UNIVERSITY OF CALIFORNIA,
IRVINE

Consistency strength of Stationary Catching

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Andres Forero Cuervo

Dissertation Committee:
Professor Martin Zeman, Chair
Professor Svetlana Jitomirskaya
Professor Sean Walsh

2015

© 2015 Andres Forero Cuervo

DEDICATION

To Xavier

TABLE OF CONTENTS

Page

LIST OF FIGURES **v**

ACKNOWLEDGMENTS **vi**

CURRICULUM VITAE **vii**

ABSTRACT OF THE DISSERTATION **viii**

1 Introduction **1**
1.1 Contents of the Thesis 8

2 Preliminaries **9**
2.1 Stationarity 9
2.2 Ideals 11
2.3 Self-genericity 14
2.3.1 Our situation of interest 18
2.4 Large cardinals and extenders 19
2.4.1 Extenders 22
2.5 Inner models 24
2.5.1 Premice 25
2.5.2 Iteration trees 27
2.5.3 Comparing premice 32
2.5.4 Q-structures 34
2.5.5 Sharps 35
2.5.6 Threadability and mouse reflection 39
2.5.7 The Core Model 41
2.6 Consistency results 46

3 Stationary Catching and $M_n^\#$ **49**
3.1 H_{ω_2} is closed under $M_n^{\#\#}$ 52
3.2 H_{ω_3} is closed under $M_n^{\#\#}$ 59
3.3 H_{ω_2} is closed under $M_{n+1}^\#$ 63
3.4 H_{ω_3} is closed under $M_{n+1}^\#$ 63
3.5 Closure of H_{ω_2} under $M_{n+1}^\#$ 64

4 Future directions **79**

Bibliography **84**

A Appendix **86**
A.1 Countably complete ultrafilters 86
A.2 Homogeneous forcing 87

LIST OF FIGURES

Page

1.1 Generic Ultrapowers 2
1.2 Q-structures for limit models. 6

2.1 $M_1^{\#}(a)$: In the picture, δ is the Woodin cardinal of $M_1^{\#}(a)$ and α is the height of $M_1^{\#}(a)$. Its top extender, E_α, is a (κ, α)-extender induced by a measure, where $\kappa > \delta$. If $\vec{E}$ denotes the extender sequence of $M_1^{\#}(a)$, then $E_\beta = \varnothing$ for any ordinal $\beta \in [\delta, \alpha)$. 27
2.2 The figure illustrates the copy construction starting from a map $\pi : \mathcal{M} \to \mathcal{N}$ (so $\pi_0 = \pi$). The maps of the form π_γ are called *copy maps*. 29
2.3 The common part model $\mathcal{M}(\mathcal{T})$. 32

ACKNOWLEDGMENTS

The completion of my PhD was aided in part by The Miguel Velez Scholarship during the academic years 2009-2010 and 2013-2014. I am also very grateful to the UCI Mathematics Department for providing financial support through several Teaching Assistant-ships, and to the UCI Staff for their valuable support, in particular to Donna McConnell.

I want to thank mi advisor Martin Zeman, whose invaluable comments and insights and commitment to teaching helped me enormously. His patience, willingness to cooperate and generous time disposition throughout the entire process are, among many other, things that I appreciated. Hearing him elaborate his views on the field was and will always be a privilege to me.

I want to also thank Trevor Wilson for his useful comments and clarifications, and to Monroe Eskew, whose conversations about his research provided great motivation to my own.

CURRICULUM VITAE

Andres Forero Cuervo

EDUCATION

Doctor of Philosophy in Mathematics **2015**
University of California, Irvine *Irvine, California*

Master of Science of Mathematics **2007**
Los Andes University *Bogota, Colombia*

Bachelor of Mathematics **2004**
Los Andes University *Bogota, Colombia*

RESEARCH EXPERIENCE

Graduate Research Assistant **2009–2015**
University of California, Irvine *Irvine, California*

TEACHING EXPERIENCE

Teaching Assistant **2009–2014**
University of California, Irvine *Irvine, California*

ABSTRACT OF THE DISSERTATION

Consistency strength of Stationary Catching

By

Andres Forero Cuervo

Doctor of Philosophy in Mathematics

University of California, Irvine, 2015

Professor Martin Zeman, Chair

The purpose of this thesis is to use the tools of Inner Model Theory to the study of notions relative to generic embeddings induced by ideals. We seek to apply the Core Model Induction technique to obtain lower bounds in consistency strength for a specific Stationary Catching principle called $StatCatch^*(I)$, related to the saturation of an ideal I of ω_2. This principle involves the central notion of *self-genericity* in its formulation, introduced by Foreman, Magidor and Shelah. In particular, we show that assuming $StatCatch^*(I)$ (plus some additional hypothesis in the universe), we can obtain, for every $n \in \omega$, an inner model with n Woodin Cardinals.

Chapter 1

Introduction

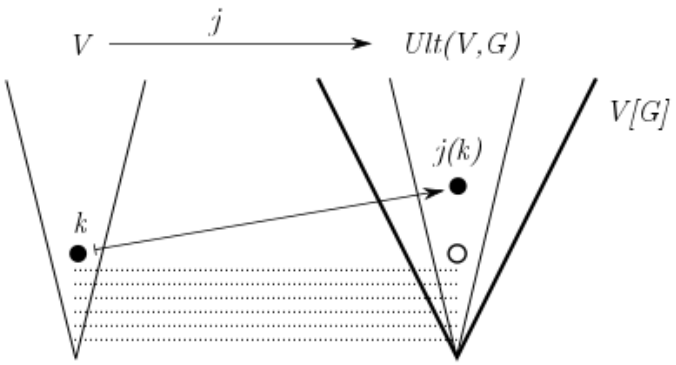
In a broad sense, the purpose of this work is to contribute to bring together the fields of Inner Model Theory and notions related to generic embeddings. More specifically, we seek to apply the Core Model Induction technique, introduced by Woodin and developed by Woodin, Steel and others to obtain lower bounds (in consistency strength) for principles related to the saturation of ideals in small cardinals, and natural weakenings that involve the notion of *self-genericity* in their formulation. In particular, these lower bounds materialize in specific models having many Woodin Cardinals that arise as part of the Core Model machinery.

Generic elementary embeddings

The area of generic embeddings deals with constructions in which a forcing $\mathbb{P} \in V$ (although V can be replaced with any transitive model, in future applications) produces a generic elementary embedding $j : V \to Ult(V, G)$ whose critical point[1] is a small cardinal (ω_1 or ω_2 are prime examples); assuming that the target model is well-founded, we can investigate several axioms in terms of properties of the forcing in question, which translate to properties of the induced elementary map. The model $Ult(V, G)$ is called the *generic ultrapower of* V

[1]The critical point of j is the first ordinal α such that $j(\alpha) \neq \alpha$. It can be easily seen that $j(\alpha) > \alpha$ and α is not in the range of j.

Figure 1.1: Generic Ultrapowers



by G. One of the motivations to consider generic ultrapowers comes from Solovay's work in the early seventies, in which he used these to prove that the consistency of the existence of a real valued measurable cardinal implies the consistency of the existence of a measurable cardinal (see [10]).

The theory of ideals comes into play in the following way: given an ideal I on a small cardinal κ[2], we can consider the induced forcing given by the quotient $\mathbb{P}_I = \mathcal{P}(\kappa)/I$. Combinatorial properties of the ideal such as strongness, presatutation, saturation, etc., that can induce properties on the associated generic elementary embedding $j : V \to Ult(V, G)$, whose existence and properties carry considerable consistency strength, going beyond a measurable cardinal. As mentioned above a basic requirement here is that $Ult(V, G)$ is well-founded (for every $\mathbb{P}_I$-generic filter G). Ideals where this is the case are called *precipitous*. We should mention that the consistency strength of the existence of a precipitous ideal is that of a measurable cardinal (see [13]).

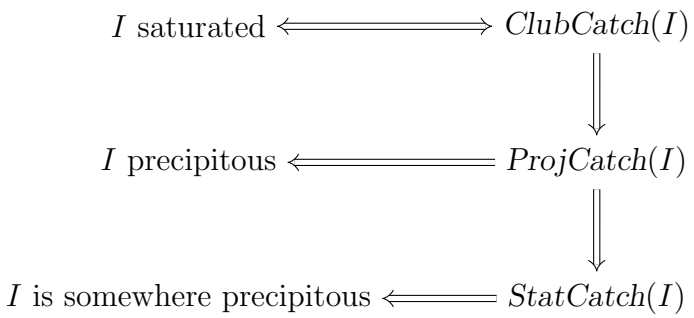
A notable result concerning the interaction between large cardinals and ideals is the following (see [4]): starting with a supercompact cardinal, and forcing to collapse it to ω_2, it is proved that the non-stationary ideal in ω_1 is presaturated. Later Shelah showed that starting with

[2]More generally one works with an ideal on a set Z, so that $I \subseteq \mathcal{P}(Z)$

a Woodin cardinal, one can produce a model in which the non-stationary ideal in ω_1 is saturated. This in turn induces a generic map $j : V \to M$ having critical point ω_1, and such that M has certain closure properties. A central notion used in the proof techniques is that of "antichain catching", which is a property of an elementary substructures of H_θ (θ sufficiently large) being able to see antichains of the forcing induced by the ideal. This can be reformulated in terms of the notion of "self-genericity" of the structure, which basically stipulates that the ultrafilter induced by the inverse of the transitive collapsing map is generic over the transitive collapse of the structure. These notions can be naturally adapted for towers of ideals, most notably to the stationary tower, which has desirable properties under certain large cardinal assumptions. For example, if one considers the stationary tower below a Woodin cardinal, and assuming that the supports of the ideals are relatively simple, then the associated Stationary Tower forcing is presaturated (see 9.24 in [3]). This forcing has several applications in Descriptive Set Theory.

There is a characterization of the notion of saturation of an ideal I in terms of self-generic structures associated to it, which roughly stipulates that an ideal is saturated exactly when there are club-many self-generic structures[3]. In other words, the frequency of self-generic structures is rather high. This principle is called $ClubCatch(I)$ (*club catching* of I) By lowering the requirement on the frequency of these structures (for example, demanding them to form only a stationary set), one can obtain interesting principles that are weaker than saturation. Two important such principles are $ProjCatch(I)$ (*projective catching* of I) and $StatCatch(I)$ (*stationary catching* of I). The word "catching" makes reference to meeting certain antichains. The reader can see how these principles compare in the figure below.

[3]We will make this statement more precise in Chapter 3.



In our project we will focus on a strenghtening of $StatCatch(I)$ called $StatCatch^*(I)$.

From large cardinals to properties on ideals

Kunen and Laver obtained saturated ideals on successor cardinals from huge cardinals. Magidor used methods similar to those of Kunen and Laver to obtain models with saturated ideals on successor cardinals from models with an almost huge cardinal, via forcing and the use of almost huge towers. In [2], Zeman and Cox show that assuming the existence of a supercompact cardinal up to an inaccessible cardinal (a weaker assumption than an almost huge cardinal), and using a "tower method" similar to the previous one, obtain an ideal I on ω_2 which satisfies the Projective Catching property but is not strong.

Inner models

In Set Theory one encounters many statements that go beyond the usual set of axioms (ZF, ZFC) in the sense that cannot be proved from those. The task is to find models for these axioms, or compare their logical consistency strength. A representative example is the Continuum Hypothesis. Gödel introduced the constructible universe L, which is an Inner Model, that is, a proper class model contained in the universe V. The model L satisfies many statements, including the Continuum Hypothesis, and some important combinatorial principles. However, L is not suited for large cardinal axioms: one can prove that L cannot have measurable cardinals.

However, one can build models inside the universe which are L-like in the sense that they are carefully built level by level, while at the same time incorporating what is called an *extender sequence*: an extender collects substantial information from large cardinals. The sequence obeys strong requirements to guarantee that the final model is well-behaved so that it resembles L in many situations. But, unlike L, the model will possibly have large cardinals.

The models described above form part of the various building blocks used in Inner Model Theory, which are called *mice*. An important case is K, the "Core Model below a Woodin Cardinal". In order to define this model one needs to assume in the first place that the universe V has no inner models with Woodin cardinals. Under this "anti-large cardinal" hypothesis, the model K exists; it is a maximal model which is close to V; by means of its extender sequence, the core model absorbs all possible large cardinals appearing in the universe. It also satisfies several useful properties: for example, it is forcing absolute for sufficiently small forcings.

This setting provides a proof method for obtaining strength from axioms in terms of Woodin Cardinals: if a statement A contradicts one of the properties of K, then it must be that we could not build K in the first place, so there must be an inner model with a Woodin Cardinal. So the statement "there is a Woodin cardinal" is a lower bound in consistency strength for the statement A.

One of the key properties of K consists of its iterability, a property involving forming iterations of ultrapowers of models in a non-linear way (using *iteration trees*), using the extenders in the extender sequence. Iterability is a key tool for comparing different models and proving many useful properties. Iterability can be formulated as the existence of a winning strategy for Player II in a specific 2-player game, in which Player I plays successor stages, and Player II must choose limits of structures previously constructed (by selecting a branch through the tree) such that these limits are well-founded models.

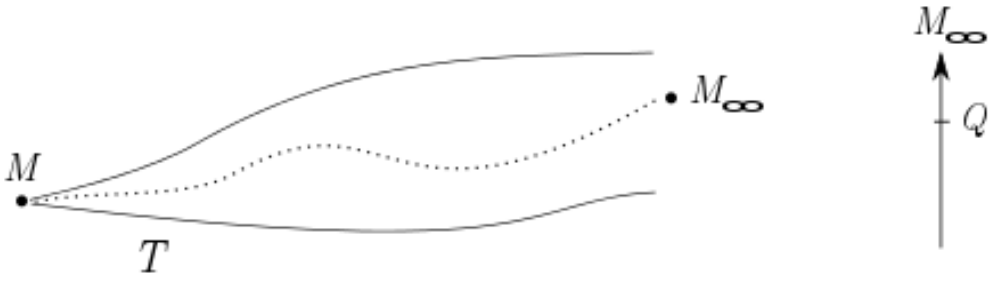


Figure 1.2: Q-structures for limit models.

It turns out that the non-existence of Inner Models with Woodin cardinals makes it possible to define iteration strategies for the models that approximate K. Roughly, some reflection arguments that turn a strategy for countable trees into a strategy for longer trees need the uniqueness of choice of branches producing well-founded limit models for Player II. A key Theorem states that if the choice is not unique, then one obtains an inner model with a Woodin cardinal.

Core model induction

What if we want to allow inner models with one Woodin cardinal? Under these circumstances the iteration strategies referred above might not be unique: nevertheless one can consider a generalization of the approach explained above, by modifying the setting: this involves considering "Q-guided strategies" to prove iterability of a structure $\mathcal{M}$. One proves that there is some structure Q that establishes the failure in a canonical way of some specific ordinal δ associated to the tree to be a Woodin cardinal, and then proves that there is a unique way for Player II to play to produce a limit model $\mathcal{M}_\infty$ having Q as an initial segment. Thus exactly one limit model is able to identify Q correctly, and this provides an iteration strategy for Player II.

The idea sketched in the previous paragraph can be captured and developed by the *Core Model Induction* technique, in which starting from some axiom A one proves by induction in

$n \in \omega$ that there are inner models with n Woodin cardinals, or even an inner model with a sequence of order type α consisting of Woodin cardinals, where $\alpha \geq \omega$. These inner models have explicit descriptions by means of certain mouse operators, as described below.

Roughly speaking, a mouse operator is a partial function F that assigns to a set a a model $M = F(a)$ having a inside at its bottom, in the sense that all extenders in the extender sequence of M appear above the ordinal level containing a. Some technical requirements must be satisfied (see [8]). An important example is the so called $M_n^{\#}$ mouse operator ($n < \omega$), that produces a minimal iterable model with n many Woodin cardinals, together with some other properties that make it well-defined. In the initial steps of a core model induction one shows the following: using the axiom A being assumed, there is a certain domain D (say H_θ for some θ) such that if $M_n^{\#}$ is defined in all D, then $M_{n+1}^{\#}$ is also defined in D. To prove this "inductive step", one argues in a similar vein as in the basic case: If $M_{n+1}^{\#}(a)$ were not to exist for some $a \in D$, one would then be able to construct a core model K (the iterability of K would be established by using the operator $M_n^{\#}$ as a guiding tool for limit models, together with the non-existence of $M_{n+1}^{\#}(a)$), and then show that A contradicts one of the properties of K. The reader can see that in this new setting, the anti-large cardinal hypothesis of "no inner model with a Woodin cardinal" has been replaced with "$M_{n+1}^{\#}(a)$ does not exist"[4].

At the finite level of a Core Model Induction, the iteration strategies are simple enough, involving simple relations between the Q-structures of the limit model and models of the form $M_n^{\#}(P)$, where P is a natural bottom part structure associated with the tree. The successor steps of the induction will have a similar flavor. However at other stages, the complexity of such strategies (which can be represented by a subset of ω, and then pulled back using reflection arguments) must be taken into consideration, and that is where Descriptive Set Theory comes into place.

[4]It can be seen that "$M_{n+1}^{\#}(a)$ exists" implies that there is an inner model (in particular, a proper class model) with $n+1$ Woodin cardinals.

In the project carried out in this thesis we will only consider finite core model induction. The axiom we are interested is "There is an ideal I in ω_2 such that StatCatch$^*(I)$ holds" (along with an extra assumption on the universe). We will obtain, for every $n < \omega$, a model with n Woodin cardinals.

1.1 Contents of the Thesis

The following provides a brief description of how the contents of this work are organized:

In chapter 2 we give some preliminaries, covering notions from stationarity, ideals, large cardinals and extenders and Inner Model Theory.

In chapter 3 we formulate the main result of this work, and we give a proof of the problem proposed in chapter 3, under an additional hypothesis, namely the threadability of ω_3, which will provide useful mouse reflection during the induction.

In chapter 4 we identify future research directions to build in or improve the results obtained.

Chapter 2

Preliminaries

In this chapter we introduce basic notions and results of importance for our project. First we study the general notion of stationarity for subsets of $\mathcal{P}(X)$. Then we consider properties of ideals, generic embeddings and the notion self-genericity. Next we mention some large cardinals and introduce (short) extenders. The last parts of the chapter are dedicated to Inner Model Theory and some consistency results.

For a basic background in set theoretical notions and terminology, the reader can consult [7].

2.1 Stationarity

Recall that if κ is an uncountable regular cardinal, one can define a notion of stationarity: a subset A of κ is stationary if and only if A has a non-empty intersection with every closed and unbounded set $C \subseteq \kappa$.

In our project we are interested in a notion of stationarity for subsets of $\mathcal{P}(Z)$ where Z is an arbitrary set. In what follows we will define stationarity for subsets of $\mathcal{P}(Z)$ where Z is an arbitrary set, by using the strongly closed unbounded filter, we we simply call the *club filter*.

Definition 2.1. Let X be a set and κ be a cardinal. We define:

- $[X]^{\kappa} = \{A \subseteq X : |A| = \kappa\}$.
- $[X]^{<\kappa} = \{A \subseteq X : |A| < \kappa\}$.

The set $[X]^{<\kappa}$ is also called $\mathcal{P}_{\kappa}(\lambda)$.

Definition 2.2. Given a function $F : [X]^{<\omega} \to X$ and $a \subseteq X$, we say that a is *closed under* F if and only if $Ran(F \restriction [a]^{<\omega}) \subseteq a$. In other words, for every $b \in [a]^{<\omega}$, $F(b) \in a$. We define the set C_F as:

$$C_F = \{a \subseteq X : \ a \text{ is closed under } F\}.$$

It can be seen that the collection of sets of the form C_F (where $F : [X]^{<\omega} \to X$) generate a filter $\mathcal{F}$ on $P(X)$, which we call the *club filter*:

$$\mathcal{F} = \{A \subseteq P(X) : \ \textit{there is some } F : [X]^{<\omega} \to X \textit{ such that } C_F \subseteq A \ \}.$$

A set $A \subseteq \mathcal{P}(X)$ is *weakly stationary*[1] *in* $\mathcal{P}(X)$ if and only if for every function $F : [X]^{<\omega} \to X$, there is some $a \in A$ such that $Ran(F \restriction [a]^{<\omega}) \subseteq a$. We will use the convention that when we say that a set A is stationary in $\mathcal{P}(X)$, we mean that A is weakly stationary in in $\mathcal{P}(X)$. The following Lemma is a very useful pigeonhole principle to reduce a stationary set S to a smaller stationary subset S', by fixing choices of the elements in S' under certain functions:

[1] See [3] for a discussion regarding the difference between this notion of stationarity and the one by Jech.

Theorem 2.3 (Fodor's Lemma)**.** *Let S be stationary in $\mathcal{P}(X)$, and let $f : S \to Y$ be a regressive function (that is, for every nonempty $a \in S$, $f(a) \in a$). Then there is a set $S' \subseteq S$ such that S' is stationary in $\mathcal{P}(X)$ and $f \restriction S'$ is constant.*

To read more about stationarity, consult section 3.1 in [3].

2.2 Ideals

Let Z be an infinite set. Recall that an ideal on Z is a nonempty set $I \subseteq \mathcal{P}(Z)$ satisfying: (i) if $a, b \in I$ then $a \cup b \in I$, and (ii) if $a \in I$ and $b \subseteq a$ then $b \in I$. The set Z is called the *universe* of I, and $X := \cup Z$ is called the *support* of I. We define $\check{I} = \{a \subseteq Z : Z - a \in I\}$. It is easy to see that $\check{I}$ is a filter (called the *dual of* I). We define $I^+ = \{a \subseteq Z : a \notin I\}$. A set $a \in I^+$ is called an *I-positive* set. Given any $a \in I^+$, we define the *restriction of I to a*, in symbols $I \restriction a$, to be the ideal on Zgenerated by $I \cup \{Z - a\}$.

Example 2.1. Here are some examples of ideals:

- For an infinite regular cardinal κ, the *bounded ideal* on κ:

$$I = [\kappa]^{<\kappa} = \{A \subseteq \kappa : |A| < \kappa\}.$$

- For an infinite regular cardinal κ, the *non-stationary ideal* on κ:

$$I = NS_\kappa = \{A \subseteq \kappa : A \text{ is non-stationary in } \kappa\}.$$

  Here we intend the usual notion of stationarity for subsets of κ: $A \subseteq \kappa$ is non-stationary if and only if there is a closed unbounded set $C \subseteq \kappa$ such that $A \cap C = \varnothing$.

- The meager ideal on the unit interval:

$$I = \{A \subseteq [0,1] : \ A \text{ is meager}\}.$$

Another example: if X is a set, let $Z = \mathcal{P}(X)$, and define NS_X to be the collection of all $A \subseteq P(X)$ that are non-stationary in Z (see section 2.1 for definitions). Then NS_X is an ideal on Z.

