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TEMPORAL METRIC SPACES IN RADIATIVE TRANSFER THEORY

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Temporal Metric Spaces in Radiative Transfer Theory

III. Characteristic Spheroids and Ellipsoids

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INTRODUCTION

A pulse of light is directed into a medium of known optical properties. As the pulse proceeds into the medium some of its energy is absorbed, some is redirected by scattering, and the scattered energy goes on to be scattered further. We may ask: How much scattered light from the pulse is expected to exist in the form of primary scattered light, secondary scattered light, etc., at a given time after the pulse enters the medium? Or we may ask: What is the expected radiance of that part of the energy scattered from the pulse, as seen at the transmitter (or any other observation point), at a given time after the pulse enters the medium? These questions are fundamental in the experimental and theoretical study of natural aerosols and hydrosols. For by knowing the answers to these questions, one can better understand the results of actual experiments carried out with time-dependent light fields. Furthermore, the order of these questions and answers may be inverted so that a measured response of a medium to a pulse of known characteristics can lead to an estimate of the optical properties of the medium.

The questions posed above may be answered by using the set theoretical tools developed in the paper; specific illustrations are given in the section on Applications. The main emphasis of this paper, and the others of this series is, however, on the development of a new set of concepts in radiative transfer theory which may eventually broaden its scope of applicability and draw it closer to the discipline of topological dynamics - the mathematics of future physics.

CHARACTERISTIC SUBSETS

The characteristic function χ defined in paper II yields a criterion for deciding whether a given point in a general carrier space is being irradiated at a certain time by flux (either directly transmitted or scattered) from a given point source. We now consider in detail two closely related problems: for a given local source and local epoch time for that source, what is the totality of points which may receive radiant flux (either directly transmitted or scattered) from that source? Also: for the same fundamental source and a given point in the associated irradiated set of points, what subset of the preceding totality of points may actually scatter flux from the fundamental source to the given point? These subsets will now be defined and studied.

Characteristic Spheroid

<u>Definition 1</u>. Let p^* in a carrier space X be a fixed local source for the point p' in X. The set of all points p'' in X for which p' is a local source at local epoch time $T_{p^*}(p)$ and which satisfy the characteristic function equation

$$\chi(p', p''; T_{p^*}(p')) = 1$$

defines a subset $\mathcal{S}(p'; T_{p^*}(p'))$ of X called the <u>characteristic spheroid</u> of p' relative to p^*.

<u>Theorem 1</u>. <u>$\mathcal{S}(p', T_{p^*}(p'))$ in X, with respect to the metric defined by least local epoch time, is</u>:

(i) <u>A spheroidal body</u>,

(ii) <u>with center p'</u>,

(iii) <u>of radius $T_{p^*}(p')$</u>.

<u>Proof</u>. By definition, $\mathcal{S}(p'; T_{p^*}(p'))$ is the set of all points p'', such that $\chi(p', p'', T_{p^*}(p')) = 1$, which is equivalent to the condition (see Figure 1a)

$$T_{p'p''} \leq T_{p^*}(p').$$

But this is precisely the metrical definition of a spheroidal body with center p' and radius $T_{p^*}(p')$, which completes the proof.

The spheroidal body referred to in the preceding theorem is thus a true sphere plus its internal points, in the space X with metric $T_{\rho\rho'}$. Hence, if one were to draw a three dimensional graph of $\mathcal{S}(\rho', T_{\rho^*}(\rho'))$ in some classical carrier space one would obtain a subregion of X' which, with respect to $T_{\rho\rho'}$, would be precisely a spheroidal body. However, if one were to adopt the usual metric d in X, the set $\mathcal{S}(\rho'; T_{\rho^*}(\rho'))$ would generally be of irregular shape and multiply connected (have holes in it).

Characteristic Ellipsoid

To set the stage for the next definition, let (ρ', ρ) be a fixed pair of points in X. Let ρ' be a local source for ρ; let ρ^* in X be a local source for ρ'. That is, ρ^* emits radiant flux and thereby induces ρ' (after a sufficient interval of time) to emit radiant flux. ρ^* may be the fundamental source ρ_0 for X, but for generality, ρ^* need not be ρ_0. We consider the effects at ρ induced only by ρ'; and we consider effects at ρ' induced only by ρ^*. With this chain of events in mind we now can state:

<u>Definition 2</u>. The subset $\mathcal{E}(\rho', \rho; T_{\rho^*}(\rho'))$ of X consisting of all points ρ'' such that

$$\chi(\rho'', \rho; T_{\rho'}(\rho'')) = 1$$

is called the <u>characteristic ellipsoid</u> of the point pair (ρ', ρ) relative to the local source ρ^* .

There are two local sources involved in the definition of $\mathcal{E}$. The local source ρ' serves, with ρ , to generate $\mathcal{E}$ by means of the above equation, $\chi = 1$. The local source ρ^* gives a means of establishing local epoch time at ρ'.

Suppose X has an index of refraction function n which is constant over a classical carrier space X . Then Definition 2 implies that the totality of points ρ'' form an actual ellipsoid of revolution with ρ' and ρ at the foci. This interpretation is actually valid for any space X when the least local epoch time metric $T_{\rho\rho'}$ is used as a yardstick, as is proved in:

<u>Theorem 2.</u> <u>The characteristic ellipsoid $\mathcal{E}(\rho', \rho; T_{\rho^*}(\rho'))$ in X with respect to the least local epoch time metric is:</u>

(i) <u>An ellipsoidal body,</u>
(ii) <u>with foci at ρ' and ρ,</u>
(iii) <u>of major diameter $T_{\rho^*}(\rho')$,</u>
(iv) <u>of eccentricity $\epsilon = T_{\rho'\rho} / T_{\rho^*}(\rho')$</u> .

<u>Proof</u>: By definition of $\mathcal{E}$, any point ρ'' in $\mathcal{E}$ satisfies the characteristic equation

$$\chi(\rho'', \rho; T_{\rho'}(\rho'')) = 1$$

which implies

$$T_{p''p} \leq T_{p'}(p'') = T_{p^*}(p') - T_{p'p''}$$

so that, at local epoch time $T_{p^*}(p')$, $\mathcal{E}$ consists of all points p'' in X satisfying the inequality (see Figure 1b):

$$T_{p'p''} + T_{p''p} \leq T_{p^*}(p')$$

But this is precisely the metric definition of an ellipsoidal body with the point pair (p', p) as foci, and with major diameter $T_{p^*}(p')$. The metric definition of eccentricity is: the quotient of the distance between the foci, by the major diameter of the ellipsoid. Thus (iv) follows, and the proof is complete.

From the definition of the characteristic ellipsoid $\mathcal{E}(p', p; T_{p^*}(p'))$ it is clear that this subset of X contains all points p'' which, at a certain given epoch time, are contributing at least primary scattered radiant flux to p with respect to the flux leaving p'. Conversely, if a point p'' is contributing at least primary scattered radiant flux to p, it must be in the characteristic ellipsoid $\mathcal{E}(p', p; T_{p^*}(p'))$. Thus at any instant $T_{p^*}(p')$, the characteristic ellipsoid $\mathcal{E}(p', p; T_{p^*}(p'))$ defines that subset of X which can pass on flux to p which originated at p'.

Some Observations for Classical Carrier Spaces

In classical carrier spaces of spatially and temporally uniform index of refraction there is a particularly simple connection between the temporal metrics and the usual metric d . In such spaces the least local epoch time metric $T_{p'p}$ coincides with the metric t (Theorem 3, paper II). As a consequence, if p' and p are any two points in X , the natural path between p' and p is a straight line (Example 1, paper I), and $T_{p'p} = t(p',p) = d(p',p)/v$, where v is the speed of light in X . Suppose, for simplicity, that $p' = p^* = p_o$, then $T_{p^*}(p') = T$, the epoch time in X . It follows that the temporal diameter of the characteristic ellipsoid $\mathcal{E}(p',p\,;T_{p^*}(p')) = \mathcal{E}(p_o,p\,;T)$ is simply T, and the geometrical length of the diameter is $D = vT$. Thus the eccentricity of $\mathcal{E}(p_o,p\,;T)$ is:

$$\epsilon = \frac{t(p_o,p)}{T} = \frac{d(p_o,p)}{D} .$$

According to the preceding formula for ϵ , when T is just equal to $t(p_o,p)$, $\epsilon = 1$, implying that the characteristic ellipsoid at this instant is a straight line in X , namely the straight line connecting p_o and p (see Figure 2a). The instant the wavefront from p_o passes p , $\mathcal{E}(p_o,p\,;T)$ starts fattening out, which is mathematically reflected in a decreasing ϵ (Figure 2b). As epoch time T increases, the eccentricity of $\mathcal{E}$ continues to decrease so that, in the limit as $T \to \infty$, $\epsilon \to 0$, and the characteristic spheroid is eventually spherical in shape.

At any epoch time $T > t(p_o, p)$, the surface of the ellipsoid may be expressed in polar coordinate form:

$$r(\theta) = \frac{D^2 - d^2}{2(D - d\cos\theta)} .$$

Here $r(\theta)$ is the geometrical distance from p_o to a boundary point of $\mathcal{E}$ where θ is the angle between the directions $p_o p$, $p_o p''$ (Figure 2c). Further, $D = vT$ and $d = d(p_o, p)$. If, for example, $d = 0$, then $p_o = p$, and

$$r(\theta) = \frac{D}{2} = \frac{vT}{2}$$

for all θ. That is, the characteristic ellipsoid $\mathcal{E}(p_o, p_o; T)$ associated with the special pair of points (p_o, p_o) is a sphere (with respect to either the temporal or spatial metrics) with a radius half that of the characteristic spheroid $\mathcal{S}(p_o; T)$. Hence, under these conditions, the boundary of $\mathcal{E}(p_o, p_o; T)$ is moving away from p with a speed half that of $\mathcal{S}(p_o; T)$. In general, the rate of recession of a point on the boundary of $\mathcal{E}(p_o, p; T)$ from either p_o or p depends on θ. However, for large T, this rate always approaches $vT/2$ for all θ:

$$r(\theta) \longrightarrow \frac{D}{2}$$

as $T \rightarrow \infty$, so that

$$\frac{d\,r(\theta)}{dt} \longrightarrow \frac{v}{2}$$

which follows from the general expression for $r(\theta)$.

