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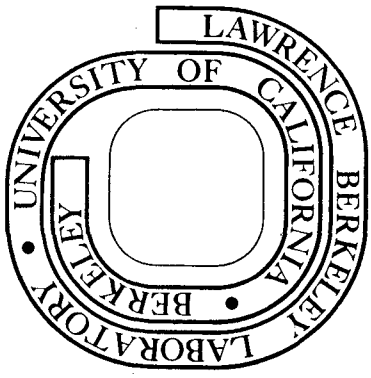

DISCLAIMER

ON THE GENERAL FORM OF THE CROSS-SECTION OF DEEP INELASTIC COLLISIONS*

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ABSTRACT

The general form of the energy-averaged "macroscopic" cross-section for deep inelastic (DI) collisions is derived on the basis of semi-classical approximations. The amplitude for DI reactions is related to the incompletely relaxed part of the fluctuating S-matrix. The possibility of diffraction effects modifying the DI cross-section is investigated. In the limit of a classical treatment of the external variables, the average DI cross-section is shown to be uniquely determined by a classical distribution function.

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1. Introduction

In studies of collisions between heavy ions, a new type of nuclear reactions was discovered[1-4]) and termed "deep inelastic (DI)", "Quasi-fission (QF)" or "heavily damped" reaction. In fact, the first indication of this reaction mechanism had been seen in a rather early experiment[5]) in 1959 which was not followed by more extensive investigations.

The most prominent features of the DI reactions are:

i) A substantial part of the initial kinetic energy of relative motion is transferred into other degrees of freedom, so that the final kinetic energy of the outgoing fragments is close to their reciprocal Coulomb energy in a contact configuration — which is the situation encountered in nuclear fission ("quasi-fission") — .

ii) The angular distribution of the final nuclei is not symmetric with respect to 90^{o} (CMS), thus ruling out the formation of a compound nucleus in the conventional sense. Instead it is peaked sideways or in forward angles depending on the system, on the energy of the incident particles referred to the interaction barrier, and on the final channels considered.

iii) In recent experiments[6]), a gradual transition to isotropic angular distributions $\frac{d\sigma}{d\Omega}$ has been observed in many systems as one moves to final states which involve a large transfer of mass and apparently a longer lifetime of the intermediate complex.[7])

Recent surveys of the rapidly growing wealth of experimental and theoretical work are put together in ref. 8. Apparently, there is general agreement that we ought to look at the deep inelastic reactions as a phenomenon of partial relaxation of certain external, "macroscopic" variables, which, in a lowest approximation, can be treated classically.

Thus several authors[9-14]) described the reactions in terms of classical equations of motion for certain "external" degrees of freedom with the inclusion of friction forces to describe the coupling to the "internal" variables. Derivations of these equations have been given from perturbation theory of the coupling between external and internal degrees,[15]) linear response theory[16]) and the Mori formalism.[17]) These theories yield only the mean values of the external variables.

Distributions of these variables were obtained[18-20]) on the basis of Master- or Fokker-Planck equations invoking diffusion or transport processes between the nuclei in contact. Different derivations of these equations were presented[21,22]) also leading to different types of Fokker-Planck equations.*

A somewhat different picture is pursued by Broglia et al and Glas and Mosel[23]). Here the usual semi-classical theory of nuclear reactions is extended so as to include the excitation of a large number of intrinsic excitations. In the case of the work of Broglia et al, the excited modes are harmonic vibrations. This is reminiscent of the theory of atomic and molecular collisions where the excitation of vibrational modes is known to be the main origin of the damping.

In all the theories, the cross-section for DI-reactions involves only squares of amplitudes, i.e. probabilities. It is, of course a prerequisite of any description based on classical statistical mechanics that the cross-section can be formulated as a function of probabilities only.

It is the purpose of this paper (i) to investigate the conditions for which the cross-section of Dl reactions is a function of "coarse

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*footnote: In this respect we also wish to draw the reader's attention to a very recent preprint by D. Agassi, C. M. Ko and H. A. Weidenm ller, MPI for Kernphysik, Heidelberg, which arrived after completion of our paper.

probabilities" only,

(ii) to establish the connection between the coarse probability and the basic S-matrix,

(iii) to find out whether and how diffraction effects may influence the DI cross-section.

In any case, the actual experiments involve an averaging over the energy of the incident beam and a summation over all the microscopic final channels which are compatible with a few measured "macroscopic" observables like the scattering angle, masses, charges, and excitation energies. We shall argue that interference terms may be disregarded on account of this double averaging procedure.

Our final result will be that the measured average cross-section for DI reactions can indeed be written as a function of a macroscopic probability distribution only. In order to achieve this result we proceed as follows:

(i) We define the quantal amplitude for DI reactions as a function of the incompletely equilibrated part of the fluctuating S-matrix (§2.1).

(ii) We evaluate this amplitude within the semi-classical theory using, separately, the pure "stationary phase approximation" (SPA) (§2.2) and the SPA with sharp lower and upper angular momentum cut-offs (§2.3). The sharp cut-offs are to represent the effect of a sudden onset of direct and compound reactions (and thus "absorption") as a function of the orbital angular momentum while the pure SPA may be applied if this transition turns out to be very smooth.

The cut-offs result in diffraction effects which are consecutively smoothened by the double averaging process.

(iii) In § 3, we perform a summation of the microscopic cross-section within "coarse cells" defined by external variables only and show that the resulting "macroscopic cross-section" may be calculated, once a classical distribution function for the external variables is known.

As will be seen, the theory leans on the use of the "Poisson representation" of the reaction amplitude[24] whose different discrete terms are semi-classically related to the number of revolutions of the intermediate system. We assume that at least that part of the DI reactions which shows a strongly peaked angular distribution $(\frac{d\sigma^{DI}}{d\vartheta})$ is due to the "no orbiting term" of the Poisson representation. Therefore, only this term is considered in the chapters 2 and 3, whereas the generalization to arbitrary numbers of orbitings is presented in appendix A1. Appendix A2 contains an improved Airy treatment of the rainbow region. In § 4 we summarize the results and lay emphasis on critical physical assumptions as well as open problems.

2. The General Form of the Amplitude for Deep Inelastic Heavy Ion Reactions

2.1 QUANTAL FORM OF THE DI AMPLITUDE

We only consider experiments with unpolarized beam and target particles and without measurement of the spin direction of the outgoing particles. The cross-section of any reaction with two final particles is given as a function of the reaction amplitude $f_{\beta I_{1\beta} M_{1\beta} I_{2\beta} M_{2\beta}; \alpha I_{1\alpha} M_{1\alpha} I_{2\alpha} M_{2\alpha}}$ by

$$\frac{d\sigma_{\beta I_{1\beta} I_{2\beta}; \alpha I_{1\alpha} I_{2\alpha}}}{d\Omega} = \frac{1}{(2I_{1\alpha}+1)(2I_{2\alpha}+1)} \cdot \sum_{M_{1\alpha}, M_{2\alpha}, M_{1\beta}, M_{2\beta}} \frac{k_\beta m_\alpha}{k_\alpha m_\beta} \left| f_{\beta I_{1\beta} M_{1\beta} I_{2\beta} M_{2\beta}; \alpha I_{1\alpha} M_{1\alpha} I_{2\alpha} M_{2\alpha}} \right|^2 \quad (2.1)$$

where $I_{1\alpha}, I_{2\alpha}$ $(I_{1\beta}, I_{2\beta})$ and $M_{1\alpha}, M_{2\alpha}$ $(M_{1\beta}, M_{2\beta})$ are the intrinsic spins and the corresponding magnetic quantum numbers of particle 1 and 2 in the entrance (exit) channel α (β); k_α, m_α (k_β, m_β) are the relative wave number and reduced mass in the entrance (exit) channel α (β). Transforming into the representation where the intrinsic spins are coupled to the channel spin s_α (s_β) and its projection $M_\alpha(M_\beta)$ we find:

$$\frac{d\sigma_{\beta I_{1\beta} I_{2\beta}; \alpha I_{1\alpha} I_{2\alpha}}}{d\Omega} = \frac{1}{(2I_{1\alpha}+1)(2I_{2\alpha}+1)} \sum_{s_\alpha s_\beta M_\alpha M_\beta} \frac{k_\beta m_\alpha}{k_\alpha m_\beta} \left| f_{\beta I_{1\beta} I_{2\beta} s_\beta M_\beta; \alpha I_{1\alpha} I_{2\alpha} s_\alpha M_\alpha} \right|^2 \quad (2.1')$$

Referring to the coordinate system S defined in section 2.2 and to the CM system, the reaction amplitude in the channel spin representation is given by [25])

$$f_{\beta I_{1\beta} I_{2\beta} s_\beta M_\beta ; \alpha I_{1\alpha} I_{2\alpha} s_\alpha M_\alpha} = \frac{i\sqrt{\pi}}{\sqrt{k_\alpha k_\beta}} \sum_{l_\alpha l_\beta I m_\beta} (2l_\alpha + 1)^{\frac{1}{2}} \cdot$$

$$\langle l_\alpha 0 s_\alpha M_\alpha | I M_\alpha \rangle \langle l_\beta m_\beta s_\beta M_\beta | I M_\alpha \rangle \cdot$$

$$\cdot i^{l_\alpha - l_\beta} Y_{l_\beta m_\beta}(\vartheta, \varphi) \left[\delta_{\alpha\beta} \delta_{l_\alpha l_\beta} - S^{I}_{\beta I_{1\beta} I_{2\beta} s_\beta l_\beta ; \alpha I_{1\alpha} I_{2\alpha} s_\alpha l_\alpha} \right] \tag{2.2}$$

where $l_\alpha (l_\beta)$ is the orbital angular momentum in the entrance (exit) channel and I is the total angular momentum. The symbol $\alpha(\beta)$ denotes all the quantum numbers not explicitly listed which are needed to completely specify an entrance (exit) channel. Since we do not consider the elastic channel, we, henceforth, drop the term with $\delta_{\alpha\beta}\, \delta_{l_\alpha l_\beta}$ in (2.2). Furthermore, in what follows, we omit the explicit listing of the quantum nrs $I_{1\alpha}$, $I_{2\alpha}$, $I_{1\beta}$, $I_{2\beta}$ for the amplitudes.

For heavy ion reactions, the summation in (2.2) extends effectively over a wide range of orbital angular momenta, and, at the same time, the main contributions originate from angular momenta which are large compared to 1. We may, therefore, use the following asymptotic approximations for the spherical harmonics and the Clebsch-Gordan coefficients[26]):

$$Y_{\lambda\mu}(\vartheta,\varphi) \approx \frac{e^{i\mu\varphi}}{\sqrt{2\pi}\sqrt{\sin\vartheta}} \cos\left[(\lambda+\tfrac{1}{2})\vartheta + \mu\tfrac{\pi}{2} - \tfrac{\pi}{4}\right] \tag{2.3}$$

valid for $\lambda \gg 1$ in the angular range $\frac{|\mu|}{\lambda} \leq \vartheta \leq \pi - \frac{|\mu|}{\lambda}$

$$\langle l_1 m_1 s m | l_2 m_2 \rangle \approx D^{s}_{l_1-l_2,\,-m}(0,\alpha,0) \tag{2.4}$$

valid for $l_{1,2} \gg s$ with α defined by

$$\cos\alpha = \frac{m_1}{\sqrt{l_1(l_1+1)}} \tag{2.5}$$

In heavy ion reactions, the orbital angular momenta l_α, l_β which effectively contribute to the cross-section are large compared to the channel spins in the entrance and exit channels: $l_\alpha \gg s_\alpha$; $l_\beta \gg s_\beta$. Since the projection M_β is of the order of the channel spin quantum nrs s_α, s_β we have

$$\cos\alpha = \frac{M_\beta}{\sqrt{l_\beta(l_\beta+1)}} \approx 0 \quad \text{or} \quad \alpha \approx \frac{\pi}{2} \tag{2.5'}$$

We, therefore, use the asymptotic formula (2.4) with $\alpha = \frac{\pi}{2}$ for both the CG-coefficients in (2.2) . Inserting (2.3)-(2.5) into (2.2), introducing the new summation variables $M'_\beta = (I - l_\beta)$; $M'_\alpha = (I - l_\alpha)$; $l = l_\beta$ and using the Poisson representation[24]), we arrive at the following form of the reaction amplitude:

$$f_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha}(\vartheta,\varphi) = \frac{1}{i\sqrt{2\pi k_\alpha k_\beta \sin\vartheta}} \sum_{p=-\infty}^{+\infty} e^{-ip\pi + i(M_\alpha - M_\beta)\varphi} \cdot$$

$$\sum_{M'_\alpha M'_\beta} D^{s_\alpha *}_{M'_\alpha M_\alpha}\left(\frac{\pi}{2},\frac{\pi}{2},-\frac{\pi}{2}\right) D^{s_\beta}_{M'_\beta M_\beta}\left(\frac{\pi}{2},\frac{\pi}{2},-\frac{\pi}{2}\right) \cdot$$

$$\cdot \left\{ \tilde{I}^{(+)}_{M'_\alpha M'_\beta p} + (-)^{M_\alpha - M_\beta} \tilde{I}^{(-)}_{M'_\alpha M'_\beta p} \right\} \qquad (2.6)$$

We use the D-functions in the convention of ref. 27. The integrals $\tilde{I}^{(\pm)}_{M'_\alpha M'_\beta p}$ are defined by

$$\tilde{I}^{(\pm)}_{M'_\alpha M'_\beta p} := (-)^{M'_\beta - M'_\alpha} e^{\pm i\frac{\pi}{4}} \int_0^\infty d\ell\, (\ell+\tfrac{1}{2})^{\frac{1}{2}} \cdot$$

$$\cdot e^{\mp i(\ell+\frac{1}{2})(\vartheta \mp 2\pi p)} S_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(\ell) \qquad (2.7)$$

In (2.7), $S_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(\ell)$ is the quantity obtained by replacing the variables I, ℓ_α, ℓ_β in $S^{I}_{\beta s_\beta \ell_\beta ; \alpha s_\alpha \ell_\alpha}$ by the variables $M'_\alpha, M'_\beta, \ell$

$$S^{I}_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(\ell) \equiv S^{I}_{\beta s_\beta \ell_\beta ; \alpha s_\alpha \ell_\alpha} \qquad (2.7')$$

So far the only approximations used are the asymptotic expressions (2.3) for the spherical harmonics and (2.4), (2.5) for the Clebsch-Gordan coefficients, both of them valid for large orbital angular momentum. Therefore, the reaction amplitude (2.6) still describes direct, deep inelastic, as well as compound nuclear reactions. The summation index p in the Poisson representation may be classically interpreted as the number of orbitings (see eq. (A1- 12)). Consequently, one expects that for compound nuclear processes all the terms of the p-sum in (2.6) are of comparable importance. On the other hand, semi-quantal treatments of direct reactions have shown that contributions from orbiting trajectories ($p \neq 0$) are unimportant except close to resonances of given partial waves where anyhow different approaches are adequate[28]).

For DI reactions, the question whether the terms with $p \neq 0$ in (2.6) are of importance is still open to discussion (see appendix A1). In order to single out the DI reactions from the total amplitude we proceed as follows: First we decompose the S-matrix into an energy-averaged matrix $\langle S \rangle$, which describes the direct reactions, and the fluctuating part

$$S_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(\ell) = \langle S_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(\ell) \rangle + S^{ff}_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(\ell) \qquad (2.8)$$

The average $\langle A \rangle$ over a quantity A(E) depending on the total energy E of the system is defined by

$$\langle A(E) \rangle := \frac{1}{\Delta} \int_{E-\frac{\Delta}{2}}^{E+\frac{\Delta}{2}} dE' \, A(E') \tag{2.9}$$

The energy interval appropriate for defining the direct amplitude is of the order of 1 MeV. The fluctuating part of the S-matrix contains both the compound nuclear and the deep inelastic reactions. We define the formation of a "compound nucleus" (occasionally referred to as "complete fusion") to be described by that part S^{CN} of S^{fl} which corresponds to a statistical equilibrium of all the degrees of freedom of the intermediate system. The remainder S^{DI} represents the DI collisions where only a partial equilibration is achieved, more precisely, where a limited number of external (macroscopic) degrees of freedom do not reach statistical equilibrium:

$$S^{fl}_{\beta s_\beta M_\beta'; \alpha s_\alpha M_\alpha'}(\ell) = S^{CN}_{\beta s_\beta M_\beta'; \alpha s_\alpha M_\alpha'}(\ell) + S^{DI}_{\beta s_\beta M_\beta'; \alpha s_\alpha M_\alpha'}(\ell) \tag{2.10}$$

The decomposition (2.8) and (2.10) entails a corresponding one for the total amplitude $f_{\beta s_\beta M_\beta; \alpha s_\alpha M_\alpha}$:

$$f_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} = \langle f_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} \rangle + f^{fl}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} \tag{2.11}$$

$$= f^{CN}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} + f^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} + \tag{2.11'}$$

$$+ \langle f_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} \rangle$$

The explicit form of the amplitudes f^{fl}, f^{CN} and f^{DI} is obtained by replacing the matrix S in (2.6), (2.7) by S^{fl}, S^{CN}, and S^{DI} resp.

As already mentioned, only the term with p=0 is of importance for the evaluation of the direct amplitude and all the terms in the p-sum are expected to contribute for the compound nuclear amplitude.

As for the number of terms of the p-sum which contribute to the amplitude for DI-reactions, the following simple classical estimate may be useful[8,18]: The number p of orbitings is simply related to the lifetime τ of the intermediate nuclear system :
The stationary phase condition for the integrals (2.7) will be shown (appendix A1) to be (ϑ= scattering angle)

$$\Theta_{\alpha\beta} = \pm\vartheta - 2\pi p$$

where $\Theta_{\alpha\beta}$, as defined in eq. (2.29), is classically interpreted as the deflection angle. This angle measures the orientation of the

vector joining the centres of the outgoing nuclei with respect to the beam axis including full rotations during the collision. Apparently, $p \gg 0$ determines the number of orbitings. Multiplying the classical relation

$$I \cdot \hbar = \mathcal{J} \omega$$

($I \cdot \hbar$ = angular momentum, $\mathcal{J}$ = moment of inertia, ω = angular frequency) by the contact time τ and putting $\Delta\varphi_{\alpha\beta} = \omega\tau$, we find the relation

$$\Delta\varphi_{\alpha\beta} = \pm\vartheta - 2\pi p = \frac{I\hbar}{\mathcal{J}}\tau$$

between the lifetime τ and the number of orbitings.

Moretto et al[8]) find that for final channels with large mass transfer the angular distribution approaches the one of a compound nuclear reaction. If, on the other hand, the final masses are far from the equilibrated values (i.e. if only a small mass transfer has occurred), the angular distributions are strongly peaked. The contact time τ is an important parameter in the diffusion model[18,19]) and can be determined from the measured mass transfer and the angular distribution. Since the angular momentum $I \cdot \hbar$ and the momentum of inertia $\mathcal{J}$ are grossly known, this leads to an estimate of p. From this analysis it appears that the "least relaxed" part which is characterized by strongly peaked angular distributions, is described by the term with p = 0 of the Poisson representation. It may even be so, with less certainty, for the strongly relaxed part. In appendix A1 we shall discuss in more detail different ways for the system to acquire an almost isotropic angular distribution $\frac{d\sigma}{d\vartheta}^{DI}$.

In the main chapters, we treat the case that the DI amplitude is described by the p = 0 term only deferring the more general case to the appendix A1. Henceforth, the amplitude for deep inelastic reactions is thus defined to be

$$f^{DI}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha} = \frac{1}{i\sqrt{2\pi k_\alpha k_\beta \sin\vartheta}} e^{i(M_\alpha - M_\beta)\varphi} \cdot$$

$$\cdot \sum_{M'_\alpha M'_\beta} D^{J_\alpha *}_{M'_\alpha M_\alpha}\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}, -\tfrac{\pi}{2}\right) D^{J_\beta}_{M'_\beta M_\beta}\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}, -\tfrac{\pi}{2}\right) \cdot$$

$$\cdot \left\{ I^{(+)}_{M'_\alpha M'_\beta 0} + (-)^{M_\alpha - M_\beta} I^{(-)}_{M'_\alpha M'_\beta 0} \right\} \qquad (2.12)$$

where the quantities $I^{(\pm)}_{M'_\alpha M'_\beta 0}$ are obtained from $\tilde{I}^{(\pm)}_{M'_\alpha M'_\beta \rho}$

by replacing the total S-matrix by the incompletely relaxed part S^{DI} of the fluctuating S-matrix. We extract from the matrix S^{DI} a factor which describes elastic scattering by the real potentials in the entrance and exit channels, thus defining a matrix R^{DI} by[25])

$$S^{DI}_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(l) = e^{i\left[\delta_\beta(l) + \delta_\alpha(l + M'_{\beta\alpha})\right]} \cdot$$

$$\cdot R^{DI}_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(l) \qquad (2.13)$$

where

$$M'_{\beta\alpha} := M'_\beta - M'_\alpha = l_\alpha - l_\beta \tag{2.13'}$$

We implicitly assume that the elastic phase shifts $\delta_\alpha(l_\alpha)$, $\delta_\beta(l_\beta)$ vary insignificantly as a function of the energy E in the interval Δ which serves to define the direct reaction part. This is incorrect in the region of narrow shape resonances which we believe to be of minor importance for the dominant part of DI reactions.

