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Stabilization via Distributed Control

Tetsuya Iwasaki and Laqshya Taneja

Abstract—We consider the linear plant with multiple pairs of input and output channels, and solve the problem of designing a stabilizing controller consisting of a set of control stations, one for each channel pair, connected with an arbitrarily fixed network topology and communication dynamics. It is shown that such stabilizing distributed controller exists if and only if the fixed modes of the plant, augmented with the network communication, are stable; this leverages and extends the classical fixed mode result for decentralized control. Moreover, a condition is given to reveal exactly what connections are needed between control stations to achieve stabilizability via a distributed control. Thus, the feasibility of optimal distributed control synthesis can readily be checked or enforced before applying computationally expensive optimization methods. Stabilizing distributed controllers, if feasible, can be designed using decentralized control theories.

Index Terms—Distributed control, Decentralized control, Network analysis and control, Stabilization, Optimal control

I. INTRODUCTION

The control design with a distributed architecture is a fundamental problem for which decades of research has provided only partial solutions, especially with optimality requirements. Control theories for such designs will have practical significance for e.g. localized control of large scale systems and resilient control of multi-degree-of-freedom robotic systems.

The classical theory of decentralized control solved the problem of stabilizing a linear time-invariant (LTI) plant by an LTI controller with a (block) diagonal transfer function. Every control design with an arbitrary sparsity constraint on the controller *transfer matrix* can be formulated in this way [1], and therefore categorized as a decentralized control. A stabilizing decentralized controller is shown to exist if and only if the fixed modes of the plant are stable [2], and one such controller can be found as an (almost) arbitrary decentralized static feedback plus a dynamic feedback through any one of the input-output channels [3]. In contrast, the optimal decentralized control has been proven to be difficult in general, and can be reduced to convex optimization over the Youla parameter [4] if and only if the plant satisfies a foundational but rather restrictive property called quadratic invariance (QI) [5], [6]. In some special cases for which QI holds, the optimal decentralized control problem can be solved in the state space, using Riccati equations for poset-causal systems [7], [8], and linear matrix inequalities for block-triangular systems [9].

The distributed control, as we define here, is a relaxation of the decentralized control in that control stations do not operate completely independently but are allowed to communicate with neighbors. These controllers may have non-sparse, fully populated transfer matrices even when the communication

topology is sparse. While the vast majority of the literature focused on decentralized control, the distributed control appears broadly applicable yet more tractable due to the additional information available to control stations through the network.

Optimal distributed control problems have been solved for some special classes of plants and controllers, including spatially invariant systems [10] and a platoon of vehicles [11]. A more recent result on the system-level synthesis (SLS) [12] and its variations e.g. [13] have provided novel parametrizations of all stabilizing controllers for general LTI plants. The new formulations are still restricted by QI when used for decentralized control due to their equivalence to the Youla parametrization [14], but arbitrary communication constraints (sparsity, delay, etc.) can be directly imposed on the free parameters with some conservatism, thereby rendering the distributed control synthesis a convex optimization.

However, no tractable method has been established for checking the feasibility of distributed control with the given communication architecture. For example, the SLS-based optimization is over the infinite dimensional space and has to rely on the premise that effective basis functions can be chosen for a finite dimensional approximation. When the optimization problem is found infeasible after intensive computation, it is difficult to determine whether the infeasibility is caused by the poor choice of basis functions, or by the fundamental lack of stabilizability via the imposed distributed control architecture.

In this paper, we will present a necessary and sufficient condition for stabilizability via distributed controllers. For a general LTI plant with multiple input-output pairs, we consider a distributed controller formed by multiple control stations, each closing a feedback loop around each input-output pair, while exchanging information with neighboring stations. Given the topology and dynamics (e.g. delay) of the communication network, we show that a stabilizing distributed controller exists if and only if the fixed modes of the plant, augmented by the communication network, are stable.

We first provide a review of some essential results in decentralized control, with a new formulation in terms of transfer functions rather than the classical state space formulation [3]. This formulation makes it more transparent that a fixed mode of the plant may arise from the dynamics of the propagation from a strongly connected subnetwork of the plant to another. To make it movable by distributed control, we need additional connection(s) through the communication between control stations so that a feedback loop is formed to contain the propagation dynamics, as illustrated by a design example.

This idea is formalized to give a further result – a systematic way to identify the additional network connections needed for stabilization when the plant has unstable fixed modes. While many formulations in the literature [7]–[9], [15], [16] assume the same network structure for the plant and controller, a

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control network complementary to the plant network can be helpful or even necessary to stabilize the plant in general. We will illustrate this point by a vehicle platoon example.

Finally, we mention that our approach is inspired by [17] which reduced the distributed control problem to the decentralized control problem through a plant augmentation in the state space. We extend the idea and propose a systematic framework for the reduction using the linear fractional transformation (LFT) in the frequency domain, adding clarity, simplicity, and generality to the augmentation process with arbitrary communication topology and dynamics. Also, while [17] focused on conditions to achieve arbitrarily fast convergence with a strongly connected control network, we characterize the communication topology necessary and sufficient for stabilization.

Notation: The empty set is denoted by $\emptyset$, and $\mathbb{I}_q := \{1, 2, \dots, q\}$ for a positive integer q . The complementary set of $\kappa \subseteq \mathbb{I}_q$, denoted by $\bar{\kappa}$, is the subset of $\mathbb{I}_q$ such that $\kappa \cap \bar{\kappa} = \emptyset$ and $\kappa \cup \bar{\kappa} = \mathbb{I}_q$. Note that $\mathbb{I}_q$ is decomposed into κ and $\bar{\kappa}$, and $(\kappa, \bar{\kappa}) = (\mathbb{I}_q, \emptyset)$ and $(\emptyset, \mathbb{I}_q)$ are valid decompositions.

II. PROBLEM FORMULATION

Consider the plant

$$\eta = \mathcal{P}(s)u, \quad \varpi := (m_1, \dots, m_q; p_1, \dots, p_q), \quad (1)$$

where the input and output are partitioned as specified by ϖ :

$$u(t) \in \mathbb{R}^m, \quad u = \text{col}(u_1, \dots, u_q), \quad u_i(t) \in \mathbb{R}^{m_i}, \\ \eta(t) \in \mathbb{R}^p, \quad \eta = \text{col}(\eta_1, \dots, \eta_q), \quad \eta_i(t) \in \mathbb{R}^{p_i}.$$

Accordingly, denote by $\mathcal{P}_{ij}(s)$ the $p_i \times m_j$ transfer matrix on the (i, j) block of $\mathcal{P}(s)$ for $i, j \in \mathbb{I}_q$. We seek to stabilize the plant by a distributed controller $u = \mathcal{K}(s)\eta$ described by

$$\begin{bmatrix} \mu_i \\ u_i \end{bmatrix} = K_i(s) \begin{bmatrix} \eta_i \\ \eta_i \end{bmatrix}, \quad \eta_i = \sum_{j \in \mathbb{N}_i} N_{ij}(s) \mu_j, \quad i \in \mathbb{I}_q, \quad (2)$$

where $K_i(s)$ is the i^{th} control station, $N_{ij}(s)$ is a transfer function capturing the dynamics of signal transmission from $K_j(s)$ to $K_i(s)$, $\mu_i(t) \in \mathbb{R}^{m_i}$ and $\eta_i(t) \in \mathbb{R}^{p_i}$ are the signals transmitted from and received by station $K_i(s)$, respectively, and $\mathbb{N}_i \subseteq \mathbb{I}_q$ is the set of indices specifying the neighbors of $K_i(s)$. The network dynamics and topology are captured by

$$\eta = N(s)\mu, \quad \hat{\varpi} := (\hat{m}_1, \dots, \hat{m}_q; \hat{p}_1, \dots, \hat{p}_q), \quad (3)$$

where $N(s)$ is formed by placing $N_{ij}(s)$ in its (i, j) block if $j \in \mathbb{N}_i$ and zero otherwise. Multiple control stations may be made to share a sensor measurement in a neighborhood by defining η_i appropriately (i.e. $\mathcal{P}(s)$ may have repeated rows). For simplicity, the plant $\mathcal{P}(s)$ and network $N(s)$ are assumed strictly proper. This restriction on $N(s)$ prevents instantaneous communication, which is more practical [16]. We can essentially extend this paper to proper $\mathcal{P}(s)$ and $N(s)$ using Theorem 2.2 of [18]. The LFT framework (2) captures existing classes of distributed controllers (e.g. the SLS and structured state space formulations) and was considered for optimal control [16], [19] although they had to narrow down the design space to make the problem tractable.

