϕ is an involution, so $|\mathcal{D}_n| = |\mathrm{Fix}(\phi)| = |\mathcal{D}'_n| \bmod 2$. Combining lemmas 2.5.2 and 2.5.3, we get $|\mathcal{D}'_n| = G(\Gamma, n)$. Thus, $C_{cn+d}(\mathcal{F}) - C_{cn+d}(\mathcal{F}') = G(\Gamma, n) \bmod 2$, as desired.

□

2.6 Example

Let us illustrate the construction in a simple case. Consider a two-stack automaton Γ_3 given in Figure 2.2. Note that Γ_3 has a unique balanced path $\gamma = v_1 v_3 v_5 v_3 v_6 v_4 v_2 v_4 v_2$.

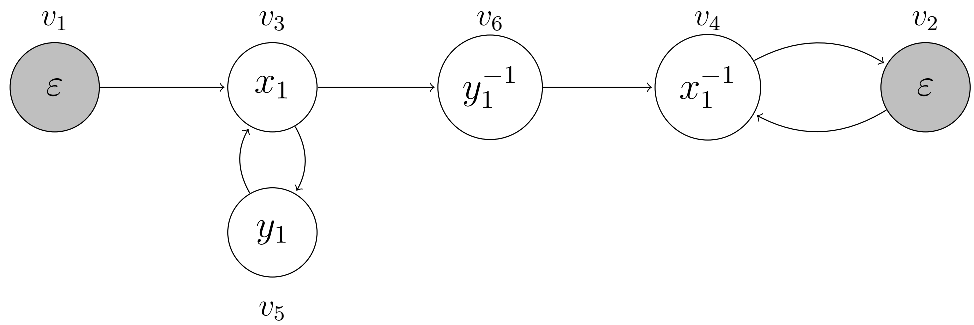
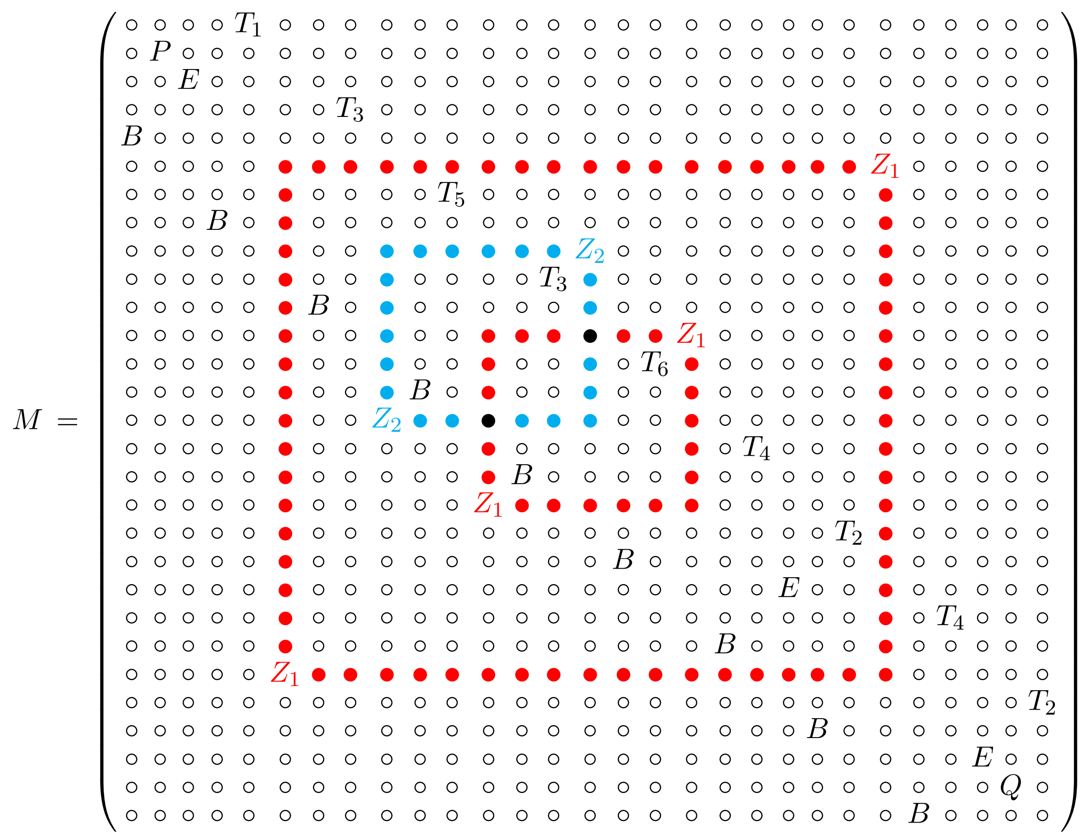


Figure 2.2: The two-stack automaton Γ_3.

Let us show that the following matrix $M = M(\gamma, \pi_\gamma)$ is unique in the set of fixed points $\mathcal{D}'_9$. Here we have $s(Z_1) = x_1$ and $s(Z_2) = y_1$.



As in the definition of $M(\gamma, \pi_\gamma)$, observe that M has a diagonal of B entries below the main diagonal, and a diagonal of T_i entries above the main diagonal.

The T_i entries give the vertices of the path γ, in order. Observe that M also has a P in the top left and Q in the bottom right, as in the definition of $M(\gamma, \pi_\gamma)$.

We have here the involution $\pi_\gamma = (2\,8)(3\,5)(4\,6)$, and the locations of the E, Z_1 and Z_2 matrices form the permutation matrix for π_γ. Each matrix E corresponds to a time when the path γ visited a vertex labelled ε. Similarly, each Z_p above the diagonal corresponds to a pair of times when a given instance of a label was written and removed from one of the stacks.

The red and blue squares in M connect each Z_p with the corresponding times along γ that the label was written and removed. Red represents X, while blue represents Y, as defined in Section 2.2. Notice that when two of these squares cross (marked black), it means that the first label written was also the first label removed. This can only happen when the two labels are written on different stacks, so squares of the same color cannot cross.

The matrix M avoids $\mathcal{F}_2$, since the only copes of P and Q are near the top left and bottom right corner. Similarly, matrix M avoids $\mathcal{F}_3$, since no T_i is too far up and to the right of any B. Clearly, M avoids B', so M avoids $\mathcal{F}_4$.

Now recall the matrix L in the definition of the alphabet $\mathcal{A}_g$:

$$L = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Observe that M avoids $\mathcal{F}_1$, since there is no submatrix where each 1 comes from a different $g \times g$ block. Finally, the fact that M avoids $\mathcal{F}_5$ corresponds to the fact that the lines coming out of Z_p and Z_q never cross when $Z_p \sim Z_q$.

Clearly, matrix M contains B and avoids B'. One can verify that every submatrix B in M is a marked submatrix in some matrix $\mathcal{F}_4'$. Since M satisfies all of the conditions of Lemma 2.5.1, we conclude that $M \in \mathcal{D}_n'$.

2.7 Proofs of technical lemmas

2.7.1 Proof of Lemma 2.5.1.

First, let us show that every matrix M which satisfies the following properties, is a fixed point of ϕ :

(1) matrix M avoids $\mathcal{F}$,

(2) matrix M avoids B',

(3) matrix B is a submatrix of M, and

(4) every submatrix B in M is a marked submatrix in some matrix $\mathcal{F}_4'$.

Observe that (1) and (3) imply that M avoids $\mathcal{F}$ but not $\mathcal{F}' = \mathcal{F} \cup \{B, B'\}$. Therefore, we have $M \in \mathcal{D}_n$. Further, (2) and (4) imply that M is a fixed point of ϕ, since M has no B', and replacing any B with a B' will create a matrix in $\mathcal{F}_4$.

Next, we show that every matrix in $\mathcal{D}_n$ which violates the above criteria is fixed by ϕ^2 but not by ϕ. Observe that a matrix $M \in \mathcal{D}_n$ which satisfies (2) also satisfies (1) and (3). Consider now a matrix $M \in \mathcal{D}_n$ which violates either (2) or (4). Clearly, we have $\phi(M) \in \mathcal{D}_n$. It suffices to show that $\phi(M) \neq M$ and that $\phi^2(M) = M$.

Let N be a matrix in $\mathcal{D}_n$. Denote by A a submatrix B or B' in N. Let N' be the matrix formed by replacing A with B' or B, respectively. Similarly, let A' denote this submatrix B or B' in N'.

Assume that N' did not avoid $\mathcal{F}_1$. Then there would exist a submatrix $S \in \mathcal{A}_g$ of N' which is not a consecutive $g \times g$ block. Note that S must intersect A'. Consider the submatrix $S \cap A'$. This must be a permutation matrix, since S and A' are permutation matrices. Furthermore, $S \cap A'$ must be a consecutive submatrix of S, since A' is a consecutive submatrix of N'. Since S is simple, we have that $S \cap A'$ is a single 1 entry. This entry might be a 0 in N, but it does not matter because we could replace it with another 1 entry from A, to form a non-consecutive submatrix S in N. Thus, N does not avoid $\mathcal{F}_1$, contradicting the assumption that $N \in \mathcal{D}_n$.

Since N' avoids $\mathcal{F}_1$, any instance of a matrix from $\mathcal{F}_2$, $\mathcal{F}_3$, or $\mathcal{F}_5$ intersecting A' in N' must contain A' or intersect A' in a single row or column. In either case, replacing A with A' gives another matrix in $\mathcal{F}_2$, $\mathcal{F}_3$, or $\mathcal{F}_5$, contradicting the fact that N' avoids $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_5$.

Therefore, if the submatrix A is blocked it must be because A' is contained in a matrix in $\mathcal{F}_4$. Since $\mathcal{F}_4$ is closed under replacing any B with a B', it must be that $A = B$ and $A' = B'$. Any matrix which violates (2) therefore contains a instance of B', which is necessarily unblocked, and so is not fixed by ϕ. For matrices in $\mathcal{D}_n$ which satisfy (2), containing an unblocked instance of B is equivalent to violating (4). Therefore, any matrix which violates (4) is not fixed by ϕ, so $\phi(M) \neq M$.

Let $\eta(N)$ denote the leftmost unblocked instance of B or B' in N, and let $A = \eta(N)$. Observe that A' is also clearly unblocked in N'. Assume that $A' \neq \eta(N')$. There must be another unblocked instance S of B or B', which is further to the left than A'. However, S would also be unblocked in N, contradicting the assumption that $A = \eta(N)$. Therefore, $A' = \eta(N')$, which implies that $\phi(N') = N$. We conclude that $\phi(\phi(M)) = M$ for all M.

In summary, every matrix M satisfying (1) through (4) is fixed by ϕ, and every matrix in $\mathcal{D}_n$ not satisfying (1) through (4) is fixed by ϕ^2 but not ϕ. Therefore, map ϕ is an involution whose fixed points are exactly the matrices satisfying (1) through (4). $\square$

2.7.2 Proof of Lemma 2.5.2.

Let M be a matrix in $\mathcal{D}'_n$. Lemma 2.5.1 shows that

(1) matrix M avoids $\mathcal{F}$,

(2) matrix M avoids B',

(3) matrix B is a submatrix of M, and

(4) every instance of B in M is the marked submatrix in some matrix in $\mathcal{F}'_4$.

Observe that every matrix in $\mathcal{F}'_4$ has a B or a P above the marked submatrix $\boxed{B}$. Therefore, for every instance of B in M, there must be another B or a P somewhere above it. Since there is at least one B in M, there must be at least one P in M. This P must have at least g rows above it and g columns to the left, since it is contained in some matrix in $\mathcal{W}'_4$. Note that matrix P cannot have any more rows above it or columns to the left, since M avoids $\mathcal{F}_3$. Therefore, $M^{2,2}$ is the unique instance of P in M. Similar analysis shows that $M^{3n+1,3n+1}$ is the unique instance of Q in M.

Similarly, every matrix in $\mathcal{F}'_4$ has a T_j at least $4g$ rows above and $4g$ columns to the right of the $\boxed{B}$. Since M avoids $\mathcal{F}_2$, there can be no more rows or columns inserted in between this B and T_j. Therefore, for every instance of B in M, there must be some T_j exactly $4g$ rows above and $4g$ columns to the right. In particular, this means that if $M^{i,j} = B$, then $M^{i-4,j+4} = T_j$ for some j.

The T_1 above the P must be $4g$ rows above and $4g$ columns to the right of the B to the left of the P. This is only possible if this B is in location $M^{5,1}$, and the T_1 is in location $M^{1,5}$.

If $M^{i,j} = B$ and $i \neq n$, then there must another instance of B exactly $3g$ columns to the right, and some T_k exactly g rows above, since every matrix in $\mathcal{W}'_1$, $\mathcal{W}'_2$, $\mathcal{W}'_3$ or $\mathcal{W}'_4$ has this pattern. We know that these submatrices cannot be further away because they are squeezed between the B and the T_j. The new T_k must be $4g$ rows below and $4g$ columns to the right of the new instance of B. This can only be achieved by letting $M^{i+3,j+3} = B$ and $M^{i-1,j+7} = T_k$.

Thus, the instances of B in a matrix M in D'_n are exactly the matrices $M^{3i+2,3i-2}$, for $1 \leq i \leq n$. Similarly, we have $M^{3i-2,3i+2} = T_j$ for some j. Let γ_i denote the vertex in Γ corresponding to the matrix $M^{3i-2,3i+2}$. The restrictions in $\mathcal{F}'_4$ about adjacent v_i ensure that $\gamma = \gamma_1 \ldots \gamma_n$ is a path from v_1 to v_2.

When we specify this path γ, we uniquely define all of the submatrices $M^{i,j}$ except for those of the form $M^{3i,3j}$. Further, the restrictions from $\mathcal{F}'_4$ tell us that each $M^{3i,3j}$ is either a matrix of zeros or equal to E or Z_p. If the matrix at location $M^{3i,3j}$ is nonzero, then it is uniquely determined by the restrictions from $\mathcal{F}'_4$ on the rows of M.

Let π be the permutation such that $\pi(i)$ is the unique j, such that $M^{3i,3j}$ is E or Z_p. Let us prove that $M = M(\gamma, \pi)$. We already showed that $M^{2,2} = P$, $M^{3n+1,3n+1} = Q$, and $M^{3i+2,3i-2} = B$ for all i. We know that $M^{3i-2,3i+2} = T_j$ where $\gamma_i = v_j$, by the definition of γ.

By definition, submatrix $M^{3i,3j}$ is non-zero if and only if $\pi(i) = j$. The restrictions on E and Z_p from $\mathcal{F}'_4$ imply that $M^{3i,3j} = E$ for $\rho(\gamma_i) = \varepsilon$, $M^{3i,3j} = Z_p$ for $\rho(\gamma_i) = s(Z_p)$, and $M^{3i,3j} = Z_p$ for $\rho(\gamma_i) = (s(Z_p))^{-1}$. We have a total of m entries $1's$, so all remaining $M^{i,j}$ are zeros matrices. This implies that $M = M(\gamma, \pi)$. $\square$

2.7.3 Proof of the *only if* part of Lemma 2.5.3.

First, we show that if $M(\gamma, \pi) \in \mathcal{D}'_n$, then γ is balanced and $\pi = \pi_\gamma$. By Proposition 2.2.2, it suffices to show that the permutation π satisfies the following four properties:

(1) $\rho(\gamma_i) = \varepsilon$ for all $\pi(i) = i$,

(2) $\rho(\gamma_i) \in X \cup Y$ and $\rho(\gamma_{\pi(i)}) = \rho(\gamma_i)^{-1}$, for all $\pi(i) > i$,

(3) there are no i and j with $\rho(\gamma_i) \sim \rho(\gamma_j)$ such that $i < j < \pi(i) < \pi(j)$, and

(4) permutation π is an involution.

Proof of (1): Consider the case where $\pi(i) = i$. The submatrix $M^{3i,3i}$ is non-zero then. In this case, the B at $M^{3i+2,3j-2}$ must be the marked submatrix $\boxed{B}$ in some instance of a matrix in $\mathcal{W}'_3$, $\mathcal{W}'_4$ or $\mathcal{W}'_5$. Thus, $M^{3i,3i} = E$ and $M^{3i-2,3j-2} = T_j$, where $\rho(\gamma_i) = \rho(v_j) = \varepsilon$, so π satisfies (1).

Proof of (2): Consider the case where $\pi(i) > i$, so $M^{3i,3\pi(i)}$ is non-zero. In this case, the submatrix B at $M^{3i+2,3i-2}$ must be $\boxed{B}$ in some instance of a matrix in $\mathcal{W}'_1$. Thus, $M^{3i,3\pi(i)} = Z_p$, and $M^{3i-2,3i+2} = T_j$, where $\rho(\gamma_i) = \rho(v_j) = s(Z_p)$. Similarly, the submatrix B at $M^{3\pi(i)+2,3\pi(i)-2}$ must be a marked submatrix in some matrix in $\mathcal{W}'_2$. Thus, $M^{3\pi(i)-2,3\pi(i)+2} = T_k$, where $\rho(\gamma_{\pi(i)}) = \rho(v_k) = (s(Z_p))^{-1}$. We conclude that $\rho(\gamma_i) \in X \cup Y$ and $\rho(\gamma_{\pi(i)}) = \rho(\gamma_i)^{-1}$, so π satisfies (2). Similar analysis shows that if $\pi(i) < i$, then $\rho(\gamma_{\pi(i)}) \in X \cup Y$ and $\rho(\gamma_i) = \rho(\gamma_{\pi(i)})^{-1}$, so π^{-1} satisfies (2) as well.

Proof of (3): Assume that there exist i and j with $\rho(\gamma_i) \sim \rho(\gamma_j)$ such that $i < j < \pi(i) < \pi(j)$. Let $s(Z_p) = \rho(\gamma_i)$ and $s(Z_q) = \rho(\gamma_j)$. We have $M^{3i,3\pi(i)} = Z_p$ and $M^{3j,3\pi(j)} = Z_q$. Observe that $M^{3j+2,3j-2} = B$. Since $3i < 3j < 3j+2$ and $3j-2 < 3\pi(i) < 3\pi(j)$, these three matrices together form a pattern which is in $\mathcal{F}_5$, a contradiction. This implies that π satisfies (3). Similar analysis shows that there are no i and j with $\rho(\gamma_i) \sim \rho(\gamma_j)$, such that $\pi(i) < \pi(j) < i < j$, so π^{-1} satisfies (3) as well.

