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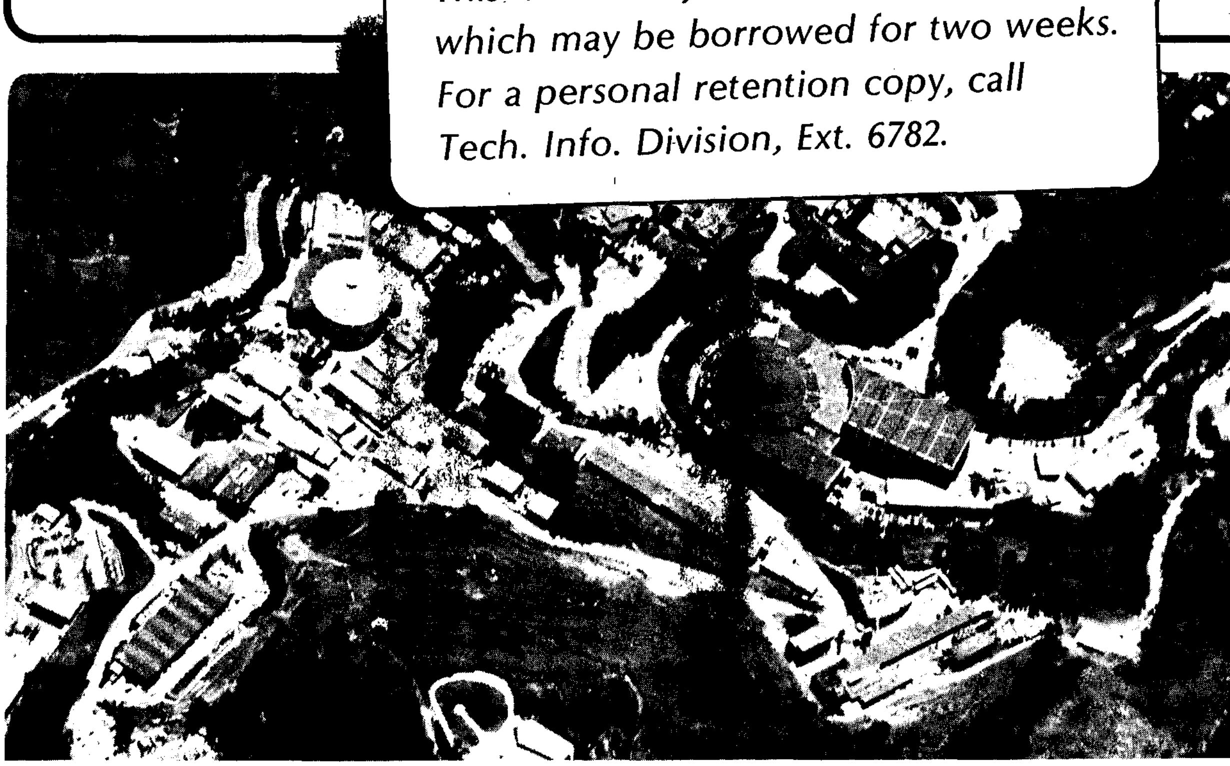

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ON THE EXTREMALS OF A SUBSIDIARY CAPILLARY PROBLEM[1]

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ON THE EXTREMALS OF A SUBSIDIARY CAPILLARY PROBLEM

Paul Concus and Robert Finn

This note refers to the (subsidiary) variational problem for the functional $\Phi[\Gamma]$, as introduced in [1] and described further in the preceding paper [2]. We adopt here the notation and definitions of [2], without further elaboration.

1. It was proved in [2, §5] that whenever $\theta_2 - \theta_1 < \pi$, the second variation $I[\eta]$ of Φ is minimized by a rigid translational motion

$$\eta = a \cos(\theta - \sigma) \tag{1}$$

in a direction σ, which is one of four translational directions along which I is stationary.

In the original capillary problem, the boundary angle γ is prescribed. The extremals for the subsidiary problem must meet the boundary curve Σ with the same angle γ. In general it cannot be expected that a rigid motion (1) will leave γ unvaried; there are however situations in which that occurs, and these situations have a special interest. We prove here:

Theorem. *The second variation* I *of* Φ *vanishes for any rigid motion of an extremal that leaves* γ *unvaried. Further,* I *is stationary (its first variation vanishes) in any such motion.*

Proof: The extremal is a circular arc of radius $R = \Omega/\Sigma \cos\gamma$; since γ is unvaried, so is R_γ. The extremal meets Σ in two points, denoted as 1 and 2 in Figure 1, with intersection angle γ as indicated.

