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University of California
Los Angeles

A Statistical Modeling Methodology for the Analysis of
Term Structure of Credit Risk and its Dependency

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Statistics

by

Jiashen You

2013

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Abstract of the Dissertation

A Statistical Modeling Methodology for the Analysis of
Term Structure of Credit Risk and its Dependency

by

Jiashen You

Doctor of Philosophy in Statistics

University of California, Los Angeles, 2013

Professor Yingnian Wu, Co-chair

Professor Tomohiro Ando, Co-chair

Many traditional mathematical finance models attempt to evaluate the time-varying credit risk term structure through various pricing formulae that assume certain stochastic dynamics. The Black-Scholes-Merton model has not only been recognized as one of the most significant contributions in economics with the Nobel Prize, but seen numerous extensions over the years. In naming a ‘fair’ price of any financial instrument, a measure of risk or uncertainty needs to be carefully specified. Common mathematical models rely on partial differential equations and can conveniently express drift and volatility explicitly in a geometric Brownian motion. Such models are widely used today for their simplicity, easy interpretation and robust estimates. However, some major limitations, such as the assumption of a stationary process or market equilibrium which is unrealistic, persist despite various extensions of the basic model.

In this dissertation, we propose a statistical methodology for modeling credit risk in a financial market. Without specifying the dynamics in a partial differential equation approach, we attempt to model the time-varying implied default probability of a firm in a complete market through the use of hazard term structure. While our pricing formulae are not complicated and additive in nature, they are

intuitive, theoretically sound and easy to generalize. Contrary to most other statistical approaches, normality is not directly assumed in the pricing error. We do, however, propose methods to effectively capture the correlation structure among residuals, both in cross-sectional studies and when time-series data is involved. Such implied default correlation structure could be useful in many practical applications such as, and not limited to, credit risk management, portfolio selection and stress tests. Empirical studies have been conducted using financial market data in Japan. We report findings by specifically comparing the market behavior before and after the 2007 - 2009 financial crisis. Various computational and theoretical advantages of our methodology are discussed.

The dissertation of Jiashen You is approved.

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2013

*To my mother ...
thanks for watching over me from above,
my father ...
life gets better from now onwards,
and my brother ...
cannot have done it without you.*

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PUBLICATIONS AND PRESENTATIONS

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CHAPTER 1

Introduction

Systematic credit default risk for corporate bonds has conventionally been studied using term structure of hazard rates. Much like the term structure of interest rates, the name suggests the dependency on time to maturity of, and hints at an implicit pricing model for, the underlying bonds. Many mathematical finance models for estimating term structure have placed emphases on market equilibrium or the no-arbitrage assumption as well as been in pursuit of Markovian properties through stochastic processes. While economic theories are crucial to the interpretability of some model parameters, there has been an increasing demand for practical and adaptive modeling procedures as the 2008 financial crisis had shown that markets do not always behave as theoretical models predict.

In this dissertation, we employ the statistical modeling framework of Ando (2012) for the analysis of the term structures of risk-free interest rate and hazard, as well as the loss given default (LGD). In addition to the existing model, we attempt to model the credit risk dependency structure and implied default correlation. The LGD of corporate bonds, in the form of a fraction of unity maturity value, is modeled by a discrete recovery rate upon a default event. We recognize the importance of the validity of key economic theories while seeking applicability and ease of expansion for our approach. When estimating credit risk from corporate bonds, following Ando (2012), we incorporate diverse information such as credit ratings, term structure for interest rate as well as the financial strength of the companies. Credit risk dependency structure between industries is estimated

from the aggregated residuals. Other parameters are divided into different credit rating classes which allows us to capture the mean structure of hazard functions for each rating class.

Modeling the term structure of interest rates from bonds with various time to maturity have long been of interest to researchers in financial econometrics and related disciplines. On the mathematical finance front, Vasicek (1977) and Cox et al. (1985) pioneered using a Markov process with averaging continuous interest rate paths. As in many similar applications, the yield curve cannot be determined exogenously. While these models are capable of ensuring desirable properties such as mean-reversion, the theoretical nature, however, also exposes them to less explainable yet possible scenarios such as negative interest for the Vasicek model.

McCulloch (1971, 1975) was the first to propose a statistical estimation technique, using spline functions to model the discount and yield curves from observations on prices of bonds with varying maturities and coupon rates. The use of the least squares method greatly speeds up the estimation process of spline coefficients, provided that the pricing formula for non-defaultable bonds is additive. Some notable extensions and successive improvements of such approximation models can be found in Vasicek (1982), Shea (1985), Chambers et al. (1984) and Krivobokova et al. (2006).

There is an extensive amount of finance literature on modeling risk structure on defaultable bonds. The Black-Scholes model (Black and Scholes (1973)) for option pricing and Merton's immediately-followed extensions (Merton (1973, 1974)) have been treated by many as sacrosanct in this field. Most of the similar stochastic models with equilibrium-theory assumption have reaped the benefit of Itô's Lemma (Itô (1951)). While mathematical models may be solved analytically and, in some cases, even closed-form solutions are obtained (Amin and Jarrow (1991), Heston (1993)), oversimplifying the dynamics could hinder expressing complex

market reality.

When a default event occurs, bond holders typically receive a fraction of the par value and write the remaining amount at a loss. By introducing hazard rate into the term structure for interest rate, Duffie and Singleton (1999), Jarrow and Turnbull (1995) laid the foundation for the reduced-form model in a continuous-time Markovian setting. However, the LGD in this model is usually a given parameter.

From the situation-dependent industry perspective, making predictions for distance to defaults or LGD can be as important as estimating the numbers and probabilities of defaults. Both statistical (Altman et al. (2005), Covitz and Han (2004)) and mathematical (Hamilton et al. (2001), Jarrow et al. (2006)) models have been employed. It is also worth noting that in recent artificial intelligence literature, various attempts have been made to model interest rate and default risk using neural network models (Jeong et al. (2012), Khashman (2010)), fuzzy-knowledge based models (Ju et al. (1997), Streit and Borenstein (2012)) or support vector machine (Li et al. (2012), Perko et al. (2011), Ribeiro et al. (2012), Yu et al. (2010, 2011)). These new approaches offer insights from unconventional angles, but their estimates may suffer loss on interpretability.

The second chapter employs a recently proposed model (Ando (2012), Kariya (2012)) that can significantly reduce the model mis-specification risk of underlying stochastic processes for corporate bond pricing. For ease of computation as well as its smoothing effect, B-splines are used to estimate the mean hazard rate structure separated by each rating class. Covariates such as company financial strength are used to boost the practicality of our model. LGD is approximated by using a logistic function of current model estimates and its value is well-defined.

In practice, bonds within the same corporation; companies within the same sector; industries within the same economy are all correlated. Therefore, it is imperative to model such dependency. Since this implied dependency is model-

specific and is not observed directly, many studies have attempted to quantify default correlation, either analytically (Zhou (1997)) or empirically (Cowan and Cowan (2004), You and Ando (2012), Zhou (2001)). For the model discussed in this chapter, industry effect is controlled as a covariate. Dimension-reduction technique is introduced to efficiently model the correlation structure in the error terms. Empirical results reveal that within-industry dependency is more influential than between-industry dependency.

The remaining chapters of this dissertation is organized as follows. Chapter 2 describes the full model for the term structure of the forward instantaneous interest rate and hazard function separately. A basic model for dependency structure and its parametrization is specified. Several key assumptions for the estimation procedure are discussed. Empirical cross-sectional analyses on bond data from Japan are presented to conclude the chapter. Chapter 3 describes a plausible pricing model for credit default swaps (CDS) rates by combining the interest and hazard functions into a discount term structure. We derive an analytic form for estimating default correlation from time-series data using a dynamic Bayesian procedure. The results section lists our findings in comparing the default correlation structure both at market level and individual corporate level before and after the 2007 - 2009 financial crisis. We conclude this dissertation in Chapter 4 with some final remarks and a discussion on future extensions and practical applications.

CHAPTER 2

A statistical modeling methodology for the analysis of term structure of credit risk and its dependency

In this chapter, we adopt a recently established statistical scheme by Ando (Ando 2012) for joint estimation of hazard term structure, interest rate term structure, and LGD without the need to explicitly state the normality assumption for error terms. A framework for modeling the dependency structure is also discussed. Situation dependent, this modeling scheme can be used to set up an analytic analysis for implied default correlation as illustrated in Chapter 3. The empirical findings from related models in this chapter are explained in Section 2.3.

2.1 Model specification

We first provide an overview for the pricing of corporate bonds as well as define the term structure of hazard and interest. The main objective here is not on improving any well-studied mathematical pricing formula, but rather to integrate scattered information together with economically sound statistical models. We strive for the ease of computation, generalization and interpretability of our model parameters.

2.1.1 Term structure for risk-free instantaneous forward interest rate

To begin, the theoretical pricing formula of a non-defaultable bond is examined. We assume for a risk-free government bond with fixed coupon amount C and a par value of M redeemable at maturity. A total of L successive payments for the bond are made at time $\mathbf{t} = (t_1, \dots, t_L)'$, measured in years from the current time $t_0 = 0$. An example of cashflow for such a bond is shown in Figure 2.1

Figure 2.1: A graphical illustration of cashflow for bond holders

Let $r(t)$ be the term structure of the risk-free instantaneous forward rate, the present value (PV) of a government bond with a maturity date t_L years from now can be computed from the sum of its discounted cash flow:

$$\text{PV}(r(\cdot), \mathbf{t}) = M \cdot \exp \left[- \int_{t_0}^{t_L} r(u) du \right] + \sum_{\gamma=1}^L C \cdot \exp \left[- \int_{t_0}^{t_\gamma} r(u) du \right]. \quad (2.1)$$

Denoting by $D_g(t)$ the attribute-dependent stochastic discount function,

$$D_g(t) = \exp \left\{ - \int_{t_0}^t r(u) du \right\}, \quad (2.2)$$

Equation (2.1) becomes

$$\text{PV}(D_g(\cdot), \mathbf{t}) = M \cdot D_g(t_L) + \sum_{\gamma=1}^L C \cdot D_g(t_\gamma). \quad (2.3)$$

We assume that $D_g(t)$ can be decomposed into the group mean function $\bar{D}_g(t)$ and the stochastic deviation term $\Delta_g(t)$ simply by:

$$D_g(t) = \bar{D}_g(t) + \Delta_g(t). \quad (2.4)$$

It is common knowledge that theoretical pricing models such as Equation (2.3) may not give accurate predictions for the prices of bonds. Therefore, error terms must be included in the corresponding statistical model to account for those variation.

$$\text{PV}(D_g(\cdot), \mathbf{t}) = M \cdot \bar{D}_g(t_L) + \sum_{\gamma=1}^L C \cdot \bar{D}_g(t_\gamma) + \varepsilon, \quad (2.5)$$

where $\varepsilon = M \cdot \Delta_g(t_L) + \sum_{\gamma=1}^L C \cdot \Delta_g(t_\gamma)$ is regarded as a noise term.

2.1.2 Term structure for hazard

Let T be a random variable that measures the time to default. The survival function is defined as

$$D_h(t) = \Pr(T \geq t). \quad (2.6)$$

A hazard function that specifies the instantaneous rate of default at $T = t$ conditioned on the survival up to time T can be defined as

$$h(t) = \lim_{\Delta t \rightarrow 0^+} \frac{\Pr(t \leq T < t + \Delta t | T \geq t)}{\Delta t} = -\frac{d \ln D_h(t)}{dt} \quad (2.7)$$

This $h(t)$, hereafter called as the hazard term structure (an instantaneous default intensity), can be considered as the standardization of the conditional probability of a bond which has not defaulted up till time t defaulting within the next time period $(t, t + \Delta t)$, where Δt is a small dimensionless measure of time. The survival

probability with the above hazard term structure of a bond up to time t is therefore

$$D_h(t) = \exp \left\{ - \int_{t_0}^t h(u) du \right\}. \quad (2.8)$$

If we assume the hazard term structure as in Equation (2.8), the present value of a corporate bond can be expressed as

$$\begin{aligned} \text{PV}(D_c(\cdot), h(\cdot), \delta, \mathbf{t}) &= \sum_{\gamma=1}^L C \cdot \exp \left[- \int_{t_0}^{t_\gamma} \{r(u) + h(u)\} du \right] \\ &\quad + M \cdot \exp \left[- \int_{t_0}^{t_L} \{r(u) + h(u)\} du \right] \\ &\quad + M \cdot \delta \cdot \exp \left[- \int_{t_0}^{t_L} r(u) du \right] \left[1 - \exp \left(- \int_{t_0}^{t_L} h(u) du \right) \right] \\ &= \sum_{\gamma=1}^L C \cdot D_c(t_\gamma) + M \cdot D_c(t_L) \\ &\quad + M \cdot \delta \cdot D_g(t_L) [1 - D_h(t_L)], \end{aligned} \quad (2.9)$$

where δ is the proportion of the par value recoverable upon default of the bond. The first two terms in Equation (2.9) are the discount present value of coupon payments and the face-value redemption payment. The third term is the discounted present value of the amount recovered in the case of default. In such an event, there will be no more coupon payments following the incident. Since the actual LGD is unobserved and may take a long time to settle on a concrete value, we attempt to estimate the implied recovery rate. Compared to the risk-free government bonds, corporate bonds with a default risk tend to have a higher discount term, namely,

$$D_c(t) = D_g(t) \cdot D_h(t) = \exp \left\{ - \int_{t_0}^t (r(u) + h(u)) du \right\}, \quad (2.10)$$

where $r(t)$ and $h(t)$ are the term structure of the risk-free instantaneous forward rate and the hazard term structure, respectively. Using this expression, we assume that the hazard term structure is independent of the risk-free interest rate term structure.

