Title
EQUILIBRIUM FLUID INTERFACES IN THE ABSENCE OF GRAVITY

Author
Concus, P.

Publication Date
1986-10-01

LBL-22388 c.2

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Physics Division

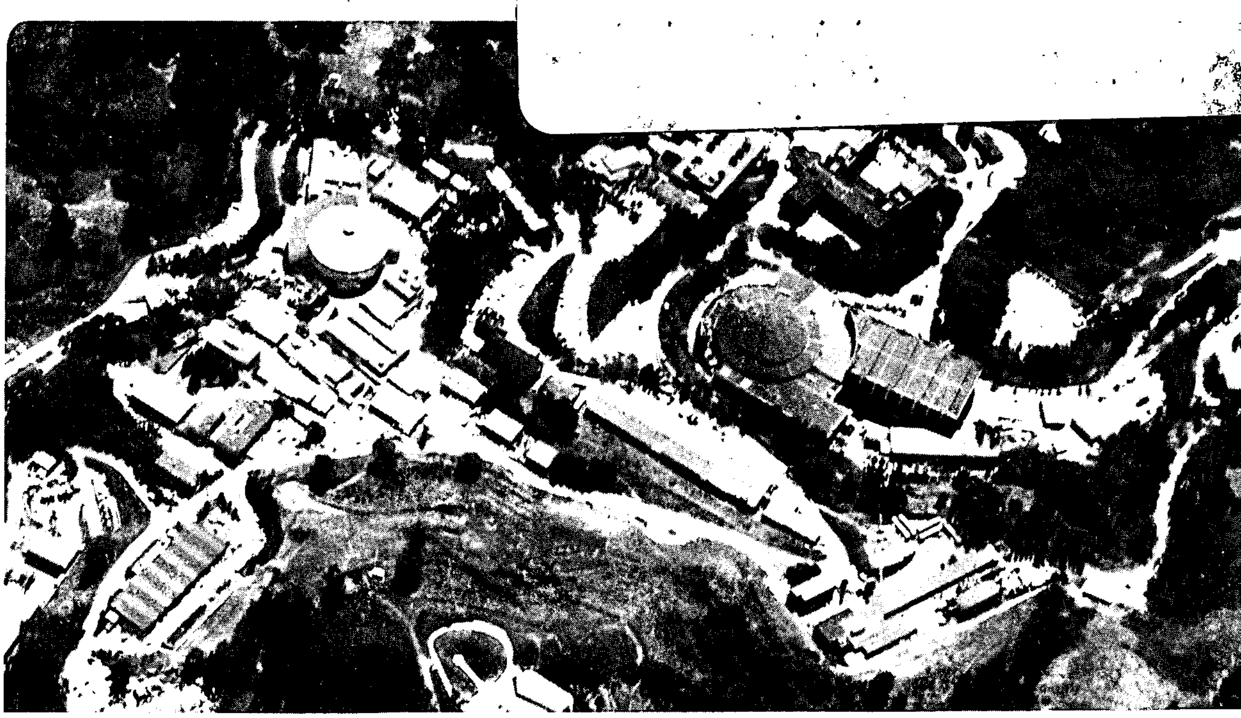
Mathematics Department

To be presented at the ASME Microgravity Fluid Mechanics Symposium, Anaheim, CA, December 11-18, 1986

EQUILIBRIUM FLUID INTERFACES IN THE ABSENCE OF GRAVITY

P. Concus

October 1986



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

DISCLAIMER

LBL-22388

EQUILIBRIUM FLUID INTERFACES IN THE ABSENCE OF GRAVITY[1]

Paul Concus

Lawrence Berkeley Laboratory
and
Department of Mathematics
University of California
Berkeley, California 94720

[1]Supported in part by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, U.S. Department of Energy under contract DE-AC03-76SF00098.