We may characterize the motion as a composition of a rigid translation (1) of the center 0 in direction σ, and a rotation about 0. Since the rotation leaves everything invariant, it can be neglected.

We focus attention first on the point 1, and adopt as parameter to describe the motion the arc length s on Σ. Referring to Figure 1, we find

$$\gamma + \xi + \tau + \delta = \pi/2 \tag{2}$$

$$P_s = -\cos\xi + \sin\xi \cot\sigma \tag{3}$$

$$P = \frac{\sin(\sigma + \tau + \delta)}{\sin\sigma} R_\gamma \tag{4}$$

and thus

$$\cos\xi - \sin\xi \cot\sigma = \frac{-R_\gamma}{\sin\sigma}\left\{(\tau+\delta)_s \cos(\sigma+\tau+\delta) + \gamma_s \tan\gamma \sin(\sigma+\tau+\delta)\right\}$$

so that

$$\sin(\xi - \sigma) = R_\gamma\left\{(\tau+\delta)_s \cos(\sigma+\tau+\delta) + \gamma_s \tan\gamma \sin(\sigma+\tau+\delta)\right\} . \tag{5}$$

For the curvature k_1 of Σ at 1 we find from (2)

$$k_1 = -\xi_s = \gamma_s + (\tau+\delta)_s . \tag{6}$$

We thus obtain from (5) and (2)

$$k_1 = \frac{\cos(\sigma+\gamma+\tau+\delta)}{R_\gamma \cos(\sigma+\tau+\delta)} + \gamma_s[1 - \tan\gamma \tan(\sigma+\tau+\delta)] . \tag{7}$$

An analogous discussion now yields

(8) $$k_2 = \frac{\cos(\sigma - \gamma + \tau - \delta)}{R_\gamma \cos(\sigma + \tau - \delta)} + \gamma_s[1 + \tan\gamma \tan(\sigma + \tau - \delta)] .$$

These relations hold for any translation in the direction σ. In the special case that γ_s vanishes at both contact points, we obtain simplified expressions for k_1, k_2, depending only on R_γ and on the angles $\sigma, \gamma, \tau, \delta$.

We now normalize (as in [2]) by a rotation of coordinates so that $\tau = 0$, and we place the resulting expressions for k_1, k_2 into (29) of [2]. A tedious but formal calculation then yields $I[\eta] = 0$, which was to be proved. Placing the indicated σ into (27) of [2], we verify directly that it provides one of the four solutions of that relation, and hence is an extremal direction for I.

2. The question remains, whether I is minimized by the above choice. We examine the question first in the particular case, for which Σ is a unit circle and Γ an interior circular arc (Figure 2; we note that Γ always contains the center of Σ). The rigid motion for which the center of Γ moves on an arc concentric to Σ then yields $\gamma_s = 0$, $I = 0$ (trivially).

Since $k_1 = k_2 = 1$, we find for a motion of the form (1)

(9) $$I = \frac{\cot\gamma}{R_\gamma}\left(1 - \frac{R_\gamma}{\cos\gamma}\right) - \frac{\sin 2\delta \cos 2\sigma}{1 + \cos 2\delta \cos 2\sigma}$$

which follows by formal calculation from (29) of [2]. In the configuration indicated, (27) of [2] has the roots $\sigma = 0, \frac{\pi}{2}, -\frac{\pi}{2}, \pi$, the roots $\pm\frac{\pi}{2}$ being those that leave γ invariant. For the roots $0, \frac{\pi}{2}$ we obtain for the corresponding I_o, $I_{\pi/2}$

$$I_{\pi/2} - I_o = \frac{2 \sin 2\delta}{1 - \cos^2 2\delta}$$

which is positive if $2\delta < \pi$, so that the "γ-invariant" direction fails to minimize. If $2\delta > \pi$, then I is in fact minimized among rigid motions, and thus $I \geq 0$ for any such motion. Nevertheless, Γ contains a semicircle in this case and hence--as shown in §5 of [2]--there exist other variations for which $I < 0$.

We remark that in the indicated configuration there holds $2\gamma + \delta = \pi$, hence for all situations that occur, we have $\delta + \gamma > \pi/2$ (cf. Theorem 7, Lemma 5 in [2]). Geometrically, this means that on the line L joining the centers of the two circles, the center of Γ lies between the intersection of L with Γ, and the intersection point of L with the line tangent to Σ at the intersection of Σ with Γ.