WORLD-LINE REPRESENTATIONS OF CHARACTERISTIC SUBSETS

General Remarks

The preceding discussion of the characteristic subsets in the classical settings threw much light on the essential geometric structure of these subsets because the spatial and temporal metrics could be simply compared. However, general carrier spaces usually do not possess a "geometrical" metric to describe the characteristic subsets; or if there is such a metric we are generally unable to compare it with temporal metric in a simple way. Nevertheless, for practical applications, it is essential to have some relatively simple way to represent both the spatial and temporal features of the characteristic subsets. We will show that a particularly fruitful solution to this problem lies in the <u>world-line representation</u> of characteristic subsets. This mode of representation of space-time events has been used to advantage for example in the special relativity theory. It has two particularly useful features: it places the time parameters on an equal footing with the space parameters; it allows a purely set-theoretic description of physical events associated with space-time phenomena.

The basic idea behind the world-line representation is to consider not just trajectories of photons in the space X, but their trajectories in the cartesian product $X \times T$ of X and T, where T is the time domain. A point of $X \times T$ is an ordered pair (p, t) of points, the p belonging to X, the t to T. Thus we no longer consider just the point p in X, but the point p at time t. Therefore, the world-line representation contains temporal information about events in addition to the usual spatial information.

As an illustration, consider the trajectory $A_oW_oB_o$ in X of a photon or a ray of light (Figure 3). The two path segments A_oW_o and W_oB_o give all the information about the spatial configuration of the trajectory. However, there is no information in the diagram about the times of travel from A_o to W_o, and from W_o to B_o. If a time axis is erected "perpendicular" to X (where X is represented for simplicity as a two dimensional space) we can include the temporal information associated with the trajectory $A_oW_oB_o$. Suppose, for example, that the ray starts at A_o at time t_o and arrives at W_o at time t'. The trajectory in space-time of the ray is given by the segment A_oW' in the space $X \times T$. The point W' is actually an ordered pair (W_o, t'), i.e., the point W_o in X at time t', and similarly A_o should be written as the ordered pair (A_o, t_o). If the ray then proceeds from W' and arrives at B_o at time t_1, this leg of the photon's space-time journey is represented by the segment $W'B_1$ in space-time. We say that $A_oW'B_1$ in $X \times T$ is the world-line of the trajectory $A_oW_oB_o$ in X.

The trajectory $A_oW_oB_o$ in X is clearly a "projection" of $A_oW'B_1$ on X. A projection P_X on X is a mapping of $X \times T$ onto X, defined as:

$$P_X(p,t) = p,$$

i.e., P_X takes a point (p,t) in $X \times T$ and assigns it to the point p in X. From this it is clear that two distinct points in $X \times T$ may have the same projected image in X. For example, $W' = (W_o, t')$ and

$W^{2} = (W_0, t'')$ are distinct points in $X \times T$, but

$$P_X(W') = P_X(W'') = W_0 \in X$$

In just the same way, two subsets of $X \times T$ may have the same projection in X. Thus the two world-lines $A_0 W' B_1$, and $A_0 W'' B_1$, have the same projection $A_0 W_0 B_0$ in X. Symbolically:

$$P_X(A_0 W' B_1) = P_X(A_0 W'' B_1) = A_0 W_0 B_0 .$$

Characteristic Cone

In Figure 4, let point p_0 be the fundamental source of radiant energy which begins to emit radiant energy at epoch $T = 0$. The <u>future nappe</u> of the characteristic cone associated with p_0 is defined as the set $C_+(p_0, 0)$ of all points (p', T') in $X \times T$ with the following property:

$$C_+(p_0, 0) = \{(p', T') : p' \in \mathcal{S}(p_0, T')\}.$$

That is, a point (p', T') in $X \times T$ is in $C_+(p_0, 0)$ if p' the spatial component of (p', T') is in the characteristic spheroid $\mathcal{S}(p_0, T')$ with center p_0 and radius T'. If X were a classical carrier space with constant index of refraction on $X \times T$, then

$\mathcal{C}_+(\rho_0, 0)$ would be a four-dimensional conical solid in the usual sense. This is the basis for the diagrammatic conical representation of $\mathcal{C}_+(\rho_0, 0)$ in Figure 4.

The concept of a future nappe $\mathcal{C}_+(\rho_0, 0)$ associated with ρ_0 need not be restricted to ρ_0. We may wish to extend our attention to the radiant flux leaving some local source ρ at some finite epoch $T > 0$. The future nappe $\mathcal{C}_+(\rho, T)$ associated with (ρ, T) is then defined as (see Figure 5):

$$\mathcal{C}_+(\rho, T) = \{(\rho'', T'') : \rho'' \varepsilon \mathcal{S}(\rho; T'' - T)\}.$$

The future nappe $\mathcal{C}_+(\rho, T)$ represents the totality of points in $X \times T$ which may receive radiant flux from ρ either directly or indirectly (by scattering, etc.). On the other hand, ρ itself may be receiving radiant flux from other local sources in X. The totality of such points makes up the past nappe of the characteristic cone associated with ρ, and is defined as

$$\mathcal{C}_-(\rho, T) = \{(\rho', T') : \rho \varepsilon \mathcal{S}(\rho', T - T')\}.$$

Another way of defining $\mathcal{C}_-(\rho, T)$ is directly in terms of future nappes (see Figure 5):

$$\mathcal{C}_-(\rho, T) = \{(\rho', T') : (\rho, T) \varepsilon \mathcal{C}_+(\rho', T')\}.$$

The characteristic cone $\mathcal{C}(p,T)$ associated with the point (p,T) in $X \times T$ is defined as:

$$\mathcal{C}(p,T) = \mathcal{C}_-(p,T) \cup \mathcal{C}_+(p,T).$$

It should be emphasized that the symbol T in the notation for $\mathcal{C}$, $\mathcal{C}_+$, and $\mathcal{C}_-$ refers to epoch time associated with the fundamental source p_o, whereas the times entering into the symbols for $\mathcal{S}$ are of course local epoch times with respect to the local sources included in the $\mathcal{S}$-notation, and these local epoch times are in turn referred, without exception, to some fixed local source such as p_o, or in general some point p^*. In this way we can conveniently drop the p^* and p from the notation for $T_{p^*}(p)$ in $\mathcal{S}(p;\, T_{p^*}(p))$ without necessarily incurring any ambiguity in the notation; the basic local source will always be explicitly given in any specific context requiring such information.

Representation of the Characteristic Spheroid

From the point of view of the preceding discussion, the notion of characteristic spheroid was basic in the sense that the concepts of $\mathcal{C}_+$, $\mathcal{C}_-$, and hence $\mathcal{C}$ were all ultimately defined in terms of $\mathcal{S}$. This is a natural consequence of the present order of development of the temporal semimetric theory. It is conceivable, however, to begin the theory in such a way as to come upon the concept of $\mathcal{C}_+$ before that of X, and hence, before that of $\mathcal{S}$. But little is gained in such an alternate approach especially since the theoretical machinery gets into full swing only after the definitions of $\mathcal{S}$ and $\mathcal{E}$ have been made. Nevertheless, even in the present order of development, the representations of $\mathcal{S}$ and $\mathcal{E}$ in terms of the characteristic cone $\mathcal{C}$ supply deeper insight into the spatial-temporal structures of these concepts, and we will continue to use the newer concept $\mathcal{C}$ in the interpretation of $\mathcal{S}$ and $\mathcal{E}$. Thus, for example, the characteristic spheroid $\mathcal{S}(p_0; T')$ may be represented as the projection on X of the intersection of $\mathcal{C}_+(p_0, 0)$ and the subset $X \times T'$ of $X \times T$:

$$\mathcal{S}(p_0; T') = P_X \left[\mathcal{C}_+(p_0, 0) \cap (X \times T') \right], \quad T' \geq 0.$$

This is illustrated in Figure 4. In general,

$$\mathcal{S}(p, T'' - T) = P_X \left[\mathcal{C}_+(p, T) \cap (X \times T'') \right], \quad T'' \geq T,$$

which may be verified on the basis of the geometrical relations depicted in Figure 5.

Representation of the Characteristic Ellipsoid

A purely set-theoretical representation of $\mathcal{E}$ in terms of $\mathcal{C}_+$ and $\mathcal{C}_-$ can be made in a way similar to that of $\mathcal{S}$. Consider a local source p' in X which begins to emit at epoch time T'. Let (p, T) be in $\mathcal{C}_+(p', T')$; thus $p \in \mathcal{S}(p', T - T')$ and p is receiving flux from p' (See Figure 6).

Moreover, ρ is also receiving flux from all those points ρ'' in its past nappe $\mathcal{C}_-(\rho,T)$ whose points themselves have been irradiated by flux from ρ'. Thus, with respect to the local source ρ' and the receiving point ρ, the set of all points ρ'' in X which receive flux from ρ' and pass it on to ρ is defined by the region in $X \times T$ common to both $\mathcal{C}_+(\rho',T')$ and $\mathcal{C}_-(\rho,T)$:

$$\mathcal{C}_+(\rho',T') \cap \mathcal{C}_-(\rho,T) .$$

Hence the projection of this space-time region into X must be none other than $\mathcal{E}(\rho',\rho\,;T-T')$:

$$\mathcal{E}(\rho',\rho\,;T-T') = P_X\left[\mathcal{C}_+(\rho',T') \cap \mathcal{C}_-(\rho,T)\right] .$$

The general proof of this set-theoretic equality is straightforward. The proof is particularly illuminating in the context of classical carrier spaces with constant index of refraction on $X \times T$. We will therefore outline the proof in this context.