Proof of (4): Note that the conditions on π only mention values i for which $\pi(i) \geq i$. Therefore, even if π were not an involution, the permutation π would have to agree with π_γ at all i such that $\pi(i) \geq i$, by the uniqueness of the involution π_γ. However, a similar analysis shows that π^{-1} must also satisfy conditions (1), (2) and (3), so π^{-1} also agrees with π_γ at all i such that $\pi^{-1}(i) \geq i$. Combining these two observations, we conclude that $\pi = \pi^{-1} = \pi_\gamma$. □

2.7.4 Proof of the *if* part of Lemma 2.5.3.

It remains to prove that $M(\gamma, \pi_\gamma) \in \mathcal{D}'_n$. Clearly, matrix B is a submatrix of $M(\gamma, \xi\pi_\gamma)$. In addition, matrix $M(\gamma, \pi_\gamma)$ has exactly n submatrices B, and each of them is the marked submatrix $\boxed{B}$ in some matrix in $\mathcal{F}'_4$.

Recall that there is a unique P and Q in $M(\gamma, \pi_\gamma)$, and they are sufficiently close to the edge to ensure that $M(\gamma, \pi_\gamma)$ avoids $\mathcal{F}_2$. Similarly, we know the locations of all submatrices B and T_i, and they ensure that $M(\gamma, \pi_\gamma)$ avoids $\mathcal{F}_3$. Clearly, matrix $M(\gamma, \pi_\gamma)$ avoids B', and therefore also avoids $\mathcal{F}_4$. Also, if $M(\gamma, \pi_\gamma)$ did not avoid $\mathcal{F}_5$, this would contradict the fact that there are no i and j with $\rho(\gamma_i) \sim \rho(\gamma_j)$ such that $i < j < \pi_\gamma(i) < \pi_\gamma(j)$.

Finally, we need to show that $M(\gamma, \pi_\gamma)$ avoids $\mathcal{F}_1$. Assume that $M(\gamma, \pi_\gamma)$ does not avoid $\mathcal{F}_1$, and that A is a $g \times g$ submatrix of $M(\gamma, \pi_\gamma)$, such that $A \in \mathcal{A}_g$ and A is not a consecutive $g \times g$ block in $M(\gamma, \pi_\gamma)$. Since A is simple, every 1 in A comes from a different block $M^{i,j}$. By definition of $\mathcal{A}_g$, matrix L is a submatrix of A (see §2.4.1). Assume the 1 in the center of L is above or on the main diagonal of M. It must therefore be in an E, T_j, or Z_p. The top three 1's in L must be in matrices of the form Z_q, since there is not enough room for any of them to be in a T_k.

Since the relation "$\sim$" partitions the set of Z_p's into two equivalence classes, two of these 1's must be in some Z_p and Z_q with $Z_p \sim Z_q$. However, there is a submatrix B that is below and to the left of both Z_p and Z_q. Together they form a matrix in $\mathcal{F}_5$, contradicting the fact that $M(\gamma, \pi_\gamma)$ was shown above to avoid $\mathcal{F}_5$. Similar analysis gives a contradiction if the 1 in the center of L is below the main diagonal.

We conclude that $M(\gamma, \pi_\gamma)$ satisfies all four conditions from Lemma 2.5.1. This implies $M(\gamma, \pi_\gamma) \in \mathcal{D}'_n$, as desired. $\square$

2.8 Decidibility

2.8.1 Simulating Turing Machines

It is well known and easy to see that two-stack automata can simulate (nondeterministic) *Turing machines*. This simulation works by using the two-stacks to represent the tape of the Turing machine (see e.g. [HMU]). To recap, let one stack contain everything written on the tape to the left of the head, while the other stack contains everything written to the right. Moving the head of the Turing machine left or right corresponds to removing a symbol from one stack and writing a (possibly different) symbol on the other stack. This simulation is direct and can be done in polynomial time. We refer to [HMU, Sip] for the definitions and details.

2.8.2 Proof of Theorem 2.1.3

Let M be an arbitrary deterministic Turing machine, with no input. Consider the two-stack automaton which simulates M. If M does not halt, $G(\Gamma, n) = 0$ for all n. If M eventually halts, then Γ will have a single balanced path, and $G(\Gamma, n) = 1$ for some n.

Construct the F and F' associated with this Γ as in Lemma 2.3.2. By analogy, we then have $C_n(\mathcal{F}) = C_n(\mathcal{F}') \bmod 2$ for all $n = d \bmod c$ *if and only if* M does not halt. Recall also that the *halting problem* is undecidable (see e.g. [Sip]).

Therefore, by the argument above, the construction in the proof of Lemma 2.3.2 can be emulated by a Turing machine, and the construction satisfies $C_n(\mathcal{F}) = C_n(\mathcal{F}') \bmod 2$ for all $n \neq d \bmod c$. Thus, given M, we can construct finite sets $\mathcal{F}$ and $\mathcal{F}'$ permutation patterns, such that $C_n(\mathcal{F}) = C_n(\mathcal{F}')$ for all $n \geq 1$ if and only if M does not halt. Therefore, it is undecidable whether $C_n(\mathcal{F}) = C_n(\mathcal{F}')$ for all $n \geq 1$. □

2.8.3 Implications and speculations

Theorem 2.1.3 has rather interesting implications for our understanding of pattern avoidance (cf. §2.9.7). For example, a standard argument (see e.g. [Poon, p. 2]), gives the following surprising result:

Corollary 2.8.1. *There exist two finite sets of patterns $\mathcal{F}$ and $\mathcal{F}'$, such that the problem of whether $C_n(\mathcal{F}) = C_n(\mathcal{F}') \bmod 2$ for all $n \in \mathbb{N}$, is independent of ZFC.*

These results give us confidence in the following open prolems, further extending Theorem 2.1.3.

Conjecture 2.8.2 (Parity problem)**.** *The problem of whether $C_n(\mathcal{F}) = 0 \bmod 2$ for all $n \in \mathbb{N}$, is undecidable.*

We believe that our tools may prove useful to establish this conjecture, likely with a great deal more effort. This goes beyond the scope of the paper.

Conjecture 2.8.3 (Wilf–equivalence problem)**.** *The problem of whether $C_n(\mathcal{F}_1) = C_n(\mathcal{F}_2)$ for all $n \in \mathbb{N}$ is undecidable.*

This conjecture is partly motivated by the recently found unusual examples of

Wilf–equivalence [BP]. It is more speculative than Conjecture 2.8.2, since in our construction $C_n(\mathcal{F}) - C_n(\mathcal{F}')$ grows rather rapidly.

In a different direction, it would be also very interesting to resolve the parity and the Wilf–equivalence problems when only one permutation is avoided. In this case, we believe that both are likely to be decidable.

Conjecture 2.8.4. *For forbidden sets with a single permutation* $|\mathcal{F}| = |\mathcal{F}'| = 1$, *both the parity and the Wilf-equivalence problems are decidable.*

To clarify and contrast the conjectures, let us note that non-P-recursiveness is quite possible and rather likely for permutation class avoiding one or two permutations (see §2.9.4). However, the undecidability is a much stronger condition and we do not believe one permutation has enough room to embed all logical axioms. This "small size is hard to achieve" phenomenon is somewhat similar to aperiodicity and undecidability for plane tilings (see §2.9.6).

2.9 Final remarks and open problems

2.9.1

Despite a large literature on pattern avoidance, relatively little is known about general sets of patterns. Notably, the *Stanley–Wilf conjecture* proved by Marcus and Tardos shows that $C_n(\mathcal{F})$ are at most exponential [MT], improving on an earlier near-exponential bound by Alon and Friedgud [AF]. Most recently, Fox showed that for a random permutation $\omega \in S_n$, we have $C_n(\mathcal{F}) = \exp\bigl(k^{\Theta(1)} n\bigr)$ [Fox] (see also [V3, §2.5]).

2.9.2

For an integer P-recursive sequence $\{a_n\}$, the generating series $\mathcal{A}(t) = \sum_{n=0}^{\infty} a_n t^n$ is *D-finite* (*holonomic*), i.e. satisfies a linear ODE (see e.g. [FS, Sta1]). In the case $a_n = e^{O(n)}$, the series $\mathcal{A}(t)$ is also a *G-function*, an important notion in Analytic Number Theory (see e.g. [Gar]).

2.9.3

The most celebrated example of Wilf-equivalence is $(123) \sim (213)$. It follows from results of MacMahon (1915) and Knuth (1973), that $C_n(123) = C_n(213) = \frac{1}{n+1}\binom{2n}{n}$, the *n-th Catalan number*, see e.g. [Kit, Sta1]. Many other Wilf-equivalent classes are known now, see e.g. [Kit].

2.9.4

There are three Wilf-equivalence classes of patterns in S_4. Two of them are known to give P-recursive sequences, but whether $\{C_n(1324)\}$ is P-recursive remains a long-standing open problem in the area (see e.g. [Ste, V3]).

We should mention that there seems to be some recent strong experimental evidence *against* sequence $\{C_n(1324)\}$ being P-recursive. Numerical analysis of the values for $n \leq 36$ given in [CG] suggest the asymptotic behavior

$$C_n(1324) \sim A \cdot \lambda^n \cdot \mu^{\sqrt{n}} \cdot n^{\alpha} \,,$$

where $A \approx 8$, $\lambda \approx 11.6$, $\mu \approx 0.04$, and $\alpha \approx -1.1$. If this asymptotics holds, this would imply non-P-recursiveness by Theorem 1.5.1. in Chapter 1, which forbids $\mu^{\sqrt{n}}$ terms.

For larger sets, the leading candidates for non-P-recursiveness are the sequences $\{C_n(4231, 4123)\}$ and $\{C_n(4123, 4231, 4312)\}$ which were recently computed for $n \leq 1000$ and 5000, respectively [A+].

2.9.5

The bound $|\mathcal{F}| < 10^{119}$ in Corollary 2.4.4 can easily be improved down to a bit under 30,000 by refining the proof of Theorem 2.4.3 and using small technical tricks. While it is possible that a single permutation suffices (see above), it is unlikely that the tools from this paper can be used.

2.9.6

It is worth comparing the non-P-recursiveness of pattern avoidance with the history of *aperiodic tilings* in the plane. Originally conjectured by Wang to be impossible [Wang], such tilings were first constructed by Berger in 1966 by an undecidability argument [Ber]. Note that Berger's construction used about 20,000 tiles. In 1971, by a different undecidability argument, Robinson was able to reduce this number to six [Rob]. Most recently, Ollinger established the current record of five polyomino tiles [Oll]. Without undecidability, Penrose famously used only two tiles to construct aperiodic tilings (see e.g. [GS]). Whether there exists one tile which forces aperiodicity remains open. This is the so-called *einstein problem* (cf. [ST] and [Yang, §7.5]).

2.9.7

In [EV], Zeilberger asks to characterize pattern avoiding permutation classes (of single permutations) with g.f.'s rational, algebraic, or D-finite. We believe a complete characterization of the latter class is impossible.

Open Problem 2.9.1. *The problem of whether* $\{C_n(\mathcal{F})\}$ *is* P-recursive *is undecidable.*

Let us note that our results offer only a weak evidence in favor of the open problem, since Theorem 2.2.1 gives only a necessary condition on P-recursiveness. It would be interesting to see if any of Zeilberger's problems are decidable. A rare decidability result in this direction is given in [BRV] (see also a discussion in [V3, §3]).

References

[AAK] M. H. Albert, M. D. Atkinson and M. Klazar, The enumeration of simple permutations, *J. Integer Seq.* **6** (2003), Art. 03.4.4, 18 pp.

[A+] M. H. Albert, C. Homberger, J. Pantone, N. Shar and V. Vatter, Generating permutations with restricted containers, in preparation.

[AF] N. Alon and E. Friedgut, On the number of permutations avoiding a given pattern, *J. Combin. Theory, Ser. A* **89** (2000), 133–140.

[Atk] M. D. Atkinson, Restricted permutations, *Discrete Math.* **195** (1999), 27–38.

[Ber] R. Berger, The undecidability of the domino problem, *Memoirs AMS* **66** (1966), 72 pp.

[Bóna] M. Bóna, *Combinatorics of permutations*, CRC Press, Boca Raton, FL, 2012.

[BHPV] M. Bóna, C. Homberger, J. Pantone and V. Vatter, Pattern-avoiding involutions: exact and asymptotic enumeration; arXiv:1310.7003.

[BRV] R. Brignall, N. Ruškuc and V. Vatter, Simple permutations: decidability and unavoidable substructures, *Theor. Comput. Sci.* **391** (2008), 150–163.

[BP] A. Burstein and J. Pantone, Two examples of unbalanced Wilf–equivalence, *J. Combin.*, to appear.

[CG] A. R. Conway and A. J. Guttmann, On 1324-avoiding permutations, *Adv. Applied Math.* **64** (2015), 50–69.

[EV] M. Elder and V. Vatter, Problems and Conjectures presented at Third Permutation Patterns Conference (2005); arXiv:math/0505504.

[Eli] S. Elizalde, A survey of consecutive patterns in permutations, to appear in *Recent Trends in Combinatorics*, IMA, 2015.

[FS] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, Cambridge, 2009.

[Fox] J. Fox, Stanley–Wilf limits are typically exponential, to appear in *Advances in Mathematics.*

[Gar] S. Garoufalidis, G-functions and multisum versus holonomic sequences, *Advances Math.* **220** (2009), 1945–1955.

[Ges] I. Gessel, Symmetric Functions and P-Recursiveness, *J. Combin. Theory, Ser. A* **53** (1990), 257–285.

[GS] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman, New York, 1989.

[HMU] J. E. Hopcroft, R. Motwani and J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation* (Third Ed.), Addison Wesley, Reading, MA, 2003.

[Kit] S. Kitaev, *Patterns in Permutations and Words*, Springer, Berlin, 2011.

[Kla] M. Klazar, Some general results in combinatorial enumeration, in *Proc. Permutation Patterns* (*St. Andrews 2007*), Cambridge Univ. Press, 2010, 3–40; arXiv:0803.4292.

[MT] A. Marcus and G. Tardos, Excluded permutation matrices and the Stanley-Wilf conjecture, *J. Combin. Theory, Ser. A* **107** (2004), 153–160.

[NZ] J. Noonan and D. Zeilberger, The enumeration of permutations with a prescribed number of forbidden patterns, *Adv. Appl. Math.* **17** (1996), 381–407.

[Odl] A. M. Odlyzko, Asymptotic enumeration methods, in *Handbook of Combinatorics*, Vol. 2, Elsevier, Amsterdam, 1995, 1063–1229.

[Oll] N. Ollinger, Tiling the plane with a fixed number of polyominoes, in *Lecture Notes Comput. Sci.* **5457**, Springer, Berlin, 2009, 638–647.

[Pak] I. Pak, Tile Invariants: New Horizons, *Theor. Comp. Sci.* **303** (2003), 303–331.

[Poon] B. Poonen, Undecidable problems: a sampler, in *Interpreting Gödel: Critical essays* (J. Kennedy, editor), Cambridge Univ. Press, 2014, 211–241.

[Rob] R. M. Robinson, Undecidability and nonperiodicity for tilings of the plane, *Invent. Math.* **12** (1971), 177–209.

[Rio] J. Riordan, *Combinatorial Identities*, John Wiley, New York, 1968.

[Sag] B. E. Sagan, Pattern avoidance in set partitions, *Ars Combin.* **94** (2010), 79–96.

[Sip] M. Sipser, *Introduction to the Theory of Computation*, PWS, Boston, 1997.

[OEIS] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org.

[ST] J. E. S. Socolar and J. M. Taylor, An aperiodic hexagonal tile, *J. Combin. Theory, Ser. A* **118** (2011), 2207–2231.

[Sta] R. P. Stanley, *Enumerative combinatorics*, Vol. 1, 2, Cambridge Univ. Press, Cambridge, UK, 1997/9.

[Ste] E. Steingrímsson, Some open problems on permutation patterns, in *Surveys in combinatorics*, Cambridge Univ. Press, Cambridge, 2013, 239–263.

[V1] V. Vatter, Problems and conjectures presented at the problem session, in *Permutation Patterns*, LMS Lecture Notes Series 376, Cambridge Univ. Press, 2010, 339–345.

[V2] V. Vatter, Enumeration schemes for restricted permutations, *Combin. Probab. Comput.* **17** (2008), 137–159.

[V3] V. Vatter, Permutation Classes, in *The Handbook of Enumerative Combinatorics*, CRC Press, Boca Raton, FL, 2015, 753–834.

[Wang] H. Wang, Proving theorems by pattern recognition II, *Bell System Tech. J.* **40** (1961), 1–41.

[Wilf] H. S. Wilf, What is an answer?, *Amer. Math. Monthly* **89** (1982), 289–292.

[Yang] J. Yang, Rectangular tileability and complementary tileability are undecidable, *Europ. J. Combin.* **41** (2014), 20–34.

[Zei] D. Zeilberger, Enumerative and Algebraic Combinatorics, in *The Princeton companion to mathematics* (T. Gowers, J. Barrow-Green and I. Leader, editors), Princeton Univ. Press, Princeton, NJ, 2008, 550–561.

CHAPTER 3

Irrational Tiles

3.1 Introduction

The study of combinatorial objects enumerated by rational generating functions (GF) is classical and goes back to the foundation of Combinatorial Theory. Rather remarkably, this class includes a large variety of combinatorial objects, from integer points in polytopes and horizontally convex polyominoes, to magic squares and discordant permutations (see e.g. [Sta1, FS]). Counting the number of tilings of a *strip* (rectangle $[k \times n]$ with a fixed height k) is another popular example in this class, going back to Golomb, see [Gol, §7] (see also [BL, CCH, KM, MSV]). The nature of GFs of such tilings is by now completely understood (see Theorem 3.1.1 below).

In this paper we present an unusual generalization to tile counting functions with irrational tiles, of rectangles $[1 \times (n+\varepsilon)]$, where $\varepsilon \in \mathbb{R}$ is fixed. This class of functions turns out to be very rich and interesting; our main result (theorems 3.1.2 and 3.1.3 below) is a complete characterization of these functions. We then use this result to construct a number of tile counting functions useful for applications.

Let us first illustrate the notion of tile counting functions with several examples. Start with Fibonacci number F_n which count the number of tilings of $[1 \times n]$ with the set T of two rectangles $[1 \times 1]$ and $[1 \times 2]$, see Figure 3.1.