In all configurations considered, there holds $\Phi[\Gamma] > 0$. In fact, $\Phi[\Gamma] > 0$ is a necessary condition for existence of a solution, and in this case a solution can be obtained explicitly for any γ, as a lower spherical cap.

3. We consider the configuration of Figure 3, in which the smaller circular arc on Σ has radius 1, and the larger arc has radius 1.974, for which one computes easily $\frac{\Omega}{\Sigma} = 1$, independent of h_0. Corresponding to the arc Γ indicated, we have $R_\gamma = \frac{\Omega}{\Sigma \cos \gamma}$ and

$$\Phi = \frac{(\pi - 2\gamma)(2\cos\gamma - 1)}{2\cos^2\gamma} - \pi\cos\gamma + \tan\gamma + \frac{\pi}{2} \tag{10}$$

which is independent of h. A horizontal translation of Γ thus yields $I = 0$. Again we have $k_1 = k_2$ $(= 0)$, so (27) of [2] yields once more the four roots $0, -\frac{\pi}{2}, \frac{\pi}{2}, \pi$. However, in this case the roles of I_0 and of $I_{\pi/2}$ are interchanged, and thus it is now the "γ-invariant" motion that minimizes I.

One verifies easily that in $0 < \gamma < \pi/2$ the value Φ determined by (10) is positive. The only other extremal arcs are the reflections of the indicated ones, for which Φ is still more positive; we thus conclude from the nonexistence-existence property that (as in §2) for any $\gamma \in (0, \frac{\pi}{2})$ a corresponding capillary surface exists. (We note that Φ vanishes when $\gamma = 0$, so that--regardless of h--no surface exists in that case.)

In the configuration indicated, one has $\delta + \gamma = \pi/2$.

4. We consider finally the case of an ellipse. Computer calculations were made for the configuration, for which the ratio of minor to major axis is 0.3 (see Figure 4). It was found that for $\gamma \approx 25.2°$, there is an extremal Γ_o such that $\gamma_s = 0$ for horizontal displacement. Again we have $k_1 = k_2$, we obtain the same four roots of (27) in [2], and we find that the γ-invariant motion minimizes I. We again have $\Phi[\Gamma_o] > 0$; we verify easily that all extremals are symmetric with respect to reflection in an axis of the ellipse, and that for the given γ the only other possibility is a shifted arc Γ_o' (as indicated in the figure) for which again $\Phi > 0$. Thus, a solution of the capillary problem at this value of γ exists.

Corresponding to Γ_o, we have $\delta + \gamma < \pi/2$, however, for Γ_o' there holds $\delta + \gamma > \pi/2$.

5. The calculations for the ellipse have an independent interest extending beyond the above considerations. For each point p of the ellipse, those values of γ were sought, for which an arc Γ through p (not exceeding a semicircle),

of radius $R_\gamma = \frac{\Omega}{\Sigma \cos \gamma}$, will meet the ellipse in two points with angle γ. The results are illustrated, qualitatively in Figure 4, quantitatively in Figure 5, for an ellipse of semi-major axis $a = 1$ and semi-minor axis $b = 0.3$. For each p, a unique γ was found. A unique point p_o yielded $\gamma = 0$, corresponding to an inscribed circle of radius $R_o = \frac{\Omega}{\Sigma}$. From each side of this circle emanates a family of extremals with varying γ: on the left, γ increases from zero until $\pi/2$ is attained on the minor axis; on the right, γ increases to a maximum $\gamma_m \approx 25.2°$ at a corresponding extremal Γ_m, then decreases back to zero (which is not attained) at the right vertex. The entire configuration is repeated by reflection in the minor axis.

At Γ_m we have $d\gamma/ds = 0$, hence also $dR_\gamma/ds = 0$. The analysis of §1 thus applies at this point, and forms the basis for §4.

Figure 5 shows I and Φ as functions of γ; the corresponding x-coordinates on the major axis are indicated on the curves.

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[2] Finn, R.: A subsidiary variational problem and existence criteria for capillary surfaces, preceding in this issue.