Similar to the discount function for a government bond, we assume that $D_c(t)$ can be decomposed into the mean function of default probability $\bar{D}_c(t)$ and the stochastic deviation function $\Delta_c(t)$ according to:

$$D_c(t) = \bar{D}_c(t) + \Delta_c(t) = \bar{D}_g(t) \cdot \bar{D}_h(t) + \Delta_c(t), \quad \text{and} \quad D_h(t) = \bar{D}_h(t) + \Delta_h(t),$$

where

$$\bar{D}_h(t) = \bar{P}(T > t) = \exp \left\{ - \int_{t_0}^t \bar{h}(u) du \right\}, \quad (2.11)$$

and $\bar{h}(t)$ is the mean structure of the hazard function to be specified.

Therefore, Equation (2.9) becomes

$$\begin{aligned} \text{PV}(D_c(\cdot), h(\cdot), \delta, \mathbf{t}) &= \sum_{\gamma=1}^L C \cdot \bar{D}_c(t_\gamma) + M \cdot \bar{D}_c(t_L) \\ &\quad + M \cdot \delta \cdot \bar{D}_g(t_L) [1 - \bar{D}_h(t_L)] + \varepsilon, \end{aligned} \quad (2.12)$$

where ε is random error. Kariya considered a more sophisticated model in his recent working paper (Kariya 2012). For the time being we use Equation (2.12) for pricing corporate bonds in order to simplify the procedure for updating parameter values.

2.1.3 Model parametrization

Without loss of generality, we may assume the following structure for the mean risk-free interest term structure $\bar{D}_g(t)$ (Kariya and Tsuda (1996, 2000)):

$$-\log [\bar{D}_g(t)] = \sum_{k=1}^m w_k \phi_k(t) = \mathbf{w}'_0 \boldsymbol{\phi}_0(t), \quad (2.13)$$

where $\boldsymbol{\phi}_0(t) = (\phi_1(t), \dots, \phi_m(t))'$ is a vector consisting of suitable basis functions with m nodes and $\mathbf{w}_0 = (w_1, \dots, w_m)'$ are the unknown risk-free interest rate parameters to be estimated.

The government bond pricing model can now be expressed as:

$$P_G = \sum_{\gamma=1}^L C_G \cdot \exp \{ -\mathbf{w}'_0 \boldsymbol{\phi}_0(t_\gamma) \} + M \cdot \exp \{ -\mathbf{w}'_0 \boldsymbol{\phi}_0(t_L) \} + \varepsilon_G. \quad (2.14)$$

A similar assumption is made on the structure of the mean hazard function, except in this case, we also insert company-specific covariates such as balance sheet and profit/loss information. While the mean risk-free interest rate refers to the fact that coefficients from the assumed structure are obtained from averaging all non-defaultable government bonds, the coefficients within each mean hazard function, in contrast, is from averaging from all corporate bonds of the same credit rating division. For such a mean hazard function with industry indicator i and rating indicator j , we assume:

$$\begin{aligned} -\log [\bar{D}_c(t)] &= -\log [\bar{D}_g(t)] - \log [\bar{D}_h(t)] \\ &= \mathbf{w}'_0 \phi_0(t) + \mathbf{w}'_j \phi_j(t) \times \exp \left\{ \sum_{k=1}^I \alpha_{hk} \delta(i, k) + \sum_{\ell=1}^p \beta_{h\ell} x_\ell \right\}, \end{aligned} \quad (2.15)$$

where $\phi_j(t) = (\phi_{j1}(t), \dots, \phi_{jm}(t))'$ is a vector of basis functions (and its length may vary), $\mathbf{w}_j = (w_{j1}, \dots, w_{jm})'$ are the unknown mean hazard function parameters for the j -th credit rating class, $\beta_h = (\beta_{h1}, \dots, \beta_{hp})'$ are the coefficients for the company-level financial information covariates, and finally $\alpha_h = (\alpha_{h1}, \dots, \alpha_{hI})'$ are the parameters for industry indicators ($i = 1, \dots, I$). This means that $\delta(i, k)$ is the delta function whereby $\delta(i, k) = 1$ if $i = k$ and 0 otherwise. We use these indicators to control the industry-level effect in the pricing model. The p -dimensional vector $\mathbf{x} = (x_1, \dots, x_p)'$ denotes the set of variables related to the credit risk of the bond issuers.

Combining Equation (2.11) and (2.15) yields

$$\int_{t_0}^t \bar{h}(u) du = -\log [\bar{D}_h(t)] = \mathbf{w}'_j \phi_j(t) \times \exp \left\{ \sum_{k=1}^I \alpha_{hk} \delta(i, k) + \sum_{\ell=1}^p \beta_{h\ell} x_\ell \right\}. \quad (2.16)$$

Hence, we may regard the derivative of the above term as the mean structure of the hazard function simply using the Fundamental Theroem of Calculus. Let

$\phi_j(t) = \int_{t_0}^t \psi_j(u) du$ (and note that this is easily achieved computationally when, for example, cubic or B-spline basis functions are used), then the mean structure of the hazard function for the j -th credit rating class can be explicitly expressed as

$$\bar{h}(t) = \mathbf{w}'_j \psi_j(t) \times \exp \left\{ \sum_{k=1}^I \alpha_{hk} \delta(i, k) + \sum_{\ell=1}^p \beta_{h\ell} x_\ell \right\}. \quad (2.17)$$

One may consider the first inner product term as the baseline mean hazard structure for this rating class over time, while different industry level and/or the most up-to-date company's financial information variable $\mathbf{x}$ may simply shift the smooth curve vertically.

Since the implied LGD can only be endogenously determined, we apply a logistic function using a different group of covariates:

$$\delta(\mathbf{x}) = \frac{\exp \left\{ \sum_{k=1}^I \alpha_{\delta k} \delta(i, k) + \sum_{\ell=1}^p \beta_{\delta \ell} x_\ell \right\}}{1 + \exp \left\{ \sum_{k=1}^I \alpha_{\delta k} \delta(i, k) + \sum_{\ell=1}^p \beta_{\delta \ell} x_\ell \right\}}, \quad (2.18)$$

where $\boldsymbol{\beta}_\delta = (\beta_{\delta 1}, \dots, \beta_{\delta p})'$ and $\boldsymbol{\alpha}_\delta = (\alpha_{\delta 1}, \dots, \alpha_{\delta I})'$ are the parameters to be estimated.

For the empirical analysis section, we fit our proposed model only on the corporate bonds with a complete set of data. That is, we assume for every observed corporate bond price P_C in our data set, we observe its coupon rate C_C , its latest credit rating class $j \in \{1, \dots, J\}$, and its industry classifier $i \in \{1, \dots, I\}$. The company's latest financial information $\mathbf{x} = (x_1, \dots, x_p)'$ is also available for the issuer of each corporate bond. Under the premise of structure defined by (2.12), (2.17) and (2.18), the value of a corporate bond with credit rating j and industry

code i can be expressed as:

$$\begin{aligned}
P_{j,C} = & \sum_{\gamma=1}^L C_{j,C} \times \\
& \exp \left[-\mathbf{w}'_0 \boldsymbol{\phi}_0(t_\gamma) - \mathbf{w}'_j \boldsymbol{\phi}_j(t_\gamma) \times \exp \left\{ \sum_{k=1}^I \alpha_{hk} \delta(i, k) + \sum_{\ell=1}^p \beta_{h\ell} x_\ell \right\} \right] \\
& + M \times \exp \left[-\mathbf{w}'_0 \boldsymbol{\phi}_0(t_L) - \mathbf{w}'_j \boldsymbol{\phi}_j(t_L) \times \exp \left\{ \sum_{k=1}^I \alpha_{hk} \delta(i, k) + \sum_{\ell=1}^p \beta_{h\ell} x_\ell \right\} \right] \\
& + M \times \frac{\exp \left\{ \sum_{k=1}^I \alpha_{\delta k} \delta(i, k) + \sum_{\ell=1}^p \beta_{\delta \ell} x_\ell \right\}}{1 + \exp \left\{ \sum_{k=1}^I \alpha_{\delta k} \delta(i, k) + \sum_{\ell=1}^p \beta_{\delta \ell} x_\ell \right\}} \times \exp \{ -\mathbf{w}'_0 \boldsymbol{\phi}_0(t_L) \} \\
& \times \left[1 - \exp \left[-\mathbf{w}'_j \boldsymbol{\phi}_j(t_L) \times \exp \left\{ \sum_{k=1}^I \alpha_{hk} \delta(i, k) + \sum_{\ell=1}^p \beta_{h\ell} x_\ell \right\} \right] \right] + \varepsilon_C.
\end{aligned} \tag{2.19}$$

Note that $\mathbf{x}$ and $\mathbf{t} = (t_1, \dots, t_L)$ vary over different bonds, but we use the common terms $\mathbf{x}$ and $\mathbf{t}$ for simplicity. The unknown parameters so far are $\mathbf{w}_j = (w_{j1}, \dots, w_{jm})'$ ($j = 1, \dots, J$), $\boldsymbol{\alpha}_h$ and $\boldsymbol{\beta}_h$ in the hazard function, and $\boldsymbol{\alpha}_\delta$ and $\boldsymbol{\beta}_\delta$ for the implied recovery rate.

The final step of parametrization is applied to the variance-covariance matrix of the error terms in order to capture any remaining correlation structure left unexplained by our model. The task lies with modeling the correlation structure of $\boldsymbol{\Sigma}$, the covariance-variance matrix of error terms, as a means to explain the expected correlation between defaultable bonds. Due to the concern on the size of the matrix in optimization procedure as well as overfitting, we choose to only model between-industry and within-industry effects. That is, for a total of n observed corporate bonds $P_{j,C}$, we first sort them into a vector in ascending order of their industry indicator variable so that $P_c = (\mathbf{p}_1, \dots, \mathbf{p}_I)$, where $\mathbf{p}_i$ is the n_i -dimensional vector of observed prices for bonds from industry i . We may effectively reduce the dimension of the covariance matrix by modeling it with a

block matrix form. In particular, we assume that

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} & \cdots & \mathbf{\Sigma}_{1I} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} & \cdots & \mathbf{\Sigma}_{2I} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Sigma}_{I1} & \mathbf{\Sigma}_{I2} & \cdots & \mathbf{\Sigma}_{II} \end{pmatrix}, \quad (2.20)$$

where each diagonal block $\mathbf{\Sigma}_{ii}$ is indeed variance-covariance matrices for bonds in the i -th industry and is parametrized by

$$\mathbf{\Sigma}_{ii} = \begin{pmatrix} \sigma_i^2 & \rho_{ii}\sigma_i^2 & \cdots & \rho_{ii}\sigma_i^2 \\ \rho_{ii}\sigma_i^2 & \sigma_i^2 & \cdots & \rho_{ii}\sigma_i^2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{ii}\sigma_i^2 & \rho_{ii}\sigma_i^2 & \cdots & \sigma_i^2 \end{pmatrix} = \sigma_i^2 \begin{pmatrix} 1 & \rho_{ii} & \cdots & \rho_{ii} \\ \rho_{ii} & 1 & \cdots & \rho_{ii} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{ii} & \rho_{ii} & \cdots & 1 \end{pmatrix}. \quad (2.21)$$

The off-diagonal blocks $\mathbf{\Sigma}_{ij}$'s are not square matrices. We model them simply by

$$\mathbf{\Sigma}_{ij} = \begin{pmatrix} \rho_{ij}\sigma_i\sigma_j & \rho_{ij}\sigma_i\sigma_j & \cdots & \rho_{ij}\sigma_i\sigma_j \\ \rho_{ij}\sigma_i\sigma_j & \rho_{ij}\sigma_i\sigma_j & \cdots & \rho_{ij}\sigma_i\sigma_j \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{ij}\sigma_i\sigma_j & \rho_{ij}\sigma_i\sigma_j & \cdots & \rho_{ij}\sigma_i\sigma_j \end{pmatrix} = \rho_{ij}\sigma_i\sigma_j \mathbf{1}_{n_i \times n_j}. \quad (2.22)$$

The set of all parameters in the above should be updated with appropriate conditions to maintain the interpretability. We may collect the correlation coefficients from $\mathbf{\Sigma}$ and form the dependency matrix:

$$\Lambda_{I \times I} = (\rho_{ij}) \quad (2.23)$$

Readers need to be reminded that although this is not the only method to model the correlation structure of $\mathbf{\Sigma}$, it is a viable and efficient way to set up the parametrization and guarantees that the corresponding matrix is positive-definite.