Note that the phase shifts should correspond to elastic scattering at the kinetic energy $E^{kin}_\beta := E - E_\beta$, where E_β is the sum of the intrinsic energies of the two nuclei in channel β and E the total energy.

The explicit form of the integrals $I^{(\pm)}_{M'_\alpha M'_\beta 0}$ is thus

$$I^{(\eta)}_{M'_\alpha M'_\beta 0} = \int_0^\infty dl\, (l+\tfrac{1}{2})^{\frac{1}{2}}\, e^{i\chi^\eta_{\alpha\beta}(l, M'_{\beta\alpha})} \cdot (-)^{M'_{\beta\alpha}} \cdot R^{DI}_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(l) \tag{2.14}$$

where $\eta = \pm 1$ and the phase function $\chi^\eta_{\alpha\beta}$ is defined by

$$\chi^\eta_{\alpha\beta}(l, M'_{\beta\alpha}) := \delta_\alpha(l + M'_{\beta\alpha}) + \delta_\beta(l) - \eta\left[(l+\tfrac{1}{2})\vartheta - \tfrac{\pi}{4}\right] \tag{2.15}$$

The energy-averaged cross-section is given by

$$\left\langle \frac{d\sigma_{\beta I_{1\beta} I_{2\beta};\alpha I_{1\alpha} I_{2\alpha}}}{d\Omega} \right\rangle = \frac{d\sigma_{\beta\alpha}^{DIR}}{d\Omega} + \frac{d\sigma_{\beta\alpha}^{CN}}{d\Omega} + \frac{d\sigma_{\beta\alpha}^{DI}}{d\Omega} \qquad (2.16)$$

+ interference terms

with the following definition for the different parts:

$$\frac{d\sigma_{\beta\alpha}^{DIR}}{d\Omega} = \frac{1}{(2I_{1\alpha}+1)(2I_{2\alpha}+1)} \sum_{s_\alpha s_\beta M_\alpha M_\beta} \frac{k_\beta m_\alpha}{k_\alpha m_\beta} \left| \left\langle f_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} \right\rangle \right|^2 \qquad (2.17)$$

$$\frac{d\sigma_{\beta\alpha}^{CN}}{d\Omega} = \frac{1}{(2I_{1\alpha}+1)(2I_{2\alpha}+1)} \sum_{s_\alpha s_\beta M_\alpha M_\beta} \frac{k_\beta m_\alpha}{k_\alpha m_\beta} \left\langle \left| f^{CN}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} \right|^2 \right\rangle \qquad (2.17')$$

$$\frac{d\sigma_{\beta\alpha}^{DI}}{d\Omega} = \frac{1}{(2I_{1\alpha}+1)(2I_{2\alpha}+1)} \sum_{s_\alpha s_\beta M_\alpha M_\beta} \frac{k_\beta m_\alpha}{k_\alpha m_\beta} \left\langle \left| f^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} \right|^2 \right\rangle \qquad (2.17'')$$

$$\text{interference terms} = \frac{1}{(2I_{1\alpha}+1)(2I_{2\alpha}+1)} \sum_{s_\alpha s_\beta M_\alpha M_\beta} \frac{k_\beta m_\alpha}{k_\alpha m_\beta} \cdot$$

$$\cdot \left\langle \left[f^{CN}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} \; f^{DI\,*}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha} + C.C. \right] \right\rangle \qquad (2.17''')$$

As usual, it is implied that the wave numbers k_α, k_β may be treated as constant within the energy interval Δ. The index $\beta(\alpha)$ includes the intrinsic spins $I_{1\beta}$, $I_{2\beta}$ ($I_{1\alpha}$, $I_{2\alpha}$) whenever these are not explicitly noted. Because the phase of the amplitudes f^{CN} and f^{DI} are uncorrelated, the energy average is expected to make the interference terms very small. This will be even more so in the actually measured coarse cross-section (see § 3) due to the summation over many microscopic channels β.

The experimental separation of the DI and truly compound cross-section is problematic only in the case of almost completely relaxed DI processes. The separation of the direct cross-section from the DI is difficult whenever both cross-sections are non-zero for given final channels β .

We now turn to a more detailed evaluation of the cross-section for DI reactions.

As in ref. 25, we wish to interpret the summation over M'_α M'_β in (2.12) as a rotation from the "focal" coordinate frame S_F into the ordinarily used system S. These coordinate systems are defined as follows:

S_F(unit vectors $\vec{e}_{x_F}$, $\vec{e}_{y_F}$, $\vec{e}_{z_F}$):

$\vec{e}_{x_F}$ and $\vec{e}_{y_F}$ lie in the reaction plane, $\vec{e}_{z_F}$ is perpendicular to it. $\vec{e}_{x_F}$ points in the direction of the symmetry axis of the classical trajectory defined by $\theta_{\alpha\beta}$, such that the point of closest distance (A in fig. 2.3) has a positive x_F - value. The direction of $\vec{e}_{z_F}$

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* footnote on page 18.

is chosen such that the asymptotic initial point of the trajectory has a positive x_S-value. In fig. 2.3 we show the two frames and a trajectory. The two coordinate systems differ only by the direction of the coordinate axes, the origins being the same. The orientation of the reaction plane in space is defined by the azimuthal angle φ of $\vec{e}_{x_S}$ and the wave vector $\vec{k}_\alpha$. If the common origin of S_F and S is at rest in the laboratory (moves with the center of mass of the system) S_F and S are laboratory (CM) systems.

The proof that the sum over M'_α, M'_β can be interpreted as a rotation proceeds in 3 steps:

(i) Using again that the angular momenta are large compared to the channel spins ($l_\alpha \gg s_\alpha$; $l_\beta \gg s_\beta$) we may interpret the quantities M'_α, M'_β as projections of the channel spins $\vec{s}_\alpha$, $\vec{s}_\beta$ on the Z-axis of the focal system S_F[25,33]).

This can be seen as follows:
The Z-component of $\vec{\hat{s}}_\alpha$ in a state $|l_\alpha s_\alpha I M_\alpha\rangle$ of the coupled representation is defined as $\langle l_\alpha s_\alpha I M_\alpha |(\hat{s}_\alpha)_{z_F}| l_\alpha s_\alpha I M_\alpha\rangle$. The asymptotic form (2.4) of the CG-coefficients leads to

$$\langle l_\alpha s_\alpha I M_\alpha |(\hat{s}_\alpha)_{z_F}| l_\alpha s_\alpha I M_\alpha\rangle =$$

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* (footnote from page 17)
We note that the orientation of the focal system depends on the choice of the EULER angles and the definition of the D-function. Our choice of S_F implies the definition used in ref. 27:

$$D^J_{mm'}(\alpha,\beta,\gamma) := \langle jm | e^{-i\alpha \hat{I}_z} e^{-i\beta \hat{I}_y} e^{-i\gamma \hat{I}_z} | jm'\rangle^*$$

$$= \sum_{m_{l_\alpha} + m_{s_\alpha} = M_\alpha} \langle l_\alpha s_\alpha I M_\alpha / l_\alpha m_{l_\alpha} s_\alpha m_{s_\alpha} \rangle^2 m_{s_\alpha}$$

$$\approx \sum_{m_{s_\alpha} + m_{l_\alpha} = M_\alpha} m_{s_\alpha} \cdot \left(D^{s_\alpha}_{l_\alpha - I, -m_{s_\alpha}} (0, \alpha, 0) \right)^2$$

Since, in the focal system, we have

$$\cos\alpha := \frac{m_{l_\alpha}}{\sqrt{l_\alpha (l_\alpha + 1)}} \approx 1$$

we may put $\alpha \approx 0$ and thus arrive at

$$\langle l_{\alpha'} s_{\alpha'} I M_{\alpha'} / (\hat{S}_{\alpha'})_{Z_F} / l_\alpha s_\alpha I M_\alpha \rangle \approx$$

$$\approx \sum_{m_{s_\alpha} + m_{l_\alpha} = M_\alpha} m_{s_\alpha} \left(D^{s_\alpha}_{l_\alpha - I, -m_{s_\alpha}} (0, 0, 0) \right)^2$$

$$= \sum_{m_{s_\alpha} + m_{l_\alpha} = M_\alpha} m_{s_\alpha} \, \delta_{l_\alpha - I, -m_{s_\alpha}} = I - l_\alpha = M'_\alpha \qquad (2.18)$$

(ii) Within the validity of semi-classical approximations the amplitude $R^{DI}_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(\ell)$ which itself is independent of the choice of the coordinate system can be shown to be equal, apart from a phase, to the semi-classical reaction amplitude $C^{DI(S_F)}_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha}(l ; t \to +\infty)$

calculated in the focal coordinate system[25]), i.e., we have

$$R^{DI}_{\beta s_\beta M'_\beta;\alpha s_\alpha M'_\alpha}(\ell)\cdot(-)^{M'_{\beta\alpha}} = C^{DI(S_F)}_{\beta s_\beta M'_\beta;\alpha s_\alpha M'_\alpha}(\ell;\, t\to+\infty) \tag{2.19}$$

The quantity on the righthand side of (2.19) is referred to the focal system and transforms under rotations as the direct product of irreducible tensors of rank s_β and s_α.

(iii) The phase function $\chi^{\eta}_{\alpha\beta}$ defined in (2.15) depends on $M'_{\beta\alpha} = \ell_\alpha - \ell_\beta$. An interpretation of the sum $\sum_{M'_\alpha M'_\beta}$ as a rotation is only possible if this dependence may be approximated to be linear. Since for DI processes the transfer from orbital into intrinsic angular momentum may be quite large (say of the order of 10 units $\hbar$) we expand $\chi^{\eta}_{\alpha\beta}$ not around $M'_{\beta\alpha}=0$ but around an average value

$$\overline{M'_{\beta\alpha}} = \overline{(\ell_\alpha - \ell_\beta)} = \overline{\Delta\ell} \neq 0 \tag{2.20}$$

of the "angular momentum mismatch." A coarse average of $(\ell_\alpha - \ell_\beta)$ can be calculated from distribution function d_0 to be introduced in chapter 3. An expansion of $\chi^{\eta}_{\alpha\beta}(\ell, M'_{\beta\alpha})$ around $M'_{\beta\alpha}=\overline{\Delta\ell}$ up to linear order leads to

$$\chi^{\eta}_{\alpha\beta}(\ell, M'_{\beta\alpha}) \approx \chi^{\eta}_{\alpha\beta}(\ell, \Delta\bar{\ell}) + (M'_{\beta\alpha} - \Delta\bar{\ell}) \left(\frac{\partial \chi^{\eta}_{\alpha\beta}}{\partial M'_{\beta\alpha}}\right)_{M'_{\beta\alpha} = \Delta\bar{\ell}} \tag{2.21}$$

From (2.15) we have

$$\left(\frac{\partial \chi^{\eta}_{\alpha\beta}}{\partial M'_{\beta\alpha}}\right)_{M'_{\beta\alpha} = \Delta\bar{\ell}} = \frac{1}{2}\, \Theta_{\alpha}(\ell + \Delta\bar{\ell}) \tag{2.22}$$

where the "quantal deflection functions" Θ_{α} are defined as

$$\Theta_{\alpha}(\ell) := 2\, \frac{\partial \delta_{\alpha}(\ell)}{\partial \ell} \tag{2.23}$$

and analogously for $\alpha \to \beta$.

Substituting (2.19), (2.21), and (2.22) into (2.12) and (2.13) we find

$$f^{DI}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha} = \frac{e^{i(M_\alpha - M_\beta)\varphi}}{i\sqrt{2\pi k_\alpha k_\beta \sin\vartheta}} \left\{ \mathcal{J}^{+}_{M_\alpha M_\beta} + (-)^{M_\alpha - M_\beta}\, \mathcal{J}^{-}_{M_\alpha M_\beta} \right\} \tag{2.24}$$

with the definitions ($\eta = \pm 1$):

$$\mathcal{J}^{\eta}_{M_\alpha M_\beta} := \int_0^{\infty} d\ell\, (\ell + \tfrac{1}{2})^{\frac{1}{2}}\, e^{i\left[\chi^{\eta}_{\alpha\beta}(\ell, \Delta\bar{\ell}) - \frac{\Delta\bar{\ell}}{2}\Theta_{\alpha}(\ell + \Delta\bar{\ell})\right]} \cdot C^{DI}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha}\left(\ell, \Theta_{\alpha}(\ell + \Delta\bar{\ell})\right) \tag{2.25}$$

$$C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(\ell, \theta_\alpha(\ell+\Delta\bar{\ell})) := $$

$$= \sum_{M'_\alpha M'_\beta} D^{s_\beta *}_{M'_\beta M_\beta}\left(-\frac{\pi+\theta_\alpha(\ell+\Delta\bar{\ell})}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) \cdot \qquad (2.26)$$

$$\cdot C^{DI\,(S_F)}_{\beta s_\beta M'_\beta;\alpha s_\alpha M'_\alpha}(\ell)\, D^{s_\alpha}_{M'_\alpha M_\alpha}\left(-\frac{\pi+\theta_\alpha(\ell+\Delta\bar{\ell})}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$$

By definition, the amplitude $C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(\ell, \theta_\alpha(\ell+\Delta\bar{\ell}))$ is obtained from $C^{DI(S_F)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(\ell)$ by a 3-dimensional rotation with the Euler angles $\alpha = -\frac{\pi+\theta_\alpha(\ell+\Delta\bar{\ell})}{2}$, $\beta = \frac{\pi}{2}$ and $\gamma = \frac{\pi}{2}$. In deriving (2.24), use is made of the relations

$$D^{J}_{m m'}(\alpha+\alpha', \beta, \gamma) = e^{im\alpha'} D^{J}_{m m'}(\alpha, \beta, \gamma)$$

$$D^{J}_{m m'}(-\alpha, \beta, -\gamma) = D^{J*}_{m m'}(\alpha, \beta, \gamma)$$

Since the first Euler angle $\alpha = -\frac{\pi+\theta_\alpha(\ell+\Delta\bar{\ell})}{2}$ depends on the variable ℓ, the amplitude defined in (2.26) has no simple physical meaning. In the subsequent sections of this chapter we shall compute the integrals $\mathcal{J}^{\eta}_{M_\alpha M_\beta}$ in various approximations.

2.2 SEMI-CLASSICAL FORM OF THE DI AMPLITUDE

If the matrix $C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(\ell, \Theta_\alpha(\ell+\Delta\bar{\ell}))$ depends smoothly on the orbital angular momentum ℓ throughout the whole integration domain, we may evaluate the integral $\mathcal{J}^{\eta}_{M_\alpha M_\beta}$ by the "stationary phase approximation (SPA)." A detailed discussion of the validity of this method is found in refs. 29.

The main contribution to the integral (2.25) originates from the vicinity of the stationary points $\ell^S_{\nu\eta}(\vartheta;\beta)$ of the phase

$$A^{\eta}_{\alpha\beta}(\ell) := \chi^{\eta}_{\alpha\beta}(\ell,\Delta\bar{\ell}) - \frac{\Delta\bar{\ell}}{2}\Theta_\alpha(\ell+\Delta\bar{\ell}) \tag{2.27}$$

which satisfy the condition

$$\left(\frac{\partial A^{\eta}_{\alpha\beta}}{\partial \ell}\right)_{\ell^S_{\nu\eta}} = 0 = \Theta_{\alpha\beta}(\ell^S_{\nu\eta}) - \eta\vartheta. \tag{2.28}$$

Here we have defined the microscopic average deflection function $\Theta_{\alpha\beta}(\ell)$ by

$$\Theta_{\alpha\beta}(\ell) := \frac{1}{2}\left[\Theta_\alpha(\ell+\Delta\bar{\ell}) - \Delta\bar{\ell}\,\Theta'_\alpha(\ell+\Delta\bar{\ell}) + \Theta_\beta(\ell)\right] \tag{2.29}$$

If the "average angular momentum mixmatch" $\Delta\bar{\ell}$ is small, $\Theta_{\alpha\beta}(\ell)$ becomes the mean value of Θ_α and Θ_β independent of $\Delta\bar{\ell}$:

$$\Theta_{\alpha\beta}(\ell) \approx \frac{1}{2}\left[\Theta_\alpha(\ell) + \Theta_\beta(\ell)\right] \tag{2.29'}$$

For each scattering angle ϑ and each sign η, eq. (2.28) defines in general a finite number of solutions $l^s_{\nu\eta}(\vartheta;\beta)$. Restriction to real stationary points implies that we neglect $\mathcal{J}^+_{M_\alpha M_\beta}$ for attractive scattering $(\Theta_{\alpha\beta}<0)$ and $\mathcal{J}^-_{M_\alpha M_\beta}$ for repulsive scattering $(\Theta_{\alpha\beta}>0)$

$$\mathcal{J}^+_{M_\alpha M_\beta} = 0 \quad \text{for} \quad \Theta_{\alpha\beta}<0 \tag{2.30}$$

$$\mathcal{J}^-_{M_\alpha M_\beta} = 0 \quad \text{for} \quad \Theta_{\alpha\beta}>0 \tag{2.30'}$$

This signifies that the index η is henceforth given by

$$\eta = \operatorname{sgn} \Theta_{\alpha\beta}(l^s_{\nu\eta}) \tag{2.30''}$$

Let $l_{\alpha\beta}(\Theta)$ be the inverse of the real deflection function $\Theta_{\alpha\beta}(l)$. In fig. 2.2 we show a typical case of deflection functions Θ_α, Θ_β, and $\Theta_{\alpha\beta}$ at energies E above the interaction barrier E_I; in fig. 3.1 an inverse deflection function is seen* in a case where negative (attractive) deflection angles also occur. Whenever there are rainbow angles (defin. see (2.41), (2.41')), the inverse deflection function is multivalued. We denote the different branches by $l^\nu_{\alpha\beta}(\Theta)$ where the index $\nu = 1,\cdots,N$ designs the branches of $l_{\alpha\beta}(\Theta)$ in increasing order of the angular momentum. The number N of branches is equal to N_r+1 where N_r is the number of rainbow angles. Restricting ourselves to <u>real</u> stationary points only, we may denote the solutions of (2.28) by the same index ν.

*footnote, see p. 24

Since the scattering angle ϑ is by definition positive, real solutions $l^s_{\nu\eta}(\vartheta;\beta)$ of (2.28) occur only for $\eta=+1$ if $\Theta_{\alpha\beta}>0$ and for $\eta=-1$ if $\Theta_{\alpha\beta}<0$. In general, for given ϑ and η, not each branch of the inverse deflection function contains a stationary point $l^s_{\nu\eta}(\vartheta;\beta)$.

Henceforth, the symbol $\sum_{\nu=1}^{N}{}'$ signifies a summation over a the real solutions $l^s_{\nu\eta}(\vartheta;\beta)$ of (2.28) (or, later, of (3.26)).

In evaluating the integral $J^{\eta}_{M_\alpha M_\beta}$ we distinguish the case (A) that the derivative $\Theta'_{\alpha\beta}(l^s_{\nu\eta})$ is substantially different from zero and the case (B) that it is close to zero:

$$(A)\qquad \Theta'_{\alpha\beta}(l^s_{\nu\eta}) \neq 0 \tag{2.31}$$

$$(B)\qquad \Theta'_{\alpha\beta}(l^s_{\nu\eta}) \approx 0 \tag{2.32}$$

In case A we expand the phase $A^{\eta}_{\alpha\beta}$ up to quadratic order around $l^s_{\nu\eta}$

$$A^{\eta}_{\alpha\beta}(l) \approx A^{\eta}_{\alpha\beta}(l^s_{\nu\eta}) + \frac{\Theta'_{\alpha\beta}(l^s_{\nu\eta})}{2}\,(l-l^s_{\nu\eta})^2 \tag{2.33}$$

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*footnote from p. 24.

The function of fig. 3.1 is to represent the inverse of an "macroscopic" deflection function $\Theta(l;a)$. Its features are the same as for the "microscopic" functions $l_{\alpha\beta}$.