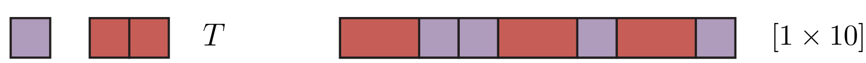


Figure 3.1: Fibonacci tiles T and a tiling of $[1 \times 10]$.

Consider now a more generic set of tiles as in Figure 3.2, where each tile has height 1 and rational side lengths. Note that the dark shaded tiles here are *bookends*, i.e. every tiling of $[1 \times n]$ must begin and end with one, and they are not allowed to be in the middle. Also, no reflections or rotations are allowed, only parallel translations of the tiles. We then have exactly $f_T(n) = \binom{n-2}{2}$ tilings of $[1 \times n]$, since the two light tiles must be in this order and can be anywhere in the sequence of $(n-2)$ tiles.

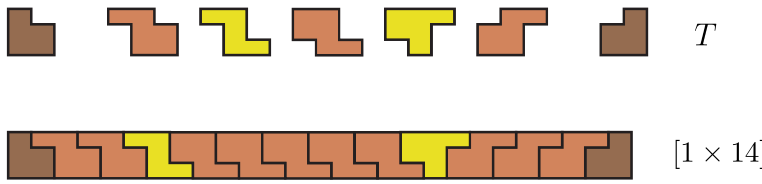


Figure 3.2: Set T of 5 rational tiles and two bookends; a tiling of $[1 \times 14]$ with T.

More generally, let $f_T(n)$ be the number of tilings on $[1 \times n]$ with a fixed set of rational tiles of height 1 and two bookends as above.[1] Denote by $\mathcal{F}_1$ the set of all such functions. It is easy to see via the transfer-matrix method (see e.g. [Sta1, §4.7]), that the GF $F_T(x) = f(0) + f(1)x + f(2)x^2 + \ldots$ is rational:

$$F_T(x) \, = \, \frac{P(x)}{Q(x)} \qquad \text{for some} \quad P, Q \in \mathbb{Z}[x]\,.$$

In the two examples above, we have GFs $1/(1-x-x^2)$ and $x^4/(1-x)^3$, respectively.

Note, however, that the combinatorial nature of $f(n)$ adds further constraints on possible GFs $F_T(x)$. The following result gives a complete characterization of such GFs. Although never stated in this form, it is well known in a sense that it follows easily from several existing results (see §3.11.2 for references and details).

Theorem 3.1.1. *Function $f(n)$ is in $\mathcal{F}_1$, i.e. equal to $f_T(n)$ for all $n \geq 1$ and some rational set of tiles T as above, if and only if its GF $F(x) = f(0) + f(1)x + f(2)x^2 + \ldots$ is $\mathbb{N}$-rational.*

Here the class $\mathcal{R}_1$ of $\mathbb{N}$-*rational functions* is defined to be the smallest class of GFs $G(x) = g(0) + g(1)x + g(2)x^2 + \ldots$, such that:

[1] For simplicity, we allow bookends in T to be *empty tiles*. In general, bookends play the role of boundary coloring for Wang tilings [Wang] (cf. [GaP, PY]). Note that irrational tilings are agile enough not to require them at all. This follows from our results, but the reader might enjoy finding a direct argument.

(1) $0, x \in \mathcal{R}_1$,

(2) $G_1, G_2 \in \mathcal{R}_1 \quad \Longrightarrow \quad G_1 + G_2, G_1 \cdot G_2 \in \mathcal{R}_1$,

(3) $G \in \mathcal{R}_1,\ g(0) = 0 \quad \Longrightarrow \quad 1/(1-G) \in \mathcal{R}_1$.

This class of rational GFs is classical and closely related to *deterministic finite automata* and *regular languages*, fundamental objets in the Theory of Computation (see e.g. [MM, Sip]), and Formal Language Theory (see e.g. [BR1, SS]).[2]

We are now ready to state the main result. Let T be a finite set of tiles as above (no bookends), which all have height 1 but now allowed to have irrational length intervals in the boundaries. Denote by $f(n) = f_{T,\varepsilon}(n)$ the number of tilings with T of rectangles $[1 \times (n+\varepsilon)]$, where $\varepsilon \in \mathbb{R}$ is fixed. Denote by $\mathcal{F}$ the set of all such functions.

Observe that $\mathcal{F}$ is much larger than $\mathcal{F}_1$. For example, take 2 irrational tiles $\bigl[1 \times (\frac{1}{2} \pm \alpha)\bigr]$, for some $\alpha \notin \mathbb{Q}$, $0 < \alpha < 1/2$, and let $\varepsilon = 0$ (see Figure 3.3). Then $f(n) = \binom{2n}{n}$, and the GF equal to $F(x) = 1/\sqrt{1-4x}$.

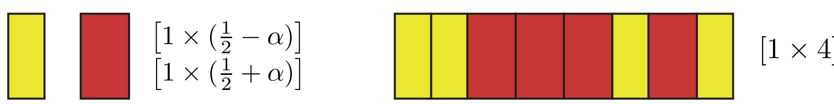


Figure 3.3: Set of 2 irrational tiles; a tiling of $[1 \times 4]$ with 8 tiles.

Let $\mathcal{R}_k$ denote the multivariate $\mathbb{N}$-rational functions defined as a as the smallest class of GFs $F \in \mathbb{N}[[x_1, \ldots, x_k]]$, which satisfies condition

(1′) $0, x_1, \ldots, x_k \in \mathcal{R}_1$.

and conditions (2), (3) as above.

Main Theorem 3.1.2. *Function $f = f(n)$ is in $\mathcal{F}$ if and only if*

$$f(n) \ = \ \bigl[x_1^n \ldots x_k^n\bigr]\, F(x_1, \ldots, x_k) \quad \textit{for some} \quad F \in \mathcal{R}_k\,.$$

[2]Although we never state the connection explicitly, both theories give a motivation for this work, and are helpful in understanding the proofs (cf. §3.11.2).

The theorem can be viewed as a multivariate version of Theorem 3.1.1, but strictly speaking it is not a generalization; here the number k of variables is not specified, and can in principle be very large even for small $|T|$ (cf. §3.11.7). Again, proving that $\mathcal{F}$ is a subset of diagonals of rational functions $F \in \mathbb{Z}[x_1, \ldots, x_k]$ is relatively straightforward by an appropriate modification of the transfer-matrix method, while our result is substantially stronger.

Main Theorem 3.1.3. *Function $f = f(n)$ is in $\mathcal{F}$ if and only if it can be written as*

$$f(n) \,=\, \sum_{(v_1,\ldots,v_d)\in\mathbb{Z}^d} \; \prod_{i=1}^{r} \binom{a_{i1}v_1 + \ldots + a_{id}v_d + a'_i n + a''_i}{b_{i1}v_1 + \ldots + b_{id}v_d + b'_i n + b''_i},$$

for some $r, d \in \mathbb{N}$, and $a_{ij}, b_{ij}, a'_i, b'_i, a''_i, b''_i \in \mathbb{Z}$, for all $1 \le i \le r$, $1 \le j \le d$.[3]

The *binomial multisums* (multidimensional sums) as in the theorem is a special case of a very broad class of *holonomic functions* [PWZ], and a smaller class of *balanced multisums* defined in [Gar] (see §3.11.6). For examples of binomial multisums, take the *Delannoy numbers* D_n (sequence A001850 in [OEIS]), and the *Apéry numbers* A_n (sequence A005259 in [OEIS]) :

$$(\lozenge) \qquad D_n \,=\, \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{2k}{k}, \qquad A_n \,=\, \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3.$$

In summary, Main Theorems 3.1.2 and 3.1.3 give two different characterizations of tile counting functions $f_{T,\varepsilon}(n)$, for some fixed $\varepsilon \in \mathbb{R}$ and an irrational set of tiles T. Theorem 3.1.2 is perhaps more structural, while Theorem 3.1.3 is easier to use to give explicit constructions (see Section 3.3). Curiously, neither direction of either main theorem is particularly easy.

The proof of the main theorems occupies much of the paper. We also present a number of applications of the main theorems, most notably to construction of

[3]The binomial coefficients here are defined to be zero for negative parameters (see §3.2.1 for the precise definition); this allows binomial multisums in the r.h.s. to be finite.

tile counting function with given asymptotics (Section 3.4). This requires the full power of both theorems and their proofs. Specifically, we use the fact that this class of functions are closed under addition and multiplication – this is easy to see for the tile counting functions and the diagonals, but not for the binomial multisums.

The rest of the paper is structured as follows. We begin with definitions and notation (Section 3.2). In the next key Section 3.3, we expand on the definitions of classes $\mathcal{F}$, $\mathcal{B}$ and $\mathcal{R}_k$, illustrate them with examples and restate the main theorems. Then, in Section 3.4, we give applications to asymptotics of tile counting functions and to the Catalan numbers conjecture (Conjecture 3.4.6). In the next four sections 3.5–3.8 we present the proof of the main theorems, followed by the proofs of applications (sections 3.9 and 3.10). We conclude with final remarks in Section 3.11.

3.2 Definitions and notation

3.2.1 Basic notation

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{P} = \{1, 2, \ldots\}$, and let $\mathbb{A} = \overline{\mathbb{Q}}$ be the field of algebraic numbers. For a GF $G \in \mathbb{Z}[[x_1, \ldots, x_k]]$, denote by $\left[x_1^{c_1} \ldots x_k^{c_k}\right] G$ the coefficient of $x_1^{c_1} \ldots x_k^{c_k}$ in G, and by $[1]G$ the constant term in G.

For sequences $f, g : \mathbb{N} \to \mathbb{R}$, we use notation $f \sim g$ to denote that $f(n)/g(n) \to 1$ as $n \to \infty$. Here and elsewhere we only use the $n \to \infty$ asymptotics.

We assume that $0! = 1$, and $n! = 0$ for all $n < 0$. We also extend binomial coefficients to all $a, b \in \mathbb{Z}$ as follows:

$$\binom{a}{b} = \begin{cases} \frac{a!}{(a-b)!b!} & \text{if} \quad 0 \le b \le a, \\ 1 & \text{if} \quad a = -1, \; b = 0, \\ 0 & \text{otherwise.} \end{cases}$$

CAVEAT: This is not the way binomial coefficients are normally extended to negative inputs; this notation allows us to use $\binom{a+b-1}{b}$ to denote the number of ways to distribute b identical objects into a distinct groups, for all $a, b \ge 0$.

3.2.2 Tilings

For the purposes of this paper, a *tile* is an axis-parallel simply connected (closed) polygon in $\mathbb{R}^2$. A *region* is a union of finitely many axis-parallel polygons. We use $|\tau|$ to denote the area of tile τ.

We consider only finite sets of tiles $T = \{\tau_1, \ldots, \tau_r\}$. A *tiling* of a region Γ with the set of tiles T, is a collection of non-overlapping translations of tiles in T (ignoring boundary intersections), which covers Γ. We use $\mathcal{T}(\Gamma)$ to denote the number of tilings of Γ with T.

A set of tiles T is called *tall* if every tile in T has height 1. We study only tilings with tall tiles of rectangular regions $\mathrm{R}_a = [1 \times a]$, where $a > 0$.

3.2.3 Graphs

Throughout the paper, we consider finite *directed weighted multi-graphs* $\mathrm{G} = (V, E)$. This means that between every two vertices $v, v' \in V$ there is a finite number of (directed) edges $v \to v'$, each with its own weight. A *path* γ in G is a sequence of oriented edges (v_1, v_2), (v_2, v_3), $\ldots$, $(v_{\ell-1}, v_\ell)$; vertices v_1 and v_ℓ are called start and end of the path. A *cycle* is a path with $v_1 = v_\ell$. The *weight* of a path or a cycle, denoted $w(\gamma)$, is defined to be the sum of the weights of its edges.

3.3 Three classes of functions

3.3.1 Tile counting functions

Fix $\varepsilon \geq 0$ and let T be a set of tall tiles. In the notation above, $f(n) = \mathcal{T}(\mathrm{R}_{n+\varepsilon})$ is the number of tiling of of rectangles $[1 \times (n+\varepsilon)]$ with T. We refer to $f(n)$ as the *tile counting function*. In notation of the introduction, $\mathcal{F}$ is the set of all such functions.

Example 3.3.1. We define functions $g_1, \ldots, g_6 : \mathbb{N} \to \mathbb{N}$ as follows:

$$g_1(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \qquad g_2(n) = 2\,, \qquad g_3(n) = n^2\,,$$

$$g_4(n) = 2^n \qquad g_5(n) = F_n \qquad g_6(n) = \binom{2n}{n}\,,$$

where F_n is the n-th Fibonacci number. Let us show that these functions are all in $\mathcal{F}$.

First, function g_1 counts tilings of a length n rectangle by a single rectangle R_2. Second, consider a set of six tiles T_2 as in Figure 3.4, with dark shaded tiles of area $\alpha > 0$, $\alpha \notin \mathbb{Q}$, the light shaded tiles of area 1, and set $\varepsilon = 2\alpha$. Now observe that $\mathrm{R}_{n+\varepsilon}$ rectangle can be tiled with T_2 in exactly two ways: one way using either the first or the second triple of tiles.

Third, take any two rationally independent irrational numbers $\alpha > \beta > 0$, and set $\varepsilon = \alpha + \beta$. Consider the set of three rectangles $T_3 = \{\mathrm{R}_1, \mathrm{R}_{1+\alpha}, \mathrm{R}_{1+\beta}, \mathrm{R}_{1+\alpha+\beta}\}$. Now observe that there are exactly n^2 tilings of $\mathrm{R}_{n+\alpha+\beta}$. Fourth, take a set T_4 with one unit square and two tiles which can only form a unit square, and observe that R_n has exactly 2^n tilings. The remaining two examples are given in the introduction.

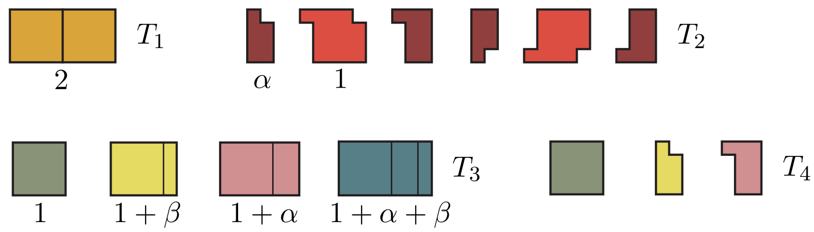


Figure 3.4: Tile sets $T_1, \ldots, T_4$ in the example.

3.3.2 Diagonals of $\mathbb{N}$-rational generating functions

As in the introduction, let $\mathcal{R}_k$ be the smallest class of GFs in k variables $x_1, \ldots, x_k$, satisfying

(1) $0, x_1, \ldots, x_k \in \mathcal{R}_k$,

(2) If $F, G \in \mathcal{R}_k$, then $F+F$ and $F \cdot G \in \mathcal{R}_k$.

(3) If $F \in \mathcal{R}_k$, and $[1]F = 0$, then $\frac{1}{1-F} \in \mathcal{R}_k$.

A GF in $\mathcal{R}_k$ is called an $\mathbb{N}$-*rational generating function in* k *variables.* Note that if $G(x_1, \ldots, x_k) \in \mathcal{R}_k$, then so is $G(x_1^m, \ldots, x_k^m)$,for all integer $m \geq 2$.

A *diagonal* of $G \in \mathbb{N}[[x_1, \ldots, x_k]]$ is a function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) \ = \ \left[x_1^n \ldots x_k^n\right] G(x_1, \ldots, x_k) \, .$$

Denote by $\mathcal{N}$ the set of diagonals of all $\mathbb{N}$-rational generating functions, over all $k \in \mathbb{P}$.

Example 3.3.2. In notation of Example 3.3.1, let us show that $g_1, \ldots, g_6 \in \mathcal{N}$:

$$g_1(n) = \left[x^n\right] \frac{1}{1-x^2} \, , \ g_2(n) = \left[x^n\right] \frac{1}{1-x} + \frac{1}{1-x}$$

$$g_3(n) = [x^n y^n]\, x\left(\frac{1}{1-x}\right)^2 y\left(\frac{1}{1-y}\right)^2, \quad g_4(n) = [x^n]\,\frac{1}{1-2x}$$

$$g_5(n) = [x^n]\,\frac{1}{1-x-x^2}\,, \quad g_6(n) = [x^n y^n]\,\frac{1}{1-x-y}\,.$$

3.3.3 Binomial multisums

Following the statement of Main Theorem 3.1.3, denote by $\mathcal{B}$ the set of all functions $f : \mathbb{N} \to \mathbb{N}$ that can be expressed as

$$f(n) \;=\; \sum_{v\in\mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

for some $\alpha_i = \mathbf{a}_i v + a_i' n + a_i''$, $\beta_i = \mathbf{b}_i v + b_i' n + b_i''$, where $r, d \in \mathbb{N}$, $\mathbf{a}_i, \mathbf{b}_i : \mathbb{Z}^d \to \mathbb{Z}$ are integer linear functions, and $a_i', b_i', a_i'', b_i'' \in \mathbb{Z}$, for all i.

Note that the summation over all $v \in \mathbb{Z}^d$ is infinite, so it is unclear from the definition whether the multisums $f(n)$ are finite. However, the binomial coefficients are zero for the negative values of β_i and $\alpha_i - \beta_i$, so the summation is in fact over integer points in a convex polyhedron defined by these inequalities.

Example 3.3.3. In notation of Example 3.3.1, it follows from the definition that $g_2, g_6 \in \mathcal{B}$. To see $g_1, g_3, g_4, g_5 \in \mathcal{B}$, note that

$$g_1(n) \;=\; \sum_{v\in\mathbb{Z}} \binom{n}{2v}\binom{2v}{n}, \quad g_3(n) \;=\; \binom{n}{1}\binom{n}{1},$$

$$g_4(n) \;=\; \sum_{v\in\mathbb{Z}} \binom{n}{v}, \quad g_5(n) \;=\; \sum_{v\in\mathbb{Z}} \binom{n-v}{v}.$$

For the last formula for the Fibonacci numbers $g_5(n) = F_n$ is classical, see e.g. [Rio, p. 14] or [Sta1, Exc. 1.37].

3.3.4 Main theorems restated

Surprisingly, the class $\mathcal{B}$ of binomial multisums as above coincides with both tile counting functions and diagonals of $\mathbb{N}$-rational functions, and plays an intermediate role connecting them.