2.2 Parameter estimation

In this section we make a series of five assumptions that, in practice, do not impede model flexibility nor jeopardize the interpretability of model estimates; while in theory, provide the support needed for convergence and even asymptotic normality. Details of the updating scheme are given in Appendix A.

A typical method for estimating the parameters is the least-squares method, which minimizes

$$\sum_{i=1}^{n_g} (\varepsilon_{G,i})^2 + \sum_{j=1}^J \sum_{i=1}^{n_j} (\varepsilon_{j,i})^2 := \sum_{k=1}^n \{\varepsilon_k(\boldsymbol{\theta})\}^2 = \boldsymbol{\varepsilon}(\boldsymbol{\theta})' \boldsymbol{\varepsilon}(\boldsymbol{\theta}) \quad (2.24)$$

Assumption 1: The squared loss function $\ell(\boldsymbol{\theta})/n$ in (2.24) converges to a population limit $\ell_0(\boldsymbol{\theta})$ when the sample size n increases. There exists a function $\ell_0(\boldsymbol{\theta})$ that has a unique minimum in a compact subset of Euclidean space at $\boldsymbol{\theta}_0$.

We opt to minimize the squared loss function

$$\ell(\boldsymbol{\theta}) = \boldsymbol{\varepsilon}(\boldsymbol{\theta})' \boldsymbol{\Sigma}^{-1} \boldsymbol{\varepsilon}(\boldsymbol{\theta}) = \left[\boldsymbol{\Sigma}^{-1/2} \boldsymbol{\varepsilon}(\boldsymbol{\theta}) \right]' \left[\boldsymbol{\Sigma}^{-1/2} \boldsymbol{\varepsilon}(\boldsymbol{\theta}) \right], \quad (2.25)$$

where $\boldsymbol{\Sigma}$ is the variance-covariance matrix for the error terms by our specification and $\boldsymbol{\theta} = (\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_J, \boldsymbol{\alpha}_h, \boldsymbol{\alpha}_\delta, \boldsymbol{\beta}_h, \boldsymbol{\beta}_\delta, \boldsymbol{\rho}, \boldsymbol{\sigma})'$.

Assumption 2: For any $\delta > 0$, there exists $\varepsilon > 0$ such that

$$\lim_{n \rightarrow \infty} \inf P \left(\sup_{|\boldsymbol{\theta} - \boldsymbol{\theta}_0| \geq \delta} \frac{1}{n} (\ell(\boldsymbol{\theta}) - \ell(\boldsymbol{\theta}_0)) \leq -\varepsilon \right) = 1.$$

Assumption 3: There exist $\Delta_n, R_n, \Omega = O(1)$ such that $R_n \rightarrow -\Omega$ as $n \rightarrow \infty$. If we write

$$\ell(\boldsymbol{\theta}) = \ell(\boldsymbol{\theta}_0) + (\boldsymbol{\theta} - \boldsymbol{\theta}_0)' \Delta_n - \frac{1}{2} (\boldsymbol{\theta} - \boldsymbol{\theta}_0)' (n R_n) (\boldsymbol{\theta} - \boldsymbol{\theta}_0) + K_n(\boldsymbol{\theta}),$$

then the following holds for any sequence of $\delta_n \rightarrow 0$:

$$\sup_{|\boldsymbol{\theta} - \boldsymbol{\theta}_0| \leq \delta_n} \frac{K_n(\boldsymbol{\theta})}{1 + n(\boldsymbol{\theta} - \boldsymbol{\theta}_0)'(\boldsymbol{\theta} - \boldsymbol{\theta}_0)} = o_p(1).$$

Assumption 4: $n^{-1}\ell(\boldsymbol{\theta}_0) - \ell_0(\boldsymbol{\theta}_0) = o_p(1)$.

Assumption 5: $R_n^{-1/2}\Delta_n/\sqrt{n} \rightarrow N(0, I)$, where I is the unit diagonal matrix.

Under Assumptions 1 $\sim$ 5, Ando (2012) investigated an asymptotic behavior of $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$. For more details, see Ando (2012) and references therein. Using the results of Ando (2012), $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$ has an asymptotic multivariate normal distribution with a mean of zero and the covariance matrix $\{V(\hat{\boldsymbol{\theta}})\}^{-1}U(\hat{\boldsymbol{\theta}})\{V(\hat{\boldsymbol{\theta}})\}^{-1}$, which can be estimated by

$$U(\hat{\boldsymbol{\theta}}) = \frac{1}{n} \sum_{k=1}^n \frac{\partial e_k(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial e_k(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \quad \text{and} \quad V(\hat{\boldsymbol{\theta}}) = -\frac{1}{n} \sum_{k=1}^n \frac{\partial^2 e_k(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}},$$

with $\Sigma^{-1/2}\boldsymbol{\varepsilon}(\boldsymbol{\theta}) = \mathbf{e}(\boldsymbol{\theta})$. Here ε_j is each of the individual error term in (2.25), and n is the number of observations. Statistical significance tests of model parameters can be performed if needed.

2.3 Empirical analysis

Many countries are still dealing with the aftermath of the recent major economic recession caused by the subprime crisis in 2007. In the United States, the end of the financial crisis was announced in 2009. However, default risk structure in credit market has undergone noticeable changes as shown in You and Ando (2012). To this day, several European countries and many corporates in different parts of the world are struggling to prevent massive defaults that could drag the global economy into another major retreat. While our forward-looking model could be used for making predictions, we do not intend to assess its predictive power. Rather, we are interested in exploring any structural change in hazard as well as recovery rates before and after a major economic event.

To investigate such impact, we implemented our model using the financial market data from Japan. Bond data were obtained from the Japan Securities Dealers Association. We used credit rating data from Rating and Investment

Figure 2.2: Daily close for Nikkei225 from July 2007 to February 2011 (source: Yahoo Finance)

Information, Inc. (R&I). Balance sheet and P/L sheet information was obtained from the Nikkei NEEDS database.

In Japan, the effect of the subprime crisis did not escalate until late 2007. During September and October of 2008, the Japanese stock market took a nosedive and Nikkei225 index dropped roughly 33% in two months (Figure 2.2). The index has not been able to recover to its pre-September-2008 level since then. We picked April 4th and December 1st of 2008 for our empirical analysis in hoping to reveal and explain possible difference in the term structure for PD and LGD as well as the interdependency among various industries.

For our analysis on hazard term structure, it is necessary to specify the total number of credit ratings (J) for the analysis. In this study, firms of investment-grade bonds were split into three ($J = 3$) groups:

- group 1: AAA, AA+, AA and AA- ratings;
- group 2: A+, A and A- ratings;
- group 3: BBB+, BBB, BBB- ratings.

In addition to the industry indicators, we used two corporate-specific financial variables: the capital adequacy ratio (x_1 ; $CAR = \text{capital} / (\text{capital} + \text{debt})$), which

measures the possibility of a capital deficit, and net income (NI, in million Yen), which is a ratio used to measure profitability and efficiency. We transformed net income to reduce its magnitude such that $x_2 = \text{sign}(\text{NI}) \times \log_{10}|\text{NI}|$.

Table 2.1 lists the number of observed bond prices on those two dates.

Industry	Index i	08-04-04	08-12-01
Government	0	282	290
Construction	1	24	23
Food	2	44	46
Chemicals, Pulps, Pharmaceuticals	3	86	83
Oil and steel	4	106	109
Machinery	5	148	133
General merchants	6	74	66
Retail	7	19	17
Financial (Banking, Insurance, Real estate)	8	204	196
Utility (Transportation, Electricity, Gas)	9	331	335
Service, Others	10	21	18
total (defaultable)		1057	826

Table 2.1: Number of observed bonds

For the 2007-2009 global recession, the financial industry was blamed as the main culprit for overselling subprime mortgages and trading risky derivatives such as credit default swaps. Some major financial firms were hit the hardest and many more smaller-sized firms have gone bankrupt. Figure 2.3 shows the implied PD for all bonds in the financial industry for the selected dates. The coefficients in the hazard functions are collectively estimated for each individual credit rating class. While the same baseline values may force curves from the same rating class to assume similar shape, our term structure model will still reveal any major difference in the overall estimated PD. The implied probability of default (PD) for

each bond belonging to the first, second and third rating class is shown in solid black, dashed red, and dotted green lines, respectively. In general, companies with higher credit ratings tend to be able to sell bonds for longer maturity terms. Firms with lower ratings are considered more volatile and some issue bonds that mature in less than 5 years. To avoid potential extrapolation, we plot only the implied PD for all bonds up to 3 years.

We should first note that on average, the implied PD curves have elevated from April (Figure 2.3a) to December (Figure 2.3b). Under our pricing model assumptions, bonds issued by financial firms are seen by investors to have an overall higher chance of default after the major stock market crash in the fall of 2008. With respect to each credit rating class, top-rated financial corporations' bonds have the lowest premium over risk-free government bonds. Although there is little difference in the implied PD between bonds issued by the 2nd and 3rd rating class in April, the disparity becomes more obvious in the December chart as some bonds issued by financial firms from the bottom rating class are seen to have roughly 1 in 4 chance to default three years from the date of purchase.

Another industry that deserves close attention is construction. In the US, as the subprime mortgage bubble burst, housing market crashed and consequently put heavy strain on the liquidity conditions for many construction companies. Figure 2.4 shows the implied PD for corporate bonds in the construction industry. Compared to those in the financial industry, these curves indicate lower projected probabilities of default. The difference between the rating classes from April to December become more noticeable as this industry contains fewer companies. It implies that when the state of an economy is relatively stable there seems little to no difference between the 3-year term structure of default probability from a company in the second rating class versus one that is in the third rating class. Soon after the economy takes a bad turn, the impact of the credit rating on a financial or construction firm's implied PD is more evident. Similar shift can be

observed in the estimated hazard term structures for the machinery and chemical industries, which are shown in Figure 2.5, and Figure 2.6 respectively.

For a final comparison, we show the implied PD for the utility industry in Figure 2.7, which is commonly believed to be the most resilient to major market movements. Overall, these bonds are seen the safest among all industries during a financial crisis as they exhibit the least amount of disparity between the two cross-sectional dates. Most of the bonds issued by utility companies are projected to default less frequently than 1 in 20 within 3 years of the purchase date. The change in implied PD among rating classes is still observable, but the variability within each rating class seems to have become smaller from April to December.

Figure 2.8 shows the histograms for the aggregate estimated recovery rates for all corporate bonds. A surprisingly sharp contrast can be found in this comparison. Most of the estimated recovery rates from April 2008 spread out in the range between 0.35 and 0.7. A handful of corporate bonds were estimated to retain above 90% of its par value if default were to take place while roughly 12% of all defaultable bonds were expected to recover less than 5% of its value. The December 2008 chart for the implied LGD shows much less variation. The overwhelming majority of corporate bonds were estimated to recover roughly half of their face values. About the same percentage of bonds were projected to retain close to nothing. This means that under our model assumptions, the market reacts to an unstable economic environment in terms of different LGD of corporate bonds being unified. We may possibly interpret this result as fear entering the market and investors temporarily losing the ability to objectively evaluate the recovery rates.