, replace l by $l^s_{\nu\eta}$ in the remainder of the integrand and evaluate the resulting integral. One obtains

$$\mathcal{J}^{\eta}_{M_\alpha M_\beta} \approx \sum_{\nu} \left(l^s_{\nu\eta} + \tfrac{1}{2}\right)^{\frac{1}{2}} e^{i A^{\eta}_{\alpha\beta}(l^s_{\nu\eta})} \; {}^{DI}C_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}\left(l^s_{\nu\eta}, {}^{+}\Theta^{\prime}_{\alpha}(l^s_{\nu\eta}+\Delta\bar{l})\right) \cdot \mathcal{J}_{\alpha\beta}(l^s_{\nu\eta}) \tag{2.34}$$

with the definitions:

$$\mathcal{J}_{\alpha\beta}(\lambda) := \int_0^{\infty} dl\, e^{i\frac{{}^{+}\Theta^{\prime}_{\alpha\beta}(\lambda)}{2}(l-\lambda)^2} \tag{2.35}$$

or

$$\mathcal{J}_{\alpha\beta}(\lambda) = \frac{\sqrt{\pi}}{2C_0(\lambda)} e^{i\varepsilon\frac{\pi}{4}} \operatorname{erfc}\left[-e^{-i\varepsilon\frac{\pi}{4}} C_0(\lambda)\cdot\lambda\right] \tag{2.35'}$$

and

$$\varepsilon := \operatorname{sgn} {}^{+}\Theta^{\prime}_{\alpha\beta}(\lambda) \tag{2.36}$$

$$C_0(\lambda) := \sqrt{\frac{|{}^{+}\Theta^{\prime}_{\alpha\beta}(\lambda)|}{2}} \tag{2.36'}$$

The complementary error function is defined by[30])

$$\operatorname{erfc} z := \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt$$

and has the asymptotic limit

$$\lim_{|z|\to\infty} \operatorname{erfc} z = \begin{cases} 0 & \text{for } |\arg z| < \frac{3\pi}{4} \\ 2 & \text{for } |\arg z| = \frac{3\pi}{4} \end{cases} \tag{2.37}$$

For $\sqrt{\frac{|\Theta'_{\alpha\beta}(l^s_{\nu\eta})|}{2}}\, l^s_{\nu\eta} \gg 1$ we thus have the simpler result

$$\mathcal{J}_{\alpha\beta}(\lambda) \approx \frac{\sqrt{\pi}}{C_0(\lambda)}\, e^{i\varepsilon\frac{\pi}{4}} \tag{2.35''}$$

which is equivalent to extending the integration in (2.35) from $-\infty$ to $+\infty$.

Our basic assumption is that the amplitude $C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(l, \Theta_\alpha)$ is a smoothly varying function of the orbital angular momentum l. It is consistent to assume in addition that it depends smoothly on the rotation angle $\Theta_\alpha(l + \Delta\bar{l})$. We may, therefore, approximately replace the angle $\Theta_\alpha(l^s_{\nu\eta} + \Delta\bar{l})$ by $\Theta_{\alpha\beta}(l^s_{\nu\eta}) = \eta\vartheta$:

$$C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}\left(l^s_{\nu\eta}, \Theta_\alpha(l^s_{\nu\eta} + \Delta\bar{l})\right) \approx$$

$$\approx C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(l^s_{\nu\eta}, \eta\vartheta) \tag{2.38}$$

$$=: C^{DI(S)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(l^s_{\nu\eta}) \tag{2.39}$$

The amplitude $C^{DI(S)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(l^s_{\nu\eta})$ defined by (2.39), and more explicitly by

$$C^{DI(S)}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha}(l^S_{\nu\eta}) = \sum_{M'_\alpha M'_\beta} D^{s_\beta *}_{M'_\beta M_\beta}\left(-\frac{\pi+\eta\vartheta}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$C^{DI(S_F)}_{\beta s_\beta M'_\beta ; \alpha s_\alpha M'_\alpha} D^{s_\alpha}_{M'_\alpha M_\alpha}\left(-\frac{\pi+\eta\vartheta}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) \qquad (2.40)$$

can indeed be interpreted as the original amplitude (2.19) transformed from the focal system S_F into the system S defined in section 2.1. This is demonstrated in fig. 2.3.

We now turn to the case of eq. (2.32) implying that the angular momentum $l^S_{\nu\eta}(\vartheta;\beta)$ as defined by the SP condition (2.28) is equal or close to an angular momentum $l^r_\beta(\beta)$ corresponding to an extremum of the average deflection function (2.29):

$$\Theta'_{\alpha\beta}(l^r_\beta) = 0 \qquad (2.41)$$

The corresponding deflection angles

$$\Theta^r_\beta(\beta) := \Theta_{\alpha\beta}(l^r_\beta) \qquad (2.41')$$

are called "rainbow angles".

There may be a finite number $N_r \gg 0$ of rainbow angles; in cases of practical interest $N_r \approx 0, 1$ or 2 dependent on the nuclear interaction. A discussion of different types of deflection functions is found in ref. 31.

In the vicinity of an angular momentum l_ς^r, the expansion (2.33) must obviously be carried to higher order which leads to the well-known "Airy approximation": Expanding $A^\eta_{\alpha\beta}(l)$ up to 3rd order around l_ς^r

$$A^\eta_{\alpha\beta}(l) \approx A^\eta_{\alpha\beta}(l_\varsigma^r) + (l-l_\varsigma^r)(\Theta_\varsigma^r - \eta\vartheta) + \frac{(l-l_\varsigma^r)^3}{6}\Theta''_{\alpha\beta}(l_\varsigma^r) \qquad (2.42)$$

, substituting l_ς^r for l in the remainder of the integrand of (2.25), and, subsequently, replacing $\Theta_\alpha(l_\varsigma^r+\Delta\bar{l})$ by $\Theta_{\alpha\beta}(l_\varsigma^r)=\Theta_\varsigma^r$ in the argument of $C^{DI}_{\beta\beta M_\beta;\alpha\alpha M_\alpha}$ leads to the following result for the integral $\mathcal{J}^\eta_{M_\alpha M_\beta}$

$$\mathcal{J}^\eta_{M_\alpha M_\beta} \approx (l_\varsigma^r+\tfrac{1}{2})^{\frac{1}{2}} e^{iA^\eta_{\alpha\beta}(l_\varsigma^r)} C^{DI}_{\beta\beta M_\beta;\alpha\alpha M_\alpha}(l_\varsigma^r,\Theta_\varsigma^r)\cdot\tilde{\mathcal{J}}^\varsigma_{\alpha\beta}\cdot\zeta_{\eta\,sgn\,\Theta_\varsigma^r} \qquad (2.43)$$

$$\tilde{\mathcal{J}}^\varsigma_{\alpha\beta} := \int_0^\infty dl\, e^{i\left\{(l-l_\varsigma^r)(\Theta_\varsigma^r-\eta\vartheta)+\frac{(l-l_\varsigma^r)^3}{6}\Theta''_{\alpha\beta}(l_\varsigma^r)\right\}} \qquad (2.44)$$

or

$$\tilde{\mathcal{J}}^\varsigma_{\alpha\beta} = \tilde{\varepsilon}\cdot 2\pi\left[\frac{2}{|\Theta''_{\alpha\beta}(l_\varsigma^r)|}\right]^{\frac{1}{3}} Ai(X,Y) \qquad (2.45)$$

where

$$\tilde{\varepsilon} := sgn\,\Theta''_{\alpha\beta}(l_\varsigma^r) \qquad 2.45')$$

The "incomplete Airy function" is defined by[31])

$$Ai(X,Y) := \frac{1}{2\pi} \int_{Y}^{\infty} dt \; e^{i\left(Xt + \frac{1}{3}t^3\right)} \tag{2.46}$$

and the arguments X, Y by

$$X := \tilde{\varepsilon} \left[\frac{2}{|\Phi''_{\alpha\beta}(l^r_\sigma)|} \right]^{\frac{1}{3}} \left(\Phi^r_\sigma - \eta^r \vartheta \right) \tag{2.47}$$

$$Y := -\tilde{\varepsilon} \left[\frac{|\Phi''_{\alpha\beta}(l^r_\sigma)|}{2} \right]^{\frac{1}{3}} l^r_\sigma \tag{2.47'}$$

Using the stationary phase condition (2.28) we may write the argument X as

$$X^r(l^s_{\nu\eta}) = \tilde{\varepsilon} \cdot \left[\frac{2}{|\Phi''_{\alpha\beta}(l^r_\sigma)|} \right]^{\frac{1}{3}} \left(\Phi^r_\sigma - \Phi_{\alpha\beta}(l^s_{\nu\eta}) \right) \tag{2.47''}$$

This is the form which lends itself readily to the general case with orbiting (appendix A1).

Some properties of the incomplete Airy function are discussed in ref. 31. If the stationary points of the phase ($Xt + \frac{1}{3}t^3$) in (2.46) are far away from the lower integration limit Y, one may replace Y by $-\infty$, in which case the incomplete Airy function becomes the ordinary Airy function

$$Ai(X) = \lim_{Y \to -\infty} Ai(X,Y) \tag{2.48}$$

where

$$Ai(X) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, e^{i\left(Xt + \frac{t^3}{3}\right)} \tag{2.49}$$

or, equivalently,

$$Ai\left[(3a)^{-\frac{1}{3}}X\right] = (3a)^{\frac{1}{3}}\frac{1}{\pi}\int_0^{\infty}\cos\left[at^3 \pm Xt\right] dt \tag{2.49'}$$

$(a>0,\ X = real)$. Properties of the Airy-function are given for instance in ref. 30. A plot of the Airy-function is shown in fig. 3.2.

The Airy approximation is valid only in a small angular range $\pm\delta\vartheta$ around the rainbow angles ϑ_σ^r or, equivalently, in a small range $\pm\delta l$ around the rainbow angular momenta l_σ^r. Within the accuracy of the Airy method, we may replace the arguments l_σ^r and ϑ_σ^r by $l_{\nu\eta}^s(\vartheta;\beta)$ and $\vartheta_{\alpha\beta}(l_{\nu\eta}^s) = \eta\vartheta$, respectively, in the first three smoothly varying factors of (2.43). Within the rainbow region, the quantity $\mathcal{J}^{\eta}_{M_\alpha M_\beta}$ thus assumes the form

$$\mathcal{J}^{\eta}_{M_\alpha M_\beta} \approx \left(l_{\nu\eta}^s + \tfrac{1}{2}\right)^{\frac{1}{2}} e^{iA_{\alpha\beta}^{\eta}(l_{\nu\eta}^s)}\, \overline{DL}(S)\, C_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(l_{\nu\eta}^s)\, \tilde{\mathcal{J}}_{\alpha\beta}^{\delta}\, \delta_{\eta\, sgn\vartheta_{\alpha\beta}^{i}(l_{\nu\eta}^s)} \tag{2.50}$$

In order to write the general result in a concise form we introduce the following function:

$$M_\sigma(l_{\nu\eta}^s) := \theta_0\left[l_{\nu\eta}^s(\vartheta) - (l_\sigma^r - \delta l)\right]\theta_0\left[(l_\sigma^r + \delta l) - l_{\nu\eta}^s(\vartheta)\right] \tag{2.51}$$

θ_0 is the ordinary "Heaviside function"

$$\theta_0(z) = \begin{cases} 1 \text{ for } z > 0 \\ 0 \text{ for } z < 0 \end{cases} \qquad (2.51')$$

We imply that $\mu(l^s_{\nu\eta})$ vanishes for all angles ϑ, if the deflection function it refers to contains no rainbow angle.

Then the quantity $\mathcal{J}^{\eta}_{M_\alpha M_\beta}$ may be given the form

$$\mathcal{J}^{\eta}_{M_\alpha M_\beta} = \sum_{\nu=1}^{N'} (l^s_{\nu\eta} + \tfrac{1}{2})^{\frac{1}{2}} e^{i A^{\eta}_{\alpha\beta}(l^s_{\nu\eta})} C^{DI(S)}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha}(l^s_{\nu\eta})$$

$$\cdot \delta_\eta \operatorname{sgn} \Theta^{\eta}_{\alpha\beta}(l^s_{\nu\eta}) \cdot \Big\{ \sum_{\varrho=1}^{N_r} \mu_\varrho(l^s_{\nu\eta}) \tilde{\mathcal{F}}^{\varrho}_{\alpha\beta} + \prod_{\varrho=1}^{N_r} (1 - \mu_\varrho(l^s_{\nu\eta})) \mathcal{F}_{\alpha\beta}(l^s_{\nu\eta}) \Big\} \qquad (2.52)$$

where $A^{\eta}_{\alpha\beta}$, $\mathcal{F}_{\alpha\beta}$ and $\tilde{\mathcal{F}}^{\varrho}_{\alpha\beta}$ are given by (2.27), (2.35') and (2.45) resp. The stationary points $l^s_{\nu\eta}(\vartheta; \beta)$ are the <u>real</u> solutions of (2.28) and η is defined by (2.30''). The reaction amplitude $f^{DI}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha}(\vartheta, \varphi)$ is obtained by substituting (2.52) into (2.24). The summation over ν can be interpreted as the superposition of the contributions of all trajectories (repulsive for $\eta = +1$, attractive for $\eta = -1$) which asymptotically end up with the same scattering angle ϑ. Whenever more than one term is of appreciable magnitude, the cross-section $\frac{d\sigma_{\alpha\alpha}^{DI}}{d\Omega}$ (see (2.17'')) contains interference terms. The question whether these interference terms are of importance for the measured energy-averaged cross-section will be treated in § 3.

2.3 SEMI-CLASSICAL FORM OF THE DI AMPLITUDE INCLUDING DIFFRACTION EFFECTS.

So far we neglected the influence that direct as well as compound nuclear processes may have on the amplitude f^{DI} of DI processes. Physically, the main effect of these channels is to produce the imaginary part of an optical potential which acts in the subspace of the DI channels. This absorptive potential may generate diffractive phenomena in close analogy with the scattering from a black body. Such diffraction effects are known to be a dominant feature in direct reactions as described by the DWBA and have been beautifully visualized in the theory of W. E. Frahn [31,33-35]. In what follows we apply the same methods in the description of DI reactions.

We assume that the effect of the imaginary potential is to confine contributions to the amplitude f^{DI} to a limited range of orbital angular momenta. A drastic way to achieve this is to introduce sharp cut-offs. Let us thus assume that DI reactions only occur if the orbital angular momentum l_α in the entrance channel is smaller than a critical value Λ_α and if the orbital angular momentum l_β in the exit channel is larger than a critical angular momentum Λ_β^{ex} :

$$l_\alpha < \Lambda_\alpha \tag{2.53}$$

$$l_\beta \equiv l > \Lambda_\beta^{ex} \tag{2.54}$$

The cut-off parameter Λ_α depends on the kinetic energy E_α^{kin} in the entrance channel α. The meaning of the upper cut-off is that partial waves with $\ell_\alpha > \Lambda_\alpha$ do not penetrate anymore into the region of strong peripheral overlap and thus do not contribute to DI reactions.

The lower limit Λ_β^{fu} might be defined as follows: Let us assume that for all, the angular momenta $\ell_\beta < \Lambda_\beta$ the total average potential in the exit channel β exhibits a "quasimolecular valley" with the top of the outer barrier being $E_I(\ell_\beta;\beta)$.

Let us then define Λ_β^{fu} by the requirement that the kinetic energy E_β^{kin} in the exit channel is to be smaller than the top of this barrier measured with respect to the asymptotic intrinsic energy

$$E_\beta^{kin} < E_I(\ell_\beta,\beta) - E_\beta$$

for all $\ell_\beta < \Lambda_\beta^{fu}$. If defined in this way, the cut-off parameter Λ_β^{fu} is at most equal to the limit Λ_β $(\Lambda_\beta^{fu} \ll \Lambda_\beta)$ and depends on the final kinetic energy.

The underlying picture is: Whenever, as a result of energy dissipation, the final kinetic energy falls below the outer barrier of the effective final potential, the lifetime of the intermediate complex is large enough to achieve complete relaxation. Thus for $\ell_\beta < \Lambda_\beta^{fu}$ the S-matrix $S^{DI}(\ell)$ is zero while the S-matrix $S^{CN}(\ell)$ is finite. We note that, in a large variety of cases,[10,13]) the measured fusion cross-section can indeed be understood on the basis of this picture.

The relation between ℓ_α and $\ell_\beta \equiv \ell$ is

$$\ell_\alpha = \ell_\beta + M'_{\beta\alpha} = \ell + M'_{\beta\alpha}$$

Replacing $M'_{\beta\alpha}$ by the average angular momentum mismatch $\overline{\Delta\ell}$, the restriction (2.53) takes the form

$$\ell \leq \Lambda_\alpha(E_\alpha^{kin}) - \overline{\Delta\ell} \equiv \tilde{\Lambda}_\alpha(E_\alpha^{kin}) \tag{2.53'}$$

The condition that $S^{DI}(\ell)$ and thus $C^{DI(S)}_{\beta\alpha}(\ell)$ is only unequal to zero for $\Lambda_\beta^{fu} \leq \ell \leq \tilde{\Lambda}_\alpha$ can be taken into account by redefining the integrals (2.25) as

$$\mathcal{J}^{\eta}_{M_\alpha M_\beta} = \int_{\Lambda_\beta^{fu}}^{\tilde{\Lambda}_\alpha} d\ell\,(\ell+\tfrac{1}{2})^{\frac{1}{2}}\, e^{i\left[\chi^{\eta}_{\alpha\beta}(\ell,\overline{\Delta\ell}) - \frac{\overline{\Delta\ell}}{2}\Theta_\alpha(\ell+\overline{\Delta\ell})\right]} \cdot C^{DI(S)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(\ell, \Theta_\alpha(\ell+\overline{\Delta\ell})) \tag{2.55}$$

As in section (2.2) we evaluate this integral under the assumption that the amplitude $C^{DI(S)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(\ell, \Theta_\alpha(\ell+\overline{\Delta\ell}))$ varies smoothly as a function of ℓ and Θ_α in the whole domain of integration.

A) In the region far from rainbow angles we obtain again the result (2.34), but with the integral $\mathcal{J}_{\alpha\beta}(\lambda)$ now being defined as

$$\mathcal{J}_{\alpha\beta}(\lambda) := \int_{\Lambda_\beta^{fu}}^{\tilde{\Lambda}_\alpha} d\ell\; e^{i\frac{\Theta'_{\alpha\beta}(\lambda)}{2}(\ell-\lambda)^2} \tag{2.56}$$

and evaluated as

$$\mathcal{J}_{\alpha\beta}(\lambda) = \frac{\sqrt{\pi}}{2C_0(\lambda)}\, e^{i\varepsilon\frac{\pi}{4}} \left\{ \mathrm{erfc}\left[e^{-i\varepsilon\frac{\pi}{4}} C_0(\lambda)\left(\Lambda_\beta^{ku} - \lambda\right)\right] \right. \tag{2.57}$$

$$\left. - \mathrm{erfc}\left[e^{-i\varepsilon\frac{\pi}{4}} C_0(\lambda)\left(\tilde{\Lambda}_\alpha - \lambda\right)\right] \right\}$$

The quantities ε and $C_0(\lambda)$ are defined by (2.36) and (2.36') resp.

B) In the region close to a rainbow angle Θ_s^r, we retrieve the result (2.43), but with the integral $\tilde{\mathcal{J}}_{\alpha\beta}^s$ being defined as

$$\tilde{\mathcal{J}}_{\alpha\beta}^s := \int_{\Lambda_\beta^{ku}}^{\tilde{\Lambda}_\alpha} d\ell\; e^{i\left\{(\ell-\ell_s^r)(\Theta_s^r - \eta\vartheta) + \frac{(\ell-\ell_s^r)^3}{6} A_{\alpha\beta}''(\ell_s^r)\right\}} \tag{2.58}$$

and evaluated as

$$\tilde{\mathcal{J}}_{\alpha\beta}^s = 2\pi\tilde{\varepsilon}\left[\frac{2}{|\Theta_{\alpha\beta}''(\ell_s^r)|}\right]^{\frac{1}{3}} \left\{ Ai\left(X_1, Y_1\right) - Ai\left(X_2, Y_2\right) \right\} \tag{2.59}$$

Here, $\tilde{\varepsilon}$ and χ^{ν} are again defined by (2.45') and (2.47'') resp.

The second arguments have the form

$$y_1 := \tilde{\varepsilon}\left[\frac{|\Theta''_{\alpha\beta}(l_\beta^{\nu})|}{2}\right]^{\frac{1}{3}}(\Lambda_\beta^{\mu} - l_\beta^{\nu}) \tag{2.60}$$

$$y_2 := \tilde{\varepsilon}\left[\frac{|\Theta''_{\alpha\beta}(l_\beta^{\nu})|}{2}\right]^{\frac{1}{3}}(\tilde{\Lambda}_\alpha - l_\beta^{\nu}) \tag{2.61}$$

The general form of the quantity $\mathcal{F}^{\eta}_{M_\alpha M_\beta}$ persists to be (2.52) but with $\mathcal{F}_{\alpha\beta}$ and $\tilde{\mathcal{F}}^{\beta}_{\alpha\beta}$ being given by (2.57) and (2.59), resp.

While the sharp cut-off approximation has the virtue of simplicity, its quantitative predictions may be altered appreciably by the more realistic assumption of a "smooth cut-off"[31]). On the other hand, the smooth cut-off prescription implies a knowledge of the reaction amplitude in the vicinity of the cut-off parameters or at least a parametrization of it. This seems to preclude a form of the final (coarse-gained) cross-section depending on probabilities only. The coarse average leads to an averaging over the cut-off parameters we introduced. We presume that thereby the defects of the sharp cut-off assumption, are alleviated.