Definition 2.4. Let I be an ideal on Z. We say that I is:

- *principal* if there is some $a \subseteq Z$ such that $I = \{b \subseteq Z : b \subseteq a\}$.
- *fine* if for every $x \in \cup Z$ there is some $a \in I^+$ such that $x \in \cap a$.
- *normal* if for every regressive function $f : Z \to V$ there is some $a \in I^+$ such that $f \restriction a$ is constant.
- *maximal* if and only if $I \cup \check{I} = \mathcal{P}(Z)$. That is, for any $a \subseteq Z$, $a \in I$ or $Z - a \in I$.

From now on we restrict our attention to ideals that are non-principal, fine and normal.

An ideal I on Z induces a partial ordering $\mathbb{P}_I = \mathcal{P}(Z)/\sim_I$, where $A \sim_I B$ if and only if $A \triangle B \in I$. By removing the smallest element $[\varnothing]$, we can force over V with this partial ordering. Given a filter $G \subseteq \mathbb{P}_I$, sometimes we shall abuse notation and identify G with the set $\{A \subseteq Z : [A] \in G\} \subseteq \mathcal{P}(Z)$.

Definition 2.5. An ideal $I \subseteq \mathcal{P}(Z)$ is called *precipitous* if and only if whenever G is a $\mathbb{P}_I$-generic filter, then $Ult(V, G)$ is well-founded. We say that I is *somewhere precipitous* if and only if there is some $A \in I^+$ such that $I \restriction A$ is precipitous.

We remark that in the previous definition, the ultrapower $Ult(V, G)$ is not defined in V, but in the generic extension $V[G]$.

Example 2.2. Let κ be a regular uncountable cardinal. Consider the *bounded ideal* on κ, defined by:

$$I = \{A \subseteq \kappa : |A| < \kappa\}.$$

Then I is not precipitous. (See [6].)

Let I be a precipitous ideal, and let G be a $\mathbb{P}_I$-generic filter. The model $Ult(V, G)$ is called the *generic ultrapower* of V by G. As usual, we work with its transitive collapse. Note that unlike the case with measures that are elements of V, here it is not the case that $Ult(V, G)$ is a subclass of V; however, it is a subclass of $V[G]$.

Definition 2.6 (Saturation)**.** Let $I \subseteq \mathcal{P}(Z)$ be an ideal on Z, let λ be a cardinal and let $\kappa = |\cup Z|$.

- I is λ-*saturated* if and only if $\mathbb{P}_I$ has the λ chain condition.
- I is *saturated* if and only if $\mathbb{P}_I$ has the κ^+ chain condition.

In other words, I is λ-saturated if and only if, for any collection $\{A_i : i < \lambda\} \subseteq I^+$, there are $i < j < \lambda$ such that $A_i \cap A_j \in I^+$. Note that I is a maximal ideal if and only if I is 2-saturated. Also note that an ideal I on κ is saturated if it is κ^+-saturated.

Solovay showed that every regular cardinal κ can be partitioned into κ-many starionary sets in κ. In other words, the non-stationary ideal NS_κ is not κ-saturated. Therefore it is natural to ask if NS_κ is κ^+-saturated, in other words, if NS_κ is saturated. For the case of ω_1, it turns out that the saturation of NS_{ω_1} is equiconsistent with the existence of one Woodin cardinal. For more, see Section .

Definition 2.7. Let $I \subseteq \mathcal{P}(Z)$ be an ideal on Z.

- I is κ-*preserving* if and only if forcing with $\mathbb{P}_I$ preserves the cardinal κ.

- I is *presaturated* if and only if I is precipitous and $|Z|^+$-preserving.
- I is *strong* if and only if I is precipitous and for every $\mathbb{P}_I$- generic filter G over V, $j_G(\mu) = \mu^{+V}$, where μ is the completeness of I.
- I is λ-*dense* if $\mathbb{P}_I$ has a dense set of cardinality λ.

Note that if $\mu < \lambda$ and I is μ-dense, then I is λ-saturated.

Theorem 2.8. *For any regular cardinal $\kappa \geq \omega_2$, the ideal NS_κ is not κ-saturated.*

Proof. See [5]. □

2.3 Self-genericity

Let I be a normal ideal on Z, and let $X = \cup Z$.[2]. We will study elementary substructures of H_θ and how they can behave nicely with respect to the forcing $\mathbb{P}_I$. Having many structures behave nicely (in ways that we specify below) gives rise to different principles, which can be seen as properties of I itself, carrying certain consistency strength.

Let θ be a regular uncountable cardinal such that $\theta \geq (2^{|Z|})^+$. We shall focus on elementary substructures $M \preceq H_\theta$ that satisfy the following conditions:

- $I \in M$ (and so $\mathbb{P}_I \in M$).
- For every $A \in \check{I} \cap M$, $i_M := M \cap X \in A$.

The second condition is called *goodness* (we say that M is *I-good*). For convenience, let $\mathcal{C}(I, \theta)$ denote the collection of all elementary substructures $M \preceq H(\theta)$ satisfying the two conditions above. In what follows we fix I and denote $\mathcal{C}(I, \theta)$ simply by $\mathcal{C}$.

[2]We will later focus on the case $Z = \omega_2$, so that $I \subseteq \mathcal{P}(\omega_2)$, and $X = \omega_2$.

Let H_M be the transitive collapse of M, and let $\sigma_M : H_M \to H(\theta)$ be the corresponding uncollapsing map. Let $Z_M = \sigma^{-1})M(Z)$.

The map σ_M induces the following ultrafilter $U_M \subseteq \mathcal{P}(\alpha_M)$ (over H_M):

$$U_M = \{A \in H_M : A \subseteq Z_M \text{ and } i_M \in \sigma_M(A)\}.$$

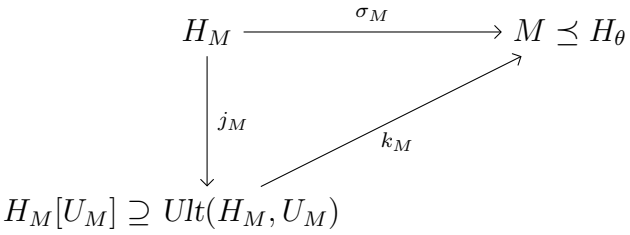
Let $\mathbb{P}_I^M = \sigma_M^{-1}(\mathbb{P}_I)$. We observe that U_M induces a well-defined ultrafilter on $\mathbb{P}_I^M$: if $A \in U_M$ and $B \in \mathcal{P}(Z_M)^{H_M}$ such that $B \sim_{I_M} A$, then $i_M \in \sigma_M(A)$ and $\sigma_M(A) \triangle \sigma_M(B) \in I$. By the assumption of I-goodness (and noting that $(\sigma_M(A) \cap \sigma_M(B)) \cup (\sigma_M(A) \cup \sigma_M(B))^c \in \check{I}$), we conclude:

$$i_M \in (\sigma_M(A) \cap \sigma_M(B)) \cup (\sigma_M(A) \cup \sigma_M(B))^c,$$

so $i_M \in (\sigma_M(A) \cap \sigma_M(B))$, and thus $B \in U_M$.

Thus, by abusing notation, we view U_M both as a filter on α_M and a filter for the poset $\mathbb{P}_I$. In the latter view, it is natural to ask wether this ultrafilter is generic (over H_M). We make this into a definition:

Definition 2.9 (Self-genericity)**.** A structure $M \in \mathcal{C}$ is *I-self-generic* if and only if U_M is $\mathbb{P}_I^M$-generic over H_M.



Self-genericity (also known as antichain catching) was introduced by Foreman, Magidor and Shelah, and later used by Woodin in the development of Stationary Tower Forcing.

Definition 2.10. Let $SG(I) = \{M \in \mathcal{C} : M \text{ is } I\text{-self-generic } \}$.

The collection $SG(I)$ is a subset of $\mathcal{P}(H(\theta))$. It is natural to ask "how big" the set $SG(I)$ is by making reference to stationarity in $\mathcal{P}(H(\theta))$. We formalize this idea in what follows, and will assume for simpliticy that Z is a (regular uncountable) cardinal.

Definition 2.11. Given an ideal I on a regular uncountable cardinal κ, and letting $SG(I)$ be as above, we define the following axioms:

- *ClubCatch*(I): $SG(I) \in \mathcal{F}_I$, where $\mathcal{F}_I$ is the conditional club filter relative to I[3].
- *ProjCatch*(I): $\forall A \in I^+$, the following set is stationary in $\mathcal{P}(H(\theta))$:

$$\{M \in \mathcal{C} : M \text{ is self-generic with respect to } I \text{ and } M \cap \kappa \in A \}.$$

- *StatCatch*(I): $SG(I)$ is stationary in $\mathcal{P}(H(\theta))$.

ClubCatch(I) implies *ProjCatch*(I), and *ProjCatch*(I) implies *StatCatch*(I). The following theorem gives a connection between the catching axioms on the one hand, and saturation and precipitousness on the other:

Theorem 2.12. *Let I be an ideal on a set Z. We have:*

- *ClubCatch*(I) *holds if and only if I is saturated.*
- *ProjCatch*(I) *implies that I is precipitous. [The converse holds if every $A \in Z$ is countable.]*
- *If StatCatch*(I) *holds, then there is some $A \in I^+$ such that $I \cap \mathcal{P}(A)$ is precipitous. [The converse holds if every $A \in Z$ is countable.]*

Proof. See [2]. □

[3]The conditional club filter relative to I is the club filter of H_θ restricted to $\{M \preceq H_\theta : M \cap \kappa \in \kappa$ and M is I-good$\}$.

It is easy to see that $\mathit{ClubCatch}(I)$ implies $\mathit{ProjCatch}(I)$, which in turn implies $\mathit{StatCatch}(I)$. Since $\mathit{ClubCatch}(I)$ is equivalent to saturation, then we can view $\mathit{ProjCatch}(I)$ and $\mathit{StatCatch}(I)$ as weakenings of saturation.

Let $SG^*(I) = \{M \in SG(I) : ORD \cap M \text{ is } \omega\text{-closed }\}$. The statement "$M \cap ORD$ is ω-closed" means that given an increasing sequence of ordinals $\langle \alpha_n : n < \omega \rangle$ such that for each $n < \omega$ $\alpha_n \in M$, then $\sup_n \alpha_n \in M$. The set $SG^*(I)$ allows us to strengthen the previous axioms:

Definition 2.13. Given an ideal I on a regular and uncountable cardinal κ and letting $SG^*(I)$ be as above, we define the following axioms:

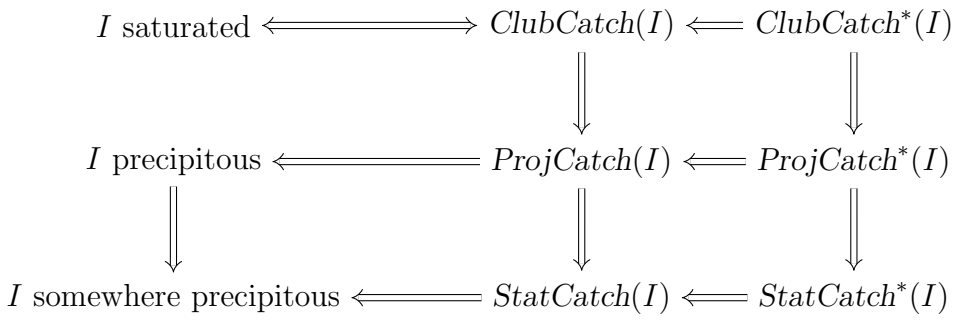
- $\mathit{ClubCatch}^*(I)$: $SG^*(I) \in \mathcal{F}_I$, where $\mathcal{F}_I$ is the conditional club filter relative to I.
- $\mathit{ProjCatch}^*(I)$: $\forall A \in I^+$, the following set is stationary in $\mathcal{P}(H(\theta))$:

$$\{M \in SG^*(I) : M \cap \kappa \in A\}.$$

- $\mathit{StatCatch}^*(I)$: $SG^*(I)$ is stationary in $\mathcal{P}(H(\theta))$.

The $*$-condition is crucial to obtain strength from these axioms: this condition allows that certain cardinals of our small structures have uncountable cofinalities, which allows to apply frequent extension of embeddings arguments.

We summarize the logical relations between the axioms in the following diagram:



2.3.1 Our situation of interest

In our specific situation (the one relevant for our project, see Chapter 3), we shall fix an ideal I on $Z = \omega_2$ (so $X = \cup Z = \omega_2$), and consider structures $M \in SG^*(I)$ such that the set $M \cap \omega_2$ is an ordinal less than ω_2 (thus by the $*$-condition, $M \cap \omega_2$ must have uncountable cofinality). We define $\alpha_M := M \cap \omega_2$. Note that $crit(\sigma_M) = \alpha_M$, and $\alpha_M = (\omega_2)^{H_M}$. Recall that σ_M induces the following H_M-ultrafilter:

$$U_M = \{A \in H_M : A \subseteq \alpha_M \text{ and } \alpha_M \in \sigma_M(A)\}.$$

We also observe that $[\alpha_M]^\omega \subseteq M$ (this fact will be used later): Let $a \in [\alpha_M]^\omega$. Fix some bijection $g : \omega_1 \to \alpha_M$, such that $g \in M$. Then $b = g^{-1}[a] \in [\omega_1]^\omega$. Let $\beta < \omega_1$ be such that $b \subseteq \beta$. Fix a bijection $h : \omega \to \beta$ such that $h \in M$. Then $c = h^{-1}[b] \subseteq \omega$, so $c \in M$. But then $a \in M$, as $a = g[h[c]]$.

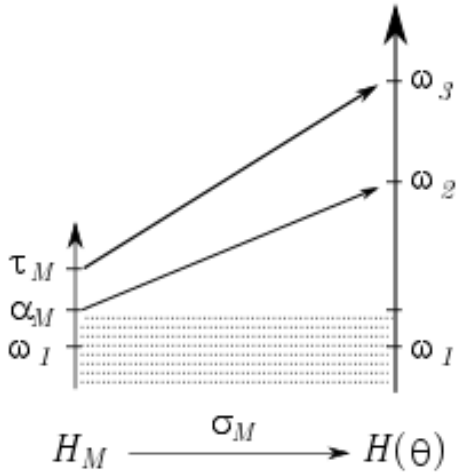
We now describe our project in detail. We start with our universe V satisfying:

- $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$ and $2^{\omega_2} = \omega_3$.
- ω_3 is threadable (see definition 2.52).

Let I be a normal ideal on ω_2. Recall that $SG^*(I)$ denotes the collection of all elementary structures $M \preceq H(\theta)$ such that:

- $I \in M$ (and so $\mathbb{P}_I \in M$).
- For every $A \in \check{I} \cap M$, $M \cap \omega_2 \in A$.
- The set $\alpha_M := M \cap \omega_2$ is an ordinal less than ω_2, and $\sup(M \cap \omega_3) < \omega_3$.
- M is I-self-generic.

- $ORD \cap M$ is ω-closed.



Assuming $StatCatch^*(I)$ to hold (that is, that $SG^*(I)$ is a stationary subset of $\mathcal{P}(H(\theta))$), we shall prove the following:

For every $n < \omega$ and for every $a \in H_{\omega_4}$, $M_n^{\#}(a)$ exists and is iterable for trees $T \in H_{\omega_4}$.

In particular, it follows that for every $n < \omega$ there is an inner model with n Woodin cardinals: take $M_n^{\#}(\varnothing)$ and iterate its top measure through the ordinals to obtain a model denoted by $M_n(\varnothing)$: the n Woodin cardinals of $M_n^{\#}$ are below the measure being iterated and remain Woodin in $M_n(\varnothing)$, and $ORD^{M_n(\varnothing)} = ORD$. Thus $M_n(\varnothing)$ is an inner model with n Woodin cardinals.

2.4 Large cardinals and extenders

In this section, we recall for reference the definitions of some large cardinals which are related to our project, in various degrees: measurable, Woodin, superstrong and supercompact. The reader can consult [7] for more information. The following is the "consistency strength" chain

that relates them:

$$\text{Measurable} \leftarrow \text{Woodin} \leftarrow \text{Superstrong} \leftarrow \text{Supercompact} \leftarrow \text{Huge}$$

Definition 2.14 (measurable cardinal)**.** Let κ be an uncountable cardinal. We say that κ is *measurable* if there is an elementary embedding $j : V \to M$ (where M is transitive) such that κ is the critical point of j (that is, κ is the smallest ordinal α such that $j(\alpha) \neq \alpha$; in symbols, $\text{crit}(j) = \kappa$).

The property "κ is measurable" is equivalent to the existence of a non-principal, κ-complete ultrafilter (also called *measure*) U on the set κ. If $j : V \to M$ has critical point κ, then we can define U as follows:

$$U = \{A \subseteq \kappa : \kappa \in j(A)\}.$$

We call U the *ultrafilter derived from* κ. We should say that many of the properties of j and M may be "lost" when we pass to U. In order to deal with this, we will introduce *extenders*, which can be thought as generalizations of ultrafilters, and will be better combinatorial objects that capture large cardinal properties given by $j : V \to M$.

On the other hand, if we start with a non-principal, κ-complete ultrafilter U on κ, then we can construct a model $M = Ult(V, U)$ (called the *ultrapower of* V *by* U), which is a U-quotient of functions $f : \kappa \to V$. Then we can naturally embed V into M (where we identify M with its transitive collapse) via a map $j : V \to M$, and prove that $crit(j) = \kappa$.

One can see that whenever $j : V \to M$ is induced by a measurable cardinal κ (so that $M = Ult(V, U)$ and $j = j_U$, for some measure U on κ), then $U \notin M$ and so $V_{\kappa+2} \not\subseteq M$. In other words, the model M does not agree with the universe below $\kappa + 2$. We would like to consider embeddings that do this (and much more), which leads to large cardinals stronger than a measurable. Kunen proved that M cannot, however, be equal to V itself.

Now we turn to Woodin cardinals, which are cardinals which witness lots of reflection below them. A key theorem (the Martin-Steel Theorem) tells us that a Woodin cardinal can be thought of as an "obstacle" for iterability (see Theorem 2.29). A Woodin cardinal marks the first place where one does not have uniqueness of cofinal wellfounded branches for certain iteration trees, and so one needs to impose extra conditions on wellfounded branches to obtain uniqueness (this is made precise with "guided strategies"). Uniqueness is important as we will use reflection to lift iterations on countable trees to iterations in longer trees. We will return to this point in Section 2.5.

Definition 2.15 (Woodin cardinal)**.** Let δ be an uncountable cardinal. We say that δ is *Woodin* if for every function $f : \delta \to \delta$ there is a cardinal $\kappa < \delta$ with $\{f(\alpha) : \alpha < \kappa\} \subseteq \kappa$, and an elementary embedding $j : V \to M$ (where M is transitive) such that

1. $crit(j) = \kappa$, and
2. $V_{j(f)(\kappa)} \subseteq M$.

Definition 2.16 (superstrong cardinal)**.** Let κ be an uncountable cardinal. We say that κ is *superstrong* if there is an elementary embedding $j : V \to M$ (where M is transitive) such that $\mathrm{crit}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

Lemma 2.17. *Every superstrong cardinal is Woodin.*

Proof. See Proposition 26.12 in [7]. □

Definition 2.18 (δ-supercompact cardinal)**.** Let κ be an uncountable cardinal, and let $\delta > \kappa$. We say that κ is a *δ-supercompact* cardinal if and only if there is an elementary embedding $j : V \to M$ (M transitive) with critical point κ such that $j(\kappa) > \delta$ and ${}^{\delta}M \subseteq M$.

Definition 2.19 (supercompact cardinal)**.** Let κ be an uncountable cardinal. We say that κ is *supercompact* if and only if for every ordinal $\delta > \kappa$, κ is δ-supercompact.

2.4.1 Extenders

An extender can be viewed as a directed system of measures, which produce some elementary embedding $j : V \to M$ that can produce an agreement between V and M considerably higher than the agreement produced by a single measure. We will, for simplicity take a different and simpler approach, which is useful for computations and manipulation of extender. The reader interested in the other approach can consult [7].

Let $j : V \to M$ be a nontrivial elementary map with $\kappa = crit(j)$, and let λ be an ordinal such that $\kappa \leq \lambda < j(\kappa)$. We define the (κ, λ)-extender derived from j to be the collection E of pairs (a, X), where:

- a is a finite subset of λ (say $n = |a|$),
- $X \in [\kappa]^n$ (that is, X is a subset of κ of cardinality n),
- $a \in j(X)$.

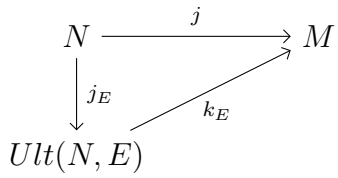
One can think that E codes some properties of the map j "up to λ": the bigger λ is, the more information we may recover from j. For example, if $\lambda = \kappa + 1$, then the extender we obtain is essentially the measure derived from j.

Definition 2.20. We say that E is a (κ, λ)-*extender* if there is an elementary embedding $j : V \to M$ (M not necessarily transitive) such that $\lambda \subseteq wfp(M)$, $\lambda < j(\kappa)$ and E is the (κ, λ)-extender derived from j. We call κ the *critical point* of E (denoted by $crit(E)$), and λ the *length* of E (denoted by $lh(E)$).

There are other equivalent ways to define the notion of extender as a combinatorial object without appealing directly to elementary embeddings (see, for example, [7]). We have chosen, for brevity, to introduce them in terms of embeddings.

In the exposition above one can replace V with any transitive model N satisfying a sufficient fragment of ZFC. Since extenders are generalizations of measures, one can expect to form the ultrapower of N by an (κ, λ)-extender E over N, which we call $Ult(N, E)$. One can view $Ult(N, E)$ as a quotient of the collection of pairs of the form (a, f), where a is a finite subset of λ, and $f \in N$ is a function $f : [\kappa]^{|a|} \to N$. The elements of the model $Ult(N, E)$ are of the form $[(a, f)]_E$. In some cases $Ult(N, E)$ is not wellfounded, but it can be seen that it is always the case that $\lambda \subseteq wfp(Ult(N, E))$.

Define $j_E : N \to Ult(N, E)$ to be the map sending x to the equivalent class of $(\{0\}, f_x)$, where $f_x : [\kappa]^1 \to N$ is the function with constant value x. Then one can see that j_E is an elementary embedding, called the *ultrapower embedding*.



The following is a useful result which specifies certain agreement between N and $Ult(N, E)$:

Lemma 2.21. *Let E be a (κ, λ)-extender over N. Then $V_\lambda^N \subseteq Ult(V_{\kappa^{+N}}^N, E)$.*

Proof. See [7]. □

Definition 2.22. Let E be a (κ, λ)-extender over a model M.