Assume that $\rho \in \mathcal{C}_+(\rho',T')$. We consider first only rays which go from (ρ',T') to (ρ,T) in $X \times T$ along the boundaries of $\mathcal{C}_+(\rho',T')$ and $\mathcal{C}_-(\rho,T)$. Figure 6 shows such a world-line defined by the segments $(\rho',T') \to (\rho'',T'')$ and $(\rho'',T'') \to (\rho,T)$. The first of these segments lies in $\mathcal{C}_+(\rho',T')$, the second lies in $\mathcal{C}_-(\rho,T)$. Therefore (ρ'',T'') lies in the space-time curve E which is the intersection of the boundaries of $\mathcal{C}_+(\rho',T')$ and $\mathcal{C}_-(\rho,T)$. Since the index of refraction function n is constant on $X \times T$, the geometric lengths of the projections of these world-line segments is a fixed factor times the temporal lengths associated with the segments, i.e.,

$$d(p', p'') = v(T'' - T'), \quad d(p'', p) = v(T - T'').$$

Hence

$$d(p', p'') + d(p'', p) = v(T - T'),$$

which is independent of (p'', T''). Therefore p'', the projection of (p'', T'') on X traces out an ellipsoidal surface $P_X(E)$ on X whose major diameter is clearly $v(T - T')$, and whose foci are at $p' = P_X(p', T')$ and $p = P_X(p, T)$. If, finally, (p'', T'') were properly contained within $C_+(p', T') \cap C_-(p, T)$, then the length of the projection of the segment $(p', T') \to (p'', T'')$ would be <u>less</u> than $v(T'' - T')$, similarly, the length of the projection of the segment $(p'', T'') \to (p, T)$ would be <u>less</u> than $v(T - T'')$. Therefore the projection of $C_+(p', T') \cap C_-(p, T)$ is contained within $P_X(E)$ that is, within $\mathcal{E}(p', p; T - T')$. Conversely, any trajectory in the region of X bounded by $P_X(E)$ with beginning and end points corresponding to (p', T') and (p, T) in $X \times T$ must necessarily map, under P_X^{-1}, (the inverse of P_X) into $C_+(p', T') \cap C_-(p, T)$. This completes the outline of the proof.

An Important Special Case of the Characteristic Ellipsoid

An important special case of the characteristic ellipsoid arises when the source point ρ' coincides with the receiving point ρ in a classical carrier space. This case is illustrated in Figure 7. The real counterpart to this situation occurs in the operation of an instrument which is designed to both send out radiant energy and receive radiant energy scattered back from the medium. There are three features of the diagram which should be observed: (a) the characteristic ellipsoid $\mathcal{E}(\rho_0, \rho_0; T)$ is a spheroid of radius $vT/2$; (b) the characteristic spheroid $\mathcal{S}(\rho_0; T)$ is a spheroid of radius vT ; (c) all primary scattered radiant flux received at (ρ_0, T) is necessarily generated on E, the intersection of $C_+(\rho_0, 0)$ and $C_-(\rho_0, T)$.

The preceding observation on the space-time locale of the generation of primary scattered flux received at ρ_0 applies also to the general case where the transmitting and receiving points are distinct. The curve E in Figure 6 is the corresponding locale in this more general case. Thus the world-line of a primary scattered photon originally emitted at ρ' (Figure 7) and received at ρ is a two-segmented line with vertex on E . The world-lines of secondary and higher order scattered photons may be easily represented on diagrams such as those in Figures 6 and 7.

APPLICATIONS

We will now discuss two applications of the concepts of characteristic spheroid and ellipsoid. The characteristic spheroid is especially useful in the derivation of formulas which describe the radiant energy content of various subsets at given epoch times. This fact will be illustrated by deriving the formula for the primary scattered radiant energy in X at epoch time T. The characteristic ellipsoid is helpful in formulating expressions for the time-dependent radiance function. This will be illustrated for the case of primary scattered radiance.

Time-Dependent Radiant Energy

For this illustration the carrier space X is to have the following properties: its location space component X_0 is the entire euclidean three-space. The volume attenuation function α, and index of refraction function n are constant on $X_0 \times T$. The volume scattering function $\sigma(\underset{\sim}{x};t;\underset{\sim}{\xi};\underset{\sim}{\xi}')$ will thus be independent of $\underset{\sim}{x} \in X_0$ and $t \in T$, but may depend in any way on the direction variables $\underset{\sim}{\xi}$ and $\underset{\sim}{\xi}'$. The fundamental source p_0 will be at the origin of X_0, and will emit radiant flux in all directions $\underset{\sim}{\xi} \in \Xi$ according to some given general pattern. Furthermore, its total radiant flux output $P^0(t)$ will vary with time in a given general manner over the time interval $(0, T_0)$. For values of t outside this interval, $P^0(t)$ will be zero.

Figure 8 shows the world-region occupied by the photons from p_0. This region is contained between the two future nappes $C_+(p_0,0)$ and $C_+(p_0,T_0)$. At any time $T \geq T_0$ the region of X_0 occupied by the emitted energy is given by

$$\bar{\mathcal{S}}(p_0,T) = \mathcal{S}(p_0,T) - \mathcal{S}(p_0,T-T_0) .$$

This region of X_0 is depicted in Figure 8 as the shaded annulus on X_0. If $T < T_0$ then the region of X_0 occupied by the radiant energy is given by

$$\bar{\mathcal{S}}(p_0,T) = \mathcal{S}(p_0,T) ,$$

since by definition $\mathcal{S}(p_0,T') = \phi$, the empty set, if $T' < 0$.

We begin our calculations with a derivation of the formula for the reduced radiant energy $U^0(T)$ in $\bar{\mathcal{S}}(p_0,T)$. The reduced radiant energy is carried by photons which have not been scattered or absorbed. Let $u^0(\underset{\sim}{x},T)$ denote the reduced radiant density (reduced radiant energy per unit volume of X_0) at $\underset{\sim}{x} \varepsilon X_0$ and at time T. Then, clearly

$$U^0(T) = \int_{\bar{\mathcal{S}}(p_0,T)} u^0(\underset{\sim}{x},T)\, dV(\underset{\sim}{x}) . \qquad (1)$$

The multiple integral (1) may be written as an iterated integral:

$$U^{\circ}(T) = \int_{\max(0,\, v(T-T_0))}^{vT} \left[\int_{A(r)} u^{\circ}(\underset{\sim}{x}, T)\, dA \right] dr$$

where $A(r)$ is a sphere of radius $r < vT$.

It is easy to see that

$$\int_{A(r)} \mu^{\circ}(\underset{\sim}{x}, T)\, dA = \frac{1}{v} P^{\circ}(T - \frac{r}{v})\, e^{-\alpha r} ,$$

which follows from the fact that the integral over $A(r)$ may be interpreted as (reduced) radiant energy content of a spherical shell of radius r and unit thickness. Hence

$$U^{\circ}(T) = \frac{1}{v} \int_{\max(0,\, v(T-T_0))}^{vT} P^{\circ}(T - \frac{r}{v})\, e^{-\alpha r}\, dr . \tag{2}$$

If, for example, $P^{\circ}(t) = P^{\circ}$ for $0 \leq t \leq T_0$, then $U^{\circ}(T)$ is of the form:

$$U^{\circ}(T) = \begin{cases} P_0 T_\alpha \left[1 - e^{-T/T_\alpha} \right], & 0 \leq T \leq T_0 , \\ P_0 T_\alpha\, e^{-T/T_\alpha} \left[e^{T_0/T_\alpha} - 1 \right], & T \geq T_0 , \end{cases} \tag{3}$$

where $T_\alpha = 1/v\alpha$ is the time constant for U° in X_0.

A graph showing the general profile of $U^o(T)$ for this simple example appears in Figure 9a. If the source emitted uniformly in time for all $T > 0$, (i.e., $T_0 = \infty$), then the resulting expression for $U^o(T)$ is given by the first equation in (3). A graph of the corresponding $U^o(T)$ is shown in Figure 9b. The limiting (or asymptotic) value of $U^o(T)$ in this case is:

$$U^o(\infty) = \lim_{T \to \infty} U^o(T) = P^o T_\alpha . \tag{4}$$

We note parenthetically that the formula for $U^o(T)$ in (3) is reminiscent of the formula in elementary circuit theory for the charge on a capacitor in a simple capacitance-resistance DC circuit. The quantity $U^o(T)$ takes the role of the charge, $1/v$ is analogous to resistance, $1/\alpha$ is analogous to capacitance, and $T_\alpha = 1/v\alpha$ is analogous to the time constant of the circuit.

We now go on to derive the formula for $U^1(T)$, the primary radiant energy in X_0 at T. Every point p_1 in $S(p_0, T)$ is a potential source of primary scattered energy: as the wavefront from p_0 passes p_1, the element of volume about p_1 begins to emit scattered radiant energy. If p_1 is at a distance r_1 from p_0, the region of X_0 permeated by the scattered energy from p_1 at time T is clearly $S(p_1, T-T_1)$, where $T_1 = r_1/v$ is the epoch time p_1 begins to emit (see Figure 8). Suppose $u_*^1(x, T)$ is the <u>radiant density</u> at point x and time T in $S(p_1, T-T_1)$ of <u>primary scattered radiant energy</u> produced per unit volume around the local source ρ_1 . Then

$$\mu'(\rho_1, T) = \int_{\mathcal{S}(\rho_1, T-T_1)} u'_*(\underline{x}, T)\, dV(\underline{x}) , \tag{5}$$

represents the total primary scattered radiant energy in X_o at time T produced by a unit volume at about ρ_1 . The total primary scattered radiant energy in X_o at time T is then

$$U'(T) = \int_{\mathcal{S}(\rho_o, T)} \mu'(\rho_1, T)\, dV(\rho_1) . \tag{6}$$

As an explicit example of the use of (5) and (6), we assume that ρ_o radiates uniformly for all $T > 0$ $(T_o = \infty)$. Then (6) reduces to

$$U'(T) = s P^o T_\alpha \int_0^{vT} (e^{-\alpha r} - e^{-\alpha v T})\, dr$$

or

$$U'(T) = \omega_o U^o(\infty) \left[1 - \left(1 + \frac{T}{T_\alpha} \right) e^{-T/T_\alpha} \right], \tag{7}$$

where $\omega_o = s/\alpha$ is the <u>albedo for single scattering</u>. Hence the limiting (or asymptotic) value of $U'(T)$ is:

$$U'(\infty) = \lim_{T \to \infty} U'(T) = \omega_o U^o(\infty) = \omega_o P^o T_\alpha .$$

Time-Dependent Radiance

The present setting is associated with a one-dimensional location space X_o (Figure 10). We assume that n is constant on $X \times T$, but that α and σ may vary spatially in an arbitrary manner on X_o and be constant on T. The volume scattering function then has only two components in this space at each point z' : $\sigma_+(z')$ the forward scattering component and $\sigma_-(z')$ the backward scattering component. The fundamental source directs radiant flux in the upward (increasing z) direction in an arbitrary manner over a time interval $(0, T_o)$, such that the inherent radiance is given by $N^o f(t)$. The radiance is zero outside this time interval. This setting corresponds closely to the real case of a narrow beam of radiant flux sent into a spatially inhomogeneous scattering-absorbing medium such as the atmosphere. Thus the formulas developed below for the down and upwelling primary radiance $N'(z, \pm, t)$, are applicable to the problem of predicting primary scattered radiance from clouds or haze layers when they are irradiated by a narrow vertical pulse of photons.