This paper will solve the following problem for the continuous-time case (an extension to discrete-time systems

is trivial). Given the plant $\mathcal{P}(s)$ with input-output partition ϖ in (1) and the network topology and dynamics specified by $N(s)$ with partition $\hat{\varpi}$ in (3), find a necessary and sufficient condition under which the plant can be stabilized by the distributed controller (2) with an appropriate choice of $K_i(s)$, $i \in \mathbb{I}_q$. Characterize all sets of network connections between control stations which are necessary and sufficient for stabilizability via distributed control. When feasible, provide a simple method for designing stabilizing distributed controllers.

The system (2) captures a general class of distributed controllers with various communication constraints, without imposing “hard zeros” in certain entries of the controller transfer matrix $\mathcal{K}(s)$ as in decentralized control. For example, consider the distributed controller with three ($q = 3$) stations connected in all-to-all fashion by $N(s)$ of the following form:

$$N = \left[\begin{array}{cc|cc|cc} 0 & 0 & \delta_{12}I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_{13}I & 0 \\ \hline \delta_{21}I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_{23}I \\ \hline 0 & \delta_{31}I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_{32}I & 0 & 0 \end{array} \right], \quad (4)$$

where the partitions indicate the $N_{ij}(s)$ blocks, $\delta_{ij}(s)$ captures the transmission delay from station K_j to station K_i , and the dimension of each identity matrix I specifies the dimension of the signals transmitted/received. Each station transmits two signals, each of which is received by one of the other two stations. If each station broadcast one signal and receives the individual transmission signals from the other stations, then the two columns in each partition can be replaced by their sum. If there are constraints restricting direct communication from station l to k , then $N(s)$ can be redefined by deleting the row and column containing δ_{kl} , or by setting $\delta_{kl}(s) = 0$. If desired, self-loops can be added on the diagonal blocks.

III. DECENTRALIZED CONTROL

The stabilization problem has been essentially solved in the decentralized setting [1]–[3]. This section reviews the classical theory of decentralized control, which forms a basis for our development of the distributed control result. In addition to a tutorial value, we intend to provide a new frequency-domain perspective which appears missing in the literature.

Let the plant with input-output partition

$$y = P(s)u, \quad \pi := (m_1, \dots, m_q; p_1, \dots, p_q) \quad (5)$$

be given, which is defined analogously to (1). The problem is the design of a decentralized controller

$$u = K(s)y, \quad K(s) := \text{diag}(K_1(s), \dots, K_q(s)) \quad (6)$$

that makes the closed-loop system stable, where $K_i(s)$ is an $m_i \times p_i$ transfer matrix representing the i^{th} control station.

The concept of the fixed mode is central for decentralized stabilization. Let a state space realization of $P(s)$ be given by

$$P(s) = C(sI - A)^{-1}B, \quad B = \text{row}(B_1, \dots, B_q), \quad (7) \\ C = \text{col}(C_1, \dots, C_q),$$

where B and C are partitioned according to π . The set of fixed modes of (A, B, C) with respect to partition π is defined by

$$\text{fm}(A, B, C; \pi) := \bigcap_{L \in \mathbb{L}(\pi)} \text{spec}(A + BLC), \quad (8)$$

$$\mathbb{L}(\pi) := \{\text{diag}(L_1, \dots, L_q) : L_i \in \mathbb{R}^{m_i \times p_i}, i \in \mathbb{I}_q\}$$

where $\text{spec}(\cdot)$ denotes the spectrum of a matrix. A fixed mode is an eigenvalue of A (since $0 \in \mathbb{L}(\pi)$) that cannot be moved by any decentralized static output feedback $L \in \mathbb{L}(\pi)$. We define $\lambda \in \mathbb{C}$ to be a fixed mode of $P(s)$ with respect to partition π if it is a fixed mode of its minimal state space realization. We shall refer to an eigenvalue of A or a pole of $P(s)$ as a movable mode if it is not a fixed mode.

The fundamental result in [2] states that the plant (7) with partition π can be stabilized by a dynamic output feedback decentralized controller (6) if and only if all the fixed modes of (A, B, C) are stable. While the fixed mode is defined in terms of the static feedback as in (8), a mode is fixed/movable by static feedback if and only if it is fixed/movable by *dynamic* feedback [1]. Clearly, any stabilizable and detectable realization (A, B, C) can be stabilized by a decentralized control if and only if all the fixed modes of $P(s)$ are stable.

The following result of [20] characterizes the fixed modes.

Lemma 1: Consider the system (5) with partition π and a state space realization (7). Fix $\lambda \in \mathbb{C}$. Then

$$\min_{L \in \mathbb{L}(\pi)} g(\lambda, L) = \max_{\kappa \subseteq \mathbb{I}_q} \tau(\lambda, \kappa) \quad (9)$$

holds, where

$$\begin{aligned} g(\lambda, L) &:= \text{nullity}(\lambda I - A - BLC), \\ \tau(\lambda, \kappa) &:= n - \text{rank} \begin{bmatrix} \lambda I - A & B_{\kappa} \\ C_{\bar{\kappa}} & 0 \end{bmatrix}, \end{aligned} \quad (10)$$

with n being the dimension of A , and

$$\begin{aligned} B_{\kappa} &:= \text{row}(B_{i_1}, \dots, B_{i_k}), \quad \kappa := \{i_1, \dots, i_k\}, \\ C_{\bar{\kappa}} &:= \text{col}(C_{i_{k+1}}, \dots, C_{i_q}), \quad \bar{\kappa} := \{i_{k+1}, \dots, i_q\}. \end{aligned}$$

Proof. The result is a restatement of Theorem 4.1 in [20], where we show an explicit relationship between the rank in (10) and the dimension $g(\lambda, L)$ of the eigenspace. ■

By definition, λ is a fixed mode of $(A, B, C; \pi)$ if and only if $g(\lambda, L) \geq 1$ holds for all $L \in \mathbb{L}(\pi)$, i.e., the left hand side of (9) is greater than or equal to one. Note that $g(\lambda, L)$ is the geometric multiplicity of the fixed mode λ as an eigenvalue of $A + BLC$. Lemma 1 characterizes the fixed modes $\lambda \in \text{fm}(A, B, C; \pi)$ by the rank property $\tau(\lambda, \kappa) \geq 1$ for some $\kappa \subseteq \mathbb{I}_q$. As a special case with $\kappa = \emptyset$ or $\mathbb{I}_q$, $\lambda \in \text{spec}(A)$ is a fixed mode if it is unobservable or uncontrollable, respectively. If $\tau(\lambda, \kappa) > 1$ for some $\kappa \subseteq \mathbb{I}_q$, then the eigenvalue λ of $A + BLC$ is repeated with geometric multiplicity $\tau(\lambda, \kappa)$ or more for any $L \in \mathbb{L}(\pi)$. We show later that $\tau(\lambda, \kappa)$ is related to the number and location of connections between control stations, needed to make λ movable via distributed control.