- Given an ordinal α, we say that α is a *generator* of E if and only if α is not represented in $wfp(Ult(M, E))$ by a pair of the form (a, f) where $a \subseteq \alpha$. In other words:

  For every $a \in [\alpha]^{<\omega}$ and $f : [\kappa]^{|a|} \to M$ with $f \in M$, $\alpha \neq [(a, f)]_E$.

- The *support* of E, denoted by ν_E, is the following ordinal:

$$\nu_E := \sup(\kappa^{+M} \cup \{\alpha + 1 : \alpha \text{ is a generator of } E\}).$$

If E is a (κ, λ)-extender over M and $\lambda' \in (\kappa, \lambda]$, we define $E \restriction \lambda'$ to be:

$$E = \{(a, X) : (a, X) \in E \text{ and } a \subseteq \lambda'.$$

One can verify that $E \restriction \lambda'$ is a (κ, λ')-extender over M. Furthermore, there is an elementary map $\sigma : Ult(M, E \restriction \lambda') \to Ult(M, E)$ sending the equivalence class of a pair (a, f) modulo $E \restriction \lambda'$ to the equivalence class of (a, f) modulo E.

Let E is a (κ, λ)-extender over M, and let $\lambda' \in [\lambda, j_E(\kappa))$. We define the *trivial completion of* E *to* λ' to be the (κ, λ')-extender F over M derived from j_E. It is easy to see that $F \restriction \lambda = E$ and $\nu_E = \nu_F$.

Definition 2.23. Let E be an extender over M and F be an extender over N. We say that E and F are *compatible* if and only if for some ordinal β, E is the trivial completion of $F \restriction \beta$ to $lh(E)$, or F is the trivial completion of $E \restriction \beta$ to $lh(F)$.

Note that if E is an extender over M, F is an extender over N and E and F are compatible, then $crit(E) = crit(F)$, say κ, and $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^N$.

2.5 Inner models

Premice are structures that in a way behave well, like levels of L, but they can also witness large cardinals through extenders. However, in order to be able to define canonical premouse, one needs to have a comparison mechanism since, unlike in the case of levels of L, two given

premice may not line up in the sense that one is an initial segment of the other. The key for this comparison is iterability, and we will call mice those premice that are iterable. We will then define the $M_n^{\#}(a)$ mice, which have n Woodin cardinals, and formulate the Core Model Dichotomy to see how we can (roughly speaking) pass from $M_n^{\#}$ to $M_{n+1}^{\#}$. The main tool is a canonical mouse called the *Core Model* K: if K cannot be defined, then one can obtain $M_{n+1}^{\#}$.

2.5.1 Premice

A *self-wellordered set* consists of a transitive set a together with a well-ordering Δ on a such that $\Delta \in J_1(a)$. We use the notation $\sup(a)$ to denote the rank of a.

When we define premice over a (also called a-premice), it is convenient for technical reasons to assume that a is self-wellordered.

A *premouse* over a self-wellordered set a is a structure of the form

$$\mathcal{M} = (J_\alpha^{a,\vec{E}}, \in, \vec{E}, E_\alpha)$$

such that $Dom(\vec{E}) = \alpha$, and $\vec{E} \cup \{E_\alpha\} = \{E_\beta : \beta \leq \alpha\}$ is a *fine extender sequence*, and every proper initial segment of it is ω-sound. The reader can consult [11] for the definition of a fine extender sequence and ω-soundness. We recall that in particular, either $E_\beta = \varnothing$, or E_β is a (κ, β)-extender (for some κ) over $J_\beta^{\mathcal{M}}$ (not necessarily over the entire structure $\mathcal{M}$). When $\beta < \alpha$ and $E_\beta \neq \varnothing$, then $\beta = lh(E_\beta)$ is not a cardinal in $\mathcal{M}$. In other words, the indices of extenders in the sequence are not cardinals in the premouse $\mathcal{M}$.

If $E_\alpha = \varnothing$ then we say that $\mathcal{M}$ is *passive*. Otherwise $\mathcal{M}$ is called *active*.

A *mouse* is a premouse that is iterable; however there are several degrees of iterability, so we caution the reader that there are different notions of mice, depending on the iterability that is needed in a particular situation. In the context of our project, it would be sufficient to define a mouse as a countably iterable premouse (see the section on iterabiliy for definitions).

Example 2.3. The following are basic examples of premice:

- Initial segments of L are premouse: Any initial segment $N = J_\alpha$ of L can be viewed as a premouse: we can let $\vec{E}$ be the sequence with constant value $\varnothing$, and also define $E_\alpha = \varnothing$. So there are no extenders in the extender sequence.

- Sharps: let a be a self-wellordered set. Then $a^\#$ is a minimal canonical premice $a^\# = (J_\alpha^{a,\vec{E}}, \in, \vec{E}, E_\alpha)$, where $\vec{E}$ is the sequence with constant value $\varnothing$, and E_α is an extender of critical point $\kappa > \sup(a)$ coding a measure U over $J_\alpha^{a,\vec{E}}$. The existence of $a^\#$ carries consistency strength beyond ZFC. For instance, if $V = L$, then for every a, $a^\#$ does not exist.

- $M_n^\#$: let $n \in \omega$, and let a be a be a self-wellordered set. Then $M_n^\#$ is a minimal canonical premice $M_n^\#(a) = (J_\alpha^{a,\vec{E}}, \in, \vec{E}, E_\alpha)$, where $M_n^\#(a)$ has n Woodin cardinals $\delta_0 < \ldots \delta_{n-1}$, $\sup(a) < \delta_0$ and E_α is an extender of critical point $\kappa > \delta_{n-1}$ coding a measure U over $J_\alpha^{a,\vec{E}}$. As the reader might expect, $M_0^\#(a) = a^\#$.

Definition 2.24 (Lower part model)**.** Let a be a self-wellordered set. We define $Lp(a)$ as the union of all a-premice N that are sound, project to a, and are countably iterable. The premice $Lp(a)$ is called the *lower part model relative to* a.

The model $Lp(a)$ is a mouse that has no total extenders, and it is a natural place to look for important a-premice appearing as initial levels of it.

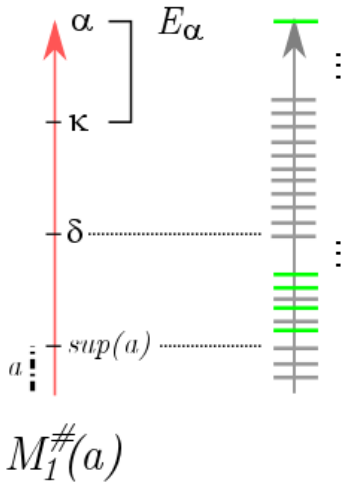


Figure 2.1: $M_1^\#(a)$: In the picture, δ is the Woodin cardinal of $M_1^\#(a)$ and α is the height of $M_1^\#(a)$. Its top extender, E_α, is a (κ, α)-extender induced by a measure, where $\kappa > \delta$. If $\vec{E}$ denotes the extender sequence of $M_1^\#(a)$, then $E_\beta = \varnothing$ for any ordinal $\beta \in [\delta, \alpha)$.

2.5.2 Iteration trees

We assume that the reader is familiar with the basic theory of iteration trees, which can be found in [11]. For purposes of reference, we recall the main properties of an iteration tree $\mathcal{T}$ on a premouse $\mathcal{M}$ (for presentation purposes one can think that the tree order structure of $\mathcal{T}$ is built as a two-player game progresses (called the *iteration game*), along with models $\mathcal{M}_\alpha$ and extenders E_α, for $\alpha < lh(\mathcal{T})$, as well as iteration maps between some of the structures):

- Let $\mathcal{M}_0 = \mathcal{M}$. The ordinal 0 is the root of $\mathcal{T}$.
- Construction of $M_{\alpha+1}$: At stage α (assuming $\{M_\beta : \beta \leq \alpha\}, \{E_\beta : \beta < \alpha\}$ are defined), Player I must choose an extender from the M_α sequence, call it E_α. E_α must satisfy: for each $\beta < \alpha$, $lh(E_\beta) < lh(E_\alpha)$.
  - Go to the first ordinal $\beta \leq \alpha$ such that $crit(E_\alpha) < \nu(E_\beta)$.
  - Declare $\alpha + 1$ to be the $\mathcal{T}$-successor of β. In other words: $\beta = Pred_T(\alpha + 1)$.

- If E_α is a total extender on $\mathcal{M}_\beta$, let $\mathcal{M}_{\alpha+1} = Ult(M_\beta, E_\alpha)$, and let $i_{\beta,\alpha+1} : \mathcal{M}_\beta \to \mathcal{M}_{\alpha+1}$ be the ultrapower embedding.
- If E_α is not a total extender on $\mathcal{M}_\beta$, let $\mathcal{M}^*_{\alpha+1}$ be the largest initial segment of M_β having E_α as a total extender (it can be seen that such segment exists and has height at least $lh(E_\beta)$). Let $\mathcal{M}_{\alpha+1} = Ult(\mathcal{M}^*_{\alpha+1}, E_\alpha)$, and let $i^*_{\beta,\alpha+1} : \mathcal{M}_\beta \to \mathcal{M}_{\alpha+1}$ be the ultrapower embedding. We also say that there is a *drop of model* at stage α.

- Limit stage: Assume $\lambda > 0$ is limit (assume $\{\mathcal{M}_\beta : \beta < \lambda\}, \{E_\beta : \beta < \lambda$ defined). Player II must choose a cofinal branch b through $[0, \lambda)_T$ such that the limit of $\mathcal{M}_\alpha$ ($\alpha \in b$) is well-founded. Call this limit model $\mathcal{M}_\lambda$, and we declare $\alpha <_{\mathcal{T}} \lambda$ for each $\alpha \in b$.

- Whenever $\alpha < \beta$, $lh(E_\alpha)$ is a cardinal in M_β.

Recall that $\mathcal{P}$ is λ-*iterable* if there is a winning strategy Σ for player II with respect to iteration trees $\mathcal{T}$ of length λ. For example, if $\mathcal{M}$ is an ω_1-iterable premouse, player II must be able to pick cofinal well-founded branches for trees of arbitrary countable limit length, but need not be able to pick a cofinal well-founded branch of a tree of length ω_1. In contrast, if $\mathcal{M}$ is an $\omega_1 + 1$-iterable premouse, then given a tree $\mathcal{T}$ of length ω_1 played according to Σ, Player II is able to pick a cofinal wellfounded branch $b = \Sigma(\mathcal{T})$ and thus construct an iteration tree of length $\omega_1 + 1$ extending $\mathcal{T}$.

Definition 2.25 (Countably iterable premouse)**.** A premouse $\mathcal{M}$ is *countably iterable* if the following holds: for every countable premouse $\bar{\mathcal{M}}$ that elementarily embeds into $\mathcal{M}$, $\bar{\mathcal{M}}$ is $\omega_1 + 1$-iterable.

Definition 2.26 (Simple trees)**.** An iteration tree $\mathcal{T}$ is *simple* if for all sufficiently large κ, $V^{Col(\omega,\kappa)} \models \mathcal{T}$ has at most one cofinal wellfounded branch.

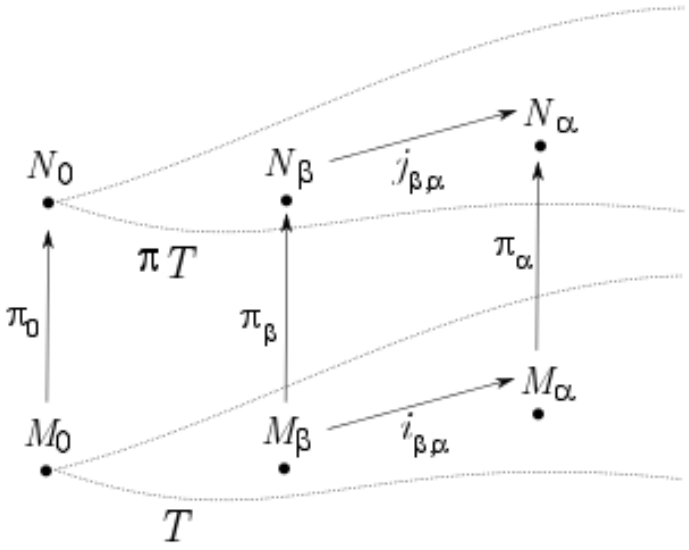


Figure 2.2: The figure illustrates the copy construction starting from a map $\pi : \mathcal{M} \to \mathcal{N}$ (so $\pi_0 = \pi$). The maps of the form π_γ are called *copy maps*.

Copying constructions

If M and N are models of set theory such that N is wellfounded and M embeds elementary into N (even Σ_0-elementarity suffices), then M must also be wellfounded. A similar property holds with respect to iterability.

Given an elementary embedding between premice $\pi : \mathcal{M} \to \mathcal{N}$, we can "copy" an iteration tree $\mathcal{T}$ on $\mathcal{M}$ to an iteration tree $\mathcal{S}$ on $\mathcal{N}$. We shall denote S by $\pi\mathcal{T}$. To see the complete details, the reader may consult [11]. Here we sketch the construction with several simplifications. In particular we will assume that there are no type III premice appearing as models of $\mathcal{T}$ (see [11]), which guarantees that $\pi\mathcal{T}$ and $\mathcal{T}$ have the same tree order (which may not hold in general).

Let us denote the models associated to $\mathcal{T}$ by $\mathcal{M}_\alpha$, and the elementary maps by $i_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ ($\alpha T \beta$). We shall denote the models of $\mathcal{S}$ (which we define inductively) by $\mathcal{N}_\alpha$, and the elementary maps by $j_{\alpha,\beta} : \mathcal{N}_\alpha \to \mathcal{N}_\beta$ ($\alpha S \beta$). Along the way we also define auxiliary "copy" maps $\pi_\alpha : \mathcal{M}_\alpha \to \mathcal{M}_\beta$.

- To start, we define $\mathcal{N}_0 := \mathcal{N}$, and $\pi_0 = \pi : \mathcal{M}_0 \to \mathcal{N}_0$.

- The extender E_0 is chosen from the sequence of $\mathcal{M}_0$. Let $F_0 = \pi_0(E_0)$, so F_0 is in the extender sequence of $\mathcal{N}_0$. Assuming for simplicity that there is no drop when using E_0, then $\mathcal{M}_1 = Ult(\mathcal{M}_\prime, E_0)$, and . Using π_0 we can define a map $\pi_1 : Ult(\mathcal{M}_0, E_0) \to Ult(\mathcal{N}, F_0)$ by $\pi_1([a, f]_{E_0}) = [\pi_0(a), \pi_0(f)]_{F_0}$ (the definition is actually more complicated for technical reasons having to do with fine structure, so again the reader can check the details in [11] for a more precise definition).

- At successor stages $\alpha \mapsto \alpha + 1$, we imitate the idea above in an obvious way.

- If $\lambda < lh(\mathcal{T})$ is a limit ordinal, we define $\mathcal{N}$ to be the direct limit of $\mathcal{N}_\alpha$, for $\alpha \in [0, \lambda)_\mathcal{T}$ (if the limit is ill-founded, we stop the construction). We define π_λ by: $\pi_\lambda(i_{\alpha,\lambda}(x)) = j_{\alpha,\lambda}(\pi_\alpha(x))$. It can be verified that this map is well-defined.

$$\begin{array}{ccc} \mathcal{N}_0 & \xrightarrow{j_0} & \mathcal{N}_1 = Ult(\mathcal{N}_0, F_0) \\ \uparrow{\scriptstyle \pi_0} & & \uparrow{\scriptstyle \pi_1} \\ \mathcal{M}_0 & \xrightarrow{i_0} & \mathcal{M}_1 = Ult(\mathcal{M}_0, E_0) \end{array}$$

Lemma 2.27. *Let $\pi : \mathcal{M} \to \mathcal{N}$ be an elementary embedding between premice, and let λ be an infinite ordinal. Assume that $\mathcal{N}$ is λ-iterable. Then $\mathcal{M}$ is λ-iterable.*

Proof. Fix a strategy Σ for player II in the iteration game relative to trees on $\mathcal{N}$ of length $< \lambda$. We let Σ^π to be the unique strategy for player II in the iteration game relative to trees on $\mathcal{M}$ of length $< \lambda$, satisfying the following property: for any tree $\mathcal{T}$ on $\mathcal{M}$,

$$\mathcal{T} \text{ is played according to } \Sigma^\pi \text{ if and only if } \pi\mathcal{T} \text{ is played according to } \Sigma.$$

It can be seen that Σ^π is a winning strategy for player II. $\square$

Roughly, the idea in the previous Lemma is the following: if $\mathcal{T}$ is a tree on $\mathcal{M}$ of limit length β, we can construct a tree $\pi\mathcal{T}$ on $\mathcal{N}$, using iterability of $\mathcal{N}$ to find a cofinal well-founded branch $b = \Sigma(\pi\mathcal{T})$ through $\pi\mathcal{T}$, and then essentially pick b itself (now viewed as a branch in the tree $\mathcal{T}$)[4]. Since we have an embedding $\pi_\beta : \mathcal{M}_\beta \to \mathcal{N}_\beta$ and $\mathcal{N}_\beta$ is wellfounded, then $\mathcal{M}_\beta$ is also well-founded, so that b is a cofinal well-founded branch through $\mathcal{T}$.

Lemma 2.28. *Let $\pi : P \to Q$ be an elementary embedding between premice, and assume that Q is countably iterable. Then P is countably iterable.*

Proof. Let $\sigma : \bar{P} \to P$ be elementary. Then $\pi \circ \sigma : \bar{P} \to Q$ is elementary, and since Q is countably iterable, then $\bar{P}$ is $\omega_1 + 1$-iterable. $\square$

If $\mathcal{T}$ is an iteration tree of limit length on a premouse $\mathcal{M}$, we can define the following objects:

- The *common part model* (associated to $\mathcal{T}$) is the model $\mathcal{M}(\mathcal{T}) = \cup\{M_\alpha^\mathcal{T} | lh(E_\alpha) : \alpha < lh(\mathcal{T})\}$.
- $\delta(\mathcal{T}) = \sup\{lh(E_\alpha) : \alpha < lh(\mathcal{T})\}$.

Note that $\mathcal{M}(\mathcal{T})$ is a premouse of height $\delta(\mathcal{T})$, which is an initial segment of every well-founded limit model $M_b^\mathcal{T} := \lim_{\alpha\in b} M_\alpha^\mathcal{T}$ (where b is a cofinal well-founded branch through $\mathcal{T}$). The model $\mathcal{M}(\mathcal{T})$ is a useful parameter to construct certain mouse operators over it in order to guide certain iteration strategies. The ordinal $\delta(\mathcal{T})$ is an ordinal whose non-Woodiness (over certain local models) is connected with the uniqueness of branches, as indicated in the following key theorem, called the Martin-Steel Theorem:

[4]In some cases where $\mathcal{T}$ and $\pi\mathcal{T}$ do not have the same tree structure, one picks some b' very close to b, and the argument will still be valid.

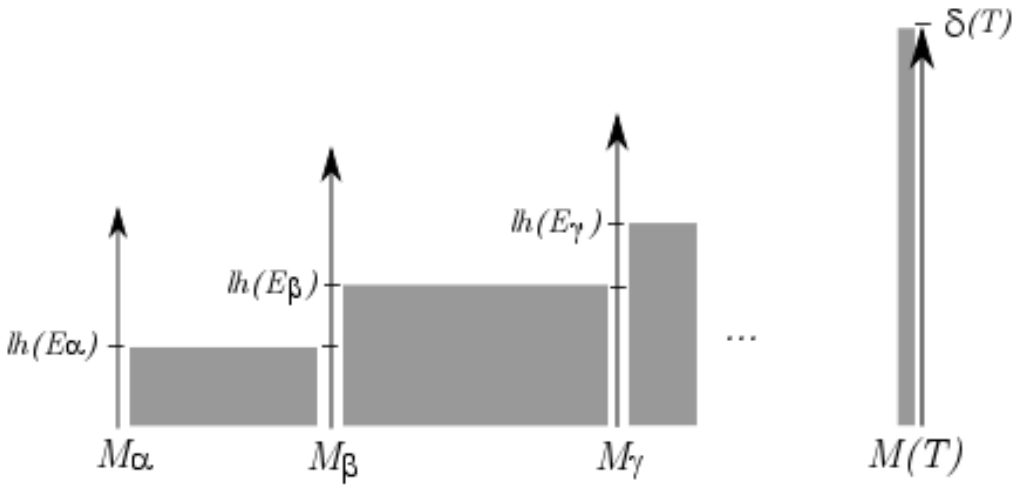


Figure 2.3: The common part model $\mathcal{M}(\mathcal{T})$.

Theorem 2.29 (Martin-Steel)**.** *Let $\mathcal{T}$ be an iteration tree of limit length on a premouse $\mathcal{M}$. Let $b \neq c$ be two cofinal branches through $\mathcal{T}$. Then $\delta(\mathcal{T})$ is Woodin over $\mathcal{M}(\mathcal{T})$ with respect to all $A \in \mathcal{M}_b^{\mathcal{T}} \cap \mathcal{M}_c^{\mathcal{T}}$.*

Proof. See [11]. □

We remark that Theorem 2.29 is an essential for the successor step in a core model induction.

2.5.3 Comparing premice

Given a premouse $\mathcal{M}$, we use $\mathcal{M}||\alpha$ to denote the initial segment of $\mathcal{M}$ up to α, including the interpretation of its top predicate E_α (which could be empty or not). We use $\mathcal{M}|\alpha$ to denote the initial segment of $\mathcal{M}$ up to α, removing the interpretation of its top predicate E_α.

It is not always the case that given premice $\mathcal{M}$ and $\mathcal{N}$, one is an initial segment of the other one. However, assuming that we have enough iterability for both $\mathcal{M}$ and $\mathcal{N}$ we can still compare them in a specific sense:

Theorem 2.30 ("Comparison Lemma"). *Let $\mathcal{P}, \mathcal{Q}$ be sound premice, and assume that both are $\kappa^+ + 1$-iterable, where $\kappa = max(|\mathcal{P}|, |\mathcal{Q}|)$. Then there are iteration trees $\mathcal{T}$ and $\mathcal{U}$ on $\mathcal{P}$ and $\mathcal{Q}$ respectively, having last models $\mathcal{M}_\infty^\mathcal{T}$ and $\mathcal{M}_\infty^\mathcal{U}$ induced by cofinal branches b and c such that:*

(a) b does not drop and $\mathcal{M}_\infty^\mathcal{T} \trianglelefteq \mathcal{M}_\infty^\mathcal{U}$, or

(b) c does not drop and $\mathcal{M}_\infty^\mathcal{U} \trianglelefteq \mathcal{M}_\infty^\mathcal{T}$.