3. The Cross-Section for DI Reactions

3.1 THE AVERAGE OVER COARSE CELLS

In an actual experiment, a complete determination of all the quantum numbers $\beta, I_{1\beta}, I_{2\beta}$ of a final channel is usually not possible. At an energy of some 100 MeV or more above the interaction barrier, one will usually only determine a limited number of properties of the final state. Using the terminology of statistical mechanics we may say that in the HI experiment we determine only the distribution of the system over "macroscopic" or "coarse" cells of phase-space which are defined by a limited number of "macroscopic variables" $a_1 \ldots a_f \equiv a$. The most important examples of such variables are: scattering angle, charges, masses, kinetic energy of relative motion, intrinsic angular momenta of the outgoing fragments.

In order to write the formulae in a concise form we define the first observables $a_1 \ldots a_4$ to be the polar and azimuthal scattering angle, the wave number k_β and the reduced mass m_β in the final channel:

$$a_1 := \vartheta \tag{3.1}$$

$$a_2 := \varphi \tag{3.1'}$$

$$a_3 := k_\beta \tag{3.1''}$$

$$a_4 := m_\beta \tag{3.1'''}$$

The measured "coarse" cross-section for DI reactions is obtained by summing the energy-averaged microscopic cross-section (2.17'') over all the final channels which lie in a given interval (a,a+Δa) of the measured macroscopic variables:

$$\frac{d^f Q^{DI}}{d\cos\alpha_1\, da_2 \cdots da_f} \cdot \Delta\cos\alpha_1\, \Delta a_2 \cdots \Delta a_f \equiv Q^{DI}(a, \Delta a) := \tag{3.2}$$

$$= \sum_{\beta \in (a, a+\Delta a)} \frac{d\sigma_{\beta\alpha}^{DI}}{d\Omega}$$

The symbol (a, a+Δa) signifies the interval between the macroscopic observables $a_1 \ldots a_f$ and $a_1+\Delta a_1, \ldots, a_f+\Delta a_f$. The summation over the "coarse cell" in (3.2) as well as the energy-averaging which is already implicit in the definition of $\frac{d\sigma_{\beta\alpha}^{DI}}{d\Omega}$ implies that interferences from different trajectories annihilate each other to zero due to the randomness of their phases. This also holds for the interferences (2.17''') between compound and deep inelastic amplitudes.

We thus obtain from substitutions of (2.17''), (2.24), and (2.52) into (3.2):

$$Q^{DI}(a,\Delta a)=\frac{1}{(2I_\alpha+1)(2I_{\alpha'}+1)}\sum_{\beta\in(a,a+\Delta a)}\sum_{\Lambda_\alpha\Lambda_\beta M_\alpha M_\beta}\sum_{\nu}{}'$$

$$\frac{m_\alpha\, l^s_{\nu\eta}}{2\pi a_\beta k_\alpha \sin a_i}\Big\langle\Big|C^{DI(S)}_{\beta\Lambda_\beta M_\beta;\alpha\Lambda_\alpha M_\alpha}(l^s_{\nu\eta})\Big|^2\cdot\delta_{\eta\,sgn\,\Theta_{\alpha\beta}(l^s_{\nu\eta})}$$

$$\cdot\Big\{\sum_{\rho}w_\rho(l^s_{\nu\eta})\,\Big|\tilde{\mathcal{F}}^\rho_{\alpha\beta}(l^s_{\nu\eta})\Big|^2+\prod_{\rho=1}^{N_r}\big(1-w_\rho(l^s_{\nu\eta})\big)\,\Big|\mathcal{F}_{\alpha\beta}(l^s_{\nu\eta})\Big|^2\Big\}\Big\rangle \qquad (3.3)$$

The quantities $\mathcal{F}_{\alpha\beta}$ and $\tilde{\mathcal{F}}^\rho_{\alpha\beta}$ are given by (2.57) and (2.59) if diffraction effects are important, and by (2.35') and (2.45) resp. if they are unimportant. We have replaced $(l^s_{\nu\eta}+\frac{1}{2})$ by $l^s_{\nu\eta}$ in (3.3) since the orbital angular momenta at stake are always much larger than 1. Furthermore, we have assumed that the variation of the stationary angular momentum $l^s_{\nu\eta}(\bar{v},\beta)$ within an interval of length Δ of the total energy is negligible.

Analogously, we consider the quantities l^τ_ρ, Θ^τ_ρ, $\Theta_{\alpha\beta}$, $\Theta'_{\alpha\beta}$ and $\Theta''_{\alpha\beta}$ as being constant within the averaging interval of the energy except if these quantities appear in the argument of rapidly oscillating functions.

If one finally assumes that the energy averages of the probability $\Big|C^{DI(S)}_{\beta\Lambda_\beta M_\beta;\alpha\Lambda_\alpha M_\alpha}\Big|^2$ and of the remaining factor are uncorrelated, one arrives at the following result:

$$q^{DI}(a,\Delta a)=\frac{1}{\sin a_1}\sum_{\beta\in(a,a+\Delta a)}{\sum_{\nu}}'\,\omega_{\alpha\beta}\left(l^{s}_{\nu\eta}(a_1;\beta)\right)\,v_{\alpha\beta}\left(l^{s}_{\nu\eta}(a_1;\beta)\right) \tag{3.4}$$

with the definitions

$$\omega_{\alpha\beta}(\lambda):=\frac{1}{(2I_\alpha+1)(2I_\alpha+1)}\sum_{s_\alpha s_\beta M_\alpha M_\beta}\left\langle\left|C^{DI(S)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(\lambda)\right|^2\right\rangle \tag{3.5}$$

$$v_{\alpha\beta}(\lambda):=\frac{m_\alpha\lambda}{a_4 k_\alpha^2}\left\{\sum_{\rho}\mu_\rho(\lambda)A_{\alpha\beta}(\lambda)+\prod_{\rho=1}^{N_r}(1-\mu_\rho(\lambda))B_{\alpha\beta}(\lambda)\right\} \tag{3.6}$$

$$A_{\alpha\beta}(\lambda):=\left[\frac{2}{|t\Theta''_{\alpha\beta}(l^r_\rho)|}\right]^{\frac{2}{3}}\left\{\left\langle\left|Ai(X_1,Y_1)\right|^2\right\rangle+\left\langle\left|Ai(X_1,Y_2)\right|^2\right\rangle\right.$$

$$\left.-2\,\mathrm{Re}\left\langle Ai(X_1,Y_1)Ai(X_1,Y_2)\right\rangle\right\} \tag{3.7}$$

$$B_{\alpha\beta}(\lambda):=\frac{1}{|t\Theta'_{\alpha\beta}(\lambda)|}\left\{\left\langle F_r\left[c_0(\lambda)(\Lambda^{lu}_\beta-\lambda)\right]\right\rangle+\left\langle F_r\left[c_0(\lambda)(\tilde{\Lambda}_\alpha-\lambda)\right]\right\rangle\right.$$

$$\left.-\frac{1}{2}\,\mathrm{Re}\left\langle erfc\left[e^{-i\varepsilon\frac{\pi}{4}}(\Lambda^{lu}_\beta-\lambda)\right]erfc\left[e^{-i\varepsilon\frac{\pi}{4}}(\tilde{\Lambda}_\alpha-\lambda)\right]\right\rangle\right\} \tag{3.8}$$

In $l^{s}_{\nu\eta}(a_1;\beta)$ the index η is given by (2.30'')

The Fresnel function is defined by[31]

$$Fr(x) := \frac{1}{4}\left|erfc\left[e^{-i\varepsilon\frac{\pi}{4}}x\right]\right|^2 = \frac{1}{4}\left|erfc\left[e^{i\frac{\pi}{4}}x\right]\right|^2$$

Furthermore, $C_0 = C_0(\lambda)$ is given by (2.36') and the arguments X, Y_1, Y_2 by (2.47''), (2.60), (2.60'), resp.

Making use of the relation

$$1-erfc\left[e^{-i\frac{\pi}{4}\varepsilon}x\right] = \left[C\left(\sqrt{\tfrac{2}{\pi}}x\right)+S\left(\sqrt{\tfrac{2}{\pi}}x\right)\right]+i\varepsilon\left[-C\left(\sqrt{\tfrac{2}{\pi}}x\right)+S\left(\sqrt{\tfrac{2}{\pi}}x\right)\right]$$

where the "Fresnel integrals" are defined by[30]

$$C(z) := \int_0^z dt\, \cos\left(\frac{\pi}{2}t^2\right)$$

$$S(z) := \int_0^z dt\, \sin\left(\frac{\pi}{2}t^2\right)$$

one finds the alternative form of $B_{\alpha\beta}$:

$$B_{\alpha\beta} = \frac{1}{4C_0^2}\left\langle\left\{\left[C(z_1)-C(z_2)\right]^2+\left[S(z_1)-S(z_2)\right]^2\right\}\right\rangle \qquad (3.8')$$

$$z_1 := \sqrt{\tfrac{2}{\pi}}\,C_0\left(\Lambda_\beta^{fu}-\lambda\right)$$

$$z_2 := \sqrt{\tfrac{2}{\pi}}\,C_0\left(\tilde{\Lambda}_\alpha-\lambda\right)$$

The Fresnel function is shown in fig. (3.4) and the Fresnel integrals in fig. (3.5).

We introduce the number $\mathcal{N}(a, \Delta a)$ of microscopic channels in the coarse interval (a,a+Δa) and a "macroscopic" ("coarse") probability density $P_v(a)$:

$$\mathcal{N}(a, \Delta a) := \sum_{\beta \in (a, a+\Delta a)} 1 = \sum_{\beta \in (a, a+\Delta a)} \int_{-\infty}^{+\infty} d\lambda \, \delta(\lambda - l_{\nu\eta}^{s}(a, \beta)) \tag{3.9}$$

$$P_v(a)\Delta a \equiv P_v(a_1, \dots, a_f)\Delta\omega a_1 \Delta a_2 \cdots \Delta a_f := $$

$$= \sum_{\beta \in (a, a+\Delta a)} w_{\alpha\beta}(l_{\nu\eta}^{s}(a, \beta))$$

$$= \sum_{\beta \in (a, a+\Delta a)} \int_{-\infty}^{+\infty} d\lambda \, \delta[\lambda - l_{\nu\eta}^{s}(a, \beta)] w_{\alpha\beta}(\lambda) \tag{3.10}$$

The average of the function $w_{\alpha\beta}$ in the coarse cell (a,a+Δa) is given by

$$w_v(a) := \frac{1}{\mathcal{N}(a, \Delta a)} \sum_{\beta \in (a, a+\Delta a)} w_{\alpha\beta}(l_{\nu\eta}^{s}(a, \beta))$$

$$= \frac{1}{\mathcal{N}(a, \Delta a)} \sum_{\beta \in (a, a+\Delta a)} \int d\lambda \, \delta[\lambda - l_{\nu\eta}^{s}(a, \beta)] w_{\alpha\beta}(\lambda) \tag{3.11}$$

and the deviations of the functions $v_{\alpha\beta}$ and $w_{\alpha\beta}$ from the average values in the coarse cell by

$$\Delta v_{\alpha\beta} := v_{\alpha\beta}\left(l^{s}_{v\eta}(a_1;\beta)\right) - v_v(a) \tag{3.12}$$

$$\Delta w_{\alpha\beta} := w_{\alpha\beta}\left(l^{s}_{v\eta}(a_1;\beta)\right) - \frac{P_v(a)\Delta a}{\mathcal{N}(a,\Delta a)} \tag{3.13}$$

Using these definitions, the measured coarse transition probability (3.4) may be written as a sum of a term which contains only the product of the average quantities and a second term which expresses a correlation between the fluctuations (3.12) and (3.13):

$$Q^{DI}(a,\Delta a) = \frac{1}{\sin a_1}\left\{ \sum_{v}{}' P_v(a)\, v_v(a)\Delta a + \sum_{\beta\epsilon(a,a+\Delta a)} \sum_{v}{}' \Delta w_{\alpha\beta}\, \Delta v_{\alpha\beta} \right\} \tag{3.14}$$

We assume that the fluctuations of the functions $v_{\alpha\beta}$ and $w_{\alpha\beta}$ are uncorrelated and that the summation is over sufficiently many states to make the second term in (3.14) negligibly small. Our final result is thus

$$Q^{DI}(a,\Delta a) = \frac{1}{\sin a_1} \sum_{v}{}' P_v(a)\, v_v(a)\, \Delta a \tag{3.15}$$

We are now left with the problem of evaluating the average quantities $P_v(a)$ and v_v from a theory which does not involve a detailed knowledge of the dynamics on the microscopic scale.

3.2 THE EVALUATION OF THE COARSE CROSS-SECTION FROM A CLASSICAL DISTRIBUTION FUNCTION

We assume that the system is described by f_0 external (macroscopic) variables. We denote the operators which represent these variables and their conjugate momenta by $\hat{A}_i$ and $\hat{\Pi}_i$ $(\hat{A}_1, \ldots, \hat{A}_{f_0} \equiv \hat{A};\ \hat{\Pi}_1, \ldots, \hat{\Pi}_{f_0} \equiv \hat{\Pi})$ and the corresponding classical variables by A_i and Π_i.

The decomposition of the degrees of freedom of a system into external and internal ones is based on their time-dependence: The development in time of the internal variables must be so fast that the "memory" of any initial state is lost after a time interval which is short compared to the collision time, while this is not so for the external variables. Examples of external variables are: the vector $\vec{r}$ joining the fragment centers, mass and charge of the fragments, shape variables and the corresponding conjugate momenta.

We assume that there are f commuting observables $\hat{\Omega}_\nu(\hat{A}, \hat{\Pi})$ which are functions of the external variables; in the simplest case they are equal to one of the $\hat{A}_i$ or $\hat{\Pi}_i$. The eigenvalues a_ν of these observables represent the "macroscopic quantum numbers" which we have used to define the coarse cells.

The evaluation of the reaction amplitude by the stationary phase method implied already that a classical description of the relative motion was a valid lowest approximation. We now hypothesize that the classical description holds also for all the other external degrees of freedom of our system as well. Consequently, we may assume to know a classical time-dependent distribution function $\alpha_0(A, \Pi, t)$ which provides the probability $\alpha_0(A, \Pi, t)\Delta^{f_0}A\,\Delta^{f_0}\Pi$

for finding the system at time t in the $2f_0$-dimensional volume element $\Delta^{f_0}A\,\Delta^{f_0}\pi$ of the phasespace of external variables. The probability density $P_o^{class}(a_1, \ldots, a_f)$ to find the system asymptotically at the macroscopic observables $a_1, \ldots, a_f \equiv a$ is given as a function of the distribution $d_0(A,\pi,t)$ by

$$P_o^{class}(a) := \int d^{f_0}A\, d^{f_0}\pi \prod_{k=1}^{f} \delta[\Omega_k(A,\pi) - a_k]\, d_0(A,\pi; t \to +\infty) \tag{3.16}$$

We agree that A_ν, π_ν, $\nu = 1,2,3$ should represent the vector $\vec{r}$ joining the centers of the nuclei and the relative momentum $\vec{p}$. It will be convenient to use as a variable A_1 the "classical deflection angle" θ_r

$$A_1 \equiv \theta_r \tag{3.17}$$

rather than the polar angle of $\vec{r}$. The classical deflection angle θ_r is defined as the angle between the direction of the z-axis of the coordinate system S (see § 2.1) and the direction of the vector $\vec{r}$ inclusive of any number p of full revolutions. We define θ_r to be positive for repulsive and negative for attractive scattering. As already noticed, we choose the observable Ω_1 to be the polar scattering angle ϑ

$$\Omega_1 \equiv \vartheta \tag{3.18}$$

which implies the relation (see appendix A1)

$$\Theta_r + 2\pi p = \eta \vartheta \tag{3.19}$$

where $\eta = sgn(\Theta_r + 2\pi p)$

and p = 0,1,2 ...

Since, so far, we treated only the case of no orbiting in DI reactions, we put p = 0 in what follows.

Classically, the orbital angular momentum $|\vec{r} \times \vec{p}|$ can be measured at the same time as the scattering angle ϑ. Quantum-mechanically, these variables are not commensurable and thus $|\vec{r} \times \vec{p}|$ does not qualify for one of the observables $\hat{\Omega}_\nu$. Nevertheless, it will turn out to be useful to define a classical distribution function $d(\Theta, a_2, \cdots, a_f; l)$ which depends on the deflection angle Θ as well as on the angular momentum l:

$$d(\Theta, a_2 \cdots a_f; l) := \int dl^6 A \, d^6\Pi \, \delta(\Theta_r - \Theta) \, \delta[|\vec{r} \times \vec{p}| - \hbar l] \cdot \prod_{k=2}^{f} \delta[\Omega_k(A,\Pi) - a_k] \, d_0(A, \Pi; t \to +\infty) \tag{3.20}$$

We use it to introduce a "macroscopic (classical) deflection function" $\Theta(l; a)$ by*

$$\Theta(l; a_2 \cdots a_f) \equiv \Theta(l; a) := \int d\Theta \, \Theta \, d(\Theta, a_2, \cdots, a_f; l) \tag{3.21}$$

In analogy to (2.28), a given scattering angle $a_1 \equiv \vartheta$ gives rise to a finite number of angular momenta $l^s_{\nu\eta}(\vartheta, a_2 \cdots a_f) \equiv l^s_{\nu\eta}(a)$ by

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*(see footnote p. 47)

$$\Theta\left[\ell^{s}_{\nu\eta}(\vartheta, a_2 \cdots a_f); a\right] = \eta\vartheta$$

$$= \mathrm{sgn}\,\Theta' \cdot \vartheta \equiv \mathrm{sgn}\,\Theta' \cdot a_1 \qquad (3.22)$$

The index ν specifies again the branch of the inverse deflection $\ell(a)$. function $\Theta(a)$. For a deflection function with 2 rainbows (ℓ_1^r, ℓ_2^r) the different "branches" refer to the angular momenta $\ell<\ell_1^r$, $\ell_1^r<\ell<\ell_2^r$, and $\ell>\ell_2^r$. The number of branches is called N and the rainbow angular momenta

$\ell_\rho^r(a)$ $(\rho = 1, \cdots, N_r)$ are defined by

$$\Theta'(\ell_\rho^r(a); a) = 0 \qquad (3.23)$$

and the rainbow angles

$$\Theta_\rho^r(a) := \Theta(\ell_\rho^r(a); a) \qquad (3.24)$$

corresponding to the "macroscopic" deflection function $\Theta(\ell; a)$.

With the specifications (3.17) (3.18) we may write the probability density $p_c^{\mathrm{class}}(a)$ as

$$P_0^{class}(\vartheta, a_2, \cdots a_f) \equiv P_0^{class}(a) = \int d^{f_0}A\, d^{f_0}\Pi\, \delta\left[\Theta_r - \mathrm{sgn}\,\Theta_r' \cdot \vartheta\right] \cdot$$

$$\cdot \prod_{\kappa=2}^{f} \delta\left[\Omega_\kappa(A,\Pi) - a_\kappa\right] \omega_0(A,\Pi; t\to+\infty) \qquad (3.25)$$

*footnote p. 46. Note that the integration $d\Theta_r$ extends over all possible deflection angles, i.e. in general from $-\infty$ to π. If orbitings are excluded its range is from $-\pi$ to $+\pi$.

or

$$P_0^{class}(a) = \int d\Theta_1 dl\, \delta[\Theta_1 - sgn\Theta_1 \cdot \vartheta]\, d(\Theta_1, a_2, \cdots, a_f; l) \tag{3.25'}$$

We use the function $\Theta_1(\lambda; a)$ of (3.21) for introducing a new integration variable λ instead of the deflection angle Θ_1:

Using the relation

$$\delta[\Theta_1(\lambda; a) - sgn\Theta_1 \cdot a_1] = \sum_{\nu=1}^{N'} \frac{1}{|\Theta_1'(l_{\nu\eta}^s(a); a)|}\, \delta[\lambda - l_{\nu\eta}^s(a)]$$

where, as before, the prime indicates that the summation is extended over those branches of the inverse deflection function which contain <u>real</u> stationary points $l_{\nu\eta}^s(a)$ we find:

$$P_0^{class}(a) = \sum_{\nu=1}^{N'} \frac{P^{class}(l_{\nu\eta}^s(a); a_2 \cdots a_f)}{|\Theta_1'(l_{\nu\eta}^s(a); a)|} \tag{3.26}$$

The probability density $P^{class}(l_{\nu\eta}^s(a); a_2 \cdots a_f)$ is defined by

$$P^{class}(l_{\nu\eta}^s(a); a_2 \cdots a_f) := \int d\lambda\, d^{f_0-1}A\, d^{f_0}\Pi \cdot \delta[\lambda - l_{\nu\eta}^s(a)] \cdot \prod_{\kappa=2}^{f} \delta[\Omega_\kappa(A, \Pi) - a_\kappa]\, d_0(A, \Pi; t \to +\infty) \tag{3.27}$$

or, equivalently

$$P^{class}(l^s_{\nu\eta}(a); a_2 \cdots a_f) = \int d\lambda \int dl\, \delta[\lambda - l^s_{\nu\eta}(a)] \cdot \cdot \tilde{d}(\lambda, a_2 \cdots a_f; l) \tag{3.28}$$

with

$$\tilde{d}(\lambda, a_2 \cdots a_f; l) := d(\Theta(\lambda; a), a_2 \cdots a_f; l) \tag{3.28'}$$

Comparing (3.27) with (3.10) we see that $P^{class}(l^s_{\nu\eta}(a), a_2 \cdots a_f)$ is indeed the classical limit of the average probability $P_\nu(a)$: The summation over microscopic transition probabilities $\mathcal{W}_{\alpha\beta}$ within the macro-cell (a,a+Δa) is replaced by an integration over the classical distribution function $d_0(A, \Pi; \tau \to \pm\infty)$ which describes the fluctuations of the external variables A, Π due to the coupling to the internal degrees of freedom. The integration is restricted to the macro-cell (a,a+Δa) by the δ-functions appearing in (3.25).