In our model, we choose to control industry effect by employing covariate terms in functions for both hazard term structure and recovery rate. Readers may refer to Tables 2.2 through 2.5 for a list of tables containing values for these estimated coefficients. 95% confidence intervals are computed for the corresponding model

parameters under the aforementioned assumptions as an alternative to statistical tests. We shall summarize briefly our findings as follows:

1. The size of a corporation's net income (in signed log scale) seems to be insignificant in estimating both PD and LGD. Capital adequacy ratio seems to be statistically positive in estimating hazard functions in April, but becomes insignificant in December 2008. This minor inconsistency does not trigger any alarm because even in log scale, the net income values are often much larger than the ratios measured in percentages. Rather, we may conclude that the selected covariates for company financial strength are overall insignificant in estimating PD and LGD.
2. In April 2008, most industry indicators are either negative or insignificant, except α_{h7} , $\alpha_{\delta 6}$ and $\alpha_{\delta 7}$, which tend to be positive. Since covariate estimates are averaged over all bonds, a positive estimate for hazard coefficient indicates that the corresponding industry tends to have a higher probability of default than its peers. A positive estimate for LGD coefficient indicates that the industry outperforms others in recovery rate upon default. The industries indexed at $i = 6$ and 7 are general merchants and retail, respectively. However, the retail industry has a low share of bonds (1.80% in April and 2.06% in December) and therefore its significance may be due to chance. One notable significance is $\alpha_{h5} = -2.05$ and $\alpha_{\delta 5} = -3.80$, both the largest in absolute value among other hazard and LGD covariate estimates, respectively. With a large market share (14%) of bonds, it indicates that machinery industry ($i = 5$) in Japan tends to have below-average hazard term function, but is predicted to suffer a higher than average loss when default occurs.
3. In December 2008, the machinery industry maintains its lead from April in terms of the rank of magnitude of the corresponding estimates. The only

statistically significant positive estimate takes place in the financial industry (23.73% market share of bonds) with $\alpha_{h8} = 0.77$. That is, the predicted PD for bonds issued by financial firms tend to be higher than its peers. This finding makes sense when we consider the fact that Lehman Brothers, one of the largest global financial services firm at the time, filed for Chapter 11 bankruptcy in September of 2008, sending ripple effects to many related companies around the world.

Due to the size of the correlation matrix, we resort to modeling the aggregate between-industry correlation coefficients in the form of a dependency matrix, $(\rho_{IJ})_{I,J \in \{1, \dots, 10\}}$, where ρ_{II} is the within-industry correlation coefficient for I -th industry and ρ_{IJ} for $(I \neq J)$ is the homogeneous block correlation coefficient between the I -th industry and J -th industry. For April 2008, most except three coefficients are non-zero. They are $\rho_{22} = 0.00471$, $\rho_{55} = 0.00349$ and $\rho_{66} = 0.00025$. All of them are within-industry correlation coefficients, indicating a presence of weak positive correlation among the food ($i = 2$), machinery and general merchants blocks among the assumed covariance structure of the residuals. More importantly, the vast majority of zero-valued estimated coefficients in the dependency matrix means that our use of industry indicator variables may have successfully accounted for most of the correlation in the covariance matrix.

The estimates from December 2008 contain more non-zero elements. They are $\rho_{11} = 0.01698$, $\rho_{44} = 0.00614$, $\rho_{55} = 0.09330$, $\rho_{59} = 0.00259$, $\rho_{66} = 0.09052$, $\rho_{77} = 0.21136$ and $\rho_{99} = 0.06683$. The addition of the single positive off-diagonal element between machinery industry (16.10%) and utility industry (40.56% of all bonds) is interesting. This means that residuals in bond pricing for those industries are positively correlated and the use of industry indicators in the covariates for both PD and LGD is not enough to capture this particular interdependency. It should be noted that our proposed method is only one of many ways to model such dependency structure in the covariance matrix. A possible area for future study

on similar dimension-reduction correlation structure is to analyze the asymptotic behavior of such dependency matrices.

2.4 Conclusion and final remarks

In this chapter, we have proposed a feasible statistical model platform for simultaneous estimation of implied PD and LGD from corporate bonds. Consistency of estimates is provided through a few realistic assumptions on the size of successive updates. The model integrates information related to each firm from separated sources such as credit-rating agencies, bond markets and balance sheet data. Such flexibility in using meta-data is particularly useful as more information about firms or market becomes available and is one the advantages over existing mathematical models. Empirical results using Japanese bond market and related financial data have shown that the proposed model is capable of making close predictions (zero mean residuals) for the price of corporate bonds. For the covariance matrix of residuals in bond prices, we apply dimension-reduction technique and use a dependency matrix to model the correlation structure. Estimates seem to converge rapidly under the iterative minimization of a square loss function. In cases where statistical significance of parameter values is sought, an additional reasonable normality assumption would provide an easy passage.

Our model can see many practical uses due to the ease of implementation, flexible model assumptions, as well as ample theoretical support for testing model parameters. The study on dependency matrix can be applied in portfolio analysis in order to assess the overall risk of a fund. Numerical results have indicated the importance of credit rating data in pricing most bonds as well as estimating their implied PD and LGD. Due to the scarcity of credit rating data, certain estimates may not reveal its true value at those pre-selected cross-sectional dates. This, in turn, suggests us a means to interpolate credit ratings as missing values when

appropriate using our model. Although situation dependent, the significance on financial information parameters could be used as a tool to perform stress tests. With the normality assumption of error terms, this model can be readily adapted to time-series data with Bayesian-type updates. We would also like to generalize our modeling of the variance-covariance matrix to provide a theoretical treatment to the implied correlation coefficients.

(a) Default term structure for 2008-04-04 (204 bonds)

(b) Default term structure for 2008-12-01 (196 bonds)

Figure 2.3: Estimated probability of default for bonds in financial industry

(a) Default term structure for 2008-04-04 (24 bonds)

(b) Default term structure for 2008-12-01 (23 bonds)

Figure 2.4: Estimated probability of default for corporate bonds in construction industry

(a) Default term structure for 2008-04-04 (148 bonds)

(b) Default term structure for 2008-12-01 (133 bonds)

Figure 2.5: Estimated probability of default for corporate bonds in machinery industry

(a) Default term structure for 2008-04-04 (86 bonds)

(b) Default term structure for 2008-12-01 (83 bonds)

Figure 2.6: Estimated probability of default for corporate bonds in chemical industry

(a) Default term structure for 2008-04-04 (331 bonds)

(b) Default term structure for 2008-12-01 (335 bonds)

Figure 2.7: Estimated probability of default for corporate bonds in utility industry

(a) 2008-04-04 (total: 1057 bonds)

(b) 2008-12-01 (total: 826 bonds)

covariate	estimate	s.e.	95% lower	95% upper
β_{h1}	1.7842	0.2143	1.3642	2.2043
β_{h2}	0.0000	0.0132	-0.0260	0.0260
α_{h1}	-0.8046	0.2250	-1.2457	-0.3636
α_{h2}	-1.2668	0.1523	-1.5655	-0.9681
α_{h3}	-1.7062	0.1585	-2.0169	-1.3956
α_{h4}	-1.8409	0.1506	-2.1361	-1.5457
α_{h5}	-2.0543	0.1346	-2.3183	-1.7904
α_{h6}	-0.2541	0.3979	-1.034	0.5257
α_{h7}	0.5475	0.1712	0.2119	0.8832
α_{h8}	-0.4811	0.2078	-0.8884	-0.0738
α_{h9}	-1.3926	0.1823	-1.750	-1.0352
α_{h10}	0.0000	0.3768	-0.7385	0.7385

Table 2.2: Hazard coefficient estimates for 2008-04-04

covariate	estimate	s.e.	95% lower	95% upper
$\beta_{\delta 1}$	0.4564	0.2083	0.0480	0.8647
$\beta_{\delta 2}$	0.0095	0.0305	-0.0503	0.0695
$\alpha_{\delta 1}$	0.4107	0.2625	-0.1038	0.9252
$\alpha_{\delta 2}$	-0.1841	0.1560	-0.4899	0.1217
$\alpha_{\delta 3}$	-0.6196	0.1950	-1.0019	-0.2373
$\alpha_{\delta 4}$	-0.7691	0.1513	-1.0657	-0.4725
$\alpha_{\delta 5}$	-3.7988	0.3882	-4.5598	-3.0378
$\alpha_{\delta 6}$	1.6959	0.1665	1.3695	2.0224
$\alpha_{\delta 7}$	2.7348	0.1806	2.3807	3.0890
$\alpha_{\delta 8}$	-0.3020	0.2148	-0.723	0.1190
$\alpha_{\delta 9}$	0.0000	0.1926	-0.3775	0.3775
$\alpha_{\delta 10}$	0.4405	0.3696	-0.2840	1.1651

Table 2.3: LGD coefficient estimates for 2008-04-04

covariate	estimate	s.e.	95% lower	95% upper
β_{h1}	0.4386	0.4936	-0.5290	1.4062
β_{h2}	-0.0286	0.0187	-0.0653	0.0080
α_{h1}	0.2152	0.3718	-0.5136	0.9441
α_{h2}	-0.2909	0.2767	-0.8334	0.2514
α_{h3}	-0.7306	0.2507	-1.2220	-0.2391
α_{h4}	-0.7822	0.2633	-1.2984	-0.2660
α_{h5}	-1.1115	0.2837	-1.6677	-0.5553
α_{h6}	-0.5644	0.3337	-1.2185	0.0896
α_{h7}	0.1156	0.3065	-0.4852	0.7165
α_{h8}	0.7699	0.3522	0.0794	1.4603
α_{h9}	-0.8193	0.3759	-1.5561	-0.0824
α_{h10}	0.0000	10.0198	-19.6389	19.6389

Table 2.4: Hazard coefficient estimates for 2008-12-01

covariate	estimate	s.e.	95% lower	95% upper
$\beta_{\delta 1}$	0.0000	0.2575	-0.5048	0.5048
$\beta_{\delta 2}$	0.0000	0.0181	-0.0356	0.0356
$\alpha_{\delta 1}$	0.0241	0.2278	-0.4223	0.4706
$\alpha_{\delta 2}$	0.0144	0.2134	-0.4040	0.4328
$\alpha_{\delta 3}$	0.0247	0.2274	-0.4210	0.4706
$\alpha_{\delta 4}$	0.0344	0.2090	-0.3753	0.4442
$\alpha_{\delta 5}$	-3.4165	0.5252	-4.4459	-2.3871
$\alpha_{\delta 6}$	0.0167	0.2106	-0.3960	0.4295
$\alpha_{\delta 7}$	0.0072	0.2942	-0.5695	0.5839
$\alpha_{\delta 8}$	0.0381	0.2233	-0.3996	0.4759
$\alpha_{\delta 9}$	0.1582	0.3208	-0.4706	0.7872
$\alpha_{\delta 10}$	0.0000	0.8956	-1.7555	1.7555

Table 2.5: LGD coefficient estimates for 2008-12-01

CHAPTER 3

Bayesian dynamic modeling of implied default correlation for exploring the impact of recent financial crisis on Japan credit default swap market

3.1 Background

Recent years have seen the rapid development of credit default swap (CDS) market. CDS, a derivative contract that enables financial institutions the transfer of credit risk without transferring their actual asset of the underlying reference entity, has become a useful tool for credit risk management. According to the International Swaps and Derivatives Association, it was estimated that notional amounts outstanding of CDS exceeded \$62 trillion by the end of 2007. Although this figure had fallen after the subprime mortgage crisis, the notional amount outstanding still remained over \$30 trillion by the end of 2009. Regardless of its total current notional amount, the CDS market, together with stock/bond markets and credit ratings has become a main source of information for measuring credit risk.

Recently, it is realized that the CDS market played an important role in the recent global economic recession which started with the outbreak of the subprime mortgage crisis. When CDS was first invented as a means of hedging and diversifying credit risk, its main role was to allow a commercial bank to shift the risk of default to a third-party and the shifted risk did not count against their regu-

latory capital requirements by entering into a CDS contract. Before 2000, most protection buyers held the underlying credit asset and viewed CDS contracts as a protection mechanism. However, both buyers and sellers of CDS contracts started to consider them more as a tool for trading speculation. Over the years leading toward 2007, the gross notional amount outstanding of all CDS has increased drastically. In this chapter, we would not comment on the impact of the CDS market during this recession. Rather, we consider it as a valuable source of information for our proposed model on default correlation and compare numerical results from before, during, and after the latest recession.

Along with probability of default and loss given default, market participants have realized that default correlation is a central premise of credit risk modeling. Default correlation, a measure of the dependence among risk, is an important piece of information for credit risk management. Many studies report that corporate defaults are correlated (for example, Lang et al. (1992)). Therefore, an accurate quantification of default correlation not only helps bring forth a fair pricing of collateralized debt obligations, but also provides useful information for overall portfolio management. Since default correlation plays an important role in the estimation of the tails of the portfolio loss distributions, a failure in quantification of default correlation might lead to an underestimate of credit risk.

