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Local Well-Posedness of the Skew Mean Curvature Flow for Small Data in $d \geq 2$ Dimensions

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Abstract

The skew mean curvature flow is an evolution equation for d dimensional manifolds embedded in \mathbb{R}^{d+2} (or more generally, in a Riemannian manifold). It can be viewed as a Schrödinger analogue of the mean curvature flow, or alternatively as a quasilinear version of the Schrödinger Map equation. In an earlier paper, the authors introduced a harmonic/Coulomb gauge formulation of the problem, and used it to prove small data local well-posedness in dimensions $d \geq 4$. In this article, we prove small data local well-posedness in low-regularity Sobolev spaces for the skew mean curvature flow in dimension $d \geq 2$. This is achieved by introducing a new, heat gauge formulation of the equations, which turns out to be more robust in low dimensions.

Mathematics Subject Classification: Primary: 35Q55; Secondary: 53E10

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1. Introduction

In this article we continue our study of the local well-posedness for the skew mean curvature flow (SMCF). This is a nonlinear Schrödinger type flow modeling the evolution of a d dimensional oriented manifold embedded in a fixed oriented $d + 2$ dimensional manifold; it can be seen as a Schrödinger analogue of the well studied mean curvature flow.

In an earlier work [9], we considered the (SMCF) flow in higher dimension $d \geq 4$, and proved local well-posedness for small initial data in low regularity Sobolev spaces. This was achieved by developing a suitable harmonic/Coulomb gauge formulation of the equations, which allowed us to reformulate the problem as a quasilinear Schrödinger evolution.

In this article, we consider the small data local well-posedness for the skew mean curvature flow in low dimensions $d \geq 2$, also for low regularity initial data. As the earlier harmonic/Coulomb gauge formulation has issues in low dimensions, here we introduce an alternative heat gauge, which resolves these difficulties.

1.1. The (SMCF) Equations

Let Σ^d be a d -dimensional oriented manifold, and $(\mathcal{N}^{d+2}, g_{\mathcal{N}})$ be a $d + 2$ -dimensional oriented Riemannian manifold. Let $I = [0, T]$ be an interval and $F : I \times \Sigma^d \rightarrow \mathcal{N}$ be a one parameter family of immersions. This induces a time dependent Riemannian structure on Σ^d . For each $t \in I$, we denote the submanifold by $\Sigma_t = F(t, \Sigma)$, its tangent bundle by $T\Sigma_t$, and its normal bundle by $N\Sigma_t$ respectively. For an arbitrary vector Z at F we denote by Z^\perp its orthogonal projection onto $N\Sigma_t$. The mean curvature $\mathbf{H}(F)$ of Σ_t can be identified naturally with a section of the normal bundle $N\Sigma_t$.

The normal bundle $N\Sigma_t$ is a rank two vector bundle with a naturally induced complex structure $J(F)$ which simply rotates a vector in the normal space by $\pi/2$ positively. Namely, for any point $y = F(t, x) \in \Sigma_t$ and any normal vector $v \in N_y \Sigma_t$, we define $J(F) \in N_y \Sigma_t$ as the unique vector with the same length so that

$$J(F)v \perp v, \quad \omega(F_*(e_1), F_*(e_2), \dots, F_*(e_d), v, J(F)v) > 0,$$

where ω is the volume form of \mathcal{N} and $\{e_1, \dots, e_d\}$ is an oriented basis of Σ^d . The skew mean curvature flow (SMCF) is defined by the initial value problem

$$\begin{cases} (\partial_t F)^\perp = J(F)\mathbf{H}(F), \\ F(\cdot, 0) = F_0, \end{cases} \quad (1.1)$$

which evolves a codimension two submanifold along its binormal direction with a speed given by its mean curvature.

The (SMCF) was derived both in physics and mathematics. The one-dimensional (SMCF) in the Euclidean space \mathbb{R}^3 is the well-known vortex filament equation (VFE)

$$\partial_t \gamma = \partial_s \gamma \times \partial_s^2 \gamma,$$

where γ is a time-dependent space curve, s is its arc-length parameter and \times denotes the cross product in \mathbb{R}^3 . The (VFE) was first discovered by Da Rios [4] in 1906 in the study of the free motion of a vortex filament.

The (SMCF) also arises in the study of asymptotic dynamics of vortices in the context of superfluidity and superconductivity. For the Gross–Pitaevskii equation, which models the wave function associated with a Bose–Einstein condensate, physics evidence indicates that the vortices would evolve along the (SMCF). An incomplete verification was attempted by Lin [18] for the vortex filaments in three space dimensions. For higher dimensions, Jerrard [11] proved this conjecture when the initial singular set is a codimension two sphere with multiplicity one.

The other motivation is that the (SMCF) naturally arises in the study of the hydrodynamical Euler equation. A singular vortex in a fluid is called a vortex membrane in higher dimensions if it is supported on a codimension two subset. The law of locally induced motion of a vortex membrane can be deduced from the Euler equation by applying the Biot–Savart formula. Shashikanth [23] first investigated the motion of a vortex membrane in \mathbb{R}^4 and showed that it is governed by the two

dimensional (SMCF), while Khesin [15] then generalized this conclusion to any dimensional vortex membranes in Euclidean spaces.

From a mathematical standpoint, the (SMCF) equation is a canonical geometric flow for codimension two submanifolds which can be viewed as the Schrödinger analogue of the well studied mean curvature flow. In fact, the infinite-dimensional space of codimension two immersions of a Riemannian manifold admits a generalized Marsden–Weinstein symplectic structure, and hence the Hamiltonian flow of the volume functional on this space is verified to be the (SMCF). Haller–Vizman [8] noted this fact where they studied the nonlinear Grassmannians. For a detailed mathematical derivation of these equations we refer the reader to the article [26, Section 2.1].

The one dimensional case of this problem has been extensively studied. This is because the one dimensional (SMCF) flow agrees the classical Schrödinger Map type equation, provided that one chooses suitable coordinates, i.e. the arclength parametrization. As such, it exhibits many special properties (e.g. complete integrability) which are absent in higher dimensions. For more details we refer the reader to the survey article of Vega [27] as well as [1] and [8].

The study of higher dimensional (SMCF), on the other hand, is far less developed. Song–Sun [26] proved the local existence of (SMCF) with a smooth, compact oriented surface as the initial data in two dimensions, then Song [25] generalized this result to compact oriented manifolds for all $d \geq 2$ and also proved a corresponding uniqueness result. Song [24] also proved that the Gauss map of a d dimensional (SMCF) in \mathbb{R}^{d+2} satisfies a Schrödinger Map type equation but relative to the varying metric. More recently, Li [16, 17] considered a class of transversal small perturbations of Euclidean planes under the (SMCF) and proved a global regularity result for small initial data.

This article is instead concerned with the case when $\Sigma^d = \mathbb{R}^d$, i.e. where Σ_t has a trivial topology. We will further restrict to the case when \mathcal{N}^{d+2} is the Euclidean space \mathbb{R}^{d+2} . Thus, the reader should visualize Σ_t as an asymptotically flat codimension two submanifold of \mathbb{R}^{d+2} .

Such manifolds $\Sigma = \mathbb{R}^d$ with $d \geq 4$ were already considered in our earlier work [9], where we proved the local well-posedness for small data in low-regularity Sobolev spaces. Here we consider instead the lower dimensional case, namely the dimensions $d = 2, 3$. A key role in both [9] and in this article was played by our gauge choices, which are discussed next.

1.2. Gauge Choices for (SMCF)

There are two components for the gauge choice, which are briefly discussed here and in full detail in Section 2:

- (1) The choice of coordinates on $I \times \Sigma$.
- (2) The choice of an orthonormal frame on $I \times N\Sigma$.

Indeed, as written above in (1.1), the (SMCF) equations are independent of the choice of coordinates in $I \times \Sigma$; here we include the time interval I to emphasize that coordinates may be chosen in a time dependent fashion. The manifold Σ^d simply

serves to provide a parametrization for the moving manifold Σ_t ; it determines the topology of Σ_t , but nothing else. Thus, the (SMCF) system written in the form (1.1) should be seen as a geometric evolution, with a large gauge group, namely the group of time dependent changes of coordinates in $I \times \Sigma$. One may think of the gauge choice here as having two components, (i) the choice of coordinates at the initial time, and (ii) the time evolution of the coordinates. One way to describe the latter choice is to rewrite the equations in the form

$$\begin{cases} (\partial_t - V\partial_x)F = J(F)\mathbf{H}(F), \\ F(\cdot, 0) = F_0, \end{cases}$$

where the vector field V can be freely chosen, and captures the time evolution of the coordinates. Indeed, some of the earlier papers [26] and [25] on (SMCF) use this formulation with $V = 0$. This would seem to simplify the equations, however it introduces difficulties at the level of comparing solutions. This is because the regularity of the map F is no longer determined by the regularity of the second fundamental form, and instead there is a loss of derivatives which may only be avoided if the initial data is assumed to have extra regularity. This loss is what prevents a complete low regularity theory in that approach.

In our earlier work [9] in dimension $d \geq 4$, we chose *harmonic coordinates* on Σ , separately at each time. This implicitly fixes V , which may be obtained as the solution of an appropriate elliptic equation. The same approach could be made to work in dimension $d = 3$, if one uses a more careful study of the linearized equation as in the present paper. Unfortunately this does not seem to work well in two dimensions, essentially due to a lack of sufficient control on the metric at low regularity, which is caused by a lack of decay of the fundamental solution for the Laplacian.

To rectify this issue, in the present paper we use instead a *heat gauge*, where the coordinates and implicitly the metric are determined dynamically via a heat flow. This in particular requires also a good choice of coordinates at the initial time; there, we fall back to the harmonic coordinates. In dimension three and higher, this is all that is needed, and in effect both gauge choices, i.e. the heat gauge and the harmonic gauge, work equally well. However, in two dimensions the harmonic coordinates fail to yield the needed low frequency decay of the metric. We rectify this by adding an a-priori low frequency assumption on the metric in suitable coordinates, and then propagate this in time via the heat gauge.

We now discuss the second component of the gauge choice, namely the orthonormal frame in the normal bundle. Such a choice is needed in order to fix the second fundamental form for Σ ; indeed, the (SMCF) is most naturally interpreted as a nonlinear Schrödinger evolution for the second fundamental form of Σ . In our earlier paper [9] we use the Coulomb gauge. But that seems to no longer be well behaved in two dimensions, so we replace it again with a heat flow. In this context, this strategy is reminiscent of the work of the second author and collaborators for the Chern–Simons–Schrödinger flow in [20].

1.3. Scaling and Function Spaces

To understand what are the natural thresholds for local well-posedness, it is interesting to consider the scaling properties of the solutions. As one might expect, a clean scaling law is obtained when $\Sigma^d = \mathbb{R}^d$ and $\mathcal{N}^{d+2} = \mathbb{R}^{d+2}$. Then we have the following:

Proposition 1.1. (Scale invariance for (SMCF)) *Assume that F is a solution of (1.1) with initial data $F(0) = F_0$, then $F_\mu(t, x) := \mu^{-1}F(\mu^2t, \mu x)$ is a solution of (1.1) with initial data $F_\mu(0) = \mu^{-1}F_0(\mu x)$.*

The above scaling would suggest the critical Sobolev space for our moving surfaces Σ_t to be $\dot{H}^{\frac{d}{2}+1}$. However, instead of working directly with the surfaces, it is far more convenient to track the regularity at the level of the curvature $\mathbf{H}(\Sigma_t)$, which scales at the level of $\dot{H}^{\frac{d}{2}-1}$.

For our main result we will use instead inhomogeneous Sobolev spaces, and it will suffice to go one derivative above scaling. There is also a low frequency issue, precisely in two space dimensions where the L^2 norm is critical. There we will need to make a slightly stronger assumption on the low frequency part of the initial data.

1.4. The Main Result

Our objective in this paper is to establish the local well-posedness of skew mean curvature flow for small data at low regularity. A key observation is that providing a rigorous description of fractional Sobolev spaces for functions (tensors) on a rough manifold is a delicate matter, which a-priori requires both a good choice of coordinates on the manifold and a good frame on the vector bundle (the normal bundle in our case). This is done in the next section, where we fix the gauge and write the equation as a quasilinear Schrödinger evolution in a good gauge. At this point, we content ourselves with a less precise formulation of the main result.

Theorem 1.2. (Small data local well-posedness in dimensions $d \geq 3$) *Let $d \geq 3$, $s > \frac{d}{2}$ and $\sigma_d = \frac{d}{2} - \delta$. Then there exists $\epsilon_0 > 0$ sufficiently small such that, for all initial data Σ_0 with metric g_0 and mean curvature \mathbf{H}_0 satisfying*

$$\| |D|^{\sigma_d}(g_0 - I_d) \|_{H^{s+1-\sigma_d}} \leq \epsilon_0, \quad \| \mathbf{H}_0 \|_{H^s(\Sigma_0)} \leq \epsilon_0, \tag{1.2}$$

relative to some parametrization of Σ_0 , the skew mean curvature flow (1.1) for maps from \mathbb{R}^d to the Euclidean space $(\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ is locally well-posed on the time interval $I = [0, 1]$ in a suitable gauge.

With a slight adjustment, a similar result holds in dimension $d = 2$.

Theorem 1.3. (Small data local well-posedness in dimension $d = 2$) *Let $d = 2$, $s > \frac{d}{2}$ and $\sigma_d = \frac{d}{2} - \delta$. Then there exists $\epsilon_0 > 0$ sufficiently small such that, for all initial data Σ_0 with metric g_0 and mean curvature \mathbf{H}_0 satisfying*

$$\| |D|^{\sigma_d}(g_0 - I_d) \|_{H^{s+1-\sigma_d}} \leq \epsilon_0, \quad \| \mathbf{H}_0 \|_{H^s(\Sigma_0)} \leq \epsilon_0,$$

as well as a low frequency bound for g_0

$$\|g_0 - I_d\|_{Y_0^{lo}} < \epsilon_0, \tag{1.3}$$

relative to some parametrization of Σ_0 , the skew mean curvature flow (1.1) for maps from \mathbb{R}^d to the Euclidean space $(\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ is locally well-posed on the time interval $I = [0, 1]$ in a suitable gauge.

We continue with some comments on the function spaces in the above theorems:

- For the metric g_0 , we use the difference $g_0 - I_d$ in the above statements in order to emphasize the normalization $g_0 \rightarrow I_d$ at infinity.
- In dimension $d \geq 3$, the $g_0 - I_d$ norm in (1.2) only plays a qualitative role, namely to place us in a regime where, in harmonic coordinates, g_0 is uniquely determined by the mean curvature \mathbf{H}_0 .
- The Y_0^{lo} norm in (1.3), defined in Section 3, captures low frequency l^1 summability properties for g_0 with respect to cube lattice partitions of \mathbb{R}^d . Similar norm appear in our analysis in dimensions $d \geq 3$. The main difference is that in higher dimension, the Y norms of $g_0 - I_d$ can be estimated in terms of the H^s norm of \mathbf{H} in harmonic coordinates. In two dimensions, this estimate borderline fails, so we instead include the Y_0^{lo} bound in the hypothesis.

Following the spirit of our earlier work [9], in these results we consider rough data and provide a full, Hadamard style well-posedness result based on a more modern, frequency envelope approach and using a paradifferential form for both the full and the linearized equations. For an overview of these ideas we refer the reader to the expository paper [10]. This is unlike any of the prior results, which prove only existence and uniqueness for smooth data.

The favourable gauge mentioned in the theorem is defined in the next section in two steps:

- (a) at the initial time, where we proceed as in [9], and use
 - Harmonic coordinates on the manifold Σ_0 .
 - The Coulomb gauge for the orthonormal frame on the normal bundle $N\Sigma_0$.
- (b) dynamically for $t > 0$, where we use
 - The heat coordinates on the manifolds Σ_t .
 - The heat gauge for the orthonormal frame on the normal bundle $N\Sigma$.

One simple example of initial data allowed by our theorem consists of graph submanifolds with defining functions u_1, u_2 , of the form

$$\Sigma_0 = \{x, u_1(x), u_2(x); x \in \mathbb{R}^d\}$$

Here one may simply take u_1 and u_2 to be small in H^{s+2} , with the added low frequency control in the Y_0^{lo} space in dimension two. However, the H^{s+2} control is only needed at high frequency, while at low frequency it suffices to have control only in homogeneous norms $\dot{H}^{\frac{d}{2}+1-\delta}$ with $\delta > 0$. This allows for perturbations which are not small in any uniform norm.

Example 1.3.1. (Bump-like sub-manifolds) Let ϕ_i , $i = 1, 2$ be Schwartz functions. Then for small $\epsilon > 0$ and $\delta > 0$, the manifold Σ_0 given by the defining functions

$$u_j = \epsilon^{\frac{d}{2}-2+\delta} \phi_j(\epsilon x)$$

satisfies the hypotheses of our theorem. with $\epsilon > 0$ sufficiently small. This manifold is not a small perturbation of the Euclidean plane in low dimension.

Example 1.3.2. (Sub-manifolds with nontrivial asymptotics) For small $\epsilon_j > 0$ and $\delta > 0$, the manifold Σ_0 given by the defining functions

$$u_j = \epsilon_j (1 + x^2)^{1-\frac{d}{4}-\delta}$$

satisfies the hypotheses of our theorem. with $\epsilon_j > 0$ sufficiently small. This manifold is also not a small perturbation of the Euclidean plane.

In the next section we reformulate the (SMCF) equations as a quasilinear Schrödinger evolution for good scalar complex variable λ , which is exactly the second fundamental form but represented in the good gauge. There we provide an alternate formulation of the above result, as a well-posedness result for the λ equation. In the final section of the paper we close the circle and show that one can reconstruct the full (SMCF) flow starting from the good variable λ .

Once our problem is rephrased as a nonlinear Schrödinger evolution, one may compare its study with earlier results on general quasilinear Schrödinger evolutions. This story begins with the classical work of Kenig–Ponce–Vega [12–14], where local well-posedness is established for more regular and localized data. Lower regularity results in translation invariant Sobolev spaces were later established by Marzuola–Metcalfe–Tataru [20–22]. The local energy decay properties of the Schrödinger equation, as developed earlier in [2,3,5,6] play a key role in these results. While here we are using some of the ideas in the above papers, the present problem is both more complex and exhibits additional structure. Because of this, new ideas and more work are required in order to close the estimates required for both the full problem and for its linearization.

1.5. An Overview of the Paper

Our first objective in this article will be to provide a self-contained formulation of the (SMCF) flow, interpreted as a nonlinear Schrödinger equation for a well chosen variable. This variable, denoted by λ , represents the second fundamental form on Σ_t , in complex notation. We remark that in our earlier paper [9] we have used instead the complex representation ψ of the mean curvature \mathbf{H} as the good variable, and λ was uniquely determined by ψ via an elliptic div-curl system. However, solving this system in two dimensions is a delicate matter, which is why here we switch to λ . The slight downside of this strategy is that the components of λ are not independent, and instead satisfy a set of compatibility conditions which need to be propagated along the flow.

In addition to the main variable λ , we will use several dependent variables, as follows:

- The Riemannian metric g on Σ_t .
- The magnetic potential A , associated to the natural connection on the normal bundle $N\Sigma_t$.

These additional variables will be viewed as uniquely determined by our main variable λ and initial metric g_0 in a dynamical fashion. This is first done at the initial time by choosing harmonic coordinates on Σ_0 , respectively the Coulomb gauge on $N\Sigma_0$. Finally, our dynamical gauge choice also has two components:

- (i) The choice of coordinates on Σ_t ; here we use heat coordinates, with suitable boundary conditions at infinity.
- (ii) The choice of the orthonormal frame on $N\Sigma_t$; here we use the heat gauge, again assuming flatness at infinity.

To begin this analysis, in the next section we describe the gauge choices, so that by the end we obtain

- (a) A nonlinear Schrödinger equation for λ , see (2.28).
- (b) A parabolic system (2.29) for the dependent variables $\mathcal{S} = (g, A)$, together with suitable compatibility conditions (constraints).

Setting the stage to solve these equations, in Section 3 we describe the function spaces for both λ and \mathcal{S} . This is done at two levels, first at fixed time, which is needed in order to track data sets, and then in the space-time setting, which is needed in order to solve both the heat flows (2.29) and the Schrödinger evolution (2.28). The fixed time spaces are classical Sobolev spaces, with matched regularities for all the components. The main space-time norms are the so called local energy spaces associated to the Schrödinger evolution, as developed in [20–22]. In addition, we also use parabolic mixed norm spaces, which capture the regularity gain in the heat flows.

We begin our analysis in Section 4, where we place the initial data in the harmonic/Coulomb gauge. In higher dimension this analysis was already carried out in our earlier paper [9]. Thus our emphasis here is on the two dimensional case, where some additional low frequency issues arise in connection with the Y norms for the metric g . Compared to our earlier article [9], here we are able to improve the analysis and relax the low frequency component of the Y norm. This suffices in dimension three, but is only borderline in dimension two, which is why we add the low frequency Y bound to the hypothesis of Theorem 1.3.

Next, in Section 5, we consider the solvability of the parabolic system (2.29). We will do this in two steps. First we prove that this system is solvable in the space \mathcal{E}^s . Then we prove space-time bounds for the metric h in local energy spaces; the latter will be needed in the study of the Schrödinger evolution (2.28).

Finally, we turn our attention to the Schrödinger system (2.28), whose study may be compared with earlier results on general quasilinear Schrödinger evolutions. This begins with the classical work of Kenig–Ponce–Vega [12–14], where local well-posedness is established for more regular and localized data. Lower regularity results in translation invariant Sobolev spaces were later established by Marzuola–Metcalfe–Tataru [20–22]. The local energy decay properties of the Schrödinger equation, as developed earlier in [2, 3, 5, 6] play a key role in these results. Here we

are following a similar track, though the present problem is both more complex and exhibits additional structure. Because of this, new ideas and more work are required in order to close the estimates required for both the full problem and for its linearization.

We divide our approach in several steps. In Section 6 we establish several multilinear and nonlinear estimates in our space-time function spaces. These are then used in Section 7 in order to prove local energy decay bounds first for the linear paradifferential Schrödinger flow, and then for a full linear Schrödinger flow associated to the linearization of our main evolution.

The analysis is completed in Section 8, where we combine the linear heat flow bounds and the linear Schrödinger bounds in order to (i) construct solutions for the full nonlinear Schrödinger flow, and (ii) to prove the uniqueness and continuous dependence of the solutions. The solutions are initially constructed without reference to the constraint equations, but then we prove that the constraints are indeed satisfied, by propagating them from the initial time.

Last but not least, in the last section we prove that the full set of variables (λ, g, A) suffice in order to uniquely reconstruct the defining function F for the evolving surfaces Σ_t , as H_{loc}^{s+2} manifolds. More precisely, with respect to the parametrization provided by our chosen gauge, F has regularity

$$\partial_t F, \partial_x^2 F \in C[0, 1; H^s].$$

2. The Differentiated Equations and the Gauge Choice

The goal of this section is to introduce our main independent variable λ , which represents the second fundamental form in complex notation, as well as the following auxiliary variables: the metric g , the connection coefficients A for the normal bundle. For λ we start with (1.1) and derive a nonlinear Schrödinger type system (2.28), with coefficients depending on $\mathcal{S} = (g, A)$. Under suitable gauge conditions, the auxiliary variables \mathcal{S} are shown to satisfy a parabolic system (2.29), as well as a natural set of constraints. We conclude the section with a gauge formulation of our main result, see Theorem 2.5. Here we will introduce the heat coordinates and heat gauge in detail. For some of the detailed derivations, we refer to section 2 in [9].

2.1. The Riemannian Metric g and the Second Fundamental Form

Let (Σ^d, g) be a d -dimensional oriented manifold and let $(\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ be $(d+2)$ -dimensional Euclidean space. Let $\alpha, \beta, \gamma, \dots \in \{1, 2, \dots, d\}$. Considering the immersion $F : \Sigma \rightarrow (\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$, we obtain the induced metric g in Σ ,

$$g_{\alpha\beta} = \partial_{x_\alpha} F \cdot \partial_{x_\beta} F. \quad (2.1)$$

We denote the inverse of the matrix $g_{\alpha\beta}$ by $g^{\alpha\beta}$, i.e.

$$g^{\alpha\beta} := (g_{\alpha\beta})^{-1}, \quad g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta.$$

Let ∇ be the canonical Levi–Civita connection on Σ associated with the induced metric g . A direct computation shows that on the Riemannian manifold (Σ, g) we have the Christoffel symbols

$$\Gamma_{\alpha\beta}^\gamma = g^{\gamma\sigma} \Gamma_{\alpha\beta,\sigma} = g^{\gamma\sigma} \partial_{\alpha\beta}^2 F \cdot \partial_\sigma F.$$

For any tensor $T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_r} dx^{\beta_1} \otimes \dots \otimes dx^{\beta_s} \otimes \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_r}}$, we define its *covariant derivative* as follows

$$\nabla_\gamma T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_r} = \partial_\gamma T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_r} - \sum_{i=1}^s \Gamma_{\gamma\beta_i}^\sigma T_{\beta_1 \dots \beta_{i-1} \sigma \beta_{i+1} \dots \beta_s}^{\alpha_1 \dots \alpha_r} + \sum_{j=1}^r \Gamma_{\gamma\delta}^{\alpha_j} T_{\beta_1 \dots \beta_s}^{\alpha_1 \dots \alpha_{j-1} \delta \alpha_{j+1} \dots \alpha_r}. \tag{2.2}$$

Hence, the Laplace–Beltrami operator Δ_g can be written in the form

$$\Delta_g f = \text{tr } \nabla^2 f = g^{\alpha\beta} (\partial_{\alpha\beta}^2 f - \Gamma_{\alpha\beta}^\gamma \partial_\gamma f),$$

for any twice differentiable function $f : \Sigma \rightarrow \mathbb{R}$. The curvature R on the Riemannian manifold (Σ, g) is given by

$$R_{\gamma\alpha\beta}^\sigma = \partial_\alpha \Gamma_{\beta\gamma}^\sigma - \partial_\beta \Gamma_{\alpha\gamma}^\sigma + \Gamma_{\beta\gamma}^m \Gamma_{\alpha m}^\sigma - \Gamma_{\alpha\gamma}^m \Gamma_{\beta m}^\sigma.$$

We also have

$$R_{\sigma\gamma\alpha\beta} = \partial_\alpha \Gamma_{\beta\gamma,\sigma} - \partial_\beta \Gamma_{\alpha\gamma,\sigma} + \Gamma_{\beta\sigma}^m \Gamma_{\alpha\gamma,m} - \Gamma_{\alpha\sigma}^m \Gamma_{\beta\gamma,m}, \tag{2.3}$$

and the Ricci curvature

$$\text{Ric}_{\alpha\beta} = R_{\alpha\sigma\beta}^\sigma = g^{\sigma\gamma} R_{\gamma\alpha\sigma\beta}.$$

Next, we derive the second fundamental form for Σ . Let $\bar{\nabla}$ be the Levi–Civita connection in $(\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ and let \mathbf{h} be the second fundamental form for Σ as an embedded manifold. For any vector fields $u, v \in T_*\Sigma$, the Gauss relation is

$$\bar{\nabla}_u F_* v = F_*(\nabla_u v) + \mathbf{h}(u, v).$$

Then we have

$$\mathbf{h}_{\alpha\beta} = \mathbf{h}(\partial_\alpha, \partial_\beta) = \bar{\nabla}_{\partial_\alpha} \partial_\beta F - F_*(\nabla_{\partial_\alpha} \partial_\beta) = \partial_{\alpha\beta}^2 F - \Gamma_{\alpha\beta}^\gamma \partial_\gamma F.$$

This gives the mean curvature \mathbf{H} at $F(x)$,

$$\mathbf{H} = \text{tr}_g \mathbf{h} = g^{\alpha\beta} \mathbf{h}_{\alpha\beta} = g^{\alpha\beta} (\partial_{\alpha\beta}^2 F - \Gamma_{\alpha\beta}^\gamma \partial_\gamma F) = \Delta_g F.$$

Hence, the F -equation in (1.1) is rewritten as

$$(\partial_t F)^\perp = J(F) \Delta_g F = J(F) g^{\alpha\beta} (\partial_{\alpha\beta}^2 F - \Gamma_{\alpha\beta}^\gamma \partial_\gamma F).$$

This equation is still independent of the choice of coordinates in Σ^d .

2.2. The Complex Structure Equations

Here we introduce a complex structure on the normal bundle $N\Sigma_t$. This is achieved by choosing $\{v_1, v_2\}$ to be an orthonormal basis of $N\Sigma_t$ such that

$$Jv_1 = v_2, \quad Jv_2 = -v_1.$$

Such a choice is not unique; in making it we introduce a second component to our gauge group, namely the group of sections of an $SU(1)$ bundle over $I \times \mathbb{R}^d$.

The vectors $\{F_1, \dots, F_d, v_1, v_2\}$ form a frame at each point on the manifold (Σ, g) , where F_α for $\alpha \in \{1, \dots, d\}$ are defined as

$$F_\alpha = \partial_\alpha F.$$

We define the tensors $\kappa_{\alpha\beta}$, $\tau_{\alpha\beta}$, the connection coefficients A_α and the temporal component B of the connection in the normal bundle by

$$\kappa_{\alpha\beta} := \partial_\alpha F_\beta \cdot v_1, \quad \tau_{\alpha\beta} := \partial_\alpha F_\beta \cdot v_2, \quad A_\alpha = \partial_\alpha v_1 \cdot v_2, \quad B = \partial_t v_1 \cdot v_2.$$

Then we complexify the normal frame $\{v_1, v_2\}$ and second fundamental form as

$$m = v_1 + iv_2, \quad \lambda_{\alpha\beta} = \kappa_{\alpha\beta} + i\tau_{\alpha\beta}.$$

Here we can define the *complex scalar mean curvature* ψ to be

$$\psi := \text{tr } \lambda = g^{\alpha\beta} \lambda_{\alpha\beta}. \tag{2.4}$$

Our objective for the rest of this section will be to interpret the (SMCF) equation as a nonlinear Schrödinger evolution for λ , by making suitable gauge choices. We remark that the action of sections of the $SU(1)$ bundle is given by

$$\psi \rightarrow e^{i\theta\psi}, \quad \lambda \rightarrow e^{i\theta\lambda}, \quad m \rightarrow e^{i\theta} m, \quad A_\alpha \rightarrow A_\alpha - \partial_\alpha \theta. \tag{2.5}$$

for a real valued function θ .

If we differentiate the frame, we obtain a set of structure equations of the following type

$$\begin{cases} \partial_\alpha F_\beta = \Gamma_{\alpha\beta}^\gamma F_\gamma + \text{Re}(\lambda_{\alpha\beta} \bar{m}), \\ \partial_\alpha^A m = -\lambda_\alpha^\gamma F_\gamma, \end{cases} \tag{2.6}$$

where $\partial_\alpha^A = \partial_\alpha + iA_\alpha$.

2.3. The Gauss and Codazzi Relations

The Gauss and Codazzi equations are derived from the equality of second derivatives $\partial_\alpha \partial_\beta F_\gamma = \partial_\beta \partial_\alpha F_\gamma$ for the tangent vectors on the submanifold Σ and for the normal vectors respectively. Here we use the Gauss and Codazzi relations to derive the Riemannian curvature, the first compatibility condition and a symmetry.

By the structure equations (2.6), we get

$$\begin{aligned} \partial_\alpha \partial_\beta F_\gamma &= (\partial_\alpha \Gamma_{\beta\gamma}^\sigma + \Gamma_{\beta\gamma}^\mu \Gamma_{\alpha\mu}^\sigma - \operatorname{Re}(\lambda_{\beta\gamma} \bar{\lambda}_{\alpha}^\sigma)) F_\sigma \\ &\quad + \operatorname{Re}[(\partial_\alpha^A \lambda_{\beta\gamma} + \Gamma_{\beta\gamma}^\sigma \lambda_{\alpha\sigma}) \bar{m}]. \end{aligned} \quad (2.7)$$

Then in view of $\partial_\alpha \partial_\beta F_\gamma = \partial_\beta \partial_\alpha F_\gamma$ and equating the coefficients of the tangent vectors, we obtain

$$\partial_\alpha \Gamma_{\beta\gamma}^\sigma + \Gamma_{\beta\gamma}^\mu \Gamma_{\alpha\mu}^\sigma - \partial_\beta \Gamma_{\alpha\gamma}^\sigma - \Gamma_{\alpha\gamma}^\mu \Gamma_{\beta\mu}^\sigma = \operatorname{Re}(\lambda_{\beta\gamma} \bar{\lambda}_\alpha^\sigma - \lambda_{\alpha\gamma} \bar{\lambda}_\beta^\sigma).$$

This gives the Riemannian curvature

$$R_{\sigma\gamma\alpha\beta} = \operatorname{Re}(\lambda_{\beta\gamma} \bar{\lambda}_{\alpha\sigma} - \lambda_{\alpha\gamma} \bar{\lambda}_{\beta\sigma}), \quad (2.8)$$

which is a complex formulation of the Gauss equation. Correspondingly we obtain the the Ricci curvature

$$\operatorname{Ric}_{\gamma\beta} = \operatorname{Re}(\lambda_{\gamma\beta} \bar{\psi} - \lambda_{\gamma\alpha} \bar{\lambda}_\beta^\alpha). \quad (2.9)$$

After equating the coefficients of the vector m in (2.7), we obtain

$$\partial_\alpha^A \lambda_{\beta\gamma} + \Gamma_{\beta\gamma}^\sigma \lambda_{\alpha\sigma} = \partial_\beta^A \lambda_{\alpha\gamma} + \Gamma_{\alpha\gamma}^\sigma \lambda_{\beta\sigma},$$

By the definition of covariant derivatives (2.2), we obtain the complex formulation of the Codazzi equation, namely

$$\nabla_\alpha^A \lambda_{\beta\gamma} = \nabla_\beta^A \lambda_{\alpha\gamma}. \quad (2.10)$$

Next, we use the relation $\partial_\alpha \partial_\beta m = \partial_\beta \partial_\alpha m$ in order to derive a compatibility condition between the connection A in the normal bundle and the second fundamental form. Indeed, from $\partial_\alpha \partial_\beta m = \partial_\beta \partial_\alpha m$ we obtain the commutation relation

$$[\partial_\alpha^A, \partial_\beta^A] m = i(\partial_\alpha A_\beta - \partial_\beta A_\alpha) m. \quad (2.11)$$

By (2.6) we have

$$\partial_\alpha^A \partial_\beta^A m = -\partial_\alpha^A (\lambda_\beta^\gamma F_\gamma) = -(\partial_\alpha^A \lambda_\beta^\sigma + \lambda_\beta^\gamma \Gamma_{\alpha\gamma}^\sigma) F_\sigma - \lambda_\beta^\gamma \operatorname{Re}(\lambda_{\alpha\gamma} \bar{m}).$$

Then multiplying (2.11) by m yields

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha = \operatorname{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\beta\gamma}).$$

This gives the compatibility condition for the curvature A ,

$$\nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \operatorname{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\beta\gamma}), \quad (2.12)$$

which can be seen as the complex form of the Ricci equations. We remark that, by equating the coefficients of the tangent vectors in (2.11), we also obtain the relation (2.10) again.

2.4. The Motion of the Frame $\{F_1, \dots, F_d, m\}$ under (SMCF)

Here we derive the equations of motion for the frame, assuming that the immersion F satisfying (1.1).

We begin by rewriting the SMCF equations in the form

$$\partial_t F = J(F)\mathbf{H}(F) + V^\gamma F_\gamma,$$

where V^γ is a vector field on the manifold Σ , which in general depends on the choice of coordinates. By the definition of m and $\lambda_{\alpha\beta}$, the above F -equation is rewritten as

$$\partial_t F = -\text{Im}(\psi \bar{m}) + V^\gamma F_\gamma. \tag{2.13}$$

Then by (2.13), the structure equations (2.6) and the orthogonality relation $m \perp F_\alpha = 0$ we obtain the following equations of motion for the frame

$$\begin{cases} \partial_t F_\alpha = -\text{Im}(\partial_\alpha^A \psi \bar{m} - i \lambda_{\alpha\gamma} V^\gamma \bar{m}) + [\text{Im}(\psi \bar{\lambda}_\alpha^\gamma) + \nabla_\alpha V^\gamma] F_\gamma, \\ \partial_t^B m = -i(\partial^{A,\alpha} \psi - i \lambda_\gamma^\alpha V^\gamma) F_\alpha. \end{cases} \tag{2.14}$$

where covariant derivative $\partial_t^B = \partial_t + iB$ and $B = \langle \partial_t v_1, v_2 \rangle$ is the temporal component of the connection in the normal bundle.

From this we obtain the evolution equation for the metric g . By the definition of the induced metric g (2.1) and (2.14), we have

$$\partial_t g_{\alpha\beta} = 2 \text{Im}(\psi \bar{\lambda}_{\alpha\beta}) + \nabla_\alpha V_\beta + \nabla_\beta V_\alpha. \tag{2.15}$$

So far, the choice of V has been unspecified; it depends on the choice of coordinates on our manifold as the time varies.

2.5. The Motion of A and λ Under (SMCF)

Here we use the equations of motion for the frame in (2.14) in order to repeat the computations of Section 2.3 with respect to time differentiation, with the aim of computing the time derivative of both λ and A . We start from the commutation relation

$$[\partial_t^B, \partial_\alpha^A]m = i(\partial_t A_\alpha - \partial_\alpha B)m.$$

In order, for the left-hand side, by (2.6) and (2.14) we have

$$\partial_t^B \partial_\alpha^A m = -[\partial_t^B \lambda_\alpha^\sigma + \lambda_\alpha^\gamma (\text{Im}(\psi \bar{\lambda}_\gamma^\sigma) + \nabla_\gamma V^\sigma)] F_\sigma + \lambda_\alpha^\gamma \text{Im}(\partial_\gamma^A \psi \bar{m} - i \lambda_{\gamma\sigma} V^\sigma \bar{m}),$$

and

$$\partial_\alpha^A \partial_t^B m = -i \nabla_\alpha^A (\partial^{A,\sigma} \psi - i \lambda_\gamma^\sigma V^\gamma) F_\sigma - i(\partial^{A,\sigma} \psi - i \lambda_\gamma^\sigma V^\gamma) \text{Re}(\lambda_{\alpha\gamma} \bar{m}).$$

Then by the above three equalities, equating the coefficients of the tangent vectors and the normal vector m , we obtain the evolution equation for λ

$$\partial_t^B \lambda_\alpha^\sigma + \lambda_\alpha^\gamma (\text{Im}(\psi \bar{\lambda}_\gamma^\sigma) + \nabla_\gamma V^\sigma) = i \nabla_\alpha^A (\partial^{A,\sigma} \psi - i \lambda_\gamma^\sigma V^\gamma), \tag{2.16}$$

as well as the compatibility condition (curvature relation)

$$\partial_t A_\alpha - \partial_\alpha B = \text{Re}(\lambda_\alpha^\gamma \bar{\partial}_\gamma^A \psi) - \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\gamma\sigma}) V^\sigma. \tag{2.17}$$

2.6. The Equations for the Connection A in the Coulomb Gauge and the Heat Gauge

Here we take the first step towards fixing the gauge, and consider the choice of the orthonormal frame in $N\Sigma$. Our starting point consists of the curvature relations (2.12) at fixed time, respectively (2.17) dynamically, together with the gauge group (2.5). We will fix the gauge in two steps, first in a static, elliptic fashion at the initial time, and then dynamically, using a heat flow, for later times.