We now summarize the results for design of stabilizing decentralized controllers. The results have been obtained in [1], [3] within the state space framework. These developments may be made more transparent and streamlined by a recent result (implicit in [1]) which provided an explicit proof that a block-triangular transfer matrix $P(s)$ admits a block-triangular, minimal state space realization (Lemma 9, [21]). In what follows, we will provide an alternative frequency-domain approach which, in our view, simplifies some development and gives an additional insight useful for distributed control.

For the purpose of the decentralized stabilization, it is important to understand the structure of the information flow in the closed-loop system since the major challenge is the lack of communication between the control stations. With a decentralized controller (6) consisting of q stations, the closed-loop system can be viewed as a network with q nodes, where each control station represents a node. There is a direct path from station j to station i if and only if the plant transfer function from u_j to y_i is not identically zero, i.e., $P_{ij}(s) \not\equiv 0$. The closed-loop network is said to be strongly connected if there is a directed path from station j to station i (possibly after going around the feedback loop multiple times) for all $i, j \in \mathbb{I}_q$ such that $i \neq j$. The plant $P(s)$ is said to be strongly connected if the closed-loop network is strongly connected.

An arbitrary plant $P(s)$ can be decomposed into “one-way connections” of strongly connected subnetworks. Define the adjacency matrix $S \in \mathbb{R}^{q \times q}$ of the closed-loop network such that its diagonal entries are zero and the off-diagonal (i, j) entry is 1 if $P_{ij}(s) \not\equiv 0$ and is zero if $P_{ij}(s) \equiv 0$. The network is strongly connected if and only if S is irreducible, that is, no permutation of rows and columns can make S block triangular [22]. If S is reducible, then there exists a permutation of rows and columns that makes S block-triangular with each of the diagonal blocks irreducible [23]. Applying the permutation to the inputs u_i and outputs y_i , the transfer matrix $P(s)$ can be rearranged into the following structure:

$$\begin{bmatrix} y_{\kappa_1} \\ \vdots \\ y_{\kappa_r} \end{bmatrix} = \begin{bmatrix} P_{11}(s) & & 0 \\ & \ddots & \\ P_{r1}(s) & \cdots & P_{rr}(s) \end{bmatrix} \begin{bmatrix} u_{\kappa_1} \\ \vdots \\ u_{\kappa_r} \end{bmatrix}, \quad (11)$$

$$u_{\kappa_k} := \text{col}(u_{i_1^k}, \dots, u_{i_{q_k}^k}), \quad \kappa_k := \{i_1^k, \dots, i_{q_k}^k\}, \quad k \in \mathbb{I}_r,$$

where y_{κ_k} is defined in a manner similar to u_{κ_k} , $\kappa_k \subseteq \mathbb{I}_q$ with $k \in \mathbb{I}_r$ are disjoint partitions of $\mathbb{I}_q$, r is the number of strongly connected subnetworks $P_{kk}(s)$, $k \in \mathbb{I}_r$, each of which has q_k nodes specified by the indices in κ_k . Hence, it can be assumed without loss of generality that the plant has the block-triangular structure, and the structure allows for the decomposition of the stabilization problem as follows.

Theorem 1: Let the plant $P(s)$ with input-output partition π be given as in (5). The following statements are equivalent.

- (i) There exists a decentralized controller $K(s)$ in (6) that stabilizes the plant $P(s)$.
- (ii) The plant $P(s)$ has no unstable fixed modes with respect to the partition π in (5).

Suppose (ii) holds. Let $P(s)$ be permuted into the block triangular form (11), where $P_{kk}(s)$ is strongly connected for each $k \in \mathbb{I}_r$. Let a decentralized controller $u = K(s)y$ in (6) be given, and define its subsystems by a permutation and grouping of inputs and outputs as follows:

$$\begin{aligned} u_{\kappa_k} &= K_{\kappa_k}(s)y_{\kappa_k}, \quad k \in \mathbb{I}_r, \\ K_{\kappa_k}(s) &:= \text{diag}(K_{i_1^k}^k(s), \dots, K_{i_{q_k}^k}^k(s)). \end{aligned} \quad (12)$$

Then the following statements are equivalent.

- (iii) $K(s)$ stabilizes $P(s)$.
- (iv) $K_{\kappa_k}(s)$ stabilizes $P_{kk}(s)$ for all $k \in \mathbb{I}_r$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is proven in [2]. We show the equivalence (iii) $\Leftrightarrow$ (iv). Let $\tilde{P}(s)$ be the transfer matrix in (11), and $\tilde{K}(s)$ be the transfer matrix defined by stacking $K_{\kappa_k}(s)$ for $k \in \mathbb{I}_r$ on the diagonal. Then the closed-loop system of $P(s)$ and $K(s)$ is identical to that of $\tilde{P}(s)$ and $\tilde{K}(s)$. By definition, it is stable if

$$H(s) := \begin{bmatrix} I & -\tilde{P}(s) \\ -\tilde{K}(s) & I \end{bmatrix}^{-1}$$

has no pole in the closed right half plane. Note that

$$H(s) = \begin{bmatrix} M(s) & M(s)\tilde{P}(s) \\ \tilde{K}(s)M(s) & I + \tilde{K}(s)M(s)\tilde{P}(s) \end{bmatrix},$$

where $M(s) := (I - \tilde{P}(s)\tilde{K}(s))^{-1}$ is lower block-triangular with the (ℓ, k) block for $k \in \mathbb{I}_r$, $\ell = k + 1, \dots, r$, given by

$$M_{\ell k} = M_{\ell \ell} \sum_{i=k}^{\ell-1} P_{\ell i} K_{\kappa_i} M_{ik}, \quad M_{kk} = (I - P_{kk} K_{\kappa_k})^{-1}.$$

Hence, $H(s)$ is formed by terms that are some products of

$$I, \quad M_{kk}, \quad K_{\kappa_k} M_{kk}, \quad M_{kk} P_{kk}, \quad K_{\kappa_k} M_{kk} P_{kk}, \quad P_{\ell k}. \quad (13)$$

Note that each of the 2×2 blocks of $H(s)$ is block-triangular, and assembling the k^{th} diagonal blocks of the four gives

$$H_k(s) := \begin{bmatrix} M_{kk}(s) & M_{kk}(s)P_{kk}(s) \\ K_{\kappa_k}(s)M_{kk}(s) & I + K_{\kappa_k}(s)M_{kk}(s)P_{kk}(s) \end{bmatrix},$$

which is the closed-loop transfer matrix for $P_{kk}(s)$ and $K_{\kappa_k}(s)$. Now, suppose (iii) holds. Stability of that $H_k(s)$ is stable for all $k \in \mathbb{I}_r$. Thus we have (iv). To show the converse, suppose (iv) is true but (iii) is false. In this case, stability of $H_k(s)$ implies that all the transfer matrices listed in (13) are stable except for $P_{\ell k}(s)$. Therefore, instability of $H(s)$ must be due to an unstable pole of an off-diagonal block $P_{\ell k}(s)$. We complete the proof by showing that this pole, denoted by λ , is a fixed mode of $P(s)$. This contradicts the supposition (ii) and we can conclude that (iv) implies (iii).