Although default correlation is sought as a key factor for credit risk management, it is difficult to observe default correlation directly. Both from theoretic and empirical perspectives, many studies have attempted to quantify default correlation (e.g., Banachewicz et al. (2008), Cowan et al. (2004), Crowder et al. (2005), Das et al. (2001), Das et al. (2007), De Servigny et al. (2002), Overbeck et al. (2005), Qi et al. (2010), Zeng et al. (2002), and Zhou (2001)). Among them, structural models and reduced-form models are often employed. In the context of structural models pioneered by Merton (1974), for example, Zhou (1997) derived an analytic result for default correlation. Reduced-form approaches can also han-

default correlation by allowing hazard rates for individual bonds (firms) to be correlated with each other (for example, Hull et al. (2001)). Other approaches to estimate default correlation are also available; for instance, by using panel data on defaults and rating migration information Das et al. (2006); to quantify it directly from historical data on defaults Demey et al. (2004) and Koopman et al. (2005), or by assuming that there is a contagion effect such that the default of one firm either directly triggers the default of other firms or increases their default probabilities Azizpour et al. (2008), Davis and Lo (2001), Giesecke (2003), Giesecke (2006). In practice, however, there is no consensus on how market participants estimate default correlation.

One of the most common ideas would be to estimate default correlation by extracting information from a CDS market. In general, the value of a CDS depends on the credit risk of the corresponding reference entity. Thus, intuitively, it is natural to consider that the correlation of CDS returns reflect default correlation. In addition, several empirical studies have found that default correlation is time-varying Das et al. (2002), Das et al. (2007). Nickell et al. (2000) and many others linked time-varying default frequencies to some economic variables, including business cycle indicators.

In the context of Bayesian dynamic modeling framework West (1999), we propose a new method to develop the joint probability distribution of defaults from the CDS market along with information from other financial markets, including some economic variables that would simultaneously affect the default probabilities of many firms. The method dynamically updates the default correlation based on the information from a CDS market and utilizes it as an estimate of the implied default correlation.

One of the advantages of the proposed method is that it dynamically extracts default correlation from the hazard levels and co-movements of CDS spreads by including economy-wide factors that might cause large numbers of firms to default

simultaneously. Therefore, it is possible to check the exposure of hazard rate to these economic variables. Furthermore, the relationship between the changes in hazard level and the CDS returns is investigated analytically. It is shown that the increase in hazard causes the increase in CDS premium. This finding fits the consensus of market participants.

Data from the Japan CDS market is analyzed using the proposed method. We divided the data span into the following three periods: (1) August 1, 2006 to June 30, 2007 denoting period before the outbreak of the subprime crisis, (2) August 1, 2007 to February 28, 2009 during the world financial chaos, (3) April 1, 2009 to March 31, 2010 beginning of economic recovery. Based on the proposed method, we found that the magnitude of implied default correlations surged substantially after the outbreak of the subprime crisis. Other interesting findings are also described.

The flow of our chapter is as follows: In Section 2, we briefly overview basic ingredients of a CDS contract, including pricing, hazard function and so on. We also investigate analytically the relationship between the change in hazard level and the CDS log-return. A new method is developed in Section 3 for modeling dynamic default correlation in the context of Bayesian framework. In section 4, we apply the proposed method to Japan CDS data. Section 5 provides conclusions and remarks.

3.2 Overview and CDS premium determination

In this section, we first present an overview of a CDS contract. Then, we use the hazard function to develop the relationship between defaults from the CDS market along with information from other financial markets.

3.2.1 Basic model

This section overviews a basics of CDS pricing model (See for e.g., Hull (2008)). We assume a time-varying CDS reference rate $C(t)$ measured in basis points (bps) per annum. For any given reference entity, let t_0 denote the effective date, usually the first payment date of a CDS contract on this financial instrument. Assume that premium payments for this CDS occur at $t_1, t_2, \dots, t_n$; often measured in years time that has elapsed since t_0 , and the notional principal for the CDS is N with a recovery rate of R . Therefore, at any time before the expiration of this CDS contract, if the reference entity suffers a “credit event”, the protection seller pays the buyer $N(1 - R)$ in settlement and thereafter the contract is terminated. For simplicity of terms, we may refer to these qualifying credit events as simply ‘default’.

We denote a random variable τ as the time to default for a reference entity and let $p_i = P(\tau > t_i | \tau > t_{i-1})$ be the conditional survival probability for the duration (t_{i-1}, t_i) . Therefore, $1 - p_i$ is the probability of reference entity incurring a credit event or default during the time interval (t_{i-1}, t_i) . At time t_i , if there has been no credit event and the CDS contract is still active, the protection buyer pays the protection seller an additional $NC(t_i)$. On the other hand, if a qualified credit event takes place within that time interval, the protection buyer stops making further payments and is entitled to obtain $N(1 - R)$ from the seller. The latter happens with a probability of $p_1 p_2 \dots p_{i-1} (1 - p_i)$ and the protection seller still keeps all previous premium payments.

Table 3.1 illustrates the discounted cashflow for a protection seller. In Table 3.1, δ_i denotes the risk-free discount rate for a cashflow at t_i . The total present value of the CDS can be found by multiplying the probability of each outcome by its present value of the cashflow and sum over all time t_i .

One of the problems is how to calculate the probability of default. For this

purpose, the notion of hazard function is often employed. In the next section, we describe how the hazard function is linked to the CDS pricing.

event	cash outflow	cash inflow (cumulated)	probability
default at time t_i	$-N(1 - R)\delta_i$	$N \sum_{k=1}^{i-1} C(t_k)\delta_k$	$(1 - p_i) \prod_{k=1}^{i-1} p_k$
no default at all	0	$N \sum_{k=1}^n C(t_k)\delta_k$	$\prod_{k=1}^n p_k$

Table 3.1: Cashflow for a protection seller.

3.2.2 Hazard rate and CDS pricing

We introduce now the notion of the hazard function. Let $\mathbf{X}_t$ be the p -dimensional current state of a reference entity's default covariates that may include firm-specific variable, economic variables and so on. Depending on data availability, market participants can prepare a set of variables. The hazard function is given by

$$h(t_0|\mathbf{x}) = \lim_{s \rightarrow 0} \frac{P(t_0 < \tau \leq t_0 + s | \tau > t_0, \mathbf{X}_t = \mathbf{x})}{s}. \quad (3.1)$$

We would like to point out that the hazard function in (3.1) is not a stochastic process but a deterministic function of time and states. Thus our model follows a statistical framework, for e.g., Ando (2009), Ando and Yamashita (2009), Kariya and Tsuda (2000). The probability of default at time t is then

$$P(\tau \leq t | \mathbf{X}_t = \mathbf{x}) = 1 - \exp \left\{ - \int_{t_0}^t h(u|\mathbf{x}) du \right\}. \quad (3.2)$$

We also define further the cumulative hazard by

$$H(t, T | \mathbf{X}_t = \mathbf{x}) = H(t, T | \mathbf{x}) = \int_t^T h(s|\mathbf{x}) ds. \quad (3.3)$$

As shown below, we can write the present value of a CDS contract using a hazard function. Assuming that the risk-free interest rate is r and the recovery rate is R , we can determine the value of the coupon at the present time t , $C(t; t_0) :=$

$C_t(t_0)$, by solving the equality between the expected payoff for the protection buyer in (3.4) and that for the seller in (3.5):

$$\begin{aligned} \text{E}(\text{payoff for the buyer}) &= \sum_{j=1}^n N(1-R) \exp\left\{-\int_{t_0}^{t_j} r(t)dt\right\} \exp\left\{-\int_{t_0}^{t_{j-1}} h(t|\mathbf{x})dt\right\} \\ &\quad \times \left[1 - \exp\left\{-\int_{t_{j-1}}^{t_j} h(t|\mathbf{x})dt\right\}\right], \end{aligned} \quad (3.4)$$

$$\begin{aligned} \text{E}(\text{payoff for the seller}) &= \sum_{j=1}^n C_t(t_0) \exp\left\{-\int_{t_0}^{t_j} r(t)dt\right\} \exp\left\{-\int_{t_0}^{t_{j-1}} h(t|\mathbf{x})dt\right\} \\ &\quad . \end{aligned} \quad (3.5)$$

Then the expected value of the CDS rate can be expressed as

$$C_t(t_0) = \frac{N(1-R) \sum_{j=1}^n \exp\left\{-\int_{t_0}^{t_j} r(t)dt\right\} (\exp\{-H(t_0, t_{j-1}|\mathbf{x})\} - \exp\{-H(t_0, t_j|\mathbf{x})\})}{\sum_{j=1}^n \exp\left\{-\int_{t_0}^{t_j} r(t)dt\right\} \exp\{-H(t_0, t_{j-1}|\mathbf{x})\}}. \quad (3.6)$$

Next, the relationship between the change in hazard level and the CDS return is investigated analytically. Let t be the present time and C_t be CDS premium at t and t_j is the time point from time 0, or equivalently, from the present time. Also, let $r_t(\cdot)$, $h_t(\cdot)$ be the risk free and hazard rate at time t . The log-return of the CDS premium during the time period $[t, t + \Delta]$, where Δ is a small time period, is approximated as

$$\log\{C_{t+\Delta}/C_t\} \approx \sum_{j=1}^n \left[\int_0^{t_j} [\hat{h}_{t+\Delta}(s|\mathbf{x}_{t+\Delta}) - \hat{h}_t(s|\mathbf{x}_t)] ds \right], \quad (3.7)$$

where $\hat{h}_t(s|\mathbf{x}_t) = r_t(s) + h_t(s|\mathbf{x}_t)$ is the term structure at time t . The derivation of this formula can be found in Appendix B. It can be seen that as the hazard rate increases, the reference rate of that CDS would increase as well. Also, this function can be used for discounting a set of future cashflows to the present values for the reference entity.

Although this equation does not establish a direct relationship between the rate of change of CDS rates and that of the hazard function or the default probability,

it can be considered as providing a means to estimate a combination of the latter two. More importantly, this result provides a foundation for the use of time-varying correlation of a set of CDS return series as a default correlation.

Mathematically, suppose that we have a set of m CDS return series. Let

$$\mathbf{y}_t = (\log(C_{1t}/C_{1,t-1}), \dots, \log(C_{mt}/C_{m,t-1}))'$$

be the m -dimensional CDS return vector for m reference entities observed at time t . Then, the resulting time-varying correlation coefficient of $\mathbf{y}_t$ would be referred to from now on as an estimate of the implied default correlation for the respective underlying reference entities.

Given the matrix $\Sigma_t = (\sigma_t(i, j)) = \text{Cov}(\log(C_{it}/C_{i,t-1}), \log(C_{jt}/C_{j,t-1}))$, we can calculate the correlation of the CDS reference rates by

$$\text{Default correlation of reference entities } i \text{ and } j = \frac{\sigma_t(i, j)}{\sqrt{\sigma_t(i, i)}\sqrt{\sigma_t(j, j)}}. \quad (3.8)$$

This quantity is used as a proxy of the implied default correlation. We can estimate the correlation of CDS reference rates Σ_t in the dynamic Bayesian linear regression framework.

3.3 Bayesian dynamic modeling of implied default correlation

For computing this implied default correlation, we use the standard multi-response normal regression model with dynamic Bayesian updating (see for e.g., West (1999)).

At time t , suppose we have a set of n past observations $\{(\mathbf{y}_{t-1}, \mathbf{x}_{t-1}), \dots, (\mathbf{y}_{t-n}, \mathbf{x}_{t-n})\}$, where $\mathbf{y}_{t-j}$ are m -dimensional response variables that store the CDS returns for m reference entities and $\mathbf{x}_{t-j}$ are p -dimensional covariates observed at time $t - j$.

The linear regression model, expressed in matrix form, is

$$Y_{tn} = X_{tn}B_t + E_{tn}, \quad (3.9)$$

where $Y_{tn} = (\mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-n})'$, $X_{tn} = (\mathbf{1}_n, W_{tn})$, $W_{tn} = (\mathbf{x}_{t-1}, \dots, \mathbf{x}_{t-n})'$, $B_t = (\boldsymbol{\alpha}_t, \Gamma_t')$, $\boldsymbol{\alpha}_t = (\alpha_{t1}, \dots, \alpha_{tm})'$, $\Gamma_t = (\boldsymbol{\beta}_{t1}, \dots, \boldsymbol{\beta}_{tm})$ consisting of p -dimensional unknown parameters $\boldsymbol{\beta}_{tj} = (\beta_{tj1}, \dots, \beta_{tjp})'$, and $E_{tn} = (\boldsymbol{\epsilon}_{t1}, \dots, \boldsymbol{\epsilon}_{tn})'$ with $\boldsymbol{\epsilon}_{t\alpha} \sim N(\mathbf{0}, \Sigma_t)$. Note that Σ_t is time varying and used to estimate the implied default correlation for the respective underlying reference entities. For a chosen sliding window size n , the response matrix Y_{tn} contains the log-return of the m CDS premiums for n consecutive trading days, ending at the current time t . Each $\mathbf{x}$ is a p -dimensional covariate vector for a day within the current time frame. $\boldsymbol{\beta}_{jk}$ carries the value of regression coefficient for the j -th reference entity on the k -th explanatory variable for this duration.