Main Theorem 3.3.4. $\mathcal{F} = \mathcal{N} = \mathcal{B}$.

The proof of Main Theorem 3.3.4 is split into three parts. Lemmas 3.5.1, 3.6.1 and 3.7.1 state $\mathcal{F} \subseteq \mathcal{B}$, $\mathcal{N} \subseteq \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{N}$, respectively. Each is proved in a separate section, and together they imply the Main Theorem.

Corollary 3.3.5. *The classes of functions $\mathcal{F} = \mathcal{N} = \mathcal{B}$ are closed under addition and (pointwise) multiplication.*

This follows from the Main Theorem 3.3.4 and Lemma 3.7.2, which proves the claim for diagonals $f \in \mathcal{N}$.

Before we proceed to further applications, let us obtain the following elementary corollary of the Main Theorem 3.3.4. Note that each of these tile counting five functions can be constructed directly via ad hoc argument in the style of Example 3.3.1. We include it as an illustration of the versatility of the theorem.

Corollary 3.3.6. *The following functions $f_1, \ldots, f_5 : \mathbb{N} \to \mathbb{N}$ are tile counting functions:*

(i) f_1 *has finite support,*

(ii) f_2 *is periodic,*

(iii) $f_3(n) = a_p n^p + \ldots + a_1 n + a_0$*, where* $a_i \in \mathbb{N}$*,*

(iv) $f_4(n) = m^n$*, where* $m \in \mathbb{N}$*,*

(v) $f_5(n) = m^n - 1$*, where* $m \in \mathbb{N}$*,* $m \geq 1$*.*

Proof. By the main theorem, it suffices to show that each function f_i is in $\mathcal{N}$. Clearly, function f_1 is the diagonal of a polynomial, so $f_1 \in \mathcal{N}$.

The functions

$$f_{k,p}(m) = \begin{cases} 1 & \text{if } m = k \bmod p \\ 0 & \text{otherwise} \end{cases}$$

are the diagonals of the generating functions $\frac{x^k}{1-x^p}$, for all $0 \le k < p$, so are clearly in $\mathcal{N}$, and f_2 can be expressed as a sum of these $f_{k,p}$ functions. Since $\mathcal{N}$ is closed under addition, this implies $f_2 \in \mathcal{N}$. Similarly, the polynomial $f(n) = 1$ and $f(n) = n$ are the diagonals of $1/(1-x)$ and $x/(1-x)^2$ respectively, and thus in $\mathcal{N}$. Since $\mathcal{N}$ is closed under addition and multiplication, we have $f_3 \in \mathcal{N}$.

The function f_4 is the diagonal of $\frac{1}{1-mx}$, and therefore in $\mathcal{N}$. Similarly, the function f_5 satisfies the recurrence $f_5(n+1) = mf_5(n) + (m-1)$, and thus the diagonal of the generating function $G(x)$ satisfying $G = mxG + (m-1)/(1-x)$. Note that

$$G(x) \;=\; (m-1) \cdot \frac{1}{(1-x)} \cdot \frac{1}{(1-mx)}\,.$$

Therefore, $G \in \mathcal{R}_1$, which implies $f_5 \in \mathcal{N}$. $\square$

3.3.5 Two more examples

Recall that our definition of binomial coefficients is modified to have $\binom{-1}{0} = 1$, see §3.2.1. This normally does not affect any (usual) binomial sums, e.g. the Delannoy and Apéry numbers defined in the introduction remain unchanged when the summations in ($\Diamond$) are extended to all integers. Simply put, whenever $\binom{-1}{0}$ appears there, some other binomial coefficient in the product is equal to zero.

The proof of Main Theorem 3.3.4 is constructed by creating a large number of auxiliary variables for the $\mathbb{N}$-rational functions, and auxiliary indices for the binomial multisums. These auxiliary indices are often constrained to a small range, and $\binom{-1}{0}$ does appear in several cases.

Example 3.3.7. Denote by L_n the *Lucas numbers* $L_n = L_{n-1} + L_{n-2}$, where $L_1 = 1$ and $L_2 = 3$, see e.g. [Rio, §4.3] (sequence A000204 in [OEIS]). They have a combinatorial interpretation as the number of matchings in an n-cycle, and are closely related to Fibonacci numbers F_n :

$$(\circledcirc) \qquad L_n = F_n + F_{n-2} \quad \text{for } n \geq 2\,.$$

From Corollary 3.3.5, the function $f(n) := L_n$ is in $\mathcal{F}$. In fact, it is immediate that $L_n \in \mathcal{R}_1$:

$$L_n = [x^n]\,\frac{1+x^2}{1-x-x^2}\,.$$

To see directly that Lucas numbers are in $\mathcal{F}$, take five tiles as in Figure 3.5, with two right bookends, emulating ($\circledcirc$). On the other hand, finding a binomial sum is less intuitive, as $\mathcal{B}$ is not obviously closed under addition. In fact, we have:

$$L_n = \sum_{(k,i)\in\mathbb{Z}^2} \binom{n-k-2i}{k}\binom{1}{i},$$

where we use ($\circledcirc$), the formula for $g_5(n)$ in Example 3.3.3, and make i constrained to $\{0,1\}$. Note that we avoid using $\binom{-1}{0}$.

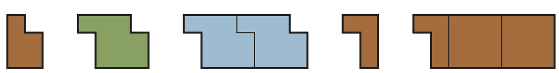


Figure 3.5: Five tiles giving Lucas numbers L_n.

Example 3.3.8. Let $f(n) = 2^n + 3^n$. Checking that $f \in \mathcal{F}$ and $f \in \mathcal{N}$ is straightforward and similar to $g_4(n)$ in the examples above. However, finding a binomial multisum is more difficult:

$$f(n) = \sum_{(i,j,k,\ell,m)\in\mathbb{Z}^5} \binom{n}{i}\binom{m}{j}\binom{1}{k}\binom{m-k}{m}\binom{\ell+k-1}{\ell}\binom{i}{m+\ell}\binom{m+\ell}{i}.$$

Note here that the term $\binom{1}{k}$ gives $k \in \{0,1\}$. Also, $\binom{i}{m+\ell}\binom{m+\ell}{i}$ terms give $m+\ell = i$. Similarly, $\binom{m-k}{m}\binom{\ell+k-1}{\ell}$ give that $m = 0$ if $k = 1$, and $\ell = 0$ if $k = 0$. Therefore,

$$f(n) \,=\, \sum_{(j,m)\in\mathbb{Z}^2} \binom{n}{m}\binom{m}{j} \,+\, \sum_{\ell\in\mathbb{Z}} \binom{n}{\ell} \,=\, 2^n \,+\, 3^n,$$

where two sums correspond to the cases $k = 0$ and $k = 1$, respectively. Note that $\binom{-1}{0} = 1$ is essential in this calculation. It would be interesting to see if Theorem 3.1.3 holds without modification.

3.4 Applications

3.4.1 Balanced multisums

Define a *positive multisum* to be a function $g : \mathbb{N} \to \mathbb{N}$ that can be expressed as

$$g(n) \,=\, \sum_{v\in\mathbb{Z}^d} \prod_{i=1}^{r} \frac{\alpha_i(v,n)!}{\beta_i(v,n)!\,\gamma_i(v,n)!},$$

for some $\alpha_i = \mathbf{a}_i v + a_i' n + a_i''$, $\beta_i = \mathbf{b}_i v + b_i' n + b_i''$, $\gamma_i = \mathbf{c}_i v + c_i' n + c_i''$, where $r, d \in \mathbb{N}$, $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i : \mathbb{Z}^d \to \mathbb{Z}$ are integer linear functions, and $a_i', \ldots, c_i'' \in \mathbb{Z}$, for all i. Here the sum is over all $v \in \mathbb{Z}^d$ for which $\alpha_i(v,n), \beta_i(v,n), \gamma_i(v,n) \geq 0$, for all i.

Positive multisum is called *balanced* if $\alpha_i = \beta_i + \gamma_i$ for all i. Denote by $\mathcal{B}'$ the set of finite sums of balanced positive multisums:

$$f(n) \,=\, g_1(n) + \ldots + g_k(n).$$

Theorem 3.4.1. $\mathcal{B} = \mathcal{B}'$.

The Delannoy and Apéry numbers defined in equation ($\Diamond$) in the introduction are examples of balanced multisums, as are Lucas numbers, see Example 3.3.7. These formulas use only one balanced positive multisum, i.e. have $k = 1$. However, as Example 3.3.8 suggests, the sums $f(n) = 2^n + 3^n$ can we written with $k = 2$, as the lengthy binomial multisum for $f(n)$ involves using the $\binom{-1}{0} = 1$ notation. Therefore, one can think of Theorem 3.4.1 as a *tradeoff*: we prohibit using the $\binom{-1}{0}$ notation, but now allow taking finite sums of balanced multisums (cf. §3.11.6).

We give direct proof of the theorem in Section 3.10. Note that $\mathcal{B}'$ is trivially closed under addition and multiplication, so Theorem 3.4.1 together with the main theorem immediately implies Corollary 3.3.5.

3.4.2 Growth of tile counting functions

We say that a function f is *eventually polynomial* if there exist an $N \in \mathbb{N}$ and a polynomial q such that for all $n \geq N$, we have $f(n) = q(n)$. We say that a function f *grows exponentially*, if there exist $c_1, c_2 > 0$ and $N \in \mathbb{N}$, such that for all $n \geq N$, we have $e^{c_1 n} \leq f(n) \leq e^{c_2 n}$.

Theorem 3.4.2. *Let $f \in \mathcal{F}$ be a tile counting function. There exists an integer $m \geq 1$, such that every function $f_i(n) := f(nm + i)$ either grows exponentially or is eventually polynomial, where $0 \leq i \leq m - 1$.*

In particular, Theorem 3.4.2 implies that the growth of f is at most exponential. Further, if the growth of f is subexponential, then f must have polynomial growth. This rules out many natural combinatorial and number theoretic sequences, e.g. the number of partitions $p(n)$, or the n-th prime p_n, cf. [FGS].

The proof of Theorem 3.4.2 uses the geometry of integer points in convex polyhedra; it is given in Section 3.9. The theorem should be contrasted with the following asymptotic characterization of diagonals of rational functions, which follows from several known results:

Theorem 3.4.3 (See §3.11.3)**.** *Let $f(n)$ be a diagonal of P/Q, where $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$. Suppose further that $f(n) = \exp O(n)$ as $n \to \infty$. Then there exists an integer $m \geq 1$, s.t.*

$$f(n) \sim A\lambda^n n^\alpha (\log n)^\beta, \quad \textit{for all} \quad n = i \ \textit{mod} \ m, \quad 0 \leq i \leq m-1,$$

where $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$, and $\lambda \in \mathbb{A}$.

In our case, the subexponential growth implies $\lambda = 1$, which gives asymptotics $An^\alpha(\log n)^\beta$. Theorem 3.4.2 implies further that $\alpha \in \mathbb{N}$, $\beta = 0$, and $A \in \mathbb{Q}$ in that case.

Example 3.4.4. The following binomial sums show that nontrivial exponents $\alpha \notin \mathbb{Z}$ and $\beta > 0$ can indeed appear for $f \in \mathcal{F}$ and $\lambda > 1$:

$$\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 4^n n^{-1/2}, \qquad \sum_{k=1}^{n} \binom{2k}{k}^2 16^{n-k} \sim \frac{1}{\pi} 16^n \log n \,.$$

Following these examples, we conjecture that α is always half-integer:

Conjecture 3.4.5. *Let $f \in \mathcal{F}$ be a tile counting function. Then there exists an integer $m \geq 1$, s.t.*

$$f(n) \sim A\lambda^n n^\alpha (\log n)^\beta, \quad \textit{for all} \quad n = i \ \textit{mod} \ m, \quad 0 \leq i \leq m-1,$$

where $\alpha \in \mathbb{Z}/2$, $\beta \in \mathbb{N}$, and $\lambda \in \mathbb{A}$.

See §3.11.3 for a brief overview of related asymptotic results.

3.4.3 Catalan numbers

Recall the Catalan numbers:

$$C_n \,=\, \frac{1}{n+1}\binom{2n}{n}.$$

We make the following mesmerizing conjecture.

Conjecture 3.4.6. *The Catalan numbers C_n is not a tile counting function.*

Several natural approaches to the conjecture can be proved not to work. First, we show that the naive asymptotic approach cannot be used to prove Conjecture 3.4.6.

Proposition 3.4.7. *For every $\varepsilon > 0$, there exists a tile counting function $f \in \mathcal{F}$, s.t.*

$$f(n) \,\sim\, A \cdot C_n \qquad \textit{for some} \qquad A \in (1-\varepsilon, 1+\varepsilon)\,.$$

In a different direction, we show that Conjecture 3.4.6 does not follow from elementary number theory considerations.

Proposition 3.4.8. *For every $m \in \mathbb{N}$, there exists a tile counting function $f \in \mathcal{F}$, s.t. $f(n) = C_n$ mod m.*

Proposition 3.4.9. *For every prime p, there exists a tile counting function $f \in \mathcal{F}$, s.t. $ord_p\big(f(n)\big) = ord_p(C_n)$, where $ord_p(m) = \max\{d \,:\, p^d | m\}$.*

The results in this subsection are proved in Section 3.10. See §3.11.10 for more on the last proposition.

3.4.4 Hypergeometric functions

We use the following special case of the *generalized hypergeometric function*:

$$ {}_{p+1}F_p(a_1, \ldots, a_p, 1; b_1, \ldots, b_p; r) \;=\; \sum_{m=0}^{\infty} \prod_{k=0}^{m-1} \frac{(k+a_1)(k+a_2)\ldots(k+a_p)r}{(k+b_1)(k+b_2)\ldots(k+b_p)}. $$

Let p be a positive integer and $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash p$ be a partition of p. Denote by Υ_λ the following multiset of p rational numbers:

$$ \Upsilon_\lambda \;=\; \bigcup_{i=1}^{\ell} \left\{ \frac{1}{\lambda_i}, \frac{2}{\lambda_i}, \ldots, \frac{\lambda_i - 1}{\lambda_i}, 1 \right\}. $$

For example, if $\lambda = (5,4,2,1) \vdash 12$, then

$$ \Upsilon_\lambda \;=\; \left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{1}{2}, 1, 1 \right\}. $$

Theorem 3.4.10. *Let $\mu = (\mu_1, \ldots, \mu_k) \vdash p$, and let $\nu = (\nu_1, \ldots, \nu_\ell) \vdash p$ be a refinement of μ. Write*

$$ \Upsilon_\mu = \{a_1, \ldots, a_p\}, \quad \Upsilon_\nu = \{b_1, \ldots, b_p\}, $$

and fix $r = r_1/r_2 \in \mathbb{Q}$. Denote $A = {}_{p+1}F_p(a_1, \ldots, a_p, 1; b_1, \ldots, b_p; r)$, and suppose that $A < \infty$ is well defined. Finally, let $c \in \mathbb{N}$ be a multiple of all prime factors of $\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_k \cdot r_2$. Then, there exists a tile counting function $f \in \mathcal{F}$, s.t. $f(n) \sim Ac^n$.

The proof of Theorem 3.4.10 is given in Section 3.10.

Corollary 3.4.11. *There exists a tile counting function $f \in \mathcal{F}$, such that*

$$ f(n) \sim \frac{\sqrt{\pi}}{\Gamma(5/8)\Gamma(7/8)}\, 128^n. $$

Proof. Let $p = 4$, let $\mu = (4)$, $\nu = (2,1,1)$, and set $r = 1/2$. Then $\Upsilon_\mu = \{1/4, 1/2, 3/4, 1\}$ and $\Upsilon_\nu = \{1/2, 1, 1, 1\}$. Since any even c is allowed, we can take $c = 128$. Then, by Theorem 3.4.10, there exists $f \in \mathcal{F}$, s.t.

$$\frac{f(n)}{128^n} \sim {}_5F_4\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1; \frac{1}{2}, 1, 1, 1; \frac{1}{2}\right) = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma(5/8)\Gamma(7/8)},$$

as desired. $\square$

Since the proof of Theorem 3.4.10 is constructive, we can obtain an explicit tile counting function $f(n)$ as in the corollary:

$$f(n) = \sum_{k=0}^{n} \binom{4k}{k}\binom{3k}{k} 128^{n-k} \sim \frac{\sqrt{\pi}}{\Gamma(5/8)\Gamma(7/8)} 128^n.$$

The corollary and the theorem suggest that there is no easy characterization of constants A in Conjecture 3.4.5, at least not enough to obtain Conjecture 3.4.6 this way. Here is yet another quick variation on the theme.

Corollary 3.4.12. *There exists a tile counting function $f \in \mathcal{F}$, such that*

$$f(n) \sim \frac{\Gamma(3/4)^3}{\sqrt[3]{2}\,\pi} 6^n.$$

Proof. Take $p = 3$, $\mu = (3)$, $\nu = (1,1,1)$, and proceed as above. $\square$

3.5 Tile counting functions are binomial multisums

In this section, we prove the following result towards the proof of Main Theorem 3.3.4.

Lemma 3.5.1. *$\mathcal{F} \subseteq \mathcal{B}$.*

The proof first restates the lemma in the language of counting cycles in multi-graphs G (see §3.2.3), and then uses graph theoretic tools to give a binomial-multisum formula for the latter.

3.5.1 Cycles in graphs

We first show how to compute tile counting functions in the language of cycles in weighted graph.

Lemma 3.5.2. *For every tile counting function $f(n)$ there exists a finite weighted directed multi-graph G_T with vertices $v_0, \ldots, v_N$, such that $f(n)$ is the number of paths of weight $n + \varepsilon$, which start and end at v_0.*

Proof. Fix an $f(n) = \mathcal{T}(\mathrm{R}_{n+\varepsilon})$. Recall that each tile $\tau \in T$ has height 1. Denote by $\partial_L(\tau)$ and $\partial_R(\tau)$ the left boundary and right boundary curves of τ of height 1, respectively. A sequence of tiles $(\tau_0, \ldots, \tau_\ell)$ is a tiling of $\mathrm{R}_{n+\varepsilon}$ if and only if

(1) $\partial_R(\tau_i) = \partial_L(\tau_{i+1})$ for $0 \leq i \leq \ell - 1$,

(2) $\partial_L(\tau_0)$ is a vertical line,

(3) $\partial_R(\tau_\ell)$ is a vertical line,

(4) $|\tau_0| + \ldots + |\tau_\ell| = n + \varepsilon$.