Consider a perturbation of the controller $K_{\kappa_k}(s) + \varepsilon L_k$, $k \in \mathbb{I}_r$, where L_k are arbitrary matrices and $\varepsilon > 0$. When ε is sufficiently small, stability of $H_k(s)$ and instability of $H(s)$ are both maintained because the eigenvalues of their minimal realizations depend continuously on ε . However, all unstable poles of the perturbed $H(s)$ must come from $P_{\ell k}(s)$ for the same reason as explained above. Therefore, the pole at λ when $\varepsilon = 0$ must remain at λ for sufficiently small ε . We claim that the pole at λ remains there for all ε . To see this, note that the closed-loop system of $P(s)$ and $K(s)$ is described by $\dot{x} = Ax$ with A depending linearly on εL_k . Hence, for each fixed L_k , $\det(\lambda I - A)$ is a polynomial of ε . At the pole λ , this polynomial is zero for infinitely many values of ε near 0 and therefore it must be independent of ε and is identically zero. Thus, λ remains to be a closed-loop eigenvalue for all ε . Since this holds for an arbitrary L_k , λ is a fixed eigenvalue of A regardless of the value of εL_k . Finally, since a fixed mode cannot be moved by dynamic decentralized controllers, we see that λ is fixed when the additional term εL_k in the controller is replaced by $N_k - K_{\kappa_k}(s)$ with an arbitrary N_k . Thus we conclude that λ is a fixed mode of $P(s)$. ■

Assuming no unstable fixed modes in $P(s)$, Theorem 1 shows that off-diagonal blocks of (11) can be ignored and stabilizing each strongly connected subnetwork on the diagonal is necessary and sufficient to stabilize the whole. This fact has been known in the state space [3], but the above statement and proof, built directly on transfer functions, is new and adds further clarity for why this is the case. This is because an unstable pole of an off-diagonal block is either a fixed mode for which we can do nothing, or a mode also included in a diagonal block of the same row or column, which can be stabilized by the corresponding station of the decentralized controller. Let us illustrate this by an example.

Example 1: Consider the following system:

$$P(s) = \begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ \frac{1}{s-2} & \frac{1}{s-2} & 0 \\ \frac{1}{s+4} & \frac{1}{s-1} & \frac{1}{s-3} \end{bmatrix}.$$

The eigenvalues of a minimal realization of $P(s)$ are $\lambda = 1, 1, 2, 3$ and -4 , where the two poles at 2 can be consolidated since they appear in the same row of $P(s)$. With diagonal controllers $K = \text{diag}(K_1, K_2, K_3)$, the fixed modes are $\lambda = 1$ and -4 which come from the off-diagonal entries. In this case, $P(s)$ cannot be stabilized due to the fixed mode $\lambda = 1$. With block-diagonal controllers $K = \text{diag}(K_1, K_2)$ with scalar K_1 and 2×2 K_2 , the fixed mode is $\lambda = -4$. Since the fixed mode is stable, $P(s)$ can be stabilized by a decentralized controller. In particular, $K_1(s)$ should stabilize the scalar $P_{11}(s)$ and $K_2(s)$ should stabilize the lower right 2×2 block $P_{22}(s)$ of $P(s)$. Note that the pole at 2 due to the $(2,1)$ entry of $P(s)$ is stabilized by $K_2(s)$.

Based on Theorem 1, the decentralized stabilization problem for a general plant $P(s)$ can be decomposed into r decentralized stabilization problems for strongly connected subnetworks $P_{kk}(s)$, $k \in \mathbb{I}_r$, of the plant. Hence, the following method for decentralized stabilization of a single strongly connected system solves the overall problem.

Theorem 2: [1], [3] *Consider the plant (5) with a minimal realization (7). Suppose $P(s)$ is strongly connected. Fix $\ell \in \mathbb{I}_q$ arbitrarily. Then, there exists a static decentralized gain $L \in \mathbb{L}(\pi)$ that makes the triple $(A+BLC, B_\ell, C_\ell)$ controllable and observable except for the fixed modes of $P(s)$. Consequently, the controller*

$$\begin{aligned} u_i &= L_i y_i, & i \in \mathbb{I}_q \setminus \{\ell\}, \\ u_i &= L_i y_i + M(s) y_i, & i = \ell \end{aligned} \quad (14)$$

with an additional dynamic component $M(s)$ can achieve arbitrary eigenvalue assignment except for the fixed modes.

If $P(s)$ has unstable fixed mode(s), then no decentralized controller can stabilize it. Otherwise, due to genericity [1], [3], almost any decentralized static output feedback L makes $(A+BLC, B_\ell, C_\ell)$ stabilizable and detectable. Hence L can be chosen randomly. The centralized controller $M(s)$ can then be designed using any standard method to stabilize $(A+BLC, B_\ell, C_\ell)$. The resulting overall decentralized controller has the central-relay structure where one control station acts as the central coordinator while the other stations relay the sensory information to spread it over the plant network.

IV. DISTRIBUTED CONTROL

This section will solve the distributed stabilization problem described in Section II. The problem is to design a distributed controller (2) that stabilizes the plant (1), subject to the communication constraint specified by $N(s)$ in (3).

Our approach is to reduce the problem to the decentralized stabilization problem solved in the previous section. To this end, we define an augmented plant $P(s)$ through the process visualized in Fig. 1. The distributed controller (2) is first written as an LFT of $N(s)$ and $\tilde{K}(s)$ as in Fig. 1 (left):

$$\begin{bmatrix} \mu \\ u \end{bmatrix} = \tilde{K}(s) \begin{bmatrix} \eta \\ \mathfrak{y} \end{bmatrix}, \quad \eta = N(s)\mu,$$

where $\tilde{K}(s)$ is a permutation of

$$K(s) := \text{diag}(K_1(s), \dots, K_q(s)).$$

Then define an augmented plant to be the transfer matrix $\tilde{P}(s)$ “seen” by $\tilde{K}(s)$ as in Fig. 1 (middle):

$$\begin{bmatrix} \eta \\ \mathfrak{y} \end{bmatrix} = \tilde{P}(s) \begin{bmatrix} \mu \\ u \end{bmatrix}, \quad \tilde{P}(s) := \text{diag}(N(s), \mathcal{P}(s)).$$

Then apply permutations to the input and output signals of $\tilde{P}(s)$ and $\tilde{K}(s)$ to obtain Fig. 1 (right), where the augmented plant $P(s)$ is constructed so that

$$\begin{aligned} y &= P(s)u, & i &\in \mathbb{I}_q, \\ u &= \text{col}(u_1, \dots, u_q), & u_i &:= \text{col}(\mu_i, \mathfrak{u}_i), \\ y &= \text{col}(y_1, \dots, y_q), & y_i &:= \text{col}(\eta_i, \mathfrak{y}_i), \end{aligned} \quad (15)$$

with the input-output partition $\pi := \varpi + \hat{\varpi}$. Thus, the closed-loop system of the plant $\mathcal{P}(s)$ and the distributed controller (2) is identical to the closed-loop system of an augmented plant $P(s)$ and the decentralized controller $K(s)$:

$$y = P(s)u, \quad u = K(s)y.$$

We then have the following result as a simple consequence of the decentralized control result.

Lemma 2: Consider the plant $\mathcal{P}(s)$ with partition ϖ in (1). Let the communication topology and dynamics be specified by $N(s)$ with partition $\hat{\varpi}$ in (3). Define the augmented plant $P(s)$ in (15) with partition $\pi := \varpi + \hat{\varpi}$ as a permutation of $\text{diag}(N(s), \mathcal{P}(s))$. The following statements are equivalent.