Let $\boldsymbol{\theta}_t = (\text{vec}(B_t), \text{vech}(\Sigma_t))'$ be the vector form for the set of all unknown parameters. The likelihood function is then

$$f(Y_{tn}|X_{tn}, \boldsymbol{\theta}_t) = (2\pi)^{-\frac{nm}{2}} |\Sigma|^{-\frac{n}{2}} \exp \left[-\frac{1}{2} \text{tr} \{ \Sigma^{-1} (Y_{tn} - X_{tn}B_t)' (Y_{tn} - X_{tn}B_t) \} \right].$$

Assuming $\pi(\boldsymbol{\theta}_t) = \pi(B_t|\Sigma_t)\pi(\Sigma_t)$, we update the posterior estimates in the time series data by using conjugate prior distributions. In particular, a multivariate normal prior for B_t and an inverse Wishart prior for Σ_t are used:

$$\pi(\Sigma_t|\Lambda_{t-1}, \nu_0) = W^{-1}(\Lambda_{t-1}, \nu_0) = \frac{|\Lambda_{t-1}|^{\frac{\nu_0}{2}}}{2^{\frac{m\nu_0}{2}} \Gamma_m(\frac{\nu_0}{2})} |\Sigma_t|^{-\frac{\nu_0}{2}} \exp \left[-\frac{1}{2} \text{tr}(\Lambda_{t-1} \Sigma_t^{-1}) \right],$$

$$\begin{aligned} \pi(B_t|B_{t-1}, \Sigma_t, A) &= N(B_{t-1}, \Sigma_t, A) \\ &\propto |\Sigma_t|^{-\frac{m}{2}} |A|^{-\frac{p+1}{2}} \exp \left[-\frac{1}{2} \text{tr} \{ \Sigma_t^{-1} (B_t - B_{t-1})' A^{-1} (B_t - B_{t-1}) \} \right] \end{aligned}$$

with $m \geq \nu_0$, and $|\Sigma_t| > 0$ for all t . $\Gamma_m(\cdot)$ is the multivariate Gamma function. Hyperparameters are ν_0, A , along with these hyperparameters, B_{t-1} and Λ_{t-1} specify the prior mean of B_t , and the prior mean for Σ_t , respectively. We set the

values of B_{t-1} and Λ_{t-1} so that the prior distributions reflect the current market situation. The value of Λ_{t-1} is set so that the prior mean of Σ_t is equal to Σ_{t-1} .

The posterior distributions of B_t and Σ_t are respectively multivariate normal and inverse Wishart, and it follows that

$$\begin{aligned}\pi(\Sigma_t|Y_{tn}, X_{tn}) &\sim W^{-1}(\Sigma_t|S_t + \Lambda_{t-1}, n + \nu_0), \\ \pi(B_t|\Sigma_t, Y_{tn}, X_{tn}) &\sim N(B_t|\bar{B}_t, \Sigma_t, X'_{tn}X_{tn} + A^{-1}).\end{aligned}$$

The posterior means of Σ_t and $B_t|\Sigma_t$ are

$$\begin{aligned}E[\Sigma_t|Y_{tn}, X_{tn}] &= \bar{\Sigma}_t = \frac{S_t + \Lambda_{t-1}}{\nu_0 + n - m - 1}, \\ E[B_t|\Sigma_t, Y_{tn}, X_{tn}] &= \bar{B}_t,\end{aligned}$$

where S_t and $\bar{B}_t$ are defined as:

$$\begin{aligned}S_t &= (Y_{tn} - X_{tn}\bar{B}_t)'(Y_{tn} - X_{tn}\bar{B}_t) + (\bar{B}_t - B_{t-1})'A^{-1}(\bar{B}_t - B_{t-1}), \\ \bar{B}_t &= (X'_{tn}X_{tn} + A^{-1})^{-1}(X'_{tn}X_{tn}\hat{B}_t + A^{-1}B_{t-1}),\end{aligned}$$

and $\hat{B} = (X'_{tn}X_{tn})^{-1}X'_{tn}Y_{tn}$.

It is well-known that the one-step Bayesian forecast has a marginal multivariate Student t -distribution (West (1999)). However, our current focus is not on making predictions, but estimating elements in the time-evolving Σ_t .

3.4 An empirical analysis

3.4.1 Data

The J-CDS website provides reference rate data starting from the launch date of March 26, 2004. Since then the world economy has seen the boom and crash of subprime mortgage and derivatives trading that have caused financial turmoil which still has a lasting effect today. We would like to investigate the implied

default correlation at several key stages of world economy since 2004. In particular, it may be of our interest to divide the time-series date into three segments

- 08/01/2006 - 06/30/2007 Before the outbreak of subprime mortgage crisis
- 08/01/2007 - 02/28/2009 During the world financial chaos
- 04/01/2009 - 03/31/2010 Beginning of economic recovery

Following Tsay and Ando (2010), we omitted a few months of returns between the periods because the exact dates of impact that the extreme events have on the market are not certain.

The control variables $\mathbf{x}_t$ we chose to use in the regression model are the return of the core interest rate, the return of the foreign exchange rate (USD/JPY), and the return of the Nikkei 225 index. The Nikkei index data can be obtained online from Yahoo Finance¹. The other rates are publicly available from the website that belongs to Bank of Japan². We also include the first principal component of the response variable, the return of $\log C$ to be the fourth explanatory variable.

3.4.2 Algorithm for Updating Σ_t

First, a default value of $n = 50$ days is set for the window size of time in updating the time-varying correlation coefficients. A simple random sample of reference entities is collected and those whose log return series of CDS reference rates contain an excessive number of missing values (for instance, 50 or more) are discarded from the list, giving a sample size n . In order to use a weak prior distribution for Σ_t , we set $\nu_0 = 10$, $A = 10^5 I_{p+1}$ and the initial value Σ_0 to be the covariance-variance matrix of $Y_{tn} = (\mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-n})'$ for the past 50 days, where $\mathbf{y}_{t-j}$ are m -dimensional response variables that store the CDS rates for m reference entities at time $t - j$.

The dynamic Bayesian updating scheme is given as follows:

¹<http://finance.yahoo.com>

²<http://www.stat-search.boj.or.jp/ssi/mtshtml/d.html>

- Apply PCA to extract the first principal component of log return series of CDS reference rates for the past $n = 50$ days, and use it as the fourth covariate in the multivariate linear regression.
- Calculate the posterior means of $\bar{B}_t$ and $\bar{\Sigma}_t$ to update B_t and Σ_t .
- Go to the next trading day and repeat the process.

3.4.3 Numerical Results

We first present the results on the data compiled on the unsegmented time frame from August 1, 2006 to March 31, 2010 with $m = 34$. Tables 3.2 and 3.3 list the corporations that appeared in all three selected dates. It can be seen in Figure 3.1 that the number of eigenvectors with a cumulative score of more than 0.90 exhibits different behavior over the entire time span. The dotted lines indicate the start and end for each time segment described in Section 4.1.

Based on the figure, we make the following observations. During the period before the outbreak of the subprime crisis, the number of eigenvectors is relatively stable. For the most part, the number of such factors takes on values between 7 and 9. However, we see that this number reduced substantially during the outbreak. The first turning point takes place after the Bear Stearn's credit crisis began to surface, and the number of factors plummeted to a global minimum of 4 on 26th and 27th of September 2007 after New York Times reported that Bear Stearn posted a 61% drop in net profits due to their hedge funds losses. This drop in the number of factors was followed by a monotonic surge resulting in a short-lived global maximum of 11 on November 6th of 2007. Other global minima were found at the end of October 2008 following the Lehman Brother's bankruptcy and at the end of January 2009 immediately after the protests for the Icelandic financial crisis and some of the worst projected economic measures were announced. Thus, a small number of common factors governs the fluctuation

of the CDS market after world-wide major credit events. This may imply that market participants have a tendency to become homogeneous once these credit events occur in a sequence. It is also worth noting that the global financial market started to decline after the pinnacle for DOW was reached in October 2007. Nikkei 225 attained its all-time high (since the Dotcom crash at the millennium) in July 2007 but fell alongside the U.S. markets and other major global exchanges during the same time. This decline eventually led to a global recession that lasted over a year. Therefore, the unusual surge and sharp decline in the number of factors might be a foreknowledge of a warning sign.

Figure 3.2 shows a time-series plot of the top 10 eigenvalues for the estimated implied default probability. A pair of plots for the largest eigenvalue are shown in Figure 3.3. These eigenvalues have a moderately strong negative linear correlation with the number of important factors displayed previously in Figure 3.1. The figure also shows that the magnitude of the first eigenvalue increased significantly after the outbreak of Lehman Brothers' credit crisis event.

Next, we check the correlation between the explanatory variables and the first principal component, as displayed in Table 3.4. Although a medium-sized negative correlation between the Nikkei 225 index and the first principal component was seen as we would have expected, the correlation is not significant and we would not be able to establish a causal relationship between the two. The table also indicated a small negative correlation between the first principal component and the foreign exchange rate.

The sample size for the unsegmented time frame is relatively small for a reason. The Japan CDS market was established in 2004 and while new reference entities have been continuously added to the pool, not all recorded reference entities have data available for the entire time span. Based on our selection criterion which eliminates reference entities with a large portion of missing values, many in the list of reference entities are dropped in the sampling process as we choose not

to impute any missing data. This gives us yet another reason to investigate the behavior of implied default correlation more closely for different time segments.

Figure 3.4 shows the plots for factor analysis as well as eigenvalues for the first time segment. These plots are generated based on a sample size of $m = 67$. As there are more reference entities included in the estimation, the amount of variation in the number of qualifying factors increases. Figure 3.4(a) shows a sudden steep climb towards the end of the first time segment before global credit condition turned for worse.

The plot for factor analysis for the second time segment using $m = 91$ is shown in Figure 3.5(a). Just like the previous figure, the range of factors gets wider as more data becomes available. The number of qualifying factors reaches 14 on November 1st and 2nd of 2007 and dropped to a low of 4 on February 8th of 2008. For the most part, the number of factors remains within the range of 5 to 10, which is significantly lower than that of the first time segment.

Descriptive statistics indicate that the correlations between the first principal component and both Nikkei 225 index and foreign exchange rate have increased in magnitude. These two indices show a medium negative correlation with the first principal component, which could be used to explain the rapid growth of CDS reference rates during the breakout and aftermath of the subprime mortgage crisis.

Figure 3.11 show two time-series plots that trace the reference entities which had the most and the least mean estimated implied default correlation with Mizuho Corporate Bank Ltd, respectively. We choose this company not only because of its market cap and influence in Japan, but the fact that financial companies suffered some of the worst losses from a series of major credit events during this particular period.

Among the other 90 reference entities, Acom Co. Ltd. had the largest mean

implied default correlation with Mizuho Corporate Bank with an average value of 0.7799. Mitsui O.S.K. Lines, Ltd. and Canon Inc. followed with a mean value of 0.4351 and 0.1891, respectively. Acom's main markets are Loan Business, Credit Card Business and Loan Guarantee Business. These results are consistent to business practice because these businesses are parts of Mizuho's targeting markets. Interestingly, estimated implied default correlation indicates that the credit risk between Mitsui O.S.K. Lines and Mizuho is highly correlated during the outbreak of subprime mortgage crisis. Mizuho group is a main lender for many Japanese firms that export their products to foreign countries and thus an increase in exports would partially imply a decline of Mizuho group's credit risk. Since Mitsui O.S.K. Lines is a multi-modal transport group, it is natural to consider that the increase of exports as well as imports decreased their credit risk. On the other hand, a drastic decline in exports while banks suffered illiquidity problem and/or credit events would increase the credit risks for both companies. Combining these factors together, a large value of mean implied default correlation with Mizuho Corporate Bank and Mitsui O.S.K. Lines does make sense.

For companies that had the least mean implied default correlation with Mizuho, Keisei Electric Railway Co., Ltd., Marubeni Corporation and Aiful Corporation took the top three places with values -0.2690, -0.2159 and -0.2005, respectively. Keisei Electric Railway Co. is a utility company and therefore seems to be more resilient to the economic crisis. Thus, Stocks for such companies would be held by bearish investors, especially during an economic turmoil.

Finally, the plots for the most recent time segment can be found in Figure 3.6. The graphical and numerical analyses are based on 98 reference entities as more reference entities became available to be traded in the Japan CDS market. The number of factors returns to stay mostly above 8 or 9 in this segment.