Figure 10a shows the location of the pulse in X_o at time T . The pulse is of temporal length T_o and of geometrical length $L_o = v T_o$, and varies in radiance in the given manner over the period of time $(0, T_o)$. This shape is governed by the function f defined above, and is graphically depicted in the figure.

Figure 10b shows a general receiving point at height z, along with its past nappe $\mathcal{C}_-(z, T)$. The intersection of this past nappe with the future nappes $\mathcal{C}_+(0, 0)$ and $\mathcal{C}_+(0, T_0)$ of the fundamental source define the heavily-drawn region on the z-axis. In the terminology of characteristic ellipsoids, this region is precisely representable as $\mathcal{E}(0, z; T) - \mathcal{E}(0, z; T_0)$ and has vertical extent of magnitude $vT_0/2$. An examination of the figure will show that it is this region which contributes downwelling primary scattered radiant flux to a radiance receiver at z at epoch time T. Consider a particular contributing point at altitude z' in this region. In order that a set of photons travel from $z = 0$ to $z = z'$ and then reach height z at time T, the set must start out at $z = 0$ at time $t = T + \frac{z}{v} - \frac{2z'}{v}$. The inherent radiance of this set is then $N^0 f(T + \frac{z}{v} - \frac{2z'}{v})$. The reduced radiance of this set as it reaches height z' is

$$N^0 f\left(T + \frac{z}{v} - \frac{2z'}{v}\right) T_{z'}(0, +)$$

where $T_{z'}(0, +)$ is the beam transmittance of the upward directed path of length z' with initial point at $z = 0$.

Now as the reduced radiance reaches height z its photons converge on the point in a solid angle of magnitude $\Omega(z')$, which is the solid angle subtense of the pulse transmitter as seen from height z'. Part of this set of photons is then scattered back; the exact amount scattered back per unit length of the shaded path at height z' is given by:

$$N^0 f\left(T + \frac{z}{v} - \frac{2z'}{v}\right) T_{z'}(0, +)\, \sigma_-(z')\, \Omega(z') .$$

This set of photons then travels downward a vertical distance $z'-z$ to reach level z ; the amount of radiance arriving at height z is

$$N^{\circ} f\left(T+\frac{z}{v}-\frac{2z'}{v}\right) T_{z'}(0,+)\, T_{z'-z}(z',-)\, \sigma_{-}(z')\, \Omega(z'),$$

where $T_{z'-z}(z',-)$ is the beam transmittance of a downward directed path of length $z'-z$ with initial point at z' .

This radiance is then the primary radiance arriving in a downward direction at height z at time T ; and this primary radiance has been generated by a unit length of path about height z'. Therefore the total primary radiance arriving at z at time T is obtained by integrating over the region $\mathcal{E}(0, z; T) - \mathcal{E}(0, z; T_0)$. The upper and lower limits of the region are shown in Figure 10b. If z exceeds the altitude of the lower limit of the region we must clearly choose z as the lower limit of integration. Thus:

$$N'(z,-,T) = N^{\circ} \int_{\max\left\{z,\, \left(\frac{vT+z}{2}\right)-\frac{L}{2}\right\}} f\left(T+\frac{z}{v}-\frac{2z'}{v}\right) T_{z'}(0,+)\, T_{z'-z}(z',-)\, \sigma_{-}(z')\, \Omega(z')\, dz \qquad (8)$$

The upward primary radiance $N'(z,+,T)$ is computed in a similar way. We now use Figure 10c to aid in the determination of the limits of integration. In order for a set of photons to start out at $z=0$ and reach altitude z at epoch time T , they must start at epoch time $T - \frac{z}{v}$. Assume that $T - \frac{z}{v}$ is in $(0, T_0)$. (If it is not, then (z, T) is out of the world-region of the given pulse, and $N^i(z, +, T) = 0$). Further, suppose that primary scattered photons arriving at (z, T) were scattered at (z', T') , i.e., at height z' at epoch time T' . Thus if

$$N^0 f(T - \frac{z}{v}) T_{z'}(0, +)$$

is the reduced radiance of the photons arriving at (z', T') , this radiance is then scattered forward, the amount per unit length at being

$$N^0 f(T - \frac{z}{v}) T_{z'}(0, +) \sigma_+(z') \Omega(z').$$

Finally, this set of photons is transmitted over an upward directed path of length $z - z'$, starting at height z . Thus the amount at (z, T) is:

$$N^0 f(T - \frac{z}{v}) T_{z'}(0, +) T_{z - z'}(z', +) \sigma_+(z') \Omega(z').$$

The range of integration is now over the altitude interval $(0, z)$. Therefore

$$N'(z, +, T) = N^0 f(T - \frac{z}{v}) T_z(0, +) \int_0^z \sigma_+(z') \Omega(z') dz', \tag{9}$$

where $T_z(0,+) = T_{z'}(0,+)\, T_{z-z'}(z',+)$, by the semigroup property of beam transmittances.

An important special case of (8) arises in practice when $z = 0$. The result of setting $z = 0$ in (8) is:

$$N^1(0,-,T) = N^0 \int_{\max\{0,\, \frac{vT}{2} - \frac{L_0}{2}\}} f\left(T - \frac{2z'}{v}\right) T^2_{z'}(0,+)\, \sigma(z')\, \Omega(z')\, dz' , \qquad (10)$$

where we have made use of the reciprocity property of $T_{z'}$:

$$T_{z'}(0,+) = T_{z'}(z',-)$$

We close the discussion of applications with a remark on the computation procedure for radiant flux of higher scattering orders. Consider Figure 11a, which shows the world-line of a photon which arrives at (z,T) after having been scattered four times. The points in $X_0 \times T$ "where" scattering occurred are numbered consecutively. Each of these points is the vertex of a characteristic cone $C(j)$, $j = 1,2,3,4$. Observe that the future nappe $C_+(j+1)$ is contained in the future nappe $C_+(j)$, $j = 1,2,3$. In order to find $N^4(z,+,T)$, we must integrate $N^4_*(z',+,T')$ over $C_+(4)$ $\left(N^4_* \equiv \int \sigma N^3 \, d\Omega\right)$; but in order to know $N^4_*(z',+,T')$ we must know $N^3(z'',\pm,T'')$ over $C_+(3)$, etc., down to knowing $N^1(z,\pm,T)$ over $C_+(1)$.

But these latter primary radiances were just computed in detail above. We may then retrace our steps through the sequence of integrals over $\mathcal{C}_+(j)$, $j = 2, 3, 4$, starting from these known primary radiances. We eventually end up with a 4-fold iterated integral of the known function $N^0 f(t)$. Figure 11b shows a typical photon world-line which gives rise to $N^4(z, -, T)$. This general technique can also be applied to find $U^n(T)$, the n-ary scattered radiant energy.

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Properties of $\mathcal{S}$ and $\mathcal{E}$ in Arbitrary Spaces