- (i) There exist $K_i(s)$ for $i \in \mathbb{I}_q$ such that the distributed controller (2) stabilizes the plant $\mathcal{P}(s)$.
- (ii) The augmented plant $P(s)$ with partition π has no unstable fixed mode.

Moreover, given $K_i(s)$ for $i \in \mathbb{I}_q$, the following are equivalent.

- (iii) The distributed controller (2) stabilizes the plant $\mathcal{P}(s)$.
- (iv) The decentralized controller (6) stabilizes the augmented plant $P(s)$.

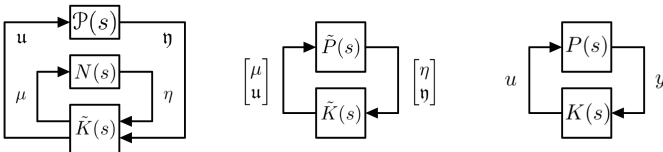


Fig. 1. Plant augmentation with the network dynamics

Proof. The result follows from the decentralized control result in Theorem 1 once we realize that the closed-loop system of the plant $\mathcal{P}(s)$ and the distributed controller (2) is identical to the closed-loop system of the augmented plant $P(s)$ and the decentralized controller $K(s)$. ■

With a decentralized controller $K(s)$ with $N(s) = 0$, the original plant $\mathcal{P}(s)$ may not be stabilizable due to some unstable fixed modes. However, with some appropriate network connection(s) between the control stations $K_i(s)$, it may be possible to stabilize the plant $\mathcal{P}(s)$. This is the case when the augmented plant $P(s)$ has no unstable fixed mode due to the additional connections through $N(s)$. We will give a condition on $N(s)$ that makes this happen.

Let $\mathcal{P}(s)$ and $N(s)$ be described by (any) state space realizations $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and (F, G, H) , respectively. Then the corresponding state space realization of $P(s)$ is given by (A, B, C) where

$$\begin{aligned} A &:= \text{diag}(F, \mathcal{A}), \\ B_i &:= \text{diag}(G_i, \mathcal{B}_i), \quad B := \text{row}(B_1, \dots, B_q), \\ C_i &:= \text{diag}(H_i, \mathcal{C}_i), \quad C := \text{col}(C_1, \dots, C_q), \end{aligned} \quad (16)$$

and $(\mathcal{B}_i, \mathcal{C}_i)$ and (G_i, H_i) are partitioned blocks of $(\mathcal{B}, \mathcal{C})$ and (G, H) in accordance with ϖ and $\hat{\varpi}$, respectively. The fixed modes of these systems are related to each other as follows.

Lemma 3: Let state space systems $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and (F, G, H) be given with input-output partitions ϖ and $\hat{\varpi}$, respectively. Define (A, B, C) by (16) with partition $\pi := \varpi + \hat{\varpi}$. Then the fixed modes are characterized by

$$\begin{aligned} \lambda \in \text{fm}(A, B, C; \pi) &\Leftrightarrow \exists \kappa \subseteq \mathbb{I}_q \text{ s.t. } r_P + r_N < n_P + n_N, \\ \lambda \in \text{fm}(\mathcal{A}, \mathcal{B}, \mathcal{C}; \varpi) &\Leftrightarrow \exists \kappa \subseteq \mathbb{I}_q \text{ s.t. } r_P < n_P, \\ \lambda \in \text{fm}(F, G, H; \hat{\varpi}) &\Leftrightarrow \exists \kappa \subseteq \mathbb{I}_q \text{ s.t. } r_N < n_N, \end{aligned}$$

where n_P and n_N are the dimensions of $\mathcal{A}$ and F , and

$$r_P := \text{rank} \begin{bmatrix} \lambda I - A & B_\kappa \\ C_{\bar{\kappa}} & 0 \end{bmatrix}, \quad r_N := \text{rank} \begin{bmatrix} \lambda I - F & G_\kappa \\ H_{\bar{\kappa}} & 0 \end{bmatrix}.$$

Moreover,

$$\text{fm}(A, B, C; \pi) \subseteq \text{fm}(\mathcal{A}, \mathcal{B}, \mathcal{C}; \varpi) \cup \text{fm}(F, G, H; \hat{\varpi}).$$

Proof. The result can be shown by applying Lemma 1 to (A, B, C) and noting that

$$\text{rank} \begin{bmatrix} \lambda I - A & B_\kappa \\ C_{\bar{\kappa}} & 0 \end{bmatrix} = r_P + r_N$$

due to the block diagonal structure of the system. The last statement about the set inclusion holds because $r_P \geq n_P$ and $r_N \geq n_N$ imply $r_P + r_N \geq n_P + n_N$. ■

Hence, any fixed mode of the augmented system (A, B, C) must be inherited from a fixed mode of the original plant $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ or the communication dynamics (F, G, H) . For most applications, it is reasonable to assume stability of the latter. In this case, any unstable fixed mode of the augmented system (A, B, C) must be an unstable fixed mode of the plant $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. However, an unstable fixed mode λ of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, which makes $r_P < n_P$, becomes movable by the augmentation of the communication channels if $N(s)$ has the property $r_N - n_N \geq n_P - r_P$ for all $\kappa \subseteq \mathbb{I}_q$ at λ . This idea leads to the following result.

Theorem 3: Consider the plant $\mathcal{P}(s)$ with input-output partition ϖ in (1). Let the communication topology and dynamics be specified by $N(s)$ with partition $\hat{\varpi}$ in (3). Define the augmented plant $P(s)$ in (15) with partition $\pi := \varpi + \hat{\varpi}$ as a permutation of $\text{diag}(N(s), \mathcal{P}(s))$. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and (F, G, H) be stabilizable and detectable realizations of $\mathcal{P}(s)$ and $N(s)$, respectively, and let (A, B, C) be the corresponding realization of $P(s)$ as defined in (16). Suppose $N(s)$ is stable. Then, the following statements are equivalent.

- (i) There exist $K_i(s)$ for $i \in \mathbb{I}_q$ such that the distributed controller (2) internally stabilizes the plant $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.
- (ii) For each unstable eigenvalue λ of $\mathcal{A}$ and for any nonempty, proper subset $\kappa \subset \mathbb{I}_q$, we have

$$\text{rank}(N_{\bar{\kappa}\kappa}(\lambda)) \geq n_P - \text{rank} \begin{bmatrix} \lambda I - A & B_\kappa \\ C_{\bar{\kappa}} & 0 \end{bmatrix}, \quad (17)$$

where n_P is the dimension of $\mathcal{A}$, and

$$N_{\bar{\kappa}\kappa}(s) := H_{\bar{\kappa}}(sI - F)^{-1}G_\kappa.$$

When (ii) holds, a stabilizing distributed controller (2) is given by selecting $K_i(s)$ for $i \in \mathbb{I}_q$ such that the decentralized controller (6) stabilizes the augmented plant (A, B, C) .