At the initial time $t = 0$ we obtain an elliptic system for A by imposing the Coulomb gauge condition

$$\nabla^\alpha A_\alpha = 0. \quad (2.18)$$

As in [9], this yields

Lemma 2.1. (Div-curl system for A) *Under the Coulomb gauge condition (2.18), the connection A solves*

$$\nabla^\alpha A_\alpha = 0, \quad \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\beta\gamma}). \quad (2.19)$$

In our previous work [9], the connection coefficients A and B were determined via the Coulomb gauge condition (2.18) at all times. Instead, in this article we only enforce the Coulomb gauge condition at the initial time $t = 0$, while for $t > 0$ we adopt from [19] a different gauge condition called the *parabolic gauge* or *heat gauge*. This is defined by the relation

$$\nabla^\alpha A_\alpha = B, \quad (2.20)$$

which in turn yields a parabolic equation for A :

Lemma 2.2. (Parabolic equations for A) *Under the heat gauge condition (2.20), the connection A solves*

$$\begin{aligned} (\partial_t - \nabla_\sigma \nabla^\sigma) A_\alpha &= \nabla^\sigma \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\sigma\gamma}) - \text{Ric}_{\alpha\delta} A^\delta \\ &\quad + \text{Re}(\lambda_\alpha^\gamma \bar{\nabla}_\gamma^A \psi) - \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\gamma\sigma}) V^\sigma. \end{aligned} \quad (2.21)$$

Proof. Since by (2.12) we have

$$\begin{aligned} \nabla_\alpha \nabla^\sigma A_\sigma &= [\nabla_\alpha, \nabla^\sigma] A_\sigma + \nabla^\sigma (\nabla_\alpha A_\sigma - \nabla_\sigma A_\alpha) + \nabla^\sigma \nabla_\sigma A_\alpha \\ &= -\text{Ric}_{\alpha\sigma} A^\sigma + \nabla^\sigma \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\sigma\gamma}) + \nabla^\sigma \nabla_\sigma A_\alpha \end{aligned}$$

Then the equations (2.21) is obtained from (2.17) and the heat gauge (2.20). \square

2.7. The Equations for the Metric g in Harmonic Coordinates and Heat Coordinates

Here we take the next step towards fixing the gauge, by choosing to work in harmonic coordinates at $t = 0$ and heat coordinates for $t > 0$. Precisely, at the initial time $t = 0$ we will require the coordinate functions $\{x_\alpha, \alpha = 1, \dots, d\}$ to be globally Lipschitz solutions of the elliptic equations

$$\Delta_g x_\alpha = 0. \tag{2.22}$$

This determines the coordinates uniquely modulo time dependent affine transformations. This remaining ambiguity will be removed later on by imposing suitable boundary conditions at infinity. After this, the only remaining degrees of freedom in the choice of coordinates at $t = 0$ will be given by translations and rigid rotations.

Here we interpret the above harmonic coordinate condition at fixed time as an elliptic equation for the metric g . The equations (2.22) may be expressed in terms of the Christoffel symbols Γ , which must satisfy the condition

$$g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma = 0, \quad \text{for } \gamma = 1, \dots, d. \tag{2.23}$$

This leads to an equation for the metric g :

Lemma 2.3. (Elliptic equations of g , Lemma 2.4 [9]) *In harmonic coordinates, the metric g satisfies*

$$\begin{aligned} g^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\gamma\sigma} &= [-\partial_\gamma g^{\alpha\beta} \partial_\beta g_{\alpha\sigma} - \partial_\sigma g^{\alpha\beta} \partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta} \partial_\sigma g^{\alpha\beta}] \\ &\quad + 2g^{\alpha\beta} \Gamma_{\sigma\alpha, \nu} \Gamma_{\beta\gamma}^\nu - 2 \operatorname{Re}(\lambda_{\gamma\sigma} \bar{\psi} - \lambda_{\alpha\gamma} \bar{\lambda}_\sigma^\alpha). \end{aligned} \tag{2.24}$$

For latter times $t > 0$ we will introduce the *heat gauge*, where we require the coordinate functions $\{x^\alpha, \alpha = 1, \dots, d\}$ to be global Lipschitz solutions of the heat equations

$$(\partial_t - \Delta_g - V^\gamma \partial_\gamma) x_\alpha = 0.$$

This can be rewritten as

$$\Delta_g x^\gamma = -V^\gamma,$$

and can also be expressed in terms of the Christoffel symbols Γ , namely,

$$g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma = V^\gamma. \tag{2.25}$$

Once a choice of coordinates is made at the initial time, the coordinates will be uniquely determined later on by this gauge condition.

With the advection field V fixed via the heat coordinate condition (2.25), we can derive a parabolic equation for the metric g .

Lemma 2.4. (Parabolic equations for metric g) *Under the condition (2.25), the metric g solves*

$$\begin{aligned} \partial_t g_{\mu\nu} - g^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\mu\nu} &= 2 \operatorname{Ric}_{\mu\nu} + 2 \operatorname{Im}(\psi \bar{\lambda}_{\mu\nu}) - 2g^{\alpha\beta} \Gamma_{\mu\beta, \sigma} \Gamma_{\alpha\nu}^\sigma \\ &\quad + \partial_\mu g^{\alpha\beta} \Gamma_{\alpha\beta, \nu} + \partial_\nu g^{\alpha\beta} \Gamma_{\alpha\beta, \mu}. \end{aligned} \tag{2.26}$$

Proof. By the relation (2.25) we have

$$\nabla_\mu V_\nu = g^{\alpha\beta} \partial_\mu \Gamma_{\alpha\beta,\nu} + \partial_\mu g^{\alpha\beta} \Gamma_{\alpha\beta,\nu} - g^{\alpha\beta} \Gamma_{\mu\nu}^\sigma \Gamma_{\alpha\beta,\sigma}$$

Using the expression for Γ and for the Riemannian curvature (2.3) we have

$$\begin{aligned} g^{\alpha\beta} (\partial_\mu \Gamma_{\alpha\beta,\nu} + \partial_\nu \Gamma_{\alpha\beta,\mu}) &= g^{\alpha\beta} \left[\partial_\mu \left(\partial_\alpha g_{\beta\nu} - \frac{1}{2} \partial_\nu g_{\alpha\beta} \right) + \partial_\nu \left(\partial_\alpha g_{\beta\mu} - \frac{1}{2} \partial_\mu g_{\alpha\beta} \right) \right] \\ &= g^{\alpha\beta} [\partial_\alpha (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) - \partial_\mu (\partial_\nu g_{\alpha\beta} + \partial_\alpha g_{\nu\beta} \\ &\quad - \partial_\beta g_{\alpha\nu}) + \partial_{\alpha\beta}^2 g_{\mu\nu}] \\ &= g^{\alpha\beta} [2\partial_\alpha \Gamma_{\mu\nu,\beta} - 2\partial_\mu \Gamma_{\alpha\nu,\beta} + \partial_{\alpha\beta}^2 g_{\mu\nu}] \\ &= 2g^{\alpha\beta} (R_{\beta\nu\alpha\mu} - \Gamma_{\nu\beta,\sigma} \Gamma_{\alpha\mu}^\sigma + \Gamma_{\alpha\beta,\sigma} \Gamma_{\mu\nu}^\sigma) + g^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\mu\nu}. \end{aligned}$$

We then obtain

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = g^{\alpha\beta} \partial_{\alpha\beta}^2 g_{\mu\nu} + 2 \text{Ric}_{\mu\nu} + \partial_\mu g^{\alpha\beta} \Gamma_{\alpha\beta,\nu} + \partial_\nu g^{\alpha\beta} \Gamma_{\alpha\beta,\mu} - 2g^{\alpha\beta} \Gamma_{\nu\beta,\sigma} \Gamma_{\alpha\mu}^\sigma.$$

Combined with (2.15), this implies (2.26). \square

2.8. Derivation of the Modified Schrödinger System from SMCF

Here we carry out the last step in our analysis of the equations, and obtain the main Schrödinger equation which governs the time evolution of λ .

Our starting point is the equations (2.16), which are rewritten as

$$i \partial_t^B \lambda_{\alpha\beta} + \nabla_\alpha^A \nabla_\beta^A \psi - i \lambda_\alpha^\gamma \text{Im}(\psi \bar{\lambda}_{\gamma\beta}) - i \lambda_\alpha^\gamma \nabla_\beta V_\gamma - i \lambda_\beta^\gamma \nabla_\alpha V_\gamma - i V^\gamma \nabla_\gamma^A \lambda_{\alpha\beta} = 0,$$

We use the compatibility conditions (2.4), (2.12) and (2.8) to write the second term as

$$\begin{aligned} \nabla_\alpha^A \nabla_\beta^A \psi &= \nabla_\alpha^A \nabla_\sigma^A \lambda_\beta^\sigma = [\nabla_\alpha^A, \nabla_\sigma^A] \lambda_\beta^\sigma + \nabla_\sigma^A \nabla_\alpha^A \lambda_{\alpha\beta} \\ &= R_{\alpha\sigma\delta} \lambda_\beta^\delta + R_{\alpha\sigma\beta\delta} \lambda^{\sigma\delta} + i \text{Im}(\lambda_{\alpha\mu} \bar{\lambda}_\sigma^\mu) \lambda_\beta^\sigma + \nabla_\sigma^A \nabla_\alpha^A \lambda_{\alpha\beta} \\ &= -\text{Ric}_{\alpha\delta} \lambda_\beta^\delta + R_{\alpha\sigma\beta\delta} \lambda^{\sigma\delta} + i \text{Im}(\lambda_{\alpha\mu} \bar{\lambda}_\sigma^\mu) \lambda_\beta^\sigma + \nabla_\sigma^A \nabla_\alpha^A \lambda_{\alpha\beta} \\ &= -\text{Re}(\lambda_{\alpha\delta} \bar{\psi}) \lambda_\beta^\delta + \text{Re}(\lambda_{\sigma\delta} \bar{\lambda}_{\alpha\beta} - \lambda_{\sigma\beta} \bar{\lambda}_{\alpha\delta}) \lambda^{\sigma\delta} + \lambda_{\alpha\mu} \bar{\lambda}_\sigma^\mu \lambda_\beta^\sigma + \nabla_\sigma^A \nabla_\alpha^A \lambda_{\alpha\beta} \end{aligned}$$

Since

$$\frac{1}{2} [-\text{Re}(\lambda_{\alpha\delta} \bar{\psi}) \lambda_\beta^\delta - \text{Re}(\lambda_{\beta\delta} \bar{\psi}) \lambda_\alpha^\delta - i \lambda_\alpha^\gamma \text{Im}(\psi \bar{\lambda}_{\gamma\beta}) - i \lambda_\beta^\gamma \text{Im}(\psi \bar{\lambda}_{\gamma\alpha})] = -\psi \text{Re}(\lambda_{\alpha\delta} \bar{\lambda}_\beta^\delta),$$

we obtain the λ -equations

$$\begin{aligned} i \partial_t^B \lambda_{\alpha\beta} + \nabla_\sigma^A \nabla_\alpha^A \lambda_{\alpha\beta} &= i V^\gamma \nabla_\gamma^A \lambda_{\alpha\beta} + i \lambda_\alpha^\gamma \nabla_\beta V_\gamma + i \lambda_\beta^\gamma \nabla_\alpha V_\gamma + \psi \text{Re}(\lambda_{\alpha\delta} \bar{\lambda}_\beta^\delta) \\ &\quad - \text{Re}(\lambda_{\sigma\delta} \bar{\lambda}_{\alpha\beta} - \lambda_{\sigma\beta} \bar{\lambda}_{\alpha\delta}) \lambda^{\sigma\delta} - \lambda_{\alpha\mu} \bar{\lambda}_\sigma^\mu \lambda_\beta^\sigma. \end{aligned} \tag{2.27}$$

In conclusion, under the heat coordinate condition $g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma = V^\gamma$ and heat gauge condition $\nabla^\alpha A_\alpha = B$, by (2.27), (2.26) and (2.21), we obtain the covariant Schrödinger equation for the complex second fundamental form tensor λ

$$\left\{ \begin{aligned} (i \partial_t + \nabla_\sigma \nabla^\sigma) \lambda_{\alpha\beta} &= i (V - 2A)^\sigma \nabla_\sigma \lambda_{\alpha\beta} - i \nabla_\sigma A^\sigma \lambda_{\alpha\beta} + i \lambda_\alpha^\gamma \nabla_\beta V_\gamma + i \lambda_\beta^\gamma \nabla_\alpha V_\gamma \\ &\quad + (B + A_\sigma A^\sigma - V_\sigma A^\sigma) \lambda_{\alpha\beta} + \psi \text{Re}(\lambda_{\alpha\delta} \bar{\lambda}_\beta^\delta) \\ &\quad - \text{Re}(\lambda_{\sigma\delta} \bar{\lambda}_{\alpha\beta} - \lambda_{\sigma\beta} \bar{\lambda}_{\alpha\delta}) \lambda^{\sigma\delta} - \lambda_{\alpha\mu} \bar{\lambda}_\sigma^\mu \lambda_\beta^\sigma, \\ \lambda(0, x) &= \lambda_0(x). \end{aligned} \right. \tag{2.28}$$

These equations are fully covariant, and do not depend on the gauge choices made earlier. On the other hand, our gauge choices imply that the advection field V and the connection coefficient B are determined by the metric g and connection A via (2.25), respectively, (2.20). In turn, the metric g and the connection coefficients A are determined in an parabolic fashion via the following equations

$$\begin{cases} (\partial_t - g^{\alpha\beta} \partial_{\alpha\beta}^2)g_{\mu\nu} = 2 \operatorname{Ric}_{\mu\nu} + 2 \operatorname{Im}(\psi \bar{\lambda}_{\mu\nu}) - 2g^{\alpha\beta} \Gamma_{\mu\beta,\sigma} \Gamma_{\alpha\nu}^{\sigma} \\ \quad + \partial_{\mu} g^{\alpha\beta} \Gamma_{\alpha\beta,\nu} + \partial_{\nu} g^{\alpha\beta} \Gamma_{\alpha\beta,\mu}, \\ (\partial_t - \nabla_{\sigma} \nabla^{\sigma})A_{\alpha} = -\nabla^{\sigma} \operatorname{Im}(\lambda_{\alpha}^{\gamma} \bar{\lambda}_{\sigma\gamma}) - \operatorname{Ric}_{\alpha\delta} A^{\delta} + \operatorname{Re}(\lambda_{\alpha}^{\gamma} \overline{\nabla_{\gamma}^A \psi}) - \operatorname{Im}(\lambda_{\alpha}^{\gamma} \bar{\lambda}_{\gamma\sigma}) V^{\sigma}, \\ V^{\gamma} = g^{\alpha\beta} \Gamma_{\alpha\beta}^{\gamma}, \quad B = \nabla^{\alpha} A_{\alpha}, \end{cases} \tag{2.29}$$

with initial data

$$g(0, x) = g_0(x), \quad A(0, x) = A_0(x). \tag{2.30}$$

These are determined at the initial time by choosing harmonic coordinates on Σ_0 , respectively the Coulomb gauge for A .

Fixing the remaining degrees of freedom (i.e. the affine group for the choice of the coordinates as well as the time dependence of the $SU(1)$ connection) we can assume that the following conditions hold at infinity in an averaged sense:

$$g(\infty) = I_d, \quad A(\infty) = 0.$$

These are needed to insure the unique solvability of the above parabolic equations in a suitable class of functions. For the metric g it will be useful to use the representation

$$g = I_d + h$$

so that h vanishes at infinity.

We have arrived at the main Schrödinger-Parabolic system (2.28)–(2.29), whose solvability is the primary objective of the rest of the paper. This system is accompanied by a family of compatibility conditions as follows:

- (i) The Gauss equations (2.8) connecting the curvature R of g and λ .
- (ii) The Codazzi equations (2.10) for λ .
- (iii) The Ricci equations (2.12) for the curvature of A .
- (iv) The compatibility condition (2.17) for the B .

We will solve the system irrespective of these compatibility conditions, but then show them to be satisfied for small solutions to the nonlinear system (2.28)–(2.29), by propagating them from the initial time $t = 0$.

Now we can restate here the small data local well-posedness result for the (SMCF) system in Theorem 1.2 in terms of the above system:

Theorem 2.5. (Small data local well-posedness in the good gauge) *Let $d \geq 2$ and $s > \frac{d}{2}$. Then there exists $\epsilon_0 > 0$ sufficiently small such that, for all initial data (λ_0, h_0, A_0) satisfying the constraints (2.10), (2.8) and (2.12) and with*

$$\|\lambda_0\|_{H^s} + \|h_0\|_{\mathbf{Y}_0^{s+2}} + \|A_0\|_{H^{s+1}} \leq \epsilon_0, \quad (2.31)$$

the modified Schrödinger system (2.28), coupled with the parabolic system (2.29) for (h, A) is locally well-posed in $L^2 X^s \times \mathcal{E}^s$ on the time interval $I = [0, 1]$. Moreover, the second fundamental form λ , the metric g and the connection coefficients A satisfy the bounds

$$\|\lambda\|_{L^2 X^s} + \|(h, A)\|_{\mathcal{E}^s} \lesssim \|\lambda_0\|_{H^s} + \|h_0\|_{\mathbf{Y}_0^{s+2}} + \|A_0\|_{H^{s+1}}. \quad (2.32)$$

In addition, the functions (λ, g, A) satisfy the constraints (2.8), (2.10), (2.12) and (2.17).

Here the solution λ satisfies in particular the expected bounds

$$\|\lambda\|_{C[0,1;H^s]} \lesssim \|\lambda_0\|_{H^s}.$$

The spaces $L^2 X^s$ and \mathcal{E}^s , defined in the next section, contain a more complete description of the full set of variables λ, h, A , which includes both Sobolev regularity and local energy bounds.

In the above theorem, by well-posedness we mean a full Hadamard-type well-posedness, including the following properties:

- (i) Existence of solutions $\lambda \in C[0, 1; H^s]$, with the additional regularity properties (2.32).
- (ii) Uniqueness in the same class.
- (iii) Continuous dependence of solutions with respect to the initial data in the strong H^s topology.
- (iv) Weak Lipschitz dependence of solutions with respect to the initial data in the weaker L^2 topology.
- (v) Energy bounds and propagation of higher regularity.

We conclude this section with several remarks concerning the result in Theorem 2.5:

Remark 2.5.1. (*The variable λ vs ψ*) In our earlier paper [9] we have worked with ψ as the main dynamic variable for the Schrödinger flow, and the full second fundamental form λ was obtained from ψ by solving an elliptic div-curl system derived from the Codazzi relations (2.10). Here we work directly with λ , because solving this elliptic system has issues at the L^2 level in two¹ space dimensions. The downside is that the components of λ are not independent, and are instead connected via the compatibility relations (2.10). Thus, these relations will have to be propagated dynamically.

¹ However, in three and higher dimensions one could still work with ψ if desired.

Remark 2.5.2. (*Initial data sets*) The harmonic/Coulomb gauge condition at the initial time plays no role in Theorem 2.5, where smallness is assumed for both λ_0 , h_0 and A_0 . However, it is useful in order to connect Theorem 2.5 with the earlier statement in Theorems 1.2, 1.3.

3. Function Spaces and Notations

The goal of this section is to define the function spaces where we aim to solve the (SMCF) system in the good gauge, given by (2.28). Both the spaces and the notation presented in this section are similar to those introduced in [20–22].

We begin with some constants. Let regularity index $s > d/2$ and $\delta > 0$ be a small² constant satisfying

$$0 < \delta \ll s - s_d.$$

We then define the constant σ_d depending on dimensions d as

$$\sigma_d = d/2 - \delta. \quad (3.1)$$

For a function $u(t, x)$ or $u(x)$, let $\hat{u} = \mathcal{F}u$ and $\check{u} = \mathcal{F}^{-1}u$ denote the Fourier transform and inverse Fourier transform in the spatial variable x , respectively. Fix a smooth radial function $\varphi : \mathbb{R}^d \rightarrow [0, 1]$ supported in $\{x \in \mathbb{R}^d : |x| \leq 2\}$ and equal to 1 in $\{x \in \mathbb{R}^d : |x| \leq 1\}$, and for any $i \in \mathbb{Z}$, let

$$\varphi_i(x) := \varphi(x/2^i) - \varphi(x/2^{i-1}).$$

We then have the spatial Littlewood–Paley decomposition,

$$\sum_{i=-\infty}^{\infty} P_i(D) = 1, \quad \sum_{i=0}^{\infty} S_i(D) = 1,$$

where P_i localizes to frequency 2^i for $i \in \mathbb{Z}$, i.e.,

$$\mathcal{F}(P_i u) = \varphi_i(\xi) \hat{u}(\xi),$$

and

$$S_0(D) = \sum_{i \leq 0} P_i(D), \quad S_i(D) = P_i(D), \quad \text{for } i > 0.$$

For simplicity of notation, we set

$$u_j = S_j u, \quad u_{\leq j} = \sum_{i=0}^j S_i u, \quad u_{\geq j} = \sum_{i=j}^{\infty} S_i u.$$

² Ideally here one would like to set $\delta = 0$, but this is only possible in dimensions three and higher.

For each $j \in \mathbb{N}$, let \mathcal{Q}_j denote a partition of \mathbb{R}^d into cubes of side length 2^j , and let $\{\chi_Q\}$ denote an associated partition of unity. For a translation-invariant Sobolev-type space U , set $l_j^p U$ to be the Banach space with associated norm

$$\|u\|_{l_j^p U}^p = \sum_{Q \in \mathcal{Q}_j} \|\chi_Q u\|_U^p$$

with the obvious modification for $p = \infty$.

Next we define the $l^2 X^s$ and $l^2 N^s$ spaces, which will be used for the primary variable λ , respectively for the source term in the Schrödinger equation for λ . Following [20–22], we first define the X -norm as

$$\|u\|_X = \sup_{l \in \mathbb{N}} \sup_{Q \in \mathcal{Q}_l} 2^{-\frac{l}{2}} \|u\|_{L^2 L^2((0,1] \times Q)}.$$

Here and throughout, $L^p L^q$ represents $L_t^p L_x^q$. To measure the source term, we use an atomic space N satisfying $X = N^*$. A function a is an atom in N if there is a $j \geq 0$ and a $Q \in \mathcal{Q}_j$ such that a is supported in $[0, 1] \times Q$ and

$$\|a\|_{L^2((0,1] \times Q)} \lesssim 2^{-\frac{j}{2}}.$$

Then we define N as linear combinations of the form

$$f = \sum_k c_k a_k, \quad \sum_k |c_k| < \infty, \quad a_k \text{ atom},$$

with norm

$$\|f\|_N = \inf \left\{ \sum_k |c_k| : f = \sum_k c_k a_k, \quad a_k \text{ atoms} \right\}.$$

For solutions which are localized to frequency 2^j with $j \geq 0$, we will work in the space

$$X_j = 2^{-\frac{j}{2}} X \cap L^\infty L^2,$$

with norm

$$\|u\|_{X_j} = 2^{\frac{j}{2}} \|u\|_X + \|u\|_{L^\infty L^2}.$$

One way to assemble the X_j norms is via the X^s space

$$\|u\|_{X^s}^2 = \sum_{j \geq 0} 2^{2js} \|S_j u\|_{X_j}^2.$$

But we will also add the l^p spatial summation on the 2^j scale to X_j , in order to obtain the space $l_j^p X_j$ with norm

$$\|u\|_{l_j^p X_j} = \left(\sum_{Q \in \mathcal{Q}_j} \|\chi_Q u\|_{X_j}^p \right)^{1/p}.$$

We then define the space $l^p X^s$ by

$$\|u\|_{l^p X^s}^2 = \sum_{j \geq 0} 2^{2js} \|S_j u\|_{l_j^p X_j}^2.$$

For the solutions of Schrödinger equation in (2.28), we will be working primarily in $l^2 X^s$.

We analogously define

$$N_j = 2^{\frac{j}{2}} N + L^1 L^2,$$

which has norm

$$\|f\|_{N_j} = \inf_{f=2^{\frac{j}{2}} f_1 + f_2} (\|f_1\|_N + \|f_2\|_{L^1 L^2}),$$

and

$$\|f\|_{l^p N^s}^2 = \sum_{j \geq 0} 2^{2js} \|S_j f\|_{l_j^p N_j}^2.$$

Here we shall be working primarily with $l^2 N^s$.

We also note that for any j , we have

$$\sup_{Q \in \mathcal{Q}_j} 2^{-\frac{j}{2}} \|u\|_{L^2 L^2([0,1] \times Q)} \leq \|u\|_X,$$

hence

$$\|u\|_N \lesssim 2^{j/2} \|u\|_{l_j^1 L^2 L^2}.$$

This bound will come in handy at several places later on.

For the parabolic system (2.29), it is natural to work in spaces of the form $L^\infty H^s$. However, in order to obtain frequency envelope bounds it is more convenient to slightly strengthen this norm. Precisely, we define the $Z^{\sigma,s}$ norm as

$$\|h\|_{Z^{\sigma,s}}^2 = \| |D|^\sigma S_0 h \|_{L^\infty L^2}^2 + \sum_{j \geq 1} 2^{2sj} \|S_j h\|_{L^\infty L^2}^2.$$

Compared to $L^\infty H^s$, here we just commute the L_t^∞ and l^2 frequency summation. For simplicity of notation, we denote $Z^s := Z^{0,s}$. In particular we have

$$Z^s \subset L^\infty H^s.$$

With these notations, we will seek the solution (h, A) to the parabolic system (2.29) in the space \mathcal{E}^s defined by

$$\|(h, A)\|_{\mathcal{E}^s} = \|h\|_{Z^{\sigma_d, s+2}} + \|A\|_{Z^{s+1}}.$$

Correspondingly, at fixed time we define the space \mathcal{H}^s as

$$\|(h, A)\|_{\mathcal{H}^s} = \| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}} + \|A\|_{H^{s+1}}.$$

In addition to the above standard norms, for the study of the Schrödinger equation for λ we will also need to control a stronger norm \mathbf{Y}^{s+2} for the metric $h = g - I_d$; this will be defined in what follows.

First, similarly to the $l_j^p X_j$ norms above, we also add the l^p spatial summation on the 2^j scale to Z_j , in order to obtain the space $l^p Z^{\sigma,s}$ with norm

$$\|h\|_{l^p Z^{\sigma,s}}^2 = \sum_{j \in \mathbb{Z}} 2^{2\sigma j^- + 2s j^+} \|P_j h\|_{l_j^p Z_j}^2 = \sum_{j \in \mathbb{Z}} 2^{2\sigma j^- + 2s j^+} \|P_j h\|_{l_j^p L^\infty L^2}^2.$$

Here we need to decompose the low frequency part, this allows us to obtain an estimate of h in $l^2 Z^{\sigma_d,s+2}$ in Proposition 5.4. Correspondingly, we will strengthen the $Z^{\sigma_d,s+2}$ norm of h to $l^2 Z^{\sigma_d,s+2}$.

More importantly, we will also introduce some additional structure which is associated to spatial scales larger than the frequency. Precisely, to measure the portion of h which is localized to frequency 2^j , this time with $j \in \mathbb{Z}$, we decompose $P_j h$ as an atomic summation of components $h_{j,l}$ associated to spatial scales 2^l with $l \geq |j|$, i.e.

$$P_j h = \sum_{l \geq |j|} h_{j,l}.$$

Then we define the Y_j -norm by

$$\|P_j h\|_{Y_j} = \inf_{P_j h = \sum_{l \geq |j|} h_{j,l}} \sum_{l \geq |j|} 2^{l-|j|} \|h_{j,l}\|_{l^1 L^\infty L^2}.$$

In the decomposition of $P_j h$ we may project and assume that all terms are also localized at frequency 2^j . However in the definition of the Y_j norms we make no such assumption.

Assembling together the dyadic pieces in an l^2 Besov fashion, we obtain the Y^s space with norm given by

$$\|h\|_{Y^s}^2 = \sum_{j \in \mathbb{Z}} 2^{2(\frac{d}{2}-\delta)j^- + 2s j^+} \|P_j h\|_{Y_j}^2.$$

Then for h -equation in (2.28), we will be working primarily in \mathbf{Y}^{s+2} , whose norm is defined by

$$\|h\|_{\mathbf{Y}^{s+2}} = \|h\|_{l^2 Z^{\sigma_d,s+2}} + \|h\|_{Y^{s+2}}.$$

Collecting all the components defined above, for the parabolic system (2.29) we define the final \mathcal{E}^s norm as

$$\|(h, A)\|_{\mathcal{E}^s} = \|h\|_{\mathbf{Y}^{s+2}} + \|A\|_{Z^{s+1}}.$$

At fixed time, we can remove the L_t^∞ in \mathbf{Y}^{s+2} and \mathcal{E}^s , and obtain the function spaces \mathbf{Y}_0^{s+2} and \mathcal{E}_0^s respectively. Precisely, we define the Y_{0j} norm corresponding to Y_j as

$$\|P_j h\|_{Y_{0j}} = \inf_{P_j h = \sum_{l \geq |j|} h_{j,l}} \sum_{l \geq |j|} 2^{l-|j|} \|h_{j,l}\|_{l^1 L^2}.$$

and obtain the Y_0^s space with norm given by

$$\|h\|_{Y_0^s}^2 = \sum_{j \in \mathbb{Z}} 2^{2(\frac{d}{2} - \delta)j^- + 2sj^+} \|P_j h\|_{Y_{0j}}^2.$$

Then we obtain the space \mathbf{Y}_0^{s+2} with norm defined by

$$\|h\|_{\mathbf{Y}_0^{s+2}} = \| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}} + \|h\|_{Y_0^{s+2}},$$

and the space \mathcal{E}_0^s defined by

$$\|(h, A)\|_{\mathcal{E}_0^s} = \|h\|_{\mathbf{Y}_0^{s+2}} + \|A\|_{H^{s+1}}.$$

Finally, to capture only the low frequency information in the Y_0^s spaces, we introduce the $Y_0^{l,o}$ norm, which is used in our main two dimensional result in Theorem 1.3:

$$\|h\|_{Y_0^{l,o}}^2 = \|P_{\geq 0} h\|_{Y_{00} \cap L^\infty} + \sum_{j < 0} 2^{2(\frac{d}{2} - \delta)j^-} \|P_j h\|_{Y_{0j}}^2.$$

Next, we define the frequency envelopes as in [20–22] which will be used in multilinear estimates. Consider a Sobolev-type space U for which we have

$$\|u\|_U^2 = \sum_{k=0}^{\infty} \|S_k u\|_U^2.$$

A frequency envelope for a function $u \in U$ is a positive l^2 -sequence, $\{a_j\}$, with

$$\|S_j u\|_U \leq a_j.$$

We shall only permit slowly varying frequency envelopes. Thus, we require $a_0 \approx \|u\|_U$ and

$$a_j \leq 2^{\delta|j-k|} a_k, \quad j, k \geq 0, \quad 0 < \delta \ll s - d/2. \tag{3.2}$$

The constant δ shall be chosen later and only depends on s and the dimension d . Such frequency envelopes always exist. For example, one may choose

$$a_j = 2^{-\delta j} \|u\|_U + \max_k 2^{-\delta|j-k|} \|S_k u\|_U. \tag{3.3}$$

Since we often use Littlewood–Paley decompositions, the next lemma is a convenient tool to see that our function spaces are invariant under the action of some standard classes of multipliers:

Lemma 3.1. *For any Schwartz function $f \in \mathcal{S}$, multiplier $m(D)$ with $\|\mathcal{F}^{-1}(m(\xi))\|_{L^1} < \infty$, and translation-invariant Sobolev-type space U , we have*

$$\|m(D)f\|_U \lesssim \|\mathcal{F}^{-1}(m(\xi))\|_{L^1} \|f\|_U.$$

Finally, we state a Bernstein-type inequality and two estimates.

Lemma 3.2. (Bernstein-type inequality, Lemma 3.2 [9]) *For any $j, k \in \mathbb{Z}$ with $j + k \geq 0$, $1 \leq r < \infty$ and $1 \leq q \leq p \leq \infty$, we have*

$$\|P_k f\|_{L^r L^p} \lesssim 2^{kd(\frac{1}{q} - \frac{1}{p})} \|P_k f\|_{L^r L^q}.$$

Proposition 3.3. (Algebra property) *For any $f, g \in Y_0^{lo}$ we have*

$$\|fg\|_{Y_0^{lo}} \lesssim \|f\|_{Y_0^{lo}} \|g\|_{Y_0^{lo}}. \tag{3.4}$$

Proof. We first note that by Bernstein’s inequality we have $Y_0^{lo} \subset L^\infty$. Then for the high-low and low-high interactions we can estimate

$$\begin{aligned} & \|P_j(P_j f P_{<j} g)\|_{Y_{0j}} + \|P_j(P_{<j} f P_j g)\|_{Y_{0j}} \\ & \lesssim \|P_j f\|_{Y_{0j}} \|P_{<j} g\|_{L^\infty} + \|P_{<j} f\|_{L^\infty} \|P_j g\|_{Y_{0j}}. \end{aligned}$$

For the high-high interactions, we have

$$\begin{aligned} & 2^{(d-\delta)j^-} \|P_j \left(\sum_{0>l>j} P_l f P_l g + P_{\geq 0} f P_{\geq 0} g \right)\|_{Y_{0j}} \\ & \lesssim 2^{(d-\delta)j^-} \left(\sum_{0>l>j} \|P_j(P_l f P_l g)\|_{l_{|j|}^1 L^1} + \|P_{\geq 0} f P_{\geq 0} g\|_{l_{|j|}^1 L^1} \right) \\ & \lesssim \sum_{0>l>j} 2^{(d-\delta)(j^- - l)} 2^{(d-\delta)l} \|P_l f\|_{L^2} \|P_l g\|_{L^2} + 2^{(d-\delta)j^-} \|P_{\geq 0} f\|_{L^2} \|P_{\geq 0} g\|_{L^2} \\ & \lesssim \sum_{0>l>j} 2^{(d-\delta)(j^- - l)} 2^{(d-\delta)l} \|P_l f\|_{Y_{0j}} \|P_l g\|_{Y_{0j}} + 2^{(d-\delta)j^-} \|P_{\geq 0} f\|_{Y_{00}} \|P_{\geq 0} g\|_{Y_{00}}. \end{aligned}$$

These two bounds imply that

$$\|P_{<0}(fg)\|_{Y_0^{lo}} \lesssim \|f\|_{Y_0^{lo}} \|g\|_{Y_0^{lo}}.$$

For the high-frequency part $P_{\geq 0}(fg)$, we bound its L^∞ norm by

$$\|P_{\geq 0}(fg)\|_{L^\infty} \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty} \lesssim \|f\|_{Y_0^{lo}} \|g\|_{Y_0^{lo}}.$$

To bound its Y_{00} norm, we further decompose it as

$$P_{\geq 0}(fg) = P_{\geq 0}(P_{\geq 0} f \cdot g) + P_{\geq 0}(P_{<0} f \cdot P_{\geq 0} g) + P_{\geq 0}(P_{<0} f \cdot P_{[-3, -1]} g).$$

The first term is bounded by

$$\|P_{\geq 0}(P_{\geq 0} f \cdot g)\|_{Y_{00}} \lesssim \|P_{\geq 0} f\|_{Y_{00}} \|g\|_{L^\infty} \lesssim \|f\|_{Y_0^{lo}} \|g\|_{Y_0^{lo}}.$$

The second term is bounded similarly. We bound the last term by

$$\|P_{\geq 0}(P_{<0} f \cdot P_{[-3, -1]} g)\|_{Y_{00}} \lesssim \|P_{<0} f\|_{L^\infty} \|P_{[-3, -1]} g\|_{Y_{00}} \lesssim \|f\|_{Y_0^{lo}} \|g\|_{Y_0^{lo}}.$$

This completes the bound for high frequency part, and thus the proof of the proposition. □

Lemma 3.4. *For any Schwartz function f , $j \in \mathbb{N}$ and $1 \leq r \leq \infty$, we have*

$$\|e^{t\Delta} f\|_{l_j^r L_t^\infty L^2} \lesssim \|f\|_{l_j^r L^2}, \tag{3.5}$$

$$\left\| \int_0^t e^{(t-s)\Delta} S_j f \, ds \right\|_{l_j^r L^\infty L^2} \lesssim 2^{-2j^+} \|S_j f\|_{l_j^r L^\infty L^2}. \tag{3.6}$$

Proof. We use the heat kernel

$$K(t, x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{x^2}{4t}}$$

which we decompose with respect to cubes $Q \in \mathcal{Q}_j$. Then from the corresponding decomposition

$$e^{t\Delta} f = \sum_{Q \in \mathcal{Q}_j} (\chi_Q(x) K(t, x)) *_x f$$

we obtain

$$\|e^{t\Delta} f\|_{l_j^r L_t^\infty L^2} \lesssim \|K\|_{l_j^1 L_t^\infty L_x^1} \|f\|_{L^2}$$

Since $t \in [0, 1]$ and $r \geq 0$, we can use the exponential off-diagonal decay for K on the unit scale to conclude that

$$\|K\|_{l_j^1 L_t^\infty L_x^1} \lesssim 1,$$

and thus (3.5) follows.

For the second bound, we separate the low frequencies and use the kernel K_0 for $S_0 e^{(t-s)\Delta}$ with a similar cube decomposition to estimate

$$\left\| \int_0^t e^{(t-s)\Delta} S_0 f \, ds \right\|_{l_0^r L^\infty L^2} \lesssim \|K_0\|_{l_j^1 L_{t,x}^1} \|f\|_{l_0^r L^\infty L^2}$$

where the K_0 norm is easily estimates using the rapid kernel decay on the unit scale.