Here the first condition implies that all the tiles fit together with no gaps, the second and third conditions imply that the union of the tiles is actually a rectangle, and the fourth condition implies that the rectangle has length $n + \varepsilon$.

We now construct a weighted directed multi-graph G_T corresponding to T as follows. The vertices of G_T are exactly the set of left or right boundaries of tiles (up to translation). Denote them $v_0, \ldots, v_N$, where v_0 is the vertical line. Let the edge $e_{ij} = (v_i, v_j)$ in G_T correspond to tile $\tau \in T$, such that $\partial_L(\tau) = v_i$,

$\partial_R(\tau) = v_j$, and let weight$(e_{ij}) = |\tau|$. Note that edges e_{ij} and e'_{ij}, corresponding to tiles τ and τ', can have different weight. By construction, the paths in G_T of weight $n + \varepsilon$, which start and end at v_0, are in bijection with tilings of $\mathrm{R}_{n+\varepsilon}$. □

3.5.2 Irreducible cycles

To count the number of cycles in a directed multi-graph starting at v_0 of weight $n + \varepsilon$, we factor the cycles into irreducible cycles.

Let $\mathrm{G} = (V, E)$ be a finite directed multi-graph, and let $V = \{v_0, \ldots, v_N\}$. A cycle γ in G is called *positive* if it starts and ends at v_i, and only passes through vertices v_j with $j \geq i$. Cycle γ is called *irreducible* if it is a positive and contains no positive shorter cycle γ'; we refer to γ' as *subcycle* of γ.

Lemma 3.5.3. *There are finitely many irreducible cycles in* G.

Proof. We proceed by induction on the number $N + 1$ of vertices in G. The claim is trivial for $N = 0$. Suppose $N \geq 1$ and let γ be an irreducible cycle in G. If γ does not contain every vertex in G, we can delete an unvisited vertex v_i and apply inductive assumption to a smaller graph $\mathrm{G}' = \mathrm{G} - v_i$. Thus we can assume that γ contains all vertices.

Since γ is positive and contains all vertices, it must start at v_0. Since γ is irreducible, it never come back to v_0 until the end. Note that γ passes through v_1 exactly one, since otherwise it is not irreducible. Identify vertices v_0 and v_1, and denote by H the resulting smaller graph. The cycle γ is then mapped into a concatenation of two irreducible cycles in H. Applying inductive assumption to H gives the result. □

3.5.3 Multiplicities of irreducible cycles

Let ρ be an irreducible subcycle of a positive cycle γ. Define $\gamma - \rho$ to be the positive cycle given by traversing γ, but skipping over ρ. The *multiplicity* of ρ in γ, denoted $m(\rho, \gamma)$, is defined to be:

$$\rho(\gamma) = \begin{cases} 1 & \text{if } \gamma = \rho, \\ 0 & \text{if } \gamma \text{ is irreducible and not equal to } \rho, \\ m(\rho, \gamma') + m(\rho, \gamma - \gamma') & \text{if } \gamma' \text{ is an irreducible positive subcycle of } \gamma. \end{cases}$$

Lemma 3.5.4. *The multiplicity $m(\rho, \gamma)$ is well defined.*

In other words, the multiplicity $m(\rho, \gamma)$ represents the number of times ρ appears in the decomposition of γ. This allows us to count cycles in G_T, which start and end at v_0, that decompose into a given list of irreducible cycles.

Proof of Lemma 3.5.4. By contradiction, assume γ is the smallest positive cycle with irreducible decompositions $\rho_1, \ldots, \rho_k$ and $\rho'_1, \ldots, \rho'_\ell$, giving different multiplicities.

We claim that ρ'_1 must appear on the first list as ρ_i and does not intersect (edge-wise) any of the previous cycles ρ_j, $j < i$. Indeed, neither ρ_j can contain ρ'_1 or vice versa since both are irreducible. However, if they have non-empty overlap, one of them must contain the end of another which contradicts positivity. Since the edges of ρ'_1 have to be eventually removed, we have the claim.

By construction, we now have a new positive cycle $\gamma' = \gamma - \rho'_1$ with irreducible decompositions $\rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_k$ and $\rho'_2, \ldots, \rho'_\ell$, giving different multiplicities. This contradicts the assumption that γ is minimal. $\square$

3.5.4 Counting cycles

Let T be a tall set of tiles and $f(n) = \mathcal{T}(\mathrm{R}_{n+\varepsilon})$. Consider graph G_T constructed in Lemma 3.5.2, and let $\rho_1, \ldots, \rho_r$ be the list of irreducible cycles in G_T, ordered lexicographically. Denote by $B_T(z_1, \ldots, z_r)$ the number of cycles γ in G_T, which start at v_0 and have multiplicity $m(\rho_i, \gamma) = z_i$.

For each $0 < j < i$, let $a_{i,j}$ be the number of times the first vertex in ρ_i is visited in ρ_j, where the first and last vertex in ρ_j is considered to be visited exactly once. Let $a_{i,0} = 1$ if the first vertex of ρ_i is v_0 and let $a_{i,0} = 0$ otherwise.

Lemma 3.5.5. *We have*

$$B_T(z_1, \ldots, z_r) \; = \; \prod_{i=1}^{r} \binom{a_{i,0} + a_{i,1} z_1 + \ldots + a_{i,i-1} z_{i-1} + z_i - 1}{z_i}.$$

Proof. Given a cycle γ in G_T starting at v_0 with $m(\rho_j, \gamma) = z_j$ for all j, we can remove irreducible subcycles of γ one at a time, until we are left with the empty cycle at v_0. Reversing the process, we can also speak of "adding" irreducible cycles to build γ.

Let i, $0 \le i \le r$, be maximal index, such that $z_i > 0$. By definition, every vertex in ρ_i has index greater than every vertex at the start of a irreducible cycle with positive multiplicity in γ. Thus, irrespectively of order in which we add the irreducible cycles, no cycles are inserted in the middle of a copy of ρ_i. Therefore, we may assume that the copies of ρ_i are added last. Further, one can take the cycle γ and determine the cycle with all of the copies of ρ_i removed, and the locations where the $\rho_i(\gamma)$ copies of ρ_i were inserted.

Let γ' be the cycle γ with all copies of ρ_i removed, and let v_k be the start of ρ_i. Note that the number of v_k in γ' is exactly

$$a_{i,0} + a_{i,1} z_1 + \ldots + a_{i,i-1} z_{i-1} \,,$$

since adding the cycle ρ_j adds $a_{i,j}$ more vertices v_k. Therefore, the number of ways to add z_i copies of ρ_i to γ' is equal to

$$\binom{a_{i,0} + a_{i,1}z_1 + \ldots + a_{i,i-1}z_{i-1} + z_i - 1}{z_i}. \tag{$\star$}$$

We conclude that $B_T(z_1, \ldots, z_r)$ is ($\star$) times the number of possible strings you can get after removing all z_i copies of ρ_i. This gives the recursive formula:

$$B_T(z_1, \ldots, z_i, 0, \ldots, 0) \ =$$

$$\binom{a_{i,0} + a_{i,1}z_1 + \ldots + a_{i,i-1}z_{i-1} + z_i - 1}{z_i} B_T(z_1, \ldots, z_{i-1}, 0, \ldots, 0).$$

Since $B_T(0, \ldots, 0) = 1$, iterating the above formula gives the result. □

We can now count all cycles γ which start at v_0 by summing over all lists of irreducible cycles as above giving decompositions of γ.

Lemma 3.5.6. *Every tile counting function $f \in \mathcal{F}$ can be written as*

$$f(n) \ = \sum \prod_{i=1}^{r} \binom{a_{i,0} + a_{i,1}z_1 + \ldots + a_{i,i-1}z_{i-1} + z_i - 1}{z_i},$$

where the sum is over all $(z_1, \ldots, z_r) \in \mathbb{Z}^r$ satisfying $c_1 z_1 + \ldots + c_r z_r = n + \varepsilon$, where all $c_i \in \mathbb{R}$ and $a_{i,j} \in \mathbb{N}$.

Proof. By Lemma 3.5.2, the function $f(n)$ counts the number of cycles γ in G_T which start at v_0 of weight $n + \varepsilon$. In notation above, we have for such γ :

$$w(\gamma) \ = \ w(\rho_1)m(\rho_1, \gamma) + \ldots + w(\rho_r)m(\rho_r, \gamma) \ = \ n + \varepsilon.,$$

Therefore,

$$f(n) \ = \sum B_T(z_1, \ldots, z_r),$$

where the summation is over all $(z_1, \ldots, z_r)$ such that $w(\rho_1)z_1 + \ldots + w(\rho_r)z_r = n + \varepsilon$. Now Lemma 3.5.5 implies the result. $\square$

3.5.5 Proof of Lemma 3.5.1

In notation above, denote by Z_n the set of of all vectors $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{Z}^r$ satisfying $c_1 z_1 + \ldots + c_r z_r = n + \varepsilon$, where all $c_i \in \mathbb{R}$ and $a_{i,j} \in \mathbb{N}$. By Lemma 3.5.6, every $f \in \mathcal{F}$ can be written as

$$f(n) = \sum_{\mathbf{z} \in \mathrm{Z}_n} \prod_{i=1}^{r} \binom{a_{i,0} + a_{i,1}z_1 + \ldots + a_{i,i-1}z_{i-1} + z_i - 1}{z_i}.$$

Without loss of generality, we may assume that $c_r = \varepsilon$ and $c_{r-1} = 1$, since if this were not the case, we could add two tiles to T of area ε and 1, each with a new boundary that only fits together with itself. This adds two disjoint loops to G, and we can label the vertices so that these two disjoint loops are the last two irreducible cycles. Note that for any n, the set Z_n is nonempty. In particular, it contains the vector $(0, 0, \ldots, n, 1)$.

Consider the set $W \subset \mathbb{Z}^r$ of all integer vectors $(w_1, \ldots w_r)$ with $c_1 w_1 + \ldots + c_r w_r = 0$. This set forms a lattice, and therefore has a basis, $\mathbf{b}_1, \ldots, \mathbf{b}_d$. Note that the set Z_n is exactly the set of all vectors $\mathbf{z} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \ldots + v_d\mathbf{b}_d + (0, \ldots, n, 1)$, with each $v_i \in \mathbb{Z}$, and each vector is expressible uniquely in this way. Thus, each coordinate $z_i = \beta_i(v_1, \ldots, v_d, n)$ is an integer coefficient affine function of $(v_1, \ldots, v_d, n)$. This implies

$$a_{i,0} + a_{i,1}z_1 + \ldots + a_{i,i-1}z_{i-1} + z_i - 1 = \alpha_i(v_1, \ldots, v_d, n),$$

where α_i is also integer coefficient affine functions of $(v_1, \ldots, v_d, n)$. Therefore,

$$f(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v, n)}{\beta_i(v, n)},$$

where α_i, β_i and $r, d \in \mathbb{P}$ are as desired. $\square$

3.6 Diagonals of $\mathbb{N}$-rational functions are tile counting functions

In this section, we make the next step towards the proof of Main Theorem 3.3.4.

Lemma 3.6.1. *$\mathcal{N} \subseteq \mathcal{F}$.*

In other words, we prove that every diagonal f of an $\mathbb{N}$-rational generating function, is also a tile counting function.

3.6.1 Paths in networks

Let $\mathrm{W} = (V, E)$ be a directed weighted multi-graph with a unique *source* v_1 and *sink* v_2. Further, assume that the edges of W are colored with k colors. We call such graph a *k-network*. We need the following technical lemma.

Lemma 3.6.2. *Let $G(x_1, \ldots, x_k) \in \mathcal{R}_k$. Then there exists a k-network W, such that for all $(n_1, \ldots, n_k) \in \mathbb{N}^k$, $n_1 + \ldots + n_k \geq 1$, the number of paths from v_1 to v_2 with exactly n_i edges of color i is equal to $\left[x_1^{n_1} \ldots x_k^{n_k}\right] G$.*

Proof. Let $\mathcal{Q}_k$ be the set of GFs, for which there is a k-network as in the lemma. We show that $\mathcal{Q}_k$ satisfies the three conditions in the definition of $\mathbb{N}$-rational generating function, which proves the result.

Condition (1) is trivial. To get $0 \in \mathcal{R}'_k$, take the graph with vertices v_1 and v_2 and no edges. Similarly, to get $x_i \in \mathcal{R}'_k$, take the graph with vertices v_1 and v_2 and a unique edge (v_1, v_2) of color i.

For (2), let $F, G \in \mathcal{Q}_k$, and let U, W be the corresponding k-networks. Attaching sinks and sources, and the rest of U and W in parallel, gives $F + G \in \mathcal{Q}_k$ (see

Figure 3.6). Similarly, if $[1]F = [1]G = 0$, attaching U and W sequentially gives GF $F \cdot G$. More generally, for $a = [1]F$ and $b = [1]G$ not necessarily zero, write $aG = G + \ldots + G$ (a times), and use

$$F \cdot G \,=\, (F - a) \cdot (G - b) \,+\, aG \,+\, bF.$$

to obtain the desired k-network.

For (3), let $F \in \mathcal{Q}_k$ and $[1]F = 0$. To obtain $1/(1 - F)$, write:

$$\frac{1}{1-F} \,=\, 1 + \,F\, +\, F \cdot F\, +\, \frac{F^3}{1-F^2}\, +\, F \,\cdot\, \frac{F^3}{1-F^2}\,.$$

For $\frac{F^3}{1-F}$, arrange four copies of k-network U as shown in Figure 3.6. The details are straightforward. □

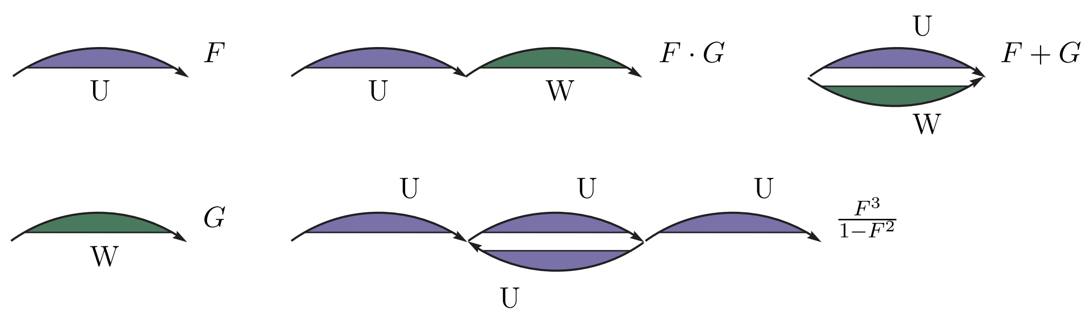


Figure 3.6: Networks giving $F \cdot G$, $F + G$ and $F^3/(1 - F^2)$.

3.6.2 Proof of Lemma 3.6.1

Let $G \in \mathcal{R}_k$ be such that $f(n) = \left[x_1^n \ldots x_k^n\right] G$. By Lemma 3.6.2, there is a k-network W with source v_1 and sink v_2, such that there are exactly $f(n)$ paths from v_1 to v_2, which pass through n edges of color i, for all i.

Let $\varepsilon, \alpha_1, \ldots, \alpha_k > 0$ be irrational numbers, such that the only rational linear dependence between them is $\alpha_1 + \ldots + \alpha_k = 1$. Assign weight α_i to each edge in W with color i. Add vertex v_0 and edges (v_0, v_1) (v_2, v_0), both of weight $\varepsilon/2$. Denote the resulting graph by W_0.

Note that cycles in W_0 which start at v_0 of weight $n+\varepsilon$ are in bijection with paths from v_1 to v_2 in W with exactly n edges of each color. Therefore, there are exactly $f(n)$ of them.

In notation of the proof of Lemma 3.5.2, we associate a different tile boundary ∂_i with vertices v_i, $1 \leq i \leq N$, and the vertical line segment with the vertex v_0. We can always ensure $\mathrm{width}(\partial_i) < \frac{1}{3}\min\{\varepsilon/2, \alpha_1, \ldots, \alpha_k\}$.

For every edge $e = (v_i, v_j)$ in W_0 with weight w_e, denote by τ_e the unique tile with height 1, $\partial_L(\tau_e) = \partial_i$, $\partial_R(\tau_e) = \partial_j$ and area $|\tau_e| = w_e$. Note that such tile exists by the width condition above. Let T be the set of tiles τ_e. From above, for all $n \geq 1$, the number of tilings of $\mathrm{R}_{n+\varepsilon}$ by T is equal to the number of cycles in W_0 starting at v_0 of weight $n+\varepsilon$, which is equal to $f(n)$ by assumption.

When $n = 0$, this tile set has zero tilings. Since $a := [1]F \in \mathbb{N}$, we can make the number of tilings of R_ε equal to a by adding a copies of a $1 \times \varepsilon$ rectangle to T. This does not change the number of tilings for any $n \geq 0$, since every tiling for $n \geq 0$ must already has two tiles of area $\varepsilon/2$, and thus cannot contain more tiles of area ε.

Finally, if T has multiple copies of the same tile, replace each copy with two tiles which only fit together with each other, to make a copy of that tile. We can always do this in such as way to make all new tiles distinct. This implies that $f \in \mathcal{F}$, as desired. $\square$

3.7 Binomial multisums are diagonals of $\mathbb{N}$-rational functions

In this section, we prove the following result towards the proof of Main Theorem 3.3.4.

Lemma 3.7.1. $\mathcal{B} \subseteq \mathcal{N}$.

The proof of the lemma follows easily from five sub-lemmas, three on diagonals and two on binomial multisums. While the former are somewhat standard, the latter are rather technical; we prove them in the next section.

3.7.1 Diagonals

We start with the following three simple results.