In particular, the restriction to a given scattering angle $a_1 \equiv \vartheta$ is achieved by the δ-functions $\delta[\lambda - l^s_{\nu\eta}(a_1; \beta)]$ in (3.10) and by $\delta[\lambda - l^s_{\nu\eta}(a)]$ in (3.27) and (3.28). The fact that in (3.27) and (3.28) only the stationary points $l^s_{\nu\eta}(a)$ of the "macroscopic" deflection function appear does not imply that fluctuations of the orbital angular momentum are neglected. In the formulation (3.28) they are seen to be contained in the integration over the last argument.

Qualitatively, the dependence of $d(\Theta a_2 \cdots a_f; l)$ on Θ and l for given values of the observables $a_2 \ldots a_f$ is shown in fig. 3.1. We expect the distribution d to be different from zero in a domain which surrounds the average deflection function. This area is shaded in fig. 3.1. Depending on the variables $a_2 \ldots a_f$, especially on the total excitation energy of the final fragments and the amount of diffused mass, the region where $d(\Theta, a_2 \cdots a_f; l)$ is large differs. Thus for <u>large</u> excitation energy, and mass transfer, one may expect d to be largest in the region of <u>small</u> angular momenta.

These considerations suggest the following decomposition of $d(\Theta, a_2 \cdots a_f; l)$

$$d(\Theta, a_2 \cdots a_f; l) = \sum_{\nu=1}^{N} d_\nu(\Theta, a_2 \cdots a_f, l) \tag{3.29}$$

where

$$d_\nu(\Theta, a_2 \cdots a_f; l) = d(\Theta, a_2 \cdots a_f; l)\, \theta_0\left[\frac{l^s_{\nu\eta}(a) + l^s_{\nu+1,\eta}(a)}{2} - l\right] \cdot \theta_0\left[l - \frac{l^s_{\nu\eta}(a) + l^s_{\nu-1,\eta}(a)}{2}\right] \delta_{\eta\, \mathrm{sgn}\,\vartheta} \tag{3.30}$$

θ_0 is the usual step function (2.51'); the scattering angle ϑ is equal to the modulus of the deflection angle: $\vartheta = |\Theta| = \eta\Theta$

By definition we have

$$l^s_{0\eta} = -l^s_{1\eta} \tag{3.30'}$$

$$l^s_{\nu_0+1,\eta} = +\infty \tag{3.30''}$$

where $\nu_0 = \begin{cases} N & \text{for } |\Theta| > \sigma \\ N-1 & \text{for } |\Theta| < \sigma \end{cases}$ (3.30''')

The evaluation of the average angular function $\mathcal{V}_\nu(a)$ from a classical distribution function is somewhat less trivial. We treat the general case including diffraction effects, since the simple SPA limit can be easily obtained from it.

The function $\mathcal{V}_{\alpha\beta}\left(l^S_{\nu\eta}(\vartheta;\beta)\right)$ depends on the microscopic channel quantum numbers β in 3 ways: (i) by the dependence of the argument $l^S_{\nu\eta}(a_1;\beta)$ on β, (ii) through the fact that the "microscopic" deflection function $\Theta_{S\beta}$ and its derivatives depend on β. (iii) by the dependence of the cut-off parameters Λ^{fu}_β, $\tilde{\Lambda}_\alpha$, and the rainbow parameters l^r_S, Θ^r_S on β.

The dependence on β causes these quantities to "fluctuate" around averages which depend only on the external variables a.

We neglect the fluctuation of the cut-off parameters, replacing them by average values Λ^{fu}_a, $\tilde{\Lambda}_a$ and, analogously, we substitute the rainbow parameters $l^r_S(\beta)$, $\Theta^r_S(\beta)$ by the corresponding average quantities $l^r_S(a)$, $\Theta^r_S(a)$ defined by (3.23)) and (3.24).

The fluctuations of the deflection angle Θ and the orbital angular momentum l are described by the distribution function $d(\Theta, a_2 \cdots a_f; l)$ (eq. (3.20) or its decomposition (3.29). This function does not contain fluctuations of Θ' and Θ'' for given values of l. By establishing the dependence of Θ' and Θ'' on the basic external variables one could construct a distribution function from $d_0(A, \vec{\pi}; t \to +\infty)$ which contains Θ' and Θ'' as arguments in addition to $(\Theta, a_2, \cdots, a_f; l)$. Since we expect that the fluctuations of Θ' and Θ'' for fixed values of $(\Theta, a_2 \cdots a_f; l)$ are small, we replace these derivatives in $\mathcal{V}_{\alpha\beta}$ by the derivatives $\Theta'(l;a)$ and $\Theta''(l;a)$ of the "macroscopic" deflection function introduced in (3.21). This results in the following

function ϑ :

$$\vartheta[\mathcal{A}, \mathcal{A}'(l;a), \mathcal{A}''(l;a), l] := \frac{m_\alpha l}{a_4 k_\alpha^2}\left\{\sum_\beta \mu_\beta(l) A(l) + \prod_{\beta=1}^{N_r}(1-\mu_\beta(l)) B(l)\right\} \quad (3.31$$

where the functions A(l) and B(l) are obtained from $A_{\alpha\beta}(l_{\nu\eta}^s(\vartheta;\beta))$ and $B_{\alpha\beta}(l_{\nu\eta}^s(\vartheta;\beta))$ as defined in (3.7), (3.8) by the following replacements:

$$\begin{aligned}
&l_{\nu\eta}^s(\vartheta;\beta) \longrightarrow l \\
&\mathcal{A}_{\alpha\beta}[l_{\nu\eta}^s(\vartheta;\beta)] \longrightarrow \mathcal{A} \\
&\mathcal{A}'_{\alpha\beta}[l_{\nu\eta}^s(\vartheta;\beta)] \longrightarrow \mathcal{A}'(l;a) \\
&\mathcal{A}''_{\alpha\beta}[l_{\nu\eta}^s(\vartheta;\beta)] \longrightarrow \mathcal{A}''(l;a) \\
&l_\beta^r(\beta),\ \tau_\beta^r(\beta) \longrightarrow l_\beta^r(a),\ \tau_\beta^r(a) \\
&\Lambda_\beta^{fu},\ \tilde{\Lambda}_\alpha \longrightarrow \Lambda_a^{fu},\ \tilde{\Lambda}_a
\end{aligned} \quad (3.31')$$

The double step function $\mu_\beta(l)$ is defined by replacing in (2.51) $l_{\nu\eta}^s(\vartheta;\beta)$ by l and $l_\beta^r(\beta)$ by $l_\beta^r(a)$.

The average of the function ϑ with the classical distribution function d_ν, with a restriction to the measured scattering angle $a_1 \equiv \vartheta$ yields a classical approximation $\vartheta_\nu^{class}(a)$ of the average function $\vartheta_\nu(a)$ defined by eq. (3.11):

$$\nu_\nu^{class}(a) := \frac{\int d\vartheta_1 \int d\ell\, v[\vartheta_1, \vartheta_1'(\ell;a), \vartheta_1''(\ell;a), \ell]\, d_\nu(\vartheta_1, a_2 \cdots a_f, \ell)\, \delta[\vartheta_1 - \operatorname{sgn}\vartheta_1 \cdot \vartheta]}{\int d\vartheta_1 \int d\ell\, d_\nu(\vartheta_1, a_2 \cdots a_f, \ell)\, \delta[\vartheta_1 - \operatorname{sgn}\vartheta_1 \cdot \vartheta]} \tag{3.32}$$

The final result for the coarse transition probability $Q^{DI}(a,\Delta a)$ is

$$Q^{DI}(a,\Delta a) = \frac{1}{\sin a_1} \sum_{\nu=1}^{N'} P^{class}(l_{\nu\eta}^s(a); a_2 \cdots a_f)\, \nu_\nu^{class}(a)\, \Delta\cos a_1\, \Delta a_2 \cdots \Delta a_f \tag{3.33}$$

and, equivalently, for the coarse cross-section

$$\frac{d^{f-1}Q^{DI}}{d\cos a_1\, da_2 \cdots da_f} = \frac{1}{\sin a_1} \sum_{\nu=1}^{N'} P^{class}(l_{\nu\eta}^s(a); a_2 \cdots a_f)\, \nu_\nu^{class}(a) \tag{3.33'}$$

We now discuss this result in more detail:

(i) Stationary phase approximation (SPA) and purely classical limit

We first consider the simplest case that the part of the fluctuating amplitude which is to represent DI reactions is everywhere a smooth function of the orbital angular momentum. The resulting reaction amplitude was investigated in section 2.2; the corresponding macroscopic cross-section is retrieved from the general formula (3.32) by choosing the cut-off parameters to be

$$\Lambda_\alpha = +\infty \tag{3.34}$$

$$\Lambda_a^{flu} = -\infty \tag{3.34'}$$

This leads to the following form of the function ϑ:

$$\vartheta[\Theta, \Theta'(\ell;a), \Theta''(\ell;a), \ell] = \delta_{\eta\,\mathrm{sgn}\Theta} \frac{m_\alpha \cdot \ell}{a_4 k_\alpha^2} \cdot$$

$$\left\{ \sum_{s=1}^{N_r} u_s(\ell) \left[\frac{2}{|\Theta''(\ell_s^r(a);a)|} \right]^{\frac{2}{3}} < |Ai(X^r)|^2 > + \right.$$

$$\left. + \prod_{s=1}^{N_r} (1 - u_s(\ell)) \frac{1}{|\Theta'(\ell;a)|} \right\} \qquad (3.35)$$

where the argument X^r is given by (see (2.47''))

$$X^r(\ell) = \mathrm{sgn}\,\Theta''(\ell_s^r(a);a) \cdot \left[\frac{2}{|\Theta''(\ell_s^r(a);a)|} \right]^{\frac{1}{3}} (\Theta_s^r(a) - \Theta) \qquad (3.35')$$

Figure 3.2 shows the Airy function and its square; in fig. 3.3 we present a deflection function for the system Ar + Th at E_{LAB} = 388 MeV together with its 1st and 2nd derivative. The deflection function of fig. 3.3 was obtained by a classical trajectory calculation including friction forces[13]) while the function $\Theta(\ell,a)$ is defined as an average deflection function for elastic scattering in the coarse cell (a,a+Δa). Nevertheless, the general features of the two functions are expected to be the same.

The averaging process (3.32) will affect the function $Ai^2(X^r)$ through softening the steep decrease for $X^r > 0$ and broadening the first maximum at $X^r = -1$. This first maximum represents the peak

of the angular distribution and is expected to occur at a deflection angle Θ_{max} and a corresponding scattering angle ϑ_{max} given by

$$\Theta_{max} = \operatorname{sgn}\Theta_{max} \cdot \vartheta_{max} = \Theta_S^r(a) + \operatorname{sgn}\Theta''(l_S^r(a), a) \cdot \left[\frac{|\Theta''(l_S^r(a), a)|}{2}\right]^{\frac{1}{3}} \qquad (3.36)$$

We note that the difference $|\Theta_{max} - \Theta_S^r(a)|$ between the peak of the angular distribution and the average rainbow angle may be as large as about 10^0. The rainbow oscillations for $\gamma < 1$ are expected to be wiped out by the averaging (3.32) whenever the width of the l-distribution is sufficiently large. In the example of fig. 3.3 we estimate this to be so if the width exceeds 20. In an actual experiment, usually only a few macroscopic observables are measured, which implies an integration of the cross-section (3.33') over all the unmeasured observables α_K. This integration has the effect of an additional smoothening of the angular distribution. Thus the fact that rainbow oscillations have as yet not been observed in DI reactions does not mean that they cannot be seen in a "maximal" experiment, i.e. one in which all the macroscopic observables are measured.

An observable feature should be that the descent on the "dark" side of the rainbow is expected to be steeper than the one on the "bright" side. If the sign of the deflection function $\Theta(l, a)$ remains the same throughout the effective integration interval

in (3.32)*, one finds that the "dark" side of the rainbow occurs for

$$\vartheta < \vartheta_{max} \quad \text{if} \quad \operatorname{sgn} \Theta''(l_\zeta^r(a)) \cdot \operatorname{sgn} \Theta_{max} = +1$$

and for

$$\vartheta > \vartheta_{max} \quad \text{if} \quad \operatorname{sgn} \Theta''(l_\zeta^r(a)) \cdot \operatorname{sgn} \Theta_{max} = -1$$

In the purely classical limit, the 2nd term in (3.35) is considered to be valid for all scattering angles ϑ. In the classical theory, one usually introduces the impact parameter b instead of the orbital angular momentum. We define the impact parameter by

$$b := \frac{l}{a_3} \tag{3.37}$$

and the "stationary" values of the impact parameter by

$$b_{\nu\eta}^s(a) := \frac{l_{\nu\eta}^s(a)}{a_3} \tag{3.37'}$$

Furthermore, we introduce a b-dependent deflection function $\tilde{\Theta}$ by

$$\tilde{\Theta}(b; a_2 \cdots a_f) := \Theta\left(\frac{l}{a_3}; a_2 \cdots a_f\right) \tag{3.38}$$

---

*footnote. This assumption cannot be made if the rainbow region is in forward direction.

Assuming that

$$\frac{\int d\vartheta_4 \int dl \; \frac{1}{|\Theta_4'(l;a)|}\, d_\nu(\vartheta_4, a_2 \cdots a_f; l)\, \delta[\vartheta_4 - \vartheta \cdot \operatorname{sgn}\vartheta_4]}{\int d\vartheta_4 \int dl \; d_\nu(\vartheta_4, a_2 \cdots a_f; l)\, \delta[\vartheta_4 - \vartheta \cdot \operatorname{sgn}\vartheta_4]} \approx \frac{1}{|\Theta_4'(l_{\nu\eta}^s(a))|} \tag{3.39}$$

we arrive at the result:

$$\left[\frac{d^f Q^{DI}}{d\cos a_1\, da_2 \cdots da_f}\right]_{class} = \frac{1}{\sin a_1}\, \frac{a_3^2\, m_\alpha}{a_4\, k_\alpha^2} \cdot$$

$$\cdot \sum_{\nu=1}^{N'} \frac{\tilde{P}^{class}(l_{\nu\eta}^s(a), a_2 \cdots a_f)\, b_{\nu\eta}^s(a)}{\left|\left(\frac{\partial \Theta_4(b,a)}{\partial b}\right)_{b_{\nu\eta}^s(a)}\right|} \tag{3.40}$$

We note that the factor $\frac{a_3^2}{k_\alpha^2}$ in (3.40) is a consequence of defining the impact parameter with the wave number a_3 instead of k_α which seems reasonable since in our theory the orbital angular momenta refer to exit channels. Analogously to (3.38), $\tilde{P}^{class}$ is defined by

$$\tilde{P}^{class}(l, a_2 \cdots a_f) := P^{class}\left(\frac{l}{a_3}, a_2 \cdots a_f\right) \tag{3.38'}$$

Due to the azimuthal symmetry of the initial state and the Hamiltonian, the quantities on the righthand side of (3.40) do not depend on $a_2 = \varphi$.

If only the scattering angles a_1, a_2 are determined, the measured cross-section is given by $\int da_3 \cdots da_f \frac{d^f Q^{DI}}{d\cos a_1\, da_2\, da_3 \cdots da_f}$. In the case of the classical limit this leads to

$$\int da_3 \cdots da_f \left(\frac{d^f Q^{DI}}{d\cos a_1 da_2 \cdots da_f}\right)_{class} = \left(\frac{d^2 Q^{DI}}{d\cos a_1 da_2}\right)_{class} =$$

$$= \sum_{\nu=1}^{N'} \frac{1}{\sin a_1} \int da_3 \cdots da_f \left(\frac{a_3}{a_3^{in}}\right)^2 \left(\frac{a_4^{in}}{a_4}\right) \cdot \qquad (3.41)$$

$$\cdot \frac{b_{\nu\eta}^s(a)\, \tilde{P}^{class}(b_{\nu\eta}^s(a), a_3 \cdots a_f)}{\left|\left(\frac{\partial \tilde{\Theta}}{\partial b}\right)_{b_{\nu\eta}^s(a)}\right|}$$

where $\eta = sgn\, \tilde{\Theta}(b_{\nu\eta}^s(a))$ and where in analogy to (3.1'') and (3.1''') we have defined

$$a_3^{in} := k_\alpha \qquad (3.41')$$

$$a_4^{in} := m_\alpha \qquad (3.41'')$$

The function $\tilde{P}^{class}$ describes a distribution of all the macroscopic observables in the final state. Let us define average values $\overline{a_K}$ of the observables $K \geq 3$ for given scattering angle a_1 :

$$\overline{a_K}(a_1) := \frac{\sum_\nu' \int da_3 \cdots da_f\, \tilde{P}^{class}(b_{\nu\eta}^s(a), a_3 \cdots a_f) a_K}{\sum_\nu' \int da_3 \cdots da_f\, \tilde{P}^{class}(b_{\nu\eta}^s(a), a_3 \cdots a_f)} \qquad (3.42)$$

If one calculates classical trajectories from Newtonian equations of motion including friction forces[10-14], one only predicts the averages $\bar{a}_K(a_1)$. The cross-section $\frac{d\sigma^{fr}}{d\Omega}$ in such a theory turns out to be

$$\frac{d\sigma^{fr}}{d\Omega} \equiv \frac{d^2\sigma^{fr}}{d\cos\alpha_1 \, d\alpha_2} = \sum_\nu \frac{b_{\nu\eta}^{fr}(a_1)}{\sin a_1} \frac{1}{\left| \left(\frac{\partial \Theta^{fr}(b)}{\partial b} \right)_{b_{\nu\eta}^{fr}(a_1)} \right|} \tag{3.43}$$

where $b_{\nu\eta}^{fr}(a_1)$ and $\Theta^{fr}(b)$ represent the impact parameters which contribute to scattering into the angle $a_1 = \vartheta \left(\zeta = sgn\, \Theta^{fr}(b_{\nu\eta}) \right)$ and the deflection function, resp. Such a theory is only meaningful if the fluctuations of the external variables are small. The cross-section (3.43) must thus be identified with the cross-section (3.41) in the limit that $\tilde{P}^{class}$ describes narrow distributions of the macroscopic variables around the mean values (3.42). The classical trajectory calculations so far performed do not allow for mass transfer. They thus only apply to systems where the factor $\frac{a_2^{in}}{a_4} \approx 1$. The average deflection function Θ^{fr}, which contains the effect of the energy loss due to friction, and its inverse $b_{\nu\eta}^{fr}(a_1)$ must thus satisfy the relation

$$\frac{b_{\nu\eta}^{fr}(a_1)}{\left| \left(\frac{\partial \Theta^{fr}(b)}{\partial b} \right)_{b_{\nu\eta}^{fr}(a_1)} \right|} = \int da_3 \cdots da_f \left(\frac{a_3}{a_3^{in}} \right) \frac{b_{\nu\eta}(a) \, \tilde{P}^{class}(b_{\nu\eta}^s(a_1), a_3 \cdots a_f)}{\left| \left(\frac{\partial \Theta(b,a)}{\partial b} \right)_{b_{\nu\eta}^s(a_1)} \right|} \tag{3.44}$$

(ii) Inclusion of diffraction effects

Let us now discuss the result (3.33') in the general case that the amplitude of the DI reaction changes rapidly in the vicinity of a lower and upper angular momentum. Then the general form of the functions A and B (see (3.7), (3.8) with the replacements (3.31')) applies.

In general, the inner rainbow region, which matters for DI reactions, corresponds to angular momenta ℓ largely different from Λ_a^{fe} and $\tilde{\Lambda}_a$. In this case one of the arguments $Y_{1,2}$ (see (2.60), (2.60')) is a large negative number and the other one a large positive number. Thus, we have again the results contained in (3.35);

$$A\left(\Theta i(\ell, a), a\right) \approx \left[\frac{2}{|\Theta''(\ell_g^r(a))|}\right]^{\frac{2}{3}} \langle |Ai(X)|^2 \rangle \qquad (3.45)$$

The function B is readily discussed for specific ranges of the argument ℓ by considering the relations

$$\lim_{X \to +\infty} C(X) = \lim_{X \to +\infty} S(X) = \frac{1}{2} \qquad (3.46)$$

$$C(-X) = -C(X) \qquad (3.47)$$

$$S(-X) = -S(X) \qquad (3.47')$$

as well as the asymptotic behavior (2.37) of the complementary error function.