Similarly, for high frequencies $j > 0$ we use the kernel K_j for $S_j e^{(t-s)\Delta}$ with a similar cube decomposition to estimate

$$\left\| \int_0^t e^{(t-s)\Delta} S_j f \, ds \right\|_{l_0^r L^\infty L^2} \lesssim \|K_j\|_{l_j^1 L_{t,x}^1} \|f\|_{l_0^r L^\infty L^2}$$

For fixed t we use the exponential symbol decay to obtain

$$\|K_j\|_{l_j^1 L_x^1} \lesssim e^{-c2^{2j}t},$$

and now the time integration yields the desired 2^{-2j} decay. This concludes the proof of (3.6). □

4. The Initial Data

Our evolution begins at time $t = 0$, where we need to make a good gauge choice for the initial submanifold Σ_0 . This has two components,

- (i) a good set of coordinates on Σ_0 , namely the global harmonic coordinates, represented via the map $F : \mathbb{R}^d \rightarrow \mathbb{R}^{d+2}$.
- (ii) a good orthonormal frame in $N\Sigma_0$, where we will use the Coulomb gauge.

Once this is done, we have the frame in the tangent space and the frame m in the normal bundle. In turn, as described in Section 2, these generate the metric g , the second fundamental form λ with trace ψ and the connection A , all at the initial time $t = 0$.

We will first carry out the construction of the global harmonic coordinates, and use them to prove bounds for the parametrization F and for the metric $g_0 = I_d + h_0$. Then we introduce the Coulomb gauge, which in turn determines λ_0 and A_0 .

The final objective of this section will be to describe the regularity and size of (λ_0, g_0, A_0) , and thus justify the smallness condition (2.31) for the Schrödinger-Parabolic system(2.28)–(2.29). The main result of this section is stated below in Proposition 4.1 for dimensions $d \geq 3$ and Proposition 4.2 for dimension 2, respectively.

In order to state the following propositions, we define some notations. Let $F : \mathbb{R}_x^d \rightarrow (\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ be an immersion with induced metric $g(x)$. For any change of coordinate $y = x + \phi(x)$, we denote

$$\tilde{F}(y) = F(x(y)),$$

and its induced metric by $\tilde{g}_{\alpha\beta}(y) = \langle \partial_{y_\alpha} \tilde{F}, \partial_{y_\beta} \tilde{F} \rangle$. We also denote its Christoffel symbol as $\tilde{\Gamma}$ and $\tilde{h}(y) = \tilde{g}(y) - I_d$. The main results are summarized as follows:

Proposition 4.1. (Harmonic coordinates and initial data in dimensions $d \geq 3$) *Let $d \geq 3, s > \frac{d}{2}$. Let $F : (\mathbb{R}_x^d, g) \rightarrow (\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ be an immersion with induced metric $g = I_d + h$. Assume that the metric h and the mean curvature \mathbf{H} satisfy the smallness conditions*

$$\| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} \leq \epsilon_0, \quad \| \mathbf{H} \|_{H^s} \leq \epsilon_0. \tag{4.1}$$

Then there exists a unique change of coordinates $y = x + \phi(x)$ with $\lim_{x \rightarrow \infty} \phi(x) = 0$ and $\nabla \phi$ uniformly small, such that the new coordinates $\{y_1, \dots, y_d\}$ are global harmonic coordinates. Moreover, we have the bound

$$\| |D|^{\sigma_d} \nabla \phi \|_{H^{s+1-\sigma_d}} \lesssim \| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} \tag{4.2}$$

and, in the new coordinates $\{y_1, \dots, y_d\}$, for the metric and mean curvature we have

$$\| |D_y|^{\sigma_d} \tilde{h} \|_{H^{s+1-\sigma_d}(dy)} + \| \mathbf{H} \|_{H^s(dy)} \lesssim \epsilon_0. \tag{4.3}$$

In addition, under the harmonic coordinate condition (2.23) for g , respectively the Coulomb gauge (2.18) for A , we have the following bounds for complex second fundamental form λ , metric $h = g - I_d$ and A :

$$\| \lambda \|_{H^s} + \| h \|_{\mathbf{Y}_0^{s+2}} + \| A \|_{H^{s+1}} \lesssim \epsilon_0. \tag{4.4}$$

Compared to the above higher dimensions cases, in dimensions 2 we would work in a smaller function space.

Proposition 4.2. (Harmonic coordinates and initial data in dimension 2) *Let $d = 2$, $s > \frac{d}{2}$, and σ_d be as in (3.1). Let $F : (\mathbb{R}_x^d, g) \rightarrow (\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ be an immersion with induced metric $g = I_d + h$. Assume that the metric h and mean curvature \mathbf{H} satisfy the smallness conditions*

$$\| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} \leq \epsilon_0, \quad \| h \|_{Y_0^{lo}} \leq \epsilon_0, \quad \| \mathbf{H} \|_{H^s} \leq \epsilon_0. \tag{4.5}$$

Then there exists a change of coordinates $y = x + \phi(x)$, with $\nabla \phi$ uniformly small and with $\lim_{x \rightarrow \infty} \nabla \phi(x) = 0$, unique modulo constants, such that the new coordinates $\{y_1, \dots, y_d\}$ are global harmonic coordinates. Moreover, we have the bound

$$\| |D|^{\sigma_d} \nabla \phi \|_{H^{s+1-\sigma_d}} \lesssim \| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}}, \tag{4.6}$$

and, in the new coordinates $\{y_1, \dots, y_d\}$, for the metric and mean curvature we have

$$\| |D_y|^{\sigma_d} \tilde{h} \|_{H^{s+1-\sigma_d}(dy)} + \| \tilde{h} \|_{Y_0^{lo}} + \| \mathbf{H} \|_{H^s(dy)} \lesssim \epsilon_0. \tag{4.7}$$

In addition, under the harmonic coordinate condition (2.23) for g , respectively the Coulomb gauge (2.18) for A , we have the following bounds for complex second fundamental form λ , metric $h = g - I_d$ and A :

$$\| \lambda \|_{H^s} + \| h \|_{Y_0^{s+2}} + \| A \|_{H^{s+1}} \lesssim \epsilon_0. \tag{4.8}$$

We remark that the bounds (4.4) respectively (4.8) are the only way the harmonic/Coulomb gauge condition at $t = 0$ enters this paper. Later, in the study of the parabolic system (2.29), we simply assume that the initial data (λ_0, h_0, A_0) satisfies the above smallness condition.

Of the three components of the initial data, λ_0 may be thought of as the fundamental one. Indeed, the initial data (g_0, A_0) for the heat flow (2.29) is determined by λ_0 via the harmonic coordinate condition (2.23) for g , respectively the Coulomb gauge (2.18) for A , which yield the elliptic equations in Lemmas 2.3 and 2.1. This was the point of view adopted in our previous paper [9] in high dimension, and it largely applies here as well. The only exception to this is in two space dimensions, where we a-priori make an additional low frequency assumption on the metric g , namely the Y_0^{lo} bound, which cannot be recovered from the λ_0 bounds.

4.1. Global Harmonic Coordinates

Here we make a change of coordinates to gain the harmonic coordinates, and then prove that in the new coordinates, the metric h and mean curvature \mathbf{H} are also small.

Step 1: Solve the ϕ equation and prove the bounds (4.2) and (4.6). To obtain harmonic coordinates, we start with the bound for metric

$$\| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} \lesssim \epsilon_0. \tag{4.9}$$

We make a change of coordinates $x + \phi(x) = y$ with $\nabla\phi$ small such that the new coordinates are harmonic. Since the operator Δ_g does not depend on the coordinates, by (2.23) we have

$$\Delta_g(x + \phi(x)) = 0,$$

which implies

$$\Delta_g\phi_\gamma = g^{\alpha\beta}\Gamma_{\alpha\beta}^\gamma, \tag{4.10}$$

and which we write schematically in the form

$$\Delta\phi = h\nabla^2\phi + g\nabla h\nabla\phi + g\nabla h.$$

Since the leading order term in the right hand side is ∇h , by the assumption on the metric $|D|^{\sigma_d}h \in H^{s+1-\sigma_d}$ we will work in the space

$$\{\phi : \| |D|^{1+\sigma_d}\phi \|_{H^{s+1-\sigma_d}} < \infty\}$$

Then by Sobolev embeddings and the smallness of h we can uniquely solve the equation (4.10) in this space using the contraction principle, obtaining a solution ϕ which satisfies the bound

$$\| |D|^{\sigma_d}\nabla\phi \|_{H^{s+1-\sigma_d}} \lesssim \| |D|^{\sigma_d}h \|_{H^{s+1-\sigma_d}} \lesssim \epsilon_0. \tag{4.11}$$

which is exactly (4.2) and (4.6) in Theorem 4.1, respectively Theorem 4.2.

Step 2: Prove the bounds (4.3) and (4.7) for \tilde{h} and \mathbf{H} in Sobolev spaces. First we prove that the desired \tilde{h} bound holds in the x -coordinates,

$$\| |D_x|^{\sigma_d}\tilde{h}(y(x)) \|_{H^{s+1-\sigma_d}(dx)} \lesssim \| |D|^{\sigma_d}h \|_{H^{s+1-\sigma_d}(dx)}. \tag{4.12}$$

By the above change of coordinate and (4.2) we have $\frac{\partial x}{\partial y} = I_d + \mathcal{P}(x)$ where \mathcal{P} is an algebraic function of $\nabla\phi$. Hence by algebra and Moser estimates we have

$$\| |D|^{\sigma_d}\mathcal{P} \|_{H^{s+1-\sigma_d}} \lesssim \| |D|^{\sigma_d}\nabla\phi \|_{H^{s+1-\sigma_d}} \lesssim \epsilon_0. \tag{4.13}$$

Then the desired bound (4.12) follows from the relation

$$\tilde{g}_{\alpha\beta}(y(x)) = g_{\mu\nu}(x)(\delta_\alpha^\mu + \mathcal{P}_\alpha^\mu)(\delta_\beta^\nu + \mathcal{P}_\beta^\nu),$$

again by using algebra bounds in the same space.

In order to complete the proof of (4.3) and (4.7), we need to be able to transfer the Sobolev norms from the x to the y coordinates. For this we will apply the following lemma:

Lemma 4.3. *Let the change of coordinates $x + \phi(x) = y$ be as in Proposition 4.1. Define the linear operator T as $T(f)(y) = f(x(y))$ for any function $f \in L^2(dx)$. Then we have*

$$\| T(f)(y) \|_{H^\sigma(dy)} \lesssim \| f(x) \|_{H^\sigma(dx)}, \quad \sigma \in [0, s + 1], \tag{4.14}$$

$$\| T(f)(y) \|_{\dot{H}^\alpha(dy)} \lesssim \| f(x) \|_{\dot{H}^\alpha(dx)}, \quad \alpha \in [0, \frac{d}{2}]. \tag{4.15}$$

Proof. The first bound is obtained from (4.13) and (4.2) using the same argument as in Lemma 8.5 in [9], It remains to prove the second bound (4.15).

By the smallness of ϕ (4.2) we have

$$\begin{aligned} \|T(f)(y)\|_{L^2(\text{d}y)} &\lesssim \|f(x)\sqrt{I + \partial_x\phi}\|_{L^2(\text{d}x)} \\ &\lesssim (1 + \| |D|^{1+\sigma_d} \|_{H^{s-\sigma_d}})^N \|f(x)\|_{L^2(\text{d}x)} \lesssim \|f(x)\|_{L^2(\text{d}x)}. \end{aligned}$$

Similarly, by (4.2) and (4.13) we also have

$$\|\partial_y T(f)(y)\|_{L^2(\text{d}y)} \lesssim \|(1 + \mathcal{P})\partial_x f(x)\sqrt{I + \partial_x\phi}\|_{L^2(\text{d}x)} \lesssim \|\partial_x f(x)\|_{L^2(\text{d}x)}.$$

Then by interpolation we obtain (4.15) for $\alpha \in [0, 1]$. This suffices in dimension $d = 2$. In higher dimension, we inductively increase the range of α by differentiating. Precisely, for $\alpha > 1$ we have

$$\|T(f)(y)\|_{\dot{H}^\alpha(\text{d}y)} = \|\partial_y T(f)(y)\|_{\dot{H}^{\alpha-1}(\text{d}y)}$$

Here

$$\partial_y T(f)(y) = T((I + \mathcal{P})\partial_x f)$$

and, by (4.13),

$$\|(I + \mathcal{P})\partial_x f\|_{\dot{H}^{\alpha-1}(\text{d}x)} \lesssim \|\partial_x f\|_{\dot{H}^{\alpha-1}(\text{d}x)}$$

Hence we have reduced the \dot{H}^α bound to the $\dot{H}^{\alpha-1}$ bound. □

Given this lemma, by (4.12), (4.14) with $\sigma = s + 1$ and (4.15) we obtain

$$\| |D|^{\sigma_d} \tilde{h} \|_{H^{s+1-\sigma_d}(\text{d}y)} \lesssim \| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}(\text{d}x)}$$

Hence the \tilde{h} bounds in (4.3) and (4.7) follow. Similarly, the \mathbf{H} bound is also directly transferred to the y coordinates by Lemma 4.3.

Step 3: Prove bounds for $\partial^2 F$ in harmonic coordinates. While this bound was not explicitly stated in Propositions 4.1, 4.2, it will play an important role later in the proof of the bounds (4.4) and (4.8).

Lemma 4.4. *Let $d \geq 2$, $s > \frac{d}{2}$, and $F : (\mathbb{R}^d, g) \rightarrow (\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ be an immersion with metric $\| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} \lesssim \epsilon_0$ and mean curvature $\| \mathbf{H} \|_{H^s} \lesssim \epsilon_0$ in some coordinates. Then we have*

$$\| \partial^2 F \|_{H^s} \lesssim \epsilon_0. \tag{4.16}$$

We note that, as a corollary, it follows that we also have the bound

$$\| \nabla \tilde{h} \|_{H^s} \lesssim \epsilon_0. \tag{4.17}$$

This bound in effect supersedes the \tilde{h} bound in (4.3), (4.7), with one exception, namely in two dimensions at low frequency.

Another corollary of this is the corresponding bound for the second fundamental form \mathbf{h} , namely

$$\| \mathbf{h} \|_{H^s} \lesssim \epsilon_0. \tag{4.18}$$

Proof of Proposition 4.4. By the smallness of $|D|^{\sigma_d}(g - I_d)$ and Sobolev embedding, we have

$$\begin{aligned} \|g^{\alpha\beta}\Gamma_{\alpha\beta}^\gamma\partial_\gamma F\|_{H^s} &\lesssim (1 + \||D|^{\sigma_d}h\|_{H^{s+1-\sigma_d}})\||D|^{\sigma_d}h\|_{H^{s+1-\sigma_d}}(\|\partial_\gamma F\|_{L^\infty\dot{H}^s}) \\ &\lesssim \epsilon_0(\|g\|_{L^\infty}^{1/2} + \|\partial^2 F\|_{H^s}) \lesssim \epsilon_0(1 + \|\partial^2 F\|_{H^s}). \end{aligned}$$

Then we can bound $\partial^2 F$ by

$$\begin{aligned} \|\partial^2 F\|_{H^s} &= \|\mathcal{R}\Delta F\|_{H^s} \lesssim \|\Delta F\|_{H^s} \\ &\lesssim \|\Delta_g F\|_{H^s} + \|h^{\alpha\beta}\partial_{\alpha\beta}^2 F\|_{H^s} + \|g^{\alpha\beta}\Gamma_{\alpha\beta}^\gamma\partial_\gamma F\|_{H^s} \\ &\lesssim \|\mathbf{H}\|_{H^s} + \epsilon_0(1 + \|\partial^2 F\|_{H^s}) \\ &\lesssim \epsilon_0(1 + \|\partial^2 F\|_{H^s}), \end{aligned}$$

which implies (4.16), and thus completes the proof of lemma. □

Step 4: Prove the $Y_0^{l_0}$ bound for the metric h in (4.7) in two dimensions. To transfer the $Y_0^{l_0}$ bounds to \tilde{h} , our starting point is the estimate

$$\|h\|_{Y_0^{l_0}} \lesssim \epsilon_0$$

Next we show that $\nabla\phi$ satisfies a similar bound,

$$\|\nabla\phi\|_{\tilde{Y}_0^{l_0}} \lesssim \epsilon_0 \tag{4.19}$$

Proof of (4.19). We use the ϕ -equations (4.10), which have the form

$$\Delta\phi = \nabla h + h\nabla^2\phi + g\nabla h\nabla\phi + h\nabla h.$$

To get (4.19) via the contraction principle it suffices to estimate the right hand side above in order to prove that

$$\|\nabla\phi\|_{Y_0^{l_0}} \lesssim \|h\|_{Y_0^{l_0}} + \epsilon_0(\|h\|_{Y_0^{l_0}} + \|\nabla\phi\|_{Y_0^{l_0}}) + \epsilon_0^2$$

First, we bound the Y_j norm of $\nabla\phi$. For the ∇h , we easily have

$$\|P_j\nabla^{-1}\nabla h\|_{Y_j} \lesssim \|P_j h\|_{Y_j},$$

which is acceptable. We will next show how to bound the most unbalanced term $h\nabla^2\phi$; the rest of the terms are estimated similarly. For the high-low interactions $P_j h\nabla^2 P_{<j}\phi$, by (4.11) we have

$$\|P_j\nabla^{-1}(P_j h\nabla^2 P_{<j}\phi)\|_{Y_{0j}} \lesssim \|P_j h\|_{Y_{0j}}\|\nabla P_{<j}\phi\|_{L^\infty} \lesssim \epsilon_0\|P_j h\|_{Y_{0j}}.$$

Similarly, for the low-high interactions $P_{<j}h\nabla^2 P_j\phi$, by (4.9) we have

$$\|P_j\nabla^{-1}(P_{<j}h\nabla^2 P_j\phi)\|_{Y_{0j}} \lesssim \|P_{<j}h\|_{L^\infty}\|\nabla P_j\phi\|_{Y_{0j}} \lesssim \epsilon_0\|\nabla P_j\phi\|_{Y_{0j}}.$$

Finally we consider the high-high interactions, $\sum_{l>j} P_j(P_l h \nabla^2 P_l \phi)$. Here we use Bernstein's inequality to obtain

$$\begin{aligned} 2^{(\frac{d}{2}-\delta)j} \left\| \sum_{l>j} \nabla^{-1} P_j(P_l h \nabla^2 P_l \phi) \right\|_{Y_{0j}} &\lesssim 2^{(\frac{d}{2}-1-\delta)j} \sum_{l>j} \|P_j(P_l h \nabla^2 P_l \phi)\|_{l_j^1 L^2} \\ &\lesssim 2^{(d-1-\delta)j} \sum_{l>j} \|P_l h\|_{L^2} \|\nabla^2 P_l \phi\|_{L^2} \\ &\lesssim \sum_{l>j} 2^{(d-1-\delta)(j-l)} \| |D|^{\frac{d}{2}-\delta} P_l h \|_{L^2} \| |D|^{\frac{d}{2}} P_l \nabla \phi \|_{L^2}, \end{aligned}$$

which in view of the bound (4.11) gives

$$\left(\sum_{j<0} 2^{2(\frac{d}{2}-\delta)j} \left\| \sum_{l>j} \nabla^{-1} P_j(P_l h \nabla^2 P_l \phi) \right\|_{Y_{0j}}^2 \right)^{1/2} \lesssim \| |D|^{\sigma_d} h \|_{L^2} \| |D|^{\frac{d}{2}} \nabla \phi \|_{L^2} \lesssim \epsilon_0^2.$$

Secondly, we bound the $Y_{00} \cap L^\infty$ norm for the high frequency part $P_{\geq 0} \nabla \phi$. The L^∞ bound follows from the H^s bound for ϕ and Sobolev embeddings. It remains to estimate its Y_{00} norm. Since the operator $P_{>0} \nabla^{-1} \nabla$ has the kernel localized to the unit spatial scale, we have

$$\|P_{\geq 0} \nabla^{-1} \nabla h\|_{Y_{00}} \lesssim \|P_{\geq 0} h\|_{Y_{00}}.$$

Here we also only discuss the term $h \nabla^2 \phi$; the contributions of the other terms are estimated similarly. We first divide this term as

$$P_{\geq 0} \nabla^{-1} (P_{\geq 0} h \nabla^2 \phi) + P_{\geq 0} \nabla^{-1} (P_{<0} h \nabla^2 P_{\geq 0} \phi) + P_{\geq 0} \nabla^{-1} (P_{<0} h \nabla^2 P_{[-3,-1]} \phi).$$

For the first term, we directly have

$$\|P_{\geq 0} \nabla^{-1} (P_{\geq 0} h \nabla^2 \phi)\|_{Y_{00}} \lesssim \|P_{\geq 0} h\|_{Y_{00}} \|\nabla^2 \phi\|_{L^\infty} \lesssim \epsilon_0 \|P_{\geq 0} h\|_{Y_{00}}.$$

The second term, we further divide it as

$$P_{\geq 0} \nabla^{-1} (P_{<0} h \nabla^2 P_{\geq 0} \phi) = P_{\geq 0} \mathcal{R}(P_{<0} h \nabla P_{\geq 0} \phi) + P_{\geq 0} \nabla^{-1} (\nabla P_{<0} h \nabla P_{\geq 0} \phi),$$

where \mathcal{R} is Riesz transform. Then we bound this by

$$\|P_{\geq 0} \nabla^{-1} (P_{<0} h \nabla^2 P_{\geq 0} \phi)\|_{Y_{00}} \lesssim \|P_{<0} h\|_{L^\infty} \|P_{\geq 0} \nabla \phi\|_{Y_{00}} \lesssim \epsilon_0 \|P_{\geq 0} \nabla \phi\|_{Y_{00}}.$$

Finally, we bound the last term by

$$\|P_{\geq 0} \nabla^{-1} (P_{<0} h \nabla^2 P_{[-3,-1]} \phi)\|_{Y_{00}} \lesssim \|P_{<0} h\|_{L^\infty} \|P_{[-3,-1]} \nabla \phi\|_{Y_{00}} \lesssim \epsilon_0 \|P_{[-3,-1]} \nabla \phi\|_{Y_0^{lo}}.$$

This concludes the proof of the $Y_{00} \cap L^\infty$ norm for $P_{\geq 0} \nabla \phi$. □

The new metric \tilde{h} expressed in the x coordinates has the cubic polynomial form

$$\tilde{h} = P(h, \nabla \phi).$$

Using the algebra property (3.4) for Y_0^{lo} and (4.19), we conclude that

$$\|\tilde{h}\|_{\tilde{Y}_0^x} \lesssim \epsilon_0$$

It remains to switch this bound to the y coordinates, i.e. show that

$$\|\tilde{h}\|_{\tilde{Y}_0^x} \approx \|\tilde{h}\|_{\tilde{Y}_0^y} \tag{4.20}$$

where the difficulty is that we need to use a Littlewood–Paley decomposition. We will circumvent this by using the following representation of Y_0^{lo} functions:

Lemma 4.5. *A function f is in Y_0^{lo} iff it admits a representation*

$$f = \sum_{j \leq 0} f_j, \quad f_j \in Y_{0j}$$

so that the following norm is finite:

$$\| (f_j) \|_{\mathbb{I}}^2 = \|f_0\|_{Y_{00} \cap L^\infty}^2 + \sum_{j < 0} 2^{2(1-\delta)j} \left(\|f_j\|_{Y_{0j}}^2 + 2^{-2j} \|\nabla f_j\|_{Y_{0j}}^2 + 2^{-4j} \|\nabla^2 f_j\|_{Y_{0j}}^2 \right)$$

Further, we have

$$\|f\|_{Y_0^{lo}} \approx \inf \{ \| (f_j) \|_{\mathbb{I}}; f = \sum_{j \leq 0} f_j \}$$

Since by Sobolev embeddings ϕ is small in C^2 , the triple norms are easily seen to be equivalent in the x and the y coordinates, therefore the relation (4.20) follows. It remains to prove the Lemma.

Proof. In one direction, we directly see that the decomposition

$$f_0 = P_{\geq 0} f, \quad f_j = P_j f, \quad j < 0$$

yields

$$\|f\|_{\tilde{Y}_0} \approx \| (f_j) \|_{\mathbb{I}}$$

Conversely, if $f = \sum f_j$, then we need to show that

$$\|f\|_{\tilde{Y}_0} \lesssim \| (f_j) \|_{\mathbb{I}} \tag{4.21}$$

For this we estimate for $k < 0$

$$\begin{aligned} \|P_k f\|_{Y_{0k}} &\lesssim \|P_k f_0\|_{Y_{0k}} + \sum_{j < 0} \|P_k f_j\|_{Y_{0k}} \\ &\lesssim 2^k \|f_0\|_{Y_{00}} + \sum_{j < 0} 2^{-|j-k|} (\|f_j\|_{Y_j} + 2^{-2j} \|\nabla^2 f_j\|_{Y_j}) \end{aligned}$$

Due to the off-diagonal decay, this implies (4.21). □

4.2. The Initial Data (λ_0, h_0, A_0)

These are determined by the initial manifold Σ_0 given a gauge choice, which consists of choosing (i) a good set of coordinates on Σ_0 , namely the harmonic coordinates, and (ii) a good orthonormal frame in $N\Sigma_0$, where we will use the Coulomb gauge.

In the previous subsection we have discussed the construction of harmonic coordinates and proved the Sobolev bound (4.17) for h_0 . Here we begin by constructing a Coulomb frame in the normal bundle. Then we can define λ_0 and A_0 and directly prove H^s bounds for them.

However, it turns out that the H^s bounds tell only part of the story for h_0 and A_0 , by treating them as linear objects. Instead, in our chosen gauge both h_0 and A_0 should be seen as quadratic objects, via the equations (2.23), respectively (2.18). In the last part of the section we use these equations to improve the bounds for both h_0 and A_0 .

Step 1: The Coulomb frame in $N\Sigma_0$ and the H^s bound for λ and A . To obtain the Coulomb gauge, we choose \tilde{v} constant uniformly transversal to $T\Sigma_0$; such a \tilde{v} exists because, by Sobolev embeddings, $\partial_x F$ has a small variation in L^∞ . Projecting \tilde{v} on the normal bundle $N\Sigma_0$ and normalizing we obtain a normalized section \tilde{v}_1 of the normal bundle with the same regularity as ∂F . Then we choose \tilde{v}_2 in $N\Sigma_0$ perpendicular to \tilde{v}_1 . We obtain the orthonormal frame $(\tilde{v}_1, \tilde{v}_2)$ in $N\Sigma_0$, which again has the same regularity and bounds as $\partial_x F$, namely (see Lemma 4.4)

$$\|\partial \tilde{v}_j\|_{H^s} \lesssim \epsilon_0. \tag{4.22}$$

This in particular implies that the associated connection \tilde{A} also satisfies

$$\|\tilde{A}\|_{H^s} \lesssim \epsilon_0. \tag{4.23}$$

Then we rotate the frame to get a Coulomb frame (ν_1, ν_2) , i.e. where the Coulomb gauge condition $\nabla^\alpha A_\alpha = 0$ is satisfied. In our complex notation, this corresponds to

$$\nu_1 + i\nu_2 = e^{ib}(\tilde{\nu}_1 + i\tilde{\nu}_2), \quad A_j = \tilde{A}_j - \partial_j b,$$

where the rotation angle b must solve

$$\Delta_g b = \nabla^\alpha \tilde{A}_\alpha.$$

This is an elliptic equation, where the metric $g_0 = I_d + h_0$ satisfies (4.17). Using the variational formulation at the H^1 level and then perturbative analysis at higher regularity, the solution is easily seen to satisfy

$$\|\partial b\|_{H^s} \lesssim \|A\|_{H^s}$$

It directly follows that ν_1, ν_2 and A also satisfy the bounds in (4.22), (4.23),

$$\|\partial \nu_j\|_{H^s} + \|A\|_{H^s} \lesssim \epsilon_0. \tag{4.24}$$

Projecting the second fundamental form \mathbf{h} and the mean curvature \mathbf{H} on the Coulomb frame as in Section 2.2 we obtain the complex second fundamental form λ and the complex mean curvature ψ . In view of (4.1), (4.5) and (4.18) both of them have the same regularity,

$$\|\lambda\|_{H^s} + \|\psi\|_{H^s} \lesssim \epsilon_0.$$

Step 2: Prove the bounds in (4.4) and (4.8) for the metric h . For this we rely on the equation (2.24). The main result is as follows:

Lemma 4.6. *Let $d \geq 2$, $s > \frac{d}{2}$ and σ_d be as in (3.1). Assume that h is a solution of (2.24) satisfying*

$$\| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} \leq \epsilon_0, \quad \|\lambda\|_{H^s} \leq \epsilon_0.$$

Then for $d \geq 3$ we have

$$\|\partial h\|_{H^{s+1}} + \|h\|_{Y_0^{s+2}} \lesssim \epsilon_0. \quad (4.25)$$

Under the additional assumption

$$\|h\|_{Y_0^{l_0}} \leq \epsilon_0, \quad (4.26)$$

in dimension $d = 2$ we have

$$\| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}} + \|P_{\geq 0} h\|_{Y_0^{s+2}} \lesssim \epsilon_0. \quad (4.27)$$

Here we remark on the key difference between dimensions two and higher. In higher dimensions $d \geq 3$, h may be seen as the unique small solution for the equation (2.24). But in two dimensions, we merely use (2.24) to improve the high frequency bound for h . At low frequency this no longer works, and instead we use the low frequency bounds on the initial metric h_0 as an assumption in our main result. We note that the assumption (4.26) in the two dimensional case could be avoided, at the expense of a considerably longer proof.

Proof. By (2.24), it suffices to write the equation for h in the shorter form

$$\Delta h = h \nabla^2 h + \nabla h \nabla h + \lambda^2.$$

From this and Sobolev embedding, we easily have

$$\begin{aligned} \|P_{\geq 0} \Delta h\|_{H^s} &\lesssim \| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} \| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}} + \|\partial_x h\|_{H^s}^2 + \|\lambda\|_{H^s}^2 \\ &\lesssim \epsilon_0 \|P_{\geq 0} |D|^{\sigma_d} h\|_{H^{s+2-\sigma_d}} + \epsilon_0^2. \end{aligned}$$

This implies that

$$\| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}} \lesssim \epsilon_0. \quad (4.28)$$

In dimension three and higher, a similar argument applies in order to improve the low frequency bound. This argument is already in [9], and we do not repeat it here.

Next, we bound the Y_0^{s+2} norm of h . For the low-frequency part, we only need to consider the higher dimensional case $d \geq 3$, as in the case $d = 2$ the low frequency bound is assumed in Theorem 1.3. We bound the high-low or low-high interactions by

$$\begin{aligned} \|\Delta^{-1} P_j(P_{\leq j-3} h \nabla^2 h)\|_{Y_j} &\lesssim 2^{-2j} \sum_{l \geq |j|} 2^{l-|j|} \|P_j(P_{\leq j-3} h \nabla^2 h_{j,l})\|_{l^1 L^2} \\ &\lesssim \|h\|_{L^\infty} \|P_j h\|_{Y_j} \\ &\lesssim \| |D|^{\sigma_d} h \|_{H^{s-\sigma_d}} \|P_j h\|_{Y_j}, \end{aligned}$$

For the high-high interactions $P_j(\nabla P_l h \nabla P_l h)$, we have

$$\begin{aligned} &\| \sum_{l \geq j} \Delta^{-1} P_j(\nabla P_l h \nabla P_l h) \|_{Y_j} \\ &\lesssim 2^{-2j} \left(\sum_{|j| \geq l \geq j} \|P_j(\nabla P_l h \nabla P_l h)\|_{l^1 L^2} + \sum_{l > |j|} 2^{l-|j|} \|P_j(\nabla P_l h \nabla P_l h)\|_{l^1 L^2} \right) \\ &\lesssim \sum_{|j| \geq l \geq j} 2^{(d/2-2)j} \|\nabla P_l h\|_{L^2} \|\nabla P_l h\|_{L^2} + \sum_{l > |j|} 2^{(d/2-2)j} 2^{l-|j|} \|\nabla P_l h\|_{L^2} \|\nabla P_l h\|_{L^2}, \end{aligned}$$

From these two bounds, for $d \geq 3$ we obtain

$$\|\Delta^{-1}(h \nabla^2 h + \nabla h \nabla h)\|_{Y_0^{s+2}} \lesssim \| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}} \|h\|_{Y_0^{s+2}} + \| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}}^2, \tag{4.29}$$

and for $d = 2$ we obtain

$$\|P_{\geq 0} \Delta^{-1}(h \nabla^2 h + \nabla h \nabla h)\|_{Y_0^{s+2}} \lesssim \| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}} \|h\|_{Y_0^{s+2}} + \| |D|^{\sigma_d} h \|_{H^{s+2-\sigma_d}}^2. \tag{4.30}$$

We then bound the contribution of the λ^2 source term. For the high-low or low-high interactions we have

$$\begin{aligned} \|\Delta^{-1} P_j(P_{< j} \lambda P_j \lambda)\|_{Y_j} &\lesssim 2^{-2j} \|P_j(P_{< j} \lambda P_j \lambda)\|_{l^1 L^2} \\ &\lesssim 2^{dj-/2-2j} \|P_{< j} \lambda\|_{H^s} \|P_j \lambda\|_{L^2}, \end{aligned}$$

For the high-high interactions we have

$$\| \sum_{l \geq j} \Delta^{-1} P_j(P_l \lambda P_l \lambda) \|_{Y_j} \lesssim \sum_{|j| \geq l \geq j} 2^{(d/2-2)j} \|P_l \lambda\|_{L^2}^2 + \sum_{l > |j|} 2^{(d/2-2)j} 2^{l-|j|} \|P_l \lambda\|_{L^2}^2.$$

These two bounds also imply for $d \geq 3$

$$\|\Delta^{-1}(\lambda^2)\|_{Y_0^{s+2}} \lesssim \|\lambda\|_{H^s}^2, \tag{4.31}$$

and for $d = 2$

$$\|P_{\geq 0} \Delta^{-1}(\lambda^2)\|_{Y_0^{s+2}} \lesssim \|\lambda\|_{H^s}^2. \tag{4.32}$$

Using (4.28), $\|h\|_{Y_0^{l_0}} \leq \epsilon_0$ and $\|\lambda\|_{H^s} \leq \epsilon_0$, by h -equation, (4.29) and (4.31) we obtain for $d \geq 3$

$$\|h\|_{Y_0^{s+2}} \lesssim \epsilon_0 \|h\|_{Y_0^{s+2}} + \epsilon_0^2.$$

and by (4.30) and (4.32) we obtain for $d = 2$

$$\|P_{\geq 0} h\|_{Y_0^{s+2}} \lesssim \epsilon_0 \|h\|_{Y_0^{s+2}} + \epsilon_0^2 \lesssim \epsilon_0 \|P_{\geq 0} h\|_{Y_0^{s+2}} + \epsilon_0^2.$$

This concludes the proof of the bounds (4.25) and (4.27). \square

Step 3: Prove the bound (4.8) for A . This is obtained by (4.24) and the following proposition. Here we solve the initial data A_0 from the elliptic div-curl system (2.19).

Proposition 4.7. (Initial data A_0) *Let $d \geq 2$, $s > d/2$ and $\delta_d = \delta$ if $d = 2$ and $\delta_d = 0$ if $d \geq 3$. Assume that*

$$\|\lambda\|_{H^s} \leq \epsilon_0, \quad \| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} \leq \epsilon_0 \quad (4.33)$$

Then the elliptic system (2.19) for A admits a unique small solution with

$$\| |D|^{\delta_d} A \|_{H^{s+1-\delta_d}} \lesssim \|\lambda\|_{H^s}. \quad (4.34)$$

Moreover, assume that p_{0k} is an admissible frequency envelope for $\lambda \in H^s$. Then we have the frequency envelope bounds

$$\| S_k |D|^{\delta_d} A \|_{H^{s+1-\delta_d}} \lesssim \epsilon_0 p_{0k}. \quad (4.35)$$

In addition, for the linearization of the solution map above we also have the bound:

$$\| |D|^{\delta_d} \delta A \|_{H^{\sigma+1-\delta_d}} \lesssim \epsilon_0 (\| |D|^{\sigma_d} \delta h \|_{H^{\sigma+1-\sigma_d}} + \|\delta \lambda\|_{H^\sigma}), \quad \sigma \in (d/2 - 2, s]. \quad (4.36)$$

Proof. Using the definition of covariant derivatives and the harmonic coordinate condition (2.23) we can rewrite the div-curl system (2.19) for A as

$$\partial^\alpha A_\alpha = 0, \quad \partial_\alpha A_\beta - \partial_\beta A_\alpha = \text{Im}(\lambda_{\alpha\gamma} \bar{\lambda}_\beta^\gamma).$$

Using these equations we derive a second order elliptic equation for A , namely

$$\begin{aligned} \partial_\gamma \partial^\gamma A_\alpha &= \partial_\gamma g^{\gamma\beta} \partial_\alpha A_\beta - \partial_\gamma g^{\gamma\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &= (\partial_\gamma g^{\gamma\beta} \partial_\alpha - \partial_\alpha g^{\gamma\beta} \partial_\gamma) A_\beta - \partial_\gamma \text{Im}(\lambda_{\alpha\sigma} \bar{\lambda}^{\gamma\sigma}) \end{aligned}$$

Here we have a leading order cancellation in the first term on the right, but we prefer to keep the divergence structure and rewrite this equation schematically in the form

$$\Delta A = \partial(\lambda^2) + \partial(h\partial A).$$

This will be well suited in order to solve this equation via the contraction principle.

Precisely, we define the map $A \rightarrow \mathcal{T}(A)$ with $\mathcal{T}(A)$ satisfying

$$\Delta \mathcal{T}(A) = \partial(\lambda^2) + \partial(h\partial A).$$

so that the solution A may be seen as a fixed point for \mathcal{T} . To use the contraction principle, it suffices to show that, under the assumption (4.33), this map is Lipschitz in the ball $\{A : \| |D|^{\delta_d} A \|_{H^{s+1-\delta_d}} \leq C \|\lambda\|_{H^s}\}$ with a small Lipschitz constant. This would yield the existence and uniqueness of solutions for A -equations and the bound (4.34).

To establish the contraction property, we consider the linearization of \mathcal{T} ,

$$\Delta \delta \mathcal{T}(A) = \nabla(\lambda \delta \lambda) + \nabla(\delta h \nabla A + h \nabla \delta A),$$

under the assumptions

$$\| |D|^{\delta_d} A \|_{H^{s+1-\delta_d}} + \| |D|^{\sigma_d} h \|_{H^{s+1-\sigma_d}} + \|\lambda\|_{H^s} \lesssim \epsilon_0.$$

Here we denote by \tilde{a}_{0k} , \tilde{c}_{0k} and \tilde{p}_{0k} are admissible frequency envelopes for $|D|^{\delta_d} \delta A \in H^{\sigma+1-\delta_d}$, $|D|^{\sigma_d} \delta h \in H^{\sigma+1-\sigma_d}$ and $\delta \lambda \in H^\sigma$ respectively. Under these assumptions we will prove that the above linearization satisfies the bound

$$\| S_k |D|^{\delta_d} \delta \mathcal{T}(A) \|_{H^{\sigma+1-\delta_d}} \lesssim \epsilon_0 (\tilde{a}_{0k} + \tilde{c}_{0k} + \tilde{p}_{0k}), \tag{4.37}$$

If the bound (4.37) is true, then by the contraction principle we immediately get a unique small solution A to our equations, as well as the linearized bound (4.36).