EQUILIBRIUM FLUID INTERFACES IN THE ABSENCE OF GRAVITY

Paul Concus

Lawrence Berkeley Laboratory
and
Department of Mathematics
University of California
Berkeley, California 94720

Some striking mathematical properties of capillary surfaces under zero gravity conditions are discussed. We consider the equilibrium interface between a liquid and a gas, or between two immiscible liquids, in a cylindrical container of general cross-section. According to the classical Laplace-Young theory, an equilibrium surface in the absence of gravity is determined by the equations

$$\text{div } Tu = \text{const.} \quad \text{in } \Omega, \tag{1}$$

$$Tu \cdot \nu = \cos\gamma \quad \text{on } \Sigma, \tag{2}$$

where

$$Tu = \frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}}.$$

In these equations, Ω is the cross section of the cylindrical container, Σ is the boundary of Ω, ν is the exterior unit normal, and $u(x,y)$ is the (single-valued) height of the surface above a reference plane, as indicated in Fig. 1. The quantity γ, the contact angle between the interface and the container wall, is a constant determined by the material properties of the physical system. The volume of the liquid in contact with the base is assumed to be sufficient to cover the base entirely. We consider here the wetting case, $0 \leq \gamma < \pi/2$; the non-wetting case can be obtained immediately by a simple transformation.

As an illustration of the striking behavior that can occur, consider a partly filled cylindrical container whose base Ω is a regular polygon having interior angles 2α and circumscribing a circle of radius a. Suppose first that there holds $\alpha + \gamma \geq \pi/2$. Then, in the absence of gravity, the unique solution (up to an additive constant) of the equilibrium equations (1), (2) is the spherical cap $u = -(a^2 - (x^2 + y^2)\cos^2\gamma)^{1/2}\sec\gamma$. However, if $\alpha + \gamma < \pi/2$, it can be proved that there is no surface satisfying the equilibrium equations.

This behavior was tested experimentally in the Zero-gravity Facility at the NASA Lewis Research Center by W. Masica. Fig. 2 depicts the results obtained in a drop-tower experiment for a cylinder with regular hexagonal cross-section, for which $\alpha = \pi/6$. The photographs show the configuration after approximately five

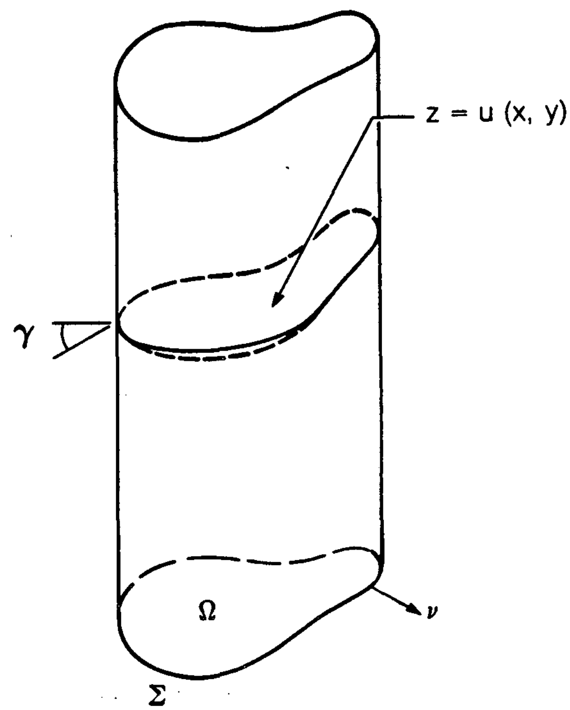
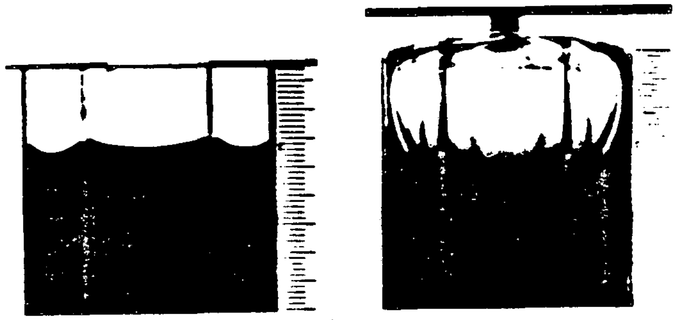


Fig. 1. A partly filled cylinder with section Ω having boundary Σ and exterior normal ν.

seconds of free fall (the maximum available), at which time the approach to equilibrium appears to be reasonably complete. In case (b), for which $\alpha + \gamma < \pi/2$, the fluid is apparently pumped up by surface forces in the corners to the top of the container. The mathematical results predict a discontinuous dependence of the solution on the data (α or γ) as the critical condition $\alpha + \gamma = \pi/2$ is traversed [1].