Due to the limited number of covariates included in the dataset, Bayes factor is used to evaluate the model selection. In hypothesis testing that compares two

linear regression models, the Bayes factor is simply the ratio of the marginal likelihood. When the proposed dynamic Bayesian updating scheme is used, the marginal likelihood can be expressed as:

$$P(\mathbf{Y}_n|X_n) = \pi^{-\frac{nm}{2}} \times \frac{|\Lambda_0|^{\frac{\nu_0}{2}} \times \Gamma_m\left(\frac{\nu_0+n}{2}\right) \times |X_n'X_n + A^{-1}|^{\frac{p+1}{2}}}{|\Lambda_0 + S|^{\frac{\nu_0+n}{2}} \times \Gamma_m\left(\frac{\nu_0}{2}\right) \times |A|^{\frac{p+1}{2}}}. \quad (3.10)$$

But everything else held equal, we only need to compute the ratio

$$\frac{|X_n'X_n + A^{-1}|^{\frac{p+1}{2}}}{|\Lambda_0 + S|^{\frac{\nu_0+n}{2}} |A|^{\frac{p+1}{2}}}$$

. Assuming even prior odds, we compare the following models, using the full model M_0 as the common reference:

M_0 : CDS Return = $\beta_0 + \beta_1 \times \text{Core Return} + \beta_2 \times \text{FX Return} + \beta_3 \times \text{Nikkei Return} + \beta_4 \times \text{1st PC}$

M_1 : CDS Return = $\beta_1 \times \text{Core Return} + \beta_2 \times \text{FX Return} + \beta_3 \times \text{Nikkei Return} + \beta_4 \times \text{1st PC}$

M_2 : CDS Return = $\beta_0 + \beta_2 \times \text{FX Return} + \beta_3 \times \text{Nikkei Return} + \beta_4 \times \text{1st PC}$

M_3 : CDS Return = $\beta_0 + \beta_1 \times \text{Core Return} + \beta_3 \times \text{Nikkei Return} + \beta_4 \times \text{1st PC}$

M_4 : CDS Return = $\beta_0 + \beta_1 \times \text{Core Return} + \beta_2 \times \text{FX Return} + \beta_4 \times \text{1st PC}$

M_5 : CDS Return = $\beta_0 + \beta_1 \times \text{Core Return} + \beta_2 \times \text{FX Return} + \beta_3 \times \text{Nikkei Return}$

The plot for the Bayes factors(M_0, M_j), $1 \leq j \leq 5$ can be found in Figure 3.10. As shown in Figure 3.10, the best model varies on time. It can be seen that, except M_{01} , the Bayes factors are under 1, which suggests negative support for the full model. We also tried different values of prior weight matrix A and found that similar results are obtained.

In Table 3.5 - 3.7, we provide correlation coefficients among covariates and the first principal component during the three subperiods. As shown in the tables, magnitudes of correlations were relatively small in the first segment. However, during the period of the recent world financial turmoil, the correlations among

FX return, Nikkei return and 1st PC increased dramatically. This could be an indication that the market structure had changed. These correlation coefficients dropped in size during the last time segment, as many analysts have been announcing it as a “recovery” stage. Regardless, those values are still relatively larger than the ones found in the first subperiod.

3.5 Conclusions and Remarks

We have been concerned with the challenge of providing a viable statistical method to estimate the implied default correlation among reference entities traded in a CDS market. In particular, we have shown the mathematical relationship between the implied default correlation and the correlation of CDS premium returns among underlying entities. We have also proposed a statistical model that estimates the implied default correlation using dynamic Bayes updating. We believe that during a major worldwide financial crisis such as the one led by subprime mortgages that started in 2007, changes in the market structure give rise to different behavior in the implied default correlation. When the market is relatively stable, the default correlation exhibits a homogeneous feature over a longer term.

Due to the fact that Japan has only adopted the CDS market within a decade, and currently CDS trades only occupy a small portion of all derivative markets worldwide, we have a limited amount of data to make a sound conclusion on the long-term behavior of implied default correlation. There is a lot of missing data in the first time segment. But as new reference entities become available for trade in the Japan CDS market, we expect to see a bigger contrast between the period of a financial crisis and the period of a relatively stable macroeconomic state. The classification of subperiods itself could become debatable as we recently saw the outbreak of the pan-Europe credit fear as Greece had been involved in a deep debt crisis. The outcome of this current issue could lead us to a new classification

rule and revealing new results. Our research may be furthered by implementing a heteroscedasticity regression model, such as autoregressive models. Private parties that have more economic data may wish to use our model to make predictions as well as to identify the key factors that were absent during the financial crisis. This chapter has provided a starting point for modeling the CDS market statistically. Possible extensions would be: (1) adding other covariates (key economic indices, qualitative variables), (2) making predictions of implied default probability with our model, (3) extending model complexity to a mixture of multivariate Gaussian linear regression, and (4) applications to portfolio selection.

Factors Plot 08/01/2006 -- 09/30/2010

Figure 3.1: Number of eigenvectors with cumulative $R^2 > 0.90$

Figure 3.2: Plot of the top 10 eigenvalues over time

Figure 3.3: Plot of the largest eigenvalue and the overlaid spline fit, 08/01/2006 - 03/31/2010. Top eigenvalues scaled by value.

Figure 3.4: Plots for the first time segment: August 1, 2006 - June 30, 2007

Eigenvalues Plot 08/01/2007 -- 02/28/2009

Figure 3.5: Plots for the second segment: August 1, 2007 - February 28, 2009

Figure 3.6: Plots for the second segment: April 1, 2009 - March 31, 2010

Figure 3.7: Implied default correlation matrix for 02/23/2007

Figure 3.8: Implied default correlation matrix for 01/06/2009

Figure 3.9: Implied default correlation matrix for 02/23/2010

Figure 3.10: Plot of the Bayes factors

Figure 3.11: Time Series of Extreme Mean Implied Default Correlations with Mizuho Corporate Bank: August 1, 2007 - February 28, 2009

Index	Company name
1	Asahi Kasei Corporation
2	Casio Computer Co., Ltd.
3	Citizen Watch Co., Ltd.
4	Daiwa Securities Group Inc.
5	Ito-Yokado Co., Ltd.
6	Itochu Corporation
7	Japan Tobacco Inc.
8	JFE Steel Corporation
9	Kajima Corporation
10	Kawasaki Heavy Industries, Ltd.
11	KDDI Corporation
12	Keihin Electric Express Railway Co., Ltd.
13	Keio Electric Railway Co., Ltd.
14	Kubota Corporation
15	Mitsubishi Chemical Corporation
16	Nikon Corporation
17	Nippon Express Co., Ltd.
18	Nippon Yusen Kabusiki Kaisha
19	Nomura Securities Co., Ltd.
20	Obayashi Corporation
21	Odakyu Electric Railway Co., Ltd.
22	Oji Paper Co., Ltd.
23	Sapporo Holdings Limited
24	Shimizu Corporation

Table 3.2: List of reference entities included in all 3 correlation plots

Index	Company name
25	Sumitomo Chemical Company, Limited
26	Sumitomo Metal Industries, Ltd.
27	Suzuki Motor Corporation
28	Taisei Corporation
29	Teijin Limited
30	Tobu Railway Co., Ltd.
31	Tokyu Corporation
32	Toray Industries, Inc.
33	Yamaha Motor Co., Ltd.
34	Yamato Holdings Co., Ltd.

Table 3.3: List of reference entities included in all 3 correlation plots (cont.)

	Core Return	FX Return	Nikkei Return	1st PC
Core Return	1.00	-0.01	-0.08	0.02
FX Return	-0.01	1.00	0.59	-0.23
Nikkei Return	-0.08	0.59	1.00	-0.33
1st PC	0.02	-0.23	-0.33	1.00

Table 3.4: Correlation among covariates and the first principal component, 08/01/2006 - 03/31/2010

	Core Return	FX Return	Nikkei Return	1st PC
Core Return	1.00	0.08	-0.11	-0.01
FX Return	0.08	1.00	0.09	-0.05
Nikkei Return	-0.11	0.09	1.00	-0.03
1st PC	-0.01	-0.05	-0.03	1.00

Table 3.5: Correlation among covariates and the first principal component, 08/01/2006 - 06/30/2007

	Core Return	FX Return	Nikkei Return	1st PC
Core Return	1.00	-0.14	-0.20	0.04
FX Return	-0.14	1.00	0.71	-0.32
Nikkei Return	-0.20	0.71	1.00	-0.38
1st PC	0.04	-0.32	-0.38	1.00

Table 3.6: Correlation among covariates and the first principal component, 08/01/2007 - 02/28/2009

	Core Return	FX Return	Nikkei Return	1st PC
Core Return	1.00	-0.04	0.03	0.00
FX Return	-0.04	1.00	0.50	-0.11
Nikkei Return	0.03	0.50	1.00	-0.29
1st PC	0.00	-0.11	-0.29	1.00

Table 3.7: Correlation among covariates and the first principal component, 04/01/2009 - 03/31/2010

CHAPTER 4

Summary and Discussions

In this dissertation, we have provided a statistical modeling methodology for analyzing term structures of instantaneous forward interest rate and hazard. As term structures are continuously changing, classical survival analysis is used to model the probability of default as a random variable rather than characterizing it using a volatility term in a stochastic differential equation setting. Smooth basis functions such as cubic or B-splines can be easily implemented even when interest and hazard term structures are estimated separately. In practice, our models reflect market investors' view of the credit risk, given the most up-to-date data. Utilizing credit rating information, we are able to plot the mean hazard term structure for each industry, averaged over every credit rating division. From the theoretical perspective, we devised a scheme to measure the dependency structure unexplained by the existing covariates. When additional assumptions are made along with grouping the interest and hazard term structures into a combined discount structure, an analytic form of implied default correlation can be defined and estimated from time-series data. Overall, the empirical analyses from Chapter 2 and Chapter 3 have shown well-interpretable results, consistent with market consensus. Both cross-sectional and time-varying data have revealed a major structural change in investors' perception of implied default dependency before and after the recent financial crisis.

The benefits of our statistical modeling methodology can be seen two-fold. From an academic point of view, our method offers a practical and efficient way

of using key financial data to obtain quick and robust estimates deterministically for model parameters. The wide range of choices for loss functions, basis functions as well as an expandable nonlinear hazard function allows users to flexibly tune the model with their own perception of credit risk. On the other hand, our modeling framework can easily see its advantages in applications such as portfolio risk analysis and stress tests. Missing value imputation could be another important practical use as company financial data and/or credit rating information is often scarce and therefore it is valuable for investors to interpolate these values from our model.

Just like the mathematical finance models, the statistical modeling methodology does have its limitations. For instance, many spline basis functions are sensitive to the location of basis nodes. It is also quite normal to observe a higher percentage of bonds in general that mature less than 10 years than those that mature in more than 25. A quick remedy for these estimation functions could come from spacing the number of bonds proportionally among the nodes for chosen basis functions. Integral splines may replace B-splines in estimating the term interest rate (yield curve) to provide monotonicity while maintaining smoothness. While the empirical results presented in this dissertation were based on the integrated panel data set without explicitly handling any data anomaly issue, we are aware that outliers caused by potential pricing errors or simply clerical mistakes may lead to uninterpretable parameter estimates.

Other possible extensions such as generalization into time-series models for the framework in Chapter 2 and establishing model selection criteria will be considered in our future work. We present, in Appendix C, some preliminary results from relaxing the multivariate normality assumption for the model in Chapter 3. Given their current performances and applicability, we are hopeful that statistical modeling methodologies in understanding credit risk and its dependency will remain an important piece in the related academic and industrial research interest groups.

APPENDIX A

Algorithm for Updating Model Parameters in Chapter 2

The main theme in this iterative updating algorithm is to minimize the loss function defined in Equation 2.25.

1. We first estimate the coefficient $\mathbf{w}_0$ from government bonds by minimizing the sum of squared errors, i.e. $\boldsymbol{\varepsilon}_G(\mathbf{w}_0)' \boldsymbol{\varepsilon}_G(\mathbf{w}_0)$.
2. Given $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$, update the remaining corporate-related parameters in $\boldsymbol{\theta}$, namely $(\mathbf{w}_1, \dots, \mathbf{w}_J, \boldsymbol{\alpha}_h, \boldsymbol{\alpha}_\delta, \boldsymbol{\beta}_h, \boldsymbol{\beta}_\delta)'$.
3. Given the current estimates of $(\mathbf{w}_1, \dots, \mathbf{w}_J, \boldsymbol{\alpha}_h, \boldsymbol{\alpha}_\delta, \boldsymbol{\beta}_h, \boldsymbol{\beta}_\delta)'$, update $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$.
4. Repeat step 2 and 3 until convergence.