Lemma 3.7.2. *The set of diagonals of an* $\mathbb{N}$*-rational generating functions is closed under addition and multiplication.*

Proof. Let $f, g \in \mathcal{N}$. We have

$$f(n) = \left[x_1^n \ldots x_k^n\right] F(x_1, \ldots, x_k), \quad g(n) = \left[y_1^n \ldots y_\ell^n\right] G(y_1, \ldots, y_\ell),$$

for some $F \in \mathcal{R}_k$ and $G \in \mathcal{R}_\ell$. Consider $A(x_1, \ldots x_k, y_1, \ldots, y_\ell)$ defined as

$$A(x_1, \ldots x_k, y_1, \ldots, y_\ell) = \left(\prod_{i=1}^{\ell} \frac{1}{1-y_i}\right) F\big(x_1, \ldots, x_k\big) + \left(\prod_{i=1}^{k} \frac{1}{1-x_i}\right) G\big(y_1, \ldots, y_\ell\big).$$

It follows from the definition of $\mathbb{N}$-rational functions, that $A \in \mathcal{R}_{k+\ell}$. We have

$$\begin{aligned} \left[x_1^n \ldots x_k^n y_1^n \ldots y_\ell^n\right] A \;=\; & \left[x_1^n \ldots x_k^n\right] F\big(x_1, \ldots, x_k\big) \\ & + \left[y_1^n \ldots y_\ell^n\right] G\big(y_1, \ldots, y_\ell\big) \;=\; f(n) + g(n). \end{aligned}$$

Similarly, define

$$B(x_1, \ldots x_k, y_1, \ldots, y_\ell) \,=\, F\big(x_1, \ldots, x_k\big) \cdot G\big(y_1, \ldots, y_\ell\big),$$

and observe that

$$\big[x_1^n \ldots x_k^n y_1^n \ldots y_\ell^n\big]\, B \,=\, f(n) \cdot g(n)\,,$$

as desired. $\square$

A function f is called a *quasi-diagonal* of an $\mathbb{N}$-rational generating function if

$$f(n) \,=\, \big[x_1^{cn} \ldots x_k^{cn}\big]\, F(x_1, \ldots, x_k),$$

for some fixed constant $c \in \mathbb{P}$.

Lemma 3.7.3. *For every function f which is the quasi-diagonal of $F \in \mathcal{R}_k$, there exists $\ell \in \mathbb{P}$ and $G \in \mathcal{R}_\ell$, such that f is the diagonal of G.*

Proof. First, we show that for every $F \in \mathcal{R}_k$, there exists a function $F_\circ \in \mathcal{R}_{k+1}$, such that for all $c_0, c_1, \ldots, c_k \in \mathbb{N}$:

$$\big[x_0^{c_0} \ldots x_k^{c_k}\big]\, F_\circ(x_0, x_1, x_2, \ldots, x_k) = \begin{cases} \big[x_1^{c_0+c_1} x_2^{c_2} \ldots x_k^{c_k}\big]\, F(x_1, \ldots, x_k) & \text{if } c_1 \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We prove this by structural induction in the definition of $\mathcal{R}_k$. For case (1), let $F_\circ = 0$ for $F = x_i$, $i \geq 2$, and $F_\circ = x_1$ for $F = x_1$. In case (2), let $F_\circ = G_\circ + H_\circ$ for $F = G + H$. For $F = G \cdot H$, let

$$F_\circ(x_0, \ldots, x_k) \,=\, G_\circ(x_0, \ldots, x_k)\, H(x_1, x_2, \ldots, x_k) + G(x_0, x_2, \ldots, x_k)\, H_\circ(x_0, \ldots, x_k).$$

For case (3), for $F = 1/(1-G)$, let

$$F_\circ(x_0,\ldots,x_k) \;=\; F(x_0,x_2,\ldots,x_k)G_\circ(x_0,\ldots,x_k)F(x_1,x_2,\ldots,x_k).$$

It is clear that this works for case (1), for $F = G + H$, and when $c_1 = 0$, so we only need to worry about the cases where $c_1 > 0$ and where $F = G \cdot H$ or $F = 1/(1-G)$.

For $F = G \cdot H$, recall that

$$\left[x_1^{c_0+c_1}x_2^{c_2}\ldots x_k^{c_k}\right] F(x_1,\ldots,x_k)$$

equals the sum over all $(d_1+e_1,\ldots,d_k+e_k) = (c_0+c_1,c_2,\ldots,c_k)$

$$\left[x_1^{d_1}x_2^{d_2}\ldots x_k^{d_k}\right] G(x_1,\ldots,x_k)\left[x_1^{e_1}x_2^{e_2}\ldots x_k^{e_k}\right] H(x_1,\ldots,x_k).$$

Note that in $F_\circ$, all instances of x_0 in $G_\circ(x_0,\ldots,x_k)H(x_1,x_2,\ldots,x_k)$, come from the $G_\circ(x_0,\ldots,x_k)$. Thus, if there are d_1 instances of x_0 or x_1 in the first summand, exactly d_1-c_0 instances of x_1 come from $G_\circ$. Similarly, if there are e_1 instances of x_0 or x_1 in the second summand, exactly e_1-c_1 instances of x_0 must come from $H_\circ$.

We break the contributions to $F_\circ$ into two cases: (a) with $d_1 > c_0$, and (b) with $d_1 \le c_0$. In the case (a), note that

$$\left[x_0^{c_0}x_1^{d_1-c_0}x_2^{d_2}\ldots x_k^{d_k}\right] G_\circ(x_0,\ldots,x_k) \;=\; \left[x_1^{d_1}x_2^{d_2}\ldots x_k^{d_k}\right] G(x_1,\ldots,x_k),$$

and

$$\left[x_0^{e_1-c_1}x_1^{c_1}x_2^{e_2}\ldots x_k^{e_k}\right] H_\circ(x_0,\ldots,x_k) \;=\; 0.$$

In the case (b), we similarly have:

$$\left[x_0^{c_0} x_1^{d_1 - c_0} x_2^{d_2} \ldots x_k^{d_k}\right] G_\circ(x_0, \ldots, x_k) \,=\, 0,$$

and

$$\left[x_0^{e_1 - c_1} x_1^{c_1} x_2^{e_2} \ldots x_k^{e_k}\right] H_\circ(x_0, \ldots, x_k) \,=\, \left[x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}\right] H(x_1, \ldots, x_k).$$

Therefore, in the case (a), we have

$$\left[x_0^{c_0} x_1^{d_1 - c_0} x_2^{d_2} \ldots x_k^{d_k}\right] G_\circ(x_0, \ldots, x_k) \left[x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}\right] H(x_1, \ldots, x_k)$$

$$= \left[x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k}\right] G(x_1, \ldots, x_k) \left[x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}\right] H(x_1, \ldots, x_k),$$

and

$$\left[x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k}\right] G(x_1, \ldots, x_k) \left[x_0^{e_1 - c_1} x_1^{c_1} x_2^{e_2} \ldots x_k^{e_k}\right] H_\circ(x_0, \ldots, x_k) \,=\, 0.$$

In the case (b), we get a similar result with the r.h.s.'s interchanged. We conclude:

$$\left[x_0^{c_0} x_1^{c_1} x_2^{c_2} \ldots x_k^{c_k}\right] F_\circ(x_0, x_1, x_2, \ldots, x_k) \,=\, \left[x_1^{c_0 + c_1} x_2^{c_2} \ldots x_k^{c_k}\right] F(x_1, \ldots, x_k).$$

For case (3), we think of any contribution to F as coming from G^r for some $r \in \mathbb{N}$, and break into cases based on how many copies of G we go through in this G^r before we have seen more than c_0 instances of x_1. We then proceed similarly.

Now, for every fixed $m \geq 2$, $F \in \mathcal{R}_k$ and a function $f(n)$ defined as

$$f(n) \,=\, \left[x_1^{mn} \ldots x_k^{mn}\right] F(x_1, \ldots, x_k),$$

we may recursively apply the above result to split each variable x_i into m variables

x_{ij}, for $j = 1, \ldots, m$. We get a function $G \in \mathcal{R}_{mk}$ satisfying

$$\left[x_{11}^{c_{11}} x_{12}^{c_{12}} \cdots x_{km}^{c_{km}}\right] G(x_{11}, x_{12}, \ldots, x_{km}) \;=\; \left[x_1^{c_1} \ldots x_k^{c_k}\right] F(x_1, \ldots, x_k),$$

whenever $c_{i1} + \ldots + c_{im} = c_i$ and $c_{ij} \geq 1$, for all i and j. In particular, for all $c_{ij} = n \geq 1$, this gives

$$\left[x_{11}^{n} x_{12}^{n} \cdots x_{km}^{n}\right] G(x_{11}, x_{12}, \ldots, x_{km}) \;=\; f(n).$$

Further $[1]G = 0$, so we can simply add the constant term $f(0)$ to get the desired function with the diagonal $f(n)$. $\square$

Lemma 3.7.4. *Let $f \in \mathcal{N}$ and $g : \mathbb{N} \to \mathbb{N}$ satisfy $f(n) = g(n)$ for all $n \geq 1$. Then $g \in \mathcal{N}$.*

Proof. The functions

$$j(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad h(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{otherwise} \end{cases}$$

are trivially diagonals of functions 1 and $x/(1-x) \in \mathcal{R}_1$. Writing

$$g(n) \;=\; g(0)j(n) \cdot f(n) \;+\; h(n) \cdot f(n)\,,$$

implies the result by Lemma 3.7.2.

3.7.2 Finiteness of binomial multisums

Next, we find a bound on which terms can contribute to a binomial multisum.

Lemma 3.7.5. *In notation of* §3.3.3, *let*

$$f(n) \;=\; \sum_{v\in\mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)}$$

be finite for all $n \in \mathbb{N}$. Then there exists a constant $c \in \mathbb{N}$, such that for all $n \in \mathbb{P}$, and $|v_i| > cn$ for all i, the product on the right hand side is zero.

Finally, we show that binomial sums bounded as in Lemma 3.7.5 are in fact quasi-diagonals of $\mathbb{N}$-rational generating functions.

Lemma 3.7.6. *In notation of* §3.3.3, *let $f(n) : \mathbb{N} \to \mathbb{N}$ be defined as*

$$f(n) \;=\; \sum_{v\in\mathbb{Z}^d, |v_i|\le cn} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

for all $n \in \mathbb{P}$. Then $f(n)$ agrees with a quasi-diagonal of an $\mathbb{N}$-rational generating function at all $n \geq 1$.

Both lemmas are proved in the next section.

3.7.3 Proof of Lemma 3.7.1

By Lemmas 3.7.5 and 3.7.6, every function $f(n)$ as in Lemma 3.7.5 agrees with a quasi-diagonal of an $\mathbb{N}$-rational generating function at all $n \geq 1$. By Lemma 3.7.3, any function of this form agrees with a diagonal of an $\mathbb{N}$-rational generating function at all $n \geq 1$. By Lemma 3.7.4, any such function is in $\mathcal{N}$. □

3.8 Proofs of lemmas 3.7.5 and 3.7.6

3.8.1 A geometric lemma

We first need the following simple result; we include a short proof for completeness.

Lemma 3.8.1. *Let $\alpha_1, \ldots, \alpha_r : \mathbb{R}^d \to \mathbb{R}$ be integer coefficient affine functions. Let $P \subset \mathbb{R}^d$ be the (possibly unbounded) polyhedron of points satisfying $\alpha_i \geq 0$ for all i. If P contains a positive finite number of integer lattice points, then P is bounded.*

Proof. Suppose P is not bounded. Without loss of generality, assume that at the origin $O \in P$. Consider the *base cone* C_P of all infinite rays in P starting at O (see e.g. [Pak, §25.5]). Since P is a rational polyhedron, the cone C_P is also rational and contains at least one ray of rational slope. This ray contains an integer point and has a rational slope, and therefore contains infinitely many integer points, a contradiction. □

3.8.2 Proof of Lemma 3.7.5

Let S be a subset of $\{1, \ldots, r\}$, and let P_S be the set of all points $(v, n) \in \mathbb{R}^{d+1}$ satisfying $\alpha_i(v, n), \beta_i(v, n) \geq 0$ for $i \notin S$, satisfying $\alpha_i(v, n) = -1$ and $\beta_i(v, n) = 0$ for $i \in S$, and satisfying $n \geq 0$. Let $|v|$ denote $\max_i |v_i|$.

Note that we have 2^r polytopes P_S, and the integer lattice points in P_S form a cover for the set of all $(v, n) \in \mathbb{Z}^{d+1}$ which contribute a positive amount to $f(n)$. For each P_S, we prove that there exists a constant c such that any integer lattice point in P_S, with $n \geq 1$ satisfies $|v_i| \leq cn$. Since f is finite, we know that P_S contains finitely many integer lattice points for any fixed value of n.

We can assume that there are two distinct values $n_1 < n_2$, such that there exist integer lattice points in P_S with $n = n_1$ and with $n = n_2$, since otherwise P_S only contains finitely many lattice points. Let (v_1, n_1) be an integer point in P_S. Consider the set of all points in P_S satisfying $n = n_2$. This is a not necessarily bounded polytope with a positive finite number of integer lattice points, so by Lemma 3.8.1, it is bounded. Thus, there exists a c_S, such that $|v_2 - v_1| < c_S$ for all (v_2, n_2) in P_S.

This implies that $|v - v_1| < c_S(n - n_1)$ for all (v, n) in P_S with $n > n_2$. Indeed, otherwise the line segment connecting (v_1, n_1) to (v, n) would intersect the hyperplane $n = n_2$ at a point (v_2, n_2) in P_S but not satisfying $|v_2 - v_1| < c_S$.

Take a c'_S such that $c'_S > c$, and all finitely many integer lattice points (v, n) in P_S with $1 \leq n < n_2$, including (v_1, n_1), satisfy $|v| \leq c'_S n$. Then, all integer lattice points (v, n) in P_S with $n \geq 1$, satisfy $|v| \leq c'_S n$. Taking $c = \max_S\{c'_S\}$, proves the result. $\square$

3.8.3 Proof of Lemma 3.7.6

Let $f(n) : \mathbb{N} \to \mathbb{N}$ be a function such that for all $n \geq 1$,

$$f(n) = \sum_{v \in \mathbb{Z}^d, |v_i| \leq cn} \prod_{i=1}^{r} \binom{\alpha_i(v, n)}{\beta_i(v, n)},$$

where α_i and β_i are integer coefficient affine functions of v and n. We construct a function g in $d + 2r + 1$ variables $x_1, \ldots, x_d, a_1, \ldots a_r, b_1, \ldots, b_r, y$. Let

$$G(x_1, \ldots, x_d, a_1, \ldots a_r, b_1, \ldots, b_r, y) \;=\; \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \cdot \Pi_4\,,$$

where

$$\Pi_1 = \prod_{j=1}^{d} \frac{1}{1 - x_j h_j}\,, \qquad \Pi_2 = \prod_{j=1}^{d} \frac{1}{1 - x_j h'_j}\,,$$

$$\Pi_3 = \prod_{i=1}^{r} \left(1 + a_i \frac{a_i + b_i a_i}{1 - (a_i + b_i a_i)}\right)\,, \quad \text{and} \quad \Pi_4 = yq \frac{1}{1 - yq'}\,.$$

The terms h_j, h'_j, q, and q' are monomials in variables a_i and b_i, to be determined later.

We consider the coefficients of terms in which the exponent on each x_j variable is $2cn$. The Π_1 part will contribute some number of these x_j factors, and the Π_2 will contribute the rest. This choice will represent the variable v_j. We will use

$cn+v_j$ to denote the number of factors of x_j coming from Π_1, so then $cn-v_j$ will be the number of factors of x_j coming from Π_2. Note that v_j can be any integer between $-cn$ and cn.

Define all the monomials in such a way that the a_i monomial needs to be repeated $\alpha_i(v,n)+1$ times in the Π_3 term, while that b_i monomial needs to be repeated $\beta_i(v,n)$ times in the Π_3 term. By the definition in §3.2.1, this is exactly $\binom{\alpha_i}{\beta_i}$.

We choose the monomials h_j, h'_j, q, and q' a follows. Let

$$\beta_i(v,n) = \beta_{i,0} + \beta_{i,1}v_1 + \ldots + \beta_{i,d}v_d + \beta_{i,d+1}n.$$

First, consider the case where $\beta_{i,j} \leq 0$, for all $0 \leq j \leq d$. In this case, we put b_i in h_j with multiplicity $|\beta_{i,j}|$, put β_i in q with multiplicity $|\beta_{0,j}|$, and not put any b_i terms in h'_j or q'. This implies that outside of the Π_3 term, the number of times b_i appears is exactly

$$-\beta_{i,0} + \sum_{j=1}^{d}(cn+v_j)(-\beta_{i,j}) \;=\; n\left(-\beta_{i,d+1} - \sum_{j=1}^{d} c\beta_{i,j}\right) - \beta(v,n).$$

Thus, for the coefficient of a term with total multiplicity $n\left(-\beta_{i,d+1} - \sum_{j=1}^{d} c\beta_{i,j}\right)$ of b_i, we have $\beta(v,n)$ of the b_i terms must come from the Π_3 term.

We only consider coefficients where the multiplicity of the y term is n. If $\beta_{i,0}$ is positive, then we can swap the multiplicity of b_i in q and q'. Since the q' term is necessarily repeated $n-1$ times in the Π_4 term, this implies that outside of the Π_3 term, b_i appears exactly

$$(n-1)\beta_{i,0} + \sum_{j=1}^{d}(cn+v_j)(-\beta_{i,j}) \;=\; n\left(\beta_{i,0} - \beta_{i,d+1} - \sum_{j=1}^{d} c\beta_{i,j}\right) - \beta(v,n)$$

times. Therefore, for the coefficient of a term with $n\left(\beta_{i,0} - \beta_{i,d+1} - \sum_{j=1}^{d} c\beta_{i,j}\right)$

total multiplicity of b_i, we again have $\beta(v,n)$ of the b_i terms must come from the Π_3 term.

If any of the $\beta_{i,j}$ terms, with $1 \le j \le d$ are actually positive, we swap the multiplicity of b_i in h_j and h'_j, which gives the same analysis with v_j negated. Using the same method, we can require that the number of a_i terms coming from the Π_3 term is $\alpha_i + 1$, for all i.