Proof. The last statement follows from Lemma 2. Suppose (i) holds. Then, from Lemma 2, (A, B, C) has no unstable fixed mode with respect to π . Let κ be a nonempty proper subset of $\mathbb{I}_q$ and let λ be an unstable fixed mode of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Since λ cannot be a fixed mode of (A, B, C) , Lemma 3 implies that $r_P + r_N \geq n_P + n_N$ holds. Since $N(s)$ is stable, λ is not an eigenvalue of F , and it can be shown that

$$T_1 \begin{bmatrix} \lambda I - F & G_\kappa \\ H_{\bar{\kappa}} & 0 \end{bmatrix} T_2 = \begin{bmatrix} \lambda I - F & 0 \\ 0 & N_{\bar{\kappa}\kappa}(\lambda) \end{bmatrix}$$

holds for appropriate nonsingular matrices T_1 and T_2 . Thus, $r_N = n_N + \text{rank}(N_{\bar{\kappa}\kappa}(\lambda))$ holds, and we have (ii) because (17) is satisfied. To show the converse, suppose (i) is violated. Then there exists an unstable fixed mode λ of (A, B, C) , and from Lemma 3, there exists a subset $\kappa \subseteq \mathbb{I}_q$ such that $r_P + r_N < n_P + n_N$. The subset κ must be nonempty and proper since (A, B, C) is stabilizable and detectable. Since λ is unstable, we have $r_N = n_N + \text{rank}(N_{\bar{\kappa}\kappa}(\lambda))$ as shown above. Hence, $\text{rank}(N_{\bar{\kappa}\kappa}(\lambda)) = r_N - n_N < n_P - r_P$. This violates (17) and also implies $n_P > r_P$, which means from Lemma 3 that λ is an unstable fixed mode of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Thus we conclude that (ii) does not hold. ■

For the plant $\mathcal{P}(s)$ and a proper nonempty subset $\kappa \subset \mathbb{I}_q$, the system $(\mathcal{A}, \mathcal{B}_\kappa, \mathcal{C}_{\bar{\kappa}})$ defines the part of the transfer function $\mathcal{P}(s)$ from the input nodes κ to the output nodes $\bar{\kappa}$. The rank property $r_P < n_P$ in Lemma 3 means that λ is a fixed mode of the plant and cannot be moved by a decentralized control due to some missing connection(s) from input u_κ to output $y_{\bar{\kappa}}$. Theorem 3 shows that an addition of the information pathway by the communication from control stations K_κ to $K_{\bar{\kappa}}$ can make λ movable/stabilizable. That is, an unstable fixed mode of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ becomes movable due to $N(s)$ if and only if (17) holds for all $\kappa \subset \mathbb{I}_q$. Note that stable fixed modes could also be made movable if desired by satisfying (17) for all $\kappa \subseteq \mathbb{I}_q$. In any case, the additional connections needed for making a fixed mode λ of $\mathcal{P}(s)$ movable are such that the rank of $N_{\bar{\kappa}\kappa}$

is greater than or equal to $\tau(\lambda, \kappa)$ for the original plant, i.e., the right hand side of the inequality in (17).

The parameter $\tau(\lambda, \kappa)$ for the plant indicates the number of signal transmissions from control stations K_κ to $K_{\bar{\kappa}}$ needed to make λ movable. If $\tau(\lambda, \kappa) \leq 0$, no additional connection from K_κ to $K_{\bar{\kappa}}$ is needed. If $\tau(\lambda, \kappa) > 0$, at least $\tau(\lambda, \kappa)$ connections are needed, and they are provided by $N(s)$ if (17) is satisfied. The number of nonzero entries of $N_{\bar{\kappa}\kappa}(\lambda)$ is the number of connections, among which the number of independent connections is defined to be the rank of $N_{\bar{\kappa}\kappa}(\lambda)$, which is related to the geometric multiplicity of λ through the general equality (9). If λ is a fixed mode of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and its smallest (among all $L \in \mathbb{L}(\pi)$) geometric multiplicity as an eigenvalue of $A + BL\mathcal{C}$ is g , then from Lemma 1, for at least one $\kappa \subset \mathbb{I}_q$, we need g independent connections from stations K_κ to $K_{\bar{\kappa}}$ through $N(s)$ to make λ movable for (A, B, C) .

A condition similar to (17) has been obtained; Theorem 2 in [17] provided a uniform lower bound on the dimensions of station-to-station communication signals, sufficient for making all decentralized fixed modes movable by a strongly connected distributed control. In contrast, Theorem 3 gives an exact characterization of network connections necessary and sufficient for distributed stabilization. We see that an increased dimension of the transmission signal (e.g. the size of the identity matrix in (4)) may help to achieve distributed stabilizability if the multiplicity of a fixed mode is more than one. Otherwise, however, it suffices to transmit scalar signals.

The overall design process is as follows. Stabilizability of the original plant $\mathcal{P}(s)$ via the distributed control (2) can be checked by the algebraic test in (17). If the right hand side of (17) is positive, then λ is a (decentralized) fixed mode of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. This fixed mode is movable by the distributed control if $N(s)$ has connection(s) from stations K_κ to $K_{\bar{\kappa}}$ and satisfies (17). If it is unstable and not movable, stabilization is impossible unless the connection(s) are added. When all unstable fixed modes of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ satisfy (17), a stabilizing distributed controller (2) can be designed by choosing $K_i(s)$, $i \in \mathbb{I}_q$, so that the decentralized control $K(s)$ in (6) stabilizes the augmented plant (A, B, C) . This is done by decomposing the problem into stabilization of the strongly connected sub-networks through the block-triangularization of the augmented plant $P(s)$ as in Theorem 1, and solving each subproblem by the central-relay control as described in Theorem 2.

V. DESIGN EXAMPLE

Consider the chain of m vehicles described by

$$\begin{aligned} \ddot{p}_\gamma &= \sigma_\gamma e_\gamma + \bar{u}_\gamma + d_\gamma, & \gamma \in \mathbb{I}_m, \\ \bar{y}_\gamma &= e_\gamma + v_\gamma, & e_\gamma := p_{\gamma-1} - p_\gamma, \end{aligned}$$

where p_γ is the displacement of the γ^{th} vehicle measured from a nominal position, σ_γ is the gain of the basic controller, $\bar{u}_\gamma$ is an additional control input, $\bar{y}_\gamma$ is the sensory output, d_γ is the disturbance, and v_γ is the sensor noise (Fig. 2). We wish to have all the vehicles return to the nominal distance ($e_\gamma = 0$) after perturbations due to the disturbances d_γ and the change in the leading vehicle position p_0 . However, some human drivers are poorly skilled and apply only the proportional feedback

σ_γ (with $\bar{u}_\gamma = 0$), which makes the response oscillatory. To stabilize a group of vehicles with unskilled human drivers, we insert several autonomous vehicles in the chain.

Suppose there are $m = 5$ vehicles, each of which is referred to as $\mathcal{V}_\gamma$, $\gamma \in \mathbb{I}_m$. Vehicles $\mathcal{V}_\gamma$ with $\gamma \in \mathbb{I}^h := \{2, 4\}$ are driven by unskilled humans with gains $\sigma_2 = 4$, $\sigma_4 = 9$, and no additional input $\bar{u}_\gamma = 0$, while vehicles $\mathcal{V}_\gamma$ with $\gamma \in \mathbb{I}^a := \{1, 3, 5\}$ are autonomously driven by a distributed controller $u = \mathcal{K}(s)\eta$ with $\sigma_\gamma = 0$ for $\gamma \in \mathbb{I}^a$, where $u(t), \eta(t) \in \mathbb{R}^q$ are the vectors obtained by stacking $\bar{u}_\gamma, \bar{\eta}_\gamma$, $\gamma \in \mathbb{I}^a$ in columns, and $q = 3$ is the number of autonomous vehicles. The plant transfer matrix is given by, with partition $\varpi = (1, 1, 1; 1, 1, 1)$,

$$\eta = \mathcal{P}(s)u, \quad \mathcal{P}(s) = \rho_o(s) \begin{bmatrix} -1 & 0 & 0 \\ \rho_2(s) & -1 & 0 \\ 0 & \rho_4(s) & -1 \end{bmatrix},$$

$$\rho_o(s) := \frac{1}{s^2}, \quad \rho_\gamma(s) := \frac{\sigma_\gamma}{s^2 + \sigma_\gamma}, \quad \gamma \in \mathbb{I}^h.$$

The objective is to design a distributed controller $u = \mathcal{K}(s)\eta$ to stabilize the closed-loop system and achieve fast convergence $e_\gamma(t) \rightarrow 0$ for $\gamma \in \mathbb{I}_m$ after temporary perturbations.