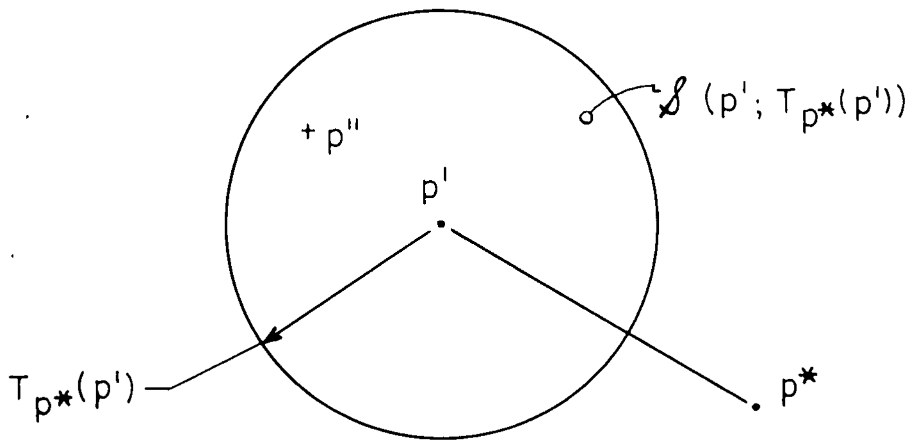

Figure 1a

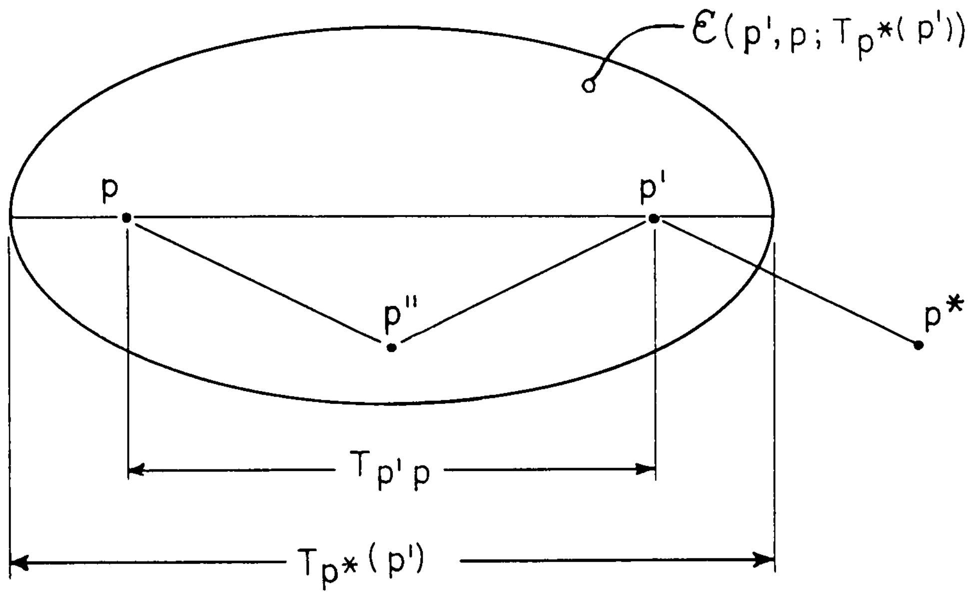

Figure 1b

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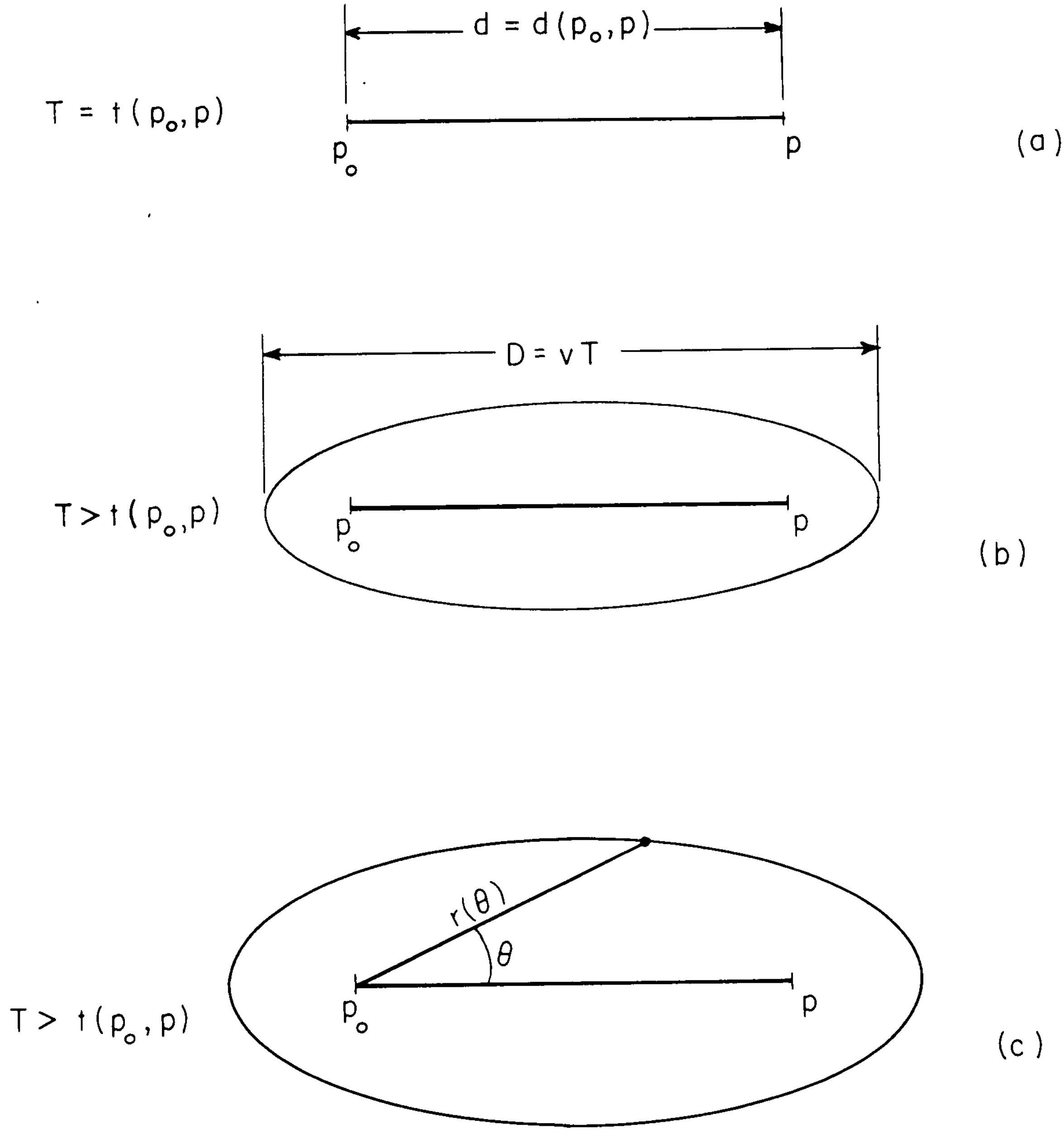
General Geometric Properties of $\mathcal{E}$ in Classical Carrier Spaces