Proof. See [11]. □

Remark: If case (a) above holds, then we say that "$\mathcal{Q}$ does not lose" in the comparison. Similarly, if case (b) holds, then we say that "$\mathcal{P}$ does not lose". If the models coiterate to a common model (that is, $\mathcal{M}_\infty^\mathcal{T} = \mathcal{M}_\infty^\mathcal{U}$), then neither model lost.

Lemma 2.31. *Let $\mathcal{P}$ and $\mathcal{Q}$ be premice, and let α be an ordinal. Assume that $\mathcal{P}$ is sound above α and projects to α, and that there is a successful terminal coiteration of $\mathcal{P}$ and $\mathcal{Q}$ above α in which $\mathcal{Q}$ does not lose. Then $\mathcal{P} \trianglelefteq \mathcal{Q}$.*

Proof. First we show that $\mathcal{P}$ does not move in the coiteration. Assume otherwise, and let E be the first extender used on the $\mathcal{P}$-side. Since the coiteration is above α, then $crit(E) > \alpha = \rho_\omega(\mathcal{P})$. By the rules of iteration trees, this implies that there must have been a drop in the $\mathcal{P}$-side, contradicting that $\mathcal{Q}$ does not lose.

Therefore $\mathcal{P} \trianglelefteq \mathcal{Q}_\infty$, where $\mathcal{Q}_\infty$ denotes the last model in the $\mathcal{Q}$-side. If the $\mathcal{Q}$-side did not move, then we are done, so assume otherwise. This implies that $\mathcal{P} \triangleleft \mathcal{Q}_\infty$. Since $\mathcal{P}$ is sound above α and projects to α, then $\mathcal{Q}_\infty \models$ "$|\mathcal{P}| = |\alpha|$". In particular, if $\beta = ON \cap \mathcal{P}$, then $|\beta| = |\alpha|$ in $\mathcal{Q}_\infty$. If F is the first extender used in the $\mathcal{Q}$-side, then $\kappa := crit(F) > \alpha$ (as the coiteration is above α).

If we let γ be the length of F, then κ is a cardinal in $J^{\mathcal{Q}}_{\gamma}$ and $J^{\mathcal{Q}}_{\gamma} = J^{\mathcal{Q}_\infty}_{\gamma}$. Also, γ is a cardinal in $\mathcal{Q}_\infty$, so $\beta < \gamma$, and there is a surjection $f : \alpha \to \beta$, $f \in J^{\mathcal{Q}_\infty}_{\gamma}$. Since $J^{\mathcal{P}}_{\gamma} = J^{\mathcal{Q}_\infty}_{\gamma}$, then $J^{\mathcal{Q}}_{\gamma} \models |\beta| \leq \alpha$, which implies that $\beta < \kappa$. But the iteration map $i : \mathcal{Q} \to \mathcal{Q}_\infty$ has critical point κ, so that $\mathcal{P}$ is also an initial segment of $\mathcal{Q}$. □

2.5.4 Q-structures

Let $\mathcal{T}$ be an iteration tree of limit length on a premouse $\mathcal{M}$, and let b be a cofinal branch though $\mathcal{T}$.

- We let $Q(\mathcal{T})$ (if it exists) be the unique premouse Q such that $\mathcal{M}(\mathcal{T}) \trianglelefteq Q$, Q is countably iterable above $\delta(\mathcal{T})$, and Q defines a minimal failure of $\delta(\mathcal{T})$ to be Woodin, via the extenders of $\mathcal{M}(\mathcal{T})$.
- We let $Q(b, \mathcal{T})$ (if it exists) to be the least initial segment of $wfp(\mathcal{M}^{\mathcal{T}}_b)$ defining a failure of $\delta(\mathcal{T})$ to be a Woodin cardinal.

Definition 2.32. a premouse $\mathcal{M}$ is *tame* if and only if for every (κ, α) extender E_α in the extender sequence of $\mathcal{M}$, $J^{\mathcal{M}}_{\alpha}$ has no Woodin cardinals larger or equal to κ.

Definition 2.33. Let $\mathcal{M}$ be a tame premouse.

- We denote by $\Sigma^t_{\mathcal{M}}$ the (possibly partial) iteration strategy defined as follows: given an iteration tree $\mathcal{T}$ on $\mathcal{M}$ of limit length, let $\Sigma^t_{\mathcal{M}}(\mathcal{T})$ be the unique cofinal branch b such that $Q(\mathcal{T}) = Q(b, \mathcal{T})$.
- Given a mouse operator J and a set Z, we say that $\Sigma^t_{\mathcal{M}}$ is *J-guided* on Z if and only if for every iteration tree of limit length $\mathcal{T}$ on $\mathcal{M}$ such that $\mathcal{T} \in Z$ and $\mathcal{T}$ is played according to $\Sigma^t_{\mathcal{M}}$, we have that $b := \Sigma^t_{\mathcal{M}}(\mathcal{T})$ exists, and $Q(b, \mathcal{T}) \trianglelefteq J(\mathcal{M}(\mathcal{T}))$.

Lemma 2.34. *Let $\mathcal{T}$ be an iteration tree of limit length on a premouse $\mathcal{P}$. Then there is at most one cofinal branch b satisfying: (i) $Q(b,\mathcal{T})$ exists and is $|\delta(\mathcal{T})|^+ + 1$ iterable, (ii) $Q(b,\mathcal{T}) \trianglelefteq wfp(M_b^{\mathcal{T}})$ and (ii) $\delta(\mathcal{T})$ is a cutpoint of $Q(b,\mathcal{T})$.*

Proof. See [11]. □

Lemma 2.35 (Q-reflection)**.** *Let $\mathcal{M}$ be a tame premouse, and Z be a transitive and rudimentary closed set such that $HC \subseteq Z$. Let J be a mouse operator such that if $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ played according to $\Sigma^t_{\mathcal{M}}$ and $\mathcal{T} \in Z$, then $J(\mathcal{M}(\mathcal{T}))$ exists.*

Suppose that for every countable premouse $\mathcal{P}$ that elementarily embeds into $\mathcal{M}$, $\Sigma^t_{\mathcal{P}}$ is J-guided on HC. Then $\Sigma^t_{\mathcal{M}}$ is J-guided on Z.

2.5.5 Sharps

Let a be a set. The statement "$a^{\#}$ exists" has several equivalent definitions. One way is assert the existence of a non-trivial elementary embedding $j : L[a] \to L[a]$. Or one can define $a^{\#}$ to be a specific theory assuming that certain class of indiscernibles exists. We shall take here another approach, which is to define $a^{\#}$ as a specific premouse. However, in order to do this, we need to assume that a is a self-wellordered set.

Definition 2.36. Let a be a self-wellordered set, with $\gamma = \sup(a)$. We define $a^{\#}$, if it exists, as the unique a-premouse $\mathcal{M}$ such that:

- $M = (J^a_\alpha, \in, \vec{E}, E_\alpha)$, with $\vec{E}_\beta = \varnothing$ for every $\beta < \alpha$; E_α is a (κ, α) extender with support $\kappa^{+J^a_\alpha}$ (so E_α is essentially a measure).
- M is sound.
- M is countably iterable.

The uniqueness of $a^\#$ follows from a comparison argument, using iterability. Since the extender sequence of $a^\#$ carries no information (except its top extender, which is essentially a measure), one can simplify notation and express $a^\#$ as $(J^a_\alpha, \in, U)$, where U is a normal and amenable J^a_α-measure on κ where $\gamma < \kappa < \alpha$.

The following theorem states that sharps for elements in the ground model cannot be added by forcing.

Theorem 2.37. *Let $\mathbb{P}$ be a forcing, and let G be a $\mathbb{P}$-generic filter over V. Let $x \in V$ and assume that $y \in V[G]$ is such that $V[G] \models$ "$y = x^\#$". Then $x^\#$ exists in V, and is equal to y.*

Theorem 2.38. *Let M be a proper class inner model, and let $a \in M$ be such that $M \models$ "$a^\#$ exists". Then $a^\#$ exists.*

Since every measurable cardinal is a Ramsey cardinal, the following Lemma guarantees that "sharps exist below a measurable":

Lemma 2.39. *Let κ be a Ramsey cardinal. Then for every $a \in V_\kappa$, $a^\#$ exists.*

Lemma 2.40. *Let G be a $\mathbb{P}$-generic filter over V, and let τ be a $\mathbb{P}$-name. If $\tau^\#$ exists, then $V[G] \models$ "(τ^G) exists".*

Lemma 2.41. *Let $n < \omega$. Let Ω be an uncountable regular cardinal, and assume that V_Ω is closed under the $M_n^\#$ operator. Let $a \in V_\Omega$ and assume that $M_{n+1}^\#(a)$ exists. Then $M_{n+1}^\#(a)$ is iterable with respect to trees in V_Ω, via a $M_n^\#$-guided strategy.*

Theorem 2.42. *Let Ω be measurable, and $\mathbb{P} \in V_\omega$ be a forcing. Let $n < \omega$ and assume that V_Ω is closed under the $M_n^\#$ operator. Then for every $a \in V_\Omega$ and $y \in V[G]$ (where G is $\mathbb{P}$-generic over V), if $V[G] \models$ "$y = M_{n+1}^\#(a)$", then in V, $M_{n+1}^\#(a)$ exists and is equal to y.*

Definition 2.43 ($M_n^{\#\#}$)**.** Let $n < \omega$, and a be a self-wellordered set. We define $M_n^{\#\#}(a)$ (if it exists) to be the least active sound a-premouse that projects to a and is closed under the $M_n^\#$ operator.

Equivalently, it can be seen that $M_n^{\#\#}(a)$ is the least active level of $Lp(a)$ that is closed under the $M_n^{\#}$ operator.

Lemma 2.44. *Let κ be a cardinal, let $n < \omega$, and assume that H_{κ^+} is closed under $M_n^{\#}$. Let $a \in H_\kappa$, and calling $\mathcal{P} = Lp(a)$, assume that $cf(\kappa^{+\mathcal{P}}) < \kappa$. Then $M_n^{\#\#}(a)$ exists.*

Proof. Let $\alpha = \kappa^{+\mathcal{P}}$, so that $cf(\alpha) < \kappa$. Let $X \preceq H_{\kappa^+}$ be such that $X \cap \alpha$ is cofinal in α, $|X| = cf(\alpha) \cdot \omega_1$. Let N be the transitive collapse of X, and $\pi : X \to N$ the corresponding uncollapsing map. Let $\beta = crit(\pi)$ (so $\beta < \kappa$). Let $\bar{\kappa} = \pi^{-1}(\kappa)$ and $\bar{\alpha} = \pi^{-1}(\alpha)$. Also, let $\mathcal{Q} = \pi^{-1}(\mathcal{P}|\alpha)$.

Claim 1: for every $\bar{\kappa}$-sound premouse $\mathcal{M}$ such that $\mathcal{Q} \triangleleft \mathcal{M}$, $\rho_\omega(\mathcal{M}) \geq \bar{\alpha}$.
Proof of claim: Assume otherwise, so let $\mathcal{M}$ be a $\bar{\kappa}$-sound premouse such that $\mathcal{Q} \triangleleft \mathcal{M}$ and $\rho_\omega(\mathcal{M}) < \bar{\alpha}$. Let $n < \omega$ such that $\rho_{n+1}(\mathcal{M}) < \bar{\alpha} \leq \rho_n(\mathcal{M})$.

Let F be the long extender derived from $\pi \restriction \mathcal{Q}$, and $\tilde{M} := Ult_n(\mathcal{M}, F)$. The fact that $X \cap \alpha$ is cofinal in α implies that $\mathcal{P}|\alpha \triangleleft \tilde{M}$. Also, note that since $\rho_{n+1}(\tilde{\mathcal{M}}) = \rho_{n+1}(\mathcal{M})$, then $\rho_\omega(\tilde{\mathcal{M}}) \leq \kappa$. This implies by the definition of lower part that $\tilde{\mathcal{M}}$ is an initial segment of $\mathcal{P}$, so that $|\alpha| \leq \kappa$ in $\mathcal{P}$, a contradiction. This completes the proof of claim 1.

Let E be the $(\beta, \beta^{++\mathcal{P}})$ $\mathcal{Q}$-extender derived from π.

Claim 2: $\beta^{+\mathcal{Q}} = \beta^{+\mathcal{P}}$.
Proof of claim: assume otherwise. By the Condensation Lemma, $\mathcal{Q}|\beta^{+\mathcal{Q}} \triangleleft \mathcal{P}$, so there must be some $\mathcal{Q}'$ such that $\mathcal{Q}' \trianglelefteq \mathcal{P}$, $\mathcal{P}$ projects to or below β and $\mathcal{Q}|\beta^{+\mathcal{Q}} \trianglelefteq \mathcal{Q}'$. If we compare $\mathcal{Q}$ and $\mathcal{Q}'$ via a coiteration, then $\mathcal{Q}$ does not move since it is a lower part model, and so we obtain a premouse $\mathcal{R}$ such that $\mathcal{R}$ is $\bar{\kappa}$-sound, $\mathcal{Q} \trianglelefteq \mathcal{R}$ and $\mathcal{R}$ projects below $\bar{\alpha}$, a contradiction.

By claim 2: E is also a $\mathcal{P}$-extender, and so $\mathcal{N} := (\mathcal{P}|\beta^{++\mathcal{P}}, \in, E)$ is a premouse. One can argue that moreover, it is countably iterable. This implies that $M_n^{\#\#}(a)$ exists (it may not be that $\mathcal{N} = M_n^{\#\#}(a)$, as $M_n^{\#\#}(a)$ may be smaller). $\square$

Lemma 2.45. *Let $P = M_n^{\#\#}(a)$ for some set a. Then no level Q of P is $M_{n+1}^{\#}(a)$-like.*

Lemma 2.46. *Let $n < \omega$. Assume that $M_n^{\#}$ is total on H_θ with θ a regular cardinal, and let $a \in H_\theta$. Assume that $P = M_n^{\#\#}(a)$ exists. Then P is iterable for trees $\mathcal{T} \in H_\theta$, via a $M_n^{\#}$-guided iteration strategy.*

Proof. Let $\pi : \bar{P} \to P$ be elementary, where $\bar{P}$ is countable transitive. Let T be an iteration tree of limit length on $\bar{P}$ such that $T \in HC$ and T is $M_n^{\#}$-guided. Since P is countably iterable, there is a cofinal well-founded branch b. We will prove that (i) $Q(b, t)$ exists, and (ii) $Q(b, T) \trianglelefteq M_n(M(T))$.

(i) Assume towards a contradiction that $Q(b, T)$ does not exist. Then b does not drop and there is an elementary map $i : \bar{P} \to M_b^T$ given by the branch embedding. The model P is closed under $M_n^{\#}$ so by elementarity of π and i, we conclude that there is a $M_n^{\#}(M(T))$-like structure N such that $N \triangleleft M_b^T$. Note that $\delta(T)$ is Woodin in N, so N is $M_n^{\#}(M(T))$-like. By elementarity, there is some N' such that N' is $M_{n+1}^{\#}(a)$-like and $N' \triangleleft P$, contradicting Lemma 2.45.

(ii) Assume for simplicity that b does not drop (the other case is similar by considering the last model to drop). If $\delta(T) = o(M_b^T)$, then $Q(b, T) = M_b^T = M(T)$, and so $Q(b, T) \trianglelefteq M_n(M(T))$. Assume that $\delta(T) \in M_b^T$. Then by elementarity, there is some N such that N is $M_n^{\#}(\delta(T))$-like and $N \trianglelefteq M_b^T$. Moreover, it can be seen that N is indeed $M_n^{\#}(M(T))$. Using Lemma 2.45 plus elementarity, it can be seen that $\delta(T)$ cannot be Woodin in N. Thus by minimality of $Q(b, T)$ we conclude that $Q(b, T) \trianglelefteq N$.

We have shown that $\bar{P}$ has a $M_n^\#$-guided iteration strategy in HC. Therefore by the Q-reflection Lemma, P has a $M_n^\#$-guided iteration strategy for trees $T \in H_\theta$.

□

Lemma 2.47. *Let $n < \omega$. Assume that $M_n^\#$ is total on H_θ with θ a regular cardinal, and let $a \in H_\theta$. Assume that $P = M_{n+1}^\#(a)$ exists. Then P is iterable for trees $T \in H_\theta$, via a $M_n^\#$-guided iteration strategy.*

Lemma 2.48. *Let $n < \omega$. Suppose that there is proper class model M with $n+1$ Woodin cardinals, such that M is set-length iterable. Then for every set a, $M_n^\#(a)$ exists and is set-length iterable.*

Lemma 2.49. *Let $n < \omega$, and let θ be an uncountable regular cardinal. Assume that H_θ is closed under the $M_n^\#$ operator. Let $\mathbb{P} \in H_\theta$ be a forcing, and let G be $\mathbb{P}$-generic over V. Then in $V[G]$, H_θ is closed under the $M_n^\#$ operator.*

Proof. Let $e \in H_\theta^{V[G]}$, say $e = \dot{e}^G$, where $\dot{e} \in H_\theta^V$. Let $N \in V$ be a transitive model of ZFC^- such that $N \in H_\theta^V$, and $\dot{e}, \mathbb{P} \in N$. It can be seen that $M_n^\#(N)[G] = M_n^\#(N')$, where $N' = N[G]$. Working in the model $M_n^\#(N')$, we can carry out the fully background certificate construction over the set e (note that $e \in M_n^\#(N')$), obtaining at least n Woodin cardinals. Therefore, $M_n^\#(e)$ exists inside $M_n^\#(N')$, and is iterable via copying from the fully background certificate construction. □

Lemma 2.50. *Let $n < \omega$ and a be a self-wellordered set. Assume that $M_{n+1}^\#(a)$ exists. Then $M_n^{\#\#}(a)$ exists.*

2.5.6 Threadability and mouse reflection

We have defined the mouse-operators $M_n^\#$ and $M_n^{\#\#}$. In general a mouse operator is a partial function J on the universe V which associates to each set $a \in Dom(J)$ a mouse (iterable

premouse) $J(a)$, and the function J is well-behaved (see [8] for details). During a Core Model induction one is sometimes interested in showing that if a mouse operator J is total in H_κ (where J and κ are specific to the situation of interest), then J is also total in H_{κ^+}. This phenomenon is an instance of what is known as *mouse reflection*. In some situations, elementary maps with critical point κ can give mouse reflection. Another property that entails mouse reflection is a compactness property relative to cardinals called *threadability*. In what follows, we introduce this notion and state the pertinent result related to the mouse operators $M_n^\#$ and $M_n^{\#\#}$.

Definition 2.51. Let κ be an uncountable regular cardinal. A *coherent sequence* (relative to κ) is a sequence $\langle C_\alpha : \alpha < \kappa \rangle$ satisfying:

(a) For all $\alpha < \kappa$, $C_\alpha \subseteq \alpha$ is a club in α.

(b) For all $\alpha < \kappa$, and for every limit point β of C_α, $C_\beta = C_\alpha \cap \beta$.

A *thread* through $\langle C_\alpha : \alpha < \kappa \rangle$ is a club $E \subseteq \kappa$ such that for all $\alpha, \beta \in E$, if $\beta < \alpha$ then $C_\beta = C_\alpha \cap \beta$.

Definition 2.52. Let κ be an uncountable regular cardinal. We say that κ is *threadable* if every coherent sequence $\langle C_\alpha : \alpha < \kappa \rangle$ has a thread $E \subseteq \kappa$.

Theorem 2.53. *Let $\kappa > \omega$ be a regular cardinal and assume that κ is threadable. Then, for every $n < \omega$, and for $J = M_n^\#$ or $J = M_n^{\#\#}$, we have:*

$$\textit{If } J \textit{ is total in } H_\kappa, \textit{ then } J \textit{ is total in } H_{\kappa^+}.$$

Proof. This is a special case of a general result for any mouse operator. See [8] for details. □

2.5.7 The Core Model

The Core Model is a premouse which exists only under certain large anti-cardinal hypotheses, and has many useful properties. The reader can consult [12] to see how the core model K is defined under the anti-large cardinal hypothesis of no inner models with a Woodin cardinal. A measurable Ω is used in the construction. The basic idea of the construction is the following:

- Construct an auxiliary premouse level by level called K^c, by searching in V for extenders that can be put in the extender sequence of K^c.
- Define K to be the transitive collapse of the intersection of all thick hulls of K^c.

Here are some important properties of K:

Theorem 2.54. *Assume that there is no inner model with a Woodin cardinal, let Ω be measurable, and let K be the core model coming from K^c below Ω. We have:*

(a) K is $\Omega + 1$-iterable.

(b) If $\mathbb{P} \in V_\Omega$ is a forcing and G is a $\mathbb{P}$-generic filter over V, then the core model $K^{V[G]}$ exists in $V[G]$ and moreover $K = K^{V[G]}$.

(c) For every singular cardinal κ, $\kappa^{+K} = \kappa^{+V}$.

(d) For every normal ultrafilter U on Ω, we have: $\{\kappa \in \Omega : \kappa^{+V} = \kappa^{+V[G]}\} \in U$.

(e) The only elementary embedding $j : K \to K$ is the identity.

Proof. See [12]. □

A phalanx is a system of several premice along with a sequence of ordinals. Iteration trees can be defined on a phalanx, sometimes called "pseudo-iteration trees", which should be thought as trees having several roots, one for each mouse in the phalanx, although they can formally be defined as a tree with one root, for convenience purposes. Phalanxes are introduced to guarantee that certain iteration maps have sufficiently large critical points.

In what follows we introduce phalanxes of length 2 (that is, consisting on only 2 premice). For a more general definition, the reader can consult [12]

Definition 2.55. A *phalanx* is a triple $\mathcal{B} = (\mathcal{P}, \mathcal{Q}, \alpha)$ where $\mathcal{P}$ and $\mathcal{Q}$ are premice and λ is an ordinal, satisfying the following conditions:

- λ is a cardinal in $\mathcal{P}$ and $\mathcal{Q}$.
- $\mathcal{P}$ and $\mathcal{Q}$ agree below λ.

The ordinal λ in the previous definition is called the *exchange ordinal* of the phalanx.