We can also use (4.37) in order to prove the frequency envelope bounds (4.35). Indeed, by (3.3) and (4.34) we have

$$\begin{aligned} a_{0k} &= 2^{-\delta k} \| |D|^{\delta_d} A \|_{H^{s+1-\delta_d}} + \max_j 2^{-\delta|j-k|} \| S_j |D|^{\delta_d} A \|_{H^{s+1-\delta_d}} \\ &\lesssim 2^{-\delta k} (\epsilon_0 \| |D|^{\delta_d} A \|_{H^{s+1-\delta_d}} + \epsilon_0 \|\lambda\|_{H^s}) + \max_j 2^{-\delta|j-k|} (\epsilon_0 a_{0j} + \epsilon_0 p_{0j}) \\ &\lesssim \epsilon_0 a_{0k} + \epsilon_0 p_{0k}. \end{aligned}$$

This implies (4.35) for ϵ_0 sufficiently small.

It remains to prove the bound (4.37). We have

$$\| S_k |D|^{\delta_d} \delta \mathcal{T}(A) \|_{H^{\sigma+1-\delta_d}} \lesssim \| S_k |D|^{-1+\delta_d} (\delta h \nabla A + h \nabla \delta A + \lambda \delta \lambda) \|_{H^{\sigma+1-\delta_d}}.$$

Here we only estimate the term $\lambda \delta \lambda$; the others are similar. Precisely, when $k = 0$, using a Littlewood–Paley decomposition, Bernstein’s inequality and (3.2) we obtain

$$\begin{aligned} \| |D|^{-1+\delta_d} S_0(\lambda \delta \lambda) \|_{L^2} &\lesssim \|\lambda\|_{L^2} \| S_0 \delta \lambda \|_{L^2} + \sum_{j \geq 0} 2^{-\sigma j} \|\lambda_j\|_{L^2} 2^{2\sigma j} \|\delta \lambda_j\|_{L^2} \\ &\lesssim \|\lambda\|_{L^2} \tilde{p}_{00} + \sum_{j \geq 0} 2^{(\delta-\sigma)j} \|\lambda_j\|_{L^2} \tilde{p}_{00} \\ &\lesssim \|\lambda\|_{H^s} \tilde{p}_{00}. \end{aligned}$$

When $k > 0$, we use Bernstein’s inequality to bound the high-low and low-high interactions by $\|\lambda\|_{H^s} \tilde{p}_{0k}$. For the high-high interaction we have

$$\begin{aligned} & \| |D|^{-1+\delta_d} \mathcal{S}_k \left(\sum_{l>k} \lambda_l \delta \lambda_l \right) \|_{H^{\sigma+1-\delta_d}} \\ & \lesssim \sum_{l>k} 2^{(\sigma+\frac{d}{2})k} \|\lambda_l\|_{L^2} \|\delta \lambda_l\|_{L^2} \\ & \lesssim \mathbf{1}_{\leq -d/2}(\sigma) \sum_{l>k} 2^{(\sigma+\frac{d}{2}-\delta)k} 2^{(\delta-\sigma)l} \|\lambda_l\|_{L^2} 2^{\sigma l} \|\delta \lambda_l\|_{L^2} \\ & \quad + \mathbf{1}_{> -d/2}(\sigma) \sum_{l>k} 2^{(\sigma+\frac{d}{2}+\delta)(k-l)} 2^{(\frac{d}{2}+\delta)l} \|\lambda_l\|_{L^2} 2^{\sigma l} \|\delta \lambda_l\|_{L^2} \\ & \lesssim \|\lambda\|_{H^s} \tilde{p}_{0k}. \end{aligned}$$

This concludes the proof of bound (4.37), and completes the proof of the lemma. \square

5. Estimates for the Parabolic Equations

Here we consider the solvability of the parabolic system (2.29). For this purpose we view $\lambda \in L^2 X^s$ as a parameter, and show that the solution $(h, A) \in \mathcal{E}^s$ exists, it is small and has a Lipschitz dependence on both the initial data and on λ .

Theorem 5.1. (a) *Let $d \geq 2, s > d/2$. Assume that $\|h_0\|_{\mathbf{Y}_0^{s+2}} \leq \epsilon, \| |D|^{\delta_d} A_0 \|_{H^{s+1-\delta_d}} \leq \epsilon$ and $\|\lambda\|_{l^2 Z^s} \leq \epsilon$. Then the parabolic system (2.29)–(2.30) admits a unique small solution $\mathcal{S} = (h, A)$ in \mathcal{E}^s , with*

$$\|\mathcal{S}\|_{\mathcal{E}^s} \lesssim \|\mathcal{S}_0\|_{\mathcal{E}_0^s} + \|\lambda\|_{l^2 Z^s}. \tag{5.1}$$

In addition this solution has a Lipschitz dependence on both \mathcal{S}_0 in \mathcal{E}_0^s and λ in $l^2 Z^s$. Moreover, assume that s_{0k} and p_k are admissible frequency envelopes for $(h_0, A_0) \in \mathcal{E}_0^s, \lambda \in l^2 Z^s$ respectively, we have the frequency envelope version

$$\|\mathcal{S}_k\|_{\mathcal{E}^s} \lesssim s_{0k} + \epsilon p_k. \tag{5.2}$$

(b) *In addition, for the linearization of the parabolic system (2.29) we have the bounds*

$$\|\delta \mathcal{S}\|_{\mathcal{E}^s} \lesssim \|\delta \mathcal{S}_0\|_{\mathcal{E}_0^s} + \epsilon \|\delta \lambda\|_{l^2 Z^s}, \tag{5.3}$$

and

$$\|\delta \mathcal{S}\|_{\mathcal{E}^\sigma} \lesssim \|\delta \mathcal{S}_0\|_{\mathcal{H}^\sigma} + \epsilon \|\delta \lambda\|_{Z^\sigma}, \tag{5.4}$$

for $\sigma \in (\frac{d}{2} - 2, s]$.

We will do this in two steps. First we prove that this system is solvable in the larger space \mathcal{E}^s . Then we improve the space-time bounds for the metric h to the stronger norm \mathbf{Y}^{s+2} ; the latter will be needed in the study of the Schrödinger evolution (2.28).

Lemma 5.2. *Let $g = I_d + h$. Assume that $\|h\|_{Z^{\sigma_d, s+1}} \leq \epsilon$ for $s > d/2$ and $d \geq 2$. Let c_k and a_k be admissible frequency envelopes for $h \in Z^{\sigma_d, s+1}$, respectively $A \in Z^s$. Then for any $d/2 - 2 < \sigma \leq s$ and a linearization operator δ we have*

$$\|\delta(hg)\|_{Z^{\sigma_d, \sigma+1}} \lesssim \|\delta h\|_{Z^{\sigma_d, \sigma+1}}, \tag{5.5}$$

$$\|\delta(Ag)\|_{Z^\sigma} \lesssim \|\delta A\|_{Z^\sigma} + \|A\|_{Z^s} \|\delta h\|_{Z^{\sigma_d, \sigma}}, \tag{5.6}$$

and hence we have

$$\|S_k(hg)\|_{Z^{\sigma_d, s+1}} \lesssim c_k, \tag{5.7}$$

$$\|S_k(Ag)\|_{Z^s} \lesssim a_k, \tag{5.8}$$

Proof. Assume that $\tilde{c}_k(\sigma)$ and \tilde{a}_k are admissible frequency envelopes for $\delta h \in Z^{\sigma_d, \sigma}$ and $\delta A \in Z^\sigma$. Using a Littlewood–Paley decomposition, Bernstein’s inequality and the smallness of h we obtain

$$\|S_k(\delta h h)\|_{Z^{\sigma_d, \sigma+1}} \lesssim \epsilon \tilde{c}_k(\sigma + 1) + \|\delta h\|_{Z^{\sigma_d, \sigma+1}} c_k.$$

This implies (5.5) and (5.7) immediately. For A we have

$$\|S_k(\delta A h)\|_{Z^\sigma} \lesssim \epsilon \tilde{a}_k,$$

and

$$\|S_k(A \delta h)\|_{Z^\sigma} \lesssim a_k \|\delta h\|_{Z^{\sigma_d, \sigma}} + \|A\|_{Z^s} \tilde{c}_k(\sigma).$$

These give (5.6) and (5.8). □

Now, we begin to solve the parabolic system (2.29) with initial data (2.30) as follows:

Proposition 5.3. (a) *Assume that $\|(h_0, A_0)\|_{\mathcal{H}^s} \leq \epsilon$ and $\|\lambda\|_{Z^s} \leq \epsilon$ for $s > d/2$ and $d \geq 2$. Then the parabolic system (2.29)–(2.30) admits a unique small solution $\mathcal{S} = (h, A)$ in \mathcal{E}^s , with*

$$\|\mathcal{S}\|_{\mathcal{E}^s} \lesssim \|\mathcal{S}_0\|_{\mathcal{H}^s} + \|\lambda\|_{Z^s}. \tag{5.9}$$

In addition this solution has a Lipschitz dependence on \mathcal{S}_0 in \mathcal{H}^s and λ in Z^s . Moreover, assume that s_{0k} and p_k are admissible frequency envelopes for $\mathcal{S}_0 \in \mathcal{H}^s$, $\lambda \in Z^s$ respectively, then we have the frequency envelope version

$$\|S_k\|_{\mathcal{E}^s} \lesssim s_{0k} + \epsilon p_k. \tag{5.10}$$

(b) *In addition, for the linearization of the parabolic system (2.29) we have the bounds*

$$\|\delta \mathcal{S}\|_{\mathcal{E}^\sigma} \lesssim \|\delta \mathcal{S}(0)\|_{\mathcal{H}^\sigma} + \epsilon \|\delta \lambda\|_{Z^\sigma}, \tag{5.11}$$

for $\sigma \in (\frac{d}{2} - 2, s]$.

Proof. First, we consider a linear equation and prove a linear estimate. Precisely, assume that the frequency localized function u_k is solution of the linear equation

$$\partial_t u_k - \Delta u_k = f_k, \quad u_k(0) = u_{0k}.$$

Then by Bernstein's inequality we have the linear estimates

$$\frac{1}{2} \frac{d}{dt} \|u_k\|_{L^2}^2 \leq -c2^{2k} \|u_k\|_{L^2}^2 + \|u_k\|_{L^2} \|f_k\|_{L^2}.$$

We cancel one $\|u_k\|_{L^2}$, then multiply both sides by $e^{c2^{2k}t}$ and integrate in time to obtain

$$\|u_k(t)\|_{L^2} \lesssim e^{-c2^{2k}t} \|u_{0k}\|_{L^2} + 2^{-2k} \|f_k(s)\|_{L^\infty L^2}. \quad (5.12)$$

In order to solve (2.29) with small initial data, it suffices to consider the following linearized equations

$$\partial_t \delta h_k - \Delta \delta h_k = \mathcal{N}_1, \quad \partial_t \delta A_k - \Delta \delta A_k = \mathcal{N}_2,$$

where the nonlinearities \mathcal{N}_1 and \mathcal{N}_2 are

$$\begin{aligned} \mathcal{N}_1 &= S_k(\delta h \nabla^2 h + h \nabla^2 \delta h + \delta h \nabla h \nabla h + g \nabla h \nabla \delta h + \lambda \delta \lambda), \\ \mathcal{N}_2 &= S_k(h \nabla^2 \delta A + \delta h \nabla^2 A + \nabla h \nabla \delta A + \nabla \delta h \nabla A \\ &\quad + \nabla^2 h \delta A + \nabla^2 \delta h A + \nabla h \nabla h \delta A + \nabla h \nabla \delta h A + \lambda \nabla \delta \lambda \\ &\quad + \delta \lambda \nabla \lambda + \lambda \delta \lambda (\nabla h + A) + \lambda^2 (\nabla \delta h + \delta A)), \end{aligned}$$

with h , A and λ satisfying $\|(h, A)\|_{\mathcal{E}^s} \lesssim \epsilon$, $\|\lambda\|_{Z^s} \leq \epsilon$. Then we will prove the bound

$$\|S_k \delta \mathcal{S}\|_{\mathcal{E}^\sigma} \lesssim \tilde{s}_{0k} + \epsilon(\tilde{s}_k + \tilde{p}_k) + (s_k + p_k)(\|\delta \mathcal{S}\|_{\mathcal{E}^\sigma} + \|\delta \lambda\|_{Z^\sigma}), \quad (5.13)$$

where \tilde{s}_{0k} , \tilde{s}_k , \tilde{p}_k and s_k are admissible frequency envelopes for $\delta \mathcal{S}(0) \in \mathcal{H}^\sigma$, $\delta \mathcal{S} \in \mathcal{E}^\sigma$, $\delta \lambda \in Z^\sigma$ and $\mathcal{S} \in \mathcal{E}^s$ respectively.

Assuming the bound (5.13) is true, then we can use the contraction mapping principle to solve the parabolic system (2.29) in the space

$$\{\mathcal{S} = (h, A) \in \mathcal{E}^s : \|\mathcal{S}\|_{\mathcal{E}^s} \leq C(\|S_0\|_{\mathcal{H}^s} + \|\lambda\|_{Z^s}) \leq 2C\epsilon\},$$

which also implies the bound (5.9).

By the definition of frequency envelopes (3.3) and (5.9), the bound (5.13) with $\sigma = s$ and $\delta = Id$ implies

$$s_k \lesssim s_{0k} + \epsilon(s_k + p_k).$$

Thus the bound (5.10) follows. By (5.9), the bound (5.13) also gives (5.11).

We now return to the proof of (5.13). By the energy estimates in (5.12) we have

$$\|S_k \delta \mathcal{S}(t)\|_{\mathcal{E}^\sigma} \lesssim \|S_k \delta \mathcal{S}_0\|_{\mathcal{H}^\sigma} + \| |D|^{\sigma_d} S_k \mathcal{N}_1 \|_{L^\infty H^{\sigma-\sigma_d}} + \|S_k \mathcal{N}_2\|_{L^\infty H^{\sigma-1}}.$$

The estimates for the nonlinearities are similar, here we only estimate the following terms.

A. *The estimate for the terms $h\nabla^2\delta h$ and $\lambda\delta\lambda$ in \mathcal{N}_1 .* Using a Littlewood–Paley decomposition we have

$$\begin{aligned} \|S_k(h\nabla^2\delta h)\|_{L^\infty H^\sigma} &\lesssim 2^{\sigma k} \|h_{\leq k}\|_{L^\infty L^\infty} \|\nabla^2\delta h_k\|_{L^\infty L^2} \\ &\quad + \sum_{l\leq k} 2^{\sigma k+(d/2+2)l} \|h_k\|_{L^\infty L^2} \|\delta h_l\|_{L^\infty L^2} \\ &\quad + \sum_{l>k} 2^{(\sigma+d/2)(k-l)+(\sigma+d/2+2)l} \|h_l\|_{L^\infty L^2} \|\delta h_l\|_{L^\infty L^2} \\ &\lesssim \| |D|^{\sigma_d} h \|_{L^\infty H^{s-\sigma_d}} \tilde{c}_k + 2^{sk+2\delta k} \|h_k\|_{L^\infty L^2} \tilde{c}_k \\ &\quad + \tilde{c}_k \| |D|^{\sigma_d} h \|_{L^\infty H^{s+1-\sigma_d}} \\ &\lesssim \tilde{c}_k \| |D|^{\sigma_d} h \|_{L^\infty H^{s+1-\sigma_d}}, \end{aligned}$$

and

$$\begin{aligned} \|S_k(\lambda\delta\lambda)\|_{L^\infty H^\sigma} &\lesssim 2^{\sigma k} \|\lambda_{\leq k}\|_{L^\infty L^\infty} \|\delta\lambda_k\|_{L^\infty L^2} + \sum_{l\leq k} 2^{\sigma k+d/2l} \|\lambda_k\|_{L^\infty L^2} \|\delta\lambda_l\|_{L^\infty L^2} \\ &\quad + \sum_{l>k} 2^{(\sigma+d/2)(k-l)+(\sigma+d/2)l} \|\lambda_l\|_{L^\infty L^2} \|\delta\lambda_l\|_{L^\infty L^2} \\ &\lesssim \|\lambda\|_{L^\infty H^s} \tilde{p}_k + p_k \|\delta\lambda\|_{L^\infty H^\sigma} + \sum_{l>k} 2^{(\sigma+d/2-\delta)(k-l)} \|\lambda_l\|_{L^\infty H^{d/2}} \tilde{p}_k \\ &\lesssim \tilde{p}_k \|\lambda\|_{L^\infty H^s} + p_k \|\delta\lambda\|_{L^\infty H^\sigma}. \end{aligned}$$

B. *The estimate for the terms $\nabla^2 h\delta A$ and $\lambda\nabla\delta\lambda$ in \mathcal{N}_2 .* The second term is estimated in the same manner as the above bound for $\lambda\delta\lambda$ in \mathcal{N}_1 . For the first term $\nabla^2 h\delta A$ we have

$$\begin{aligned} \|S_k(\nabla^2 h\delta A)\|_{L^\infty H^{\sigma-1}} &\lesssim \| |D|^{\sigma_d} h \|_{L^\infty H^{s+1-\sigma_d}} \|\delta A_k\|_{L^\infty H^\sigma} + \sum_{l\leq k} 2^{\sigma k+k} \|h_k\|_{L^\infty L^2} \|\delta A_l\|_{L^\infty H^{d/2}} \\ &\quad + \sum_{l>k} 2^{(\sigma+d/2-1)(k-l)+(\sigma+d/2+1)l} \|h_l\|_{L^\infty L^2} \|\delta A_l\|_{L^\infty L^2} \\ &\lesssim \epsilon \tilde{a}_k + c_k \|\delta A\|_{L^\infty H^{\sigma+1}} + \sum_{l>k} 2^{(\sigma+d/2-1)(k-l)} \|h_l\|_{L^\infty H^{d/2+\delta}} \tilde{a}_k \\ &\lesssim \epsilon \tilde{a}_k + c_k \|\delta A\|_{L^\infty H^{\sigma+1}}. \end{aligned}$$

This concludes the proof of the bound (5.13), and completes the proof of the theorem. \square

We continue with the bound for the $l^2 Z^{\sigma_d, s+2}$ -norm of the metric h .

Proposition 5.4. *Assume that $\|(h_0, A_0)\|_{\gamma^s} \leq \epsilon$ and $\|\lambda\|_{l^2 Z^s} \leq \epsilon$ for $s > d/2$ and $d \geq 2$. Then the solution h also belongs to $l^2 Z^{s+2}$ and satisfies the bounds*

$$\|h\|_{l^2 Z^{\sigma_d, s+2}} \lesssim \| |D|^{\sigma_d} h_0 \|_{H^{s+2-\sigma_d}} + \|\lambda\|_{l^2 Z^s}. \tag{5.14}$$

Assume that c_{0k} and p_k are admissible frequency envelopes for $|D|^{\sigma_d} h_0 \in H^{s+2-\sigma_d}$, $\lambda \in l^2 Z^s$ respectively. Then we have the frequency envelope bounds

$$\|S_k h\|_{l^2 Z^{\sigma_d, s+2}} \lesssim c_{0k} + \epsilon p_k. \tag{5.15}$$

Finally, for the linearization of the h -equations we have the bounds

$$\|\delta h\|_{l^2 Z^{\sigma_d, s+2}} \lesssim \| |D|^{\sigma_d} \delta h_0 \|_{H^{s+2-\sigma_d}} + \epsilon \|\delta \lambda\|_{l^2 Z^s}. \tag{5.16}$$

Proof of Proposition 5.4. We split the proof into two steps, where we first prove the appropriate bound for the linear constant coefficient heat flow and then we apply that bound to solve the nonlinear problem perturbatively.

Step 1. Here we consider the linear equations

$$\partial_t P_k u - \Delta P_k u = P_k f, \tag{5.17}$$

with $P_k u$ localized at frequency 2^k for $k \in \mathbb{Z}$, and prove that

$$\|P_k u\|_{l^2_{|k|} L^\infty L^2} \lesssim \|P_k u(0)\|_{L^2} + 2^{-2k^+} \|P_k f\|_{l^2_{|k|} L^\infty L^2}. \tag{5.18}$$

By Duhamel’s formula, we have

$$\|P_k u\|_{l^2_{|k|} L^\infty L^2} \lesssim \|e^{t\Delta} P_k u_0\|_{l^2_{|k|} L^\infty L^2} + \left\| \int_0^t e^{(t-s)\Delta} P_k f ds \right\|_{l^2_{|k|} L^\infty L^2}.$$

Then we use (3.5) and (3.6) to bound the above two terms respectively, then we obtain (5.18).

Step 2. Here it suffices to write the linearized h equation in the form

$$\partial_t \delta h - \Delta \delta h = \delta h \nabla^2 h + h \nabla^2 \delta h + \delta h \nabla h \nabla h + g \nabla h \nabla \delta h + \lambda \delta \lambda := \mathcal{N},$$

and to prove that

$$\begin{aligned} \|S_k \delta h\|_{l^2 Z^{\sigma_d, s+2}} &\lesssim \| |D|^{\sigma_d} S_k \delta h_0 \|_{H^{s+2-\sigma_d}} + \epsilon (\tilde{c}_k + \tilde{p}_k) \\ &\quad + (c_k + p_k) (\|\delta h\|_{l^2 Z^{\sigma_d, s+2}} + \|\delta \lambda\|_{l^2 Z^s}), \end{aligned} \tag{5.19}$$

where \tilde{c}_k , \tilde{p}_k and c_k are admissible frequency envelopes for $\delta h \in l^2 Z^{\sigma_d, s+2}$, $\delta \lambda \in l^2 Z^s$ and $h \in l^2 Z^{\sigma_d, s+2}$ respectively.

If the bound (5.19) is true, then we choose the operator $\delta = Id$ to obtain (5.14). Then by (5.19) and (3.3) we also obtain (5.15). The bound (5.19) combined with (5.14) also implies (5.16).

We now continue with the proof of (5.19). By (5.18) we have

$$\|S_k \delta h\|_{l^2 Z^{\sigma_d, s+2}} \lesssim \| |D|^{\sigma_d} \delta h_{0k} \|_{H^{s+2-\sigma_d}} + \|S_k \mathcal{N}\|_{l^2 Z^{\sigma_d, s}}.$$

For the nonlinearities, we only estimate $h \nabla^2 \delta h$ and $\lambda \delta \lambda$, the others are estimated similiarly. Indeed, using a Littlewood–Paley decomposition we have

$$\|P_k (h \nabla^2 \delta h)\|_{l^2_{|k|} L^\infty L^2} \lesssim \|h\|_{L^\infty L^\infty} 2^{2k} \|P_k \delta h\|_{l^2_{|k|} L^\infty L^2} + \|P_k h\|_{l^2_{|k|} L^\infty L^2} \|\nabla^2 \delta h\|_{L^\infty L^\infty}$$

$$\begin{aligned}
 &+ \sum_{|k| \geq l > k} 2^{\frac{d}{2}k+2l} \|P_l h\|_{L^\infty L^2} \|P_l \delta h\|_{l^2 L^\infty L^2} \\
 &+ \sum_{l > |k|} 2^{(\frac{d}{2}+2)l} \|P_l h\|_{L^\infty L^2} \|P_l \delta h\|_{l^2 L^\infty L^2}.
 \end{aligned}$$

By this estimate and Sobolev embeddings we obtain

$$\|S_k(h \nabla^2 \delta h)\|_{l^2 Z^{\sigma_d, s}} \lesssim \| |D|^{\sigma_d} h \|_{L^\infty H^{s-\sigma_d}} \tilde{c}_k + c_k \| |D|^{\sigma_d} h \|_{L^\infty H^{s+2-\sigma_d}}.$$

For the term $\lambda \delta \lambda$, we also have

$$\begin{aligned}
 2^{sk} \|S_k(\lambda \delta \lambda)\|_{l^2 L^\infty L^2} &\lesssim \|\lambda\|_{l^2 Z^s} \|\delta \lambda_k\|_{L^\infty H^s} + \|\lambda_k\|_{l^2 L^\infty H^s} \|\delta \lambda\|_{Z^s} \\
 &+ \sum_{l > k} 2^{sk} 2^{\frac{d}{2}l} \|\lambda_l\|_{L^\infty L^2} \|\delta \lambda_l\|_{l^2 L^\infty L^2} \\
 &\lesssim \epsilon \tilde{p}_k + p_k \|\delta \lambda\|_{l^2 Z^s}.
 \end{aligned}$$

This completes the proof of Proposition 5.4. □

Finally, we carry out the last step in the proof of Theorem 5.1, and establish bounds for the solutions h in the Y^{s+2} spaces:

Proposition 5.5. *Let $d \geq 2, s > d/2$. Assume that $\|(h_0, A_0)\|_{\mathcal{T}^s} \leq \epsilon$ and $\|\lambda\|_{l^2 Z^s} \leq \epsilon$. Then we have the bound*

$$\|h\|_{Y^{s+2}} \lesssim \|h_0\|_{Y_0^{s+2}} + \|\lambda\|_{l^2 Z^s}. \tag{5.20}$$

with Lipschitz dependence on the initial data in these topologies. Moreover, assume that c_{0k} and p_k are admissible frequency envelope for $h(0) \in Y_0^{s+2}$ and $\lambda \in l^2 Z^s$, then we have the frequency envelope version

$$\|S_k h\|_{Y^{s+2}} \lesssim c_{0k} + \epsilon p_k. \tag{5.21}$$

In addition, for the linearization of the elliptic system (2.29) we have the bounds

$$\|\delta h\|_{Y^{s+2}} \lesssim \|\delta h_0\|_{Y_0^{s+2}} + \epsilon \|\delta \lambda\|_{l^2 Z^s}. \tag{5.22}$$

Proof. Again it suffices to write the h equation in the form:

$$\partial_t \delta h - \Delta \delta h = \delta h \nabla^2 h + h \nabla^2 \delta h + \delta h \nabla h \nabla h + g \nabla h \nabla \delta h + \lambda \delta \lambda := \mathcal{N},$$

and to prove that

$$\|\delta h_k\|_{Y^{s+2}} \lesssim \|\delta h_{0k}\|_{Y_0^{s+2}} + \epsilon (\tilde{c}_k + \tilde{p}_k) + (c_k + p_k) (\|\delta h\|_{Y^{s+2}} + \|\delta \lambda\|_{l^2 Z^s}), \tag{5.23}$$

where \tilde{c}_k and \tilde{p}_k are admissible frequency envelopes for $\delta h \in Y^{s+2}$ and $\delta \lambda \in l^2 Z^s$ respectively.

If (5.23) is true, then the bound (5.20) is obtained by (5.23) with the operator $\delta = Id$ and the bound (5.14). We also obtain (5.21) by (5.23) and (5.15). The bound (5.23) combined with (5.16) also implies (5.22).

We now return to prove the bound (5.23). By Duhamel’s formula, (3.5) and (3.6), we have

$$\|S_k \delta h\|_{Y^{s+2}} \lesssim \|e^{t\Delta} \delta h_{0k}\|_{Y^{s+2}} + \left\| \int_0^t e^{(t-s)\Delta} S_k \mathcal{N} \, ds \right\|_{Y^{s+2}}.$$

We estimate the first term in the right hand side. For any decomposition $P_j \delta h(0) = \sum_{l \geq |j|} \delta h_{j,l}(0)$, by (3.5) we have

$$\begin{aligned} \|e^{t\Delta} P_j \delta h(0)\|_{Y_j} &\lesssim \inf_{P_j \delta h(0) = \sum_{l \geq |j|} h_{j,l}(0)} \sum_{l \geq |j|} 2^{l-|j|} \|e^{t\Delta} \delta h_{j,l}(0)\|_{l^1_{|l|} L^\infty L^2} \\ &\lesssim \inf_{P_j \delta h(0) = \sum_{l \geq |j|} \delta h_{j,l}(0)} \sum_{l \geq |j|} 2^{l-|j|} \|\delta h_{j,l}(0)\|_{l^1_{|l|} L^2} \\ &= \|\delta h(0)\|_{Y_{0j}}. \end{aligned}$$

This gives the bound for the first term.

Next, for the nonlinearities, we only estimate the Duhamel contributions of $h \nabla^2 \delta h$ and $\lambda \delta \lambda$ in detail. In order to bound the contribution of term $h \nabla^2 \delta h$, we use the Littlewood–Paley trichotomy to decompose it into three cases:

a) *Low-high interactions:* $P_j(P_{<j} h \nabla^2 P_j \delta h)$. By (3.6), for any decomposition $P_j \delta h = \sum_{l \geq |j|} \delta h_{j,l}$ we have

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\Delta} P_j(P_{<j} h \nabla^2 P_j \delta h) \, ds \right\|_{Y_j} &= \sum_{l \geq |j|} 2^{l-|j|} 2^{-2j^+} \|P_j(P_{<j} h \nabla^2 \delta h_{j,l})\|_{l^1_{|l|} L^\infty L^2} \\ &\lesssim 2^{2j-2j^+} \|P_{<j} h\|_{L^\infty L^\infty} \sum_{l \geq |j|} 2^{l-|j|} \|\delta h_{j,l}\|_{l^1_{|l|} L^\infty L^2} \\ &\lesssim 2^{2j-2j^+} \| |D|^{\sigma_d} h \|_{L^\infty H^{s-\sigma_d}} \|P_j \delta h\|_{Y_j}. \end{aligned}$$

This implies both the low-frequency part bound

$$\left\| \int_0^t e^{(t-s)\Delta} \sum_{j \leq 0} P_j(P_{<j} h \nabla^2 P_j \delta h) \, ds \right\|_{Y^{s+2}} \lesssim \epsilon \|S_0 \delta h\|_{Y^{s+2}},$$

and the high frequency part bound

$$\left\| \int_0^t e^{(t-s)\Delta} S_j(P_{<j} h \nabla^2 P_j \delta h) \, ds \right\|_{Y^{s+2}} \lesssim \epsilon \|\delta h_j\|_{Y^{s+2}}.$$

b) *The high-low interactions* $P_j(P_j h \nabla^2 P_{<j+O(1)} \delta h)$ are estimated in the same manner as the above *low-high* case, so we omit the computations.

c) *High-high interactions:* $\sum_{l > j} P_j(P_l h \nabla^2 P_l \delta h)$. This sum can be further decomposed as $\sum_{l > j} = \sum_{|j| > l > j} + \sum_{l \geq |j|}$. Then by (3.6) we bound the contribution of the first term by

$$2^{(\frac{d}{2}-\delta)j^-} \left\| \int_0^t e^{(t-s)\Delta} \sum_{|j| > l > j} P_j(P_l h \nabla^2 P_l \delta h) \, ds \right\|_{Y_j}$$

$$\begin{aligned}
 &\lesssim 2^{(\frac{d}{2}-\delta)j^-} \sum_{|j|>l>j} \|P_j(P_l h \nabla^2 P_l \delta h)\|_{l_{|j|}^1 L^\infty L^2} \\
 &\lesssim \sum_{|j|>l>j} 2^{(d-\delta)(j-l)+(d+2-\delta)l} \|P_l h\|_{l_{|j|}^2 L^\infty L^2} \|P_l \delta h\|_{l_{|j|}^2 L^\infty L^2} \\
 &\lesssim \sum_{|j|>l>j} 2^{(d-\delta)(j-l)} \|P_l h\|_{l^2 Z^{\sigma_d, s} \tilde{c}_0}.
 \end{aligned}$$

Also by (3.6) we bound the contribution of second term by

$$\begin{aligned}
 &2^{(\frac{d}{2}-\delta)j^-+(s+2)j^+} \left\| \int_0^t e^{(t-s)\Delta} \sum_{l>|j|} P_j(h_l \nabla^2 \delta h_l) ds \right\|_{Y_j} \\
 &\lesssim 2^{(\frac{d}{2}-\delta)j^-+(s+2)j^+} \sum_{l>|j|} 2^{l-|j|} \left\| \int_0^t e^{(t-s)\Delta} P_j(h_l \nabla^2 \delta h_l) ds \right\|_{l_{|j|}^1 L^\infty L^2} \\
 &\lesssim 2^{(\frac{d}{2}-\delta)j^-+(s+2)j^+} \sum_{l>|j|} 2^{l-|j|+dj/2-2j^++2l} \|h_l\|_{l_{|j|}^2 L^\infty L^2} \|\delta h_l\|_{l_{|j|}^2 L^\infty L^2} \quad (5.24)
 \end{aligned}$$

This term is further controlled by

$$\begin{aligned}
 \text{LHS}(5.24) &\lesssim 2^{(d+1-\delta)j^-+(s+\frac{d}{2}-1)j^+} \sum_{l>|j|} 2^{3l} \|h_l\|_{l_{|j|}^2 L^\infty L^2} \|\delta h_l\|_{l_{|j|}^2 L^\infty L^2} \\
 &\lesssim 2^{(d+1-\delta)j^-} \|h\|_{l^2 Z^{\sigma_d, s+1} \tilde{c}_0} + \mathbf{1}_{>0}(j) \|h\|_{l^2 Z^{\sigma_d, s+1} \tilde{c}_j}.
 \end{aligned}$$

This concludes the proof of the bound for the contribution of $h \nabla^2 \delta h$. Next we consider the term $\lambda \delta \lambda$. We also split its analysis into three cases:

a) *Low-high interactions:* $P_j(P_{<j} \lambda P_j \delta \lambda)$ and *high-low interactions:* $P_j(P_j \lambda P_{<j} \delta \lambda)$. These two cases are similar, we only estimate the first term. By (3.6), we have

$$\begin{aligned}
 \left\| \int_0^t e^{(t-s)\Delta} P_j(P_{<j} \lambda P_j \delta \lambda) ds \right\|_{Y_j} &\lesssim 2^{-2j^+} \|P_j(P_{<j} \lambda P_j \delta \lambda)\|_{l_{|j|}^1 L^\infty L^2} \\
 &\lesssim 2^{-2j^+} \|\lambda\|_{l^2 Z^s} \|P_j \delta \lambda\|_{l_{|j|}^2 L^\infty L^2},
 \end{aligned}$$

which is acceptable.

b) *High-high interactions:* $\sum_{l>j} P_j(P_l \lambda \cdot P_l \delta \lambda)$. This sum can be further decomposed as $\sum_{l>j} = \sum_{|j|>l>j} + \sum_{l \geq |j|}$. By (3.6) we bound the contribution of the first sum by

$$\begin{aligned}
 &2^{(\frac{d}{2}-\delta)j^-} \left\| \int_0^t e^{(t-s)\Delta} \sum_{|j|>l>j} P_j(P_l \lambda \cdot P_l \delta \lambda) ds \right\|_{Y_j} \\
 &\lesssim 2^{(\frac{d}{2}-\delta)j^-} \sum_{|j|>l>j} \|P_j(P_l \lambda \cdot P_l \delta \lambda)\|_{l_{|j|}^1 L^\infty L^2} \\
 &\lesssim 2^{(d-\delta)j^-} \sum_{|j|>l>j} \|\lambda_l\|_{l_{|j|}^2 L^\infty L^2} \|\delta \lambda_l\|_{l_{|j|}^2 L^\infty L^2}
 \end{aligned}$$

$$\lesssim 2^{(d-2\delta)j^-} \|\lambda\|_{l^2 Z^s} \tilde{p}_0.$$

Next we bound the contribution of the second sum by

$$\begin{aligned} & 2^{(\frac{d}{2}-\delta)j^-+(s+2)j^+} \left\| \int_0^t e^{(t-s)\Delta} \sum_{l>|j|} P_j(\lambda_l \delta \lambda_l) ds \right\|_{Y_j} \\ & \lesssim 2^{(\frac{d}{2}-\delta)j^-+(s+2)j^+} \sum_{l>|j|} 2^{l-|j|} \left\| \int_0^t e^{(t-s)\Delta} P_j(\lambda_l \delta \lambda_l) ds \right\|_{l^1 L^\infty L^2} \\ & \lesssim 2^{(\frac{d}{2}-\delta)j^-+(s+2)j^+} \sum_{l>|j|} 2^{l-|j|+dj/2-2j^+} \|\lambda_l\|_{l^2 L^\infty L^2} \|\delta \lambda_l\|_{l^2 L^\infty L^2} \\ & \lesssim 2^{(d+1-\delta)j^-+(s+\frac{d}{2}-1)j^+} \sum_{l>|j|} 2^l \|\lambda_l\|_{l^2 L^\infty L^2} \|\delta \lambda_l\|_{l^2 L^\infty L^2} \\ & \lesssim 2^{(d+1-\delta)j^-} \|\lambda\|_{l^2 Z^s} \tilde{p}_0 + \mathbf{1}_{>0}(j) \|\lambda\|_{l^2 Z^s} \tilde{p}_j. \end{aligned}$$

This concludes the proof of the bound (5.23), and completes the proof of the proposition. \square

6. Multilinear and Nonlinear Estimates

This section contains our main multilinear estimates which are needed for the analysis of the Schrödinger equation in (2.28). We begin with the following low-high bilinear estimates of $\nabla h \nabla \lambda$.

Proposition 6.1. *Let $s > \frac{d}{2}$, $d \geq 2$ and $k \in \mathbb{N}$. Suppose that $\nabla a(x) \lesssim \langle x \rangle^{-1}$, $h \in Y^{s+2}$ and $\lambda_k \in l^2 X^s$. Then for $-s \leq \sigma \leq s$ we have*

$$\|\nabla h_{\leq k} \cdot \nabla \lambda_k\|_{l^2 N^\sigma} \lesssim \min\{\|h\|_{Y^{\sigma+2}} \|\lambda_k\|_{l^2 X^s}, \|h\|_{Y^{s+2}} \|\lambda_k\|_{l^2 X^\sigma}\}, \tag{6.1}$$

$$\|h_{\leq k} \nabla a \nabla \lambda_k\|_{l^2 N^\sigma} \lesssim \min\{\|h\|_{Y^{\sigma+2}} \|\lambda_k\|_{l^2 X^s}, \|h\|_{Y^{s+2}} \|\lambda_k\|_{l^2 X^\sigma}\}. \tag{6.2}$$

In addition, if $d/2 - 2 < \sigma \leq s - 1$ then we have

$$\|\nabla h_{\leq k} \cdot \nabla \lambda_k\|_{l^2 N^\sigma} \lesssim \min\{\|h\|_{Z^{\sigma_d, \sigma+2}} \|\lambda_k\|_{Z^s}, \|h\|_{Z^{\sigma_d, s+2}} \|\lambda_k\|_{Z^\sigma}\}, \tag{6.3}$$

and if $d/2 - 2 < \sigma \leq s - 2$ then we have

$$\|h_{\leq k} \nabla^2 \lambda_k\|_{l^2 N^\sigma} \lesssim \|h\|_{Z^{\sigma_d, \sigma+2}} \|\lambda_k\|_{l^2 X^s}. \tag{6.4}$$

Proof. (a) The estimate (6.1). This is obtained by a Littlewood–Paley decomposition and the following estimate

$$\|\nabla P_j h \nabla \lambda_k\|_{l^2_k N_k} \lesssim 2^{\frac{d}{2}j+2j^+} \|P_j h\|_{Y_j} \|\lambda_k\|_{X_k}, \quad j \leq k, \quad j \in \mathbb{Z}, \quad k \in \mathbb{N},$$

which has been proved in [9, Lemma 5.1].