For $C_0(\ell - \Lambda_a^{fu}) \gg +1$ and for

$C_0(\tilde{\Lambda}_a - \ell) \gg +1$

there is no effect of diffraction

$$\lim_{\ell \ll \tilde{\Lambda}_a;\ \ell \gg \Lambda_a^{fu}} B = \frac{1}{|\Theta i'(l_{vy}^{s}(a))|} \qquad (3.48)$$

Finally, if ℓ is close to one of the cut-off parameters Λ_a^{fu}, $\tilde{\Lambda}_a$ and far from the other, one of the C-functions and one of the S-functions in (3.8') can be replaced by $\frac{1}{2}$, while the other ones describe oscillations. Of course, the averaging process (3.32) will smoothen these "Fresnel" oscillations, and so does an integration over unmeasured macroscopic variables. The question whether Fresnel and rainbow oscillations should be visible at all in DI heavy ion reactions can only be determined by careful calculations based on a realistic distribution function d_0 which we intend to carry out.

Let us discuss the expected effects on the angular distribution for the (fictitious) cut-off parameters $\Lambda_a^{fu} = 100$ (which would grossly correspond to the measured fusion corss-section) and $\tilde{\Lambda}_a = 190$ in fig. 3.3:

For $\vartheta > \vartheta^{fu} = \Theta i^{fu} \approx 65°$, the function B and thus the cross-section would have to tend to zero.

For $\vartheta^{fu} > \vartheta > \vartheta^{D} = \Theta i^{D} \approx 27°$, it is the ℓ-values

$\Lambda_a^{fu} < \ell < \Lambda_1$

and for $\vartheta < \vartheta^{D}$ it is the ℓ-values

$\Lambda_1 < \ell < \tilde{\Lambda}_a$

which contribute (for definition of $\Lambda_1, \Theta^{fu}, \Theta^{D}$ see fig. 3.3).

Let us finally comment on the problem of distinguishing DI and CN reactions: In appendix A2 we have generalized the semi-classical and the classical result to an arbitrary number N_0 of orbitings. Since the semi-classical approximation as well as the purely classical limit may also be applied to the totally equilibrated amplitude, the resulting average cross-section (A1-17) and, more specifically, its limit (A1-22) also apply for the CN cross-section with the difference that $\tilde{P}^{class}$ is then to be replaced by the corresponding transition density $\tilde{P}^{class}_{CN}$ for totally equilibrated reactions. Contrary to the transition probability for DI reactions, the distribution P^{class}_{CN} would factorize into a formation and decay probability (see A1-23). Since even the angular distribution $\frac{dQ^{DI}}{da_i}$ may approach isotropy without complete equilibration being reached (see Appendix A1), an experimental separation of CN and DI reactions may be very difficult in certain cases.

Theoretically, the problem may consist in decomposing a classical distribution function of the external variables into a part describing a fully relaxed component of the reaction and the partially relaxed remainder, which is called $d_0(A, \pi, t)$ in this paper.

4. Summary and Discussion

We defined the amplitude for deep inelastic reactions as being generated by the incompletely equilibrated part of the fluctuating S-matrix. By applying well-known semi-classical approximations, we derived a general form of the amplitude and of the average cross-section for DI reactions.

We distinguished the treatment without and with diffraction effects as limiting cases of a very smooth and a sudden onset of DI reactions as a function of the orbital angular momentum. The experimental results indicate that the deflection angle relaxes more slowly than other external variables like the radial translational motion. Therefore, the Poisson representation which achieves a decomposition of the total amplitude into terms related to given numbers of orbitings was chosen as a starting point. We believe that the contribution from events without any orbiting is at least responsible for that part of the DI reactions which exhibits a strongly peaked angular distribution, perhaps even for the entire cross-section of DI reactions (see appendix A1).

Thus the main chapters contain the results for the case without orbiting only, while the general case of an arbitrary number of orbitings is dealt with in appendix A1.

We believe that the following results should be drawn to the reader's attention:

(i) The measured cross-section involves a summation over very many microscopic channels compatible with given values of the macroscopic external observables, apart from the averaging over the energy width of the incident beam. It is shown that this measured "macroscopic cross-section" is determined by a coarse transition probability only.

(ii) For the case that the external variables of the system are amenable to a classical treatment, it is demonstrated that the macroscopic cross-section can indeed be calculated from the knowledge of a classical distribution function.

(iii) It is found that diffraction effects are produced if the amplitude for DI reactions turns out to decrease rapidly to zero in the vicinity of a lower and upper value of the orbital angular momentum. Although such diffraction effects are beyond the scope of a description by classical statistical mechanics, their evaluation is shown to involve only the above-mentioned distribution function.

(iv) We point out in the appendix A1 that an isotropic angular distribution $\frac{dQ^{DI}}{da_i}$ may be produced either if contributions from various numbers of orbitings superimpose or if the dominant part of the cross-section is due to the smooth low impact parameter part of the macroscopic deflection function. In the strongly relaxed DI reactions which were recently studied by Moretto et al, apparently the latter case seems to be realized.

(iv) Whenever any direct channels are open, the energy average of the S-matrix is not zero and thus the fluctuating part of the S-matrix is not unitary. On the other hand, the "macroscopic" probabilities

can only be expected to describe the DI and the CN reactions, not the direct reactions. The coarse probability in our final result for the DI reactions may be considered to be the partially equilibrated part of a total coarse probability describing both the DI and CN reactions. Even this total macroscopic probability does not add up to 1, if summed over all coarse cells whenever direct reactions exist. Thus, it also cannot be expected that the classical distribution function $d_0(A,\Pi;t)$ strictly satisfies the ordinary Master or Fokker-Planck equations. We cannot exclude, of course, that this will be so in a meaningful approximation.

(iv) In all the experiments hitherto performed, only a part of the macroscopic observables was actually measured. This implies that the experimental cross-section is an integral of our resulting expression (3.33') over all the unobserved variables a_ν. In the case that the scattering angle $\vartheta \equiv a_1$, the mass (charge) a_3, and the kinetic energy a_4 of the outgoing fragments are observed the measured coarse cross section is given by

$$\frac{d^4Q^{DI}}{d\omega\, da_1\, da_2\, da_3\, da_4} = \sum_\nu' \int da_5 \cdots da_f \; P^{class}(l^s_{\nu\eta}(a), a_2 \cdots a_f) \frac{\mathcal{V}_\nu(a)}{\sin a_1} \qquad (4.1)$$

Even in the case of a purely classical approximation, where

$$\mathcal{V}_\nu(a) = \frac{1}{|A'(l^s_{\nu\eta}(a); a_3 \cdots a_f)|}$$

the result of the integration is generally not of the form (3.33').

(iv) Last, not least, we wish to emphasize that the use of the SPA with or without cut-off is based on the hypothesis that the matrix $C^{DI(S)}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha}(\ell)$ depends smoothly on the orbital angular momentum. It is by no means obvious that this is correct. Indeed, if one were to make the opposite assumption that this amplitude differs from 0 only in a narrow "window" of ℓ-values around a "grazing" angular momentum" ℓ_{gr}, —— an assumption which was demonstrated to be appropriate[35]) in many cases of elastic as well as direct inelastic reactions,— a totally different result would be obtained. Then the amplitude for DI reactions would be given essentially by a sort of Fourier transform of the form

$$\int_{-\infty}^{+\infty} d\ell \; e^{i\left[(\ell+\frac{1}{2})\vartheta + \delta_\alpha(\ell+\Delta\ell) + \delta_\beta(\ell)\right]} \; C^{DI(S)}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha}(\ell)$$

The differences between this approach and the SPA were recently pointed out in a succinct and lucid way by S. Kahana[36].

We believe that two arguments may be given in favor of our smoothness assumption:

a) Only the applicability of the SPA makes the success of purely classical methods understandable.

b) As one compares DWBA amplitudes with the more complex amplitudes resulting from coupled channel calculations, the dependence of the S-matrix on ℓ tends to become smoother, as the number of participating degrees of freedom increases.

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APPENDICES

1. STRONGLY RELAXED DI REACTIONS

We have already noted in §2.1 that there may be a smooth transition between DI reactions where a few external degrees like the mass asymmetry and the deflection angle remain far from equilibration and the limit of the compound nucleus formation where all the external degrees of freedom reach equilibrium. In the chapters 2 and 3 we then treated the special case that contributions from orbiting events may be neglected, i.e. we considered the limit of DI reactions with completely unequilibrated deflection angles.

We now treat the more general case that the lifetime of the intermediate system is long enough to permit one or several full revolutions. This is expected to introduce a gradual transition to the typical features of compound nuclear reactions. In our theory, it implies that we have to study the terms with $p \neq 0$ in the Poisson distribution as well. Since the calculation is completely analogous to the one in the chapters 2 and 3 we confine ourselves to a short presentation of the results.

The general quantal form of the amplitude for partially relaxed collisions is

$$f^{DI}_{\beta s_\beta M_\beta ; \alpha s_\alpha M_\alpha} = \frac{e^{i(M_\alpha - M_\beta)\varphi}}{i\sqrt{2\pi k_\alpha k_\beta \sin\vartheta}} \sum_p \left\{ \mathcal{J}^{+}_{M_\alpha M_\beta p} + (-)^{M_\alpha - M_\beta} \mathcal{J}^{-}_{M_\alpha M_\beta p} \right\} \qquad \text{(A1-1)}$$

with

$$\mathcal{J}^{\eta}_{M_\alpha M_\beta p} := \int_0^\infty dl\,(l+\tfrac{1}{2})^{\frac{1}{2}} e^{iA^{\eta}_{\alpha\beta;p}(l)}\, C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(l, \Theta_\alpha(l,\Delta\bar{l})) \tag{A1-2}$$

$$A^{\eta}_{\alpha\beta;p}(l) := \chi^{\eta}_{\alpha\beta;p}(l,\Delta\bar{l}) - \frac{\Delta\bar{l}}{2}\Theta_\alpha(l+\Delta\bar{l}) \tag{A1-3}$$

$$\chi^{\eta}_{\alpha\beta;p}(l,\Delta\bar{l}) := \delta_\alpha(l+\Delta\bar{l}) + \delta_\beta(l) - \eta\left[(l+\tfrac{1}{2})\vartheta - \tfrac{\pi}{4}\right] + (l+\tfrac{1}{2})2\pi p \tag{A1-4}$$

The amplitude $C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}$ is again defined by (2.26). We note that it is in fact undesirable to expand around the same average angular momentum mismatch $\Delta\bar{l}$ irrespectively of the value of p. On the other hand, the resulting formulae become considerably more complicated if one were to consider $\Delta\bar{l}$ to be a function of p.

As in chapter 2, we evaluate the integrals (A1-2) by the SPA with and without cut-offs at finite values of the angular momentum. The stationary points $l^s_{\nu\eta p}(\vartheta;\beta)$ are given as solutions of the equation

$$\left[\frac{\partial A^{\eta}_{\alpha\beta;p}(l)}{\partial l}\right]_{l^s_{\nu\eta p}} = 0 = \Theta_{\alpha\beta}(l^s_{\nu\eta p}) - \eta\vartheta + 2\pi p \tag{A1-5}$$

Since we limit ourselves to real solutions of (A1-5), the index p may only assume positive values p = 0,1,2,... . Furthermore, it is easily seen that for given deflection angle $\Theta_{\alpha\beta}$ and given p, the sign η is uniquely defined as

$$\eta = sgn\left[\Theta_{\alpha\beta}\left(l^s_{\nu\eta p}(\vartheta;\beta)\right) + 2\pi p\right] \tag{A1-6}$$

As in chapter 2, we replace the angle $\Theta_\alpha(l+\Delta\bar{l})$ by $\Theta_{\alpha\beta}(l')$ in the argument of $C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}$ (see A1-2)) thus obtaining

$$C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}\left(l^s_{\nu\eta p};\Theta_\alpha(l^s_{\nu\eta p}+\Delta\bar{l})\right) \approx$$

$$\approx C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}\left(l^s_{\nu\eta p};\Theta_{\alpha\beta}(l^s_{\nu\eta p})\right)$$

$$= C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}\left(l^s_{\nu\eta p};\eta\vartheta - 2\pi p\right) =$$

$$= \sum_{M'_\alpha M'_\beta} D^{s_\beta *}_{M'_\beta M_\beta}\left(-\frac{\pi+\eta\vartheta}{2}+\pi p,\frac{\pi}{2},\frac{\pi}{2}\right)\cdot C^{DI(S_F)}_{\beta s_\beta M'_\beta;\alpha s_\alpha M'_\alpha}$$

$$\cdot D^{s_\alpha}_{M'_\alpha M_\alpha}\left(-\frac{\pi+\eta\vartheta}{2}+\pi p,\frac{\pi}{2},\frac{\pi}{2}\right) =: C^{DI(S)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}\left(l^s_{\nu\eta p}\right) \tag{A1-7}$$

Because of

$$D^{s_\alpha}_{M'_\alpha M_\alpha}(\alpha+2\pi,\beta,\gamma) = e^{i2\pi M'_\alpha} D^{s_\alpha}_{M'_\alpha M_\alpha}(\alpha,\beta,\gamma)$$

, a rotation by 2π introduces a sign-change of the amplitude $C^{DI}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(l^s_{\nu\eta p};\eta\vartheta-2\pi p)$ if one of the channel spins is odd. This has no influence on the probability $\omega_{\alpha\beta}$ (see equation (3.5)). Consequently, for even values of p, the rotation by the 1st Euler angle $\alpha = \left[-\frac{\pi+\eta\vartheta}{2}+\pi p\right]$ is equivalent to a rotation by $\alpha = -\frac{\pi+\eta\vartheta}{2}$, and for odd p, it is equivalent to one by $\alpha = \left[-\frac{\pi+\eta\vartheta}{2}+\pi\right]$. On the other hand, for trajectories with odd p, the point of closest approach is on the <u>opposite</u> side of the beam than the point at time $t = -\infty$. Thus the focal coordinate system S_F for odd p - trajectories differs from the one for even p - trajectories by just the Euler rotation $\Delta\alpha = \pi$ as provided for by eq. (A1-7).

By a straightforward calculation one finds the following general form of the integrals $\mathcal{J}^{\eta}_{M_\alpha M_\beta p}$:

$$\mathcal{J}^{\eta}_{M_\alpha M_\beta p} = \sum_{\nu=1}^{N'} (l^s_{\nu\eta p}+\tfrac{1}{2})^{\frac{1}{2}} e^{iA^{\eta}_{\alpha\beta;p}(l^s_{\nu\eta p})} \cdot$$

$$\cdot C^{DI(S)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(l^s_{\nu\eta p}) \Big\{ \sum_{\varrho=1}^{N_r} \mathcal{U}_\varrho(l^s_{\nu\eta p})\, \tilde{\mathcal{F}}^{\varrho p}_{\alpha\beta}(l^s_{\nu\eta p}) +$$

$$+ \prod_{\varrho=1}^{N_r} \left(1-\mathcal{U}_\varrho(l^s_{\nu\eta p})\right) \mathcal{F}_{\alpha\beta}(l^s_{\nu\eta p}) \Big\} \qquad \text{(A1-8)}$$

The sum Σ' extends over those branches of the inverse deflection function which contain a real stationary solution $l^{s}_{\nu\eta\rho}(\vartheta;\beta)$. The double step function u_ρ is defined in (2.51) and only serves to separate the region of the Airy approximation from the one of the ordinary SPA. The function $\mathcal{J}_{\alpha\beta}$ is defined by (2.36), if diffraction effects are negligible, and by (2.57) if diffraction effects are accounted for by sharp cut-offs. The integral $\tilde{\mathcal{J}}^{s\rho}_{\alpha\beta}$ is defined by

$$\tilde{\mathcal{J}}^{s\rho}_{\alpha\beta} := \int_{l_1}^{l_2} e^{i\left\{(l-l^r_\rho)(\Psi^r_\rho - \eta\vartheta + 2\pi\rho) + \frac{(l-l^r_\rho)^3}{6}\Psi''_{\alpha\beta}(l^r_\rho)\right\}} dl \qquad \text{(A1-9)}$$

where $l_1 = 0$ or $-\infty$; $l_2 = +\infty$ without diffraction effects and $l_1 = l^{\text{left}}_\beta$; $l_2 = l^{\text{right}}_\alpha$ including diffraction effects. The result is again given by (2.45) without and by (2.59) with diffraction effects if only we replace the argument χ^ν (see eq. (2.47)) by

$$\chi^\nu_\rho = \tilde{\varepsilon}\left[\frac{2}{|\Psi''_{\alpha\beta}(l^r_\rho)|}\right]^{\frac{1}{3}}(\Psi^r_\rho - \eta\vartheta + 2\pi\rho) \qquad \text{(A1-10)}$$

The cross-section (2.17'') contains interferences between trajectories differing by ν (i.e the branch of the inverse deflection function) and p (i.e. the number of orbitings). Since the actually measured cross-section (3.3) involves a summation of micro-channels β within a coarse cell $(a, a+\Delta a)$ as well as the energy-averaging, the contribution from the various interferences is given by the sum over a

large number of terms with random phases. It may thus be neglected. Assuming again that the fluctuations of the transition probabilities $w_{\alpha\rho}$ and angular functions $v_{\alpha\rho}$ are uncorrelated (see (3.14)) we arrive at the result ($'$ means summation over real stationary points only):

$$\frac{d^f Q^{DI}}{d\omega x_1 da_2 \cdots da_f} = \frac{1}{\sin a_1} \sum_{p=0}^{\infty\,\prime} \sum_{\nu=1}^{N\,\prime} P_{\nu p}(a)\, v_{\nu p}(a) \qquad \text{(A1-11)}$$

The macroscopic transition density $P_{\nu p}$ (a) and the average angular function $v_{\nu p}(a)$ are defined by (3.10) and (3.11) resp. with the only difference that the argument $l^s_{\nu\eta}(\vartheta;\beta)$ is replaced by $l^s_{\nu\eta p}(\vartheta;\beta)$ and the argument χ of the Airy-function by χ_p (see A1-10)).

The derivation of the average quantities $P_{\nu p}$ (a) and $v_{\nu p}(a)$ from the classical distribution function $d_0(A, \pi; t \to +\infty)$ proceeds in complete analogy to §3.2. :

Since we now allow for any number of orbitings, the argument Θ of the distribution function d (see (3.20)) may have values between $-\infty$ and π: $-\infty < \Theta \leq \pi$. It is convenient to note the number p of full revolutions as a supplementary argument of the function $d(\Theta, a_2, \cdots, a_f; l, p)$.

As for the macroscopic deflection function (3.21), we may now have stationary values $l^s_{\nu\eta p}(\vartheta; a)$ corresponding to a finite number p of orbitings, replacing (3.22) by

$$\Theta\left[l^{s}_{\nu\eta p}(\vartheta;a);a\right]=\eta\vartheta-2\pi p \tag{A1-12}$$

where η is defined to be

$$\eta=\operatorname{sgn}\left[\Theta\left(l^{s}_{\nu\eta p}(\vartheta;a)\right)+2\pi p\right] \tag{A1-12'}$$

Usually the deflection function $\Theta(l;a)$ through eq. (A1-12), only allows for a finite maximal number N_0 of orbitings which may depend on the scattering angle, i.e. we have $0\leq p\leq N_0$. Analogously to the case of $\sum_{\nu=1}^{N_1}$ we shall imply by the symbol $\sum_{p=0}^{\infty\,\prime}$ that the sum is only to be extended over those values p, which correspond to <u>real</u> solutions $l^{s}_{\nu\eta p}(\vartheta;a)$ of (A1-12).

With obvious modifications of the derivation in §3 one finds within the validity of a classical theory of external variables that the macroscopic probability density $P_{\nu p}(a)$ becomes equal to the classical probability density

$$P^{class}_{s}\left(l^{s}_{\nu\eta p}(a),a_2\cdots a_f\right)=\int d\lambda\int dl\,\delta\left[\lambda-l^{s}_{\nu\eta p}(a)\right]\tilde{d}(\lambda,a_2\cdots a_f;l,p) \tag{A1-13}$$

with

$$\tilde{d}(\lambda,a_2\cdots a_f;l,p)=d\left(\Theta(\lambda;a),a_2\cdots a_f;l,p\right) \tag{A1-13'}$$

For evaluating the average $\nu_{\nu p}(a)$ we decompose the distribution $d(\theta_1, a_2 \cdots a_f; l, p)$ into components $d_\nu(\theta_1, a_2 \cdots a_f; l, p)$ related to the different branches of the macroscopic deflection function. In complete analogy to (3.29), (3.30) we have

$$d(\theta_1, a_2 \cdots a_f; l, p) = \sum_{\nu=1}^{N} d_\nu(\theta_1, a_2 \cdots a_f; l, p) \tag{A1-14}$$

$$d_\nu(\theta_1, a_2 \cdots a_f; l, p) = d(\theta_1, a_2 \cdots a_f; l, p) \cdot \theta_0[\bar{l}_{\nu\eta p}(a) - l]\, \theta_0[l - \bar{l}_{\nu-1,\eta p}] \tag{A1-15}$$

$$\bar{l}_{\nu\eta p}(a) := \frac{l^s_{\nu\eta p}(a) + l^s_{\nu+1,\eta,p}(a)}{2} \tag{A1-15'}$$

$$l_{0\eta p} := -l_{1\eta p} \tag{A1-15''}$$

$$l_{\nu_0+1,\eta,p} := +\infty \tag{A1-15'''}$$

Where ν_0 is defined by (3.30''')

The classical approximation of the average angular function $\nu_{\nu p}(a)$ is then given by

$$\nu_{\nu p}^{class}(a) := \frac{\int d\theta \int dl\, \nu[\theta_1, \theta'(l,a), \theta''(l,a), l]\, \delta[\theta - \eta\vartheta + 2\bar{u}p]\, d_\nu(\,)}{\int d\theta \int dl\, d_\nu(\theta_1, a_2 \cdots a_f; l, p)} \tag{A1-16}$$

In (A1-16) the range of the ϑ-integration depends on p:

$$-\pi - 2\pi p < \vartheta < \pi$$

and the function ν is defined by (3.31).