When considering an iteration tree on a phalanx $\mathcal{B} = (\mathcal{P}, \mathcal{Q}, \alpha)$, the exchange ordinal α tells us to which model to apply an extender E: the first time we have choose an extender whose critical point is larger than λ, then we apply the extender to the model $\mathcal{Q}$. In more detail:

Definition 2.56. An iteration tree on a phalanx $\mathcal{B} = (\mathcal{P}, \mathcal{Q}, \alpha)$ of length θ is a system $\mathcal{T} = \langle E_\beta : 2 \leq \beta + 1 < \theta \rangle$ (along with a tree order T), premice $\mathcal{M}_\alpha$ $(\alpha < \theta)$

- $\mathcal{M}_0 = \mathcal{P}$, $\mathcal{M}_1 = \mathcal{Q}$ and $0T1$.
- $\lambda < lh(E_1)$ and if $2 \leq \beta + 1 < \gamma + 1 < \theta$, then $lh(E_\beta) < lh(E_\gamma)$.
- Given $2 \leq \beta + 1 < \theta$: the T-predecessor of $\beta + 1$ is $\mathcal{P}$ if $crit(E_\beta) < \lambda$, and otherwise it is the least ordinal γ such that $\gamma \geq 1$ and $crit(E_\beta) < \nu(E_\alpha)$. The definition of $\mathcal{M}_{\beta+1}$ as an ultrapower and corresponding embedding follows the usual rules of iteration trees.

Definition 2.57. A *phalanx embedding* from (M_0, M_1, α) to (N_0, N_1, β) is a pair of maps (σ_0, σ_1) such that:

- $\sigma_0 : M_0 \to N_0$ and $\sigma_1 : M_1 \to N_1$ are Σ_0-preserving and cardinal preserving.
- $\sigma_0 \restriction \alpha = \sigma_1 \restriction \alpha$.
- $\sigma_1'' \alpha \subseteq \beta$ and $\sigma_1(\alpha) \geq \beta$.

Under these circumstances, we write: $(\sigma_0, \sigma_1) : (M_0, M_1, \alpha) \to (N_0, N_1, \beta)$.

Lemma 2.58. *Let $(\sigma_0, \sigma_1) : (M_0, M_1, \alpha) \to (N_0, N_1, \beta)$ be a phalanx embedding. The the non-iterability of (M_0, M_1, α) implies the non-iterability of (N_0, N_1, β).*

The following is a useful criterion to determine whether an extender belongs to the extender sequence of K. It can also be seen as a "maximality" property of the extender sequence of the Core Model.

Theorem 2.59 (Maximality)**.** *Let W be a weasel witnessing that the Core Model K exists. Let F be a (κ, λ) extender that coheres with W (that is, $(J_\alpha^W, \in, E^W \restriction \lambda, F)$). Then: F is on the W-sequence if and only if the phalanx $(W, Ult(W, F), \lambda)$ is iterable.*

Proof. This is a special case of Theorem 8.6 in [12]. □

One can similarly define the core model $K(a)$ relative to a prewellordered set a.

The following is the Dichotomy Theorem for the Core Model, which will play a central role in our induction:

Theorem 2.60 (Steel)**.** *Let Ω be a measurable cardinal and let $n < \omega$. Suppose that V_Ω is closed under the "$\Omega + 1$-iterable $M_n^\#$" operator (meaning: for every $a \in V_\Omega$, $M_n^\#(a)$ exists and is $\Omega + 1$-iterable). Then one of the following holds:*

(a) V_Ω *is closed under the "*$\Omega+1$*-iterable* $M_{n+1}^{\#}$*" operator.*

(b) For some $a \in V_\omega$*,* $K^c(a)$ *is* n*-small, has no Woodin cardinals, and is* $\Omega+1$*-iterable.*

Proof. See [8]. □

Note: clause (b) implies that the Core Model $K(a)$ exists, is n-small and has no Woodin cardinals.

We remark that in the previous theorem, the conclusion in (b) can be read as the existence of the core model K seen as an F-premouse over a, where F is the $M_n^{\#}$-operator. That means that $K^c(a)$ is built using the F operator instead as the rudimentary closure operator to form its next level. For more details about this construction, the reader can consult [8].

Definition 2.61. A premouse $\mathcal{M}$ is *properly small* if and only if:

$$\mathcal{M} \models \text{“There are no Woodin cardinals, and there is a largest cardinal”}.$$

A phalanx is called *properly small* is every premouse of it is properly small.

Lemma 2.62. *Assume that the* K^c*-construction does not have any Woodin cardinals, and let* W *be an* $(\Omega+1)$*-iterable weasel such that* Ω *is thick in* W*. Then* W *does not have Woodin cardinals.*

Theorem 2.63. *If* $K^c \models$ *"There are no Woodin cardinals", then* K *elementarily embeds into every universal weasel.*

Proof. See Theorem 8.10, [12]. □

The following result is well-know:

Theorem 2.64. *Assume that inside a transitive model M, the core model K exists and is $(\omega, \Omega+1)$-iterable. Let $\kappa < \Omega$ be regular uncountable cardinal. Let $\mathbb{P} \in (H_\Omega)^M$ be a forcing. Let W be a soundness witness for $K||\kappa$. Then for every $\mathbb{P}$-generic filter G over M, if $Ult(M,G)$ is wellfounded and $crit(j) = \kappa$ (where $j : M \to Ult(M,G)$ is the ultrapower embedding), then the models W and $j(W)$ agree on the cardinal successor of κ.*

Proof. Since $M \models$ "W is thick", and $\mathbb{P}$ is small, then $M[G] \models$ "W is thick". Now we claim that $M[G] \models$ "$j(W)$ is thick". By universality of W and $j(W)$, these two models coiterate in $M[G]$ to a common model $\mathcal{M}_\infty$, and both critical points of the corresponding iteration maps are at least κ (as $crit(j) = \kappa$). This implies that $\mathcal{P}(\kappa)^W = \mathcal{P}(\kappa)^{j(W)}$, and so $\kappa^{+W} = \kappa^{+j(W)}$. □

Definition 2.65. Let W be a premouse, and α an ordinal. We define the relation $<_S$ on premice as follows: Given premice Q_1, Q_2, we say that $Q_1 <_S Q_2$ if the following holds: there is a normal iteration tree $\mathcal{T}$ on the phalanx (W, Q_2, α) such that $Q_1 \trianglelefteq \mathcal{M}_\infty^\mathcal{T}$, and one of the following holds:

(a) W is on the main branch of $\mathcal{T}$, or

(b) Q_2 is on the main branch of $\mathcal{T}$, and either this main branch truncates or $Q_1 \triangleleft \mathcal{M}_\infty^\mathcal{T}$.

Theorem 2.66. *Assume that the core model K exists and has no Woodin cardinals. If W is a soundness witness for an initial segment of K, then for every ordinal α, the relation $<_S$ is well-founded below W.*

Theorem 2.67. *Let R be a coarse premouse, closed under sharps, whose K^c-construction does not reach $M_1^\#$. Let W be a soundness witness for a level of $(K^c)^R$. Let $\mathcal{T}$ be an iteration tree on the phalanx $(W, Ult(W,F), \lambda)$, where F is a (κ,λ)-extender over W. Let $Z \preceq H_\theta^R$ be countable such that $W, F, \mathcal{T} \in Z$. Let H be the transitive collapse of Z, and let $\pi : H \to H_\theta^R$*

be the corresponding uncollapsing map. Let $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$. *Then we have:*

$$V \models \text{“}\mathcal{T} \text{ has a cofinal, well-founded branch”} ,$$

if and only if

$$H \models \text{“}\mathcal{T} \text{ has a cofinal, well-founded branch”} ,$$

2.6 Consistency results

In this section we list some important consistency results involving properties of ideals.

Theorem 2.68. *The following statements are equiconsistent (relative to ZFC):*

(a) There is a measurable cardinal.

(b) The non-stationary ideal NS_{ω_1} *is precipitous.*

(c) There is a precipitous ideal on ω_1.

(d) There is a precipitous ideal.

Proof. See [13]. □

Kunen showed that from a huge cardinal one can obtain a model with a saturated ideal on ω_1. Some years after, the large cardinal strength needed was reduced considerably to a supercompact, and later to a Woodin cardinal, obtaining the exact strength needed:

Theorem 2.69. *The following statements are equiconsistent (relative to ZFC):*

1. *There is a Woodin Cardinal.*

2. *There is a strong ideal on ω_1.*

3. *There is a saturated ideal on ω_1.*

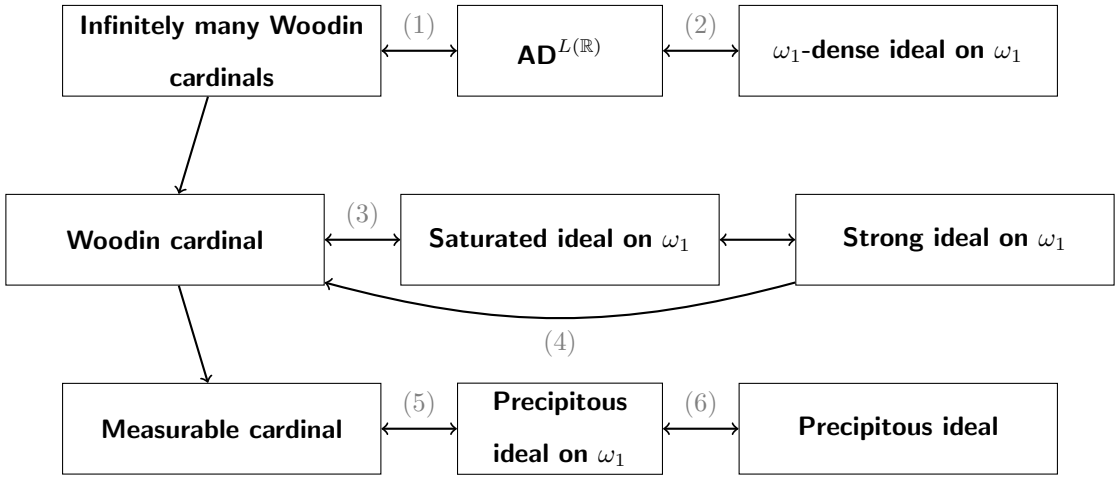
Proof. See [9], [1] and [12]. □

Theorem 2.70 (Woodin)**.**

(a) *Assume that there is an ω_1-dense ideal on ω_1. Then AD (the axiom of determinacy) holds in $L(\mathbb{R})$.*

(b) *Assume that AD holds in $L(\mathbb{R})$. Then there is a model in which there is an ω_1-dense ideal on ω_1.*

The following diagram shows the consistency strength of some ideal axioms. The arrows are not implications, but relations in terms of consistency strength.



(1) is due to Woodin, Martin-Steel. (2) is due to Woodin. (3) ($\rightarrow$) is due to Shelah, ($\leftarrow$) is due to Jensen, Steel. (4) is due to Claverie-Schindler. (5),(6) are due to Jech, Magidor, Mitchel, Prikry.

The following are two results relating the consistency strength of the axioms $ProjCatch^*(I)$ and $StatCatch^*(I)$.

Theorem 2.71 (Cox, Zeman)**.** *Let $\kappa < \delta$ be cardinals with δ inaccessible and assume that κ is δ-supercompact. Then for every $n \in \omega$ such that $n \geq 2$, there is a model having a normal ideal I on ω_n such that $ProjCatch^*(I)$ holds, and I is not strong.*

Proof. See [2] (in which a stronger result of this theorem is presented). □

Theorem 2.72 (Cox, Zeman)**.** *Let Ω be a measurable cardinal and let I be an ideal on ω_2 such that $StatCatch^*(I)$ holds. Then there is an inner model with a Woodin cardinal.*

Proof. See [2]. □

In the next chapter we shall obtain, from $StatCatch^*(I)$ (plus some additional conditions), for every finite n an inner model with n Woodin cardinals. Thus we obtain Projective Determinacy as a consistency strength lower bound.

Chapter 3

Stationary Catching and $M_n^{\#}$

In this chapter we prove the main result of this work: assuming a variant of Stationary Catching (plus some additional hypotheses on the universe), obtain, for every $n < \omega$, an inner model with n Woodin cardinals.

Theorem 3.1. *Assume $2^{\omega} = \omega_1$, $2^{\omega_1} = \omega_2$, $2^{\omega_2} = \omega_3$, and that ω_3 is threadable. Let I be an ideal on ω_2, and suppose that StatCatch$^*(I)$ holds. Then for every $n < \omega$ and for every $a \in H_{\omega_4}$, $M_n^{\#}(a)$ exists and is iterable with respect to iteration trees in H_{ω_4}.*

From $M_n^{\#}(\varnothing)$ we can obtain $M_n(\varnothing)$, an inner model having n Woodin cardinals. Thus we have:

Corollary 3.2. *Assume that $2^{\omega} = \omega_1$, $2^{\omega_1} = \omega_2$, $2^{\omega_2} = \omega_3$, ω_3 is threadable and that for some ideal I on ω_2 StatCatch$^*(I)$ holds. Then for every $n < \omega$, there is an inner model with n Woodin cardinals.*

Assume that ω_3 is threadable. This guarantees the following mouse reflection principle (see Theorem 2.53): whenever J is total on H_{ω_3}, J is also total on H_{ω_4} (where $J = M_n^{\#}$ or

$J = M_n^{\#\#}$, for some $n < \omega$).

Lemma 3.3. *Assume $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, $2^{\omega_2} = \omega_3$. Let I be an ideal on ω_2, and suppose that $StatCatch^*(I)$ holds. Then H_{ω_3} is closed under sharps, that is: for every $a \in H_{\omega_3}$, $a^\#$ exists.*

Proof. First we show that I is somewhere precipitous: By self-genericity, for stationarily many structures $M \preceq H(\theta_1)$, $H_M \models$ "Forcing with (a restriction of) $\mathbb{P}_I^M$ gives a well-founded ultrapower, and the critical point of the ultrapower embedding is α_M". So fixing any such model M, we obtain by elementarity that $V \models$ "Forcing with (a restriction of) $\mathbb{P}_I$ gives a well-founded ultrapower, and the critical point of the ultrapower embedding is ω_2". Let $\mathbb{P}'$ be such restriction. So for any $\mathbb{P}'$-generic filter G over V, we have that $Ult(V, G)$ is wellfounded and $j_G : V \to Ult(V, G)$ has critical point ω_2.

First we show that for every bounded subset a of ω_2, $a^\#$ exists. For such an a, let G be a $\mathbb{P}'$-generic filter over V, and let $j_G : V \to N$ be the generic embedding induced by G. Since $crit(j_G) = \omega_2$, then $j_G(a) = a$. So in $V[G]$, the map $j_G \restriction L[a]$ is a nontrivial elementary embedding from $L[a]$ to $L[a]$. This implies that $a^\#$ exists in $V[G]$. Since sharps cannot be added by forcing (see Lemma 2.37), then $a^\#$ exists in V.

We have proved that

$$V \models \text{ "every bounded subset of } \omega_2 \text{ has a sharp"},$$

so that by elementarity,

$$N \models \text{ "every bounded subset of } j_G(\omega_2) \text{ has a sharp"}.$$

Now let $a \subseteq \omega_2$. Since $crit(j_G) = \omega_2$, then $a = j_G(a) \cap \omega_2$, so $a \in N$. Also, a is a bounded subset of $j_G(\omega_2)$. Therefore $N \models$ "$a^{\#}$ exists". Since $N \subseteq V[G]$, this implies that $V[G] \models$"$a^{\#}$ exists". Forcing does not add sharps (and $a \in V$), so $V \models$"$a^{\#}$ exists".

Since every element in H_{ω_3} can be coded by a subset of ω_2, we are done. □

Lemma 3.3 provides the first step in the core model induction to prove Theorem 3.1. The following is the structure of the induction (where $n < \omega$):

(i) Assume that H_{ω_4} is closed under $M_n^{\#}$.

(ii) H_{ω_2} is closed under $M_n^{\#\#}$.

(iii) H_{ω_3} is closed under $M_n^{\#\#}$.

(iv) H_{ω_4} is closed under $M_n^{\#\#}$.

(v) H_{ω_2} is closed under $M_{n+1}^{\#}$.

(vi) H_{ω_3} is closed under $M_{n+1}^{\#}$.

(vii) H_{ω_4} is closed under $M_{n+1}^{\#}$.

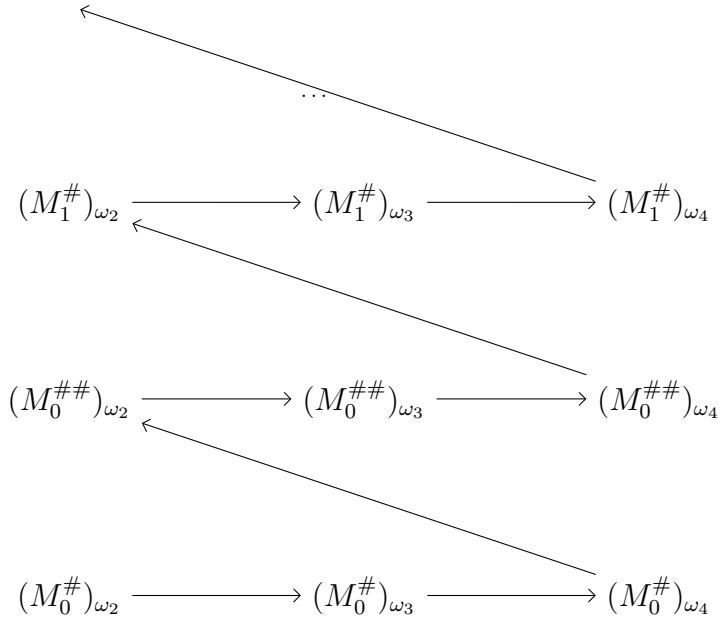
Here is the proof of the theorem in a nutshell:

Proof. (of Theorem 3.1): We proceed by induction on $n < \omega$. First let $n = 0$. By Lemma 3.3 H_{ω_3} is closed under $M_0^{\#}$; by the threadability of ω_3 and mouse reflection (Theorem 2.53), the same holds for H_{ω_4}.

Let $n < \omega$ and assume that H_{ω_4} is closed under $M_n^{\#}$. Using this hypothesis, we can prove (ii) (see 3.4), and then derive (iii) from (ii). By mouse reflection we obtain (iv). This allows

us to prove (v) (see Theorem 3.5), from which we can derive (vi). Again by mouse reflection, we obtain (vii), that is, that H_{ω_4} is closed under $M_{n+1}^{\#}$. □

During the next sections we will provide the remaining details described in the proof of Theorem 3.1. The diagram below illustrates how the induction goes through (so the arrows don't strictly indicate direct implication, but rather the order of steps of the argument): $(J)_\kappa$ abbreviates "The mouse operator J is total in H_κ".



3.1 H_{ω_2} is closed under $M_n^{\#\#}$

In this section we see how we can obtain $M_n^{\#\#}$ for sets in H_{ω_2} using some of our hypotheses, by assuming that $M_n^{\#}$ is total in H_{ω_2}.

Theorem 3.4. *Let $n < \omega$. Assume that $2^\omega = \omega_1$, that there is an ideal I on ω_2 such that StatCatch$^*(I)$ holds, and that $M_n^{\#}$ is total on H_{ω_2}. Then $M_n^{\#\#}$ is total on H_{ω_2}.*

Proof. Let $a \in H_{\omega_2}$, so w.l.o.g., let $a \subseteq \omega_1$. We want to show that $M_n^{\#\#}(a)$ exists. We assume the contrary, and seek a contradiction.

Let $Z = Lp(a)$. Let S be the collection of all structures $M \in SG^*(I)$ such that $Z|\omega_3, a \in M$, and $\alpha_M > sup(a)$. S is a stationary set in $\mathcal{P}(H(\theta_1))$. Let $N = Z|\omega_2^{+Z}$. For each $M \in S$, let $N_M = \sigma_M^{-1}(N)$, so $H_M \models$ "$N_M = Lp(a)|\alpha_M^{+Lp(a)}$" (note that $\sigma_M^{-1}(a) = a$). Note that N_M is an initial segment of N, as every initial segment of N_M is iterable in V.

Claim 1: $\alpha_M^{+N_M} = \alpha_M^{+N}$.
Proof of claim: Let $\tilde{N}_M = (Lp(a))^{\tilde{H}_M}$. Since $crit(k_M) > \alpha_M^{+H_M}$, then $\alpha_M^{+\tilde{N}_M} = \alpha_M^{+N}$. So it is enough to show that $\alpha_M^{+\tilde{N}_M} = \alpha_M^{+N_M}$. Assume towards a contradiction that $\alpha_M^{+\tilde{N}} > \alpha_M^{+N}$. Then there is some $P \in \tilde{H}_M$ (and thus $P \in H_M[U_M]$) such that $P \triangleleft \tilde{N}_M$, $\alpha_M^{+N_M} = \alpha_M^{+P}$, and there is a P-definable surjection $f : \alpha_M \to \alpha_M^{+N_M}$. Note that P cannot belong to H_M.

Let $\mathbb{Q} = \mathbb{P}_I^M \times Col(\omega, \omega_3^{H_M})$. We shall show that P is definable from a in every $\mathbb{Q}$-generic extension of H_M. Fix a $\mathbb{Q}$-generic filter over H_M. If we show that P is iterable inside $H_M[G]$, and so P can be defined as the smallest a-mouse over which a surjection $f : \alpha_M \to \alpha_M^{+N_M}$ can be defined. By homogeneity of the forcing $\mathbb{Q}$, we conclude that $P \in H_M$, which gives a contradiction.

We show the iterability of P inside $H_M[G]$, with respect to trees in H_{ω_4}. By a lemma, $H_M[G]$ is closed under $M_n^{\#}$. Working inside $H_M[G]$, let $T \in H_{\omega_4}$ be an iteration tree of limit length on P. Let X be a countable structure with $X \preceq H_\theta$ (θ sufficiently large), such that $a, P, T, \mathcal{M}(T) \in X$ and $a \subseteq X$ (note that a is countable in $H_M[G]$). Let H be the transitive collapse of X, with $\pi : H \to H_\theta$ the corresponding uncollapsing map. Let $\bar{P} = \pi^{-1}(P)$, $\bar{T} = \pi^{-1}(T)$, $\bar{M} = \pi^{-1}(\mathcal{M}(T))$, $\bar{a} = \pi^{-1}(a)$.

Since P is countably iterable in V (by a copying argument, using the map $k_M : \tilde{H}_M \to H(\theta_1)$), then there is a cofinal branch $b \in V$ through $\bar{T}$. Since $\bar{T}$ is countable and $\mathcal{P}(\omega) \subseteq H_M[G]$

(using our hypothesis $2^\omega = \omega_1$), then $b \in H_M[G]$. We now show that (i) $Q(b, \bar{T})$ exists, and (ii) $Q(b, \bar{T}) \trianglelefteq M_n^\#(\bar{M})$.

(i) $Q(b, \bar{T})$ exists: Let $\beta = \alpha_M^{+N}$. First we assume that the first extender used in $\bar{T}$ has length larger than β. Then, since P is a lower part model, and by the initial segment condition all extenders applied have critical points larger than α_M. This implies that $M_b^{\bar{T}}$ projects to or below α_M, and so $Q(b, \bar{T})$ exists. Now we assume that the first extender used in $\bar{T}$ has length at most β. Then there is a truncation in the first model of the iteration, and so $Q(b, \bar{T})$ exists.