(b) *The estimate (6.2).* Compared to [9, (5.2)], the estimate (6.2) is improved by decomposing physical space dyadically. By duality, it suffices to prove that

$$\begin{aligned}
 II_j &= \int_0^1 \langle P_j h \nabla a \nabla \lambda_k, z_k \rangle dt \\
 &\lesssim 2^{dj/2} \log(2 + |j|) \|P_j h\|_{Y_j} \|\lambda_k\|_{X_k}, \quad j \leq k, \quad j \in \mathbb{Z}, \tag{6.5}
 \end{aligned}$$

for any $z_k \in l_k^2 X_k$ with $\|z_k\|_{l_k^2 X_k} \leq 1$. For any decomposition $P_j h = \sum_{l \geq |j|} h_{j,l}$, using the bound $|\nabla a(x)| \lesssim \langle x \rangle^{-1}$, we consider the two cases $|x| \geq 2^l$ and $|x| < 2^l$ respectively and then obtain

$$\begin{aligned}
 II_j &\lesssim \sum_{l \geq |j|} \sup_{\|z_k\|_{l_k^2 X_k} \leq 1} \sum_{0 \leq l_1 \leq l} \int_0^1 \langle h_{j,l} \langle x \rangle^{-1} \mathbf{1}_{[2^{l_1-1}, 2^{l_1}]}(x) \nabla \lambda_k, z_k \rangle dt \\
 &\quad + \sum_{l \geq |j|} \sup_{\|z_k\|_{l_k^2 X_k} \leq 1} \int_0^1 \langle h_{j,l} \langle x \rangle^{-1} \mathbf{1}_{>2^l}(x) \nabla \lambda_k, z_k \rangle dt \\
 &= II_{j1} + II_{j2}.
 \end{aligned}$$

By Bernstein’s inequality we bound the first term by

$$\begin{aligned}
 II_{j1} &\lesssim \sum_{l \geq |j|} \sup_{\|z_k\|_{l_k^2 X_k} \leq 1} \sum_{0 \leq l_1 \leq l} 2^{-l_1} \|h_{j,l}\|_{L^\infty L^\infty} \|\nabla \lambda_k\|_{l_1^\infty L^2 L^2} \|z_k\|_{l_1^\infty L^2 L^2} \\
 &\lesssim \sum_{l \geq |j|} \sum_{0 \leq l_1 \leq l} 2^{\frac{dj}{2}} \|h_{j,l}\|_{L^\infty L^2} \|\lambda_k\|_{X_k} \\
 &\lesssim 2^{dj/2} \log(2 + |j|) \sum_{l \geq |j|} \frac{|l|}{|j|} \|h_{j,l}\|_{l_1^1 L^\infty L^2} \|\lambda_k\|_{X_k}.
 \end{aligned}$$

The second term is bounded by

$$\begin{aligned}
 II_{j2} &\lesssim \sum_{l \geq |j|} 2^{-l} \sup_{\|z_k\|_{l_k^2 X_k} \leq 1} \|h_{j,l}\|_{l_1^1 L^\infty L^\infty} \|\nabla \lambda_k\|_{l_1^\infty L^2 L^2} \|z_k\|_{l_1^\infty L^2 L^2} \\
 &\lesssim \sum_{l \geq |j|} \|h_{j,l}\|_{l_1^1 L^\infty L^\infty} \|\lambda_k\|_{X_k} \\
 &\lesssim 2^{dj/2} \sum_{l \geq |j|} \|h_{j,l}\|_{l_1^1 L^\infty L^2} \|\lambda_k\|_{X_k}.
 \end{aligned}$$

Finally we take the infimum over the decompositions of $P_j h$ to get the bound (6.5), which in turn implies the estimate (6.2).

c) *The estimates (6.3) and (6.4).* By duality and Sobolev embedding, for any $j < k$ we have

$$\|P_j h \nabla^2 \lambda_k\|_{l^{2N\sigma}} \lesssim 2^{\sigma k} \|P_j h \nabla^2 \lambda_k\|_{L^2 L^2} \lesssim 2^{(\sigma+2)k} \| |D|^{\sigma d} P_j h \|_{L^\infty H^{d/2-\sigma d}} \|\lambda_k\|_{L^2 L^2},$$

which gives the bound (6.4). We can also obtain the bound (6.3) similarly. This completes the proof of the lemma. \square

We next prove the remaining bilinear estimates and trilinear estimates.

Proposition 6.2. (Nonlinear estimates) *Let $s > \frac{d}{2}$ and $d \geq 2$. Assume that p_k , \tilde{p}_k , s_k and \tilde{s}_k are admissible frequency envelopes for $\lambda \in Z^s$, $\lambda \in Z^\sigma$, $S \in \mathcal{E}^s$ and $S \in \mathcal{E}^\sigma$ respectively. Then we have*

$$\|S_k(B\lambda)\|_{l^2 N^s} \lesssim s_k \|\lambda\|_{Z^s} + p_k \|B\|_{L^2 H^s}, \quad (6.6)$$

$$\|S_k(A^2\lambda)\|_{l^2 N^s} \lesssim s_k \|A\|_{Z^s} \|\lambda\|_{Z^s} + p_k \|A\|_{Z^s}^2, \quad (6.7)$$

$$\|S_k(\lambda^3)\|_{l^2 N^s} \lesssim p_k \|\lambda\|_{Z^s}^2. \quad (6.8)$$

For $-s \leq \sigma \leq s$ we have

$$\|S_k \nabla(h_{\geq k-4} \nabla \lambda)\|_{l^2 N^\sigma} \lesssim \min\{\tilde{s}_k \|\lambda\|_{Z^s}, \tilde{p}_k \|h\|_{Z^{\sigma, d, s+2}}\}, \quad (6.9)$$

$$\|S_k(A_{\geq k-4} \nabla \lambda)\|_{l^2 N^\sigma} \lesssim \min\{\tilde{s}_k \|\lambda\|_{Z^s}, \tilde{p}_k \|A\|_{Z^{s+1}}\}, \quad (6.10)$$

and for $-s \leq \sigma \leq s - \delta$ we have

$$\|S_k(B\lambda)\|_{l^2 N^\sigma} \lesssim \min\{\tilde{s}_k \|\lambda\|_{Z^s}, \tilde{p}_k \|B\|_{L^2 H^s}\}, \quad (6.11)$$

$$\|S_k(A^2\lambda)\|_{l^2 N^\sigma} \lesssim \min\{\tilde{s}_k \|A\|_{Z^s} \|\lambda\|_{Z^s}, \tilde{p}_k \|A\|_{Z^s}^2\}, \quad (6.12)$$

$$\|S_k(\lambda^3)\|_{l^2 N^\sigma} \lesssim \tilde{s}_k \|\lambda\|_{Z^s}^2. \quad (6.13)$$

If $-s \leq \sigma \leq s - 1$, then

$$\|S_k(A_{< k-4} \nabla \lambda)\|_{l^2 N^\sigma} \lesssim p_k \|A\|_{Z^{\sigma+1}}. \quad (6.14)$$

Proof. We first prove (6.9) and (6.10). These two bounds are proved by Hölder's inequality and Bernstein's inequality, here we only prove the first bound in detail. For the high-low case, by duality we have

$$\begin{aligned} \sum_{j \leq k+C} \|S_k \nabla(h_k \nabla \lambda_j)\|_{l^2 N^\sigma} &\lesssim \sum_{j \leq k+C} 2^{\sigma k} \|S_k \nabla(h_k \nabla \lambda_j)\|_{L^2 L^2} \\ &\lesssim \sum_{j \leq k+C} 2^{(\sigma+1)k} \|h_k\|_{L^2 L^2} \|\nabla \lambda_j\|_{L^\infty L^\infty} \\ &\lesssim \sum_{j \leq k+C} 2^{(\sigma+1)k+(d/2+1)j} \|h_k\|_{L^2 L^2} \|\lambda_j\|_{L^\infty L^2}. \end{aligned}$$

Then by $-s \leq \sigma \leq s$ and (3.2), we can bound this by $\min\{\tilde{s}_k \|\lambda\|_{Z^s}, \tilde{p}_k \|h\|_{L^2 H^{s+2}}\}$. For the high-high case, when $\sigma + d/2 + 1 > 0$ we have

$$\begin{aligned} &\sum_{j > k} \|S_k \nabla(h_j \nabla \lambda_j)\|_{l^2 N^\sigma} \\ &\lesssim \sum_{j_1=j_2+O(1), j_1 > k} 2^{(\sigma+1)k+dj_1/2} \|S_k(h_{j_1} \nabla \lambda_{j_2})\|_{L^2 L^1} \\ &\lesssim \sum_{j_1=j_2+O(1), j_1 > k} 2^{(\sigma+1+d/2+\delta)(k-j_1)+(\sigma+2+d/2+\delta)j_1} \|h_{j_1}\|_{L^2 L^2} \|\lambda_{j_2}\|_{L^\infty L^2} \\ &\lesssim \min\{\tilde{s}_k \|\lambda\|_{Z^s}, \tilde{p}_k \|h\|_{Z^{1, s+2}}\}, \end{aligned}$$

and when $\sigma + d/2 + 1 \leq 0$ we have

$$\begin{aligned} & \sum_{j_1=j_2+O(1), j_1>k} \|S_k \nabla(h_{j_1} \nabla \lambda_{j_2})\|_{L^2 N^\sigma} \\ & \lesssim \sum_{j_1=j_2+O(1), j_1>k} 2^{(\sigma+1+d/2-\delta)k+(\delta+1)j_1} \|h_{j_1}\|_{L^2 L^2} \|\lambda_{j_2}\|_{L^\infty L^2} \\ & \lesssim \min\{\tilde{s}_k \|\lambda\|_{Z^s}, \tilde{p}_k \|h\|_{Z^{1,s+2}}\}. \end{aligned}$$

Next, we prove the bounds (6.6)–(6.8) and (6.11)–(6.13). These are all similar, so we only prove (6.6) and (6.11) in detail. Indeed, by duality we have

$$\|S_k(B\lambda)\|_{l^2 N^\sigma} \lesssim 2^{\sigma k} \|S_k(B\lambda)\|_{L^2 L^2}.$$

Then using the Littlewood–Paley trichotomy we divide this into *low-high*, *high-low* and *high-high* cases. For the low-high interactions, by Sobolev embeddings we have for $-s \leq \sigma \leq s$

$$2^{\sigma k} \|S_k(B_{<k} \lambda_k)\|_{L^2 L^2} \lesssim \|B_{<k}\|_{L^2 L^\infty} 2^{\sigma k} \|\lambda_k\|_{L^\infty L^2} \lesssim \tilde{p}_k \|B\|_{L^2 H^s}.$$

If $-s \leq \sigma \leq s - \delta$, we use $L^2 H^\sigma$ for $B_l = S_l B$. Then by $\|B_l\|_{L^2 H^\sigma} \lesssim 2^{\delta(k-l)} \tilde{s}_k$ we also have

$$2^{\sigma k} \|S_k(B_{<k} \lambda_k)\|_{L^2 L^2} \lesssim \tilde{s}_k \|\lambda\|_{Z^s}.$$

The high-low interactions can be estimated similarly. For the high-high interactions, by Sobolev embeddings when $-d/2 - \delta \leq \sigma \leq s$ we have

$$\begin{aligned} 2^{\sigma k} \left\| \sum_{l>k} S_k(B_l \lambda_l) \right\|_{L^2 L^2} & \lesssim \sum_{l>k} 2^{(\sigma+d/2+2\delta)(k-l)} 2^{(\sigma+d/2+2\delta)l} \|B_l\|_{L^2 L^2} \|\lambda_l\|_{L^\infty L^2} \\ & \lesssim \min\{\tilde{s}_k \|\lambda\|_{Z^s}, \tilde{p}_k \|B\|_{L^2 H^s}\}, \end{aligned}$$

and when $-s \leq \sigma < -d/2 - \delta$ we have

$$\begin{aligned} 2^{\sigma k} \left\| \sum_{l>k} S_k(B_l \lambda_l) \right\|_{L^2 L^2} & \lesssim \sum_{l>k} 2^{(\sigma+d/2)k} \|B_l\|_{L^2 L^2} \|\lambda_l\|_{L^\infty L^2} \\ & \lesssim 2^{-\delta k} \min\{\|B\|_{L^2 H^\sigma} \|\lambda\|_{Z^{-\sigma}}, \|B\|_{L^2 H^{-\sigma}} \|\lambda\|_{Z^\sigma}\} \\ & \lesssim \min\{\tilde{s}_k \|\lambda\|_{Z^s}, \tilde{p}_k \|B\|_{L^2 H^s}\}, \end{aligned}$$

These imply the bounds (6.6) and (6.11).

Finally, we prove the bound (6.14). If $d/2 - 1 + \delta \leq \sigma \leq s - 1$, by duality and Sobolev embeddings, we have

$$\|A_{<k} \nabla \lambda_k\|_{l^2 N^\sigma} \lesssim 2^{(\sigma+1)k} \|A_{<k}\|_{L^2 L^\infty} \|\lambda_k\|_{L^\infty L^2} \lesssim p_k \|A\|_{Z^{\sigma+1}}.$$

If $\sigma < d/2 - 1 + \delta$, we have

$$\begin{aligned} 2^{\sigma k} \|A_{<k} \nabla \lambda_k\|_{L^2 L^2} & \lesssim \sum_{0 \leq l < k} 2^{(d/2-1-\sigma+2\delta)(l-k)} \|A\|_{L^2 H^{\sigma+1}} 2^{(d/2+2\delta)k} \|\lambda_k\|_{L^\infty L^2} \\ & \lesssim p_k \|A\|_{Z^{\sigma+1}}. \end{aligned}$$

Then the bound (6.14) follows. Hence this completes the proof of the lemma. \square

We shall also require the following bounds for commutators.

Proposition 6.3. (Commutator bounds) *Let $d \geq 2$ and $s > \frac{d}{2}$. Let $m(D)$ be a multiplier with symbol $m \in S^0$. Assume $h \in Y^{s+2}$, $\partial_x A \in L^2 H^s$ and $\lambda_k \in l^2 X^s$, frequency localized at frequency 2^k . If $-s \leq \sigma \leq s$ then we have*

$$\|\nabla[S_{<k-4}h, m(D)]\nabla\lambda_k\|_{l^2 N^\sigma} \lesssim \min\{\|h\|_{Y^{\sigma+2}}\|\lambda_k\|_{l^2 X^s}, \|h\|_{Y^{s+2}}\|\lambda_k\|_{l^2 X^\sigma}\}, \tag{6.15}$$

$$\|[S_k, A_{<k-4}]\nabla\lambda_k\|_{l^2 N^\sigma} \lesssim \min\{\|\partial_x A\|_{L^2 H^s}\|\lambda_k\|_{L^\infty H^\sigma}, \|\partial_x A\|_{L^2 H^\sigma}\|\lambda_k\|_{L^\infty H^s}\}. \tag{6.16}$$

Proof. This is similar to Proposition 5.3 in [9]. First we estimate (6.15). In [20, Proposition 3.2], it was shown that

$$\nabla[S_{<k-4}g, m(D)]\nabla S_k\lambda = L(\nabla S_{<k-4}g, \nabla S_k\lambda),$$

where L is a translation invariant operator satisfying

$$L(f, g)(x) = \int f(x + y)g(x + z)\tilde{m}(y + z) dydz, \quad \tilde{m} \in L^1.$$

Given this representation, as we are working in translation-invariant spaces, by (6.1) the bound (6.15) follows.

Next, for the bound (6.16). Since

$$[S_k, A_{<k}]\nabla\lambda = \int_0^1 \int 2^{kd} \check{\varphi}(2^k y) 2^k y \nabla A_{<k}(x - sy) 2^{-k} \nabla \lambda_{[k-3, k+3]}(x - y) dy ds,$$

By translation-invariance and the similar argument to (6.11), the bound (6.16) follows. This completes the proof of the lemma. \square

7. Local Energy Decay and the Linearized Problem

In this section, we consider a linear Schrödinger equation

$$\begin{cases} i \partial_t \lambda + \partial_\alpha (g^{\alpha\beta} \partial_\beta \lambda) + 2i A^\alpha \partial_\alpha \lambda = F, \\ \lambda(0) = \lambda_0, \end{cases} \tag{7.1}$$

and, under suitable assumptions on the coefficients, we prove that the solution satisfies suitable energy and local energy bounds.

7.1. The Linear Paradifferential Schrödinger Flow

As an intermediate step, here we prove energy and local energy bounds for a frequency localized linear paradifferential Schrödinger equation

$$i \partial_t \lambda_k + \partial_\alpha (g_{<k-4}^{\alpha\beta} \partial_\beta \lambda_k) + 2i A_{<k-4}^\alpha \partial_\alpha \lambda_k = f_k. \tag{7.2}$$

We begin with the energy estimates, which are fairly standard.

Lemma 7.1. (Energy-type estimate) *Let $d \geq 2$. Assume that λ_k solves the equation (7.2) with initial data $\lambda_k(0)$ in the time interval $[0, 1]$. For a fixed $s > \frac{d}{2}$, assume that $\partial_x A \in L^2 H^s$, $\lambda_k \in L_k^2 X_k$, and $f_k = f_{1k} + f_{2k}$ with $f_{1k} \in N$ and $f_{2k} \in L^1 L^2$. Then we have*

$$\begin{aligned} \|\lambda_k\|_{L_t^\infty L_x^2}^2 &\lesssim \|\lambda_k(0)\|_{L^2}^2 + \|\partial_x A\|_{L^2 H^s} \|\lambda_k\|_{X_k}^2 \\ &\quad + \|\lambda_k\|_{X_k} \|f_{1k}\|_{N_k} + \|\lambda_k\|_{L^\infty L^2} \|f_{2k}\|_{L^1 L^2}. \end{aligned} \tag{7.3}$$

Proof. For the proof, we refer the readers to Lemma 6.1 in [9]. Here we just replace the condition $A \in Z^{1,s+1}$ in [9] by the assumption $\partial_x A \in L^2 H^s$. \square

Next, we prove the main result of this section, namely the local energy estimates for solutions to (7.2).

Proposition 7.2. (Local energy decay) *Let $d \geq 2$. Assume that the coefficients $h = g - I_d$ and A in (7.2) satisfy*

$$\|h\|_{Y^{s+2}} + \|A\|_{Z^{s+1}} \lesssim \epsilon \tag{7.4}$$

for some $s > \frac{d}{2}$ and $\epsilon > 0$ small. Let λ_k be a solution to (7.2) which is localized at frequency 2^k . Then the following estimate holds:

$$\|\lambda_k\|_{l_k^2 X_k} \lesssim \|\lambda_{0k}\|_{L^2} + \|f_k\|_{l_k^2 N_k}. \tag{7.5}$$

Proof. The proof is closely related to that given in [9, 20, 21]. However, here the metric $g = I_d + h$ and magnetic potential A will satisfy some parabolic equations, so we need to modify the assumptions both on h and A to match our main results.

As an intermediate step in the proof, we will establish a local energy decay bound in a cube $Q \in \mathcal{Q}_l$ with $0 \leq l \leq k$:

$$\begin{aligned} 2^{k-l} \|\lambda_k\|_{L^2 L^2([0,1] \times Q)}^2 &\lesssim \|\lambda_k\|_{L^\infty L^2}^2 + \|f_k\|_{N_k} \|\lambda_k\|_{X_k} \\ &\quad + (2^{-k} + \|A\|_{Z^{1-\delta,s+1}} + \|h\|_{Y^{s+2}}) \|\lambda_k\|_{l_k^2 X_k}^2. \end{aligned} \tag{7.6}$$

The proof of this bound is based on a positive commutator argument using a well chosen multiplier \mathcal{M} . This will be first-order differential operator with smooth coefficients which are localized at frequency $\lesssim 1$. Precisely, we will use a multiplier \mathcal{M} which is a self-adjoint differential operator having the form

$$i2^k \mathcal{M} = a^\alpha(x) \partial_\alpha + \partial_\alpha a^\alpha(x), \tag{7.7}$$

with uniform bounds on a and its derivatives.

Before proving (7.5), we need the following lemma which is used to dismiss the $(g - I)$ contribution to the commutator $[\partial_\alpha g^{\alpha\beta} \partial_\beta, \mathcal{M}]$:

Lemma 7.3. *Let $d \geq 2$ and $s > \frac{d}{2}$. Assume that $h \in \mathbf{Y}^{s+2}$, $A \in Z^{1-\delta, s+1}$ and $\lambda \in l_k^2 X_k$, and let \mathcal{M} be as (7.7). Then we have*

$$\int_0^1 \langle [\partial_\alpha h^{\alpha\beta}_{\leq k} \partial_\beta, \mathcal{M}] \lambda_k, \lambda_k \rangle dt \lesssim \|h\|_{\mathbf{Y}^{s+2}} \|\lambda_k\|_{l_k^2 X_k}^2, \tag{7.8}$$

$$\int_0^1 \text{Re} \langle A^{\alpha}_{<k-4} \partial_\alpha \lambda_k, \mathcal{M} \lambda_k \rangle dt \lesssim \|A\|_{Z^{1-\delta, s+1}} \|\lambda_k\|_{X_k}^2. \tag{7.9}$$

Proof of Lemma 7.3. By (7.7) and direct computations, we get

$$[\partial_\alpha h^{\alpha\beta} \partial_\beta, \mathcal{M}] \approx 2^{-k} [\nabla(h \nabla a + a \nabla h) \nabla + \nabla h \nabla^2 a + h \nabla^3 a].$$

Then it suffices to estimate

$$2^{-k} \int_0^1 \langle (h_{\leq k} \nabla a + a \nabla h_{\leq k}) \nabla \lambda_k, \nabla \lambda_k \rangle dt + 2^{-k} \int_0^1 \langle (\nabla h_{\leq k} \nabla^2 a + h_{\leq k} \nabla^3 a) \lambda_k, \lambda_k \rangle dt.$$

The first integral is estimated by (6.1) and (6.2), while the second integral is bounded by Sobolev embeddings. Hence, the bound (7.8) follows.

For the second bound (7.9), by (7.7) and integration by parts we rewrite the left-hand side of (7.9) and bound it by

$$\begin{aligned} \text{Re} \int_0^1 \langle A^{\alpha}_{<k-4} \partial_\alpha \lambda_k, \mathcal{M} \lambda_k \rangle dt &\lesssim 2^{-k} \int_0^1 \int_{\mathbb{R}^d} |\langle \nabla \rangle A_{<k} \lambda_k \nabla \lambda_k| dx dt \\ &\lesssim \| |\nabla|^{1-\delta} A \|_{L^2 H^{s+\delta}} \|\lambda_k\|_{L^\infty L^2}^2. \end{aligned}$$

This implies the bound (7.9), and hence completes the proof of the lemma. □

Returning to the proof of (7.6), for the self-adjoint multiplier \mathcal{M} we compute

$$\begin{aligned} \frac{d}{dt} \langle \lambda_k, \mathcal{M} \lambda_k \rangle &= 2 \text{Re} \langle \partial_t \lambda_k, \mathcal{M} \lambda_k \rangle \\ &= 2 \text{Re} \langle i \partial_\alpha (g^{\alpha\beta}_{<k-4} \partial_\beta \lambda_k) - 2A^{\alpha}_{<k-4} \partial_\alpha \lambda_k - i f_k, \mathcal{M} \lambda_k \rangle \\ &= i \langle [-\partial_\alpha g^{\alpha\beta}_{<k-4} \partial_\beta, \mathcal{M}] \lambda_k, \lambda_k \rangle + 2 \text{Re} \langle -2A^{\alpha}_{<k-4} \partial_\alpha \lambda_k - i f_k, \mathcal{M} \lambda_k \rangle \end{aligned}$$

We then use the multiplier \mathcal{M} as in [20, 21] so that the following three properties hold:

- (1) Boundedness on frequency 2^k localized functions,

$$\|\mathcal{M}u\|_{L_x^2} \lesssim \|u\|_{L_x^2}.$$

- (2) Boundedness in X ,

$$\|\mathcal{M}u\|_X \lesssim \|u\|_X.$$

(3) Positive commutator,

$$i \langle [-\partial_\alpha g_{<k-4}^{\alpha\beta} \partial_\beta, \mathcal{M}]u, u \rangle \gtrsim 2^{k-l} \|u\|_{L^2_{t,x}(\{0,1\} \times \mathcal{Q})}^2 - O(2^{-k} + \|h\|_{\mathbf{Y}^{s+2}}) \|u\|_{L^2_k X_k}^2.$$

If these three properties hold for $u = \lambda_k$, then by (7.9) and (7.4) the bound (7.6) follows.

We first do this when the Fourier transform of the solution λ_k is restricted to a small angle

$$\text{supp } \widehat{\lambda}_k \subset \{|\xi| \lesssim \xi_1\}. \tag{7.10}$$

Without loss of generality due to translation invariance, $\mathcal{Q} = \{|x_j| \leq 2^l : j = 1, \dots, d\}$, and we set m to be a smooth, bounded, increasing function such that $m'(s) = \varphi^2(s)$ where φ is a Schwartz function localized at frequencies $\lesssim 1$, and $\varphi \approx 1$ for $|s| \leq 1$. We rescale m and set $m_l(s) = m(2^{-l}s)$. Then, we fix

$$\mathcal{M} = \frac{1}{i2^k} (m_l(x_1) \partial_1 + \partial_1 m_l(x_1)).$$

The properties (1) and (2) are immediate due to the frequency localization of $u = \lambda_k$ and m_l as well as the boundedness of m_l . By (7.8) it suffices to consider the property (3) for the operator

$$-\Delta = -\partial_\alpha g_{<k-4}^{\alpha\beta} \partial_\beta + \partial_\alpha h_{<k-4}^{\alpha\beta} \partial_\beta.$$

This yields

$$i2^k [-\Delta, \mathcal{M}] = -2^{-l+2} \partial_1 \varphi^2(2^{-l}x_1) \partial_1 + O(1),$$

and hence

$$i2^k \langle [-\Delta, \mathcal{M}] \lambda_k, \lambda_k \rangle = 2^{-l+2} \|\varphi(2^{-l}x_1) \partial_1 \lambda_k\|_{L^2 L^2}^2 + O(\|\lambda_k\|_{L^2 L^2}^2)$$

Utilizing our assumption (7.10), it follows that

$$2^{k-l} \|\varphi(2^{-l}x_1) \lambda_k\|_{L^2 L^2}^2 \lesssim i \langle [-\Delta, \mathcal{M}] \lambda_k, \lambda_k \rangle + 2^{-k} O(\|\lambda_k\|_{L^2 L^2}^2)$$

which yields (3) when combined with (7.8).

We proceed to reduce the problem to the case when (7.10) holds. We let $\{\theta_j(\omega)\}_{j=1}^d$ be a partition of unity,

$$\sum_j \theta_j(\omega) = 1, \quad \omega \in \mathbb{S}^{d-1},$$

where $\theta_j(\omega)$ is supported in a small angle about the j -th coordinate axis. Then, we can set $\lambda_{k,j} = \Theta_{k,j} \lambda_k$ where

$$\mathcal{F} \Theta_{k,j} \lambda = \theta_j \left(\frac{\xi}{|\xi|} \right) \sum_{k-1 \leq l \leq k+1} \varphi_l(\xi) \widehat{\lambda}(t, \xi).$$

We see that

$$\begin{aligned} & (i \partial_t + \partial_\alpha g_{<k-4}^{\alpha\beta} \partial_\beta) \lambda_{k,j} + 2i A_{<k-4}^\alpha \partial_\alpha \lambda_{k,j} \\ &= \Theta_{k,j} f_k - \partial_\alpha [\Theta_{k,j}, g_{<k-4}^{\alpha\beta}] \partial_\beta \lambda_k - 2i [\Theta_{k,j}, A_{\leq k-4}^\alpha] \partial_\alpha \lambda_k. \end{aligned}$$

By applying \mathcal{M} , suitably adapted to the correct coordinate axis, to $\lambda_{k,j}$ and summing over j , we obtain

$$\begin{aligned} & 2^{k-l} \|\lambda_k\|_{L^2 L^2([0,1] \times Q)}^2 \\ & \lesssim \|\lambda_k\|_{L^\infty L^2}^2 + \sum_{j=1}^d \int_0^1 \langle -\Theta_{k,j} f_k, \mathcal{M} \lambda_{k,j} \rangle ds \\ & \quad + \sum_{j=1}^d \int \langle [\Theta_{k,j}, \partial_\alpha g_{<k-4}^{\alpha\beta} \partial_\beta] \lambda_k + [\Theta_{k,j}, 2i A_{<k-4}^\alpha] \partial_\alpha \lambda_k, \mathcal{M} \lambda_{k,j} \rangle ds \\ & \quad + (2^{-k} + \| |D|^{1-\delta} A \|_{L^2 H^{s+\delta}} + \|h\|_{\mathbf{Y}^{s+2}}) \|\lambda_k\|_{L_k^2 X_k}^2 \\ & \lesssim \|\lambda_k\|_{L^\infty L^2}^2 + \|f_k\|_{N_k} \|\lambda_k\|_{X_k} + (2^{-k} + \| |D|^{1-\delta} A \|_{L^2 H^{s+\delta}} + \|h\|_{\mathbf{Y}^{s+2}}) \|\lambda_k\|_{L_k^2 X_k}^2. \end{aligned}$$

The commutator is done via (6.15) and (6.16). Then (7.6) follows.

Next we use the bound (7.6) to complete the proof of Proposition 7.2. Taking the supremum in (7.6) over $Q \in \mathcal{Q}_l$ and over l , we obtain

$$\begin{aligned} 2^k \|\lambda_k\|_{X^k}^2 & \lesssim \|\lambda_k\|_{L^\infty L^2}^2 + \|f_{1k}\|_{N_k} \|\lambda_k\|_{X_k} + \|f_{2k}\|_{L^1 L^2} \|\lambda_k\|_{L^\infty L^2} \\ & \quad + (2^{-k} + \| |D|^{1-\delta} A \|_{L^2 H^{s+\delta}} + \|h\|_{\mathbf{Y}^{s+2}}) \|\lambda_k\|_{L_k^2 X_k}^2 \\ & \lesssim \|\lambda_k\|_{L^\infty L^2}^2 + \|f_{1k}\|_{N_k} \|\lambda_k\|_{X_k} + \|f_{2k}\|_{L^1 L^2}^2 \\ & \quad + (2^{-k} + \| |D|^{1-\delta} A \|_{L^2 H^{s+\delta}} + \|h\|_{\mathbf{Y}^{s+2}}) \|\lambda_k\|_{L_k^2 X_k}^2. \end{aligned}$$

Combined with (7.3), we get

$$\begin{aligned} \|\lambda_k\|_{X_k}^2 & \lesssim \|\lambda_k(0)\|_{L^2}^2 + \|f_{1k}\|_{N_k}^2 + \|f_{2k}\|_{L^1 L^2}^2 \\ & \quad + (2^{-k} + \| |D|^{1-\delta} A \|_{L^2 H^{s+\delta}} + \|h\|_{\mathbf{Y}^{s+2}}) \|\lambda_k\|_{L_k^2 X_k}^2. \end{aligned} \tag{7.11}$$

We now finish the proof by incorporating the summation over cubes. We let $\{\chi_Q\}$ denote a partition via functions which are localized to frequencies $\lesssim 1$ which are associated to cubes Q of scale $M2^k$. We also assume that $|\nabla^l \chi_Q| \lesssim (2^k M)^{-l}$, $l = 1, 2$. Thus,

$$\begin{aligned} & (i \partial_t + \partial_\alpha g_{<k-4}^{\alpha\beta} \partial_\beta) \chi_Q \lambda_k + 2i A_{<k-4}^\alpha \partial_\alpha \chi_Q \lambda_k \\ &= \chi_Q f_k + [\partial_\alpha g_{<k-4}^{\alpha\beta} \partial_\beta, \chi_Q] \lambda_k + 2i A_{<k-4}^\alpha \partial_\alpha \chi_Q \cdot \lambda_k \end{aligned}$$

Applying (7.3) to $\chi_Q \lambda_k$, we obtain

$$\begin{aligned} & \sum_Q \|\chi_Q \lambda_k\|_{L^\infty L^2}^2 \\ & \lesssim \sum_Q \|\chi_Q \lambda_k(0)\|_{L^2}^2 + \|\partial_x A\|_{L^2 H^s} \sum_Q \|\chi_Q \lambda_k\|_{X_k}^2 \end{aligned}$$

$$\begin{aligned}
 & + \left(\sum_Q \|\chi_Q f_k\|_{N_k}^2 \right)^{1/2} \left(\sum_Q \|\chi_Q \lambda_k\|_{X_k}^2 \right)^{1/2} \\
 & + \sum_Q \|\partial_\alpha g_{<k-4}^{\alpha\beta} \partial_\beta \chi_Q \lambda_k + 2i A_{<k-4}^\alpha \partial_\alpha \chi_Q \cdot \lambda_k\|_{L^1 L^2}^2.
 \end{aligned}$$

But by (7.4) we have

$$\begin{aligned}
 \sum_Q \|\nabla g \nabla \chi_Q \lambda_k\|_{L^1 L^2}^2 & \lesssim \sum_Q \|\nabla g \cdot \nabla \chi_Q \cdot \lambda_k + g \nabla(\nabla \chi_Q \cdot \lambda_k)\|_{L^1 L^2}^2 \\
 & \lesssim (1 + \|D\|^{\sigma_d} h\|_{L^\infty H^{s+1-\sigma_d}}) M^{-2} \sum_Q \|\chi_Q \lambda_k\|_{L^\infty L^2}^2,
 \end{aligned} \tag{7.12}$$

and also

$$\sum_Q \|2i A_{<k-4}^\alpha \partial_\alpha \chi_Q \cdot \lambda_k\|_{L^1 L^2}^2 \lesssim (1 + \|D\|^{1-\delta} A\|_{L^2 H^{s+\delta}}) M^{-2} \sum_Q \|\chi_Q \lambda_k\|_{L^\infty L^2}^2. \tag{7.13}$$

For M sufficiently large, we can bootstrap the commutator terms, and, after a straightforward transition to cubes of scale 2^k rather than $M2^k$, we observe that

$$\|\lambda_k\|_{I_k^2 L^\infty L^2}^2 \lesssim \|\lambda_k(0)\|_{L^2}^2 + \|D\|^{1-\delta} A\|_{L^2 H^{s+\delta}} \|\lambda_k\|_{I_k^2 X_k}^2 + \|f_k\|_{I_k^2 N_k} \|\lambda_k\|_{I_k^2 X_k}. \tag{7.14}$$

We now apply (7.11) to $\chi_Q \lambda_k$, and then by (7.12) and (7.13) we see that

$$\begin{aligned}
 \sum_Q \|\chi_Q \lambda_k\|_{X_k}^2 & \lesssim \|\lambda_k(0)\|_{L^2}^2 + \sum_Q \|\chi_Q f_k\|_{N_k}^2 + M^{-2} \sum_Q \|\chi_Q \lambda_k\|_{X_k}^2 \\
 & + (2^{-k} + \|h\|_{Y^{s+2}} + \|D\|^{1-\delta} A\|_{L^2 H^{s+\delta}}) \sum_Q \|\chi_Q \lambda_k\|_{I_k^2 X_k}^2.
 \end{aligned}$$

For $M \gg 1$, we have

$$M^{-d} \|\lambda_k\|_{I_k^2 X_k}^2 \lesssim \|\lambda_k(0)\|_{L^2}^2 + \|f_k\|_{I_k^2 N_k}^2 + (2^{-k} + \|h\|_{Y^{s+2}} + \|D\|^{1-\delta} A\|_{L^2 H^{s+\delta}}) \|\lambda_k\|_{I_k^2 X_k}^2.$$

By (7.4), for k sufficiently large (depending on M), we may absorb the the last term in the right-hand side into the left, i.e

$$\|\lambda_k\|_{I_k^2 X_k}^2 \lesssim \|\lambda_k(0)\|_{L^2}^2 + \|f_k\|_{I_k^2 N_k}^2.$$

On the other hand, for the remaining bounded range of k , we have

$$\|\lambda\|_{X_k} \lesssim \|\lambda\|_{L^\infty L^2},$$

and then (7.14) and (7.4) give

$$\begin{aligned}
 \|\lambda_k\|_{I_k^2 X_k}^2 & \lesssim \|\lambda_k(0)\|_{L^2}^2 + \|D\|^{1-\delta} A\|_{L^2 H^{s+\delta}} \|\lambda_k\|_{I_k^2 X_k}^2 + \|f_k\|_{I_k^2 N_k} \|\lambda_k\|_{I_k^2 X_k} \\
 & \lesssim \|\lambda_k(0)\|_{L^2}^2 + \|f_k\|_{I_k^2 N_k}^2,
 \end{aligned}$$

which finishes the proof of (7.5). □

7.2. The Full Linear Problem

Here we use the bounds for the paradifferential equation in the previous subsection in order to prove similar bounds for the full equation (7.1).