(a) (b)

Fig. 2. Equilibrium free surface in a regular hexagonal cylinder at zero gravity.

(a) $\gamma \approx 48^\circ$, (b) $\gamma \approx 25^\circ$.

For a cylinder of general cross section, the discussion is less immediate. Even for smooth sections without corners, solution surfaces may or may not exist, depending in a non-evident way on the shape of the container. Based on the results in [3,4] one can reduce the question of whether a (two-dimensional) surface satisfying (1),(2) will exist to a mathematical procedure involving simply (one-dimensional) circular arcs. These results from [3,4] are as follows:

Let Ω^* be a portion of Ω, bounded by a curve (or curves) Γ in Ω and by a portion Σ^* of the boundary Σ of Ω. Let the functional $\Phi[\Omega^*;\gamma]$ be defined by

$$\Phi[\Omega^*;\gamma] \equiv |\Gamma| - |\Sigma^*| \cos\gamma + \frac{|\Sigma|}{|\Omega|} |\Omega^*| \cos\gamma,$$

where the symbols $|\Gamma|$, $|\Sigma|$, $\cdots$ denote the length or area of the indicated geometric quantity. Then, *a capillary surface for the section Ω exists if and only if Φ is minimized by the null set $\Omega^* = \emptyset$ and by the entire section $\Omega^* = \Omega$.* Furthermore, *if Φ is minimized by Ω^*, where $\Omega^* \neq \emptyset$ and $\Omega^* \neq \Omega$, then the boundary Γ of Ω^* in Ω consists of a finite number of subarcs of semicircles of radius $R_\gamma = \dfrac{|\Omega|}{|\Sigma| \cos\gamma}$ that meet Σ with angle γ.*

Thus it is necessary only to search for all possible arcs Γ interior to Ω satisfying the stated minimizing conditions, and for each of them to calculate Φ. If Φ is always positive, then, since $\Phi[\emptyset;\gamma] = \Phi[\Omega;\gamma] = 0$, one concludes that the derived surface exists. If $\Phi < 0$ for some such Γ, then no surface exists.

The procedure based on the above results yields, for example, that for a bathtub-shaped domain whose boundary Σ consists of two parallel sides joined by semicircular ends a capillary surface will exist for any contact angle γ, regardless of the length and width of the domain. The situation can change strikingly for this example, however, if the end circles have different radii and the two straight lines are not parallel. For sufficiently long domains, there is a unique critical angle γ_{cr} such that a surface will exist only for contact angles $\gamma_{cr} < \gamma \leq \pi/2$ and will fail to exist for angles $0 \leq \gamma \leq \gamma_{cr}$. The value of γ_{cr} can be made as close as desired to $\pi/2$ by choosing a domain with sufficiently long sides that are sufficiently close to being parallel.

In addition to the above seemingly paradoxical result, other qualitative differences can be observed when the cross-section departs from the regular polygon. As γ decreases toward γ_{cr} for the bathtub-shape configuration with different end radii, it can be proved that the solution surfaces, which exist for each $\gamma > \gamma_{cr}$, tend to infinite height throughout Ω^*, while the surfaces converge smoothly, remaining bounded, throughout the remainder of Ω. Further examples and details can be found in [1]-[4].

The author's work reported here has been carried out jointly with R. Finn. In collaboration with D. Coles and L. Hesselink we are planning a space experiment to investigate the above and other features of capillary surfaces. This work has been supported in part by the National Aeronautics and Space Administration under grant NAG3-146 and by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, Department of Energy, under contract DE-AC03-76SF00098.

REFERENCES

[1] P. Concus and R. Finn, *On capillary free surfaces in the absence of gravity,* Acta Math., 132 (1974), pp. 177-198.

[2] P. Concus and R. Finn, *Continuous and discontinuous disappearance of capillary surfaces,* Report LBL-21002, Lawrence Berkeley Laboratory, University of California, 1986.

[3] R. Finn, *Existence criteria for capillary free surfaces without gravity,* Ind. Univ. Math. J., 32 (1983), pp. 439-460.

[4] R. Finn, *A subsidiary variational problem and existence criteria for capillary surfaces,* J. Reine Angew. Math., 353 (1984), pp. 196-214.

LAWRENCE BERKELEY LABORATORY
TECHNICAL INFORMATION DEPARTMENT
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720