For the purely classical limit we find as a generalization of (3.40)

$$\left[\frac{d^f Q^{DI}}{d\cos a_1\, da_2 \cdots da_f}\right]_{class} = \frac{a_3^2 \cdot m_\alpha}{\sin a_1\, k_\alpha^2\, a_4} \sum_{p=0}^{\infty\prime} \sum_{\nu=1}^{N_1} \cdot \left\{ \frac{\tilde{P}^{class}(b, a_3 \cdots a_f)\, b}{\left| \frac{\partial \vartheta(b, a_3 \cdots a_f)}{\partial b} \right|} \right\}_{b = b^s_{\nu\eta p}(a)} \tag{A1-17}$$

where the quantities $b^s_{\nu\eta p}$, ϑ and $\tilde{P}^{class}$ are defined as in (3.37'), (3.38), and (3.38') resp.. η is defined by (A1-12').

There are various ways by which the double sum on the right hand side of (A1-17) may become independent of the scattering angle a_f. We discusss two alternative ways which represent physically opposite situations:

(i) The probability density $\tilde{P}^{class}$ is unequal zero only for p = 0 (no orbiting) and $\nu = 1$ (low impact parameter branch of the deflection function, see fig. 3-1) and is independent of l in this range of l-values.

$$\tilde{P}^{class} \approx \delta_{\nu 1}\, \delta_{p 0}\, \tilde{P}^{class}(b^s_{\nu\eta p}(a), a_3 \cdots a_f) \tag{A1-18}$$

Furthermore, the 2nd factorin (A1-17) is independent of b too:

$$\frac{b}{\left|\frac{\partial \tilde{H}(b, a_3 \cdots a_f)}{\partial b}\right|} \approx k_0(a_3 \cdots a_f) \qquad \text{(A1-19)}$$

Obviously, in this case, the coarse cross-section $\frac{d^f Q^{DI}}{d\cos a_1 da_2 \cdots da_f}$ depends on a_1 only through $\frac{1}{\sin a_1}$ which is a part of the differential space angle $(d\Omega = \sin a_1 da_1 da_2)$:

$$\frac{d^f Q^{DI}}{d\cos a_1 da_2 \cdots da_f} = \frac{a_3^2 m_2}{\sin a_1 k_2^2 a_4} \tilde{P}^{class}(l_{\eta 0}^s, a_3 \cdots a_f) k_0(a_3 \cdots a_f) \qquad \text{(A1-20)}$$

At first sight, the conditions (A1-18), (A1-19) seem to be rather artificial. Nevertheless, they may be fulfilled for the strongly relaxed component of HIR :

The macroscopic probability obtained from solving a Master – equation is indeed found to depend slowly on b for the very relaxed component of DI reactions (see L. Moretto and P. Schmitt in ref. 8). Furthermore, eq. (A1-19) holds whenever $\tilde{H}(b)$ can be approximated by a parabola

$$\tilde{\Theta}(b) = \frac{k_0}{2} b^2 + k_1 \qquad \text{(A1-20)}$$

where k_0 and k_1 may still be functions of $a_3 \ldots a_f$. For the low impact parameter branch of the deflection function this may not be a bad approximation as is seen qualitatively from figs. 3.1 and 3.3.

(ii) A large number $N_0 \gg 1$ of terms in the sum $\sum'_p$ contributes to (A1-17) to the extent that the discrete values $b_{\nu\eta p}(a_1; a_3 \ldots a_f)$ lie on a smooth interpolating curve $b(p\,; a_3 \ldots a_f)$ whose dependence on the scattering angle $a_1 \equiv \vartheta$ is negligible whenever the points $b_{\nu\eta p}$ are sufficiently closely spaced. From fig. A1-1 it is seen that this is the case for $N_0 \gg 1$.

One then has

$$\sum_{p=0}^{N_0} \sum_{\nu=1}^{N}{}' \left\{ \frac{\tilde{P}^{class}(b, a_3 \cdots a_f)\, b}{\left| \frac{\partial \tilde{\Theta}(b, a_3 \cdots a_f)}{\partial b} \right|} \right\}_{b_{\nu\eta p}^{s}(a)} \approx$$

$$\approx \int_{b_1}^{b_2} db\, \frac{\tilde{P}^{class}(b, a_3 \cdots a_f)\, b}{\left| \frac{\partial \tilde{\Theta}}{\partial b} \right|} \left| \frac{\partial p}{\partial b} \right|$$

$$= \int_{b_1}^{b_2} db\, \frac{\tilde{P}^{class}(b, a_3 \cdots a_f)\, b}{\left| \left(\frac{\partial \tilde{\Theta}}{\partial p} \right) \right|_{p(b)}} \approx \frac{1}{2\pi} \int_{b_1}^{b_2} db\, b\, \tilde{P}^{class}(b, a_3 \cdots a_f) \qquad \text{(A1-21)}$$

The function $p(b) \equiv p(b, a_3 \ldots a_f)$ is the inverse of $b(p, a_3 \ldots a_f)$. In the last step of (A1-21) we neglect the term $\eta\vartheta$ in

$$\hbar\tilde{\Theta}[b] = \eta\vartheta - 2\pi p$$

The limits b_1, b_2 of the integral depend on the scattering angle. As is seen qualitatively from fig. A1-1, this dependence is negligible if a very large number N_0 contributes. Thus the cross-section

$$\left(\frac{d^f Q^{PI}}{d\cos a_1 da_2 \cdots da_f}\right)_{class} = \frac{a_3^{\,2} m_2}{\sin a_1 \, k_2^2 a_4 \hat{\imath}\tau} \int_{b_1}^{b_2} db\, b\, \tilde{P}^{class}(b, a_3 \cdots a_f) \qquad \text{(A1-22)}$$

depends on the scattering angle $a_1 \equiv \vartheta$ only through the trivial factor $\frac{1}{\sin a_1}$.

The semi-classical approximations and the neglect of fluctuation correlations which lead to the result (A1-22) can also be upheld for the compound nucleus cross-section in which case $\tilde{p}^{class}$ is replaced by the probability density, $\tilde{p}^{class}_{CN}$ describing the macroscopic transition density for fully equilibrated external variables. It is characteristic for this case (neglecting the restrictions imposed by the conservation of angular momentum) that this probability density factorizes into a part describing the formation of the compound system and a probability for its decay

$$\tilde{P}^{class}_{CN}(b, a_3 \cdots a_f) = w_1(b) \cdot w_2(a_3 \cdots a_f) \qquad \text{(A1-23)}$$

It is typical for partly relaxed systems that the factorization (A1-23) is not possible.

A2, Improved Airy-method

The Airy-approximation holds only in a very small range of the scattering angle. In the case that the amplitude $C^{DI(S)}_{\beta s_\beta M_\beta;\alpha s_\alpha M_\alpha}(\ell)$ is a very slowly varying function of ℓ, the Airy approximation can be somewhat improved by carrying the expansion of the phase $A^{\eta}_{\alpha\beta;p}(\ell)$ (see A1-3)) to one additional order:

$$A^{\eta}_{\alpha\beta;p}(\ell) = A^{\eta}_{\alpha\beta;p}(\ell_S^r) + (\ell-\ell_S^r)\left(\Theta_S^r - \eta\vartheta + 2\pi p\right) + \frac{(\ell-\ell_S^r)^3}{6}\Theta''_{\alpha\beta}(\ell_S^r) + \frac{(\ell-\ell_S^r)^4}{24}\Theta'''(\ell_S^r) \qquad \text{(A2-1)}$$

The integral $\tilde{\mathcal{F}}^{sp}_{\alpha\beta}$ (see A1-9) is thus replaced by

$$\tilde{\mathcal{F}}^{sp}_{\alpha\beta} = \int_{\ell_1}^{\ell_2} d\ell\; e^{i\left\{(\ell-\ell_S^r)\left(\Theta_S^r-\eta\vartheta+2\pi p\right)+\frac{(\ell-\ell_S^r)^3}{6}\Theta''_{\alpha\beta}(\ell_S^r)\right\}} \cdot e^{i\frac{(\ell-\ell_S^r)^4}{24}\Theta'''_{\alpha\beta}(\ell_S^r)} \qquad \text{(A2-2)}$$

A straightforward evaluation leads to the result:

(i) $\ell_1 = 0$; $\ell_2 = +\infty$ (no diffraction)

$$\tilde{\mathcal{F}}^{sp}_{\alpha\beta} = \operatorname{sgn} \Phi''_{\alpha\beta}(\ell_s^r) \cdot 2\pi \left[\frac{2}{|\Phi''_{\alpha\beta}(\ell_s^r)|}\right]^{\frac{1}{3}} \tilde{Ai}(X_p, Y, Z) \qquad \text{(A2-3)}$$

(ii) $\ell_1 = \Lambda^{fu}_{\beta}$; $\ell_2 = \Lambda_\alpha$

$$\tilde{\mathcal{F}}^{sp}_{\alpha\beta} = \operatorname{sgn} \Phi''_{\alpha\beta}(\ell_s^r) \cdot 2\pi \left[\frac{2}{|\Phi''_{\alpha\beta}(\ell_s^r)|}\right]^{\frac{1}{3}} \left\{\tilde{Ai}(X_p, Y_1, Z) - \tilde{Ai}(X_p, Y_2, Z)\right\} \qquad \text{(A2-4)}$$

The arguments X_p, Y_1, Y_1, and Y_2 are given by (A1-10), (2.47'), (2.60), and (2.60') resp., while the new argument Z is defined by

$$Z := \frac{2^{\frac{1}{3}}}{12} \cdot \frac{\Phi'''_{\alpha\beta}(\ell_s^r)}{|\Phi''_{\alpha\beta}(\ell_s^r)|^{4/3}} \qquad \text{(A2-5)}$$

The function $\tilde{Ai}(x,y,z)$ represents the integral

$$\tilde{Ai}(x,y,z) := \frac{1}{2\pi} \int_{\gamma}^{\infty} d\tau \, e^{i\left(x\tau + \frac{1}{3}\tau^3 + z\tau^4\right)} \qquad \text{(A2-6)}$$

It is seen that the results are in all cases the same as for the ordinary Airy-method with the only difference that the incomplete Airy functions (2.46) are replaced by the functions $\tilde{Ai}$ of (A2-6). Since, in practice,

the incomplete Airy functions must be numerically evaluated, one may as well evaluate the functions (A2-6) in their place.

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Figure Captions

Fig. 2,1: Coordinate axes $\vec{e}_{x_F}$, $\vec{e}_{y_F}$, $\vec{e}_{z_F}$ (⟶) of the focal system S_F and coordinate axes $\vec{e}_{x_S}$, $\vec{e}_{y_S}$, $\vec{e}_{z_S}$ (⇢) of the system S in combination with a classical trajectory for repulsive scattering. Also shown are the x-axis ($\vec{e}_x$) of the LAB (or CM-) system and the aximuthal angle φ orienting $\vec{e}_{x_S}$ with respect to it. The vectors $\vec{e}_{x_F}$, $\vec{e}_{y_F}$, $\vec{e}_{x_S}$, and $\vec{e}_{z_S}$, and $\vec{e}_{z_S}$ lie in the reaction plane.

Fig. 2,2: Schematic plot of deflection functions Θ_α (---), Θ_β (—), $\Theta_{\alpha\beta}$ (— · — · —) for a case with two rainbow angles ($\Theta_{r,1}$, $\Theta_{r,2}$). The indicated geometrical construction of $\Theta_{\alpha\beta}$ from Θ_α and Θ_β (CA is tangent to Θ_α in A; CD = DB) shows that $\Theta_{\alpha\beta}$ usually lies between Θ_α and Θ_β.

Fig. 2,3: Display of the rotation from system S_F to system S by 3 successive (positive) Euler rotations:

$\alpha = -\frac{\pi \pm \vartheta}{2}$ around $\vec{e}_{z_F}$; $\beta = \frac{\pi}{2}$ around $\vec{e}_{y_{S1}}$;
$\gamma = \frac{\pi}{2}$ around $\vec{e}_{z_{S2}}$.

1st line, left: Systems S_F for repulsive scattering in the case with and without a full revolution.

1st line, right: System S_F for attractive scattering without orbiting.

Fig. 3,1: Schematic plot showing the regions of the (Θ, ℓ) - plane (shaded area) where the distribution $d(\Theta; a_2 \cdots a_f; \ell)$ is expected to be different from zero. Also shown are the inverse $\ell(\Theta; a_2 \cdots a_f)$ of the "macroscopic" deflection function (fully drawn line) and the stationary ℓ-values for a given scattering angle $a_1 \equiv \vartheta$. An almost isotropic angular distribution $\frac{dQ^{DI}}{da_1}$ is produced if the function $d(\Theta; a_2 \cdots a_f; \ell)$ is unequal to zero only in the flat part of $\ell(\Theta; a_2 \cdots a_f)$ (encircled by a full line).

Fig. 3,2: Airy function Ai(x) and square of Airy function $[\mathrm{Ai}(x)]^2$.

Fig. 3,3: Deflection function $\Theta(\ell)$ (—), 1st and 2nd derivative thereof (– – – represents $10 \times \Theta'(\ell)$; .... represents $100 \times \Theta''(\ell)$), and modulus of the reciprocal 1st derivative (–·–·– represents $10 \times \frac{1}{|\Theta'(\ell)|}$) $\Theta(\ell)$ is calculated[13] for the system Ar + Th at E_{LAB} = 388 MeV including friction ($\Theta \equiv \Theta^{fr}$ of equ.(3.43)).

Fig. 3,4: Fresnel integrals (from ref. 30).

Fig. A1,1: Inverse classical deflection function in a case where a large number of orbitings contribute to the cross-section. The discrete values of the impact parameter b which contribute to a scattering angle $a_1 \equiv \vartheta = \pi$ are shown by dots. Also shown is the limit b_2 (see equ. (A1-21)) for this case $(b_1 = 0)$.