Proposition 7.4. (Well-posedness) *Let $s > \frac{d}{2}$, $d \geq 2$ and $h = g - I_d$. Assume that the metric g and the magnetic potential A satisfy*

$$\|h\|_{\mathbf{Y}^{s+2}}, \||D|^{1-\delta} A\|_{L^2 H^{s+\delta}} \ll 1.$$

Then the equation (7.1) is well-posed for initial data $\lambda_0 \in H^\sigma$ with $-s \leq \sigma \leq s$, and we have the estimate

$$\|\lambda\|_{l^2 X^\sigma} \lesssim \|\lambda_0\|_{H^\sigma} + \|F\|_{l^2 N^\sigma}. \tag{7.15}$$

Proof. The well-posedness follows in a standard fashion from a similar energy estimate for the adjoint equation. Since the adjoint equation has a similar form, with similar bounds on the coefficients, such an estimate follows directly from (7.15). Thus, we now focus on the proof of the bound (7.15). For λ solving (7.1), we see that λ_k solves

$$\begin{cases} i \partial_t \lambda_k + \partial_\alpha (g_{<k-4}^{\alpha\beta} \partial_\beta \lambda_k) + 2i A_{<k-4}^\alpha \partial_\alpha \lambda_k = F_k + H_k, \\ \lambda_k(0) = \lambda_{0k}, \end{cases}$$

where $F_k := S_k F$ and

$$\begin{aligned} H_k := & -S_k \partial_\alpha (g_{\geq k-4}^{\alpha\beta} \partial_\beta \lambda) - \partial_\alpha [S_k, g_{<k-4}^{\alpha\beta}] \partial_\beta \lambda - 2i [S_k, A_{<k-4}^\alpha] \partial_\alpha \lambda \\ & - 2i S_k (A_{\geq k-4}^\alpha \partial_\alpha \lambda). \end{aligned}$$

If we apply Proposition 7.2 to each of these equations, we see that

$$\|\lambda_k\|_{l^2 X^\sigma}^2 \lesssim \|\lambda_{0k}\|_{H^\sigma}^2 + \|F_k\|_{l^2 N^\sigma}^2 + \|H_k\|_{l^2 N^\sigma}^2.$$

We claim that

$$\sum_k \|H_k\|_{l^2 N^\sigma}^2 \lesssim (\|h\|_{\mathbf{Y}^{s+2}} + \|\partial_x A\|_{L^2 H^s})^2 \|\lambda\|_{l^2 X^\sigma}^2, \text{ for } -s \leq \sigma \leq s. \tag{7.16}$$

Indeed, the bound for the terms in H_k follows from (6.9), (6.15), (6.16) and (6.10), respectively. Then by the above two bounds and the smallness of h and A , we obtain the estimate (7.15). □

7.3. The Linearized Problem

Here we consider the linearized equation

$$\begin{cases} i \partial_t \Lambda + \partial_\alpha (g^{\alpha\beta} \partial_\beta \Lambda) + 2i A^\alpha \partial_\alpha \Lambda = F + G, \\ \Lambda(0) = \Lambda_0, \end{cases} \quad (7.17)$$

where

$$G = -\nabla(\mathcal{G}\nabla\lambda) - 2i\mathcal{A}^\alpha \partial_\alpha \lambda,$$

and we prove the following:

Proposition 7.5. *Let $s > \frac{d}{2}$, $\frac{d}{2} - 2 < \sigma \leq s - 2$, $d \geq 2$ and $h = g - I_d \in \mathbf{Y}^{s+2}$, assume that Λ is a solution of (7.17), the metric g and A satisfy*

$$\|h\|_{\mathbf{Y}^{s+2}}, \| |D|^{1-\delta} A \|_{L^2 H^{s+\delta}} \ll 1.$$

Then we have the estimate

$$\|\Lambda\|_{J^2 X^\sigma} \lesssim \|\Lambda_0\|_{H^\sigma} + \|F\|_{J^2 N^\sigma} + (\|\mathcal{G}\|_{Z^{\sigma_d, \sigma+2}} + \|\mathcal{A}\|_{Z^{\delta_d, \sigma+1}}) \|\lambda\|_{J^2 X^s}. \quad (7.18)$$

Proof. For Λ solving (7.17), we see that Λ_k solves

$$\begin{cases} i \partial_t \Lambda_k + \partial_\alpha (g_{\geq k-4}^{\alpha\beta} \partial_\beta \Lambda_k) + 2i A_{< k-4}^\alpha \partial_\alpha \Lambda_k = F_k + G_k + H_k, \\ \Lambda_k(0) = \Lambda_{0k}, \end{cases}$$

where

$$G_k = -S_k(\nabla(\mathcal{G}\nabla\lambda) - 2i\mathcal{A}^\alpha \partial_\alpha \lambda),$$

$$\begin{aligned} H_k &= -S_k \partial_\alpha (g_{\geq k-4}^{\alpha\beta} \partial_\beta \Lambda) - \partial_\alpha [S_k, g_{< k-4}^{\alpha\beta}] \partial_\beta \Lambda - 2i [S_k, A_{< k-4}^\alpha] \partial_\alpha \Lambda \\ &\quad - 2i S_k (A_{\geq k-4}^\alpha \partial_\alpha \Lambda). \end{aligned}$$

The proof of (7.18) is similar to that of (7.16). Here it suffices to prove

$$\sum_k \|G_k\|_{J^2 N^\sigma}^2 \lesssim \|\mathcal{G}\|_{Z^{\sigma_d, \sigma+2}}^2 \|\lambda\|_{J^2 X^s}^2 + \|\mathcal{A}\|_{Z^{\delta_d, \sigma+1}}^2 \|\lambda\|_{J^2 X^s}^2.$$

Indeed, the bound for the terms in G_k follows from (6.9), (6.4), (6.10) and (6.14). This completes the proof of the Lemma. \square

8. Well-Posedness in the Good Gauge

In this section we use the parabolic results in Section 5, the multilinear estimates in Section 6 and the linear local energy decay bounds in Section 7 in order to prove the good gauge formulation of our main result, namely Theorem 2.5.

8.1. The Iteration Scheme: Uniform Bounds

Here we seek to construct solutions to (2.28) iteratively, based on the scheme

$$\begin{cases} i \partial_t \lambda^{(n+1)} + \partial_\alpha (g^{(n)\alpha\beta} \partial_\beta \lambda^{(n+1)}) + 2i A^{(n)\alpha} \partial_\alpha \lambda^{(n+1)} = F^{(n)}, \\ \lambda^{(n+1)}(0) = \lambda_0, \end{cases} \tag{8.1}$$

with the trivial initialization

$$\lambda^{(0)} = 0,$$

where the nonlinearities $F^{(n)}$ are the following F with $(\lambda, h, A) = (\lambda^{(n)}, h^{(n)}, A^{(n)})$

$$\begin{aligned} F = & \partial_\mu (g^{\mu\nu} \partial_\nu \lambda_{\alpha\beta}) - \nabla^\sigma \nabla_\sigma \lambda_{\alpha\beta} + i V^\sigma \nabla_\sigma \lambda_{\alpha\beta} - i \nabla_\sigma A^\sigma \lambda_{\alpha\beta} + i \lambda_\alpha^\gamma \nabla_\beta V_\gamma + i \lambda_\beta^\gamma \nabla_\alpha V_\gamma \\ & + (B + A_\sigma A^\sigma - V_\sigma A^\sigma) \lambda_{\alpha\beta} + \psi \operatorname{Re}(\lambda_{\alpha\delta} \bar{\lambda}_\beta^\delta) - R_{\alpha\sigma\beta\delta} \lambda^{\sigma\delta} - \lambda_{\alpha\mu} \bar{\lambda}_\sigma^\mu \lambda_{\beta\sigma}^\sigma, \end{aligned} \tag{8.2}$$

and $\mathcal{S}^{(n)} = (h^{(n)}, A^{(n)})$ are the solutions of parabolic system (2.29) with $\lambda = \lambda^{(n)}$ and initial data

$$h^{(n)}(0, x) = h_0(x), \quad A^{(n)}(0, x) = A_0(x). \tag{8.3}$$

We assume that (λ_0, h_0) is small in $H^s \times \mathbf{Y}^{s+2}$. Due to the above trivial initialization for $\lambda^{(0)}$, we also inductively assume that

$$\|\lambda^{(n)}\|_{L^2 X^s} \leq C \|\lambda_0\|_{H^s}, \tag{8.4}$$

where C is a large constant.

Applying the parabolic estimates (5.1) to (2.29) with $\lambda = \lambda^{(n)}$ and initial data (8.3) at each step, we obtain

$$\|\mathcal{S}^{(n)}\|_{\mathcal{E}^s} \lesssim \|(h_0, A_0)\|_{\mathcal{E}_0^s} + \|\lambda^{(n)}\|_{L^2 Z^s} \lesssim \|(h_0, A_0)\|_{\mathcal{E}_0^s} + \|\lambda_0\|_{H^s} \lesssim \epsilon_0. \tag{8.5}$$

In order to estimate $\lambda^{(n+1)}$, we bound the nonlinear terms in $F^{(n)}$ first. In the computations we would omit the superscript (n) . More precisely, for the first three terms in (8.2), by covariant derivatives (2.2) and $V^\gamma = g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma$ we have the form

$$\partial_\mu (g^{\mu\nu} \partial_\nu \lambda_{\alpha\beta}) - \nabla^\sigma \nabla_\sigma \lambda_{\alpha\beta} + i V^\sigma \nabla_\sigma \lambda_{\alpha\beta} \approx \nabla h \nabla \lambda + \nabla h \nabla h \lambda.$$

Then the first term $\nabla h \nabla \lambda$ is estimated using (6.1) and (6.9), the second term $\nabla h \nabla h \lambda$ is estimated using (6.7) with its $A = \nabla h$. We obtain

$$\|\nabla h \nabla \lambda + \nabla h \nabla h \lambda\|_{L^2 N^s} \lesssim \|h\|_{\mathbf{Y}^{s+2}} \|\lambda\|_{L^2 X^s} + \|h\|_{Z^{s+1}}^2 \|\lambda\|_{Z^s}.$$

For the fourth to seventh terms in (8.2), we have the expression

$$\begin{aligned} & -i \nabla_\sigma A^\sigma \lambda_{\alpha\beta} + i \lambda_\alpha^\gamma \nabla_\beta V_\gamma + i \lambda_\beta^\gamma \nabla_\alpha V_\gamma + (B + A_\sigma A^\sigma - V_\sigma A^\sigma) \lambda_{\alpha\beta} \\ & \approx (\nabla^2 h + \nabla A) \lambda + (\nabla h + A)^2 \lambda. \end{aligned}$$

Then these two terms are estimated using (6.6) and (6.7) respectively. We obtain

$$\|(\nabla^2 h + \nabla A) \lambda + (\nabla h + A)^2 \lambda\|_{L^2 N^s} \lesssim (1 + \|\mathcal{S}\|_{\mathcal{E}^s}) \|\mathcal{S}\|_{\mathcal{E}^s} \|\lambda\|_{Z^s}.$$

For the last three terms in (8.2), by (2.8) we have

$$\psi \operatorname{Re}(\lambda_{\alpha\delta} \bar{\lambda}_{\beta}^{\delta}) - R_{\alpha\sigma\beta\delta} \lambda^{\sigma\delta} - \lambda_{\alpha\mu} \bar{\lambda}_{\sigma}^{\mu} \lambda_{\beta}^{\sigma} \approx \lambda^3.$$

Using (6.13) we obtain

$$\|\lambda^3\|_{L^2 N^s} \lesssim \|\lambda\|_{Z^s}^3.$$

Hence, by the above estimates, (8.5) and (8.4) we bound the $F^{(n)}$ by

$$\|F^{(n)}\|_{L^2 N^s} \lesssim (1 + \|\mathcal{S}^{(n)}\|_{\mathcal{E}^s}) \|\mathcal{S}^{(n)}\|_{\mathcal{E}^s} \|\lambda^{(n)}\|_{L^2 X^s} \lesssim \epsilon_0 \|\lambda_0\|_{H^s}.$$

Now applying at each step the local energy bound (7.15) with $\sigma = s$ we obtain the estimate

$$\|\lambda^{(n+1)}\|_{L^2 X^s} \lesssim \|\lambda_0\|_{H^s} + \|F^{(n)}\|_{L^2 N^s} \lesssim \|\lambda_0\|_{H^s} + C\epsilon_0 \|\lambda_0\|_{H^s} \leq C\|\lambda_0\|_{H^s}, \tag{8.6}$$

which closes our induction.

8.2. The Iteration Scheme: Weak Convergence

Here we prove that our iteration scheme converges in the weaker H^{s-2} topology. We denote the differences by

$$\begin{aligned} \Lambda^{(n+1)} &= \lambda^{(n+1)} - \lambda^{(n)}, \\ \delta\mathcal{S}^{(n+1)} &= (\mathcal{G}^{(n+1)}, \mathcal{A}^{(n+1)}, \mathcal{B}^{(n+1)}) = \mathcal{S}^{(n+1)} - \mathcal{S}^{(n)} \end{aligned}$$

Then from (8.1) we obtain the system

$$\begin{cases} i \partial_t \Lambda^{(n+1)} + \partial_{\alpha}(g^{(n)\alpha\beta} \partial_{\beta} \Lambda^{(n+1)}) + 2i A^{(n)\alpha} \partial_{\alpha} \Lambda^{(n+1)} = F^{(n)} - F^{(n-1)} + G^{(n)}, \\ \Lambda^{(n+1)}(0, x) = 0, \end{cases}$$

where the nonlinearities $G^{(n)}$ have the form

$$G^{(n)} = -\partial_{\alpha}(\mathcal{G}^{(n)} \partial_{\beta} \lambda^{(n)}) - 2i \mathcal{A}^{(n)\alpha} \partial_{\alpha} \lambda^{(n)},$$

By (5.4) we obtain

$$\|\delta\mathcal{S}^{(n)}\|_{\mathcal{E}^{s-2}} \lesssim \|\Lambda^{(n)}\|_{L^2 X^{s-2}}. \tag{8.7}$$

Applying (7.18) with $\sigma = s - 2$ for the $\Lambda^{(n+1)}$ equation we have

$$\|\Lambda^{(n+1)}\|_{L^2 X^{s-2}} \lesssim \|F^{(n)} - F^{(n-1)}\|_{L^2 N^{s-2}} + (\|\mathcal{G}^{(n)}\|_{Z^{\sigma_d, s}} + \|\mathcal{A}^{(n)}\|_{Z^{\delta_d, s-1}}) \|\lambda^{(n)}\|_{L^2 X^s}.$$

For the nonlinear terms $F^{(n)} - F^{(n-1)}$, using (6.3), (6.9), (6.12), (6.11) and (6.13) we have

$$\begin{aligned} \|F^{(n)} - F^{(n-1)}\|_{L^2 N^{s-2}} &\lesssim (1 + \|(\mathcal{S}^{(n)}, \mathcal{S}^{(n-1)})\|_{\mathcal{E}^s})^N (\|\delta\mathcal{S}^{(n)}\|_{\mathcal{E}^{s-2}} \|\lambda^{(n)}, \lambda^{(n-1)}\|_{L^2 X^s} \\ &\quad + \|(\mathcal{S}^{(n)}, \mathcal{S}^{(n-1)})\|_{\mathcal{E}^s} \|\Lambda^{(n)}\|_{L^2 X^{s-2}}). \end{aligned}$$

Then by (8.7) and the uniform bounds (8.5), (8.6) we bound the right hand side above by

$$\begin{aligned} \|\Lambda^{(n+1)}\|_{L^2 X^{s-2}} &\lesssim (1 + \|\mathcal{S}_0\|_{\mathcal{E}_0^s} + \|\lambda_0\|_{H^s})^N \\ &\quad \cdot [\|\Lambda^{(n)}\|_{L^2 X^{s-2}} \|\lambda_0\|_{H^s} + (\|\mathcal{S}_0\|_{\mathcal{E}_0^s} + \|\lambda_0\|_{H^s}) \|\Lambda^{(n)}\|_{L^2 X^{s-2}}] \\ &\ll \|\Lambda^{(n)}\|_{L^2 X^{s-2}}. \end{aligned}$$

This implies that our iterations $\lambda^{(n)}$ converge in $L^2 X^{s-2}$ to some function λ . Furthermore, by the uniform bound (8.6) it follows that

$$\|\lambda\|_{L^2 X^s} \lesssim \|\lambda_0\|_{H^s}. \tag{8.8}$$

Interpolating, it follows that $\lambda^{(n)}$ converges to λ in $L^2 X^{s-\epsilon}$ for all $\epsilon > 0$. This allows us to conclude that the auxiliary functions $\mathcal{S}^{(n)}$ associated to $\lambda^{(n)}$ converge to the functions \mathcal{S} associated to λ , and also to pass to the limit and conclude that λ solves the (SMCF) equation (2.28). Moreover, we have the bound for \mathcal{S}

$$\|\mathcal{S}\|_{\mathcal{E}^s} \lesssim \|\mathcal{S}_0\|_{\mathbf{Y}_0^{s+2}} + \|\lambda_0\|_{H^s}. \tag{8.9}$$

Thus we have established the existence part of our main theorem.

8.3. Uniqueness via Weak Lipschitz Dependence

Consider the difference of two solutions

$$(\Lambda, \delta\mathcal{S}) = (\lambda^{(1)} - \lambda^{(2)}, \mathcal{S}^{(1)} - \mathcal{S}^{(2)}).$$

The Λ solves an equation of this form

$$\begin{cases} i\partial_t \Lambda + \partial_\alpha (g^{(1)\alpha\beta} \partial_\beta \Lambda) + 2iA^{(1)\alpha} \partial_\alpha \Lambda = F^{(1)} - F^{(2)} + G, \\ \Lambda(0, x) = \lambda_0^{(1)}(x) - \lambda_0^{(2)}(x), \end{cases}$$

where the nonlinearity G is

$$G = -\partial_\alpha (G \partial_\beta \lambda^{(2)}) - 2iA^\alpha \partial_\alpha \lambda^{(2)}.$$

By (5.4) we have

$$\|\delta\mathcal{S}\|_{\mathcal{E}^{s-2}} \lesssim \|\delta\mathcal{S}_0\|_{\mathcal{H}^{s-2}} + \|\Lambda\|_{L^2 X^{s-2}}.$$

Applying (7.18) with $\sigma = s - 2$ to the Λ equation, we obtain the estimate

$$\begin{aligned} \|\Lambda\|_{L^2 X^{s-2}} &\lesssim \|\Lambda_0\|_{H^{s-2}} + \|F^{(1)} - F^{(2)}\|_{L^2 N^{s-2}} + (\|\mathcal{G}\|_{Z^{\sigma d, s}} + \|\mathcal{A}\|_{Z^{\delta d, s-1}}) \|\lambda^{(2)}\|_{L^2 X^s} \\ &\lesssim \|\Lambda_0\|_{H^{s-2}} + C\|(\lambda_0^{(1)}, \lambda_0^{(2)})\|_{H^s} \|(\Lambda, \delta\mathcal{S})\|_{L^2 X^{s-2} \times \mathcal{E}^{s-2}}. \end{aligned}$$

Then, by the above bound for $\delta\mathcal{S}$, we further have

$$\|\Lambda\|_{L^2 X^{s-2}} \lesssim \|\Lambda_0\|_{H^{s-2}} + C\|(\lambda_0^{(1)}, \lambda_0^{(2)})\|_{H^s} (\|\delta\mathcal{S}_0\|_{\mathcal{H}^{s-2}} + \|\Lambda\|_{L^2 X^{s-2}})$$

Since the initial data $\lambda_0^{(1)}$ and $\lambda_0^{(2)}$ are sufficiently small, we obtain

$$\|\Lambda\|_{L^2 X^{s-2}} \lesssim \|\Lambda_0\|_{H^{s-2}} + \|\delta\mathcal{S}_0\|_{\mathcal{H}^{s-2}}. \tag{8.10}$$

This gives the weak Lipschitz dependence, as well as the uniqueness of solutions for (2.28).

8.4. Frequency Envelope Bounds

Here we prove a stronger frequency envelope version of estimate (8.8).

Proposition 8.1. *Let $\lambda \in l^2 X^s$, $\mathcal{S} \in \mathcal{E}^s$ be small data solution to (2.28)–(2.29), which satisfies (8.8) and (8.9). Let $\{p_{0k}\}$, $\{s_{0k}\}$ be admissible frequency envelopes for the initial data $\lambda_0 \in H^s$ and $\mathcal{S}_0 \in \mathcal{E}_0^s$. Then $\{p_{0k} + s_{0k}\}$ is also frequency envelope for (λ, \mathcal{S}) in $l^2 X^s \times \mathcal{E}^s$.*

Proof. Let p_k and s_k be the admissible frequency envelopes for solution $(\lambda, \mathcal{S}) \in l^2 X^s \times \mathcal{E}^s$. Applying S_k to the modified Schrödinger equation in (2.28), we obtain the paradifferential equation

$$\begin{cases} i \partial_t \lambda_k + \partial_\alpha (g_{<k-4}^{\alpha\beta} \partial_\beta \lambda_k) + 2i A_{<k-4}^\alpha \partial_\alpha \lambda_k = F_k + J_k, \\ \lambda(0, x) = \lambda_0(x), \end{cases}$$

where

$$\begin{aligned} J_k &= -S_k \partial_\alpha (g_{\geq k-4}^{\alpha\beta} \partial_\beta \lambda) - [S_k, \partial_\alpha g_{<k-4}^{\alpha\beta} \partial_\beta] \lambda \\ &\quad - 2i [S_k, A_{<k-4}^\alpha] \partial_\alpha \lambda - 2i S_k [A_{\geq k-4}^\alpha \partial_\alpha \lambda_k], \end{aligned}$$

and $\mathcal{S} = (h, A)$ is the solution to the parabolic system (2.29). We estimate $\lambda_k = S_k \lambda$ using Proposition 7.4,

$$\|\lambda_k\|_{l^2 X^s} \lesssim p_{0k} + \|F_k\|_{l^2 N^s} + \|J_k\|_{l^2 N^s}.$$

By Proposition 6.2, Lemma 6.1 and Lemma 6.3 we bound the nonlinear terms by

$$\|F_k\|_{l^2 N^s} + \|J_k\|_{l^2 N^s} \lesssim (1 + \|\mathcal{S}\|_{\mathcal{E}^s} + \|\lambda\|_{l^2 X^s})^N (\|\mathcal{S}\|_{\mathcal{E}^s} p_k + s_k \|\lambda\|_{l^2 X^s}).$$

Then by (8.9), (8.8), (5.10) and the smallness of initial data we obtain

$$\|\lambda_k\|_{l^2 X^s} \lesssim p_{0k} + \epsilon p_k + \epsilon (s_{0k} + p_k) \lesssim p_{0k} + s_{0k} + \epsilon p_k.$$

For metric $g = I_d + h$, by (5.2) we also have

$$\|\mathcal{S}_k\|_{\mathcal{E}^s} \lesssim s_{0k} + \epsilon p_k.$$

From the definition of frequency envelope (3.3), these two bounds imply

$$p_k + s_k \lesssim p_{0k} + s_{0k}.$$

and conclude the proof. □

8.5. Continuous Dependence on the Initial Data

Here we show that the map $(\lambda_0, \mathcal{S}_0) \rightarrow (\lambda, \mathcal{S})$ is continuous from $H^s \times \mathcal{E}_0^s$ into $L^2 X^s \times \mathcal{E}^s$. By (5.3), it suffices to prove $(\lambda_0, \mathcal{S}_0) \rightarrow \lambda$ is continuous from $H^s \times \mathcal{E}_0^s$ to $L^2 X^s$.

Suppose that $(\lambda_0^{(n)}, \mathcal{S}_0^{(n)}) \rightarrow (\lambda_0, \mathcal{S}_0)$ in $H^s \times \mathcal{E}_0^s$. Denote by $(p_{0k}^{(n)}, s_{0k}^{(n)})$, respectively (p_{0k}, s_{0k}) the frequency envelopes associated to $(\lambda_0^{(n)}, \mathcal{S}_0^{(n)})$, respectively $(\lambda_0, \mathcal{S}_0)$, given by (3.3). If $(\lambda_0^{(n)}, \mathcal{S}_0^{(n)}) \rightarrow (\lambda_0, \mathcal{S}_0)$ in $H^s \times \mathcal{E}_0^s$ then $(p_{0k}^{(n)}, s_{0k}^{(n)}) \rightarrow (p_{0k}, s_{0k})$ in l^2 . Then for each $\epsilon > 0$ we can find some N_ϵ so that

$$\|p_{0, > N_\epsilon}^{(n)}\|_{l^2} + \|s_{0, > N_\epsilon}^{(n)}\|_{l^2} \leq \epsilon, \text{ for all } n.$$

By Proposition 8.1 we obtain that

$$\|\lambda_{> N_\epsilon}^{(n)}\|_{l^2 X^s} \leq \epsilon, \text{ for all } n. \tag{8.11}$$

To compare $\lambda^{(n)}$ with λ we use (8.10) for low frequencies and (8.11) for the high frequencies,

$$\begin{aligned} \|\lambda^{(n)} - \lambda\|_{l^2 X^s} &\lesssim \|S_{< N_\epsilon}(\lambda^{(n)} - \lambda)\|_{l^2 X^s} + \|S_{> N_\epsilon} \lambda^{(n)}\|_{l^2 X^s} + \|S_{> N_\epsilon} \lambda\|_{l^2 X^s} \\ &\lesssim 2^{2N_\epsilon} \|S_{< N_\epsilon}(\lambda^{(n)} - \lambda)\|_{l^2 X^{s-2}} + 2\epsilon \\ &\lesssim 2^{2N_\epsilon} (\|S_{< N_\epsilon}(\lambda_0^{(n)} - \lambda_0)\|_{H^{s-2}} + \|S_{< N_\epsilon}(\mathcal{S}_0^{(n)} - \mathcal{S}_0)\|_{\mathcal{H}^{s-2}}) + 2\epsilon. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain

$$\limsup_{n \rightarrow \infty} \|\lambda^{(n)} - \lambda\|_{l^2 X^s} \lesssim \epsilon.$$

Letting $\epsilon \rightarrow 0$ we obtain

$$\lim_{n \rightarrow 0} \|\lambda^{(n)} - \lambda\|_{l^2 X^s} = 0,$$

which completes the desired result.

8.6. Higher Regularity

Here we prove that the solution (λ, \mathcal{S}) satisfies the bound

$$\|(\lambda, \mathcal{S})\|_{l^2 X^\sigma \times \mathcal{E}^\sigma} \lesssim \|\lambda_0\|_{H^\sigma} + \|\mathcal{S}_0\|_{\mathcal{E}_0^\sigma}, \quad \sigma \geq s, \tag{8.12}$$

whenever the right hand side is finite.

The proof of (8.12) is similar to that in [9, Section 7.6]. Here we simply repeat this process. Differentiating the original Schrödinger equation (2.28), and then using Proposition 7.4, Lemma 6.1 and Proposition 6.2 we easily obtain

$$\|\nabla \lambda\|_{l^2 X^s} \lesssim \|\nabla \lambda_0\|_{H^s} + \|(\nabla \lambda, \nabla \mathcal{S})\|_{l^2 X^s \times \mathcal{E}^s} \|(\lambda, \mathcal{S})\|_{l^2 X^s \times \mathcal{E}^s} (1 + \|(\lambda, \mathcal{S})\|_{l^2 X^s \times \mathcal{E}^s})^N.$$

For the parabolic equations, by (5.3) we obtain

$$\|\nabla \mathcal{S}\|_{\mathcal{E}^s} \lesssim \|\nabla \mathcal{S}_0\|_{\mathcal{E}_0^s} + \|\lambda\|_{l^2 X^s} \|\nabla \lambda\|_{l^2 X^s}.$$

Hence, by (8.8) and (8.9), these imply (8.12) with $\sigma = s + 1$. Inductively, we can further obtain (8.12) for any $\sigma \geq s$.

8.7. The Compatibilities Conditions

As part of our derivation of the (SMCF) equations (2.28) for the second fundamental form λ in the good gauge, coupled with the parabolic system (2.29), we have seen that the compatibility conditions are described by the equations (2.9), (2.8), (2.10), (2.12), (2.20) and (2.17). However, our proof of the well-posedness result for the Schrödinger evolution (2.28) does not a priori guarantee that these constraints hold. Here we rectify this omission:

Lemma 8.2. (Constraint conditions) *Assume that $\lambda \in C[0, T; H^s]$ solves the SMCF equation (2.28) coupled with the parabolic system (2.29). Then the relations (2.9), (2.8), (2.10), (2.12), (2.20) and (2.17) hold.*

Proof. To shorten the notations, we define

$$\begin{aligned} T_{\alpha\beta}^1 &= \text{Ric}_{\alpha\beta} - \widetilde{\text{Ric}}_{\alpha\beta}, & \widetilde{\text{Ric}}_{\alpha\beta} &:= \text{Re}(\lambda_{\alpha\beta}\bar{\psi} - \lambda_{\alpha\sigma}\bar{\lambda}^\sigma_\beta), \\ T_{\sigma\gamma\alpha\beta}^2 &= R_{\sigma\gamma\alpha\beta} - \widetilde{R}_{\sigma\gamma\alpha\beta}, & \widetilde{R}_{\sigma\gamma\alpha\beta} &:= \text{Re}(\lambda_{\gamma\beta}\bar{\lambda}_{\sigma\alpha} - \lambda_{\gamma\alpha}\bar{\lambda}_{\sigma\beta}), \\ T_{\alpha\beta,\gamma}^3 &= \nabla_\alpha^A \lambda_{\beta\gamma} - \widetilde{\nabla}_\beta^A \lambda_{\alpha\gamma}, \\ T_{\alpha\beta}^4 &= \mathbf{F}_{\alpha\beta} - \widetilde{\mathbf{F}}_{\alpha\beta}, & \mathbf{F}_{\alpha\beta} &:= \nabla_\alpha A_\beta - \nabla_\beta A_\alpha, \quad \widetilde{\mathbf{F}}_{\alpha\beta} := \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\gamma\beta}), \\ T_\alpha^5 &= \mathbf{F}_{0\alpha} - \widetilde{\mathbf{F}}_{0\alpha}, & \mathbf{F}_{0\alpha} &:= \partial_t A_\alpha - \nabla_\alpha B, \quad \widetilde{\mathbf{F}}_{0\alpha} := \text{Re}(\lambda_\alpha^\gamma \bar{\partial}_\gamma^A \bar{\psi}) + \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_{\gamma\sigma}) V^\sigma. \end{aligned}$$

Here T^3 and T^4 are antisymmetric, T^1 is symmetric and T^2 inherits all the linear symmetries of the curvature tensor.

Our goal is to show that all these functions vanish, knowing that they vanish at the initial time. We will prove this by showing that they solve a coupled linear homogeneous evolution system of the form

$$\left\{ \begin{aligned} (\partial_t - \Delta_g) T_\alpha^{1,\beta} &= \lambda^2 T^4 + T^1 \nabla V + V \nabla T^1 + T^3 \nabla \lambda + \lambda \nabla T^3, \\ \nabla_\delta T_{\sigma\gamma\alpha\beta}^2 + \nabla_\sigma T_{\gamma\delta\alpha\beta}^2 + \nabla_\gamma T_{\delta\sigma\alpha\beta}^2 &= T^1 \lambda, \\ \nabla^\sigma T_{\sigma\gamma\alpha\beta}^2 &= \nabla_\alpha T_{\gamma\beta}^1 - \nabla_\beta T_{\gamma\alpha}^1 + T^1 \lambda, \\ (i \partial_t^B - \Delta_g^A) T_{\alpha\beta,\gamma}^3 &= \lambda T^5 + T^3 (\nabla V + \lambda^2 + R) + (\nabla^A \lambda + \lambda V) (T^1 + T^2 + T^4) \\ &\quad + \lambda \nabla (T^2 + T^4) + V \nabla T^3 \\ (\partial_t - \Delta_g) T_{\alpha\beta}^4 &= \text{Ric } T^4 + \nabla^A \lambda T^3 + V \lambda T^3, \\ T_\alpha^5 &= \nabla^\sigma T_{\sigma\alpha}^4 + T_{\alpha\delta}^1 A^\delta. \end{aligned} \right.$$

Then standard energy estimates show that zero is the only solution for this system.

The formulas for T^5 are obtained directly by the equations for A (2.21) and heat gauge $B = \nabla^\alpha A_\alpha$. It remains to derive the system for (T^1, \dots, T^4) .

The equation for T^1 . This has the form

$$(\partial_t - \Delta_g) T_\alpha^{1,\beta} = \lambda^2 T^4 + T^1 \nabla V + V \nabla T^1 + T^3 \nabla \lambda + \lambda \nabla T^3.$$

Using the parabolic equations for h we recover the representation of $\partial_t g$ as

$$\partial_t g_{\mu\nu} = 2G_{\mu\nu} - 2T_{\mu\nu}^1, \quad G_{\mu\nu} := \text{Im}(\psi \bar{\lambda}_{\mu\nu}) + \frac{1}{2} \nabla_\mu V_\nu + \frac{1}{2} \nabla_\nu V_\mu, \quad (8.13)$$

and obtain

$$\partial_t \Gamma_{\alpha\beta}^\gamma = \nabla_\alpha G_\beta^\gamma + \nabla_\beta G_\alpha^\gamma - \nabla^\gamma G_{\alpha\beta} - (\nabla_\alpha T_\beta^{1,\gamma} + \nabla_\beta T_\alpha^{1,\gamma} - \nabla^\gamma T_{\alpha\beta}^1). \tag{8.14}$$

We then use the two formulas to write

$$\begin{aligned} \partial_t \text{Ric}_\alpha^\beta &= \partial_t g^{\mu\nu} R_{\mu\alpha\nu}^\beta - g^{\mu\nu} \partial_t R_{\mu\nu\alpha}^\beta \\ &= (-2G^{\mu\nu} + 2T^{1,\mu\nu}) R_{\mu\alpha\nu}^\beta + g^{\mu\nu} (\nabla_\alpha \partial_t \Gamma_{\mu\nu}^\beta - \nabla_\nu \partial_t \Gamma_{\mu\alpha}^\beta) \\ &= 2T^{1,\mu\nu} R_{\mu\alpha\nu}^\beta - 2G^{\mu\nu} R_{\mu\alpha\nu}^\beta \\ &\quad + \nabla_\alpha [2\nabla^\mu (G_\mu^\beta - T_\mu^{1,\beta}) - \nabla^\beta (G_\mu^\mu - T_\mu^{1,\mu})] \\ &\quad - \nabla^\mu [\nabla_\mu (G_\alpha^\beta - T_\alpha^{1,\beta}) + \nabla_\alpha (G_\mu^\beta - T_\mu^{1,\beta}) - \nabla^\beta (G_{\mu\alpha} - T_{\mu\alpha}^1)] \\ &= \nabla^\mu \nabla_\mu T_\alpha^{1,\beta} + 2T^{1,\mu\nu} R_{\mu\alpha\nu}^\beta + \nabla_\alpha (-2\nabla^\mu T_\mu^{1,\beta} + \nabla^\beta T_\mu^{1,\mu}) \\ &\quad + \nabla^\mu (\nabla_\alpha T_\mu^{1,\beta} - \nabla^\beta T_\mu^\alpha) \\ &\quad - 2G^{\mu\nu} R_{\mu\alpha\nu}^\beta + 2[\nabla_\alpha, \nabla^\mu] G_\mu^\beta - \nabla_\alpha \nabla^\beta G_\mu^\mu - \nabla^\mu (\nabla_\mu G_\alpha^\beta - \nabla_\alpha G_\mu^\beta - \nabla^\beta G_{\mu\alpha}). \end{aligned}$$

By the relation $\nabla^\mu T_{\mu\nu}^1 = \frac{1}{2} \nabla_\nu T_\mu^{1,\mu}$, the third term in the right hand side vanishes.