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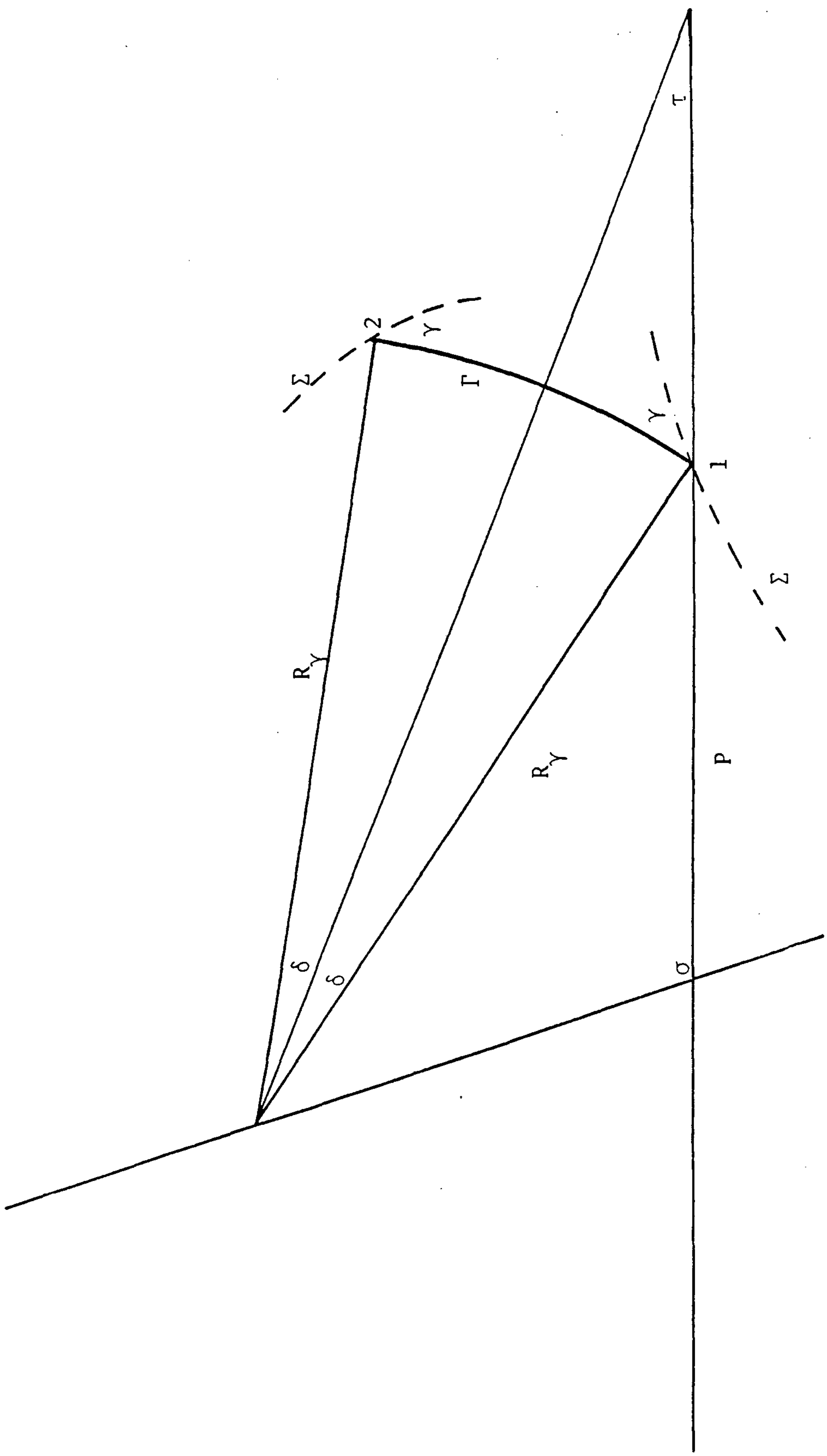