(ii) $Q(b, T) \trianglelefteq M_n^\#(\mathcal{M}(T))$: Coiterate $Q(b, T)$ against $M_n^\#(\mathcal{M}(T))$, with trees $\mathcal{U}$ and $\mathcal{V}$ respectively. This is coiteration above $\delta(T)$, meaning that all extenders used have critical point strictly above $\delta(T)$. We claim that $M_\infty^\mathcal{U} \trianglelefteq M_\infty^\mathcal{V}$: assume otherwise, so that $M_\infty^\mathcal{V} \triangleleft M_\infty^\mathcal{U}$. Note that $\delta(T)$ is Woodin in $M_\infty^\mathcal{V}$. Therefore we have an $M_{n+1}^\#(a)$-like initial segment of $M_\infty^\mathcal{V}$, from which we can obtain $M_{n+1}^\#(a)$. So by Lemma 2.50, $M_n^{\#\#}(a)$ also exists, a contradiction.

Therefore $M_\infty^\mathcal{U} \trianglelefteq M_\infty^\mathcal{V}$. Since $Q(b, T)$ is sound and projects to $\delta(T)$, then by Lemma 2.31 $Q(b, T) \trianglelefteq M_n^\#(\mathcal{M}(T))$.

Thus Claim 1 is established.

Let U_M be the N_M-measure on α_M derived from σ_M. By the previous claim N_M and N have the same subsets of α_M, so U_M is also a measure over $Z|\alpha_M^{++Z}$. Note that the ultrapower $P' := Ult(Z|\alpha_M^{++Z}, U_M)$ is well-founded, as it embeds into $Z|\omega_2^{++Z}$. Let $P_M = (Z|\alpha_M^{++P'}, U_M)$. Then P_M is an active premouse over a that is closed under the $M_n^\#$ operator.

We will now show that P_M is countably iterable at least one structure $M \in SG^*(I)$. First we prove:

Claim 2: For at least one structure $M \in S$, U_M is countably complete[1].

Proof of claim: We consider two cases:

- Case 1: For some $M \in S$, $\alpha_M^{+P_M} = \alpha_M^{+H_M}$: fix such an M. This implies since $M \in SG^*(I)$ that $cf(\alpha_M^{+P_M}) > \omega$. Let $\langle X_n : n < \omega \rangle$ be a sequence of subsets of α_M such that for every $n < \omega$, $X_n \in U_M$. We want to find an element in the intersection of the sets X_n. To each $n < \omega$, let $\gamma_n < \alpha_M^{+H_M}$ be the smallest ordinal γ such that $X_n \in P_M|\gamma$. Let $\gamma^* = \sup\{\gamma_n : n < \omega\}$, so $\gamma^* < \alpha_M^{+H_M}$. Let $f \in H_M$ be a surjection $f : \alpha_M \to P_M|\gamma$. By amenability, the set $\{i < \alpha_M : f(i) \in U_M\}$ belongs to H_M, which allows us to define a function $g : \alpha_M \to U_M$ by $g(i) = f(i)$ if $f(i) \in U_M$, and $g(i) = \alpha_M$ otherwise.

  By construction of g, for each $n < \omega$ there is some ordinal $\xi_n < \alpha_M$ such that $g(\xi_n) = X_n$. Let $\xi^* = \sup\{\xi_n : n < \omega\}$. By normality, the diagonal intersection $X = \Delta_{i<\alpha_M} g(i)$ is an element of U_M, so in particular, it is unbounded in α_M. Pick some $\xi' \in X$ such that $\xi' > \xi^*$. Then for every $n < \omega$ we conclude that $\xi' \in X_M$. [$xi' \in X$ so $\xi' \in g(i)$ for all $i < xi'$, and therefore $\xi' \in g(\xi_n) = X_n$.]

- Case 2: For every M, $\alpha_M^{+P_M} < \alpha_M^{+H_M}$: Assume towards a contradiction that for every $M \in S$, U_M is not countably complete. So for each $M \in S$, we can fix a sequence $\langle X_{i,M} : i < \omega \rangle \subseteq P_M$ such that for each $i < \omega$, $X_{i,M} \in U_M$ (so that $X_{i,M} \subseteq \alpha_M$ and $\alpha_M \in \sigma_M(X_{i,M})$), but $\bigcap_{i<\omega} X_{i,M} = \varnothing$.

  For each $M \in S$, we can find some $M^* \in S$ such that for $M^* \in M$ and for each $i \in \omega$, $\sigma_M(X_{i,M}) \in M^*$. By Fodor's Lemma, we can find a stationary set $S' \subseteq S$ and a structure $Q \in S$ such that for every $M \in S'$, $Q \in M$ and

$$\{\sigma_M(X_{i,M}) : i < \omega\} \subseteq Q.$$

[1] See A.1.

For every $M \in S'$, we can let $Y_{i,M} = \sigma_{Q,M}^{-1}(X_{i,M})$. We shall show that there is a stationary set S'' such that if $M, M' \in S''$, then $\sigma_M(X_{i,M}) = \sigma_{M'}(X_{i,M'})$.

For each $M \in S'$, let $f_M \in H_M$ be a surjection $f_M : \alpha_M \to \mathcal{P}(\alpha_M)^{P_M}$ (because we are in case 2). Let $g_M = \sigma_M(f_M) \in M$.

Let $a_M = \{(i, f_M^{-1}(X_{i,M})) : i < \omega\}$. Note that a_M can be coded by a countable subset of α_M, so $a_M \in M$. By Fodor's Lemma, there is some stationary set S'', and fixed a, f such that for every $M \in S''$, $g_M = f$ and $a_M = a$.

Given $M, M' \in S''$ and $i < \omega$, note that $(i, (i, f_M^{-1}(X_{i,M})) \in a_M$ and $(i, (i, f_{M'}^{-1}(X_{i,M'})) \in a_{M'}$, so that $f_M^{-1}(X_{i,M}) = f_{M'}^{-1}(X_{i,M'})$. But then, calling $\beta = f_M^{-1}(X_{i,M}) < \alpha_M$, we obtain:

$$g_M(\beta) = \sigma_M(f_M)(\sigma_M(\beta)) = \sigma_M(f_M(\beta)) = \sigma_M(X_{i,M}).$$

Similarly $g_{M'}(\beta) = \sigma_{M'}(X_{i,M'})$, and since $g_M = g_{M'}$, then $\sigma_M(X_{i,M}) = \sigma_{M'}(X_{i,M'})$.

Since $\sigma_M(X_{i,M}) = \sigma_{M'}(X_{i,M'})$, then $Y_{i,M} = Y_{i,M'}$ by applying σ_Q^{-1}.

Fix some $M \in S''$, and let $M' \in S''$ such that $M \in M'$ and $\alpha_M < \alpha_{M'}$. Since $\alpha_M \in \sigma_M(X_{i,M})$ and $\alpha_{M'} > \alpha_M$, then $\alpha_M \in \sigma_{M'}^{-1}(\sigma_M(X_{i,M})) = \sigma_{M'}^{-1}(\sigma_{M'}(X_{i,M'})) = X_{i,M'}$. In conclusion, for every $i < \omega$, $\alpha_M \in X_{i,M'}$, so $\bigcap_{i<\omega} X_{i,M'} \neq \varnothing$, a contradiction.

The proof of Claim 2 is complete.

Claim 3: If U_M is countably complete, then P_M is countably iterable.

Proof of claim: Let $P_M = (Q_M, U_M)$. Q_M is an initial segment of $Lp(a)$ so it is countably iterable. Let $\pi : \bar{P} \to P_M$ be elementary, where $\bar{P} = (\bar{Q}, F)$. Let's fix an iteration strategy Σ on $\bar{Q}$ for trees of length at most ω_1. We will see that Σ induces an iteration strategy Σ' on $\bar{P}$, for trees of length at most ω_1.

Let T be a iteration tree on $\bar{P}$ of limit length $\lambda \leq \omega_1$. We will denote the models M_β^T simply be M_β (where $\beta < lh(T)$). Let $C(T)$ be the set of ordinals β such that E_β is the top extender of M_β. Let $\nu^* = \sup\{\nu(E_\beta) : \beta \in C(T)\}$, and $\alpha = \min(lh(T) - C(T))$.

Subclaim: For every $\beta \geq \alpha$, $Pred_T(\beta + 1) \geq \alpha$.
Proof of subclaim: First observe that by normality, for every $\beta > \sup C(T)$ we have that $\nu_\beta > \nu^*$.

Suppose α is a limit ordinal (the case where α is a successor cardinal follows a similar argument). Then ν^* is a limit cardinal in M_α^T, so $\kappa_\alpha \geq \nu^*$.

Let $\beta \geq \alpha$. If E_β is not the top extender of M_β, then $\kappa_\beta \geq \nu^*$, as otherwise E_β would be total in M_β, contradicting that M_β is a lower part model after we remove its top extender.

Suppose now that E_β is the top extender of M_β. Let $\xi \geq \alpha$ be the ordinal such that E_ξ is applied to M_α. We claim that there is a truncation, in other words, that E_ξ is not a total extender over M_α. If $\xi = \alpha$ this is clear since $\alpha \notin C(T)$, and every extender in the M_α-sequence (except it's top extender) is not total. So we can assume that $\xi < \alpha$. Then $lh(E_\alpha) < lh(E_\xi)$. By rules of iteration trees, $\kappa_\xi < \nu_\alpha$, so that $lh(E_\alpha) \in (\kappa_\xi, lh(E_\xi))$. Since $lh(E_\alpha)$ is a cardinal in M_ξ, then E_ξ must be the top extender in M_ξ.

Let λ be the largest cardinal of M_α^T strictly below $lh(E_\alpha)$. Note that $\kappa_\xi^{+M_\xi} \leq \nu_\alpha$. Also, since E_ξ is total in M_ξ, $lh(E_\alpha)$ is a cardinal in M_ξ and M_α agrees with M_ξ below $lh(E_\alpha)$, we have that $\mathcal{P}(\kappa_\xi)^{M_\xi} \subseteq M_\alpha^T | lh(E_\alpha)$.

We observe that $\kappa_\xi \geq \lambda$: for suppose $\kappa_\xi < \lambda$. There are two cases:

- $\kappa_\xi^{+M_\xi} < \lambda$: let E be the extender-measure over M_ξ derived from E_ξ (so that $\nu(E) = \kappa_\xi^{+M_\xi}$). Then by the initial segment condition, E is an extender in the sequence of

$M_\xi|\lambda$, and is total as E_ξ is total in M_ξ. But the model $M_\xi|\lambda$ is a lower part model (as λ is a cardinal, and $\lambda < lh(E_\alpha)$. This is a contradiction.

- $\kappa_\xi^{+M_\xi} = \lambda$: let F be the extender-measure derived from E_ξ, and let η be its index in the M_ξ-sequence (note that $\eta < lh(E_\alpha)$). By coherence, $F = E_\eta^{M_\alpha}$. Also, since $\kappa_\xi^{+M_\xi}$ is a cardinal in M_α, F is a total extender in M_α (which is not its top extender), contradicting that M_α is a lower part model. In conclusion, this case cannot occur.

Since $\kappa_\xi \geq \lambda$ and $\kappa_\xi^{+M_\xi} < lh(E_\alpha)$, then by maximality of λ, $\kappa_\xi^{+M_\xi}$ cannot be a cardinal in M_α, and so $\mathcal{P}(\kappa_\xi)^{M_\alpha} \neq \mathcal{P}(\kappa_\xi)^{M_\xi}$. Therefore, E_ξ is not a total extender in M_α^T, as claimed.

Let $b = [\alpha, \beta]_T$. We have seen that there is at least one truncation in this branch. Let η be the last truncation in b and let γ be the ordinal such that E_γ is applied to M_η so that we have a branch embedding $i : M^*_{\gamma+1} \to M_\beta$. Let F be the extender such that $i(F) = E_\beta$ (so that F is the top extender in the $M^*_{\gamma+1}$-sequence). Since the iteration from α to β is above ν^*, then $lh(F) > \nu^*$, so $crit(F) \geq \nu^*$ (as M^*_γ is a lower part model). Therefore applying the map i we conclude that $crit(E_\beta) \geq \nu^*$.

In summary, we have shown: for every $\beta \in [\alpha, lh(T))$, $crit(E_\beta) \geq \nu^*$, which implies by the rules of iteration trees that E_β is not applied to models of the form M_γ where $\gamma \in C(T)$. Thus, $Pred_T(\beta + 1) \geq \alpha$. This proves the subclaim.

This analysis show that the models associated to T satisfy the following: there is some α (possibly 0) such that for each $\gamma < \alpha$ the extender E_γ is the top extender of M_γ, and is applied to M_γ itself (so that up to α, T is linear), and either $\alpha + 1 = lh(T)$, or the following holds: E_α is not the top extender of M_α, there is truncation at α, and for every $\beta \geq \alpha$, E_β is not applied to a model of the form M_γ where $\gamma < \alpha$.

We conclude that T can be expressed as the concatenation of a linear iteration (from 0 to α), and an iteration tree on the model M^* resulting from removing the top extender from M_α. Using the countably completeness of U_M and its iterates up to α, one can embed M^* into Q_M (see Theorem A.2). Since Q_M is countably iterable, then by copying, one can use the strategy for Q_M to find a cofinal wellfounded branch through $[\alpha, lh(T))_T$, which naturally induces a cofinal well-founded branch through T. This finishes the proof of claim 3.

In summary, there is at least some structure M such that P_M is countably iterable. This implies that $M_n^{\#\#}(a)$ exists.

□

3.2 H_{ω_3} is closed under $M_n^{\#\#}$

We inductively assume that H_{ω_4} is closed under the $M_n^{\#}$ operator, and that H_{ω_2} is closed under the $M_n^{\#\#}$ operator.

Let $a \in H_{\omega_3}$ and w.l.o.g. assume that $a \subseteq \omega_2$. We want to show that $M_n^{\#\#}(a)$ exists.

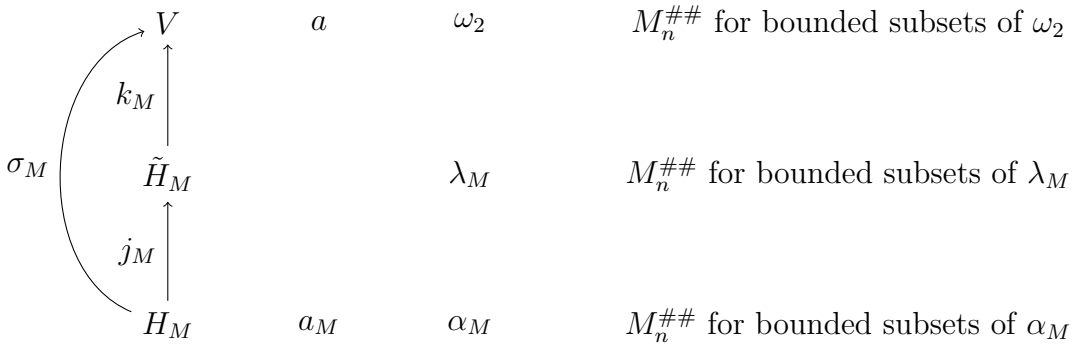
We fix a structure $M \in S_I^*$ such that $a \in M$, and let $a_M = \sigma_M^{-1}(a)$. For every bounded subset b of ω_2, $M_n^{\#\#}$ exists, so by elementarity we have:

$$H_M \models \quad \text{“For every bounded subset } b \text{ of } \alpha_M,\ M_n^{\#\#}(b) \text{ exists.”}$$

Therefore, letting $\tilde{H}_M = Ult(H_M, U_M)$ and $\lambda_M = j_M(\alpha_M) > \alpha_M$ (where $j_M : H_M \to \tilde{H}_M$ is the ultrapower embedding), we have:

$$\tilde{H}_M \models \quad \text{“For every bounded subset } b \text{ of } \lambda_M,\ M_n^{\#\#}(b) \text{ exists.”}$$

Since $a_M = j_M(a_M) \cap \alpha_M$, then $a_M \in \tilde{H}_M$ and a_M is a bounded subset of λ_M, so we have that $\tilde{H}_M \models$ "$M_n^{\#\#}(a_M)$ exists". Let P be equal to $M_n^{\#\#}(a_M)$ as computed inside $\tilde{H}_M$.



We will show that $H_M \models$ "$M_n^{\#\#}(a_M)$ exists", and in fact $P \in H_M$ and $P = (M_n^{\#}(a))^{H_M}$. We prove the following statements in order to obtain the conclusion:

- $H_M[U_M]$ is closed under $M_n^{\#}$.
- P is iterable in V.
- P is iterable inside $H_M[U_M]$.
- P is iterable in $H_M[G]$, where G is generic for $\mathbb{R} = \mathbb{P}_I^M * Col(\omega, \omega_3^{H_M})$.
- P is definable in $H_M[G]$ (where G is as above) from parameter $a_M \in H_M$, so $P \in H_M$.
- P is iterable inside H_M.

We proceed with the proof: By Lemma 2.49, $H_M[U_M]$ is closed under the $M_n^{\#}$ operator for sets in $Z = (H_{\omega_4})^{H_M[U_M]}$. Note that the existence of the factor map k_M implies that P is countably iterable in V, so that P is equal to $M_n^{\#\#}(a_M)$ as computed in V.

Claim: $P \in H_M$.

Proof of claim: Inside H_M, let $\mathbb{R} = \mathbb{P}_I^M * \mathbb{Q}$, where $\mathbb{Q} = Col(\omega, \omega_3^{H_M})$. We prove that

whenever G is a $\mathbb{R}$-generic filter over H_M, P is definable in $H_M[G]$ (with parameter a_M). To see this, we fix such a G and we claim that P is the smallest iterable a_M-premouse that is active and closed under $M_n^{\#\#}$.

First we prove that P is iterable inside $H_M[U_M]$. For this, it is enough to prove that P is "countably $M_n^{\#}$-guided". But note that since $H_M[U_M] \supseteq \mathcal{P}(\omega)$, $H_M[U_M]$ is closed under $M_n^{\#}$, and P is "countably $M_n^{\#}$-guided" (see Lemma 2.46), then the desired conclusion follows. By the Q-reflection Lemma (2.35) we conclude that P has a $M_n^{\#}$-guided iteration strategy for trees inside $(H_{\omega_4})^{H_M[U_M]}$, using the fact that this set is closed under $M_n^{\#}$.

Now we prove that P is iterable inside $H_M[G]$. Let H be a $\mathbb{Q}$-generic filter over $H_M[U_M]$ such that $H_M[U_M][H] = H_M[G]$. Let's call $W = H_M[U_M]$ so that $P \in W$. We show that inside $H_M[G] = W[H]$, P is iterable via a $M_n^{\#}$-guided strategy. Assume otherwise. Therefore, working in W, we can find some map $\pi : N \to H_\theta$, where $\theta = \omega_4^W$, N is countable and transitive, $P, \mathbb{Q} \in Ran(\pi)$ (say $\pi(\bar{P}) = P$ and $\pi(\bar{\mathbb{Q}}) = \mathbb{Q}$), such that

$$N \models \text{ "some condition in } \bar{\mathbb{Q}} \text{ forces that there is a bad } M_n^{\#}\text{-guided tree on } \bar{P}\text{".}$$

Let $G^* \in W$ be a $\bar{\mathbb{Q}}$-generic filter over N such that for some $M_n^{\#}$-guided tree $T \in N[G^*]$:

$$N[G^*] \models \text{ "}T \text{ is a bad tree on } \bar{P}\text{".}$$

By the elementarity of the map $\pi : \bar{P} \to P$, we conclude that $\bar{P}$ is iterable in W, so there is some $b \in W$ such that b is a cofinal well-founded branch through T. Moreover one can check that T has a $M_n^{\#}$-guided strategy, so such b is the unique cofinal well-founded branch, and satisfies that $Q(b, T)$ exists and is an initial segment of $M_n^{\#}(\mathcal{M}(T))$.

We shall show that $b \in N[G^*]$, which will contradict our hypothesis. Let G^{**} be generic over $N[G^*]$ for the forcing $Col(\omega, |T|+|\bar{P}|)$. We will verify that $b \in N[G^*][G^{**}]$ and is definable in $N[G^*][G^{**}]$, implying by homogeneity that $b \in N[G^*]$. Let $x \in N[G^*][G^{**}]$ be a real coding $(T, \bar{P})$. Let $\tau \in N$ be a $\bar{\mathbb{Q}}$-name such that $T = \tau^{G^*}$. Since $\tau^{\#}$ exists, this implies that $x^{\#}$ exists and belongs to $N[G^*][G^{**}]$.

Also, note that $M_n^{\#}(\mathcal{M}(T)) \in N[G^*]$: let $\dot{c} \in N$ be a name for $\mathcal{M}(T) \in N[G^*]$ such that $\dot{c} \in (H_{\omega_4})^N$. Then $\pi(\dot{c}) \in (H_{\omega_4})^{H_M[G]}$, so $M_n^{\#}(\pi(\dot{c}))$ exists. This implies using the map π that $M_n^{\#}(\dot{c})$ exists, belongs to N, and in fact N interprets it correctly. Since $\bar{\mathbb{Q}}$ is small, one can turn the extenders in $M_n^{\#}(\dot{c})$ to extenders in the sense of $N[G^*]$ to produce $M_n^{\#}(\mathcal{M}(T)) = M_n^{\#}(\mathcal{M}(T))$ (the iterability of it will follow from the iterability of $M_n^{\#}(\mathcal{M}(T))$).

Therefore, $M_n^{\#}(\mathcal{M}(T)) \in N[G^*][G^{**}]$. So by Σ^1_2-absoluteness for statements about x, plus the existence of b, we conclude:

$N[G^*][G^{**}] \models$ "there is a cofinal branch c such that $Q(c,T)$ exists, $Q(c,T) \subseteq wfp(M_b^T)$ and $Q(c,T) \trianglelefteq M_n^{\#}(\mathcal{M}(T))$".

Let $c \in N[G^*][G^{**}]$ be one such branch. This implies by Lemma 2.34 that $b = c$. Thus, $b \in M[G^*][G^{**}]$.

In summary, P is definable in $H_M[G]$ from parameter $a_M \in H_M$. By the homogeneity of $\mathbb{R}$, we conclude that $P \in H_M$. Again, since $(H_{\omega_4})^{H_M}$ is closed under $M_n^{\#}$, using reflection we conclude that P is iterable inside H_M with respect to trees in $(H_{\omega_4})^{H_M}$, so in particular, $H_M \models$ "$M_n^{\#\#}(a_M)$ exists". Therefore by elementarity of σ_M, $M_n^{\#\#}(a)$ exists in V and is iterable with respect to trees in H_{ω_4}, a contradiction to our initial assumption.