We can also rewrite the fourth term as

$$\begin{aligned} \nabla^\mu (\nabla_\alpha T_\mu^{1,\beta} - \nabla^\beta T_\mu^\alpha) &= [\nabla^\mu, \nabla_\alpha] T_\mu^{1,\beta} - [\nabla^\mu, \nabla^\beta] T_\mu^\alpha + \nabla_\alpha \nabla^\mu T_\mu^{1,\beta} - \nabla^\beta \nabla^\mu T_\mu^\alpha \\ &= R^\mu_{\alpha\mu\delta} T^{1,\delta\beta} + R^\mu_{\alpha\beta\delta} T_\mu^{1,\delta} - R^{\mu\beta}_{\mu\delta} T_\alpha^{1,\delta} \\ &\quad - R^{\mu\beta}_{\alpha\delta} T_\mu^{1,\delta} + [\nabla_\alpha, \nabla^\beta] T_\mu^{1,\mu}, \end{aligned}$$

where the last term vanishes. Commuting we compute the fifth and sixth terms as

$$\begin{aligned} -2G^{\mu\nu} R_{\mu\alpha\nu}^\beta + 2[\nabla_\alpha, \nabla^\mu] G_\mu^\beta &= -2G^{\mu\nu} R_{\mu\alpha\nu}^\beta + 2R_{\alpha\mu\nu} G^{\beta\nu} + 2R_{\alpha\mu\beta\nu} G^{\mu\nu} \\ &= -2 \text{Ric}_{\alpha\nu} G^{\beta\nu}. \end{aligned}$$

Hence, from the above three formulas and the representation of $G_{\mu\nu}$ (8.13), we rearrange $\partial_t \text{Ric}_\alpha^\beta$ as

$$\begin{aligned} \partial_t \text{Ric}_\alpha^\beta - \Delta_g T_\alpha^{1,\beta} & \tag{I_1} \\ &= RT^1 - 2 \text{Ric}_{\alpha\nu} G^{\beta\nu} - \nabla_\alpha \nabla^\beta G_\mu^\mu + \nabla^\mu (-\nabla_\mu G_\alpha^\beta + \nabla_\alpha G_\mu^\beta + \nabla^\beta G_{\mu\alpha}) \\ &= RT^1 - (\nabla^\mu V^\nu + \nabla^\nu V^\mu) R_{\mu\alpha\nu}^\beta \\ &\quad - 2 \text{Ric}_{\alpha\nu} \text{Im}(\psi \bar{\lambda}^{\beta\nu}) + \nabla^\mu (-\nabla_\mu \text{Im}(\psi \bar{\lambda}_\alpha^\beta) + \nabla_\alpha \text{Im}(\psi \bar{\lambda}_\mu^\beta) + \nabla^\beta \text{Im}(\psi \bar{\lambda}_{\mu\alpha})) \end{aligned}$$

$$\begin{aligned} &+ [\nabla_\alpha \nabla^\mu (\nabla_\mu V^\beta + \nabla^\beta V_\mu) - \nabla_\alpha \nabla^\beta \nabla^\mu V_\mu] \tag{I_2} \\ &+ \frac{1}{2} \nabla^\mu [-\nabla_\mu (\nabla_\alpha V^\beta + \nabla^\beta V_\alpha) - \nabla_\alpha (\nabla_\mu V^\beta + \nabla^\beta V_\mu) + \nabla^\beta (\nabla_\mu V_\alpha + \nabla_\alpha V_\mu)]. \end{aligned}$$

We write I_1 as

$$\begin{aligned} I_1 &= -2 \text{Ric}_{\alpha\nu} \text{Im}(\psi \bar{\lambda}^{\beta\nu}) \\ &\quad + \text{Im}(-\nabla^{A,\mu} \nabla_\mu^A \psi \bar{\lambda}_\alpha^\beta - 2\nabla^{A,\mu} \psi \overline{\nabla_\mu^A \lambda_\alpha^\beta} - \overline{\psi \nabla_\mu^A \lambda_\alpha^\beta}) \end{aligned}$$

$$\begin{aligned}
 & + \nabla^{A,\mu} \nabla_\alpha^A \psi \bar{\lambda}_\mu^\beta + \nabla^{A,\mu} \psi \overline{\nabla_\alpha^A \lambda_\mu^\beta} + \nabla_\alpha^A \psi \overline{\nabla^{A,\mu} \lambda_\mu^\beta} + \overline{\psi \nabla^{A,\mu} \nabla_\alpha^A \lambda_\mu^\beta} \\
 & + \nabla^{A,\mu} \nabla^{A,\beta} \psi \bar{\lambda}_{\mu\alpha} + \nabla^{A,\mu} \psi \overline{\nabla^{A,\beta} \lambda_{\mu\alpha}} + \nabla^{A,\beta} \psi \overline{\nabla^{A,\mu} \lambda_{\mu\alpha}} + \overline{\psi \nabla^{A,\mu} \nabla^{A,\beta} \lambda_{\mu\alpha}} \\
 = & \nabla \psi T^3 + \psi \nabla T^3 - 2 \operatorname{Ric}_{\alpha\nu} \operatorname{Im}(\psi \bar{\lambda}^{\beta\nu}) \\
 & + \operatorname{Im}(-\nabla^{A,\mu} \nabla_\mu^A \psi \bar{\lambda}_\alpha^\beta + \nabla^{A,\mu} \nabla_\alpha^A \psi \bar{\lambda}_\mu^\beta + \nabla^{A,\mu} \nabla^{A,\beta} \psi \bar{\lambda}_{\mu\alpha} + \overline{\psi \nabla^{A,\mu} \nabla^{A,\beta} \lambda_{\mu\alpha}})
 \end{aligned}$$

Here the I_1 term will be cancelled by J_1, J_2 later modulo $\{\psi \nabla T^3, \lambda \lambda T^1, \lambda \lambda T^4\}$. Using commutators we rearrange I_2 as

$$\begin{aligned}
 I_2 = & [\nabla_\alpha, \nabla^\mu] \nabla_\mu V^\beta + \frac{1}{2} \nabla^\mu [\nabla_\alpha, \nabla_\mu] V^\beta + \frac{1}{2} \nabla^\mu [\nabla^\beta, \nabla_\mu] V_\alpha \\
 & + \nabla_\alpha [\nabla^\mu, \nabla^\beta] V_\mu + \frac{1}{2} \nabla^\mu [\nabla^\beta, \nabla_\alpha] V_\mu.
 \end{aligned}$$

Then by Riemannian curvature and Bianchi identities we have

$$\begin{aligned}
 I_2 = & R_{\alpha\mu\mu\delta} \nabla^\delta V^\beta + R_{\alpha\mu\beta\delta} \nabla^\mu V^\delta \\
 & + \frac{1}{2} \nabla^\mu (R_{\alpha\mu\beta\delta} V^\delta + R_{\beta\mu\alpha\delta} V^\delta + R_{\beta\alpha\mu\delta} V^\delta) + \nabla_\alpha (R_{\mu\beta\mu\delta} V^\delta) \\
 = & -\operatorname{Ric}_{\alpha\delta} \nabla^\delta V^\beta + R_{\alpha\mu\beta\delta} \nabla^\mu V^\delta + \nabla^\mu (R_{\beta\mu\alpha\delta} V^\delta) + \nabla_\alpha (\operatorname{Ric}_{\beta\delta} V^\delta) \\
 = & -\operatorname{Ric}_{\alpha\delta} \nabla^\delta V^\beta + (R_{\alpha\mu\beta\delta} + R_{\beta\mu\alpha\delta}) \nabla^\mu V^\delta - \nabla_\alpha R_{\delta\mu\beta\mu} V^\delta - \nabla_\delta R_{\mu\alpha\beta\mu} V^\delta \\
 & + \nabla_\alpha \operatorname{Ric}_{\beta\delta} V^\delta + \operatorname{Ric}_{\beta\delta} \nabla_\alpha V^\delta \\
 = & -\operatorname{Ric}_{\alpha\delta} \nabla^\delta V^\beta + R_{\alpha\mu\beta\delta} (\nabla^\mu V^\delta + \nabla^\delta V_\mu) + \nabla_\delta \operatorname{Ric}_{\alpha\beta} V^\delta + \operatorname{Ric}_{\beta\delta} \nabla_\alpha V^\delta,
 \end{aligned}$$

which gives

$$I_2 - (\nabla^\mu V^\nu + \nabla^\nu V^\mu) R_{\mu\alpha\nu}{}^\beta = -\operatorname{Ric}_{\alpha\delta} \nabla^\delta V^\beta + V^\delta \nabla_\delta \operatorname{Ric}_{\alpha\beta} + \operatorname{Ric}_{\beta\delta} \nabla_\alpha V^\delta.$$

This term will be cancelled by J_3 modulo $\{T^1 \nabla V, V \nabla T^1\}$.

Next, we compute the expression for $-\partial_t \widetilde{\operatorname{Ric}}_\alpha{}^\beta$. From the λ -equations (2.28) and the formula (8.13) we have the evolution equation for λ_α^σ

$$\begin{aligned}
 i \partial_t^B \lambda_\alpha^\sigma + \frac{1}{2} (\nabla_\alpha^A \nabla^{A,\sigma} + \nabla^{A,\sigma} \nabla_\alpha^A) \psi + \lambda (T^1 + T^2 + T^4) + i \lambda_\gamma^\sigma \left(\frac{3}{2} \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma) + \nabla_\gamma V^\sigma \right) \\
 - i \lambda^{\gamma\sigma} \left(\frac{1}{2} \operatorname{Im}(\psi \bar{\lambda}_{\gamma\alpha}) + \nabla_\alpha V_\gamma \right) - i V^\gamma \nabla_\gamma^A \lambda_\alpha^\sigma = 0,
 \end{aligned} \tag{8.15}$$

and the evolution equation for the mean curvature ψ

$$i \partial_t^B \psi + \nabla_\sigma^A \nabla^{A,\sigma} \psi + \lambda (T^1 + T^2 + T^4) + i \lambda_\gamma^\sigma \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma) - i V^\gamma \nabla_\gamma^A \psi = 0.$$

Then for $\widetilde{\operatorname{Ric}}_\alpha{}^\beta = \operatorname{Re}(\lambda_\alpha^\beta \bar{\psi} - \lambda_\alpha^\sigma \bar{\lambda}_\sigma^\beta)$, by the above two formulas we have

$$\begin{aligned}
 -\partial_t \widetilde{\operatorname{Ric}}_\alpha{}^\beta & = -\operatorname{Re}(\partial_t^B \lambda_\alpha^\beta \bar{\psi} + \lambda_\alpha^\beta \overline{\partial_t^B \psi} - \partial_t^B \lambda_\alpha^\mu \bar{\lambda}_\mu^\beta - \lambda_\alpha^\mu \overline{\partial_t^B \lambda_\mu^\beta}) \\
 & = \operatorname{Im}(-i \partial_t^B \lambda_\alpha^\beta \bar{\psi} - \bar{\lambda}_\alpha^\beta i \partial_t^B \psi + i \partial_t^B \lambda_\alpha^\mu \bar{\lambda}_\mu^\beta + \bar{\lambda}_\alpha^\mu i \partial_t^B \lambda_\mu^\beta) \\
 & = \lambda^2 (T^1 + T^2 + T^4) + K_1 + K_2 + K_3 + K_4,
 \end{aligned}$$

where

$$\begin{aligned}
 K_1 &= \operatorname{Im} \left[\left(\frac{1}{2} (\nabla_\alpha^A \nabla^{A,\beta} + \nabla^{A,\beta} \nabla_\alpha^A) \psi + i \lambda_\alpha^\gamma \left(\frac{3}{2} \operatorname{Im}(\psi \bar{\lambda}_\gamma^\beta) + \nabla_\gamma V^\beta \right) \right. \right. \\
 &\quad \left. \left. - i \lambda^{\gamma\beta} \left(\frac{1}{2} \operatorname{Im}(\psi \bar{\lambda}_{\gamma\alpha}) + \nabla_\alpha V_\gamma \right) - i V^\gamma \nabla_\gamma^A \lambda_\alpha^\beta \right) \bar{\psi} \right], \\
 K_2 &= \operatorname{Im} \left[\bar{\lambda}_\alpha^\beta \left(\nabla_\sigma^A \nabla^{A,\sigma} \psi + i \lambda_\sigma^\gamma \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma) - i V^\gamma \nabla_\gamma^A \lambda_\sigma^\beta \right) \right], \\
 K_3 &= -\operatorname{Im} \left[\left(\frac{1}{2} (\nabla_\alpha^A \nabla^{A,\sigma} + \nabla^{A,\sigma} \nabla_\alpha^A) \psi + i \lambda_\alpha^\gamma \left(\frac{3}{2} \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma) + \nabla_\gamma V^\sigma \right) \right. \right. \\
 &\quad \left. \left. - i \lambda^{\gamma\sigma} \left(\frac{1}{2} \operatorname{Im}(\psi \bar{\lambda}_{\gamma\alpha}) + \nabla_\alpha V_\gamma \right) - i V^\gamma \nabla_\gamma^A \lambda_\alpha^\sigma \right) \bar{\lambda}_\sigma^\beta \right], \\
 K_4 &= -\operatorname{Im} \left[\bar{\lambda}_\alpha^\sigma \left(\frac{1}{2} (\nabla_\sigma^A \nabla^{A,\beta} + \nabla^{A,\beta} \nabla_\sigma^A) \psi + i \lambda_\sigma^\gamma \left(\frac{3}{2} \operatorname{Im}(\psi \bar{\lambda}_\gamma^\beta) + \nabla_\gamma V^\beta \right) \right. \right. \\
 &\quad \left. \left. - i \lambda^{\gamma\beta} \left(\frac{1}{2} \operatorname{Im}(\psi \bar{\lambda}_{\gamma\sigma}) + \nabla_\sigma V_\gamma \right) - i V^\gamma \nabla_\gamma^A \lambda_\sigma^\beta \right) \right].
 \end{aligned}$$

This can be further rearranged as

$$-\partial_t \widetilde{\operatorname{Ric}}_\alpha^\beta = \lambda^2 (T^1 + T^2 + T^4) + J_1 + J_2 + J_3,$$

where J_1, J_2 and J_3 are

$$\begin{aligned}
 J_1 &= \operatorname{Im} \left[\frac{1}{2} (\nabla_\alpha^A \nabla^{A,\beta} \psi + \nabla^{A,\beta} \nabla_\alpha^A \psi) \bar{\psi} + \bar{\lambda}_\alpha^\beta \nabla_\sigma^A \nabla^{A,\sigma} \psi \right. \\
 &\quad \left. - \frac{1}{2} (\nabla_\alpha^A \nabla^{A,\sigma} \psi + \nabla^{A,\sigma} \nabla_\alpha^A \psi) \bar{\lambda}_\sigma^\beta - \frac{1}{2} \bar{\lambda}_\alpha^\sigma (\nabla_\sigma^A \nabla^{A,\beta} \psi + \nabla^{A,\beta} \nabla_\sigma^A \psi) \right], \\
 J_2 &= \frac{3}{2} \operatorname{Re}(\lambda_\alpha^\gamma \bar{\psi}) \operatorname{Im}(\psi \bar{\lambda}_\gamma^\beta) - \frac{1}{2} \operatorname{Re}(\lambda^{\gamma\beta} \bar{\psi}) \operatorname{Im}(\psi \bar{\lambda}_{\gamma\alpha}) + \operatorname{Re}(\bar{\lambda}_\alpha^\beta \lambda_\sigma^\gamma) \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma) \\
 &\quad - \frac{3}{2} \operatorname{Re}(\lambda_\alpha^\gamma \bar{\lambda}_\sigma^\beta) \operatorname{Im}(\psi \bar{\lambda}_\gamma^\sigma) + \frac{1}{2} \operatorname{Re}(\lambda^{\gamma\sigma} \bar{\lambda}_\sigma^\beta) \operatorname{Im}(\psi \bar{\lambda}_{\gamma\alpha}) \\
 &\quad - \frac{3}{2} \operatorname{Re}(\bar{\lambda}_\alpha^\sigma \lambda_\sigma^\gamma) \operatorname{Im}(\psi \bar{\lambda}_\gamma^\beta) + \frac{1}{2} \operatorname{Re}(\bar{\lambda}_\alpha^\sigma \lambda^{\gamma\beta}) \operatorname{Im}(\psi \bar{\lambda}_{\gamma\sigma}), \\
 J_3 &= \operatorname{Re}(\lambda_\alpha^\gamma \bar{\psi}) \nabla_\gamma V^\beta - \operatorname{Re}(\lambda^{\gamma\beta} \bar{\psi}) \nabla_\alpha V_\gamma - V^\gamma \operatorname{Re}(\nabla_\gamma^A \lambda_\alpha^\beta \bar{\psi}) \\
 &\quad - V^\gamma \operatorname{Re}(\bar{\lambda}_\alpha^\beta \nabla_\gamma^A \psi) \\
 &\quad - \operatorname{Re}(\lambda_\alpha^\gamma \bar{\lambda}_\sigma^\beta) \nabla_\gamma V^\sigma + \operatorname{Re}(\lambda^{\gamma\sigma} \bar{\lambda}_\sigma^\beta) \nabla_\alpha V_\gamma + V^\gamma \operatorname{Re}(\nabla_\gamma^A \lambda_\alpha^\sigma \bar{\lambda}_\sigma^\beta) \\
 &\quad - \operatorname{Re}(\bar{\lambda}_\alpha^\sigma \lambda_\sigma^\gamma) \nabla_\gamma V^\beta + \operatorname{Re}(\bar{\lambda}_\alpha^\sigma \lambda^{\gamma\beta}) \nabla_\sigma V_\gamma + V^\gamma \operatorname{Re}(\bar{\lambda}_\alpha^\sigma \nabla_\gamma^A \lambda_\sigma^\beta).
 \end{aligned}$$

Then $I_1 + J_1 + J_2$ will vanish modulo $\{\psi \nabla T^3, \lambda^2 T^1, \lambda^2 T^4\}$. Precisely, we have

$$\begin{aligned}
 I_1 + J_1 &= \nabla \psi T^3 + \psi \nabla T^3 - 2 \operatorname{Ric}_{\alpha\nu} \operatorname{Im}(\psi \bar{\lambda}^{\beta\nu}) \\
 &\quad + \operatorname{Im} \left(\frac{1}{2} [\nabla^{A,\mu}, \nabla_\alpha^A] \psi \bar{\lambda}_\mu^\beta + \frac{1}{2} [\nabla^{A,\mu}, \nabla^{A,\beta}] \psi \bar{\lambda}_{\mu\alpha} + \overline{\psi [\nabla^{A,\mu}, \nabla^{A,\beta}] \lambda_{\mu\alpha}} \right. \\
 &\quad \left. + \overline{\psi \nabla^{A,\beta} T_{\mu\alpha}^3} + \frac{1}{2} \overline{\psi [\nabla^{A,\beta}, \nabla_\alpha^A] \psi} \right) \\
 &= \nabla \psi T^3 + \psi \nabla T^3 - 2 \operatorname{Ric}_{\alpha\nu} \operatorname{Im}(\psi \bar{\lambda}^{\beta\nu}) + \frac{1}{2} \mathbf{F}^\mu_\alpha \operatorname{Re}(\psi \bar{\lambda}_\mu^\beta) - \frac{1}{2} \mathbf{F}^{\mu\beta} \operatorname{Re}(\psi \bar{\lambda}_{\mu\alpha})
 \end{aligned}$$

$$+ \text{Ric}^{\mu\beta} \text{Im}(\psi \bar{\lambda}_{\mu\alpha}) + R_{\mu\beta\alpha\delta} \text{Im}(\psi \bar{\lambda}^{\mu\delta}) - \frac{1}{2} |\psi|^2 \mathbf{F}^\beta_\alpha.$$

We rewrite J_2 as

$$J_2 = \frac{3}{2} \text{Im}(\psi \bar{\lambda}^{\beta\gamma}) \widetilde{\text{Ric}}_{\alpha\gamma} - \frac{1}{2} \text{Im}(\psi \bar{\lambda}_{\gamma\alpha}) \widetilde{\text{Ric}}^{\gamma\beta} + \text{Im}(\psi \bar{\lambda}^{\sigma\gamma}) \tilde{R}^\beta_{\sigma\alpha\gamma}.$$

Then we obtain

$$\begin{aligned} I_1 + J_1 + J_2 &= \psi \nabla T^3 + \lambda^2 (T^1 + T^4) + \frac{1}{2} \text{Im}(\psi \bar{\lambda}_{\gamma\alpha}) \widetilde{\text{Ric}}^{\gamma\beta} - \frac{1}{2} \text{Im}(\psi \bar{\lambda}^{\beta\gamma}) \widetilde{\text{Ric}}_{\alpha\gamma} \\ &\quad + \frac{1}{2} \tilde{\mathbf{F}}^\mu_\alpha \text{Re}(\psi \bar{\lambda}^\beta_\mu) - \frac{1}{2} \tilde{\mathbf{F}}^{\mu\beta} \text{Re}(\psi \bar{\lambda}_{\mu\alpha}) - \frac{1}{2} |\psi|^2 \tilde{\mathbf{F}}^\beta_\alpha \\ &= \psi \nabla T^3 + \lambda^2 (T^1 + T^4). \end{aligned}$$

We can also show that $I_2 - (\nabla^\mu V^\nu + \nabla^\nu V^\mu) R_{\mu\alpha\nu}{}^\beta + J_3$ vanishes modulo $\{T^1 \nabla V, V \nabla T^1\}$. This is because J_3 can be written as

$$\begin{aligned} J_3 &= \text{Re}(\lambda^\gamma_\alpha \bar{\psi}) \nabla_\gamma V^\beta - \text{Re}(\lambda^{\gamma\beta} \bar{\psi}) \nabla_\alpha V_\gamma - V^\gamma \text{Re}(\nabla_\gamma \lambda^\beta_\alpha \bar{\psi}) \\ &\quad + \text{Re}(\bar{\lambda}^\beta_\alpha \lambda^\gamma_\sigma) (\nabla_\gamma V^\sigma - \nabla^\sigma V_\gamma) - V^\gamma \text{Re}(\bar{\lambda}^\beta_\alpha \nabla_\gamma^A \psi) \\ &\quad - \text{Re}(\lambda^\gamma_\alpha \bar{\lambda}^\beta_\sigma) \nabla_\gamma V^\sigma + \text{Re}(\lambda^{\gamma\sigma} \bar{\lambda}^\beta_\sigma) \nabla_\alpha V_\gamma + V^\gamma \text{Re}(\nabla_\gamma \lambda^\sigma_\alpha \bar{\lambda}^\beta_\sigma) \\ &\quad - \text{Re}(\bar{\lambda}^\sigma_\alpha \lambda^\gamma_\sigma) \nabla_\gamma V^\beta + \text{Re}(\bar{\lambda}^\sigma_\alpha \lambda^{\gamma\beta}) \nabla_\sigma V_\gamma + V^\gamma \text{Re}(\bar{\lambda}^\sigma_\alpha \nabla_\gamma^A \lambda^\beta_\sigma) \\ &= \widetilde{\text{Ric}}_{\alpha\gamma} \nabla^\gamma V^\beta - \widetilde{\text{Ric}}^{\gamma\beta} \nabla_\alpha V_\gamma - V^\gamma \widetilde{\text{Ric}}_{\alpha\gamma}{}^\beta. \end{aligned}$$

Then we have

$$I_2 - (\nabla^\mu V^\nu + \nabla^\nu V^\mu) R_{\mu\alpha\nu}{}^\beta + J_3 = -T^1_{\alpha\gamma} \nabla^\gamma V^\beta + T^{1,\gamma\beta} \nabla_\alpha V_\gamma + V^\gamma \nabla_\gamma T^1_{\alpha}{}^\beta.$$

This concludes the proof of the T^1 -equations. □

The equation for T^2 . By the second Bianchi identities for the Riemannian curvature and the following equality

$$\begin{aligned} \nabla_\delta \tilde{R}_{\sigma\gamma\alpha\beta} + \nabla_\sigma \tilde{R}_{\gamma\delta\alpha\beta} + \nabla_\gamma \tilde{R}_{\delta\sigma\alpha\beta} \\ = \text{Re}(T^3_{\delta\gamma,\beta} \bar{\lambda}_{\alpha\sigma} + T^3_{\delta\sigma,\alpha} \bar{\lambda}_{\beta\gamma} - T^3_{\delta\gamma,\alpha} \bar{\lambda}_{\beta\sigma} - T^3_{\delta\sigma,\beta} \bar{\lambda}_{\alpha\gamma} + T^3_{\sigma\gamma,\alpha} \bar{\lambda}_{\beta\delta} - T^3_{\sigma\gamma,\beta} \bar{\lambda}_{\alpha\delta}), \end{aligned}$$

we have the counterpart of the second Bianchi identities

$$\nabla_\delta T^2_{\sigma\gamma\alpha\beta} + \nabla_\sigma T^2_{\gamma\delta\alpha\beta} + \nabla_\gamma T^2_{\delta\sigma\alpha\beta} = T^1 \lambda,$$

which combine with the algebraic symmetries of the same tensor to yield an elliptic system for T^2 . Precisely, using the above relation we have

$$\nabla^\sigma T^2_{\sigma\gamma\alpha\beta} = \nabla_\alpha T^1_{\gamma\beta} - \nabla_\beta T^1_{\gamma\alpha} + T^1 \lambda,$$

which combined with the previous one yields the desired elliptic system, with T^1 viewed as a source term. □

The equations for T^3 . This has the form

$$(i\partial_t^B - \Delta_g^A)T_{\alpha\beta,\gamma}^3 = \lambda T^5 + V\nabla T^3 + T^3(\nabla V + \lambda^2 + R) + (\nabla^A \lambda + \lambda V)(T^1 + T^2 + T^4) + \nabla(T^2 + T^4)\lambda.$$

Recall the λ -equations

$$i\partial_t^B \lambda_{\beta\gamma} + \nabla_\mu^A \nabla^{A,\mu} \lambda_{\beta\gamma} - \frac{1}{2} \left(\widetilde{\text{Ric}}_{\beta\delta} \lambda_\gamma^\delta + \widetilde{\text{Ric}}_{\gamma\delta} \lambda_\beta^\delta \right) + \tilde{R}_{\beta\sigma\gamma\delta} \lambda^{\sigma\delta} + \frac{i}{2} \left(\tilde{\mathbf{F}}_{\beta\delta} \lambda_\gamma^\delta + \tilde{\mathbf{F}}_{\gamma\delta} \lambda_\beta^\delta \right) - \frac{i}{2} \lambda_\beta^\delta \left[\text{Im}(\psi \bar{\lambda}_{\delta\gamma}) + 2\nabla_\gamma V_\delta \right] - \frac{i}{2} \lambda_\gamma^\delta \left[\text{Im}(\psi \bar{\lambda}_{\delta\beta}) + 2\nabla_\beta V_\delta \right] - iV^\delta \nabla_\delta^A \lambda_{\beta\gamma} = 0.$$

Applying ∇_α^A and ∇_β^A to the above $\lambda_{\beta\gamma}$ and $\lambda_{\alpha\gamma}$ -equations respectively, we obtain the difference

$$\begin{aligned} 0 &= \left[\nabla_\alpha^A \left(i\partial_t^B \lambda_{\beta\gamma} + \nabla_\mu^A \nabla^{A,\mu} \lambda_{\beta\gamma} \right) - \nabla_\beta^A \left(i\partial_t^B \lambda_{\alpha\gamma} + \nabla_\mu^A \nabla^{A,\mu} \lambda_{\alpha\gamma} \right) \right] \\ &\quad + \left[\nabla_\alpha^A \left[-\frac{1}{2} \left(\widetilde{\text{Ric}}_{\beta\delta} \lambda_\gamma^\delta + \widetilde{\text{Ric}}_{\gamma\delta} \lambda_\beta^\delta \right) + \tilde{R}_{\beta\sigma\gamma\delta} \lambda^{\sigma\delta} + \frac{i}{2} \left(\tilde{\mathbf{F}}_{\beta\delta} \lambda_\gamma^\delta + \tilde{\mathbf{F}}_{\gamma\delta} \lambda_\beta^\delta \right) \right] \right. \\ &\quad \left. - \nabla_\beta^A \left[-\frac{1}{2} \left(\widetilde{\text{Ric}}_{\alpha\delta} \lambda_\gamma^\delta + \widetilde{\text{Ric}}_{\gamma\delta} \lambda_\alpha^\delta \right) + \tilde{R}_{\alpha\sigma\gamma\delta} \lambda^{\sigma\delta} + \frac{i}{2} \left(\tilde{\mathbf{F}}_{\alpha\delta} \lambda_\gamma^\delta + \tilde{\mathbf{F}}_{\gamma\delta} \lambda_\alpha^\delta \right) \right] \right] \\ &\quad + \left[\nabla_\alpha^A \left[-\frac{i}{2} \lambda_\beta^\delta \left[\text{Im}(\psi \bar{\lambda}_{\delta\gamma}) + 2\nabla_\gamma V_\delta \right] - \frac{i}{2} \lambda_\gamma^\delta \left[\text{Im}(\psi \bar{\lambda}_{\delta\beta}) + 2\nabla_\beta V_\delta \right] - iV^\delta \nabla_\delta^A \lambda_{\beta\gamma} \right] \right. \\ &\quad \left. - \nabla_\beta^A \left[-\frac{i}{2} \lambda_\alpha^\delta \left[\text{Im}(\psi \bar{\lambda}_{\delta\gamma}) + 2\nabla_\gamma V_\delta \right] - \frac{i}{2} \lambda_\gamma^\delta \left[\text{Im}(\psi \bar{\lambda}_{\delta\alpha}) + 2\nabla_\alpha V_\delta \right] - iV^\delta \nabla_\delta^A \lambda_{\alpha\gamma} \right] \right] \\ &:= I + II + III. \end{aligned}$$

We first compute I . We commute ∇_α^A with ∂_t^B and $\nabla_\mu^A \nabla^{A,\mu}$ to give

$$I = (i\partial_t - \Delta_g^A)T_{\alpha\beta,\gamma}^3 + i[\nabla_\alpha^A, \partial_t^B] \lambda_{\beta\gamma} + [\nabla_\alpha^A, \nabla_\mu^A \nabla^{A,\mu}] \lambda_{\beta\gamma} \quad (I_1)$$

$$\begin{aligned} &- i[\nabla_\beta^A, \partial_t^B] \lambda_{\alpha\gamma} - [\nabla_\beta^A, \nabla_\mu^A \nabla^{A,\mu}] \lambda_{\alpha\gamma} \\ &= (i\partial_t - \Delta_g^A)T_{\alpha\beta,\gamma}^3 \\ &\quad + i\partial_t \Gamma_{\alpha\gamma}^\sigma \lambda_{\beta\sigma} + \mathbf{F}_{0\alpha} \lambda_{\beta\gamma} - i\partial_t \Gamma_{\beta\gamma}^\sigma \lambda_{\alpha\sigma} - \mathbf{F}_{0\beta} \lambda_{\alpha\gamma} \\ &\quad + [[\nabla_\alpha, \nabla_\mu] \nabla^{A,\mu} \lambda_{\beta\gamma} + i\mathbf{F}_{\alpha\mu} \nabla^{A,\mu} \lambda_{\beta\gamma} + \nabla^{A,\mu} ([\nabla_\alpha, \nabla_\mu] + i\mathbf{F}_{\alpha\mu}) \lambda_{\beta\gamma} \\ &\quad - [\nabla_\beta, \nabla_\mu] \nabla^{A,\mu} \lambda_{\alpha\gamma} - i\mathbf{F}_{\beta\mu} \nabla^{A,\mu} \lambda_{\alpha\gamma} - \nabla^{A,\mu} ([\nabla_\beta, \nabla_\mu] + i\mathbf{F}_{\beta\mu}) \lambda_{\alpha\gamma}]. \end{aligned} \quad (I_2)$$

For I_1 , by the formulas for $\partial_t \Gamma$ in (8.14), for $G_{\mu\nu}$ in (8.13) and for the commutators $[\nabla_\alpha, \nabla_\beta]$ we have

$$\begin{aligned} I_1 &= i(\nabla_\alpha G_{\gamma\delta} + \nabla_\gamma G_{\alpha\delta} - \nabla_\delta G_{\alpha\gamma}) \lambda_\beta^\delta - i(\nabla_\beta G_{\gamma\delta} + \nabla_\gamma G_{\beta\delta} - \nabla_\delta G_{\beta\gamma}) \lambda_\alpha^\delta + \nabla T^1 \lambda \\ &\quad + T_\alpha^5 \lambda_{\beta\gamma} - T_\beta^5 \lambda_{\alpha\gamma} + (\text{Re}(\lambda_\alpha^\sigma \bar{\nabla}_\sigma^A \psi) - \tilde{\mathbf{F}}_{\alpha\sigma} V^\sigma) \lambda_{\beta\gamma} - (\text{Re}(\lambda_\beta^\sigma \bar{\nabla}_\sigma^A \psi) - \tilde{\mathbf{F}}_{\beta\sigma} V^\sigma) \lambda_{\alpha\gamma} \\ &= \nabla T^1 \lambda + T_\alpha^5 \lambda_{\beta\gamma} - T_\beta^5 \lambda_{\alpha\gamma} + I_{11} + I_{12}, \end{aligned}$$

where I_{11}, I_{12} are the terms containing $\lambda \nabla \lambda$ and $V \lambda$ respectively,

$$\begin{aligned} I_{11} &:= i(\nabla_\alpha \text{Im}(\psi \bar{\lambda}_{\gamma\delta}) + \nabla_\gamma \text{Im}(\psi \bar{\lambda}_{\alpha\delta}) - \nabla_\delta \text{Im}(\psi \bar{\lambda}_{\alpha\gamma})) \lambda_\beta^\delta \\ &\quad - i(\nabla_\beta \text{Im}(\psi \bar{\lambda}_{\gamma\delta}) + \nabla_\gamma \text{Im}(\psi \bar{\lambda}_{\beta\delta}) - \nabla_\delta \text{Im}(\psi \bar{\lambda}_{\beta\gamma})) \lambda_\alpha^\delta \end{aligned}$$

$$\begin{aligned}
& + \operatorname{Re}(\lambda_\alpha^\sigma \overline{\nabla_\sigma^A \psi}) \lambda_{\beta\gamma} - \operatorname{Re}(\lambda_\beta^\sigma \overline{\nabla_\sigma^A \psi}) \lambda_{\alpha\gamma}, \\
I_{12} := & \frac{i}{2} [(\nabla_\alpha \nabla_\gamma + \nabla_\gamma \nabla_\alpha) V_\delta \lambda_\beta^\delta + R_{\alpha\sigma\gamma\delta} V^\delta \lambda_\beta^\sigma + R_{\gamma\sigma\alpha\delta} V^\delta \lambda_\beta^\sigma] \\
& - \frac{i}{2} [(\nabla_\beta \nabla_\gamma + \nabla_\gamma \nabla_\beta) V_\delta \lambda_\alpha^\delta + R_{\beta\sigma\gamma\delta} V^\delta \lambda_\alpha^\sigma + R_{\gamma\sigma\beta\delta} V^\delta \lambda_\alpha^\sigma] \\
& - \tilde{\mathbf{F}}_{\alpha\sigma} V^\sigma \lambda_{\beta\gamma} + \tilde{\mathbf{F}}_{\beta\sigma} V^\sigma \lambda_{\alpha\gamma}.
\end{aligned}$$

Here, using the expressions for $\tilde{\mathbf{F}}_{\alpha\beta}$ and $\tilde{R}_{\beta\alpha\gamma\delta}$, the expression I_{11} can be rewritten as

$$I_{11} = i \nabla_\alpha \operatorname{Im}(\psi \bar{\lambda}_{\gamma\delta}) \lambda_\beta^\delta - i \nabla_\beta \operatorname{Im}(\psi \bar{\lambda}_{\gamma\delta}) \lambda_\alpha^\delta - i \nabla_\gamma^A \psi \tilde{\mathbf{F}}_{\alpha\beta} + \nabla_\delta^A \psi \tilde{R}_{\beta\alpha\gamma\delta},$$

Using commutators $[\nabla_\gamma, \nabla_\alpha]$ and the Bianchi identities, the I_{12} expression can be rewritten as

$$\begin{aligned}
I_{12} = & i \nabla_\alpha \nabla_\gamma V_\delta \lambda_\beta^\delta + i R_{\gamma\sigma\alpha\delta} V^\delta \lambda_\beta^\sigma - i \nabla_\beta \nabla_\gamma V_\delta \lambda_\alpha^\delta \\
& - i R_{\gamma\sigma\beta\delta} V^\delta \lambda_\alpha^\sigma - \tilde{\mathbf{F}}_{\alpha\sigma} V^\sigma \lambda_{\beta\gamma} + \tilde{\mathbf{F}}_{\beta\sigma} V^\sigma \lambda_{\alpha\gamma}
\end{aligned}$$

For I_2 , we use the Riemannian curvature tensor to write

$$\begin{aligned}
I_2 = & R_{\alpha\mu\mu\delta} \nabla^{A,\delta} \lambda_{\beta\gamma} + R_{\alpha\mu\beta\delta} \nabla^{A,\mu} \lambda_\gamma^\delta + R_{\alpha\mu\gamma\delta} \nabla^{A,\mu} \lambda_\beta^\delta + i \mathbf{F}_{\alpha\mu} \nabla^{A,\mu} \lambda_{\beta\gamma} \\
& + \nabla^{A,\mu} (R_{\alpha\mu\beta\delta} \lambda_\gamma^\delta + R_{\alpha\mu\gamma\delta} \lambda_\beta^\delta + i \mathbf{F}_{\alpha\mu} \lambda_{\beta\gamma}) \\
& - R_{\beta\mu\mu\delta} \nabla^{A,\delta} \lambda_{\alpha\gamma} - R_{\beta\mu\alpha\delta} \nabla^{A,\mu} \lambda_\gamma^\delta - R_{\beta\mu\gamma\delta} \nabla^{A,\mu} \lambda_\alpha^\delta - i \mathbf{F}_{\beta\mu} \nabla^{A,\mu} \lambda_{\alpha\gamma} \\
& - \nabla^{A,\mu} (R_{\beta\mu\alpha\delta} \lambda_\gamma^\delta + R_{\beta\mu\gamma\delta} \lambda_\alpha^\delta + i \mathbf{F}_{\beta\mu} \lambda_{\alpha\gamma}) \\
= & - \operatorname{Ric}_{\alpha\delta} \nabla^{A,\delta} \lambda_{\beta\gamma} + 2R_{\alpha\mu\beta\delta} \nabla^{A,\mu} \lambda_\gamma^\delta + 2R_{\alpha\mu\gamma\delta} \nabla^{A,\mu} \lambda_\beta^\delta + 2i \mathbf{F}_{\alpha\mu} \nabla^{A,\mu} \lambda_{\beta\gamma} \\
& + \nabla^\mu R_{\alpha\mu\beta\delta} \lambda_\gamma^\delta + \nabla^\mu R_{\alpha\mu\gamma\delta} \lambda_\beta^\delta + i \nabla^\mu \mathbf{F}_{\alpha\mu} \lambda_{\beta\gamma} \\
& + \operatorname{Ric}_{\beta\delta} \nabla^{A,\delta} \lambda_{\alpha\gamma} - 2R_{\beta\mu\alpha\delta} \nabla^{A,\mu} \lambda_\gamma^\delta - 2R_{\beta\mu\gamma\delta} \nabla^{A,\mu} \lambda_\alpha^\delta - 2i \mathbf{F}_{\beta\mu} \nabla^{A,\mu} \lambda_{\alpha\gamma} \\
& - \nabla^\mu R_{\beta\mu\alpha\delta} \lambda_\gamma^\delta - \nabla^\mu R_{\beta\mu\gamma\delta} \lambda_\alpha^\delta - i \nabla^\mu \mathbf{F}_{\beta\mu} \lambda_{\alpha\gamma} \\
= & 2R_{\alpha\mu\beta\delta} T^{3,\mu\delta}{}_{,\gamma} + (T^1 + T^2 + T^4) \nabla^A \lambda + \nabla(T^2 + T^4) \lambda + J_1,
\end{aligned}$$

where the terms in J_1 have the form $\lambda \lambda \nabla \lambda$ as

$$\begin{aligned}
J_1 = & \nabla^{A,\delta} \lambda_{\beta\gamma} (-\widetilde{\operatorname{Ric}}_{\alpha\delta} + 2i \tilde{\mathbf{F}}_{\alpha\delta}) + \nabla^{A,\delta} \lambda_{\alpha\gamma} (\widetilde{\operatorname{Ric}}_{\beta\delta} - 2i \tilde{\mathbf{F}}_{\beta\delta}) \\
& + i \nabla^\mu \tilde{\mathbf{F}}_{\alpha\mu} \lambda_{\beta\gamma} - i \nabla^\mu \tilde{\mathbf{F}}_{\beta\mu} \lambda_{\alpha\gamma} \\
& + 2\tilde{R}_{\alpha\mu\gamma\delta} \nabla^{A,\mu} \lambda_\beta^\delta + \nabla^\mu \tilde{R}_{\alpha\mu\beta\delta} \lambda_\gamma^\delta + \nabla^\mu \tilde{R}_{\alpha\mu\gamma\delta} \lambda_\beta^\delta \\
& - 2\tilde{R}_{\beta\mu\gamma\delta} \nabla^{A,\mu} \lambda_\alpha^\delta - \nabla^\mu \tilde{R}_{\beta\mu\alpha\delta} \lambda_\gamma^\delta - \nabla^\mu \tilde{R}_{\beta\mu\gamma\delta} \lambda_\alpha^\delta.
\end{aligned}$$

We next rewrite the III expression as

$$\begin{aligned}
III = & \nabla_\alpha^A \left[-\frac{i}{2} \lambda_\beta^\delta [\operatorname{Im}(\psi \bar{\lambda}_{\delta\gamma}) + 2\nabla_\gamma V_\delta] - \frac{i}{2} \lambda_\gamma^\delta [\operatorname{Im}(\psi \bar{\lambda}_{\delta\beta}) + 2\nabla_\beta V_\delta] - i V^\delta \nabla_\delta^A \lambda_{\beta\gamma} \right] \\
& - \nabla_\beta^A \left[-\frac{i}{2} \lambda_\alpha^\delta [\operatorname{Im}(\psi \bar{\lambda}_{\delta\gamma}) + 2\nabla_\gamma V_\delta] - \frac{i}{2} \lambda_\gamma^\delta [\operatorname{Im}(\psi \bar{\lambda}_{\delta\alpha}) + 2\nabla_\alpha V_\delta] - i V^\delta \nabla_\delta^A \lambda_{\alpha\gamma} \right] \\
= & -\frac{i}{2} T_{\alpha\beta,\delta}^3 [\operatorname{Im}(\psi \bar{\lambda}_{\delta\gamma}^\delta) + 2\nabla_\gamma V^\delta] + III_1 + III_2,
\end{aligned}$$

where

$$\begin{aligned}
 III_1 &:= -\frac{i}{2}\lambda_\beta^\delta \nabla_\alpha \operatorname{Im}(\psi \bar{\lambda}_{\delta\gamma}) - \frac{i}{2}\nabla_\alpha^A(\lambda_\gamma^\delta \operatorname{Im}(\psi \bar{\lambda}_{\delta\beta})) \\
 &\quad + \frac{i}{2}\lambda_\beta^\delta \nabla_\beta \operatorname{Im}(\psi \bar{\lambda}_{\delta\gamma}) + \frac{i}{2}\nabla_\beta^A(\lambda_\gamma^\delta \operatorname{Im}(\psi \bar{\lambda}_{\delta\alpha})), \\
 III_2 &:= -i\lambda_\beta^\delta \nabla_\alpha \nabla_\gamma V_\delta - i\nabla_\alpha^A(\lambda_\gamma^\delta \nabla_\beta V_\delta) - i\nabla_\alpha^A(V^\delta \nabla_\delta^A \lambda_{\beta\gamma}) \\
 &\quad + i\lambda_\alpha^\delta \nabla_\beta \nabla_\gamma V_\delta + i\nabla_\beta^A(\lambda_\gamma^\delta \nabla_\alpha V_\delta) + i\nabla_\beta^A(V^\delta \nabla_\delta^A \lambda_{\alpha\gamma}).
 \end{aligned}$$

The $I_{12} + III_2$ expression vanishes modulo $\{V\nabla T^3, T^3\nabla V, \lambda V(T^2 + T^4)\}$. Precisely, we can further write III_2 as

$$\begin{aligned}
 III_2 &= -iT_{\alpha\delta,\gamma}^3 \nabla_\beta V^\delta + iT_{\beta\delta,\gamma}^3 \nabla_\alpha V^\delta - iV^\delta \nabla_\alpha^A T_{\delta\beta,\gamma}^3 + iV^\delta \nabla_\beta^A T_{\delta\alpha,\gamma}^3 \quad (III_{21}) \\
 &\quad + i(-\lambda_\beta^\delta \nabla_\alpha \nabla_\gamma V_\delta + \lambda_\alpha^\delta \nabla_\beta \nabla_\gamma V_\delta - V^\delta R_{\alpha\beta\gamma\sigma} \lambda_\delta^\sigma) + V^\delta \mathbf{F}_{\alpha\beta} \lambda_{\delta\gamma}.
 \end{aligned}$$

Then replacing $R_{\alpha\beta\gamma\delta}, \mathbf{F}_{\alpha\beta}$ by $\tilde{R}_{\alpha\beta\gamma\delta}$ and $\tilde{\mathbf{F}}_{\alpha\beta}$ respectively, we have

$$\begin{aligned}
 I_{12} + III_{21} &= i(\lambda_\beta^\sigma R_{\gamma\sigma\alpha\delta} V^\delta - \lambda_\alpha^\sigma R_{\gamma\sigma\beta\delta} V^\delta - V^\delta R_{\alpha\beta\gamma\sigma} \lambda_\delta^\sigma) \quad (J_2) \\
 &\quad - \tilde{\mathbf{F}}_{\alpha\sigma} V^\sigma \lambda_{\beta\gamma} + \tilde{\mathbf{F}}_{\beta\sigma} V^\sigma \lambda_{\alpha\gamma} + V^\delta \mathbf{F}_{\alpha\beta} \lambda_{\delta\gamma} \\
 &= i(\lambda_\beta^\sigma T_{\gamma\sigma\alpha\delta}^2 V^\delta - \lambda_\alpha^\sigma T_{\gamma\sigma\beta\delta}^2 V^\delta - V^\delta T_{\alpha\beta\gamma\sigma}^2 \lambda_\delta^\sigma) + V^\delta T_{\alpha\beta}^4 \lambda_{\delta\gamma} \\
 &\quad + [i(\lambda_\beta^\sigma \tilde{R}_{\gamma\sigma\alpha\delta} V^\delta - \lambda_\alpha^\sigma \tilde{R}_{\gamma\sigma\beta\delta} V^\delta - V^\delta \tilde{R}_{\alpha\beta\gamma\sigma} \lambda_\delta^\sigma) \\
 &\quad - \tilde{\mathbf{F}}_{\alpha\sigma} V^\sigma \lambda_{\beta\gamma} + \tilde{\mathbf{F}}_{\beta\sigma} V^\sigma \lambda_{\alpha\gamma} + V^\delta \tilde{\mathbf{F}}_{\alpha\beta} \lambda_{\delta\gamma}] \\
 &= \lambda V T^2 + \lambda V T^4,
 \end{aligned}$$

where the term J_2 vanishes due to the representations of $\tilde{R}_{\gamma\sigma\alpha\delta}$ and $\tilde{\mathbf{F}}_{\alpha\sigma}$.