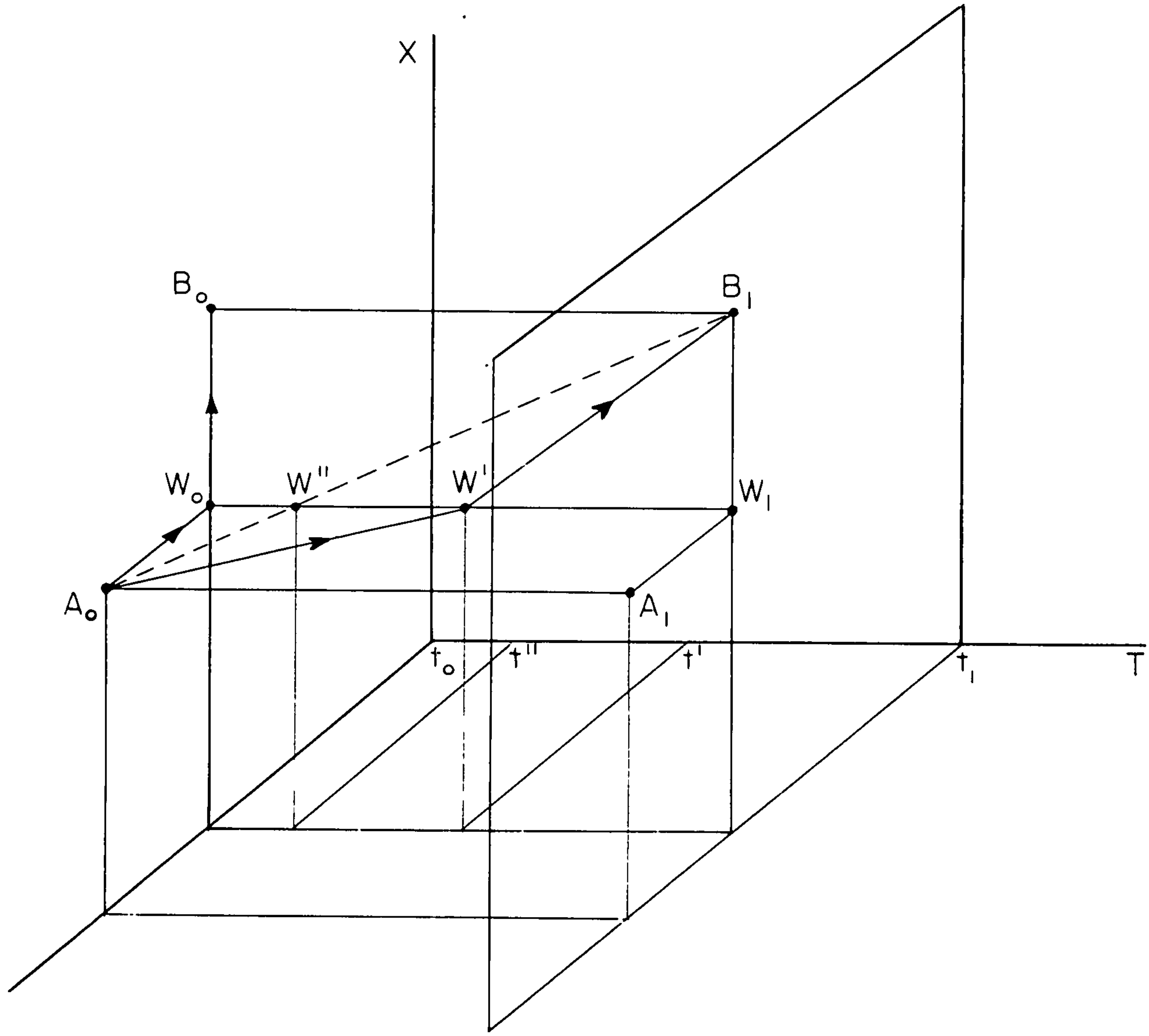


$$r(\theta) = \frac{D^2 - d^2}{2(D - d\cos\theta)}$$

Figure 2

R. Preisendorfer

World-Line Representation

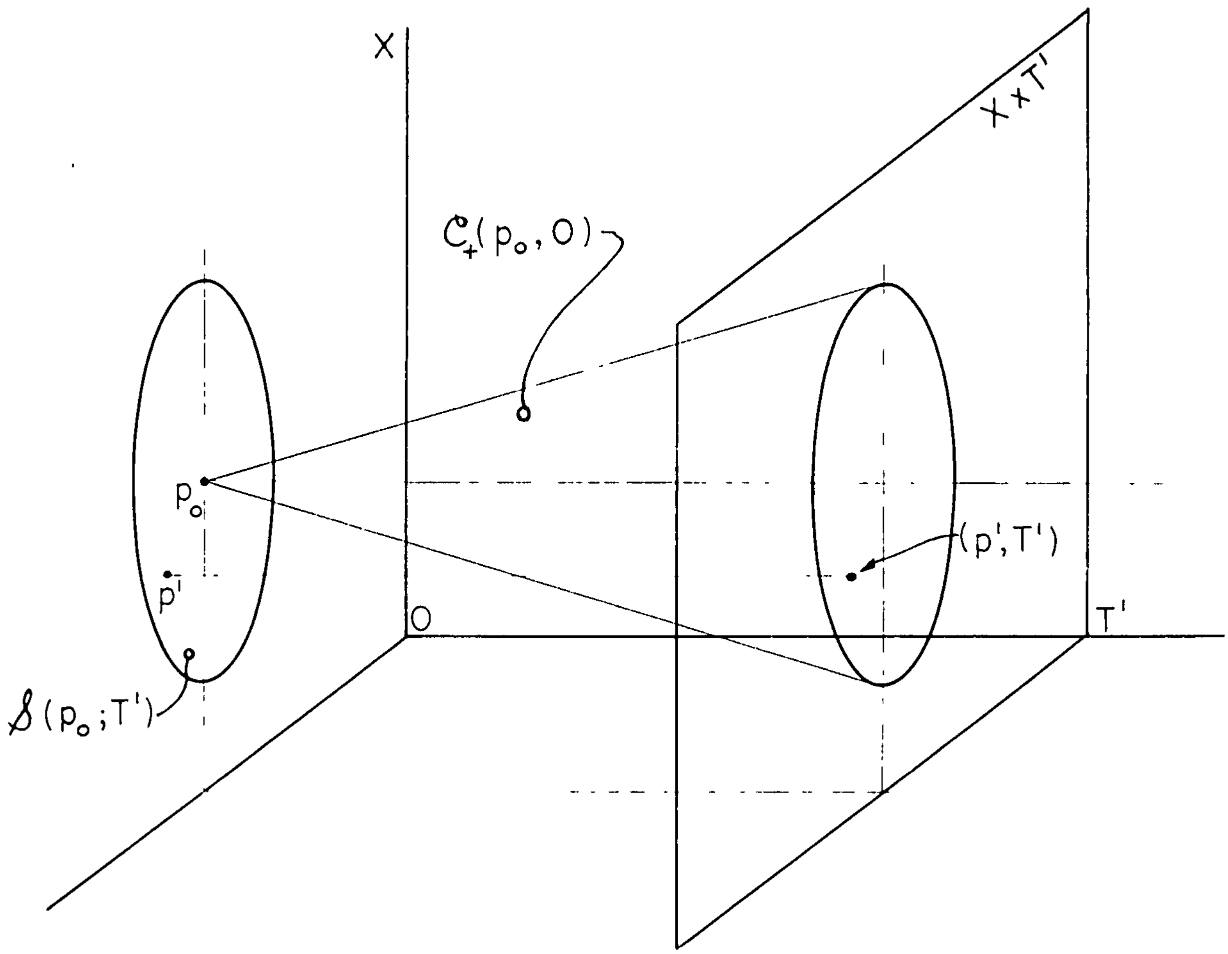


$$A_oW_oB_o = P_X\left[A_oW'B_1\right]$$

Figure 3

R. Preisendorfer

Relation Between Characteristic Spheroid $\mathcal{S}$ and Cone $\mathcal{C}_+$



$$\mathcal{S}(p_o, T') = \mathcal{C}_+(p_o, 0) \cap (X \times T')$$

$$\mathcal{C}_+(p_o, 0) = \{(p', T') : p' \in \mathcal{S}(p_o, T')\}$$

Figure 4

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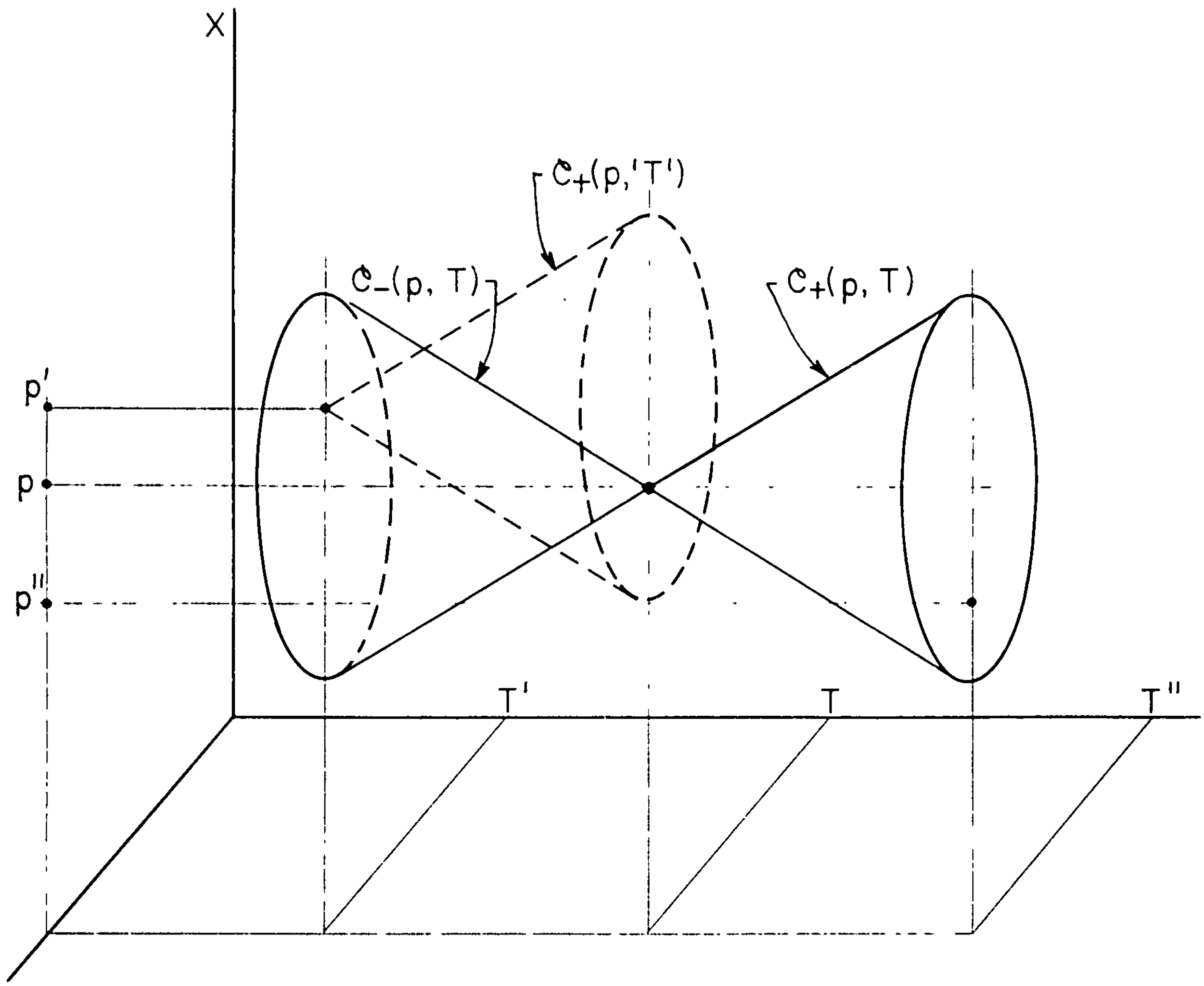
Future and Past Nappes of Characteristic Cone



$$\mathcal{C}_+(p,T) = \left\{(p'',T''):\ p'' \in \mathcal{S}(p,T''-T)\right\}$$ Future nappe

$$\mathcal{C}_-(p,T) = \left\{(p',T'):\ p \in \mathcal{S}(p',T-T')\right\}$$ Past nappe

or

$$\mathcal{C}_-(p,T) = \left\{(p',T'):\ (p,T) \in \mathcal{C}_+(p'T')\right\}$$

Figure 5

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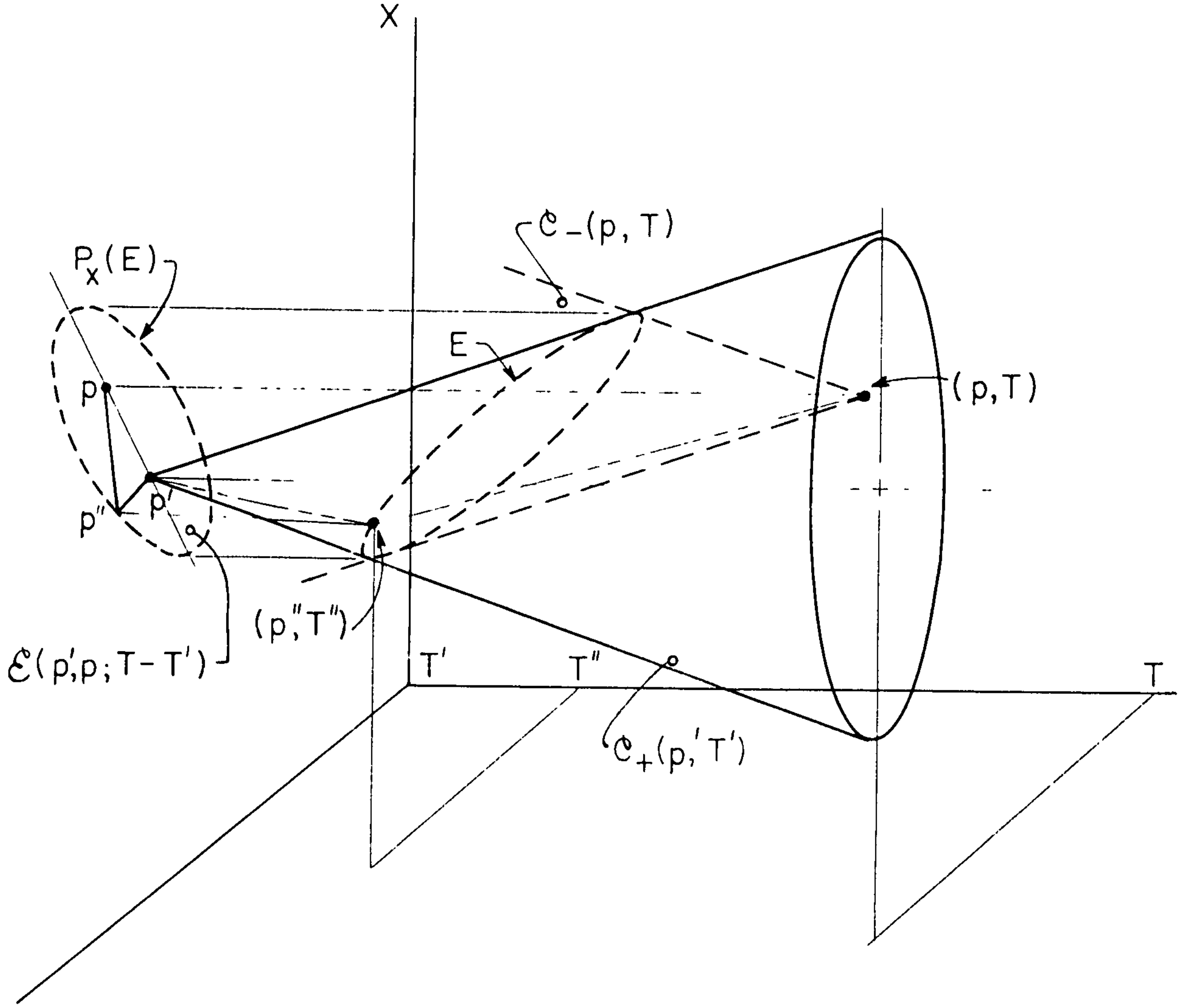
Relation Between Characteristic Ellipsoid and Cones



$$\mathcal{E}(p', p; T-T') = P_X\left[\mathcal{C}_+(p', T') \cap \mathcal{C}_-(p, T)\right]$$