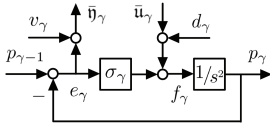


Fig. 2. Vehicle dynamics

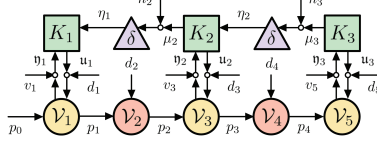


Fig. 3. Vehicle chain with distributed control

The controller $u = \mathcal{K}(s)\eta$ has the distributed structure as in (2), where there are three ($q = 3$) control stations K_i , $i \in \mathbb{I}_q$, for the three autonomous vehicles $\gamma \in \mathbb{I}^a$. The station $K_i(s)$ receives communication signal from other stations $K_j(s)$ as specified by the communication dynamics

$$\eta = N(s)(\mu + h), \quad N(s) := \delta(s)N_o, \quad \delta(s) = \frac{e^{-\tau_d s}}{1 + \tau_l s},$$

where the transmission signal $\mu(t)$ is contaminated by the noise $h(t)$, each communication channel is subject to the transmission delay τ_d (varied) and time lag $\tau_l = 0.01$ s, and the constant matrix N_o specifies the communication topology. We will use the third order Padé approximation for the delay $e^{-\tau_d s}$. An example where each circle (red: human, yellow: autonomous) represents a vehicle with dynamics captured by Fig. 2, the three squares are the control stations, and the triangles are the communication dynamics.

Let us first examine the stabilizability by the decentralized control using the rank condition for the fixed mode in Lemma 1. In particular, the decentralized stabilizability is checked by calculating $\tau(\lambda, \kappa)$ for a minimal realization

TABLE I

THE NUMBER OF INDEPENDENT CONNECTIONS $\tau(\lambda, \kappa)$ NEEDED TO MAKE THE DECENTRALIZED (FIXED) MODE MOVABLE.

κ	{1}	{2}	{3}	{1,2}	{2,3}	{3,1}
$\lambda = 0$	0	0	0	0	0	0
$\lambda = \pm j\sqrt{\sigma_2}$	-1	0	0	-1	1	-1
$\lambda = \pm j\sqrt{\sigma_4}$	-1	-1	1	-1	0	0

($\mathcal{A}, \mathcal{B}, \mathcal{C}$) of the plant $\mathcal{P}(s)$, i.e., the right hand side of the inequality in (17). The result is summarized in Table I, which shows that $\lambda = j\sqrt{\sigma_2}$ and $j\sqrt{\sigma_4}$ are fixed modes since $\tau(\lambda, \kappa) > 0$ for some κ . Thus, the spectrum of $\mathcal{A} + \mathcal{B}\mathcal{L}\mathcal{C}$ contains $\pm j\sqrt{\sigma_2}$ and $\pm j\sqrt{\sigma_4}$ for all 3×3 diagonal constant $\mathcal{L}$, and these modes cannot be stabilized by static or dynamic decentralized control. The fixed modes originate from the dynamics of vehicles $\mathcal{V}_2$ and $\mathcal{V}_4$ which are outside of the decentralized feedback loops (Fig. 3 without δ). Note that $\mathcal{P}(s)$ has the triangular structure where the unstable fixed modes in the off-diagonal entries are not contained in the diagonal entries, and hence stabilization of the transfer functions on the diagonal would not stabilize the whole system (cf. Theorem 1).

Next, we examine stabilizability of $\mathcal{P}(s)$ by a distributed control, which is equivalent to stabilizability of

$$P(s) = \begin{bmatrix} 0 & 0 & \delta N_{12} & 0 & \delta N_{13} & 0 \\ 0 & -\rho_o & 0 & 0 & 0 & 0 \\ \delta N_{21} & 0 & 0 & 0 & \delta N_{23} & 0 \\ 0 & \rho_o \rho_2 & 0 & -\rho_o & 0 & 0 \\ \delta N_{31} & 0 & \delta N_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_o \rho_4 & 0 & -\rho_o \end{bmatrix},$$

i.e., the augmented plant in (15), by a decentralized control, where N_{ij} are the (i, j) block of N_o (no self-loops assumed, $N_{ii} = 0$ for $i = 1, 2, 3$, which does not matter for stabilizability). From Theorem 3, the two entries of “1” in Table I indicate which connections are needed to make the decentralized fixed modes movable by a distributed control. In particular, the mode $\lambda = j\sqrt{\sigma_4}$ becomes movable if one connection is added from control station K_3 to station K_1 or K_2 , and the mode $\lambda = j\sqrt{\sigma_2}$ becomes movable if one connection is added from station K_2 or K_3 to station K_1 . Thus, the distributed controller is able to stabilize the plant with one or two connections, e.g.,

(N_{13}) a single connection $K_3 \rightarrow K_1$, or

(N_{12} & N_{23}) two connections $K_2 \rightarrow K_1$ and $K_3 \rightarrow K_2$.

This makes sense since such communication pathways would form a feedback loop containing the fixed modes at vehicles $\mathcal{V}_2$ and $\mathcal{V}_4$. In each case, $P(s)$ is strongly connected. Note that it suffices to have one-dimensional (i.e. scalar) transmission signal(s) since the maximum entry in Table I is one.

We now design distributed controllers for several cases of the network structure satisfying (17), where $N(s)$ is given by (4) with $I = 1$ and $\delta_{ij}(s) = \delta(s)$ or 0, depending on the presence/absence of the connection. For each case, the augmented plant $P(s)$ has the input-output partition $\pi = (3, 3, 3; 3, 3, 3)$, and we use a minimal realization (A, B, C) of $P(s)$. From Theorem 2, the triple $(A + BLC, B_\ell, C_\ell)$ is stabilizable and detectable for almost all static decentralized gains $L \in \mathbb{L}(\pi)$. For example, the property is achieved for all $\ell \in \mathbb{I}_q$ with

$$L = \text{diag}(L_1, L_2, L_3), \quad L_i = l^T l, \quad l := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad i \in \mathbb{I}_q.$$

We will use this L as the relay control to secure connectedness of the network and design a central controller $M(s)$ at a selected station K_ℓ as in (14) through optimization.

With the controller (14), consider the closed-loop transfer function $T_{zw}(s)$ from w to z , where

$$z := \text{col}(e, u), \quad u \in \mathbb{R}^q, \quad e, d \in \mathbb{R}^m, \\ w := \text{col}(\alpha \ddot{p}_0, d, v, h), \quad v \in \mathbb{R}^q, \quad \ddot{p}_0 \in \mathbb{R}, \quad h \in \mathbb{R}^{2q},$$

with $\alpha = 10$ being a design weight. We wish to design the central controller $M(s)$ by minimizing the H_2 norm of $T_{zw}(s)$. However, the norm is infinite since $T_{zw}(\infty) \neq 0$ due to the (3,3) entry of L_i . To avoid the technical difficulty, we replace the (3,3) entry of L_i by 0, and set $\sigma_\gamma = 1$ for $\gamma \in \mathbb{I}^a$ to preserve the same relay effect (see Fig. 2) while making $T_{zw}(\infty) = 0$ for strictly proper $M(s)$.