3.3 H_{ω_2} is closed under $M_{n+1}^{\#}$

We delay the proof of this step in the induction to Theorem 3.5, which is the most technical part of the induction. The proof of this theorem is an adaptation of Theorem 4.1 in [2], which uses the a "frequent extension of embeddings" argument (this is a place where the stationarity of the set $SG^*(I)$ of structures is heaviliy used). This is also the step in the induction where we use the Core Model Dichotomy (relative to $M_n^{\#}$) to argue that if the Core Model does not reach an iterable $M_{n+1}^{\#}$ then the core model K exists; at the same time we shall derive an extender E_M from the map σ_M (coming from one of our structures) and prove that E_M belongs to the extender sequence of K and witnesses that there is a superstrong cardinal in K, which is impossible.

3.4 H_{ω_3} is closed under $M_{n+1}^{\#}$

Let $a \subseteq \omega_2$. Fix a structure $M \in SG^*(I)$ such that $a, R \in M$, let $a_M = \sigma_M^{-1}(a)$ (so $a_M \in H_{\omega_2}$).

Consider a structure $M \in S_I^*$ such that $a \in M$, and let $a_M = \sigma_M^{-1}(a)$. For every bounded subset b of ω_2, $M_n^{\#\#}$ exists, so by elementarity we have:

$$H_M \models \quad \text{"For every bounded subset } b \text{ of } \alpha_M,\ M_n^{\#\#}(b) \text{ exists."}$$

Therefore, letting $\tilde{H}_M = Ult(H_M, U_M)$ and $\lambda_M = j_M(\alpha_M) > \alpha_M$ (where $j_M : H_M \to \tilde{H}_M$ is the ultrapower embedding), we have:

$$\tilde{H}_M \models \quad \text{"For every bounded subset } b \text{ of } \lambda_M,\ M_n^{\#\#}(b) \text{ exists."}$$

Since $a_M = j_M(a_M) \cap \alpha_M$, then $a_M \in \tilde{H}_M$ and a_M is a bounded subset of λ_M, so that $\tilde{H}_M \models$ "$M_n^{\#\#}(a_M)$ exists". Let P be equal to $M_n^{\#\#}(a_M)$ as computed inside $\tilde{H}_M$.

At this point we can follow the proof in section 3.2, performing the obvious adaptations. The following remarks justify why it is possible to do so:

- As in the case with $M_n^{\#\#}$, the operator $M_{n+1}^{\#}$ is $M_n^{\#}$-guided (Lemma 2.41).
- Note that $M_n^{\#}(a_M)$ can be defined as the minimal countably iterable active a_M-premouse having n Woodin cardinals.

3.5 Closure of H_{ω_2} under $M_{n+1}^{\#}$

In order to finish the proof of Theorem 3.1, we now consider the step in the induction in which we obtain $M_{n+1}^{\#}$ for elements of H_{ω_2}. The following theorem serves to this purpose.

Theorem 3.5. *Assume the hypotheses of Theorem 3.1. Let $n < \omega$ and assume that H_{ω_4} is closed under the $M_n^{\#}$ and $M_n^{\#\#}$ operator. Then for every $a \in H_{\omega_2}$, $M_{n+1}^{\#}(a)$ exists and is iterable with respect to trees in H_{ω_4}.*

Proof. Let $a \subseteq \omega_1$: we want to show that $M_{n+1}^{\#}(a)$ exists, and that is iterable with respect to trees on H_{ω_4}. We will assume this not to be the case, and obtain a contradiction. It can be seen that the threadability of ω_3 implies that it is not the case that $M_{n+1}^{\#}(a)$ exists and is iterable with respect to trees on H_{ω_3}.

Let $A \subseteq \omega_3$ code $\mathcal{P}(\omega_2)$, and $I' \subseteq \omega_3$ code I. Consider $W = M_n^{\#\#}(A, I')$. Let E be the top extender of W, and let $\Omega = crit(E)$ (so $\Omega > \omega_3$, as $sup(A) = \omega_3$). Let R be the

structure resulting from removing E from W. Note that V^R_Ω is closed under the $M^\#_n$ operator, and E provides an external R-measure U on Ω, which can be used to construct $K^c(a)$.

Observation: $H_{\omega_3} \subseteq R|\omega_3$. *Proof*: By $\in$-induction: let $a \in H_{\omega_3}$ and assume that $a \subseteq R|\omega_3$. We can assume that $a = \{a_i : i < \omega_2\}$. Let $\gamma < \omega_3$ such that $a \subseteq R|\gamma$. There is a bijection $f : \omega_2 \to R|\gamma$ such that $f \in R|\omega_3$. Let $S = f^{-1}[a]$. Since $\mathcal{P}(\omega_2) \subseteq R|\omega_3$, then $S \in R|\omega_3$, and so $a \in R|\omega_3$, as $a = f[S]$.

Claim: $R \models$ "There is no $\Omega + 1$-iterable $M^\#_{n+1}(a)$". *Proof of claim*: assume otherwise, and let P be equal to $M^\#_{n+1}(a)$ as built inside R, and such that $R \models$ "P is $\Omega + 1$-iterable". It can be seen that $V \models$ "P is equal to $M^\#_n(a)$". Moreover, we show that P is iterable in V for trees in H_{Ω_3}: let $\mathcal{T} \in H_{\Omega_3}$ be an iteration tree of limit length on P. We observed that $H_{\omega_3} \subseteq R$, so that $\mathcal{T} \in R$ and $lh(\mathcal{T}) < \Omega$. Therefore there is some $b \in R$ such that $\mathcal{M}^\mathcal{T}_b$ is well-founded. Such b is given by the iteration strategy for P inside R. This gives an iteration strategy (in V) for P with respect to trees in H_{ω_3}. We thus obtain a contradiction to our initial hypothesis. The proof of the claim is complete.

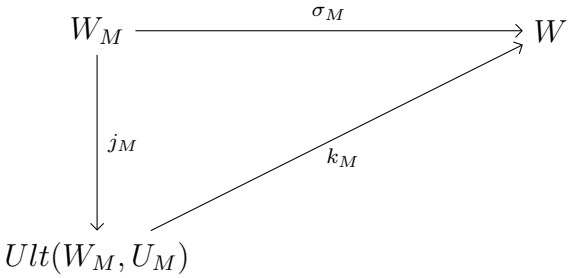
By the previous claim, we can apply the Dichotomy Theorem (Theorem 2.60) inside R to conclude that the core model $K := K(a)$ exists (as built with respect to Ω). We shall fix a soundness witness W for $K||\omega_3$.

Recall that the collection $SG^*(I)$ of I-self generic structures (along with other conditions) is stationary in $\mathcal{P}(H(\theta_1))$, where $\theta_1 = (2^{\omega_2})^{+2}$, and for each $M \in SG^*(I)$, we have an elementary map $\sigma_M : H_M \to H(\theta)$ (where H_M is the transitive collapse of M) with critical point $\alpha_M = \omega_2^{H_M}$, and $\tau_M := \omega_3^{H_M}$.

[2]Note that in fact $\theta_1 = \omega_4$, by our hypotheses.

Let $S' = \{M \in SG^*(I) : R, K, W \in M\}$; S' is stationary in $\mathcal{P}(H(\theta))$. For each $M \in S'$, we let $R_M = \sigma^{-1}(R)$, $K_M = \sigma^{-1}(K)$ and $W_M = \sigma_M^{-1}(W)$. Note that $A_M, I_M \in R_M$ so that $\mathbb{P}_I^M \in R_M$ and $\mathcal{P}(\alpha_M)^{R_M} = \mathcal{P}(\alpha_M)^{H_M}$. This implies that U_M is a $\mathbb{P}_I^M$-generic filter over R_M: if $\mathcal{A} \in R_M$ is such that ($\mathcal{A}$ is a m.a.c. $)^{R_M}$, then $\mathcal{A} \in H_M$, and ($\mathcal{A}$ is a m.a.c.)^{H_M}, as otherwise $\mathcal{P}(\alpha_M)^{H_M} - \mathcal{P}(\alpha_M)^{R_M} \neq \varnothing$. Therefore $U_M \cap \mathcal{A} \neq \varnothing$ since M is I-self-generic.

Let $\tilde{R}_M = Ult(R_M, U_M)$, with ultrapower embedding map $j_M : R_M \to \tilde{R}_M$, and factor map $k_M : \tilde{R}_M \to R$. By elementarity and noting that $\sigma_M^{-1}(a) = a$, $R_M \models$ "K_M is the core model relative to a, and it has no Woodin cardinals".



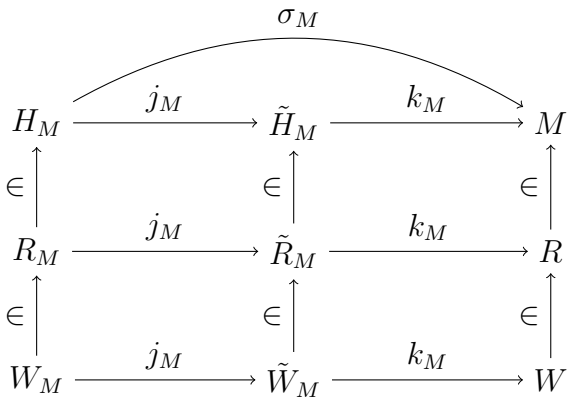
Let U_M be the H_M-ultrafilter on α_M derived from the map σ_M:

$$U_M := \{a \in H_M \cap \mathcal{P}(\alpha_M) : \alpha_M \in \sigma_M(a)\}.$$

Let $j_M : H_M \to \tilde{H}_M := Ult(H_M, U_M)$ be the ultrapower embedding. Let $\lambda_M = j_M(\alpha_M)$, and $k_M : \tilde{H}_M \to H(\theta_1)$ be the factor map. Let F_M be the (α_M, λ_M)-extender over K'_M derived from σ_M (note that F_M is also derived from j_M, since $k_M \restriction \lambda_M = id$). We shall see that F_M is also an extender over W.

Recall that $\tau'_M := \alpha_M^{+K_M}$. By Theorem 2.64 applied in the universe R'_M, we conclude that the models W_M and $\tilde{W}_M := j_M(W_M)$ agree on the cardinal successor of α_M. Also by condensation, W_M and $\tilde{W}_M$ agree up to $\tau'_M = \alpha_M^{+W_M} = \alpha_M^{+\tilde{W}_M}$, and so $\mathcal{P}(\alpha_M)^{W_M} = \mathcal{P}(\alpha_M)^{\tilde{W}_M}$. Note also that α_M must be a limit cardinal in W'_M: otherwise, say if $\alpha_M = \mu^{+W'_M}$ for some

$\mu < \alpha_M$, then by elementarity $\lambda_M = \mu^{+\tilde{W}_M}$, and since $\alpha_M < \lambda_M$, then α_M is not a cardinal in $\tilde{W}_M$. Since $\mathcal{P}(\alpha_M)^{W_M} = \mathcal{P}(\alpha_M)^{\tilde{W}_M}$, then α_M is not a cardinal in W_M, a contradiction (since $W_M \subseteq R'_M \subseteq H_M$). Therefore, by elementarity, λ_M is a limit cardinal in $\tilde{W}_M$. Since k_M is the identity in λ_M, then λ_M is a limit cardinal in W.



Now note:

$$\alpha_M^{+W} = k_M(\alpha_M^{+\tilde{W}_M}) = k_M(\tilde{\tau}_M) = \tilde{\tau}_M.$$

The last equality holds since λ_M is a limit cardinal in $\tilde{W}_M$. Again, we can conclude that $\tilde{W}_M$ and W agree up to $\tilde{\tau}_M$, and that $\mathcal{P}(\alpha_M)^{W'_M} = \mathcal{P}(\alpha_M)^W$. In summary:

- $\alpha_M^{+W_M} = \alpha_M^{+\tilde{W}_M} = \alpha_M^{+W}$.
- $W_M||\tau'_M = \tilde{W}_M||\tau'_M = W||\tau'_M$.
- $\mathcal{P}(\alpha_M)^{W_M} = \mathcal{P}(\alpha_M)^{\tilde{W}_M} = \mathcal{P}(\alpha_M)^W$.

The last clause implies that F_M is an extender over W. Since F_M is the (α_M, λ_M) extender derived from $j_M : W_M \to \tilde{W}_M$, then F_M coheres with $\tilde{W}_M$. That is, the structure $(J^{\tilde{W}}_{\lambda_M}, \in, E^{\tilde{W}}, F_M)$ is a premouse. Now, $crit(k_M) \geq \lambda_M$ and so by condensation, the extender sequences of $\tilde{W}_M$ and W agree below $\lambda_M^{+\tilde{W}_M}$. Also, since $\tau'_M = \alpha_M^{+W_M}$, then $j_M(\tilde{\tau}_M) = \lambda_M^{+\tilde{W}_M}$. From these facts, it can be seen that F_M also coheres with W.

We claim that for stationarily many structures $M \in S'$,

$$R \models \text{“The phalanx } (W, Ult(W, F_M), \lambda_M) \text{ is iterable”}.$$

Assuming the claim to be proved, we can obtain a contradiction in the following way: fix some structure M satisfying the statement in the claim. Using Theorem 5.13 in [12] we conclude that the core model $K(a)$ is defined, and witnessed by W. By 2.59 applied in the model R (here is where we use that F_M coheres with W), we conclude that F_M belongs to the extender sequence of $K(a)$. Thus F_M is a superstrong extender, as the following computation shows:

$$\begin{aligned}
H(\lambda_M)^W &= H(\lambda_M)^{\tilde{W}_M} && (k_M \restriction \lambda_M = id) \\
&\subseteq Ult(W_M||\tau'_M, F_M) && (F_M \text{ is also the } (\alpha_M, \lambda_M)\text{-extender derived from } j_M) \\
&= Ult(W||\tau'_M, F_M) && (W||\tau'_M = W_M||\tau'_M) \\
&\subseteq Ult(W, F_M) && (\tau'_M = \alpha_M^{+W})
\end{aligned}$$

So $H(j(\alpha_M))^W \subseteq Ult(W, F_M)$. Since $(K(a)||\omega_3) \triangleleft W$, and $K(a)$ does not have Woodin cardinals, then in particular W cannot have superstrong extenders below ω_3, so we obtain a contradiction.

We proceed to prove the claim: assume the contrary, which implies that there is a stationary set $S'' \subseteq S'$ such that for every $M \in S''$:

$$R \models \text{ “The phalanx } (W, Ult(W, F_M), \lambda_M) \text{ is not iterable”}.$$

For simplicity, we shall replace the extender F_M with a longer one: let G_M be the (α_M, ω_2)-extender derived from σ_M. Then we still have:

$$R \models \text{ "The phalanx } \mathcal{B}_M = (W, Ult(W, G_M), \omega_2) \text{ is not iterable"}.$$

Inside the model R, fix an iteration tree $\mathcal{T}_M$ that witnesses the non-iterability of the phalanx $\mathcal{B}'_M = (N_M, Ult(N_M, G_M), \omega_2)$, where $N_M = W||\gamma_M$, for some (sufficiently large) successor cardinal γ_M in W (it can be seen that such γ_M must exist). By weak compactness of Ω in R, we know that $lh(\mathcal{T}_M) < \Omega$.

Find a countable structure $Z_M \preceq H(\theta_M)^R$ (for some suitable θ_M) such that $\mathcal{B}'_M, \mathcal{T}_M \in Z_M$. Let H^Z_M be the transitive collapse of Z_M, with uncollapsing map $\rho_M : H^Z_M \to H(\theta_M)^{H_M}$. Let

$$\bar{N}_M = \rho_M^{-1}(N_M), \bar{F}_M = \rho_M^{-1}(F_M), \bar{\mathcal{T}}_M = \rho_M^{-1}(\mathcal{T}_M), \bar{\alpha}_M = \rho_M^{-1}(\alpha_M), \bar{\beta}_M = \rho_M^{-1}(\omega_2).$$

By elementarity,

$$H^Z_M \models \text{ "The phalanx } \bar{\mathcal{B}}' := (\bar{N}_M, Ult(\bar{N}_M, \bar{G}_M), \bar{\beta}_M) \text{ is not iterable, witnessed by } \bar{\mathcal{T}}_M".$$

Since $(K^c(\sigma_M^{-1}(a)))^{R_M}$ has no Woodin cardinals, then the phalanx $\mathcal{B}$ is properly small, which implies that $\bar{\mathcal{B}}'$ is properly small.

Let $x = (\bar{\mathcal{B}}', \bar{\mathcal{T}}_M)$. Since $\rho_M(x) \in V^R_\Omega$ and V^R_Ω is closed under the $M^\#_n$ operator, then it is easy to see that $x^\# \in H^Z_M$. This implies that $H^Z_M[G]$ is Π^1_2-correct about statements with parameter G, where G is a generic real coding x. We fix such a G.

Claim: $H^Z_M[G] \models$ "$\bar{\mathcal{T}}_M$ has no cofinal wellfounded branch".

Proof of the claim: Assume otherwise, and let $b \in H^Z_M[G]$ be a cofinal well-founded branchh. By an argument similar to lemma 6.13 in [12], we see that b must be the unique cofinal branch c in $H^Z_M[G]$ satisfying that $M_n^{\#}(\mathcal{M}(\bar{\mathcal{T}}_M)) \trianglelefteq M_c^{\bar{\mathcal{T}}_M}$. For this we use that N_M is an F-mouse where $F = M_n^{\#}$ operator, and that $\mathcal{B}'$ is properly small. Therefore:

$$H^Z_M[G] \models \text{ "}\bar{\mathcal{T}}_M \text{ has a unique cofinal branch such that } M_n^{\#}(\mathcal{M}(\bar{\mathcal{T}}_M)) \trianglelefteq M_c^{\mathcal{T}}\text{."}.$$

Let $b \in H^Z_M[G]$.

Therefore b is definable in $H^Z_M[G]$ from the parameter $M_n^{\#}(\bar{\mathcal{T}}_M)$, which belongs to H^Z_M since $\bar{\mathcal{T}}_M \in (H_{\omega_4})^{H^Z_M}$ and this set is closed under $M_n^{\#}$. By homogeneity of the forcing $Col(\omega, |\bar{\mathcal{B}}'| + |\bar{\mathcal{T}}_M|)$, $b \in H^Z_M$, a contradiction. The proof of the claim is complete.

By Π^1_2-correctness with respect to G,

$$R \models \text{"}\bar{\mathcal{T}}_M \text{ has no cofinal well-founded branch"}.$$

So in particular,

$$R \models \text{"The phalanx } \bar{\mathcal{B}}'_M = (\bar{N}_M, Ult(\bar{N}_M, G_M), \bar{\beta}_M) \text{ is not iterable"}. \tag{3.1}$$

Recalling that $\tau'_M = \alpha_M^{+W_M}$, we let $e_M = H^Z_M \cap (W_M|\tau'_M)$. Note that $\sigma''_M e_M$ is a countable subset of $W|\tau_M$.

Claim: For some $B \in M$, $\tilde{\sigma}''_M e_M \subseteq B \subseteq M$.

Proof of claim: By cases:

- Case 1: $\omega_2^{+W} < \omega_3$: then there is a surjection $f : \omega_2 \to W|\omega_2^{+W}$. By elementarity of M we can fix such a function f that belongs to M.

  Note that if we let $\tilde{\sigma}''_M e_M = \{x_i : i < \omega\}$, then for every $i < \omega$ we can find some $\gamma_i \in \omega_2 \cap M$ such that $f(\gamma_i) = x_i$. Since $\alpha_M = \omega_2 \cap M$ has uncountable cofinality, fix some $\gamma < \alpha_M$ such that $\gamma_i < \gamma$ for all $i < \omega$.

  Let $B = f''\gamma$. Then $B \in M'$, $\sigma''_M e_M \subseteq B$ and $B \subseteq M$. (Note that $B \subseteq M$ because $B \in M$, $|B| = \omega_1$ and $\omega_1 + 1 \subseteq M$.)

- Case 2: $\omega_2^{+W} = \omega_3$: Since $cf(\tilde{\tau}_M) > \omega$, then $sup(\tilde{\sigma}''_M e_M) < \tau''_M$ (where $\tau''_M = \sigma''_M \tilde{\tau}_M$). Fix $\beta \in M$ such that $sup(\tilde{\sigma}''_M e_M) < \beta < \omega_3$. Similarly as above, we can show that there is a surjection $f : \omega_2 \to W|\beta$ such that $f \in M$. We can then argue as in case 1, using the fact that $\tilde{\sigma}''_M e_M \subseteq W|\beta$ to find B.

Fix a sufficiently large regular cardinal $\theta_2 > \theta_1$ and define:

$$S_2 = \{M \preceq H(\theta_2) : S'' \in M \text{ and } M \cap H(\theta_1) \in S''\},$$

$$S_1 = \{M \cap H(\theta_1) : M \in S_2\}.$$

Since S'' is stationary, both S_1 and S_2 are stationary.

Let $M_1 \in S_1$, say $M_1 = M_2 \cap H_{\theta_1}$, where $M_2 \preceq H_{\theta_2}$. Then

$$H_{\theta_2} \models M_1 \in S'' \text{ and } B \in M_1,$$

where the set B is as in the previous claim. Therefore,

$$H_{\theta_2} \models \text{ There is some } P \in S'' \text{ such that } B \in P,$$

so by elementarity,

$$M_2 \models \text{ There is some } P \in S'' \text{ such that } B \in P.$$

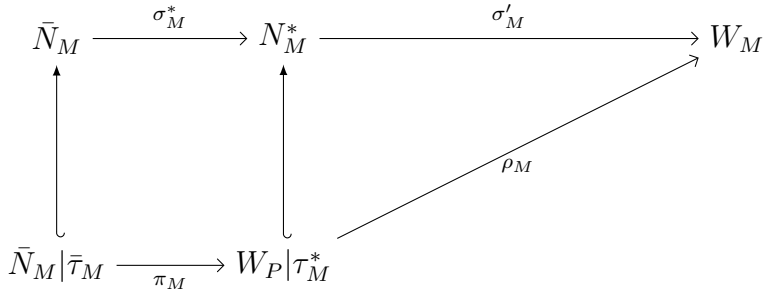
Fixing such a P, we conclude that $P \in M_1$ (since $P \in M_2$ and $P \in H_{\theta_1}$ as $P \in S''$).