In summary,

$$f(n) = \left[x_1^{nc_1} \dots x_d^{nc_d} a_1^{nc_{d+1}} \dots a_r^{nc_{d+r}} b_1^{nc_{d+r+1}} \dots b_r^{nc_{d+2r}} y^{nc_{d+2r+1}}\right] G.$$

Take c' to be a common multiple of all c_i and make a substitution $x_i \leftarrow x_i^{c'/c_i}$, $a_j \leftarrow a_j^{c'/c_{d+j}}$, etc. This gives a desired quasi-diagonal. $\square$

3.9 Proof of Theorem 3.4.2

3.9.1 Preliminaries

We start with the following simple result:

Lemma 3.9.1. *Let $f \in \mathcal{F}$ be a tile counting function. Then $f(n) \le C^n$, for all $n \in \mathbb{P}$ for some $C > 0$.*

Proof. Let $T = \{\tau_1, \dots, \tau_s\}$, and let $\mu = \min_i |\tau_i|$ be the minimum area of a tile in T. Every tiling of $\mathrm{R}_{n+\varepsilon}$ with T corresponds to a unique sequence of tiles in the tilings, listed from left to right. The length of this sequence is at most $(n+\varepsilon)/m$. Therefore, $f_T(n) \le (s+1)^{(n+\varepsilon)/\mu} = e^{O(n)}$. $\square$

Theorem 3.4.2 shows that this upper bound is usually tight, and every function growing slower than this must be eventually quasi-polynomial. The following lemma is a special case of Theorem 1.1 in [CLS];

Lemma 3.9.2 ([CLS]). *Let $g(n)$ be the number of integer points $(x_1, \ldots, x_r, n) \in \mathbb{Z}^{r+1}$ satisfying m inequalities $a_i(x, n) > c_i$ where a_i is an integer coefficient linear function and c_i is an integer for all $i = 1, \ldots, m$. Then $g(n)$ is eventually quasi-polynomial.*

The proof of the lemma uses a generalization of *Ehrhart polynomials*. We refer to [Bar] for a review of the area and further references.

3.9.2 Proof setup

From Main Theorem 3.3.4, function f can be expressed as

$$(\circledast) \qquad f(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

where each α_i and β_i is an integer coefficient affine function of v and n.

From Lemma 3.9.1, function $f \le e^{cn}$ for some c. Therefore, it suffice to show that f is either greater than e^{cn} for some $c > 0$, or is eventually polynomial. Furthermore, it suffices to show that f is either greater than e^{cn} for some c, or eventually quasi-polynomial. We can decompose f into even more functions, by multiplying p by the periods of all of the quasi-polynomials, which proves that each component function that does not grow exponentially is eventually polynomial.

Denote by k the number of indices i in $(\circledast)$, such that α_i, β_i, and $\gamma_i = \alpha_i - \beta_i$ are three non-constant functions. We use induction on k.

3.9.3 Step of Induction

Let M be a constant integer satisfying $M > |\beta_i(0,0)|, |\gamma_i(0,0)|$, for all i. We decompose f into a sum of $(M+2)^{2r}$ functions depending on the values of β_i and γ_i, for all i. For each β_i and γ_i, we either require that $\beta_i \ge M$ or that the value of

β_i be some constant $< M$. There are $M+2$ possibilities for each function, since only values ≥ -1 give non-zero binomial coefficients, giving $(M+2)^{2r}$ bound as above.

To ensure that $\beta_i = z \geq -1$ for some constant z, we replace $\beta_i(v,n)$ by z, and multiply the binomial coefficients $\binom{\beta_i(v,n)+1}{z+1}$ and $\binom{z+1}{\beta_i(v,n)+1}$ to the existing product. This works because $\binom{\beta_i(v,n)+1}{z+1}\binom{z+1}{\beta_i(v,n)+1}$ is 1 if $\gamma_i = z$, and 0 otherwise.

Similarly, to ensure that $\gamma_i = z \geq -1$, we replace $\alpha_i(v,n)$ with $\beta_i + z$, and multiply the binomial coefficients $\binom{\gamma_i(v,n)+1}{z+1}$ and $\binom{z+1}{\gamma_i(v,n)+1}$ to the existing product.

Finally, to ensure that $\beta_i \geq M$, we multiply the binomial coefficient $\binom{\beta_i - M - 1}{0}$ to the existing product. This binomial coefficient is 1 if $\beta_i - M - 1 \geq -1$, and 0 otherwise. Similarly for enforcing that $\gamma_i \geq M$.

Note that when we require that β_i or γ_i equal to a constant, we reduce k by 1, and when we specify that $\beta_i \geq M$ or $\gamma_i \geq M$, we keep k the same. Therefore of these $(M+2)^{2r}$ functions which add to g, we get by induction that all but one of them is quasi-polynomial. The only one we have to worry about is the function g in which each β_i and γ_i is specified to be at least $\mathcal{M}$, and we show that this function is identically 0.

Assume by way of contradiction that g is not identically 0. Then, there exists some (v_1, n_1) such that

$$\prod_{i=1}^{r} \binom{\alpha_i(v_1,n_1)}{\beta_i(v_1,n_1)} > 0, \quad \beta_i(v_1,n_1) - \beta_i(0,0) > 0, \quad \text{and} \quad \gamma_i(v_1,n_1) - \gamma_i(0,0) > 0,$$

for all i. Adding (v_1, n_1) to any point (v, n) would increase every $\beta_i(v,n)$ and $\gamma_i(v,n)$ by at least 1. Consider the sequence of points $(v_t, n_t) = (tv_1, tn_1)$, where t is a positive integer. Note that every $\beta_i(v_t, n_t)$ and every $\gamma_i(v_t, n_t)$ is at least t. Therefore, $\binom{\alpha_1(v_t,n_t)}{\beta_1(v_t,n_t)} \geq \binom{2t}{t}$, so $f(n_1 t) \geq \binom{2t}{t} \geq 2^t$, contradicting the fact that $f(n) < e^{cn}$ for all positive c.

3.9.4 Base of Induction:

Now consider the case $k = 0$. In notation of (⊛), this means

$$f(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

where for each i, at least one of α_i, β_i, and γ_i is constant. Without loss of generality, we can assume that either α_i or β_i are constant.

When α_i is a nonzero constant, then we can write f as a sum of functions where we condition on the value of β_i to be z, by replacing β_i with z and multiplying the existing product by $\binom{0}{\beta_i(v,n)-z}$. Since $\binom{\alpha_i}{z}$ is a constant, we can again express our functions as a sum of $\binom{\alpha_i}{z}$ copies of that function with the $\binom{\alpha_i}{z}$ term removed. Therefore, we can assume that if α_i is constant, that constant is 0.

We can also replace every $\binom{0}{\beta_i(v,n)}$ with $\binom{\beta_i(v,n)-1}{0}\binom{-\beta_i(v,n)-1}{0}$, since we are replacing the indicator that $\beta_i = 0$ with the indicators $\beta_i - 1 \geq -1$ and $-\beta_i - 1 \geq -1$. Therefore, we may assume that every β_i is a constant.

In summary,

$$f(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r_1} \binom{\alpha_i(v,n)}{0} \prod_{i=r_1+1}^{r} \binom{\alpha_i(v,n)}{z_i},$$

where each α_i and β_i is an integer coefficient affine function of v and n, and each $z_i \in \mathbb{P}$.

Note that each $\binom{\alpha_i(v,n)}{0}$ term is just an indicator function that $\alpha_i(v,n) \geq -1$. Therefore,

$$f(n) = \sum_{v \in P_n} \prod_{i=r_1+1}^{r_2} \binom{\alpha_i(v,n)}{z_i},$$

where P_n is the polytope of all integer points such that $\alpha_i(v,n) \geq -1$ for all $1 \leq i \leq d_1$.

Also, note that $\binom{\alpha_i(v,n)}{z_i}$ is equal to the number of integer points $(x_1, \ldots, x_{z_i})$, such that

$$0 \le x_1 < x_2 < \ldots < x_{z_i} < \alpha_i(v, n).$$

Therefore, $f(n)$ is equal to the number of points in the polytope

$$P_n \times \mathbb{Z}^{z_{r_1+1}} \times \ldots \times \mathbb{Z}^{z_r},$$

where v is the point in P_n, and the coordinates $(x_1, \ldots, x_{z_i})$ satisfying

$$0 \le x_1 < x_2 < \ldots < x_{z_i} < \alpha_i(v, n).$$

By Lemma 3.9.2, $f(n)$ is eventually quasi-polynomial. This proves the base of induction, and completes the proof of Theorem 3.4.2.

3.10 Proofs of applications

3.10.1 Proof of Theorem 3.4.1

First, we show that $\mathcal{B}' \subseteq \mathcal{B}$. Since $\mathcal{B}$ is closed under addition, it suffices to show that every balanced multisum

$$g(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \frac{\alpha_i(v, n)!}{\beta_i(v, n)!\gamma_i(v, n)!}$$

is in $\mathcal{B}$. This follows since

$$\frac{\alpha_i(v, n)!}{\beta_i(v, n)!\gamma_i(v, n)!} = \binom{\alpha_i(v, n)}{\beta_i(v, n)} \binom{\alpha_i(v, n) - 1}{0}.$$

The second factor ensures that $\alpha_i \ge 0$, so the first factor is never $\binom{-1}{0}$.

To show that $\mathcal{B} \subseteq \mathcal{B}'$, take

$$g(n) \,=\, \sum_{v\in\mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)}.$$

Denote by S the set of all subsets of $\{1,\ldots,r\}$, and $\gamma_i = \alpha_i - \beta_i$. For each $s \in S$, let

$$g_s(n) \,=\, \sum_{v\in\mathbb{Z}^d} e_s(v,n)\cdot \prod_{i\in s} \frac{\alpha_i(v,n)!}{\beta_i(v,n)!\gamma_i(v,n))!}\,, \qquad \text{where}$$

$$e_s(v,n) \,=\, \prod_{i\notin s} \left[\frac{0!}{(\alpha_i(v,n)+1)!(-\alpha_i(v,n)-1)!}\right]\left[\frac{0!}{\beta_i(v,n)!(-\beta_i(v,n))!}\right].$$

Observe that $e_s(v,n) = 1$ if $\alpha_i(v,n) = -1$ and $\beta_i(v,n) = 0$, and $e_s(v,n) = 0$ otherwise.

For every v, let s_v be the set of indices $i \in \{1,\ldots,r\}$ for which $\alpha_i(v,n), \beta_i(v,n) \geq 0$. Then

$$e_s(v,n)\cdot \prod_{i\in s} \frac{\alpha_i(v,n)!}{\beta_i(v,n)!\gamma_i(v,n)!} \,=\, \binom{\alpha_i(v,n)}{\beta_i(v,n)} \quad \text{when} \quad s = s_v\,,$$

and 0 otherwise. Therefore,

$$g(n) \,=\, \sum_{s\in S} g_s(n)\,.$$

This implies that $g \in \mathcal{B}'$, and completes the proof. $\square$

3.10.2 Getting close to Catalan numbers

Before we prove Proposition 3.4.7, we need the following weaker result.

Lemma 3.10.1. *There exists a tile counting function f such that*

$$f(n) \sim \frac{3\sqrt{3}}{\pi} C_n\,, \qquad \textit{as} \quad n \to \infty\,.$$

Proof. Consider the following three binomial multisums $f_1, f_2, f_3 \in \mathcal{B}$:

$$f_1(n) \;=\; \sum_{v\in\mathbb{Z}} \binom{n}{3v}\binom{3v}{n}\binom{2v}{v}^3, \quad f_2(n) \;=\; 4\sum_{v\in\mathbb{Z}} \binom{n-1}{3v}\binom{3v}{n-1}\binom{2v}{v}^3,$$

$$\text{and} \quad f_3(n) \;=\; 16\sum_{v\in\mathbb{Z}} \binom{n-2}{3v}\binom{3v}{n-2}\binom{2v}{v}^3.$$

Let $f = f_1 + f_2 + f_3$. By Main Theorem 3.3.4 and Corollary 3.3.5, we know that $f \in \mathcal{F}$.

Observe that $f_1(n) \neq 0$ only when n is a multiple of 3. We have:

$$f_1(n) \;=\; \binom{2n/3}{n/3}^3 \sim \left(\frac{4^{n/3}}{\sqrt{n/3}\sqrt{\pi}}\right)^3 \sim \frac{3\sqrt{3}}{\pi}\,C_n\,, \qquad \text{for } \; 3|n.$$

Analyzing f_2 and f_3 gives the same result when $n = 1, 2 \bmod 3$, respectively. $\square$

3.10.3 Proof of Proposition 3.4.7

Let $f \in \mathcal{F}$ be the tile counting function from Lemma 3.10.1. For each $i \in \mathbb{N}$, let $g_i(n) = f(n-i)$ if $n \geq i$, and let $g_i(n) = 0$ otherwise. Each g_i is also a tile counting function, since we can take the exact same tile set, and replace ε with $\varepsilon + i$.

Denote $\xi = 3\sqrt{3}/\pi$. Note that $g_i(n) \sim f(n)/4^i \sim C_n\xi/4^i$. Given any $\varepsilon > 0$, we can take i large enough so that $\xi/4^i < \varepsilon$, and $m \in \mathbb{P}$ such that $1-\varepsilon < m\xi/4^i < 1+\varepsilon$. This gives $mg_i(n) \sim C_n\xi m/4^i$ which is between $1-\varepsilon$ and $1+\varepsilon$, as desired. Finally, we have $mg_i \in \mathcal{F}$ since $\mathcal{B} = \mathcal{F}$ is closed under addition. $\square$

3.10.4 Proof of Proposition 3.4.8

Given an $m \geq 1$, let

$$f(n) = \binom{2n}{n} + (m-1)\binom{2n}{n+1}.$$

Note that f is a tile counting function, since it is a finite sum of binomial coefficients of affine functions of n. Since $C_n = \binom{2n}{n} - \binom{2n}{n+1}$, we have that $f(n)$ and C_n differ by $m\binom{2n}{n}$, and are therefore congruent modulo m. $\square$

3.10.5 Proof of Proposition 3.4.9

Given a prime $p \geq 2$, let

$$f(n) = \binom{2n}{n} + (p^{2n}-1)\binom{2n}{n+1}.$$

By Corollary 3.3.6, $p^{2n} - 1 \in \mathcal{B}$, and binomial coefficients are in $\mathcal{B}$ by definition. Since $\mathcal{B}$ is closed under addition and multiplication, we obtain $f \in \mathcal{B}$. Note that $p^{2n} > C_n$, so adding or subtracting an integer multiple of p^{2n} to C_n does not change the order of p. Therefore,

$$\mathrm{ord}_p(C_n) \;=\; \mathrm{ord}_p\left(C_n + p^{2n}\binom{2n}{n+1}\right) \;=\; \mathrm{ord}_p(f(n)),$$

as desired. $\square$

3.10.6 Proof of Theorem 3.4.10

We start with the case where $k = 1$ and $\ell = 2$. Let $r = r_1/r_2$, $c = \mu_1^{\mu_1} r_2$, and let

$$f(n) = \sum_{\ell=0}^{n} \binom{\mu_1 \ell}{\nu_1 \ell} c^{n-\ell} (\nu_1^{\nu_1} \nu_2^{\nu_2} r_1)^{\ell} = \sum_{\ell \in \mathbb{Z}} \binom{\ell-1}{0} \binom{n-\ell-1}{0} \binom{\mu_1 \ell}{\nu_1 \ell} c^{n-\ell} (\nu_1^{\nu_1} \nu_2^{\nu_2} r_1)^{\ell}.$$

First, let us prove that f is a tile counting function. Replace $c^{n-\ell}$ with

$$\sum_{v_1,\ldots,v_c\in\mathbb{Z}} \binom{n-\ell}{v_1}\binom{n-\ell-v_1}{v_2}\cdots\binom{n-\ell-v_1-v_2-\ldots-v_{c-1}}{v_c}.$$

We can ignore the fact that $\binom{-1}{0} = 1$, since if we take the least i such that $n-\ell-v_1-v_2-\ldots-v_i = -1$, we have $\binom{n-\ell-v_1-v_2-\ldots-v_{i-1}}{v_i} = 0$. We then make a similar replacement for $(\nu_1^{\nu_1}\nu_2^{\nu_2}r_1)^\ell$. Therefore, $f \in \mathcal{F}$ by the Main Theorem 3.3.4.

Letting

$$g(\ell) = \binom{\mu_1\ell}{\nu_1\ell} c^{-\ell}(\nu_1^{\nu_1}\nu_2^{\nu_2}r_1)^\ell,$$

we get

$$f(n) = \sum_{\ell=0}^{n} g(\ell)c^n.$$

Note that $g(0) = 1$, and

$$g(\ell{+}1)/g(\ell) = \frac{\nu_1^{\nu_1}\nu_2^{\nu_2}\, r\, \prod_{i=1}^{\mu_1}(\mu_1\ell+i)}{\mu_1^{\mu_1}\, \prod_{i=1}^{\nu_1}(\nu_1\ell+i)\, \prod_{i=1}^{\nu_2}(\nu_2\ell+i)} = \frac{(\ell+a_1)(\ell+a_2)\ldots(\ell+a_p)r}{(\ell+b_1)(\ell+b_2)\ldots(\ell+b_p)}.$$

Therefore,

$$f(n)/c^n = \sum_{\ell=0}^{n}\prod_{k=0}^{\ell-1}\frac{(k+a_1)(k+a_2)\ldots(k+a_p)r}{(k+b_1)(k+b_2)\ldots(k+b_p)} \to A \quad \text{as} \quad n\to\infty.$$

In general, let each part μ_i be subdivided into ℓ_i parts $\nu_{i,1},\ldots,\nu_{i,\ell_i}$. Let $r = r_1/r_2$, and let $c = (\mu_1)^{\mu_1}\ldots(\mu_k)^{\mu_k}r_2$. Similarly to in the previous case, we define f as

$$f(n) = \sum_{\ell=0}^{n} c^n r_1^\ell r_2^{-\ell}\prod_{i=1}^{p} g_i(\ell),$$

where each

$$g_i(\ell) = \binom{\mu_i\ell}{\nu_{i,1}}\binom{\mu_i\ell-\nu_{i,1}\ell}{\nu_{i,2}\ell}\cdots\binom{\mu_i\ell-\nu_{i,1}\ell-\ldots-\nu_{i,\ell_i-1}\ell}{\nu_{i,l_i}}(\mu_i^{\mu_i})^{-\ell}\left(\nu_{i,1}^{\nu_{i,1}}\ldots\nu_{i,\ell_i}^{\nu_{i,\ell_i}}\right)^\ell.$$

This is a tile counting function for reasons similar to in the previous case. Note that all the negative exponents are canceled out by the c^n term. We have:

$$g_i(\ell+1)/g(\ell) \;=\; \frac{\nu_{i,1}^{\nu_{i,1}} \ldots \nu_{i,\ell_i}^{\nu_{i,\ell_i}} \prod_{j=1}^{\mu_i} (\mu_i \ell + j)}{\mu_i^{\mu_i} \prod_{j=1}^{\nu_{i,1}} (\nu_{i,1}\ell + j) \cdots \prod_{j=1}^{\nu_{i,\ell_1}} (\nu_{i,\ell_i}\ell + j)}\,.$$

Therefore,

$$\frac{r_1^{\ell+1} r_2^{-(\ell+1)} \prod_{i=1}^{p} g_i(\ell)}{r_1^{\ell} r_2^{-\ell} \prod_{i=1}^{p} g_i(\ell)} = \frac{(\ell+a_1)(\ell+a_2)\,\cdots\,(\ell+a_p)r}{(\ell+b_1)(\ell+b_2)\ldots(\ell+b_p)}\,.$$

Since

$$r_1^0 r_2^0 \prod_{i=1}^{p} g_i(0) \;=\; 1\,,$$

we have

$$f(n)/c^n = \sum_{\ell=0}^{n} \prod_{k=0}^{\ell-1} \frac{(k+a_1)(k+a_2)\ldots(k+a_p)r}{(k+b_1)(k+b_2)\ldots(k+b_p)} \;\to\; A \quad \text{as} \quad n\to\infty\,.$$

The base of exponent we get from this construction is $c = (\mu_1)^{\mu_1}\ldots(\mu_k)^{\mu_k} r_2$. However, note that it is easy to multiply c by any positive integer N, simply by multiplying f by N^n. In particular, let L be the product of all primes which are factors of $\mu_1 \ldots \mu_k r_2$. Then there exists some positive integer d, such that L^d is a multiple of $(\mu_1)^{\mu_1}\ldots(\mu_k)^{\mu_k} r_2$. This implies that there exists a function $h \in \mathcal{F}$ with $h(n) \sim AL^{dn}$.