The distributed controllers are designed under the same condition except for the communication topology N_o and the location of the central control station ℓ . The design cases reported here are summarized in Table II. The closed-loop responses of vehicle velocities $\dot{p}_\gamma$, $\gamma \in \mathbb{I}_m$, to the impulse input in $\ddot{p}_0$ (with d , v , and h all zero) are shown in Fig. 4.

TABLE II
DESIGN CASES

	network connections	central station	H_2 norm $\tau_d=0.05$	H_2 norm $\tau_d=0.5$	H_2 norm $\tau_d=5$
Design 1	N_{13}	1	48.5	48.5	50.9
Design 2	N_{12}, N_{23}	1	89.3	97.1	111.2
Design 3	N_{12}, N_{23}	2	47.9	52.6	68.7
Design 4	all-to-all	3	33.4	36.9	109.8
Design 5	centralized		23.5		

The result confirms that one (Design 1) or two (Design 2) network connections are enough to stabilize the closed-loop system. For the same communication topology (Designs 2 and 3, Fig. 3), the regulation performance depends on the central controller location. When the connections are all-to-all (Design 4), the overall performance is better, provided the time delay is small. The all-to-all result is not sensitive to the location of the central control station; the H_2 norm ($\tau_d=0.05$) is 32.3 and 33.7 for $\ell = 1$ and 2, respectively. In general, the H_2 performance is degraded for a larger time delay, but the sensitivity depends on the network topology. Finally, with the optimal centralized control with no communication constraints for the plant with $\sigma_\gamma = 1$ for $\gamma \in \mathbb{I}^a$ (Design 5), the H_2 norm is 23.5, and the time response is similar to Design 4 in Fig. 4.

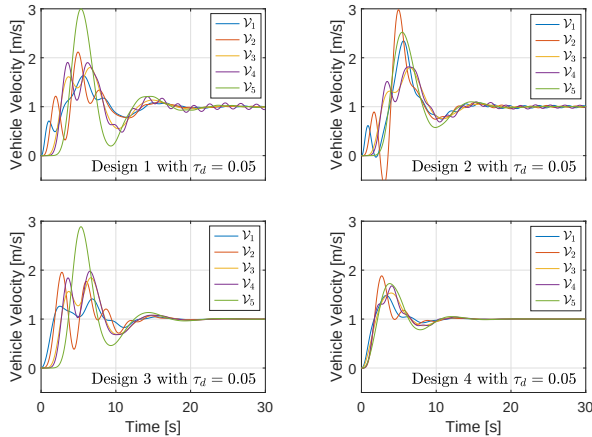


Fig. 4. Closed-loop responses to step velocity $\dot{p}_0 = 1$ of the leading vehicle.

VI. CONCLUSION

We have developed a necessary and sufficient condition for stabilizability via distributed controllers subject to communication constraints. A rank test is shown to indicate which

network connections are needed for stabilization to make unstable decentralized fixed modes movable via distributed control. Broadly speaking, the developed algebraic test can be used prior to optimal distributed control synthesis to ensure feasibility of the design process. A design example illustrated the results with optimal central-relay controllers.

Though we cannot provide details due to the page limitation, the class of structured state space controllers (e.g. [16]) can be exactly characterized as (2) by choosing a constant $N(s)$ with a sparse structure and by restricting $K_i(s)$ to be partially strictly proper from η_i to μ_i . Theorem 3 can then be extended to provide a condition for stabilizability by such controller.

REFERENCES

- [1] J.C. Willems. Decentralized control system stabilization: Time invariant and time varying feedback strategies. *Fourth IMA International Conference on Control Theory*, pages 3–23, 1985.
- [2] S.-H. Wang and E.J. Davison. On the stabilization of decentralized control systems. *IEEE Trans. Auto. Contr.*, 18(5):473–478, 1973.
- [3] J.P. Corfmat and A.S. Morse. Decentralized control of linear multivariable systems. *Automatica*, 12:479–495, 1976.
- [4] D.C. Youla, H.A. Jabr, and J.J. Bongiorno Jr. Modern Wiener-Hopf design of optimal controllers – Part II: The multivariable case. *IEEE Trans. Auto. Contr.*, 21(3):319–338, 1976.
- [5] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Trans. Auto. Contr.*, 51(2):274–286, 2006.
- [6] L. Lessard and S. Lall. Convexity of decentralized controller synthesis. *IEEE Trans. Auto. Contr.*, 61(10):3122–3127, 2016.
- [7] P. Shah and P.A. Parrilo. H_2 -optimal decentralized control over posets: A state-space solution for state-feedback. *IEEE Trans. Auto. Contr.*, 58(12):3084–3096, 2013.
- [8] J.-H. Kim and S. Lall. Explicit solutions to separable problems in optimal cooperative control. *IEEE Trans. Auto. Contr.*, 60(5):1304–1319, 2015.
- [9] C.A. Rösinger and C.W. Scherer. A flexible synthesis framework of structured controllers for networked systems. *IEEE Trans. Control Netw. Syst.*, 7(1):6–18, 2020.
- [10] B. Bamieh, F. Paganini, and M.A. Dahleh. Distributed control of spatially invariant systems. *IEEE Trans. Auto. Contr.*, 47(7):1091–1107, 2002.
- [11] Ş. Sabău, C. Oară, S. Warnick, and A. Jadbabaie. Optimal distributed control for platooning via sparse coprime factorizations. *IEEE Trans. Auto. Contr.*, 62(1):305–320, 2017.
- [12] Y.-S. Wang, N. Matni, and J.C. Doyle. A system-level approach to controller synthesis. *IEEE Trans. Auto. Contr.*, 64(10):4079–4093, 2019.
- [13] L. Furieri, Y. Zheng, A. Papachristodoulou, and M. Kamgarpour. An input-output parametrization of stabilizing controllers: Amidst Youla and system level synthesis. *IEEE Control Syst. Lett.*, 3(4):1014–1019, 2019.
- [14] Y. Zheng, L. Furieri, A. Papachristodoulou, N. Li, and M. Kamgarpour. On the equivalence of Youla, system-level, and input-output parametrizations. *IEEE Trans. Auto. Contr.*, 66(1):413–420, 2021.
- [15] R. D’Andrea and G. Dullerud. Distributed control design for spatially interconnected systems. *IEEE Trans. Auto. Contr.*, 48:1478–1495, 2003.
- [16] A.S.M. Vamsi and N. Elia. Optimal distributed controllers realizable over arbitrary networks. *IEEE Trans. Auto. Contr.*, 61(1):129–144, 2016.
- [17] F. Liu, L. Wang, D. Fullmer, and A.S. Morse. Distributed feedback control of multichannel linear systems. *IEEE Trans. Auto. Contr.*, 68(12):7164–7178, 2023.
- [18] E.J. Davison, A.G. Aghdam, and D.E. Miller. *Decentralized control of large-scale systems*. Springer-Verlag, 2020.
- [19] Y.-S. Wang, N. Matni, and J.C. Doyle. Separable and localized system-level synthesis for large-scale systems. *IEEE Trans. Auto. Contr.*, 63(12):4234–4249, 2018.
- [20] B.D.O. Anderson and D.J. Clements. Algebraic characterization of fixed modes in decentralized control. *Automatica*, 17(5):703–712, 1981.
- [21] L. Lessard, M. Kristalny, and A. Rantzer. On structured realizability and stabilizability of linear systems. *Proc. American Contr. Conf.*, pages 5784–5790, 2013.
- [22] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge, 1990.
- [23] K. Murota and M. Sugihara. *Linear Algebra II: Advanced Topics for Applications*. World Scientific, 2022.