Figure 6

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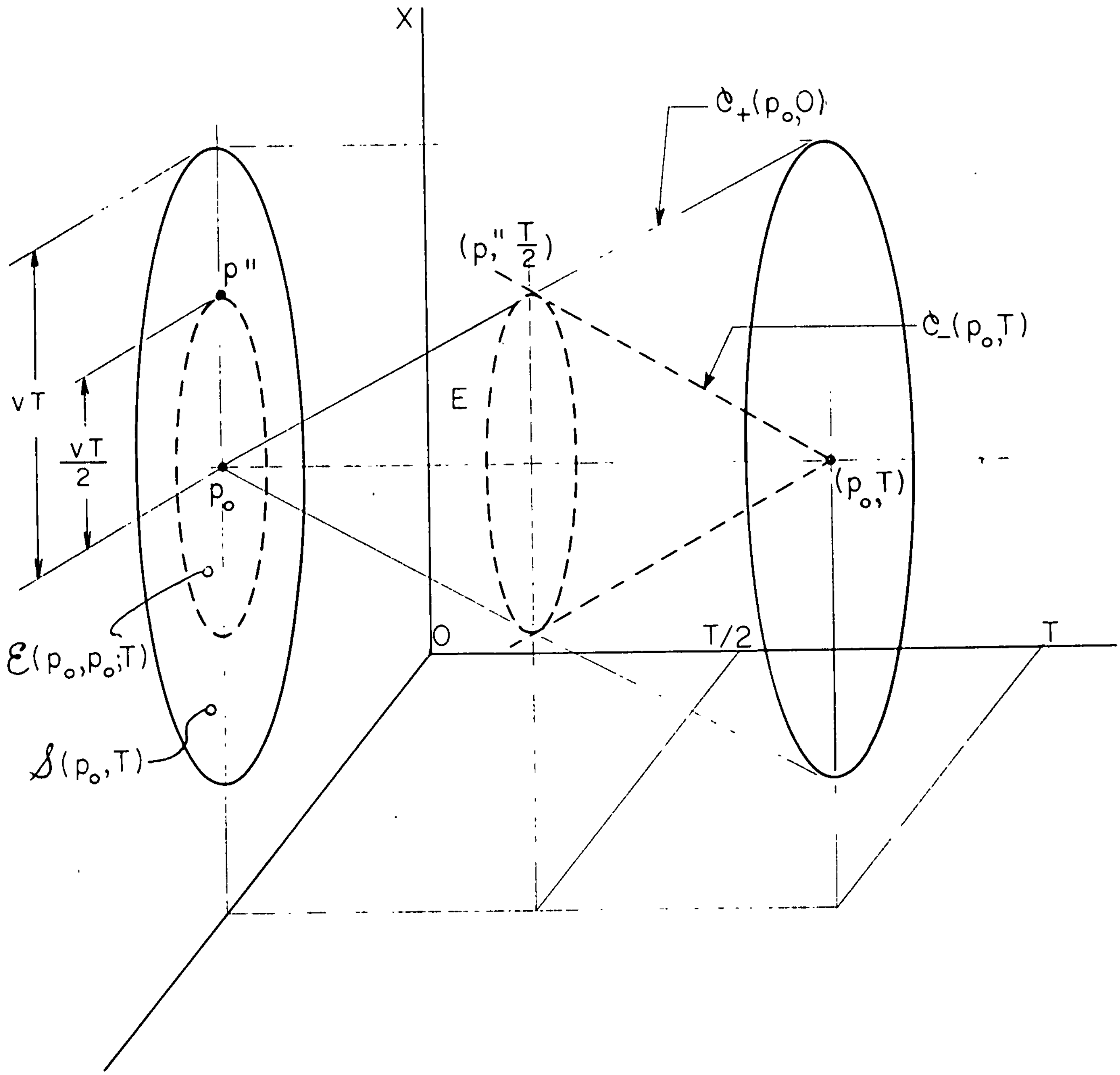
Important Special Case of $\mathcal{E}$ in Classical Carrier Spaces



$$\mathcal{E}(p_o, p_o; T) = P_X\left[\mathcal{C}_+(p_o,0) \cap \mathcal{C}_-(p_o,T)\right]$$

Figure 7

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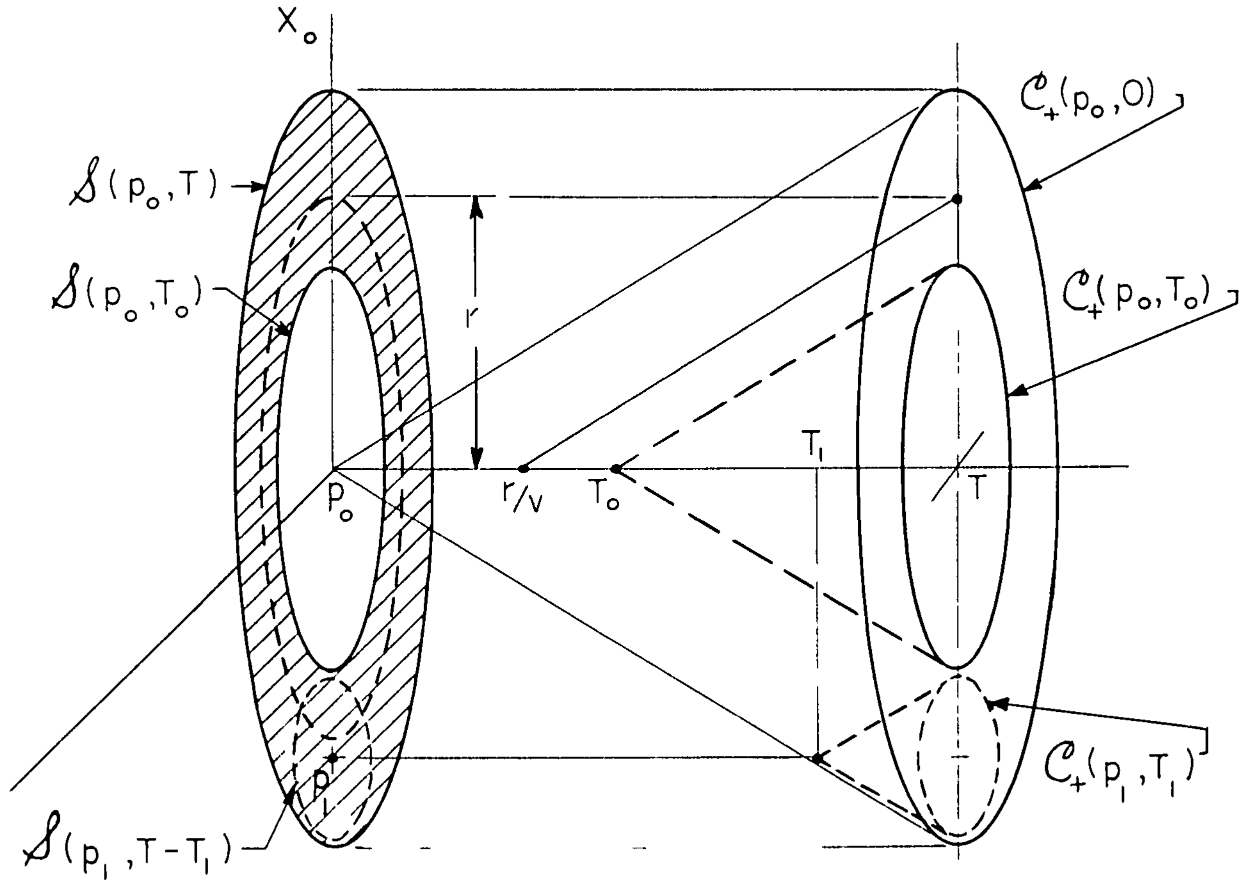

Figure 8

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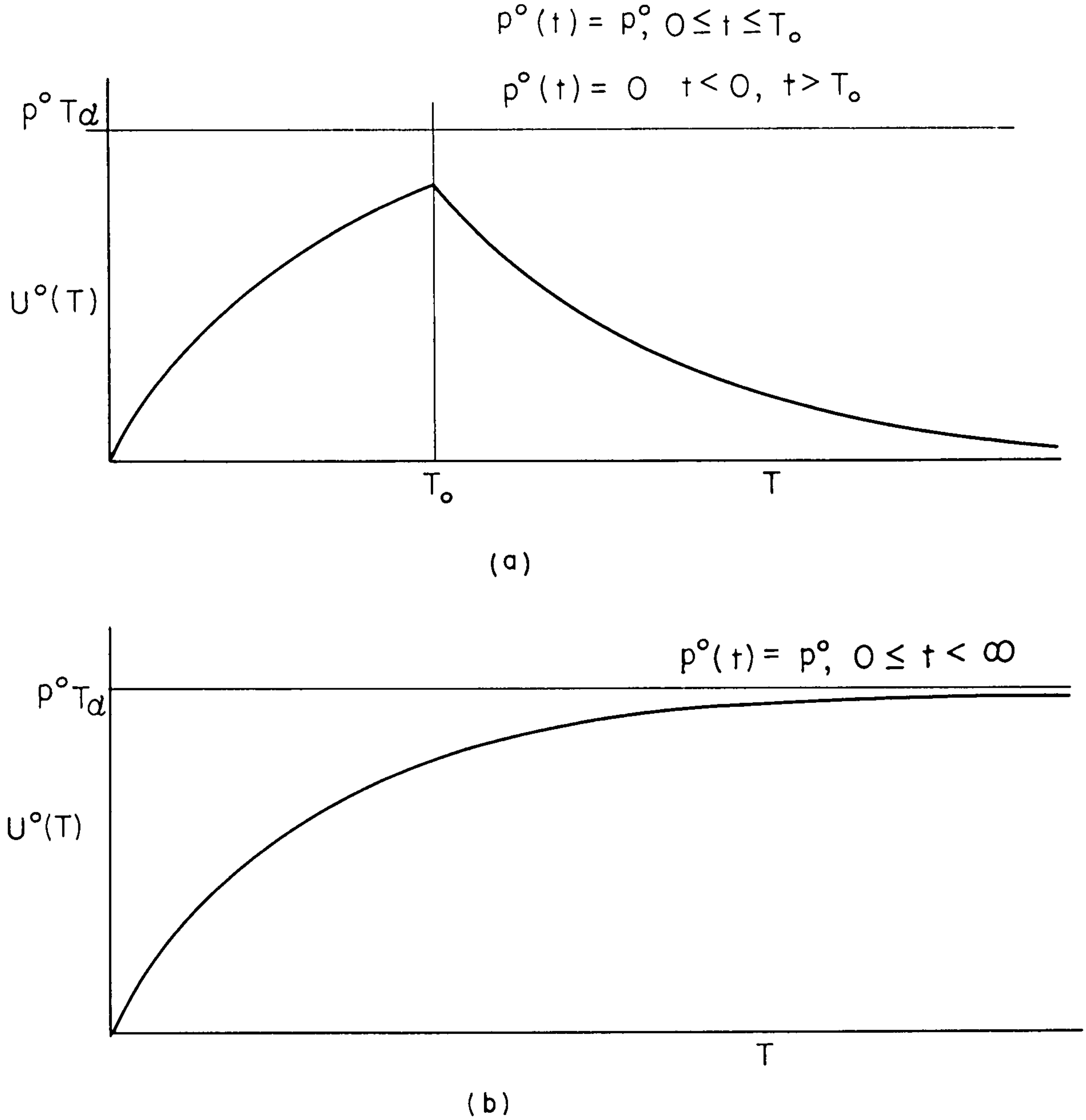