Note now, that we can scale all the tiles horizontally by d, and scale ε by d, to get a new function $f_0(n)$ such that $f_0(dn) = h(n)$. We may assume that $f_0(n) = 0$ when n is not a multiple of d, because we can multiply f_0 by the indicator that $d|n$. We can similarly get a function f_i for $i = 1,\ldots,d-1$, such that $f_i(n)$ is nonzero only when $n = i \bmod d$, and $f_i(nd+i) = L^i h(n)$. We have

$$f(n) = \sum_{i=0}^{d-1} f_i(n) \in \mathcal{F},$$

and by the way we constructed f we obtain $f(n) \sim AL^n$. Thus, we can take $c = L$ or any integer multiple of L, as desired. $\square$

3.11 Final Remarks

3.11.1

The idea of irrational tilings was first introduced by Korn, who found a bijection between *Baxter permutations* and tilings of large rectangles with three fixed irrational rectangles [Korn, §6].

3.11.2

For Theorem 3.1.1, much of the credit goes to Schützenberger [Schü] who proved the equivalence between *regular languages* and the (weakest in power) *deterministic finite automata* (DFA). He used the earlier work of Kleene (1956) and the language of semirings; the GF reformulation in the language of $\mathbb{N}$-rational functions came later, see [SS]. We refer to [BR1, SS] for a thorough treatment of the subject and connections to GFs, and to [Pin] for a more recent survey.

Now, the relationship between (polyomino) tilings of the strip, regular languages (as well as DFAs) were proved more recently in [BL, MSV]. Theorem 3.1.1 now follows as combination of these results in several different ways.

Let us mention here that in the usual polyomino tiling setting there is no height condition, so in fact Theorem 3.1.1 remains unchanged when rational tiles of smaller height are allowed. In the irrational tiling setting, the standard "finite number of cut paths" argument fails. Still, we conjecture that Theorem 3.1.2 also extends to tiles with smaller heights.

3.11.3

The history of Theorem 3.4.3 is somewhat confusing. In fact, it holds for integer G-*sequences* defined as integer D-*finite* (*holonomic*) *sequences* with at most at most exponential growth. It is stated in this form since diagonals of all rational GFs are D-finite [Ges1] and at most exponential. We refer to [FS, Sta1] for more on D-finite sequences, examples and applications, and to [DGS, Gar] for G-sequences.

The asymptotics of D-finite GFs go back to Birkhoff and Trjitzinsky (1932), and Turrittin (1960). See [FS, §VIII.7] and [Odl, §9.2] for various formulations of general asymptotic estimates, and an extensive discussion of priority and validity issues. However, for G-sequences, the result seems to be accepted and well understood, see [BRS, §2.2] and [Gar].

3.11.4

Note that D-finite sequences can be superexponential, e.g. $n!$. They can also have $\exp(n^\gamma)$ terms with $\gamma \in \mathbb{Q}$, e.g. the number a_n of involutions in S_n :

$$a_n \sim 2^{-1/2} e^{-1/4} \left(\frac{n}{e}\right)^{n/2} e^{\sqrt{n}},$$

(see [Sta1] and A000085 in [OEIS]).

In notation of Theorem 3.4.3, the $\alpha \in \mathbb{Q}$ conclusion cannot be substantially strengthened even for $k = 2$ variables. To understand this, recall Furstenberg's theorem (see [Sta1, §6.3]) that every algebraic function is a diagonal of $P(x, y)/Q(x, y)$, and that by Theorem 2 in [BD] there exist algebraic functions with asymptotics $A\lambda^n n^\alpha$, for all $\alpha \in \mathbb{Q} \smallsetminus \{-1, -2, \ldots\}$. For example, the number $g(n)$ of *Gessel walks* (see A135404 in [OEIS]) is famously algebraic [BK], and has asymptotics

$$g(n) \sim \frac{2^{2/3}\Gamma(\frac{1}{3})}{3\pi} 16^n n^{-7/3}.$$

3.11.5

There is more than one way a sequence can be a diagonal of a rational function. For example, the Catalan numbers C_n are the diagonals of

$$\frac{1-x/y}{1-x-y} \quad \text{and} \quad \frac{y(1-2xy-2xy^2)}{1-x-2xy-xy^2}.$$

The former follows from $C_n = \binom{2n}{n} - \binom{2n}{n-1}$, while the second is given in [RY].

3.11.6

There is a vast literature on binomials sums and multisums, both classical and modern, see e.g. [PWZ, Rio]. It was shown by Zeilberger [Zei] (see also [WZ]), that under certain restrictions, the resulting functions are D-finite, a crucial discovery which paved a way to *WZ algorithm*, see [PWZ, WZ]. A subclass of *balanced multisums*, related but larger than $\mathcal{B}'$, was defined and studied in [Gar]. Note that the positivity is the not the only constraint we add. For example, balanced multisums in [Gar] easily contain Catalan numbers:

$$\frac{(2n)!\,1!}{n!\,(n+1)!} = C_n\,.$$

We refer to [B+, §5.1] and [BLS] for the recent investigations of binomial multisums which are diagonals of rational functions, but without $\mathbb{N}$-rationality restriction.

3.11.7

The class $\mathcal{R}_1$ of $\mathbb{N}$-rational GFs does not contain *all* of $\mathbb{N}[[x]]$ (Berstel, 1971); see [BR2, Ges2] for some examples. These are rare, however; e.g. Koutschan investigated "about 60" nonnegative rational GFs from [OEIS], and found all of them to be in $\mathcal{R}_1$, see [Kou, §4.4]. In fact, there is a complete characterization of $\mathcal{R}_1$ by analytic means, via the Berstel (1971) and Soittola (1976) theorems. We refer to [BR1, SS] for these results and further references, and to [Ges2] for a friendly introduction.

Unfortunately, there is no such characterization of $\mathcal{N}$, nor we expect there to be one, as singularities in higher dimensions are most daunting [FS, PW]. Even the most natural questions remain open in that case (cf. Conjecture 3.4.5). Here is one such question.

Open Problem 3.11.1. *Let $f \in \mathcal{F}$ such that the corresponding GF $F(x) \in \mathbb{N}[[x]]$. Does it follow that $f \in \mathcal{F}_1$?*

Personally, we favor a negative answer. In [Ges2], Gessel asks whether there are (nonnegative) rational GF which have a *combinatorial interpretation*, but are not $\mathbb{N}$-rational. Thus, a negative answer to Problem 3.11.1 would give a positive answer to Gessel's question.[4] Of course, what's a combinatorial interpretation is in the eye of the beholder; here we are implicitly assuming that our irrational tilings or paths in graphs (see §3.5.1) are *nice* enough to pass this test (cf. [Ges2]). We plan to revisit this problem in the future.

3.11.8

There are over 200 different combinatorial interpretation of Catalan numbers [Sta2], some of them 1-dimensional such as the *ballot sequences*. A quick review suggests that none of them can be verified with a bounded memory read only TM. For example, for the ballot 0–1 sequences one must remember the running differences (#0 − #1), which can be large. This gives some informal support in favor of our Conjecture 3.4.6. Let us make following, highly speculative and priceless claim.[5]

Conjecture 3.11.2. *There is no tile counting function $f \in \mathcal{F}$ which is asymptotically Catalan:*

$$f(n) \sim C_n \quad \text{as} \quad n \to \infty.$$

We initially tried to disprove the conjecture. Recall that by Lemma 3.10.1 and the technology in Section 3.10, it suffices to obtain the constant $\frac{\pi}{3\sqrt{3}}$ as the product of values of the hypergeometric functions given in Theorem 3.4.10. While $\frac{1}{3\sqrt{3}}$ is easy to obtain,

[4]Christophe Reutenauer writes to us that according the "general metamathematical principle that goes back to Schützenberger" (see [BR2, p. 149]), the logic must be reversed: a negative answer to Gessel's question implies that the answer to Problem 3.11.1 must be positive.

[5]Cf. http://tinyurl.com/mc3h8tn.

our hypergeometric sums seem too specialized to give value π. This is somewhat similar to the conjecture that $\frac{1}{\pi}$ is not a *period* [KZ].

We should mention here that it is rare when we can say anything at all about the constant A in Conjecture 3.4.5. The constant in Corollary 3.4.12 is an exception: is known to be transcendental by the celebrated 1996 result of Nesterenko on algebraic independence of π and $\Gamma(\frac{1}{4})$, see [NP].

3.11.9

The proof of Lemma 3.5.3 uses a generalization of a standard argument in combinatorial linear algebra, for computing the number of rational tilings:

$$F(x) = \sum_{n=0}^{\infty} f_T(n) x^n = \sum_{n=0}^{\infty} (M^n)_{00}\, x^n = \left(\frac{1}{1-Mx}\right)_{00} = \frac{\det(1-M^{00}x)}{\det(1-Mx)},$$

where M is the weighted adjacency matrix of G_T. It is thus not surprising that we use a cycle decomposition argument somewhat similar but more general than that in [CF, KP].

In the same vein, the "well-defined multiplicities" argument in the proof of Lemma 3.5.4 is similar to the "cycle popping" argument in [Wil] (see also [GoP, Mar]). The details are quite different, however.

3.11.10

The values $\mathrm{ord}_p(C_n)$ in Proposition 3.4.9 were computed by Kummer (1852); see [DS] for a recent combinatorial proof.

References

[BD] C. Banderier and M. Drmota, Formulae and asymptotics for coefficients of algebraic functions, to appear in *Combin. Probab. Comput.*; available at http://tinyurl.com/kny2xum.

[Bar] A. Barvinok, *Integer points in polyhedra*, European Mathematical Society, Zürich, 2008.

[BL] K. P. Benedetto and N. A. Loehr, Tiling problems, automata, and tiling graphs, *Theoret. Comput. Sci.* **407** (2008), 400–411.

[BR1] J. Berstel and C. Reutenauer, *Rational series and their languages*, Springer, Berlin, 1988.

[BR2] J. Berstel and C. Reutenauer, *Noncommutative rational series with applications*, Cambridge Univ. Press, Cambridge, 2011.

[B+] A. Bostan, S. Boukraa, G. Christol, S. Hassani and J.-M. Maillard, Ising n-fold integrals as diagonals of rational functions and integrality of series expansions, *Journal of Physics A* **46** (2013), 185202, 44 pp.; extended version is available at arXiv:1211.6031, 100 pp.

[BK] A. Bostan and M. Kauers, The complete generating function for Gessel walks is algebraic, with an appendix by M. van Hoeij, *Proc. AMS.* **138** (2010), 3063–3078.

[BLS] A. Bostan, P. Lairez and B. Salvy, Représentations intégrales des sommes binomiales, talk slides (11 April, 2014, Séminaire Teich, Marseille); available at http://tinyurl.com/n5l7kyt.

[BRS] A. Bostan, K. Raschel and B. Salvy, Non-D-finite excursions in the quarter plane, *J. Combin. Theory, Ser. A* **121** (2014), 45–63.

[CCH] P. Callahan, P. Chinn and S. Heubach, Graphs of tilings, *Congr. Numer.* **183** (2006), 129–138.

[CF] P. Cartier and D. Foata, *Problèmes combinatoires de commutation et réarrangements*, Lecture Notes in Mathematics, No. 85, Springer, Berlin, 1969; available at http://tinyurl.com/ntklmwv

[CLS] S. Chen, N. Li and S. V. Sam, Generalized Ehrhart polynomials, *Trans. AMS* **364** (2012), 551–569.

[DS] E. Deutsch and B. E. Sagan, Congruences for Catalan and Motzkin numbers and related sequences, *J. Number Theory* **117** (2006), 191–215.

[DGS] B. Dwork, G. Gerotto and F. J. Sullivan, An introduction to G-functions, Princeton Univ. Press, Princeton, NJ, 1994.

[FGS] P. Flajolet, S. Gerhold and B. Salvy, On the non-holonomic character of logarithms, powers, and the nth prime function, *Electron. J. Combin.* **11** (2004/06), A2, 16 pp.

[FS] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, Cambridge, 2009.

[Gar] S. Garoufalidis, G-functions and multisum versus holonomic sequences, *Advances Math.* **220** (2009), 1945–1955.

[GaP] S. Garrabrant and I. Pak, Counting with Wang tiles, in preparation.

[Ges1] I. Gessel, Two theorems of rational power series, *Utilitas Math.* **19** (1981), 247–254.

[Ges2] I. Gessel, Rational functions with nonnegative integer coefficients, in *Proc. 50th Sém. Lotharingien de Combinatoire* (March 2003); available at http://tinyurl.com/krlbvvl.

[Gol] S. W. Golomb, *Polyominoes*, Scribner, New York, 1965.

[GoP] I. Gorodezky and I. Pak, Generalized loop-erased random walks and approximate reachability, *Random Structures Algorithms* **44** (2014), 201–223.

[KM] D. A. Klarner and S. S. Magliveras, The number of tilings of a block with blocks, *European J. Combin.* **9** (1988), 317–330.

[KP] M. Konvalinka and I. Pak, Non-commutative extensions of the MacMahon Master Theorem, *Advances Math.* **216** (2006), 29–61.

[KZ] M. Kontsevich and D. Zagier, *Periods*, IHES preprint M/01/22 (May 2001), 38 pp.; available at http://tinyurl.com/k7h7gvx.

[Korn] M. R. Korn, *Geometric and Algebraic properties of polyomino tilings*, Ph.D. thesis, MIT, 2004; available at http://dspace.mit.edu/handle/1721.1/16628.

[Kou] C. Koutschan, *Regular languages and their generating functions: the inverse problem*, Master thesis, University of Erlangen-Nuremberg, Germany, 2005; available at http://tinyurl.com/pj6l6gl.

[Mar] P. Marchal, Loop-erased random walks and heaps of cycles, Preprint PMA-539, Univ. Paris VI, 1999; available at http://tinyurl.com/o54deou.

[MSV] D. Merlini, R. Sprugnoli and M. Verri, Strip tiling and regular grammars, *Theoret. Comput. Sci.* **242** (2000) 109–124.

[MM] C. Moore and S. Mertens, *The nature of computation*, Oxford Univ. Press, Oxford, UK, 2011.

[NP] Yu. V. Nesterenko and P. Philippon (Eds.), *Introduction to algebraic independence theory*, Springer, Berlin, 2001.

[Odl] A. M. Odlyzko, Asymptotic enumeration methods, in *Handbook of Combinatorics*, Vol. 2, Elsevier, Amsterdam, 1995, 1063–1229.

[Pak] I. Pak, *Lectures on Discrete and Polyhedral Geometry*, monograph in preparation; available at http://www.math.ucla.edu/~pak/book.htm.

[PY] I. Pak and J. Yang, Tiling simply connected regions with rectangles, *J. Combin. Theory, Ser. A* **120** (2013), 1804–1816.

[PW] R. Pemantle and M. C. Wilson, *Analytic combinatorics in several variables*, Cambridge Univ. Press, Cambridge, UK, 2013.

[PWZ] M. Petkovšek, H. S. Wilf and D. Zeilberger, $A = B$, A K Peters, Wellesley, MA, 1996.

[Pin] J.-E. Pin, Finite semigroups and recognizable languages: an introduction, in *Semigroups, formal languages and groups*, Kluwer, Dordrecht, 1995, 1–32.

[Rio] J. Riordan, *Combinatorial identities*, John Wiley, New York, 1968.

[RY] E. Rowland and R. Yassawi, Automatic congruences for diagonals of rational functions, to appear in *J. Théor. Nombres Bordeaux*; available at arXiv:1310.8635.

[SS] A. Salomaa and M. Soittola, *Automata-Theoretic Aspects of Formal Power Series*, Springer, New York, 1978.

[Schü] M. P. Schützenberger, On the definition of a family of automata, *Information and Control* **4** (1961), 245–270

[Sip] M. Sipser, *Introduction to the Theory of Computation*, PWS, 1997.

[OEIS] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org.

[Sta1] R. P. Stanley, *Enumerative combinatorics*, Vol. 1 and 2, Cambridge Univ. Press, Cambridge, UK, 1997 and 1999.

[Sta2] R. P. Stanley, *Catalan Numbers*, monograph in preparation; an early draft titled *Catalan Addendum* is available at http://www-math.mit.edu/˜rstan/ec/catadd.pdf.

[Wang] H. Wang, Games, logic and computers, in *Scientific American* (Nov. 1965), 98–106.

[WZ] H. Wilf and D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and q) multisum/integral identities, *Inventiones Math.* **108** (1992), 575–633.

[Wil] D. B. Wilson, Generating random spanning trees more quickly than the cover time, in *Proc. 28th STOC*, ACM, 1996, 296–303.

[Zei] D. Zeilberger, Sister Celine's technique and its generalizations, *J. Math. Anal. Appl.* **85** (1982), 114–145.