Figure 1

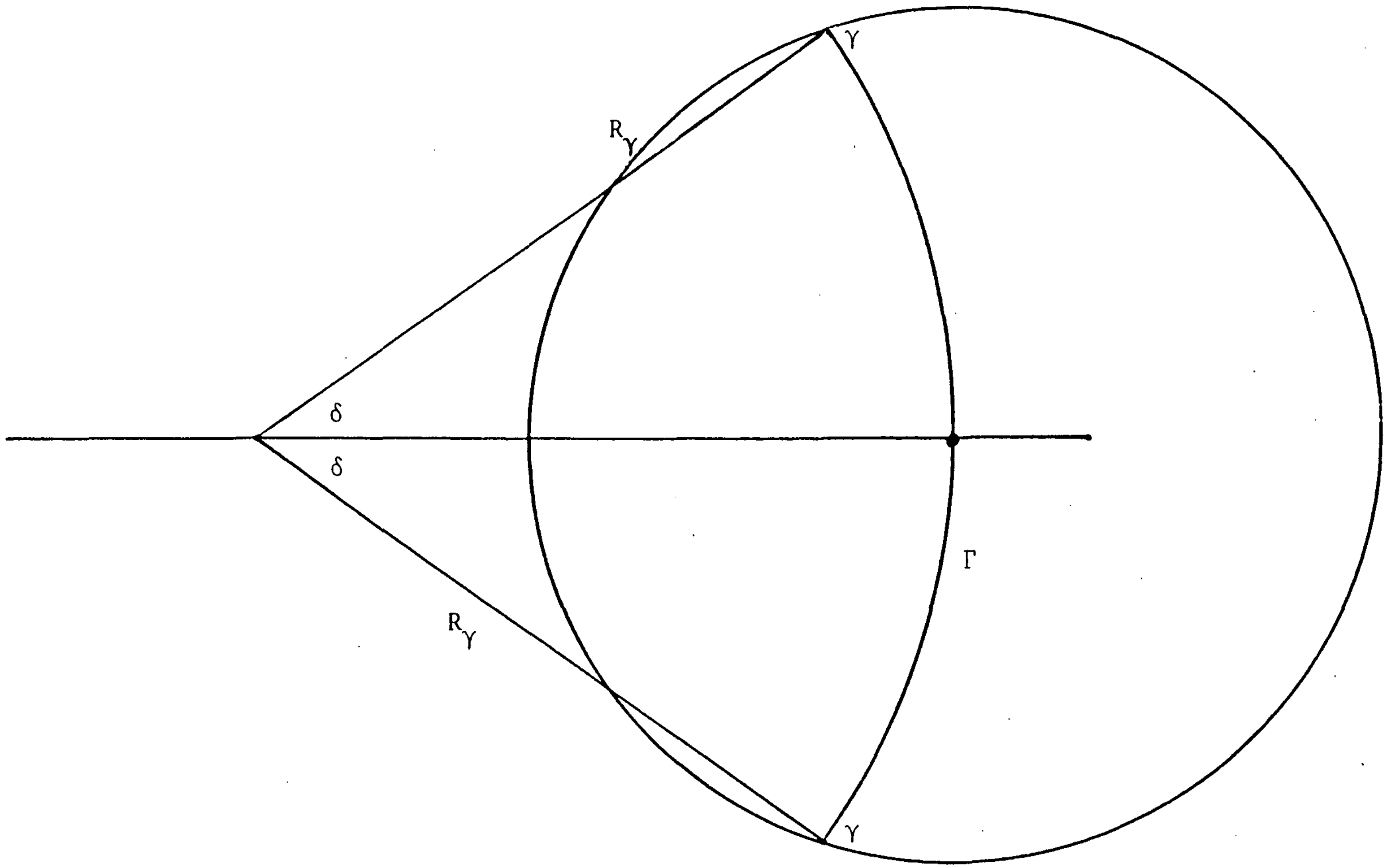

Figure 2

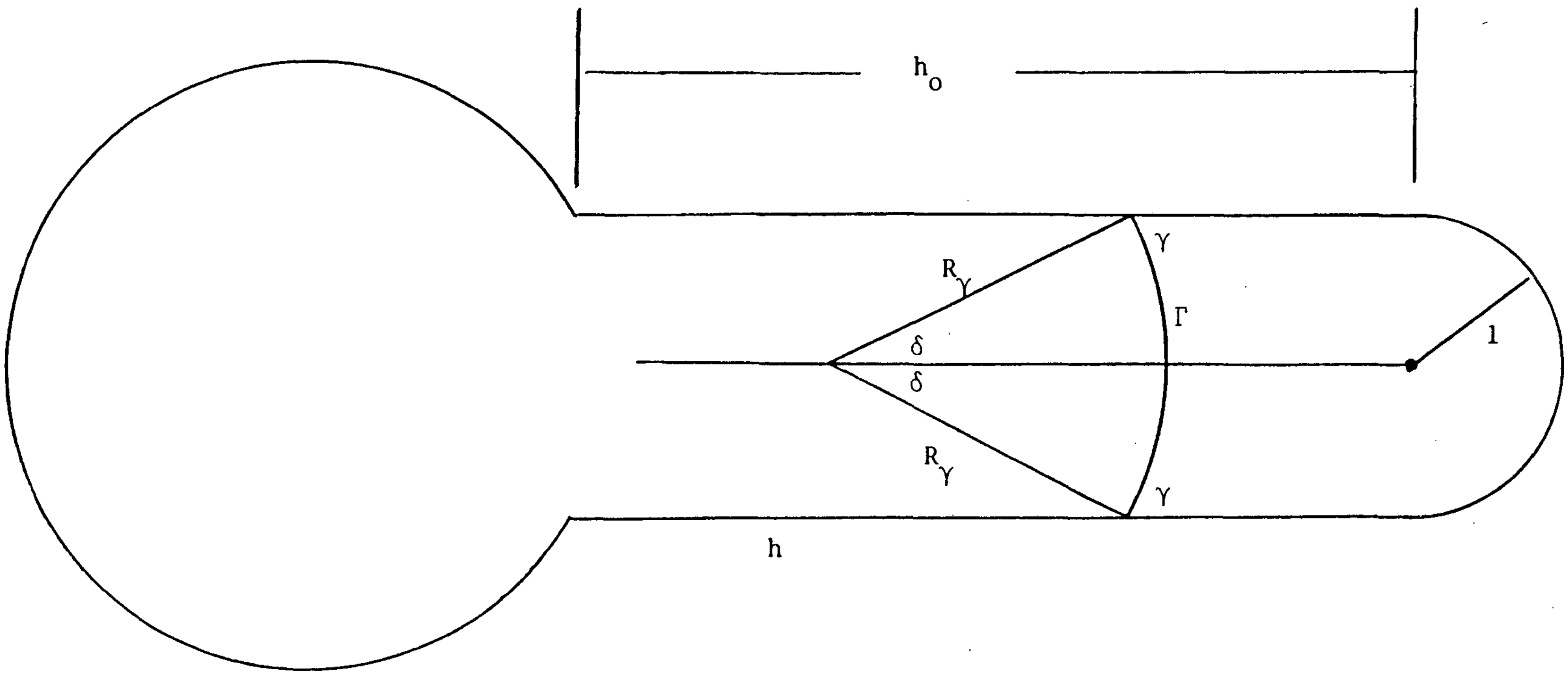