Figure 9

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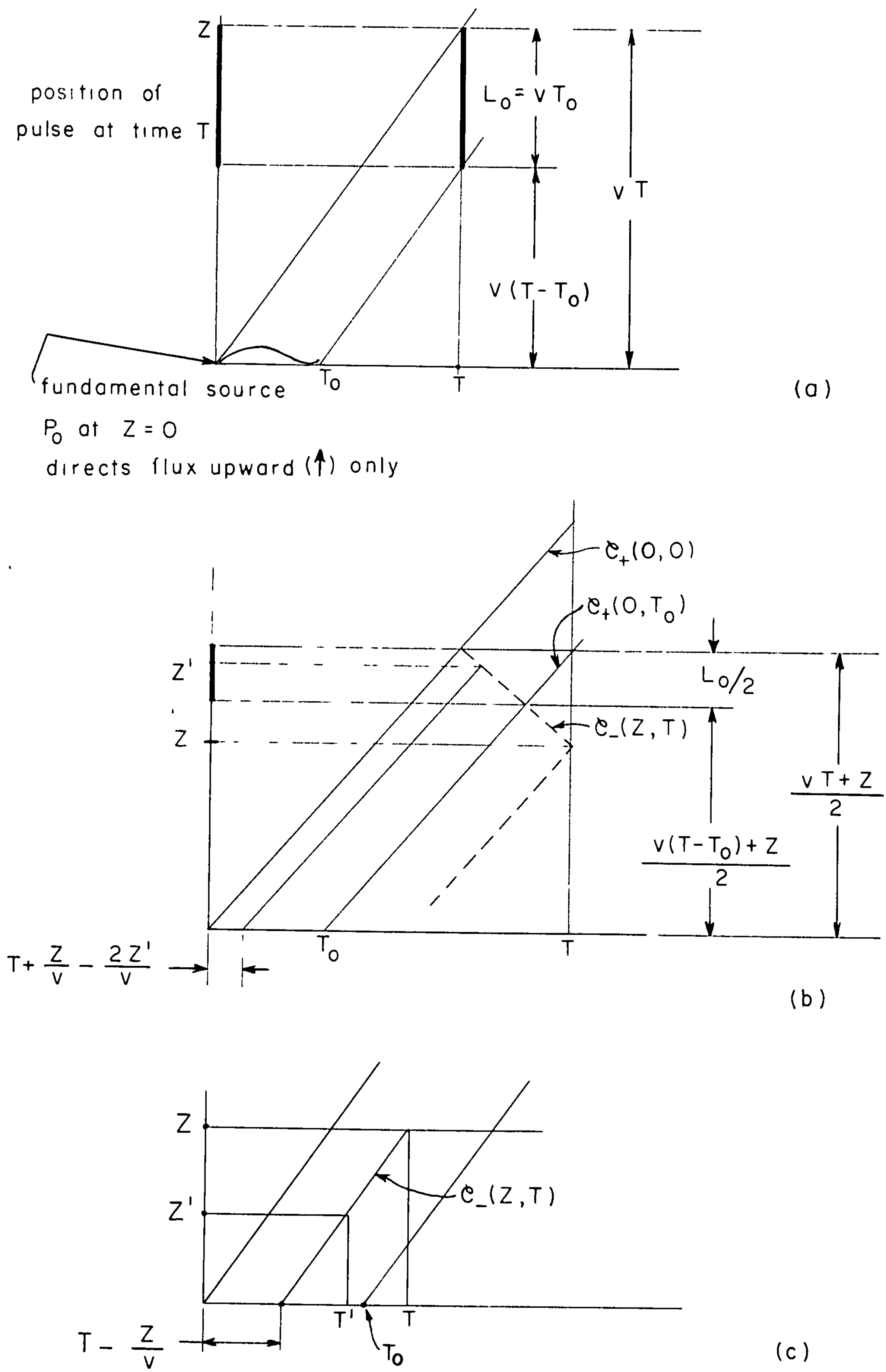
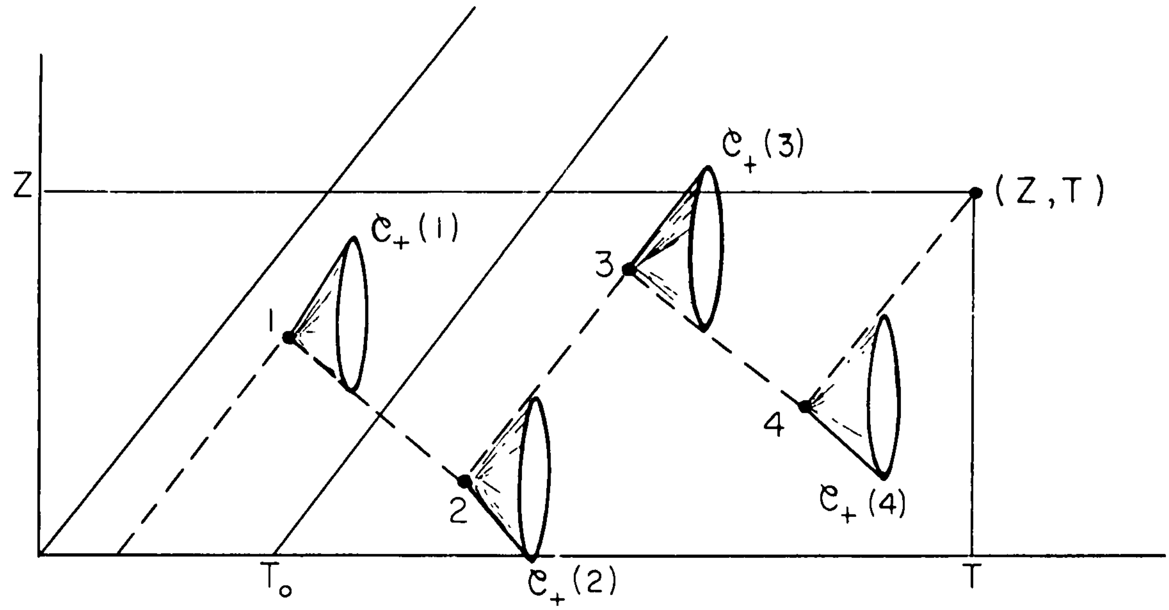
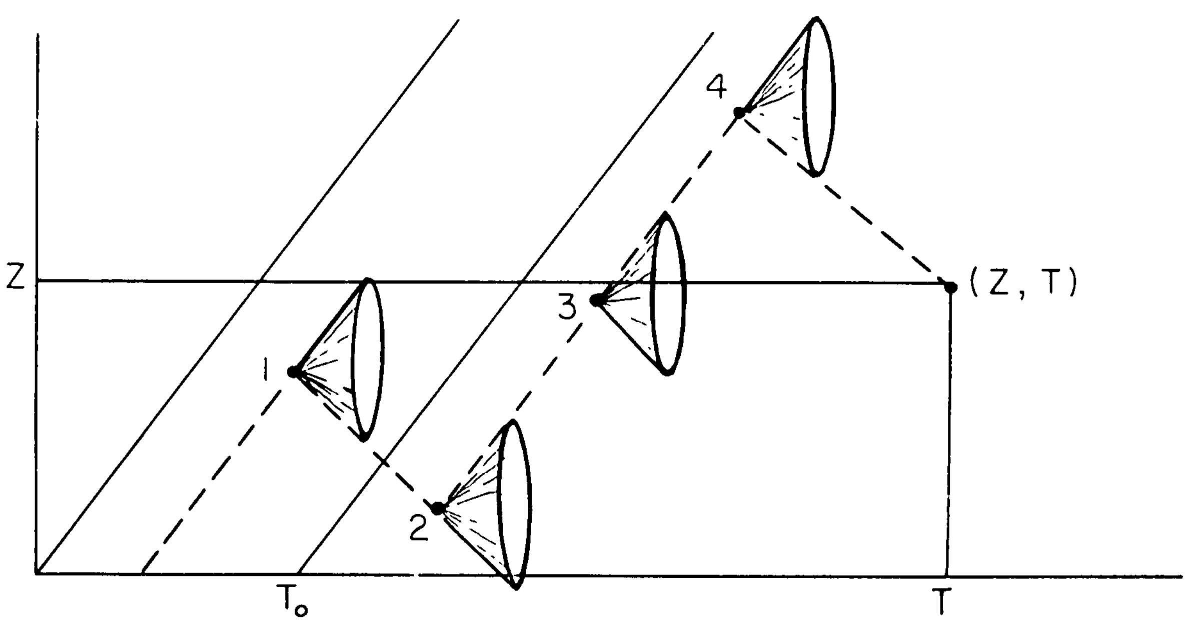


R. rreisendorfer

Figure 10



(a)



(b)

Figure II

R. Preisendorfer