APPENDIX B

Derivation of Equation (3.7)

Let t be the present time and C_t be CDS premium at t . Also, we denote $r_t(\cdot)$ and $h_t(\cdot)$ be the term structures of a risk free and hazard rate for the present time t , respectively. First we take the log on both sides of the Equation (3.6). This yields

$$\begin{aligned} \log C_t = & \log \left[\sum_{j=1}^n \exp\left\{-\int_0^{t_j} r_t(s)ds\right\} (\exp\{-H_t(0, t_{j-1}|\mathbf{x})\} - \exp\{-H_t(0, t_j|\mathbf{x})\}) \right] \\ & + \log N(1-R) - \log \sum_{j=1}^n \exp\left\{-\int_0^{t_j} r_t(s)ds\right\} \exp\{-H_t(0, t_{j-1}|\mathbf{x})\}, \quad (\text{B.1}) \end{aligned}$$

where $H(0, T|\mathbf{x}) = \int_0^T h_t(s|\mathbf{x})ds$ is the cumulative hazard. For $0 < x \leq 2$, we have from the first-order Taylor expansion,

$$\log x = -\sum_{n=1}^{\infty} \frac{(1-x)^n}{n} = x - 1 + O((1-x)^2). \quad (\text{B.2})$$

Therefore, we get

$$\begin{aligned} \log C_t = & \log N(1-R) - \sum_{j=1}^n \exp\left\{-\int_0^{t_j} r_t(s)ds\right\} \exp\{-H_t(0, t_j|\mathbf{x})\} \\ & + O\left(\left[1 - \sum_{j=1}^n \exp\left\{-\int_0^{t_j} r_t(s)ds\right\} \exp\{-H_t(0, t_{j-1}|\mathbf{x})\}\right]^2\right) \\ \approx & \log N(1-R) - \sum_{j=1}^n \exp\left\{-\int_0^{t_j} [r_t(s) + h_t(s|\mathbf{x})]ds\right\}. \quad (\text{B.3}) \end{aligned}$$

Let Δ be a small time period. Also, let $r_{t+\Delta}(t)$ and $h_{t+\Delta}(t)$ be the risk-free interest rate and hazard rate at time $t + \Delta$. In the same way above, we also obtain the approximated expression (B.3) for $\log C_{t+\Delta}$. Then, the log-return of the CDS

premium during the time period $[t, t + \Delta]$, $\log\{C_{t+\Delta}/C_t\}$ is expressed as

$$\log\{C_{t+\Delta}/C_t\} \approx \sum_{j=1}^n - \left[\exp\left\{-\int_0^{t_j} [r_{t+\Delta}(s) + h_{t+\Delta}(s|\mathbf{x}_{t+\Delta})]ds\right\} - \exp\left\{-\int_0^{t_j} [r_t(s) + h_t(s|\mathbf{x}_t)]ds\right\} \right] \quad (\text{B.4})$$

For any x , we have from the first-order Taylor expansion, $\exp(x) = 1 + x + O(x^2)$. Therefore, the log-return of the CDS premium during the time period $[t, t + \Delta]$ is approximated as

$$\begin{aligned} \log\{C_{t+\Delta}/C_t\} &\approx \sum_{j=1}^n \left[\int_0^{t_j} [r_{t+\Delta}(s) + h_{t+\Delta}(s|\mathbf{x}_{t+\Delta})]ds - \int_0^{t_j} [r_t(s) + h_t(s|\mathbf{x}_t)]ds \right] \\ &= \sum_{j=1}^n \left[\int_0^{t_j} [\hat{h}_{t+\Delta}(s|\mathbf{x}_{t+\Delta}) - \hat{h}_t(s|\mathbf{x}_t)]ds \right], \end{aligned} \quad (\text{B.5})$$

where $\hat{h}_t(s|\mathbf{x}_t) = r_t(s) + h_t(s|\mathbf{x})$ is the term structure at time t . This function can be used for discounting a set of future cashflows to the present values for the reference entity. Also, it is clear that a larger hazard rate requires a bigger CDS premium.

APPENDIX C

An MCMC Approach to Relax the Multivariate Normality Assumption

In Section 3.3, we assumed normality for the multi-response error term in order to model the time-varying correlation structure for Σ_t effectively with dynamic Bayesian updating. Without fixing the number of days in the sliding time-frame window, we now assume a multivariate student-t distribution for the error terms and present a new algorithm for sampling its degrees of freedom.

The sensitivity analysis in Section 3.3 takes the form of the following likelihood

$$f(Y|X, B, \Sigma, \nu) = \frac{\Gamma_m\left(\frac{\nu+n+m-1}{2}\right)}{\Gamma_m\left(\frac{\nu+m-1}{2}\right)(2\pi)^{\frac{nm}{2}}} |\Sigma|^{-\frac{n}{2}} \left| I - (Y - XB)' \Sigma^{-1} (Y - XB) \right|^{-\frac{\nu+n+m-1}{2}},$$

where

$$\Gamma_m(x) = \pi^{m(m-1)/4} \prod_{j=1}^m \Gamma\left(x + \frac{1-j}{2}\right)$$

is the multivariate Gamma function.

Following the updating scheme in Section 3.4.2 mostly unchanged but instead generating ν by the method specified in Watanabe's paper (Watanabe 2001). In order to do this, we express the log-likelihood as a function of ν while keeping all other estimates current.

$$\begin{aligned} f(\nu) &= \log f(Y|X, B, \Sigma, \nu) \\ &= \log \Gamma_m\left(\frac{\nu+n+m-1}{2}\right) - \log \Gamma_m\left(\frac{\nu+m-1}{2}\right) - \frac{nm}{2} \log(2\pi) \\ &\quad - \frac{n}{2} \log |\Sigma| - \frac{\nu+n+m-1}{2} \log \left| I - (Y - XB)' \Sigma^{-1} (Y - XB) \right|. \end{aligned}$$

The second-order Taylor expansion centered at an appropriately chosen value ν^* is then

$$f(\nu^*) + f'(\nu^*)(\nu - \nu^*) + \frac{f''(\nu^*)}{2}(\nu - \nu^*)^2 + o(|\nu - \nu^*|^2) = h(\nu) + o(|\nu - \nu^*|^2).$$

The first and second order derivatives in the above expression can be calculated as

$$\begin{aligned} f'(\nu) &= \frac{1}{2} \frac{\partial}{\partial \nu} \log \Gamma_m \left(\frac{\nu + n + m - 1}{2} \right) - \frac{1}{2} \frac{\partial}{\partial \nu} \log \Gamma_m \left(\frac{\nu + m - 1}{2} \right) \\ &\quad - \frac{1}{2} \log \left| I - (Y - XB)' \Sigma^{-1} (Y - XB) \right|, \\ f''(\nu) &= \frac{1}{4} \frac{\partial^2}{\partial \nu^2} \log \Gamma_m \left(\frac{\nu + n + m - 1}{2} \right) - \frac{1}{4} \frac{\partial^2}{\partial \nu^2} \log \Gamma_m \left(\frac{\nu + m - 1}{2} \right). \end{aligned}$$

Let $\psi(x) = \frac{d}{dx} \log \Gamma(x) = \Gamma'(x)/\Gamma(x)$ and $\psi'(x) = \frac{d}{dx} \psi(x)$ denote the digamma and trigamma function, respectively. Since

$$\frac{\partial}{\partial x} \log \Gamma_m(x) = \sum_{j=1}^m \psi \left(x + \frac{1-j}{2} \right)$$

and likewise, the values of $f''(\nu)$ can be expressed in terms of ψ' , we can easily calculate the value for $h(\nu)$.

In practice, we use the following numerical approximations for log-Gamma, digamma and trigamma functions to speed up the calculation (see Bhattacharjee (1970), Bernardo (1976) and Schneider (1978) for references).

$$\begin{aligned} \log \Gamma(x) &= \left(x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log(2\pi) \\ &\quad + \log \left[1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + O(z^{-5}) \right], \\ \psi(x) &= \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + O(x^{-8}), \\ \psi'(x) &= \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + O(x^{-11}). \end{aligned}$$

The normalized version of h (in log scale) is a normal density function since

$$h(\nu) = A\nu^2 - B\nu + C = \frac{1}{2} \frac{(\nu - \frac{B}{2A})^2}{\frac{1}{2A}} + \left(C - \frac{B^2}{4A} \right)$$

where $A = \frac{1}{2}f''(\nu^*)$, $B = \nu^*f''(\nu^*) - f'(\nu^*)$, and $C = f(\nu^*) - f'(\nu^*)\nu^* + \frac{1}{2}f''(\nu^*)(\nu^*)^2$. Hence, it makes sense to use a normal-candidate-generating distribution with mean $B/(2A)$ and variance $-1/(2A)$.

If updated values for ν become too large, it would indicate that the likelihood converges to a normal distribution. Therefore we may choose an initial value ν_0 from a uniform prior distribution on an truncated interval $[2, 50]$. The Metropolis-Hastings (M-H) Acceptance-Rejection (A-R) algorithm for updating the degrees of freedom ν is as follows.

- Sample a candidate x from the normalized version of h centered at the current update ν_i , and a value u_1 from the uniform distribution on $(0, 1)$. If the mean of h is above 50, truncate and reset it to 50.
- If $u_1 \leq f(x)/h(x)$ and $x \leq 50$, return the value of x ; otherwise repeat this step until a candidate draw is accepted.
- Sample a value u_2 from the uniform distribution on $(0, 1)$.
- If $f(\nu_i) < h(\nu_i)$, then let $q = 1$;
If $f(\nu_i) \geq h(\nu_i)$ and $f(x) < h(x)$, then let $q = h(\nu_i)/f(\nu_i)$;
If $f(\nu_i) \geq h(\nu_i)$ and $f(x) \geq h(x)$, then let $q = \min\{f(x)h(\nu_i)/f(\nu_i)h(x), 1\}$.
- If $u_2 \leq q$, set $\nu_{i+1} = x$. Otherwise, set $\nu_{i+1} = \nu_i$.

The first 10000 runs are regarded as the burn-in period. We compare the results between the original method and Watanabe's method, as well as the original MCMC algorithm based on one estimate taken from every 50 runs. Instead of listing separately the acceptance rates from A-R and M-H steps in Watanabe's method, we choose to calculate the value of $\alpha^* = (\# \text{ total updates})/(\# \text{ total draws})$. The number of draws for ν in the original method is fixed to be simply the number of iterations, namely 50,000 (excluding the burn-in period). That

number is different in Watanabe's method as we keep drawing until a plausible candidate value is accepted in the A-R step.

Figures C.1 through C.3 display the results from the original MCMC algorithm with a proposed normal density with mean ν_i and standard deviation 5 whose values are truncated when they exceed 50. The modified acceptance rate α^* is around 12% and successive updates are heavily correlated. By taking one update of ν for every 50 runs, we can reduce the autocorrelation among the drawn values significantly. When Watanabe's algorithm is applied, it not only reduces the sum of autocorrelation coefficients but improved the overall acceptance rate by about 20% in value. The density plots in Figure C.4 indicate that the sampled degrees of freedom tend to be left-skewed with the median around 45 on all selected dates in Chapter 3. Tables C.1 through C.3 list the 95% confidence intervals for ν for comparing the original method and our proposed method using Watanabe's algorithm.

Figure C.1: Plot of Autocorrelation coefficients and density of estimated d.f. for 01/23/2007

Figure C.2: Plot of Autocorrelation coefficients and density of estimated d.f. for 01/06/2009

Figure C.3: Plot of Autocorrelation coefficients and density of estimated d.f. for 02/23/2010

Figure C.4: Density plots for d.f.

Date	$\bar{\nu}$	s_ν	95% C.I. for ν	$\sum_{k=1}^{100} \rho(k)$	$\alpha^\star$
2007/01/23	49.23	0.774	[49.222, 49.236]	2.035	0.1229
2009/01/06	49.20	0.794	[49.194, 49.208]	1.935	0.1247
2010/02/23	49.24	0.781	[49.237, 49.250]	1.970	0.1229

Table C.1: Original method, parameters estimated from all 50,000 runs

Date	$\bar{\nu}$	s_ν	95% C.I. for ν	$\sum_{k=1}^{100} \rho(k)$
2007/01/23	49.27	0.702	[49.226, 49.313]	0.558
2009/01/06	49.24	0.759	[49.195, 49.289]	1.216
2010/02/23	49.28	0.740	[49.235, 49.327]	0.829

Table C.2: Original method, parameters estimated from one in every 50 runs

Date	$\bar{\nu}$	s_ν	95% C.I. for ν	$\sum_{k=1}^{100} \rho(k)$	$\alpha^\star$
2007/01/23	44.92	3.763	[44.885, 44.951]	0.966	0.3283
2009/01/06	44.94	3.764	[44.902, 44.968]	0.922	0.3295
2010/02/23	44.93	3.776	[44.897, 44.963]	0.861	0.3300

Table C.3: Watanabe's method, parameters estimated from all 50,000 runs

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