Figure 3

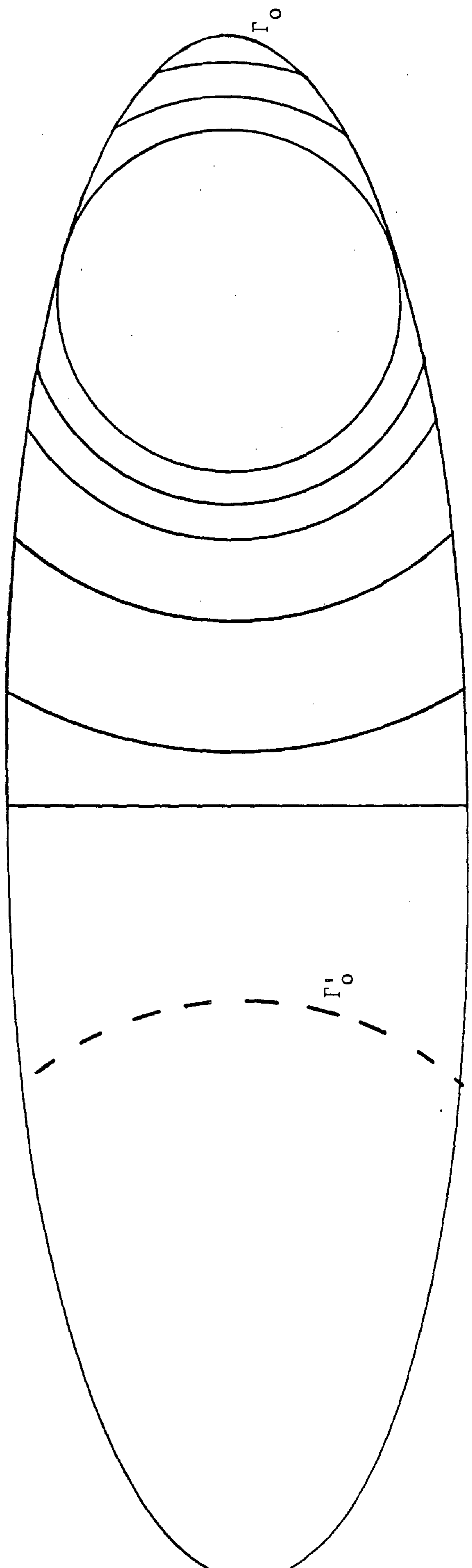

Figure 4

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