The previous analysis yields: For each $M \in S_1$, there is a set B and $P \in S'' \cap M_1$ such that $B \in P$, $B \subseteq P$ (this follows from the same argument in which we concluded that $B \subseteq M$) and

$$\tilde{\sigma}''_M e_M \subseteq B \subseteq M, \text{ where } e_M = H^Z_M \cap (W_M|\tau'_M). \tag{3.2}$$

By Fodor's lemma, we can fix a structure $P \in S''$ and a stationary set S'_1 such that for every $M \in S'_1$, the statement above holds true. This implies that there is a *total* map π_M from $\bar{N}_M|\bar{\tau}_M$ to $W_P|\tau^*_M$ given by $x \mapsto \sigma_P^{-1}(\sigma_M(\rho_M(x)))$, where $\tau^*_M = (\sigma_P^{-1} \circ \sigma_M \circ \rho_M)''\bar{\tau}_M$.

Observe that π_M is Σ_0-preserving and cofinal. By the argument in the proof of the Interpolation Lemma (see [14]), we can find a premouse N^*_M such that $W_P|\tau^*_M \triangleleft N^*_M$, and maps $\sigma^*_M : \bar{N}_M \to N^*_M$, $\sigma'_M : N^*_M \to W_M$. Also, $\sigma'_M \circ \sigma^*_M = \rho_M$.



Since $P \in S'_1$, the phalanx $(W, Ult(W, G_P), \omega_2)$ is not iterable inside R. Working in R, let Q be a $<_S$-minimal premouse below W such that $(W, Ult(Q, G_P), \omega_2)$ is not iterable. Since $Q <_S W$ or $Q = W$, from the definition of $<_S$ and the fact that W is iterable, it can be seen that (W, Q, α_P) is iterable.

The fact that $H_{\omega_3} \subseteq R$ implies that for every $M \in S'_1$, $\bar{N}_M | \bar{\tau}_M, W_P | \tau^*_M, \pi_M, N^*_M, \sigma^*_M, \sigma'_M \in R$. Furthermore, the reader will note that all the relevant premice and phalanx embeddings that we define will be in R. All iterability statements that we consider from now on are to be interpreted in the sense of R.

Claim: For every $M \in S'_1$, the phalanx $(W, Ult(N^*_M, G_P), \omega_2)$ is not iterable inside R.

Proof of claim: Let $b \in [\beta_M]^{<\omega}$ and $Y \subseteq [\bar{\alpha}_M]^{|b|}$. We have:

$$\begin{aligned}
& Y \in (\bar{G}_M)_a \\
\text{iff} \quad & \rho_M(Y) \in (G_M)_{\rho_M(b)} && (\rho_M(\bar{G}_M) = G_M) \\
\text{iff} \quad & \rho_M(b) \in \sigma_M(\rho_M(Y)) && (G_M \text{ is derived from } \sigma_M) \\
\text{iff} \quad & \rho_M(b) \in \sigma_P(\sigma^*_M(Y)) \\
\text{iff} \quad & \sigma^*_M(Y) \in (G_P)_{\rho_M(b)} && (G_P \text{ is derived from } \sigma_P)
\end{aligned}$$

To justify the third equivalence, note that $\sigma^*_M(Y) \in [\tau^*_M]^{|b|}$, so $\sigma'_M(\sigma^*_M(Y)) = \sigma_{P,M}(\sigma^*_M(Y))$. Since $\rho_M = \sigma'_M \circ \sigma^*_M$, we conclude that $\rho_M(Y) = \sigma^{-1}_M(\sigma_P(\sigma^*_M(Y))$, from which the equivalence follows.

The equivalence between $Y \in (\bar{G}_M)_b$ and $\sigma^*_M(Y) \in (G_P)_{\rho_M(b)}$ implies that the map

$$\rho'_M : Ult(\bar{N}_M, \bar{G}_M) \to Ult(N^*_M, G_P),\ [b, f]_{\bar{G}_M} \mapsto [\rho_M(b), \sigma^*_M(f)]_{G_P}$$

is a Σ_0-embedding, which is cardinal preserving. Also, note that $\rho_M \restriction \bar{\beta}_M = \rho'_M \restriction \bar{\beta}_M$ and $\rho'_M(\bar{\beta}_M) = \omega_2$. Therefore, we have a phalanx embedding:

$$(\rho_M, \rho'_M) : (\bar{N}_M, Ult(\bar{N}_M, \bar{G}_M), \bar{\beta}_M) \to (W, Ult(N^*_M, G_P), \omega_2).$$

We conclude that $(W, Ult(N^*_M, G_P), \omega_2)$ is non-iterable in R, thus proving the claim.

Work in R. Since $(W, Ult(Q, G_P), \omega_2)$ is not iterable, we can find suitable θ and λ, and an iteration tree $\mathcal{T}$ such that

$$H_\theta \models \text{“}\mathcal{T} \text{ witnesses non-iterability of } (W|\lambda, Ult(Q, G_P), \omega_2)\text{”}.$$

Find some countable $X \preceq H_\theta$ such that $\mathcal{T} \in X$. Let H be the transitive collapse of X, with corresponding uncollapsing map $\pi : H \to H_\theta$, and let $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$, $\bar{W} = \pi^{-1}(W|\lambda)$, $\bar{Q} = \pi^{-1}(Q)$, $\bar{G} = \pi^{-1}(G_P)$ and $\beta = \pi^{-1}(\omega_2)$. Then

$$H \models \text{“}\bar{\mathcal{T}} \text{ witnesses non-iterability of } (\bar{W}, Ult(\bar{Q}, \bar{G}), \omega_2)\text{”}.$$

By an absoluteness argument,

$$R \models \text{“}\bar{\mathcal{T}} \text{ witnesses non-iterability of } (\bar{W}, Ult(\bar{Q}, \bar{G}), \omega_2)\text{”}.$$

Now, in V let $M \in S'_1$ such that $\alpha_M > sup(X \cap \omega_2)$. Let $G = G_P|\alpha_M$.

We claim that the extender G restricted to sets in Q agrees with the extender E derived from σ'_M restricted to sets in Q. To see this, first observe that since $\sigma''_M \restriction \tau^*_M = id$, $\alpha_P < \tau^*_M$ and $crit(\sigma_{P,M}) = \alpha_P$, then $crit(\sigma'_M) = \alpha_P$. Let $b \in [\alpha_M]^{<\omega}$ $Y \in Q$, $Y \subseteq [\alpha_P]^{|b|}$. Assume that $Y \in (E)_b$; then $b \in \sigma'_M(Y)$. Since $Y \in [\alpha_P]^{|b|}$, $\sigma''_M(Y) = Y$, and so $b \in \sigma_{P,M}(Y) =$

$\sigma_M^{-1}(\sigma_P(Y))$, so $\sigma_M(b) \in \sigma_P(Y)$. Since $b \in [\alpha_M]^{<\omega}$, then $\sigma_M(b) = b$, so $b \in \sigma_P(Y)$, and thus $Y \in (G)_b$.

It can be easily seen that the map $\pi' : Ult(\bar{Q}, \bar{G}) \to Ult(Q, G)$ given by $[b, f]_{\bar{G}} \mapsto [\pi(b), \pi(f)]_G$ is elementary. Also note that $\pi \restriction \beta = \pi' \restriction \beta$, $\pi'[\beta] \subseteq \alpha_M$ (since $\alpha_M > sup(X \cap \omega_2)$) and $\pi'(\beta) = j_G(\alpha_P) \geq \alpha_M$ (where $j_G : Q \to Ult(Q, G)$ is the ultrapower embedding). Thus, we have a phalanx embedding

$$(\pi, \pi') : (\bar{W}, Ult(\bar{Q}, \bar{G}), \beta) \to (W|\lambda, Ult(Q, G), \alpha_M).$$

We conclude that inside R the phalanx $(W, Ult(Q, G), \alpha_M)$ is non-iterable. We shall fix the structure M for the rest of the proof.

In summary, we have:

- Q is chosen so that, inside R: if $Q' <_S Q$, then $(W, Ult(Q, G_P), \omega_2)$ is iterable. We also have:

  $$(W, Q, \alpha_P) \text{ is iterable in } R, \text{ but } (W, Ult(Q, G_P), \omega_2) \text{ is not iterable in } R.$$

- We fix some M with α_M large enough to achieve that, letting $G = G_P \restriction \alpha_M$,

  $$(W, Ult(Q, G), \alpha_M) \text{ is not iterable inside } R \tag{3.3}$$

- The pair $(\rho_M, \rho'_M) : (\bar{N}_M, Ult(\bar{N}_M, G_M), \bar{\beta}_M) \to (W, Ult(N^*_M, G_P), \omega_2)$ is a phalanx embedding, and so by (3.1):

  $$(W, Ult(N^*_M, G_P), \omega_2) \text{ is not iterable in } R. \tag{3.4}$$

- We have a phalanx embedding $(id, \sigma'_M) : (W, N^*_M, \alpha_P) \to W$, so that (W, N^*_M, α_P) is iterable in R.

Inside R, we now coiterate the (iterable) phalanxes (W, Q, α_P) and (W, N^*_M, α_P), via trees $\mathcal{U}$ and $\mathcal{V}$ respectively. We shall show that $M^{\mathcal{V}}_\infty <_S Q$.

Claim 1: Q is on the main branch of $\mathcal{U}$.

Proof of claim: Otherwise, W is on the main branch of $\mathcal{U}$, and since W is thick, then W cannot be on the main branch of $\mathcal{V}$. Therefore N^*_M is on the main branch of $\mathcal{V}$. Also, since W is universal (being thick), there is no truncation in the main branch of $\mathcal{U}$, so that $M^{\mathcal{V}}_\infty \trianglelefteq M^{\mathcal{U}}_\infty$. This implies that $M^{\mathcal{V}}_\infty <_S Q$, so by minimality we obtain:

$$\text{The phalanx } (W, Ult(M^{\mathcal{V}}_\infty, G_P), \omega_2) \text{ is iterable.}$$

The iteration map $i^* : N^*_M \to M^{\mathcal{V}}_\infty$ has critical point at least α_P, and induces the following map k:

$$k : Ult(N^*_M, G_P) \to Ult(M^{\mathcal{V}}_\infty, G_P), \;\; k([b, f]) = [b, g], \text{ where } g = i^*(f) \restriction [\alpha_P]^{|b|}.$$

Note that $crit(k) \geq \omega_2$, so that we have the following phalanx embedding:

$$(id, k) : (W, Ult(N^*_M, G_P), \omega_2) \to (W, Ult(M^{\mathcal{V}}_\infty, G_P), \omega_2).$$

Therefore the phalanx $(W, Ult(N^*_M, G_P), \omega_2)$ is iterable, a contradiction to 3.4. So Claim 1 is established.

Using the phalanx embedding $(id, \sigma'_M) : (W, N^*_M, \alpha_P) \to W$, we can copy the tree $\mathcal{V}$ to a tree $\mathcal{V}'$ on W. Note that the iteration indexes of $\mathcal{V}'$ are above α_M. Let $\pi_\infty : M^{\mathcal{V}}_\infty \to M^{\mathcal{V}'}_\infty$ be the

copy map between the last models of the corresponding trees.

Claim 2: $b^{\mathcal{U}}$ truncates or $M_\infty^{\mathcal{V}} \triangleleft M_\infty^{\mathcal{U}}$.

Proof of claim: Assume not, so that $b^{\mathcal{U}}$ does not truncate, and $M_\infty^{\mathcal{U}} \trianglelefteq M_\infty^{\mathcal{V}}$. Let $i : Q \to M_\infty^{\mathcal{U}}$ be the iteration map.

Observe that since $crit(i) \geq \alpha_P$, then $Q \cap \mathcal{P}([\alpha_P]^{<\omega}) = M_\infty^{\mathcal{U}} \cap \mathcal{P}([\alpha_P]^{<\omega})$.

Let $W^\infty = \pi_\infty(M_\infty^{\mathcal{U}})$ (stipulating $W^\infty = M_\infty^{\mathcal{V}'}$ if $M_\infty^{\mathcal{U}} = M_\infty^{\mathcal{V}}$).

Observation: Letting $r([b, f]) = \pi_\infty(f)(b)$, we have a phalanx embedding:

$$(id, r) : (W, Ult(M_\infty^{\mathcal{U}}, G), \alpha_M) \to (W, W^\infty, \alpha_M).$$

Observation: (W, W^∞, α_M) embeds into a K^c-generated phalanx, so it must be iterable. Therefore, $(W, Ult(M_\infty^{\mathcal{U}}, G), \alpha_M)$ is iterable. Also note that we have a phalanx embedding:

$$(id, r) : (W, Ult(Q, G), \alpha_M) \to (W, Ult(M_\infty^{\mathcal{U}}, G), \alpha_M), \;\; r([a, f]) = [a, i(f) \restriction [\alpha_P]^{|a|}].$$

This implies by (3.3) that $(W, Ult(M_\infty^{\mathcal{U}}, G), \alpha_M)$ is not iterable, a contradiction. This establishes Claim 2.

By Claims 1 and 2, $M_\infty^{\mathcal{V}} <_S Q$, so:

$$(W_P, Ult(M_\infty^{\mathcal{V}}, G_P), \omega_2) \text{ is iterable.}$$

Notice that we have a phalanx embedding:

$$(id, s) : (W, Ult(N_M^*, G_P), \omega_2) \to (W, Ult(M_\infty^{\mathcal{V}}, G_P), \omega_2).$$

Therefore $(W, Ult(N_M^*, G_P), \omega_2)$ is iterable, contradicting (3.4). This is the final contradiction and the proof is complete.

□

Chapter 4

Future directions

We describe some of the possible lines of direction to continue with our project.

Removal of the threadability condition

One of the hypotheses of Theorem 3.1 is the threadability of ω_3, which allows reflection for mouse operators: if J is a mouse operator that is total in H_{ω_3}, then it is total in H_{ω_4}. This reflection is key in the induction, which allows to obtain a universe R closed under $M_n^{\#}$ below an external measurable Ω, and "tall enough" so that it contains the ideal I (recall that $I \subseteq \mathcal{P}(\omega_3)$, so by the hypothesis $2^{\omega_2} = \omega_3$, I can be coded by a subset of ω_3). The model $M_n^{\#\#}(A)$ (where A is a subset of ω_3 coding the ideal I and the power set of ω_2) is a natural candidate for R. One then applies the Core Model Dichotomy in this universe, where the Core Model is built up to Ω.

If R were not tall enough as to contain I as an element, then the genericity of the measures derived from the inverse collapsing maps over the collapses of the universe becomes unclear. One could try to investigate if only a partial segment of I is needed to obtain self-genericity

from the original $StatCatch^*(I)$ hypothesis, which could lead to a possible result that would not need the mouse reflection tool.

Core model induction beyond $M_n^\#$

Using the core model induction technique we have shown that our set of hypotheses implies the existence of $M_n^\#(a)$ for every $a \in H_{\omega_4}$ and every $n < \omega$. If one wants to continue with the induction, the next step is "accumulate" the mouse operators $M_n^\#$: define F to be the following mouse operator on H_{ω_4}: given $a \in H_{\omega_4}$, let $F(a)$ be the least level of $Lp(a)$ that is closed under every $M_n^\#$ $(n < \omega)$. The operator F provides the setting for the next Core Model Dichotomy: we can define F^1 in an analogous way to how $M_1^\#$ is defined, as follows: $F^1(a)$ is the unique active F-mouse over a that has one Woodin Cardinal, but such that every proper initial segment fails to have Woodin cardinals. By an F-premouse we mean that its levels are built using F, instead of the rudimentary closure operation. We can aim to show that our hypothesis implies that F^1 is total in H_{ω_4}, by ruling out the existence of a Core Model K: this model K would be closed under the operator F, which would guide the iteration strategies for its levels.

Continuing this way, define $F^2, F^3, F^4, \ldots$ in the natural way. To simplify terminology, we can introduce the notation J^w, so that $F^{k+1} = J^w(F_k)$. J^w assigns, to an operator G, the "one-Woodin over G" operator $J^w(G)$. It is reasonable to expect that our set of hypotheses implies that every F^i is total in H_{ω_3}. We would then define G as the "union" of $\{F^n : n < \omega\}$ in the same way that we glued together the operators $M_n^\#$, $n < \omega$.

Summarizing, we can define, for $n, i < \omega$:

- $F_0 = M_0^\#$.
- $F_k^{n+1} = J^w(F_k^n)$.

- F^0_{k+1} = "union" of $\{F^n_k : n < \omega\}$.

By replacing $n + 1$ with a countable ordinal α one can define F^0_α in a similar way, by accumulating the operators F^0_β for $\beta < \alpha$. However, at some stage this approach breaks down, so one needs to take a different approach to define mouse operators. In order to have a Dichotomy Theorem for these later stages, one needs to take into account the complexity of the iteration strategy for the core model. This is a place where Descriptive Set Theory comes into place. For a more formal treatment of this, the reader can consult [8].

Index

$M_n^{\#\#}$, 36

$M_n^{\#}$, 36

Q-structures, 34

κ-preserving ideal, 13

$ClubCatch^*(I)$, 17

$ProjCatch^*(I)$, 17

$StatCatch^*(I)$, 17

active premouse, 25

club catching, 16

club filter, 10

coherent sequence, 40

common part model $\mathcal{M}(\mathcal{T})$, 31

compatible extenders, 24

conditional club filter, 16

countable iterable
    premouse, 28

countably complete filter, 86

density, 14

Dichotomy Theorem, 43

drop of model, 28

extender, 22

fine ideal, 12

generator, 23

generic ultrapower, 13

goodness, 14

homogeneous forcing, 87

ideal, 11

iterable premouse, 28

iteration trees, 27

lower part model, 26

Martin-Steel Theorem, 32

maximal ideal, 12

measurable cardinal, 20

mouse, 26

normal ideal, 12

passive premouse, 25

phalanx, 42

phalanx embedding, 43

precipitousness, 12

premouse, 25

presaturated ideal, 14

principal ideal, 12
projective catching, 16
properly small premouse, 44
restriction
    extender, 24
restriction of an ideal, 11
saturation, 13
self-genericity, 15
self-wellordered set, 25
sharps, 35
simple iteration tree, 28
somewhere precipitous, 12
stationary catching, 16
stationary set, 10
strong ideal, 14
supercompact cardinal, 21
superstrong cardinal, 21
support of an extender, 24
support of an ideal, 11
tame premouse, 34
thread, 40
threadable cardinal, 40
trivial completion
    extender, 24
universe of an ideal, 11
Woodin cardinal, 21

Bibliography

[1] Benjamin Claverie and Ralf Schindler. Woodin's axiom (*), bounded forcing axioms, and precipitous ideals on ω_1. J. Symbolic Logic 77, no. 2, pp. 475 to 498, doi 10.2178/jsl/1333566633. MR2963017, 2012.

[2] Sean Cox and Martin Zeman. Ideal Projections and Forcing Projections. The Journal of Symbolic Logic, 79, pp 1247 to 1285. doi:10.1017/jsl.2013.24, 2014.

[3] Matthew Foreman. *Ideals and Generic Elementary Embeddings*. 2012.

[4] Matthew Foreman, Menachen Magidor, and Saharon Shelah. Martin's maximum, saturated ideals, and nonregular ultrafilters. I, Ann. of Math. (2) 127 (1988), no. 1, pp. 1 to 47. MR924672 (89f:03043), 1988.

[5] Moti Gitik and Saharon Shelah. Less saturated ideals. Proceedings of the American Mathematical Society, 125(5): pp. 1523 to 1530, 1997.

[6] Thomas J. Jech and Karel L. Prikry. On ideals of sets and the power set operation. Bulletin of the American Mathematical Society, 82: pp. 593 to 596, 1976.

[7] Akihiro Kanamori. The Higher Infinite, 2nd Edition. Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[8] Ralf Schindler and John Steel. The Core Model Induction. To appear, 2014.

[9] Saharon Shelah. Iterated forcing and normal ideals on ω_1. Israel Journal of Mathematics 60 (1987), pp. 345 to 380., 1987.

[10] Robert M. Solovay. Real-valued measurable cardinals. In Axiomatic Set Theory, volume 13(1) of Proceedings of Symposia in Pure Mathematics, pp. 397 to 428. American Mathematical Society, Providence, 1971.

[11] John Steel. An Outline of Inner Model Theory. Handbook of Set Theory, Springer, 2010.

[12] John Steel. *The Core Model Iterability Problem*. 2012.

[13] William J. Mitchell Thomas J. Jech, Menachem Magidor and Karel L. Prikry. Precipitous ideals. The Journal of Symbolic Logic, 45(1): pp. 1 to 8, 1980.

[14] Martin Zeman. Inner Models and Large Cardinals. de Gruyter Series in Logic and its Applications, vol. 5, Walter de Gruyter Co., Berlin., 2002.

Appendix A

Appendix

A.1 Countably complete ultrafilters

Definition A.1. A filter is U *countably complete* if and only if for every family $\langle X_n : n < \omega \rangle \subseteq U$, $\bigcap_{n<\omega} X_n \neq \varnothing$.

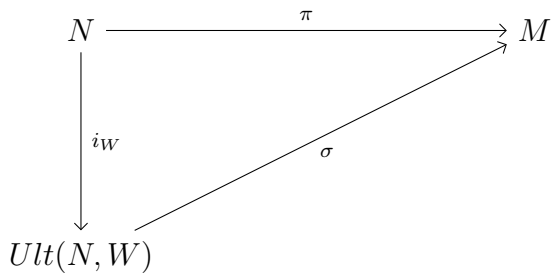
Let M be a model of ZFC^-. Recall that by an M-ultrafilter on κ we mean a filter U on κ such that for every $A \in \mathcal{P}(\kappa) \cap M$, $A \in U$ or $\kappa - A \in U$. It may not be the case that $U \in M$.

Theorem A.2. *Let M be a model of ZFC^-, and let U be a normal M-ultrafilter on a cardinal κ. The following are equivalent:*

(a) U is countably complete.

(b) For every countable model N and every elementary map $\pi : (N, W) \to (M, U)$, there is an elementary map $\sigma : Ult(N, W) \to M$ such that $\pi = \sigma \circ i_W$ (where $i_W : N \to Ult(N, W)$ is the ultrapower embedding).

A map σ as in the previous theorem is called a π-*realization* of $Ult(N, W)$ into M.



A.2 Homogeneous forcing

Definition A.3. A forcing $\mathbb{P}$ is *homogeneous* if and only if for every p and q in $\mathbb{P}$, there is an automorphism f of $\mathbb{P}$ such that $f(p)$ and q are compatible.

Note: sometimes in the literature the term *weakly homogeneous* is used to denote homogeneous forcing, reserving the term *homogeneous* for those cases in which one can actually obtain $f(p) = q$ in the definition.

Theorem A.4. *Let $\mathbb{P}$ be an homogeneous forcing, let $a \in V$, and let G be a $\mathbb{P}$-generic filter over V. Suppose that $b \in V[G]$ and b is definable over $V[G]$ from parameter a. Then $b \in V$.*

An important example of an homogeneous forcing is the forcing $Col(\lambda, S)$ (where λ is a regular cardinal and S is a set of ordinals).