Next, we show that the terms $I_{11} + J_1 + II + III_1$ vanish modulo $\lambda\lambda T^3$. We have

$$\begin{aligned}
 I_{11} + III_1 &= -i\nabla_\gamma^A \psi \tilde{\mathbf{F}}_{\alpha\beta} + \nabla_\delta^A \psi \tilde{R}_{\beta\alpha\gamma\delta} + \frac{i}{2}\lambda_\beta^\delta \nabla_\alpha \operatorname{Im}(\psi \bar{\lambda}_{\delta\gamma}) - \frac{i}{2}\lambda_\alpha^\delta \nabla_\beta \operatorname{Im}(\psi \bar{\lambda}_{\delta\gamma}) \\
 &\quad - \frac{i}{2}\nabla_\alpha^A(\lambda_\gamma^\delta \operatorname{Im}(\psi \bar{\lambda}_{\delta\beta})) + \frac{i}{2}\nabla_\beta^A(\lambda_\gamma^\delta \operatorname{Im}(\psi \bar{\lambda}_{\delta\alpha})).
 \end{aligned}$$

We rewrite II as

$$\begin{aligned}
 II &= \nabla_\alpha^A \left[-\frac{1}{2}(\widetilde{\operatorname{Ric}}_{\beta\delta} \lambda_\gamma^\delta + \widetilde{\operatorname{Ric}}_{\gamma\delta} \lambda_\beta^\delta) + \tilde{R}_{\beta\sigma\gamma\delta} \lambda^{\sigma\delta} + \frac{i}{2}(\tilde{\mathbf{F}}_{\beta\delta} \lambda_\gamma^\delta + \tilde{\mathbf{F}}_{\gamma\delta} \lambda_\beta^\delta) \right] \quad (J_3) \\
 &\quad - \nabla_\beta^A \left[-\frac{1}{2}(\widetilde{\operatorname{Ric}}_{\alpha\delta} \lambda_\gamma^\delta + \widetilde{\operatorname{Ric}}_{\gamma\delta} \lambda_\alpha^\delta) + \tilde{R}_{\alpha\sigma\gamma\delta} \lambda^{\sigma\delta} + \frac{i}{2}(\tilde{\mathbf{F}}_{\alpha\delta} \lambda_\gamma^\delta + \tilde{\mathbf{F}}_{\gamma\delta} \lambda_\alpha^\delta) \right] \\
 &= -\frac{1}{2}(\widetilde{\operatorname{Ric}}_{\gamma\delta} - i\tilde{\mathbf{F}}_{\gamma\delta}) T_{\alpha\beta}^3 \delta \\
 &\quad + \frac{1}{2}\nabla_\alpha^A(\lambda_\gamma^\delta (-\widetilde{\operatorname{Ric}}_{\beta\delta} + i\tilde{\mathbf{F}}_{\beta\delta})) - \frac{1}{2}\nabla_\beta^A(\lambda_\gamma^\delta (-\widetilde{\operatorname{Ric}}_{\alpha\delta} + i\tilde{\mathbf{F}}_{\alpha\delta}))
 \end{aligned}$$

$$+ \frac{1}{2} \lambda_\beta^\delta \nabla_\alpha (-\widetilde{\text{Ric}}_{\gamma\delta} + i\tilde{\mathbf{F}}_{\gamma\delta}) - \frac{1}{2} \lambda_\alpha^\delta \nabla_\beta (-\widetilde{\text{Ric}}_{\gamma\delta} + i\tilde{\mathbf{F}}_{\gamma\delta}) \quad (J_4)$$

$$- \nabla_\sigma \tilde{R}_{\alpha\beta\gamma\delta} \lambda^{\sigma\delta} + \tilde{R}_{\beta\mu\gamma\delta} \nabla_\alpha^\lambda \lambda^{\mu\delta} - \tilde{R}_{\alpha\mu\gamma\delta} \nabla_\beta^\lambda \lambda^{\mu\delta}. \quad (J_5)$$

Hence,

$$\begin{aligned} I_{11} + III_1 + J_3 + J_4 &= -i\nabla_\gamma^A \psi \tilde{\mathbf{F}}_{\alpha\beta} + \nabla_\delta^A \psi \tilde{R}_{\beta\alpha\gamma\delta} + \frac{1}{2} \lambda_\beta^\delta \nabla_\alpha (-\lambda_{\gamma\delta} \bar{\psi} + \lambda_{\gamma\sigma} \bar{\lambda}_\delta^\sigma) \\ &\quad + \frac{1}{2} \lambda_\alpha^\delta \nabla_\beta (\lambda_{\gamma\delta} \bar{\psi} - \lambda_{\gamma\sigma} \bar{\lambda}_\delta^\sigma) + \frac{1}{2} \nabla_\alpha^A [\lambda_\gamma^\delta (-\bar{\lambda}_{\delta\beta} \psi + \lambda_{\beta\sigma} \bar{\lambda}_\delta^\sigma)] \\ &\quad + \frac{1}{2} \nabla_\beta^A [\lambda_\gamma^\delta (\bar{\lambda}_{\delta\alpha} \psi - \lambda_{\alpha\sigma} \bar{\lambda}_\delta^\sigma)] \\ &= -i\nabla_\gamma^A \psi \tilde{\mathbf{F}}_{\alpha\beta} + \nabla_\delta^A \psi \tilde{R}_{\beta\alpha\gamma\delta} - \nabla_\alpha^A \lambda_\gamma^\delta (\widetilde{\text{Ric}}_{\beta\delta} - i\tilde{\mathbf{F}}_{\beta\delta}) + \nabla_\beta^A \lambda_\gamma^\delta (\widetilde{\text{Ric}}_{\alpha\delta} - i\tilde{\mathbf{F}}_{\alpha\delta}) \\ &\quad + \overline{\nabla_\alpha^A \lambda^{\sigma\delta}} \lambda_{\beta\sigma} \lambda_{\gamma\delta} - \overline{\nabla_\beta^A \lambda^{\sigma\delta}} \lambda_{\alpha\sigma} \lambda_{\gamma\delta} - \lambda_\gamma^\delta \text{Re}(\nabla_\alpha^A \psi \bar{\lambda}_{\beta\delta}) + \lambda_\gamma^\delta \text{Re}(\nabla_\beta^A \psi \bar{\lambda}_{\alpha\delta}) + \lambda^2 T^3. \end{aligned}$$

Since by \tilde{R} and $\tilde{\mathbf{F}}$ we also have

$$\begin{aligned} J_1 + J_5 &= -\nabla^\alpha A^\delta \lambda_{\beta\gamma} \widetilde{\text{Ric}}_{\alpha\delta} + \nabla^\alpha A^\delta \lambda_{\alpha\gamma} \widetilde{\text{Ric}}_{\beta\delta} + \lambda_\gamma^\delta \text{Re}(\nabla_\alpha^A \psi \bar{\lambda}_{\beta\delta} - \nabla_\beta^A \psi \bar{\lambda}_{\alpha\delta}) \\ &\quad - \overline{\nabla_\alpha^A \lambda^{\mu\sigma}} \lambda_{\mu\beta} \lambda_{\sigma\gamma} + \overline{\nabla_\beta^A \lambda^{\mu\sigma}} \lambda_{\mu\alpha} \lambda_{\sigma\gamma} - i\nabla^\alpha A^\delta \lambda_{\alpha\gamma} \tilde{\mathbf{F}}_{\beta\delta} + i\nabla^\alpha A^\delta \lambda_{\beta\gamma} \tilde{\mathbf{F}}_{\alpha\delta} \\ &\quad + \nabla^\alpha A^\sigma \psi \tilde{R}_{\alpha\beta\gamma\sigma} + i\nabla_\gamma^A \psi \tilde{\mathbf{F}}_{\alpha\beta} + \lambda^2 T^3. \end{aligned}$$

Then in the above two formulas all terms cancel except for $\lambda\lambda T^3$. Hence, we obtain that $I_{11} + J_1 + II + III_1$ vanishes modulo $\lambda\lambda T^3$. This concludes the proof of the T^3 -equations. \square

The equations for T^4 . These have the form

$$\begin{aligned} (\partial_t - \Delta_g) T_{\alpha\beta}^4 &= -\text{Ric}_{\alpha\delta} T^{4,\delta}_\beta + \text{Ric}_{\beta\delta} T^{4,\delta}_\alpha - R_{\beta\alpha\sigma\delta} T^{2,\sigma\delta} \\ &\quad - \text{Re}(\nabla^{A,\sigma} \psi \overline{T_{\alpha\beta,\sigma}^3}) - V^\gamma \text{Im}(\lambda_\gamma^\sigma \overline{T_{\alpha\beta,\sigma}^3}) + V^\gamma \text{Im}(T_{\gamma\alpha,\sigma}^3 \bar{\lambda}_\beta^\sigma). \end{aligned}$$

By the A -equations we have

$$\begin{aligned} (\partial_t - \Delta_g) T_{\alpha\beta}^4 &= -[\Delta_g, \nabla_\alpha] A_\beta + [\Delta_g, \nabla_\beta] A_\alpha - \nabla_\alpha (\widetilde{\text{Ric}}_{\beta\delta} A^\delta) + \nabla_\beta (\widetilde{\text{Ric}}_{\alpha\delta} A^\delta) \\ &\quad + \nabla_\alpha \nabla^\sigma \tilde{\mathbf{F}}_{\beta\sigma} - \nabla_\beta \nabla^\sigma \tilde{\mathbf{F}}_{\alpha\sigma} - \Delta_g \tilde{\mathbf{F}}_{\alpha\beta} \\ &\quad - \partial_t \tilde{\mathbf{F}}_{\alpha\beta} + \nabla_\alpha [\text{Re}(\lambda_\beta^\gamma \overline{\nabla_\gamma^A \psi}) - \tilde{\mathbf{F}}_{\beta\delta} V^\delta] - \nabla_\beta [\text{Re}(\lambda_\alpha^\gamma \overline{\nabla_\gamma^A \psi}) - \tilde{\mathbf{F}}_{\alpha\delta} V^\delta] \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

For the commutator we use the Bianchi identities to compute

$$\begin{aligned} &- [\nabla^\sigma \nabla_\sigma, \nabla_\alpha] A_\beta + [\nabla^\sigma \nabla_\sigma, \nabla_\beta] A_\alpha \\ &= -\nabla^\sigma (R_{\sigma\alpha\beta\delta} A^\delta - R_{\sigma\beta\alpha\delta} A^\delta) - (R_{\sigma\alpha\beta\delta} - R_{\sigma\beta\alpha\delta}) \nabla^\sigma A^\delta - R^\sigma_{\alpha\sigma\delta} \nabla^\delta A_\beta + R^\sigma_{\beta\sigma\delta} \nabla^\delta A_\alpha \\ &= -\nabla^\sigma R_{\beta\alpha\sigma\delta} A^\delta - 2R_{\beta\alpha\sigma\delta} \nabla^\sigma A^\delta - \text{Ric}_{\alpha\delta} \nabla^\delta A_\beta + \text{Ric}_{\beta\delta} \nabla^\delta A_\alpha \\ &= -(\nabla_\beta \text{Ric}_{\alpha\delta} - \nabla_\alpha R_{\beta\delta}) A^\delta - R_{\beta\alpha\sigma\delta} \mathbf{F}^{\sigma\delta} - \text{Ric}_{\alpha\delta} (\mathbf{F}^\delta_\beta + \nabla_\beta A^\delta) + \text{Ric}_{\beta\delta} (\mathbf{F}^\delta_\alpha + \nabla_\alpha A^\delta) \\ &= -\nabla_\beta (\text{Ric}_{\alpha\delta} A^\delta) + \nabla_\alpha (\text{Ric}_{\beta\delta} A^\delta) - R_{\beta\alpha\sigma\delta} \mathbf{F}^{\sigma\delta} - \text{Ric}_{\alpha\delta} \mathbf{F}^\delta_\beta + \text{Ric}_{\beta\delta} \mathbf{F}^\delta_\alpha. \end{aligned}$$

We commute $\nabla_\alpha, \nabla_\beta$ with ∇^σ and use $\nabla_\alpha \tilde{\mathbf{F}}_{\beta\sigma} + \nabla_\beta \tilde{\mathbf{F}}_{\sigma\alpha} + \nabla_\sigma \tilde{\mathbf{F}}_{\alpha\beta} = 0$ to compute I_2 by

$$\begin{aligned} I_2 &= R_{\alpha\sigma\beta\gamma} \tilde{\mathbf{F}}^{\gamma\sigma} + R_{\alpha\sigma\sigma\gamma} \tilde{\mathbf{F}}_\beta^\gamma - R_{\beta\sigma\alpha\gamma} \tilde{\mathbf{F}}^{\gamma\sigma} - R_{\beta\sigma\sigma\gamma} \tilde{\mathbf{F}}_\alpha^\gamma \\ &= \text{Ric}_{\alpha\delta} \tilde{\mathbf{F}}_\beta^\delta - \text{Ric}_{\beta\delta} \tilde{\mathbf{F}}_\alpha^\delta + R_{\beta\alpha\sigma\delta} \tilde{\mathbf{F}}^{\sigma\delta}. \end{aligned}$$

Then we obtain

$$I_1 + I_2 = -\text{Ric}_{\alpha\delta} T^{4,\delta}_\beta + \text{Ric}_{\beta\delta} T^{4,\delta}_\alpha - R_{\beta\alpha\sigma\delta} T^{4,\sigma\delta}$$

For I_3 we compute $\partial_t \tilde{\mathbf{F}}_{\alpha\beta}$ first.

$$\partial_t \tilde{\mathbf{F}}_{\alpha\beta} = \text{Im}(\partial_t \lambda_{\alpha\sigma} \bar{\lambda}_\beta^\sigma - \partial_t \lambda_{\beta\sigma} \bar{\lambda}_\alpha^\sigma) + \partial_t g^{\sigma\mu} \text{Im}(\lambda_{\alpha\sigma} \bar{\lambda}_{\beta\mu})$$

By the g -equations and

$$i \partial_t^B \lambda_{\alpha\beta} + \nabla_\alpha^A \nabla_\beta^A \psi - i \lambda_\alpha^\gamma \text{Im}(\psi \bar{\lambda}_{\gamma\beta}) - i \lambda_\alpha^\gamma \nabla_\beta V_\gamma - i \lambda_\beta^\gamma \nabla_\alpha V_\gamma - i V^\gamma \nabla_\gamma^A \lambda_{\alpha\beta} = 0,$$

we have

$$\begin{aligned} \text{Im}(\partial_t \lambda_{\alpha\sigma} \bar{\lambda}_\beta^\sigma) &= -\text{Re}(B \lambda_{\alpha\sigma} \bar{\lambda}_\beta^\sigma - \nabla_\alpha^A \nabla_\sigma^A \psi \bar{\lambda}_\beta^\sigma) + \text{Im}(\lambda_\alpha^\gamma \bar{\lambda}_\beta^\sigma) (\text{Im}(\psi \bar{\lambda}_{\gamma\sigma}) + \nabla_\sigma V_\gamma) \\ &\quad + \nabla_\alpha (\tilde{\mathbf{F}}_\beta^\gamma V_\gamma) - V^\gamma \text{Im}(\lambda_{\gamma\sigma} \overline{\nabla_\alpha^A \lambda_\beta^\sigma}) + V^\gamma \text{Im}(T_{\gamma\alpha,\sigma}^1 \bar{\lambda}_\beta^\sigma) \end{aligned}$$

Then we rewrite the expression $\partial_t \tilde{\mathbf{F}}_{\alpha\beta}$ as

$$\begin{aligned} \partial_t \tilde{\mathbf{F}}_{\alpha\beta} &= \nabla_\alpha \text{Re}(\nabla_\sigma^A \psi \bar{\lambda}_\beta^\sigma) - \nabla_\beta \text{Re}(\nabla_\sigma^A \psi \bar{\lambda}_\alpha^\sigma) - \text{Re}(\nabla^{A,\sigma} \psi \overline{T_{\alpha\beta,\sigma}^3}) \\ &\quad + \nabla_\alpha (\tilde{\mathbf{F}}_\beta^\gamma V_\gamma) - \nabla_\beta (\tilde{\mathbf{F}}_\alpha^\gamma V_\gamma) - V^\gamma \text{Im}(\lambda_\gamma^\sigma \overline{T_{\alpha\beta,\sigma}^3}) + V^\gamma \text{Im}(T_{\gamma\alpha,\sigma}^3 \bar{\lambda}_\beta^\sigma) \\ &= \nabla_\alpha \tilde{\mathbf{F}}_{0\beta} - \nabla_\beta \tilde{\mathbf{F}}_{0\alpha} + \nabla \psi T^3 + \lambda V T^3. \end{aligned}$$

Hence, we have

$$\begin{aligned} I_3 &= \nabla_\alpha \text{Re}(\nabla_\sigma^A \psi \bar{\lambda}_\beta^\sigma) - \nabla_\beta \text{Re}(\nabla_\sigma^A \psi \bar{\lambda}_\alpha^\sigma) - \text{Re}(\nabla^{A,\sigma} \psi \overline{T_{\alpha\beta,\sigma}^3}) \\ &\quad + \nabla_\alpha (\tilde{\mathbf{F}}_\beta^\gamma V_\gamma) - \nabla_\beta (\tilde{\mathbf{F}}_\alpha^\gamma V_\gamma) - V^\gamma \text{Im}(\lambda_\gamma^\sigma \overline{T_{\alpha\beta,\sigma}^3}) + V^\gamma \text{Im}(T_{\gamma\alpha,\sigma}^3 \bar{\lambda}_\beta^\sigma) \\ &\quad + \nabla_\alpha [\text{Re}(\lambda_\beta^\gamma \overline{\nabla_\gamma^A \psi}) - \tilde{\mathbf{F}}_{\beta\delta} V^\delta] - \nabla_\beta [\text{Re}(\lambda_\alpha^\gamma \overline{\nabla_\gamma^A \psi}) - \tilde{\mathbf{F}}_{\alpha\delta} V^\delta] \\ &= -\text{Re}(\nabla^{A,\sigma} \psi \overline{T_{\alpha\beta,\sigma}^3}) - V^\gamma \text{Im}(\lambda_\gamma^\sigma \overline{T_{\alpha\beta,\sigma}^3}) + V^\gamma \text{Im}(T_{\gamma\alpha,\sigma}^3 \bar{\lambda}_\beta^\sigma) \end{aligned}$$

This concludes the proof T^4 -equations. □

9. The Reconstruction of the Flow

In this last section we close the circle of ideas in this paper, and prove that one can start from the good gauge solution given by Theorem 2.5, and reconstruct the flow at the level of d -dimensional embedded submanifolds. For completeness, we provide here another, more complete statement of our main theorem.

Theorem 9.1. (Small data local well-posedness) *Let $d \geq 2$ and $s > \frac{d}{2}$. Consider the skew mean curvature flow (1.1) for maps F from \mathbb{R}^d to the Euclidean space $(\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$ with initial data Σ_0 which, in some coordinates, has a metric g_0 satisfying $\| |D|^{\sigma_d}(g_0 - I_d) \|_{H^{s+1-\sigma_d}} \leq \epsilon_0$ and mean curvature $\| \mathbf{H}_0 \|_{H^s(\Sigma_0)} \leq \epsilon_0$. In addition, we assume that $\| g_0 - I_d \|_{Y_0^{l_0}} \lesssim \epsilon_0$ in dimension $d = 2$.*

If $\epsilon_0 > 0$ is sufficiently small, then there exists a unique solution

$$F : \mathbb{R}^d \times [0, 1] \rightarrow (\mathbb{R}^{d+2}, g_{\mathbb{R}^{d+2}})$$

which, when represented in harmonic coordinates at the initial time and heat coordinates dynamically, has regularity

$$\partial_x^2 F, \partial_t F \in C([0, 1]; H^s(\mathbb{R}^d)),$$

and induced metric and mean curvature

$$|D|^{\sigma_d}(g - I_d) \in C([0, 1]; H^{s+2-\sigma_d}(\mathbb{R}^d)), \quad \mathbf{H} \in C([0, 1]; H^s(\mathbb{R}^d)).$$

In addition the mean curvature satisfies the bounds

$$\| \lambda \|_{L^2 X^s} + \| (h, A) \|_{\mathcal{E}^s} \lesssim \| \lambda_0 \|_{H^s} + \| h_0 \|_{Y_0^{s+2}}.$$

where λ and A are expressed using the Coulomb gauge initially and the heat gauge dynamically in the normal bundle $N\Sigma_t$.

We prove the theorem in several steps.

9.1. The Moving Frame

Once we have the initial data (h_0, A_0, λ_0) which is small in $\mathcal{H}^s \times H^s$ by Proposition 4.1 and 4.2, Theorem 2.5 yields the good gauge local solution λ , along with the associated derived variables (h, A) . But this does not yet give us the actual maps F .

Here we undertake the task of reconstructing the frame (F_α, m) . For this we use the system consisting of (2.6) and (2.14), viewed as a linear ode. We recall these equations here:

$$\begin{cases} \partial_\alpha F_\beta = \Gamma_{\alpha\beta}^\gamma F_\gamma + \text{Re}(\lambda_{\alpha\beta} \bar{m}), \\ \partial_\alpha^A m = -\lambda_\alpha^\gamma F_\gamma, \end{cases} \tag{9.1}$$

respectively

$$\begin{cases} \partial_t F_\alpha = -\operatorname{Im}(\partial_\alpha^A \psi \bar{m} - i \lambda_{\alpha\gamma} V^\gamma \bar{m}) + [\operatorname{Im}(\psi \bar{\lambda}_\alpha^\gamma) + \nabla_\alpha V^\gamma] F_\gamma, \\ \partial_t^B m = -i(\partial^{A,\alpha} \psi - i \lambda_\gamma^\alpha V^\gamma) F_\alpha. \end{cases} \quad (9.2)$$

We start with the frame at time $t = 0$, which already is known to solve (9.1), and has the following properties:

- (i) *Orthogonality*, $F_\alpha \perp m$, $\langle m, m \rangle = 2$, $\langle m, \bar{m} \rangle = 0$ and consistency with the metric $g_{\alpha\beta} = \langle F_\alpha, F_\beta \rangle$.
- (ii) *Integrability*, $\partial_\beta F_\alpha = \partial_\alpha F_\beta$.
- (iii) *Consistency* with the second fundamental form and the connection A :

$$\partial_\alpha F_\beta \cdot m = \lambda_{\alpha\beta}, \quad \langle \partial_\alpha m, m \rangle = -2i A_\alpha.$$

Next we extend this frame to times $t > 0$ by simultaneously solving the pair of equations (9.1) and (9.2).

9.1.1. The Solvability of (9.1) and (9.2) The system consisting of (9.1) and (9.2) is overdetermined, and the necessary and sufficient condition for existence of solutions is provided by Frobenius' theorem. We now verify these compatibility conditions in two steps:

- (a) Compatibility conditions for the system (9.1) at fixed time. Here, by $T_{\alpha\beta\mu\nu}^2 = 0$, $T_{\alpha\beta,\gamma}^3 = 0$, $T_{\alpha\beta}^4 = 0$ and we have

$$\partial_\alpha(\Gamma_{\beta\gamma}^\sigma F_\sigma + \operatorname{Re}(\lambda_{\beta\gamma} \bar{m})) - \partial_\beta(\Gamma_{\alpha\gamma}^\sigma F_\sigma + \operatorname{Re}(\lambda_{\alpha\gamma} \bar{m})) = 0,$$

and

$$\partial_\alpha(i A_\beta m + \lambda_\beta^\sigma F_\sigma) - \partial_\beta(i A_\alpha m + \lambda_\alpha^\sigma F_\sigma) = 0,$$

as needed.

- (b) Compatibility conditions between the system (9.1) and (9.2). By (9.1), (9.2) and (8.15) we have

$$\partial_t(i A_\alpha m + \lambda_\alpha^\sigma F_\sigma) - \partial_\alpha(i B m + i(\partial^{A,\sigma} \psi - i \lambda_\gamma^\sigma V^\gamma) F_\sigma) = 0$$

and

$$\begin{aligned} & \partial_\beta[-\operatorname{Im}(\partial_\alpha^A \psi \bar{m} - i \lambda_{\alpha\gamma} V^\gamma \bar{m}) + [\operatorname{Im}(\psi \bar{\lambda}_\alpha^\gamma) + \nabla_\alpha V^\gamma] F_\gamma] \\ & - \partial_t[\Gamma_{\beta\alpha}^\gamma F_\gamma + \operatorname{Re}(\lambda_{\beta\alpha} \bar{m})] = 0. \end{aligned}$$

9.1.2. Solving the System (9.1)–(9.2) Locally Starting from the existing frame at time $t = 0$, we want to extend it forward in time by solving (9.2), while insuring that (9.1) remains valid. The difficulty is that we lack the uniform integrability in time for the coefficients in (9.2). However, in view of the local energy decay bounds for λ and ψ , we do know that locally we have $\lambda \in L_t^2 H^{s+\frac{1}{2}}$. We choose a distinguished coordinate, say x_d , and denote the remaining coordinates by x' . Then in view of Sobolev embeddings we have the local regularity

$$\partial\lambda \in C_{x_d} L_t^2 H_{x'}^{s-1} \cap L_{x_d}^2 L_t^2 C_{x'}$$

Thus on a “good” x_d slice we have $\partial\lambda \in L_t^2 C_{x'}$ and we can extend our frame forward in time as a continuous function, with $L_t^2 L_{x'}^\infty$ time derivatives and bounded spatial derivatives.

At fixed time all the coefficients are continuous so we can start from the above x_d slice and solve the system (9.1) globally in x , obtaining a global frame (F_α, m) which is locally Lipschitz in x and continuous in t . By Frobenius’ theorem, this solution must also satisfy (9.2) on any good x_d slice, which is a.e. Thus we have obtained the desired global frame (F_α, m) for $t \in [0, 1]$.

9.1.3. Propagating the Properties (i)–(iii) Here we show that the properties (i)–(iii) above also extend to all $t \in [0, 1]$. The properties (ii) and (iii) follow directly from the equations (9.1) and (9.2) once the orthogonality conditions in (i) are verified. We denote

$$\tilde{g}_{00} = \langle m, m \rangle, \quad \tilde{g}_{\alpha 0} = \langle F_\alpha, m \rangle, \quad \tilde{g}_{\alpha\beta} = \langle F_\alpha, F_\beta \rangle.$$

The first step is to propagate (i) forward in time on a good x_d slice. Indeed, by (9.2) and (8.13) we have

$$\begin{aligned} \partial_t \tilde{g}_{\alpha 0} &= -\frac{i}{2} (\partial_\alpha^A \bar{\psi} + i \bar{\lambda}_{\alpha\gamma} V^\gamma) (\tilde{g}_{00} - 2) - i (\partial^{A,\sigma} \bar{\psi} + i \bar{\lambda}_\gamma^\sigma V^\gamma) (g_{\alpha\sigma} - \tilde{g}_{\alpha\sigma}) \\ &\quad + \frac{i}{2} (\partial_\alpha^A \psi + i \lambda_{\alpha\gamma} V^\gamma) \langle \bar{m}, m \rangle + (\text{Im}(\psi \bar{\lambda}_\alpha^\gamma) + \nabla_\alpha V^\gamma) \tilde{g}_{\gamma 0} + i B \tilde{g}_{\alpha 0}, \\ \partial_t (\tilde{g}_{00} - 2) &= 2 \text{Im}[(\partial^{A,\alpha} \psi - i \lambda_\gamma^\alpha V^\gamma) \tilde{g}_{\alpha 0}], \\ \partial_t \langle m, \bar{m} \rangle &= -2i B \langle m, \bar{m} \rangle - 2i (\partial^{A,\alpha} \psi - i \lambda_\gamma^\alpha V^\gamma) \bar{\tilde{g}}_{\alpha 0}, \\ \partial_t (g_{\alpha\beta} - \tilde{g}_{\alpha\beta}) &= (\text{Im}(\psi \bar{\lambda}_\alpha^\gamma) + \nabla_\alpha V^\gamma) (g_{\beta\gamma} - \tilde{g}_{\beta\gamma}) + (\text{Im}(\psi \bar{\lambda}_\beta^\gamma) + \nabla_\beta V^\gamma) (g_{\alpha\gamma} - \tilde{g}_{\alpha\gamma}) \\ &\quad + \text{Im}(\partial_\alpha^A \psi \tilde{g}_{\beta 0} - i \lambda_{\alpha\gamma} V^\gamma \tilde{g}_{\beta 0}) - \text{Im}(\partial_\beta^A \psi \bar{\tilde{g}}_{\alpha 0} - i \lambda_{\beta\gamma} V^\gamma \bar{\tilde{g}}_{\alpha 0}). \end{aligned}$$

Viewed as a linear system of ode’s in time, these equations allow us to propagate (i) in time, given that it is satisfied at $t = 0$.

It remains to propagate (i) spatially. Using (9.1) we compute

$$\begin{aligned} \partial_\alpha \tilde{g}_{\beta 0} &= \Gamma_{\alpha\beta}^\gamma \tilde{g}_{\gamma 0} + \frac{1}{2} \lambda_{\alpha\beta} \langle \bar{m}, m \rangle + \frac{1}{2} \bar{\lambda}_{\alpha\beta} (\tilde{g}_{00} - 2) + \bar{\lambda}_\alpha^\gamma (g_{\beta\gamma} - \tilde{g}_{\beta\gamma}) + i A_\alpha \tilde{g}_{\beta 0}, \\ \partial_\alpha (\tilde{g}_{00} - 2) &= -2 \text{Re}(\lambda_\alpha^\gamma \tilde{g}_{\gamma 0}), \\ \partial_\alpha \langle m, \bar{m} \rangle &= -2i A_\alpha \langle m, \bar{m} \rangle - 2 \text{Re} \lambda_\alpha^\gamma \bar{\tilde{g}}_{\gamma 0}, \\ \partial_\alpha (g_{\beta\gamma} - \tilde{g}_{\beta\gamma}) &= \Gamma_{\alpha\beta}^\sigma (g_{\sigma\gamma} - \tilde{g}_{\sigma\gamma}) + \Gamma_{\alpha\gamma}^\sigma (g_{\sigma\beta} - \tilde{g}_{\sigma\beta}) + \text{Re}(\bar{\lambda}_{\beta\alpha} \tilde{g}_{\gamma 0} + \bar{\lambda}_{\gamma\alpha} \tilde{g}_{\beta 0}). \end{aligned}$$

By ode uniqueness and the choice of the initial data, the desired properties (i) for the frame are indeed propagated spatially.

9.1.4. The Sobolev Regularity of the Frame Here we show that our frame has the global regularity

$$\partial_x(F_\alpha, m) \in L^\infty H^s, \quad \partial_t(F_\alpha, m) \in L^\infty H^{s-1}.$$

As a consequence of the property (i), we directly see that $(F_\alpha, m) \in L^\infty$. From (9.1) it then follows that $\partial_x(F_\alpha, m) \in L^\infty$. This allows us to differentiate further in (9.1) and bound higher derivatives of the frame, up to the H^s regularity for $\partial_x(F_\alpha, m)$, which is imposed by λ . We can directly estimate this last norm. Precisely, by (9.1), (2.32) and Sobolev embeddings we have

$$\begin{aligned} \|\partial_x F_\alpha\|_{H^s} &\lesssim \|\Gamma F_\gamma + \lambda m\|_{H^s} \\ &\lesssim \|\Gamma\|_{H^s} \|F_\gamma\|_{L^\infty \cap \dot{H}^s} + \|\lambda\|_{H^s} \|m\|_{L^\infty \cap \dot{H}^s} \\ &\lesssim \epsilon_0(\|g\|_{L^\infty}^{1/2} + \|\partial_x F_\alpha\|_{H^s}) + \epsilon_0(1 + \|\partial_x m\|_{H^s}) \\ &\lesssim \epsilon_0(1 + \|\partial_x F_\alpha\|_{H^s} + \|\partial_x m\|_{H^s}) \end{aligned}$$

and

$$\begin{aligned} \|\partial_\alpha m\|_{H^s} &\lesssim \|Am + \lambda F_\gamma\|_{H^s} \\ &\lesssim \|A\|_{H^s} \|m\|_{L^\infty \cap \dot{H}^s} + \|\lambda\|_{H^s} \|F_\gamma\|_{L^\infty \cap \dot{H}^s} \\ &\lesssim \epsilon_0(1 + \|\partial_x F_\alpha\|_{H^s} + \|\partial_\alpha m\|_{H^s}). \end{aligned}$$

These imply the uniform bound

$$\|\partial_x F_\alpha\|_{H^s} + \|\partial_x m\|_{H^s} \lesssim \epsilon_0.$$

9.2. The Moving Manifold Σ_t

Here we propagate the full map F by simply integrating (2.13), i.e.

$$F(t) = F(0) + \int_0^t -\text{Im}(\psi \bar{m}) + V^\gamma F_\gamma ds.$$

Then by (9.1), we have

$$\partial_\alpha F(t) = \partial_\alpha F(0) + \int_0^t -\text{Im}(\partial_\alpha^A \psi \bar{m} - i\lambda_{\alpha\gamma} V^\gamma \bar{m}) + [\text{Im}(\psi \bar{\lambda}_\alpha^\gamma) + \nabla_\alpha V^\gamma] F_\gamma ds,$$

which is consistent with above definition of F_α .

9.3. The (SMCF) Equation for F

Here we establish that F solves (1.1). Using the relation $\lambda_{\alpha\beta} = \partial_\alpha \partial_\beta F \cdot m$ we have

$$\begin{aligned} -\text{Im}(\psi \bar{m}) &= -\text{Im}(g^{\alpha\beta} \partial_\alpha \partial_\beta F \cdot (v_1 + i v_2) (v_1 - i v_2)) \\ &= (\Delta_g F \cdot v_1) v_2 - (\Delta_g F \cdot v_2) v_1 \\ &= J(\Delta_g F)^\perp = \mathbf{JH}(F). \end{aligned}$$

This implies that the F solves (1.1).

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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