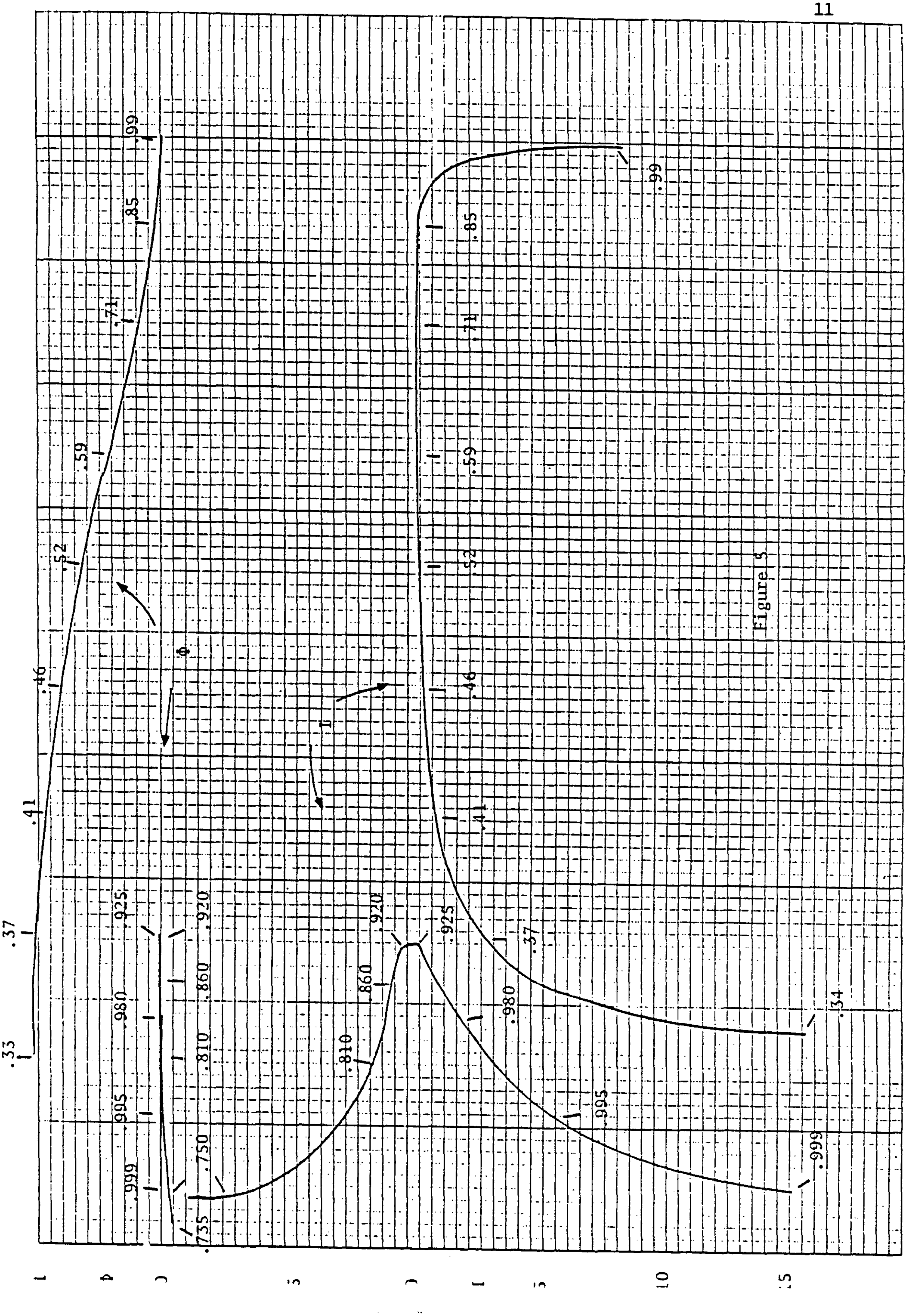

Figure 5

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