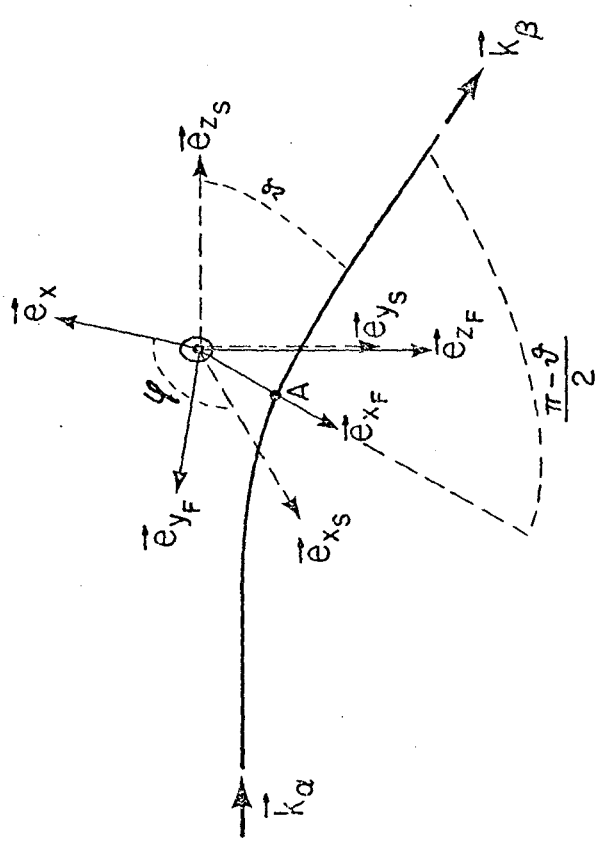

Fig. 2.1

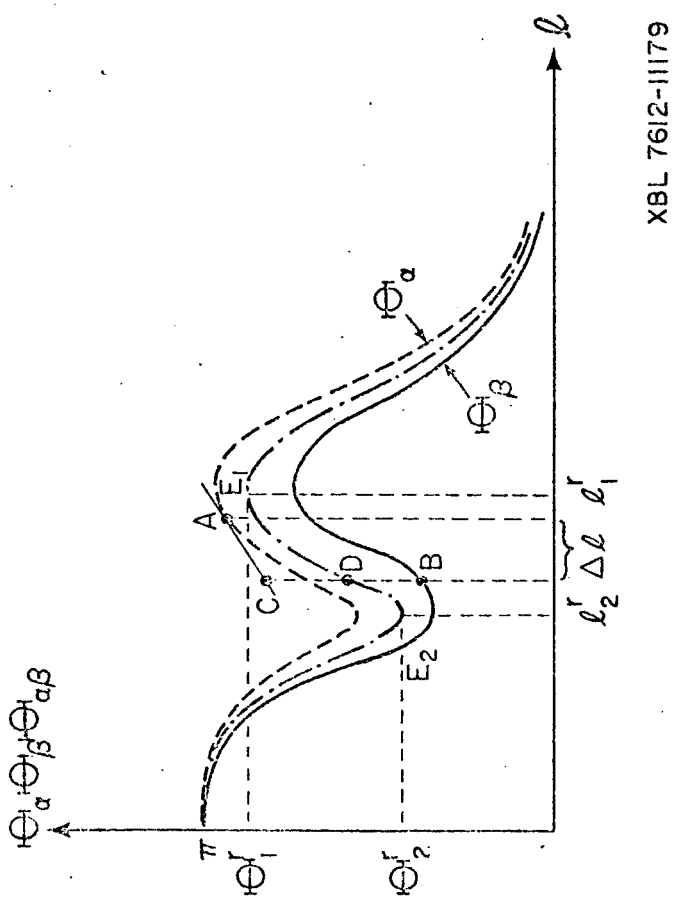

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Fig. 2.2

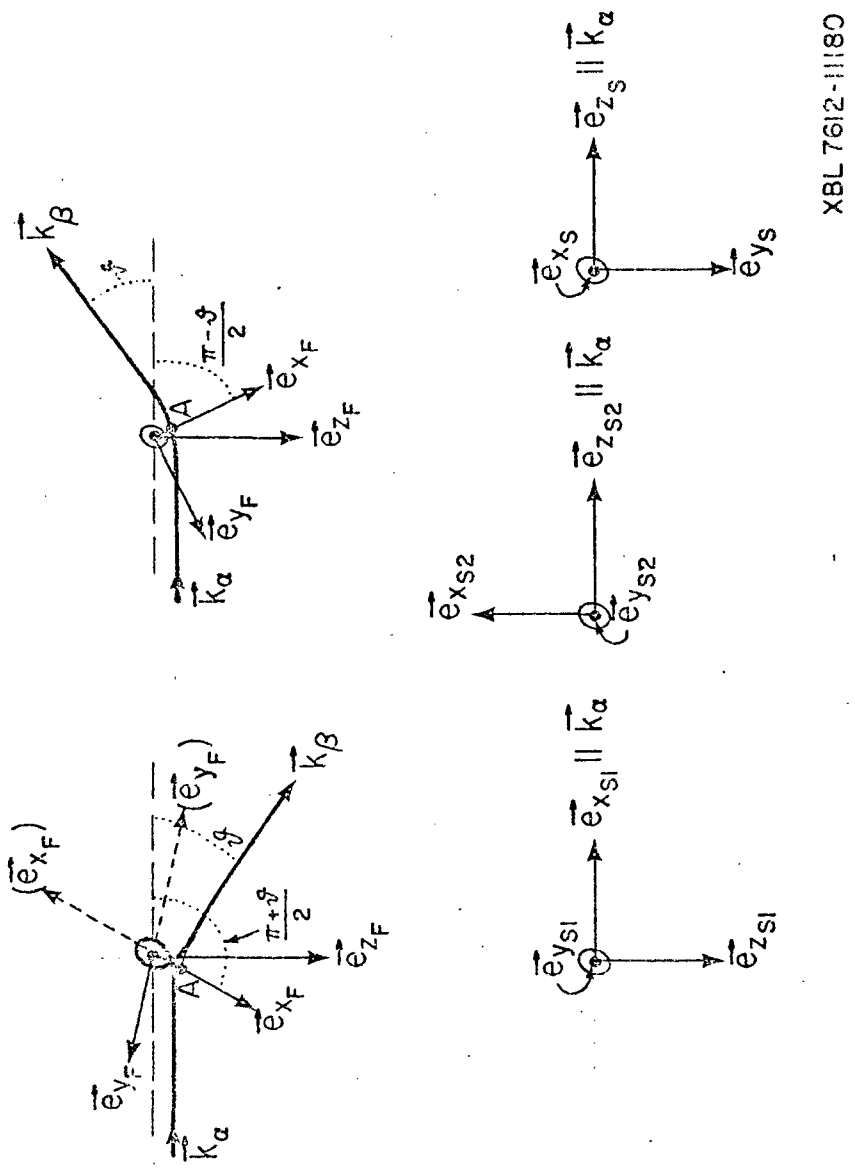

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Fig. 2.3

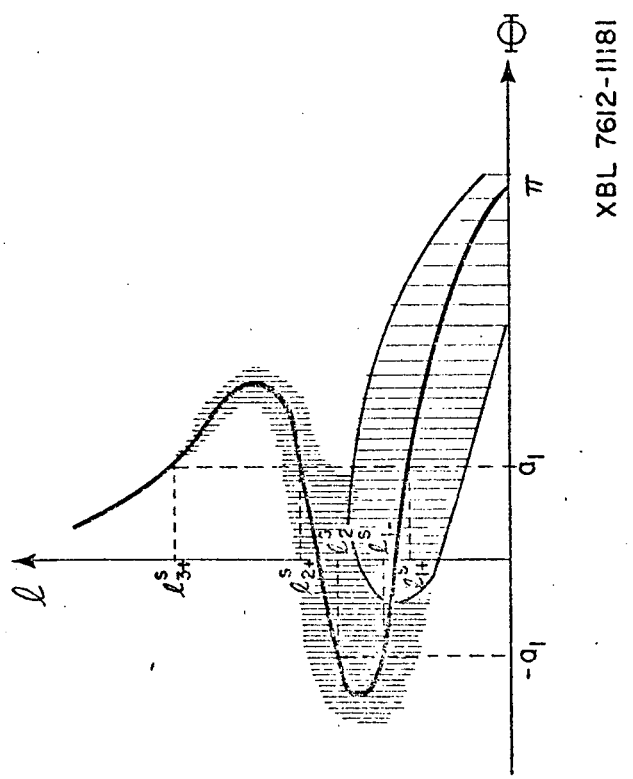
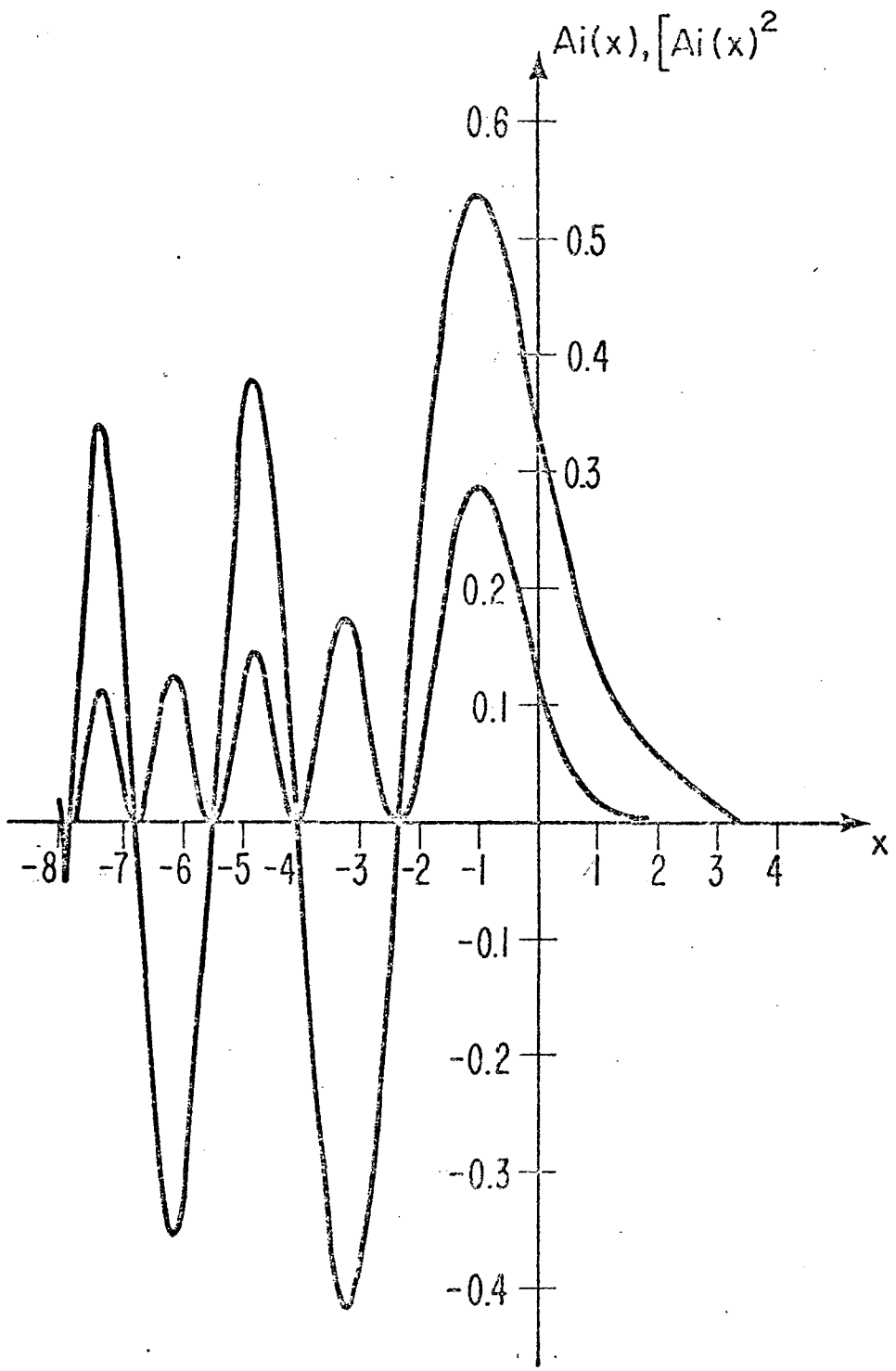
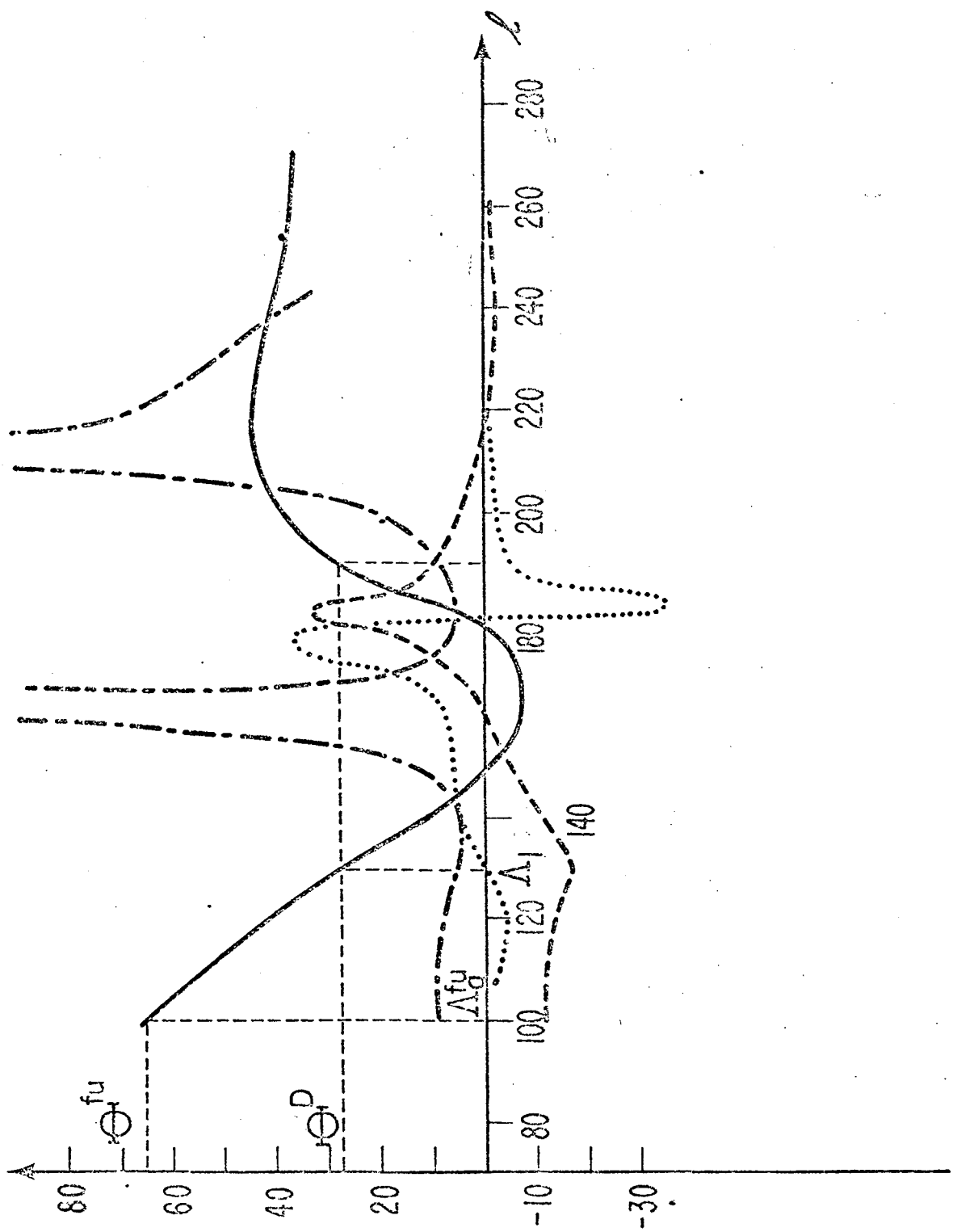


fig 3.1



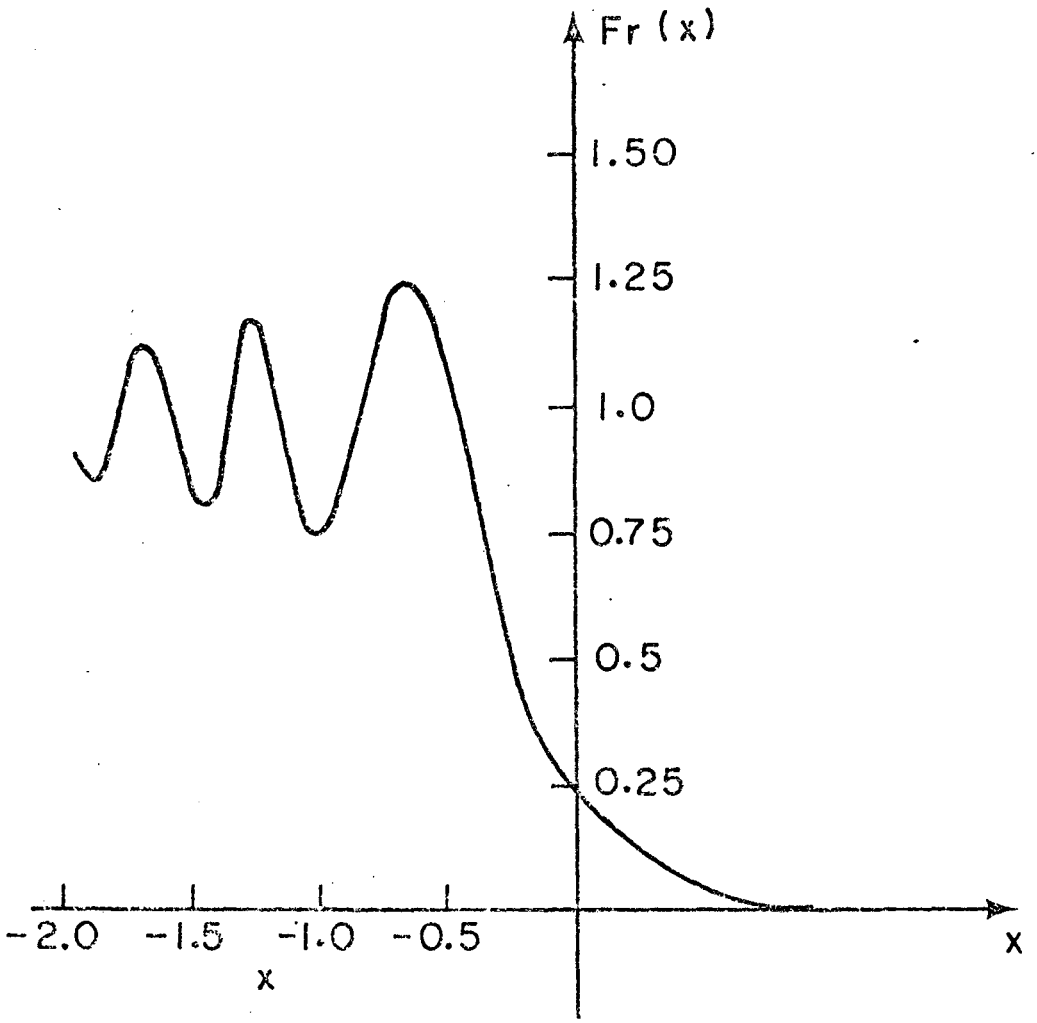
XBL 7612-11174

Fig 3.2



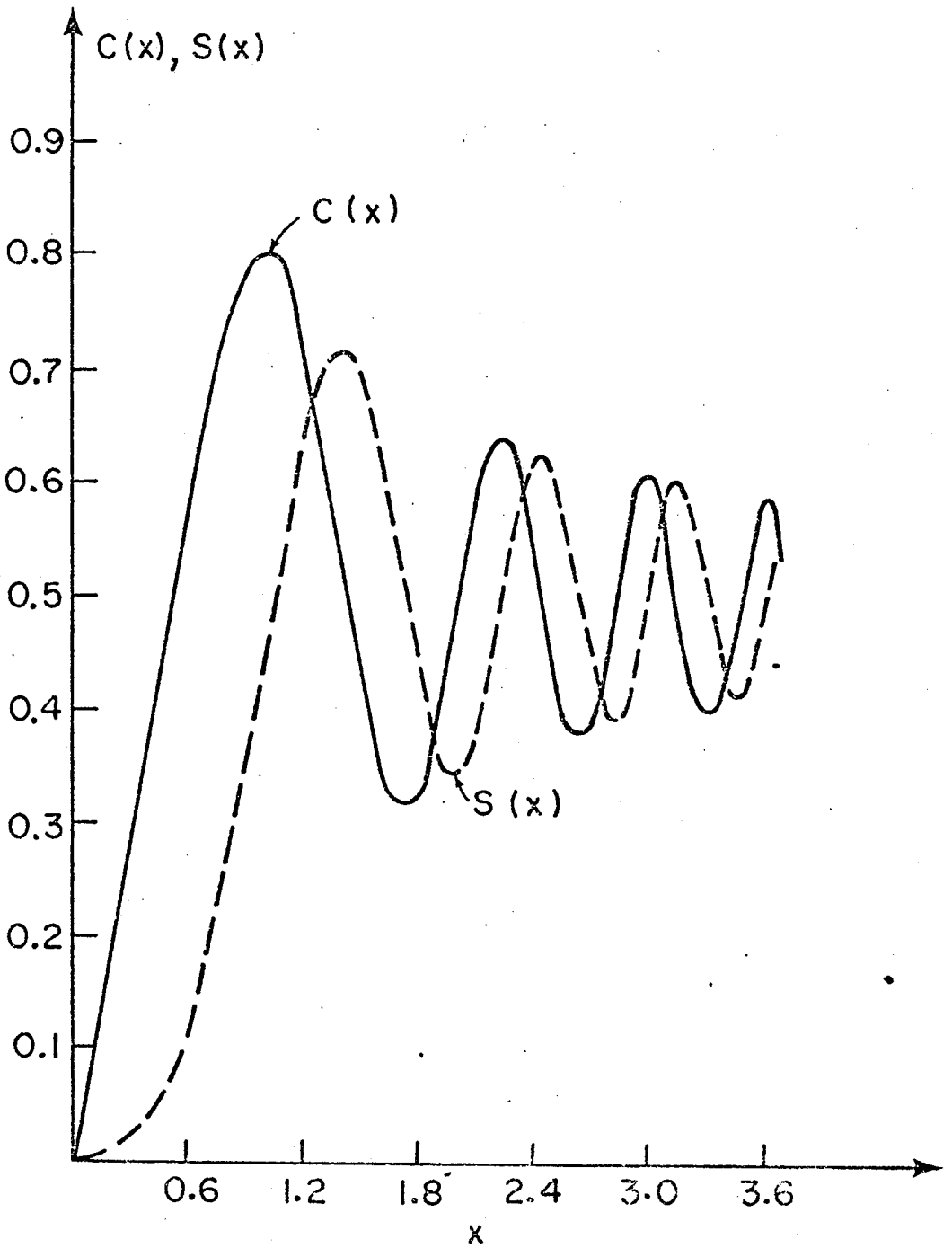
XBL 7612-11175

Fig. 3.3



XBL 7612-11176

fig. 3.4



XBL 7612-11177

Fig. 3.5

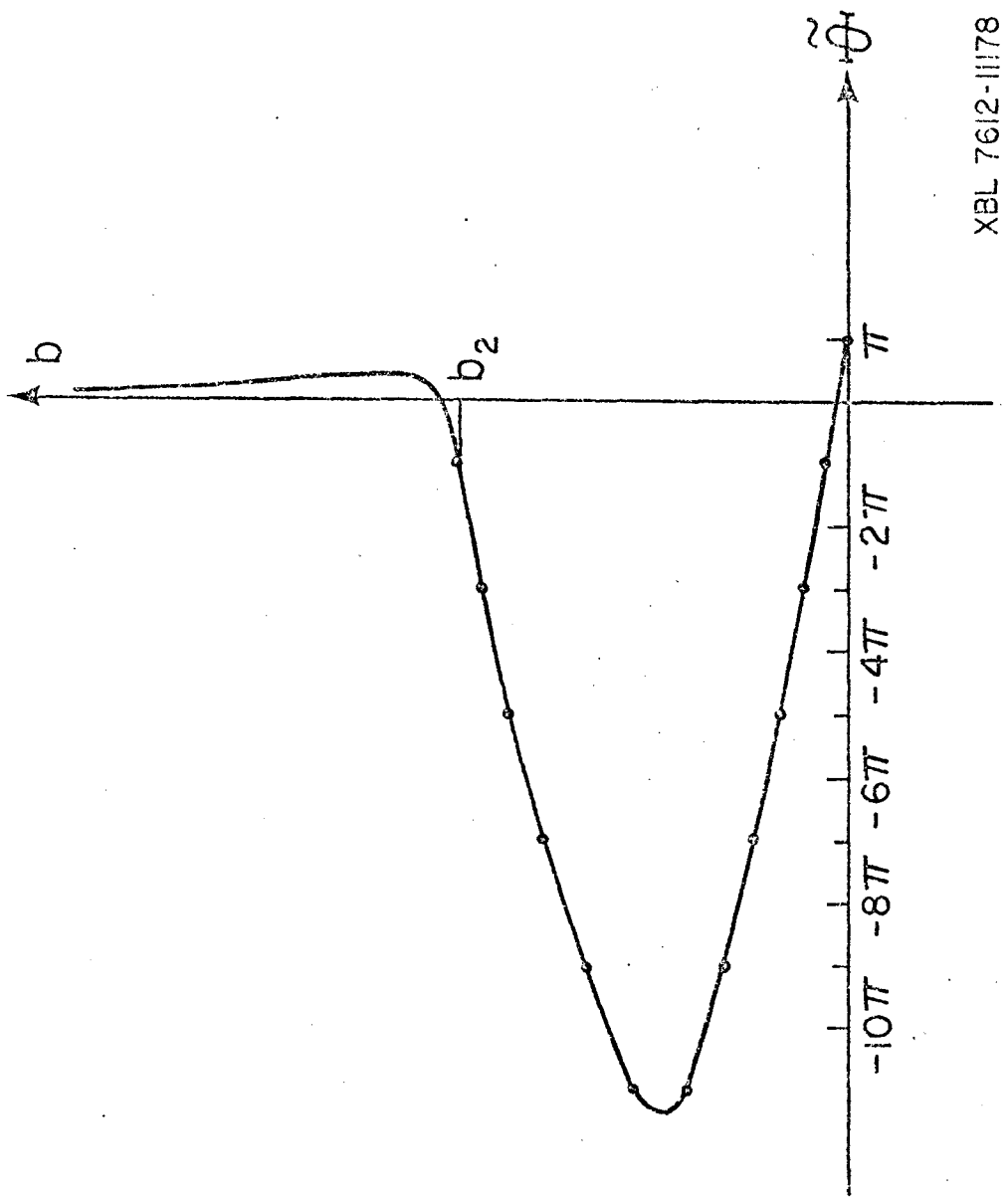


Fig A1.1

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